Quantum Electronics of Nanostructures

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Lecture 3

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- How to strictly derive quantum mechanical relations for electrical circuits.
- Flux qubit: RF-SQUID. Double-well potential.
- Three junction flux qubit

Important formulas (from Week 3)

$$H = T + U$$

$$H = \frac{\hat{Q}^2}{2C} + E_J\left(1 - \cos\hat{\varphi}\right)$$

$$\left[\hat{\varphi}, \hat{N}\right] = i$$

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

$$\hat{N} = -i\frac{\partial}{\partial \varphi}$$

This form of the charge operator can be used, when the charge is not quantised

$$H = \frac{\left(2e\hat{N} - C_g V_g\right)^2}{2C} + E_J\left(1 - \cos\hat{\varphi}\right)$$

$$\boxed{H = E_C\left(\hat{N} - n\right)^2 - E_J\cos\hat{\varphi}}$$

The Charging energy: $\boxed{E_C = \frac{4e^2}{2C}}$

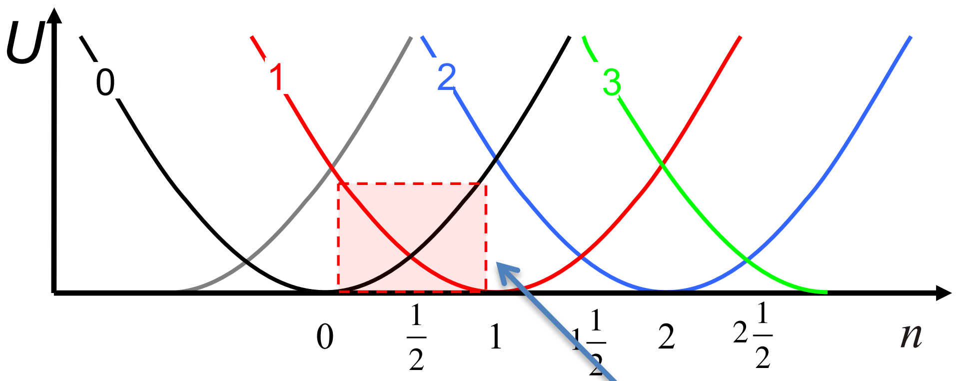
$$n = \frac{C_g V_g}{2e}$$

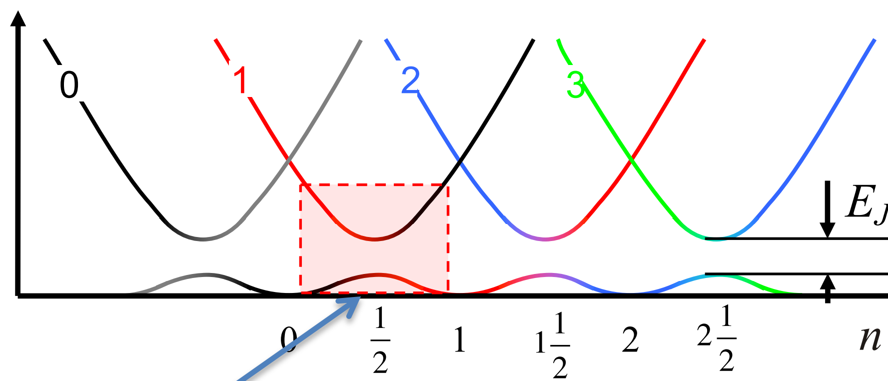
The Hamiltonian in the charge basis (from Week 3)

$$H = E_C (N-n)^2 |N\rangle\langle N| - \frac{1}{2} E_J \left(|N-1\rangle\langle N| + |N-1\rangle\langle N| \right)$$

$$H = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & E_C(-2-n)^2 & -\frac{1}{2}E_J & 0 & 0 & 0 & \cdots \\ \cdots & -\frac{1}{2}E_J & E_C(-1-n)^2 & -\frac{1}{2}E_J & 0 & 0 & \cdots \\ \cdots & 0 & -\frac{1}{2}E_J & E_C n^2 & -\frac{1}{2}E_J & 0 & \cdots \\ \cdots & 0 & 0 & -\frac{1}{2}E_J & E_C(1-n)^2 & -\frac{1}{2}E_J & \cdots \\ \cdots & 0 & 0 & 0 & -\frac{1}{2}E_J & E_C(2-n)^2 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The superconducting charge qubit (Week 3)





Two-level approximation

$$H = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & E_C(-2-n)^2 & -\frac{1}{2}E_J & 0 & 0 & 0 & \cdots \\ \cdots & -\frac{1}{2}E_J & E_C(-1-n)^2 & -\frac{1}{2}E_J & 0 & 0 & \cdots \\ \cdots & 0 & -\frac{1}{2}E_J & E_C n^2 & -\frac{1}{2}E_J & 0 & \cdots \\ \cdots & 0 & 0 & -\frac{1}{2}E_J & E_C(1-n)^2 & -\frac{1}{2}E_J & \cdots \\ \cdots & 0 & 0 & 0 & -\frac{1}{2}E_J & E_C(2-n)^2 & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

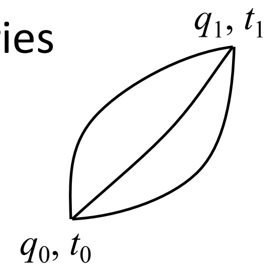
Lagrangian formalism

From Lagrangian to Hamiltonian: Why one can map mechanical systems to electrical

Lagrangian:

$$\mathcal{L} = T - V \qquad \mathcal{L}(q, \dot{q})$$

Trajectories



Hamlitonian's principle:

The dynamics of a physical system is determined by a variational problem for a functional based on a single function, the Lagrangian, which contains all physical information concerning the system and the forces acting on it.

$$\boxed{\int_{t_1}^{t_2} \delta\mathcal{L}\, dt = 0}$$

Cannonical variables:

$$q \qquad p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

Mechanical system:

$$x \qquad \mathcal{L} = \frac{m\dot{x}^2}{2} - U(x) \qquad p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

Hamiltonian

$$H = \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L}$$

$$H = \dot{x}(m\dot{x}) - \frac{m\dot{x}^2}{2} + U(x) = \frac{m\dot{x}^2}{2} + U(x)$$

In case of quadratic form of kinetic energy: $H = T + U$

How to strictly derive quantum mechanical relations for electrical circuits?

Canonical variables:

$$q \qquad p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \qquad \left[\hat{q}, \hat{p}\right] = -i\hbar$$

Mechanical system

$$x \qquad p = \frac{\partial \mathcal{L}}{\partial \dot{x}} \qquad \left[\hat{x}, \hat{p}\right] = -i\hbar$$

Electric circuit (charge-flux):

$$Q \qquad \Phi = \frac{\partial \mathcal{L}}{\partial \dot{Q}} \qquad \left[\hat{Q}, \hat{\Phi}\right] = -i\hbar$$

$$T = \frac{L\dot{Q}^2}{2} + E_J\left(1 - \cos\left(\frac{2\pi L\dot{Q}}{\Phi_0}\right)\right) \qquad U = \frac{Q^2}{2C}$$

$$H = \dot{Q}\frac{\partial \mathcal{L}}{\partial \dot{Q}} - \mathcal{L}$$

Electric circuit (flux-charge):

$$\Phi \qquad Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi} \qquad \left[\hat{\Phi}, \hat{Q}\right] = -i\hbar \qquad T = \frac{C\dot{\Phi}^2}{2} \qquad U = E_J\left(1 - \cos\left(\frac{2\pi\Phi}{\Phi_0}\right)\right)$$

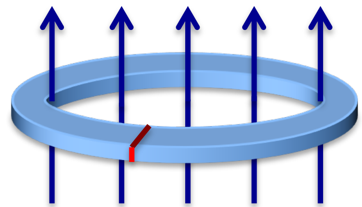
$$H = \dot{\Phi}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}} - \mathcal{L} \qquad H = \frac{C\dot{\Phi}^2}{2} + E_J\left(1 - \cos\left(\frac{2\pi\Phi}{\Phi_0}\right)\right) \qquad \mathcal{L} = \frac{C\dot{\Phi}^2}{2} - E_J\left(1 - \cos\left(\frac{2\pi\Phi}{\Phi_0}\right)\right)$$

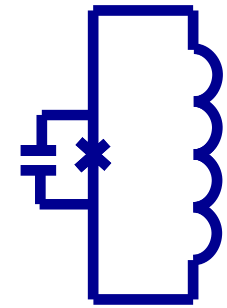
$$H = \frac{\hat{Q}^2}{2C} + E_J\left(1 - \cos\hat{\varphi}\right)$$

RF-SQUID qubit

The flux qubit: RF-SQUID

$$T = \frac{Q^2}{2C} = E_C \hat{N}^2 \qquad \hat{Q} = -i\hbar \frac{\partial}{\partial \Phi}$$





$$\hat{N} = \frac{\hat{Q}}{2e} = -i\frac{\hbar}{2e}\frac{\partial}{\partial \Phi} = -i\frac{\Phi_0}{2\pi}\frac{\partial}{\partial \Phi} = -i\frac{\partial}{\partial \varphi}$$

Condition for superconducting phases in a loop with inductance and Josephson junction:

$$H = -E_C \frac{\partial^2}{\partial \varphi_J^2} - E_J \cos \varphi_J + E_L \varphi_L^2 \qquad E_L = \left(\frac{\Phi_0}{2\pi}\right)^2 \frac{1}{2L}$$

$$\varphi_L + \varphi_J = \varphi_{ext}$$

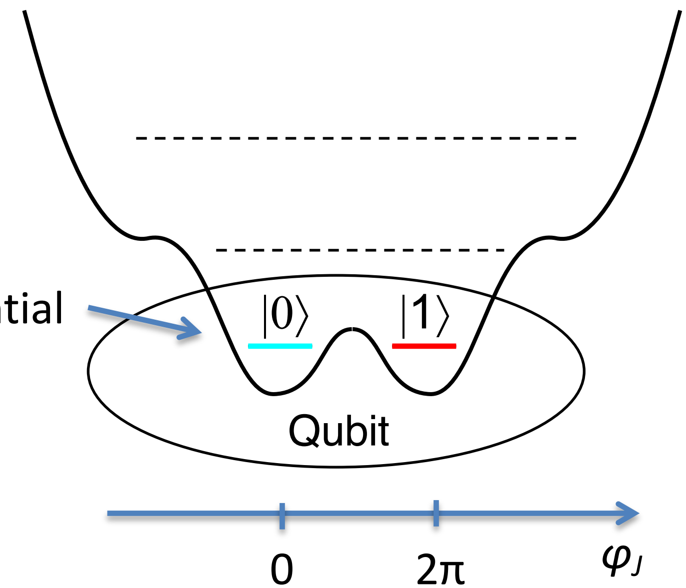
$$H = -E_C \frac{\partial^2}{\partial \varphi_J^2} - E_J \cos \varphi_J + E_L \left(\varphi_{ext} - \varphi_J\right)^2$$

The Hamiltonian of RF-SQUID qubit

$$H = -E_C \frac{\partial^2}{\partial \varphi_J^2} - E_J \cos\varphi_J + E_L \left(\varphi_{ext} - \varphi_J\right)^2$$

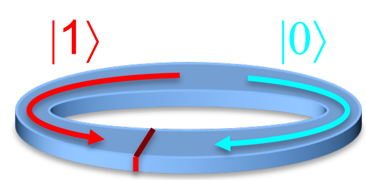
Compare with $H = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} - U(x)$ $\quad x \Leftrightarrow \varphi \quad m \Leftrightarrow C \quad \hat{N}_J = -i\frac{\partial}{\partial \varphi_J}$

Double well potential



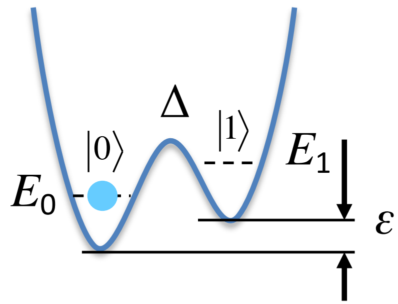
$$H|0\rangle = E_0|0\rangle$$

$$H|1\rangle = E_1|1\rangle$$

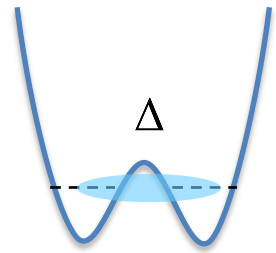


Simplified Hamiltonian

$$H = -E_C \frac{\partial^2}{\partial \varphi_J^2} - E_J \cos\varphi_J + E_L \left(\varphi_{ext} - \varphi_J\right)^2$$



$H_0 \approx -\frac{\varepsilon}{2}\sigma_z \quad \Psi_0 \approx |0\rangle$



$H_1 = -\frac{\Delta}{2}\sigma_x$

Δ is the tunneling energy between two wells

Degeneracy point: $E_L \varphi_{ext}^2 = E_L \left(2\pi - \varphi_{ext}\right)^2 \qquad \varphi_{ext} = \pi$

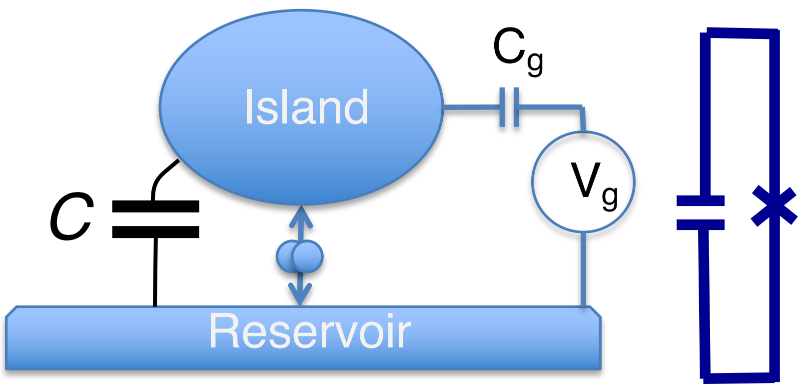
$\varepsilon = -2E_L \delta\varphi$, where $\delta\varphi = \varphi_{ext} - \pi$

$$\boxed{H \approx -\frac{\varepsilon}{2}\sigma_z - \frac{\Delta}{2}\sigma_x}$$

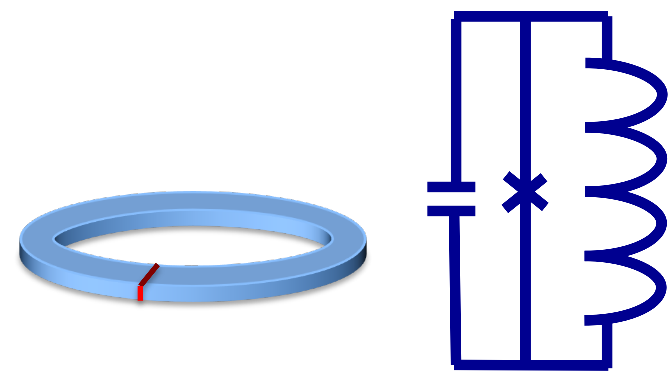
The charge vs flux qubit

$$H \approx -\frac{\varepsilon}{2}\sigma_z - \frac{\Delta}{2}\sigma_x$$

The charge qubit



The flux qubit



Charge states (0, 2e): $|0\rangle$, $|1\rangle$

Cooper pair tunneling in/out

Charging energy: E_C

Tunneling energy: E_J

Controlled by V

$\varepsilon = -2E_C\delta N$

Flux states (0, 2π): $|0\rangle$, $|1\rangle$

Flux tunneling tunneling in/out

Magnetic energy: E_L

Tunneling energy: Δ

Controlled by B

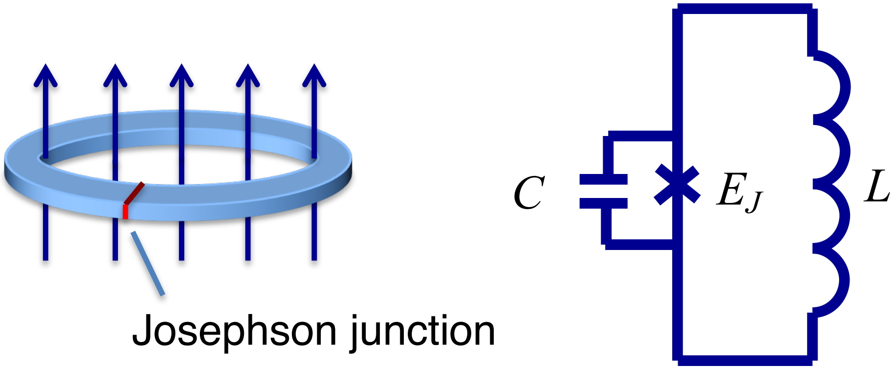
$\varepsilon = -2E_L\delta\varphi$

The flux qubit: RF-SQUID

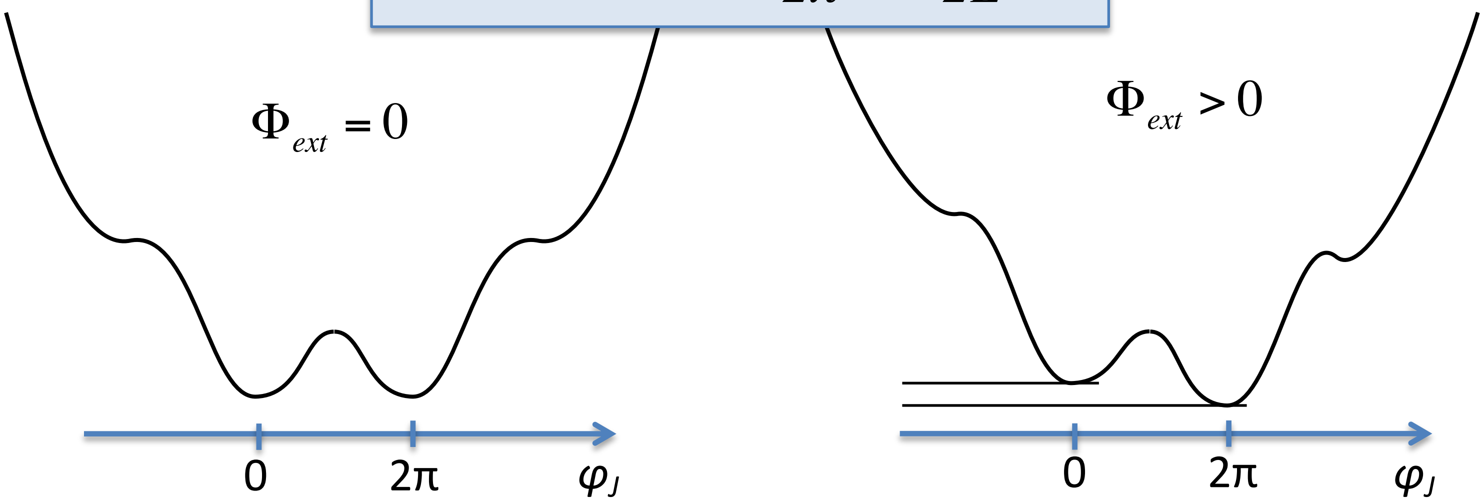
$$U = -E_J \cos\varphi_J + \frac{\Phi_L^2}{2L}$$

$$\varphi_L = \frac{2\pi\Phi_L}{\Phi_0} \qquad \varphi_J = \frac{2\pi\Phi_J}{\Phi_0}$$

$$\Phi_{ext} - \Phi_J - \Phi_L = 0 \qquad \qquad \varphi_J + \varphi_L = \varphi_{ext}$$

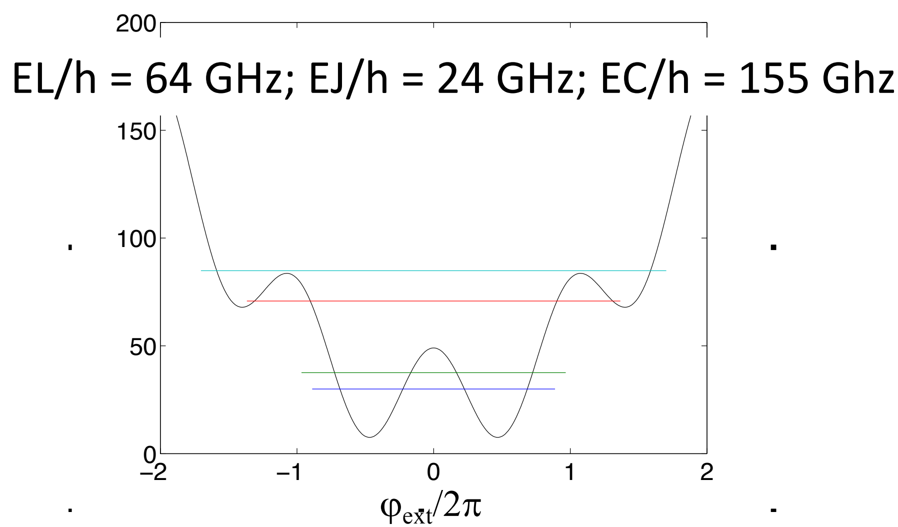


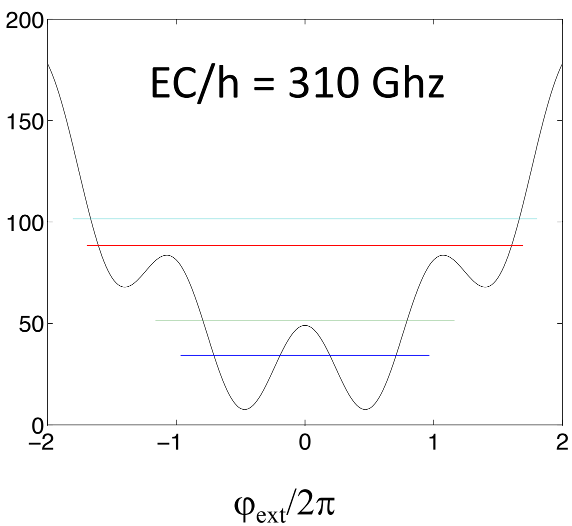
$$U = -E_J \cos\varphi_J + \frac{\Phi_0}{2\pi}\frac{(\varphi_{ext} - \varphi_J)^2}{2L}$$

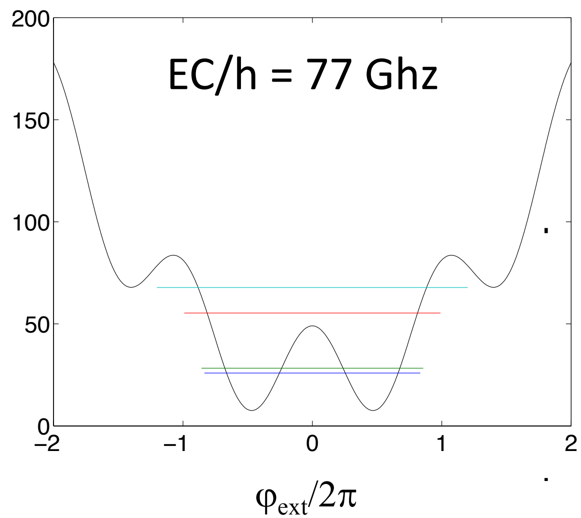


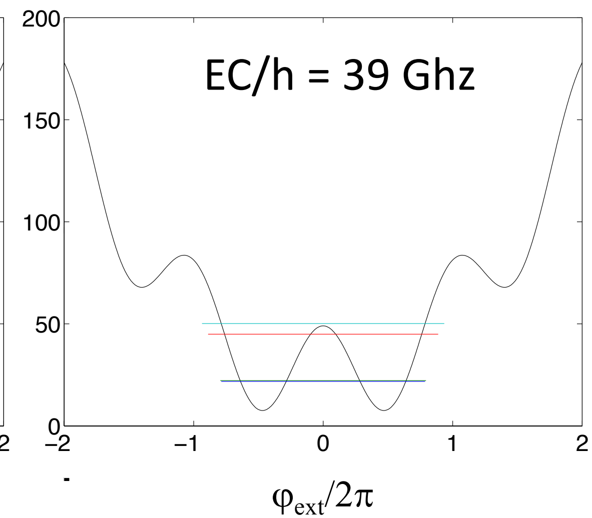
Two preferable states: $\varphi_J = 0$ and $\varphi_J = 2\pi$

The RF-SQUID flux qubit: particle mass dependence



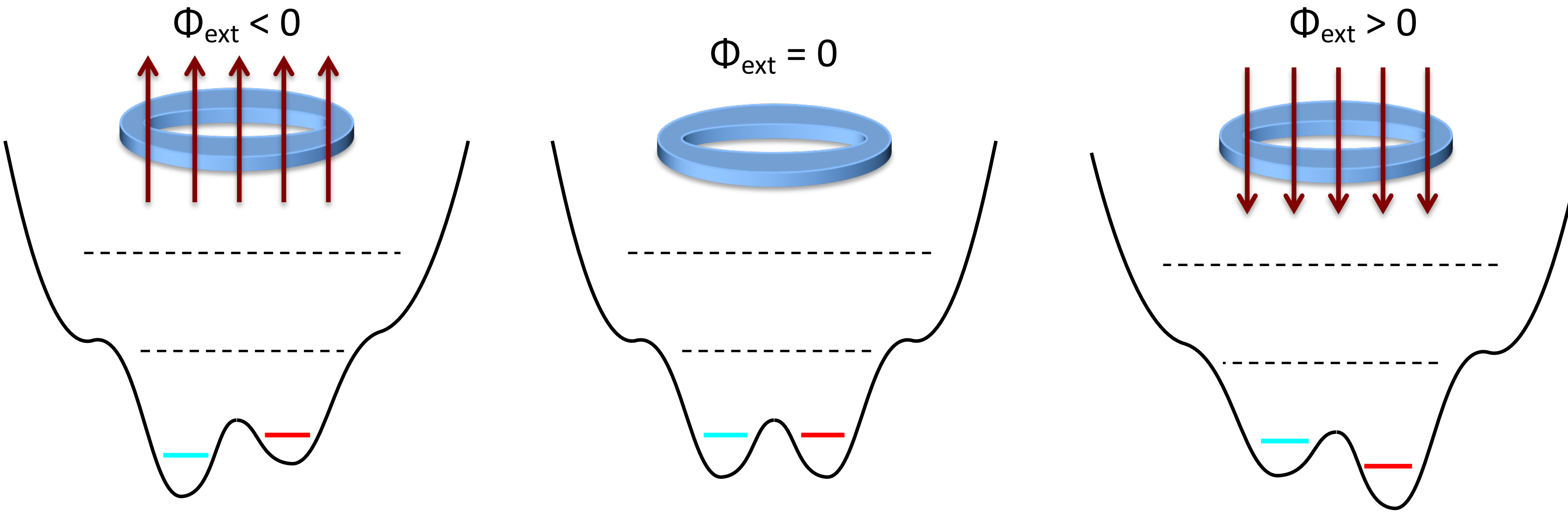






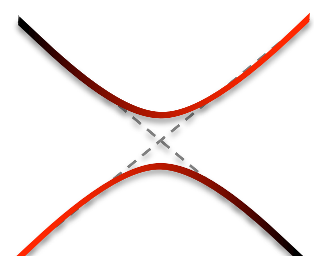
Level position and tunneling energy depend on the particle mass

The flux qubit

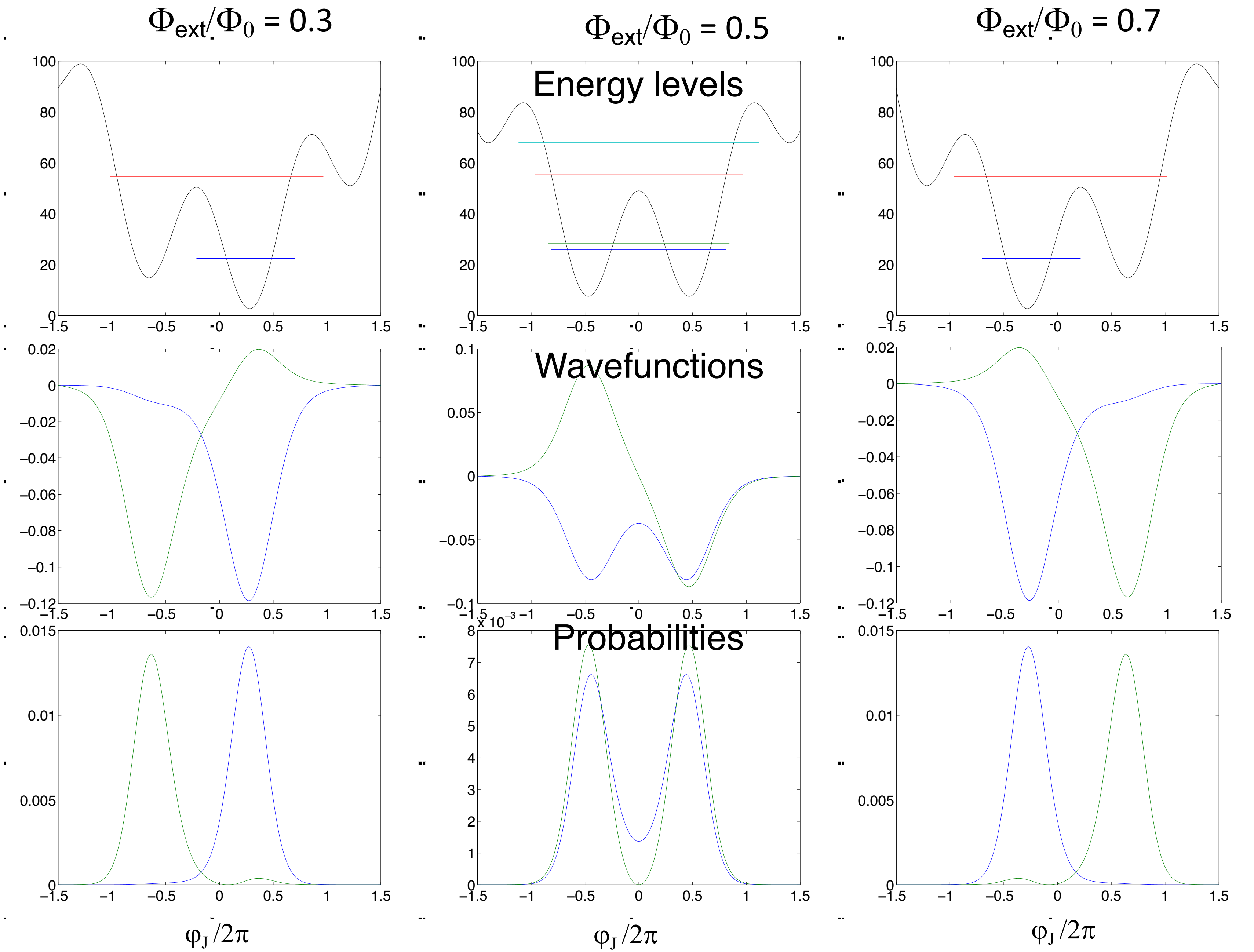


$$H = \begin{pmatrix} -E_L n_{ext} & -\Delta/2 \\ -\Delta/2 & E_L n_{ext} \end{pmatrix} = -\frac{\varepsilon}{2}\sigma_z - \frac{\Delta}{2}\sigma_x$$

$$\varepsilon = 2E_L n_{ext}$$

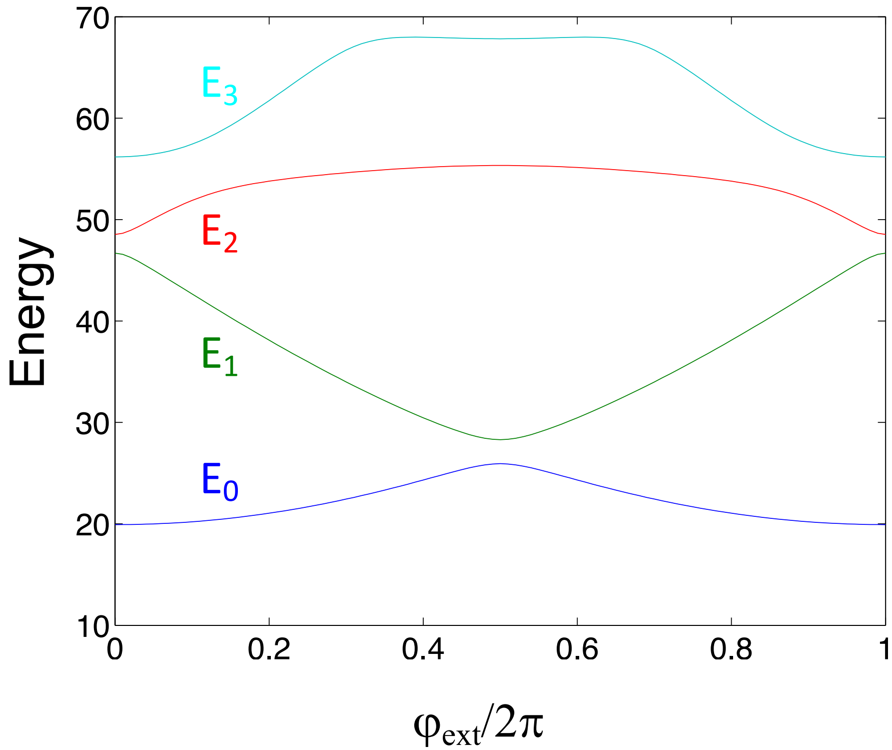


EL/h = 64 GHz; EJ/h = 24 GHz; EC/h = 77 Ghz;

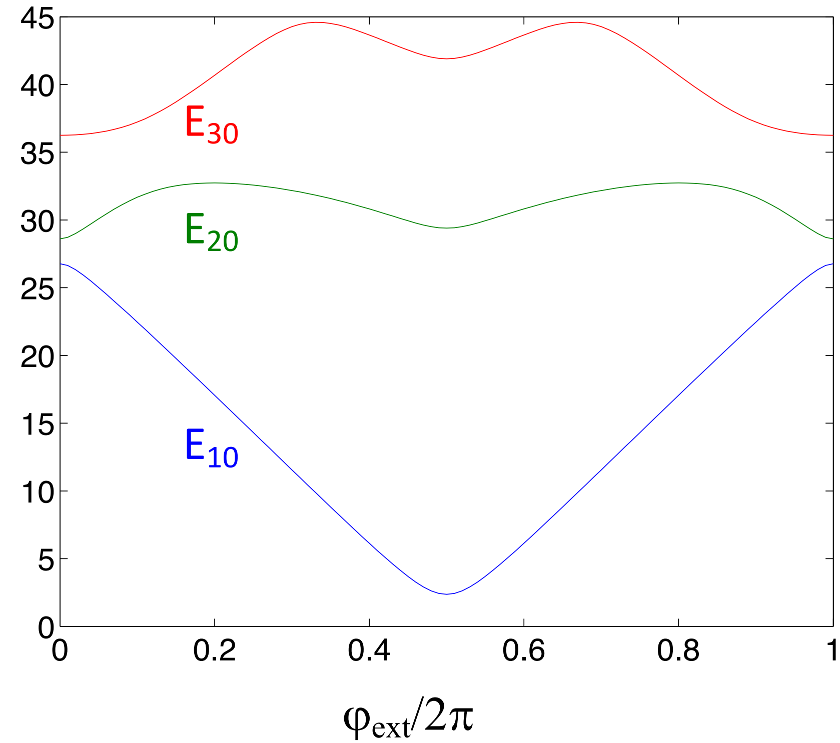


Energies versus external flux (phase)

Energies of the levels



Transition energies

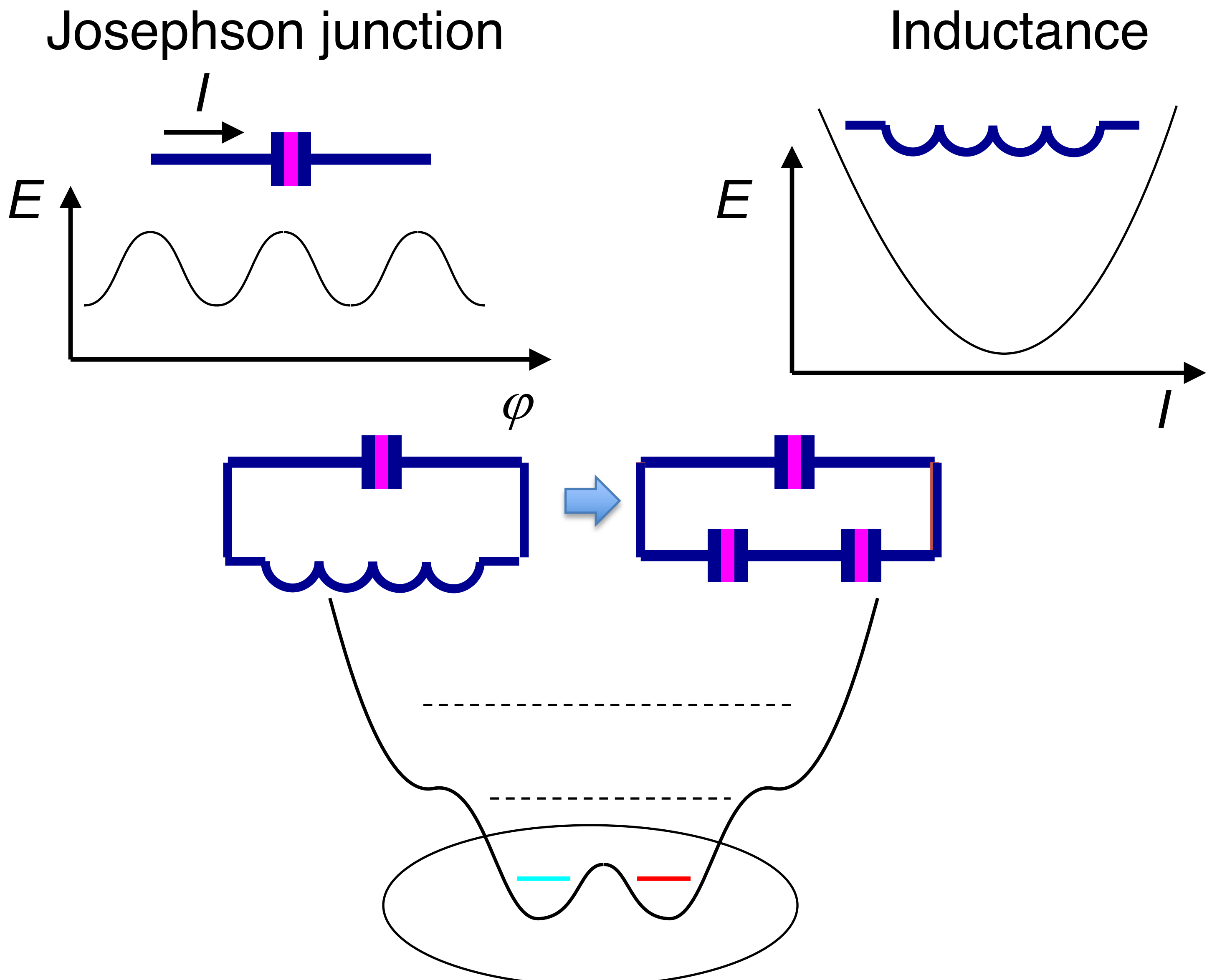


Two-level approximation:

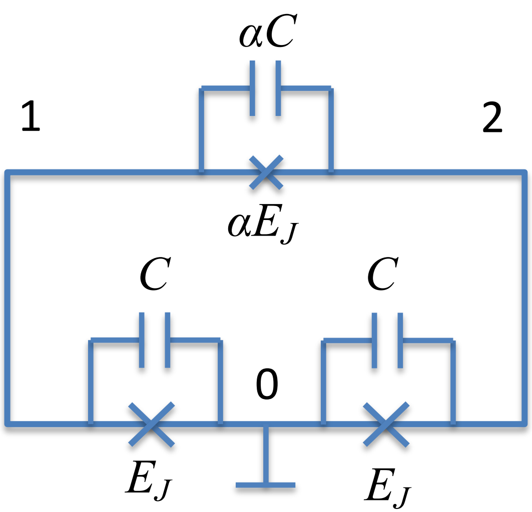
$$H \approx -\frac{\varepsilon}{2}\sigma_z - \frac{\Delta}{2}\sigma_x$$

The three-junction flux qubit

The flux qubit



The three-junction flux qubit



$$\varphi_{01} + \varphi_{12} + \varphi_{20} = \frac{2\pi}{\Phi_0}\Phi_{ext} = \varphi_{ext}$$

$$U = E_J\left(1 - \cos\varphi_{01}\right) + \alpha E_J\left(1 - \cos\varphi_{12}\right) + E_J\left(1 - \cos\varphi_{20}\right)$$

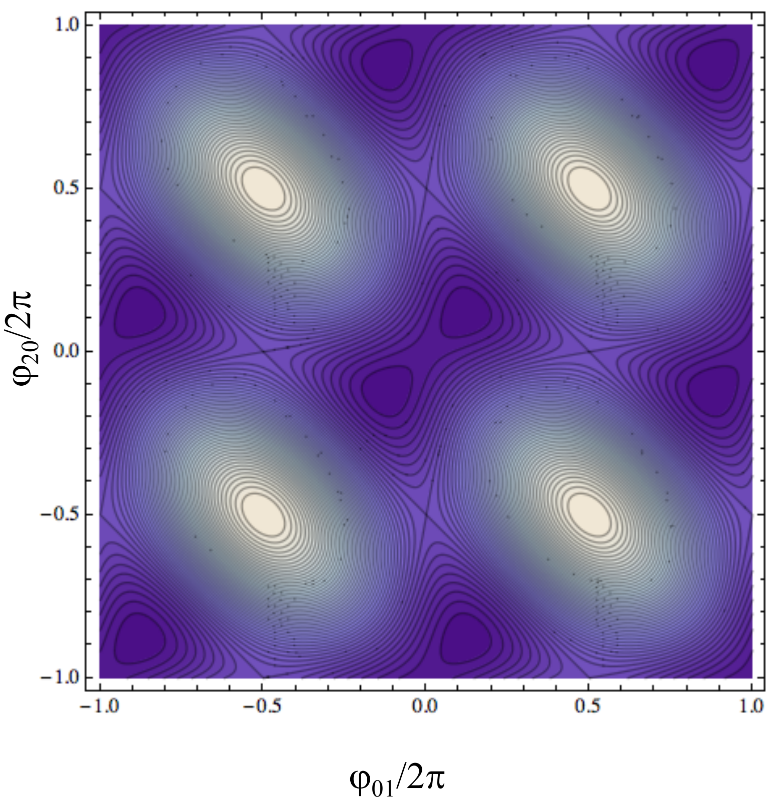
$$U = E_J\left[\left(1 - \cos\varphi_{01}\right) + \alpha\left(1 - \cos\left(\varphi_{ext} - \varphi_{01} - \varphi_{20}\right)\right) + \left(1 - \cos\varphi_{20}\right)\right]$$

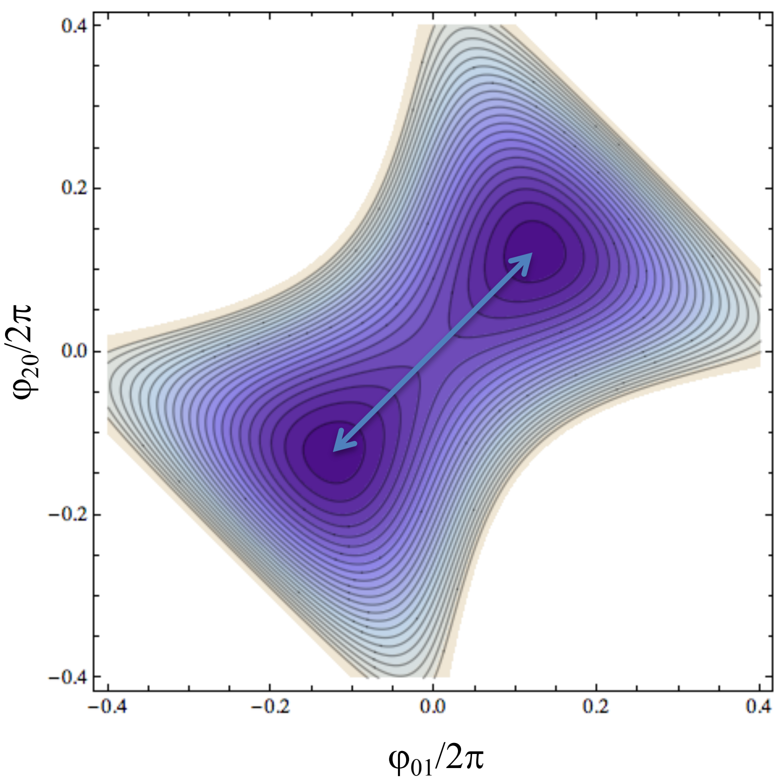
$$U = E_J\left[2 + \alpha - \cos\varphi_{01} - \cos\varphi_{20} - \alpha\cos\left(\varphi_{ext} - \varphi_{01} - \varphi_{20}\right)\right]$$

Josephson potential of the three-junction loop

$$\alpha = 0.7$$

$$U/E_J = \left[2 + \alpha - \cos\varphi_{01} - \cos\varphi_{20} - \alpha\cos\left(\varphi_{ext} - \varphi_{01} - \varphi_{20}\right)\right]$$

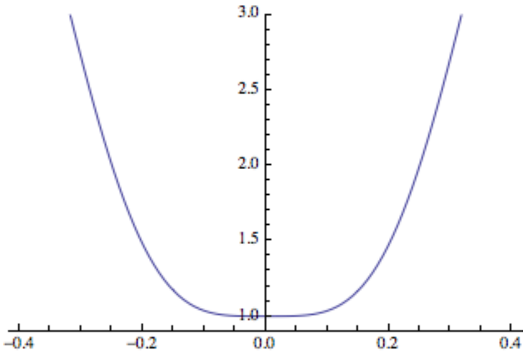




Shape of Josephson potential vs alpha

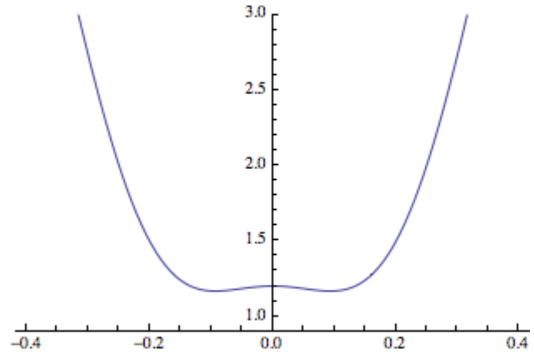
$\varphi_{01}/2\pi = \varphi_{20}/2\pi$

$\alpha = 0.5$



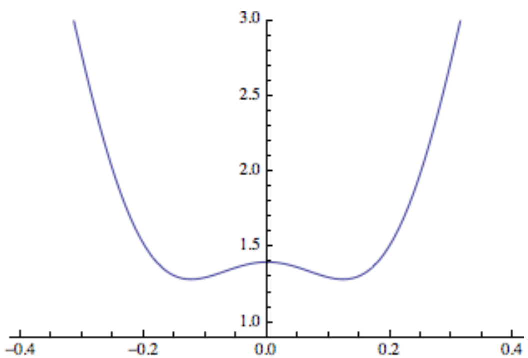
$\varphi_{01}/2\pi$

$\alpha = 0.6$



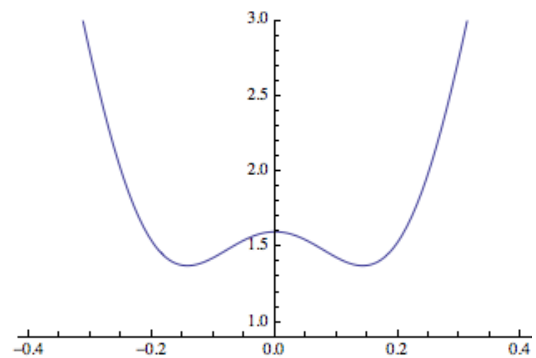
$\varphi_{01}/2\pi$

$\alpha = 0.7$



$\varphi_{01}/2\pi$

$\alpha = 0.8$

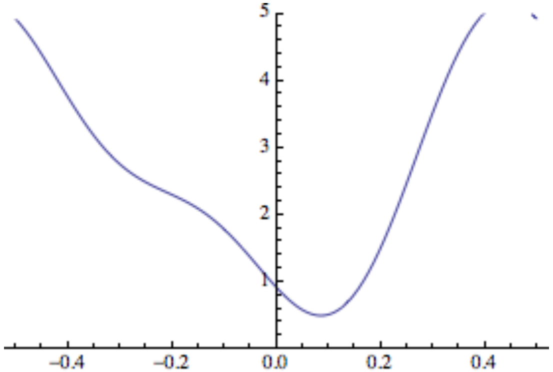


$\varphi_{01}/2\pi$

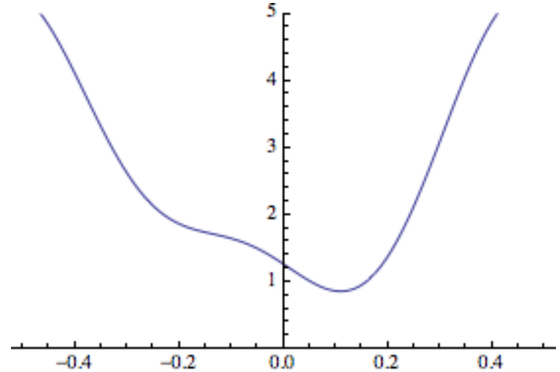
Josephson potential of the biased flux qubit

$\alpha = 0.7$

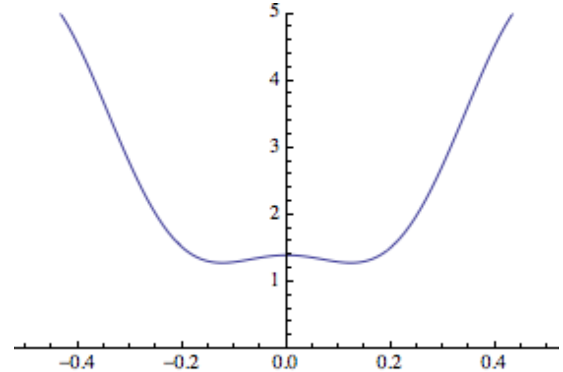
$\varphi_{ext}/2\pi = 0.3$



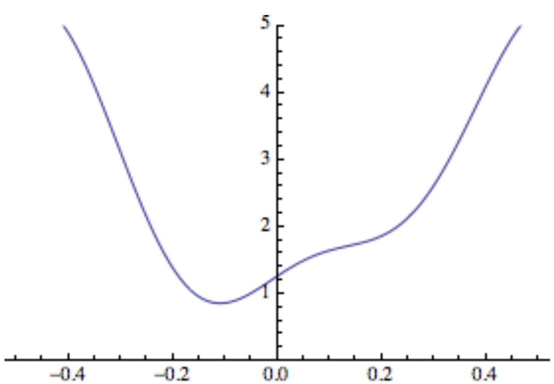
$\varphi_{ext}/2\pi = 0.4$



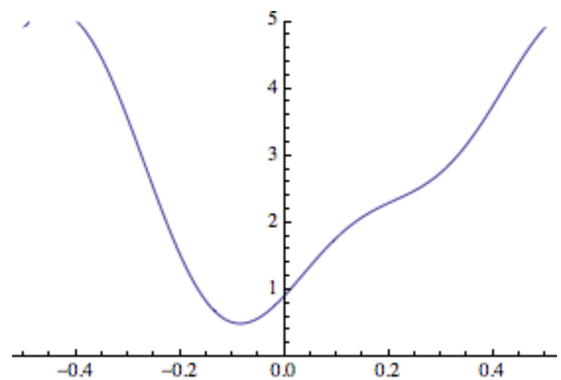
$\varphi_{ext}/2\pi = 0.5$



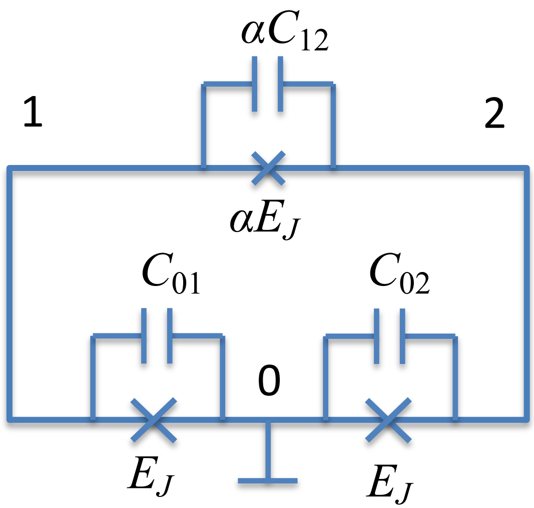
$\varphi_{ext}/2\pi = 0.6$



$\varphi_{ext}/2\pi = 0.7$



Flux biased qubit



Josephson potential:

$$U = E_J\left[2+\alpha-\cos\varphi_{01}-\cos\varphi_{20}-\alpha\cos\left(\varphi_{ext}-\varphi_{01}-\varphi_{20}\right)\right]$$

Charge:

$$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

Potential:

$$\vec{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

Capacitance matrix:

$$C = \begin{pmatrix} C_{01}+C_{12} & -C_{12} \\ -C_{12} & C_{02}+C_{12} \end{pmatrix}$$

$$\vec{n} = \frac{C\vec{V}}{2e} \qquad \vec{V} = 2eC^{-1}\vec{n}$$

Electrostatic energy: $T = \dfrac{(2e)^2}{2}\vec{n}C^{-1}\vec{n}$

The Hamiltonian:

$$\boxed{H = \frac{(2e)^2}{2}\vec{n}C^{-1}\vec{n} + E_J\left[2+\alpha-\cos\varphi_{01}-\cos\varphi_{20}-\alpha\cos\left(\varphi_{ext}-\varphi_{01}-\varphi_{20}\right)\right]}$$

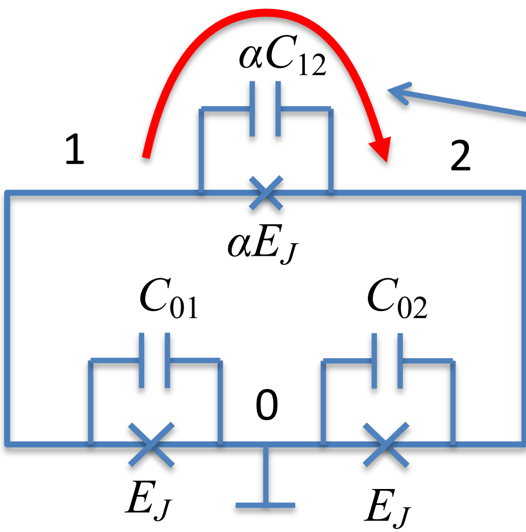
Phase operators

$$\cos\hat{\varphi} = \frac{e^{i\hat{\varphi}} + e^{-i\hat{\varphi}}}{2} \qquad e^{i\hat{\varphi}} = |N\rangle\langle N-1| \qquad e^{-i\hat{\varphi}} = |N-1\rangle\langle N|$$

$$\boxed{\cos\hat{\varphi}_{01} = \frac{1}{2}\left(|N_1\rangle\langle N_1-1| + |N_1-1\rangle\langle N_1|\right)} \qquad \boxed{\cos\hat{\varphi}_{20} = \frac{1}{2}\left(|N_2\rangle\langle N_2-1| + |N_2-1\rangle\langle N_2|\right)}$$

$$\cos\left(\varphi_{ext} - \hat{\varphi}_{01} - \hat{\varphi}_{20}\right) = \frac{e^{i(\varphi_{ext}-\hat{\varphi}_{01}-\hat{\varphi}_{20})} + e^{-i(\varphi_{ext}-\hat{\varphi}_{01}-\hat{\varphi}_{20})}}{2}$$

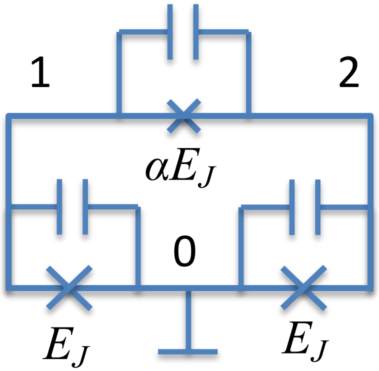
$$e^{i(\varphi_{ext}-\hat{\varphi}_{01}-\hat{\varphi}_{20})} = e^{i\varphi_{ext}} e^{-i\hat{\varphi}_{01}} e^{-i\hat{\varphi}_{20}} = e^{i\varphi_{ext}} |N_1-1\rangle\langle N_1||N_2\rangle\langle N_2-1| = e^{i\varphi_{ext}} \boxed{|N_1-1, N_2\rangle\langle N_1, N_2-1|}$$



Physical meaning: tunneling of a charge quantum (Cooper pair) from island 1 to island 2

$$\boxed{\cos\left(\varphi_{ext} - \hat{\varphi}_{01} - \hat{\varphi}_{20}\right) = \frac{1}{2}\left(e^{i\varphi_{ext}} |N_1-1, N_2\rangle\langle N_1, N_2-1| + e^{-i\varphi_{ext}} |N_1, N_2-1\rangle\langle N_1-1, N_2|\right)}$$

The flux qubit Hamiltonian in the charge basis



$$\boxed{H=\frac{(2e)^2}{2}\vec{n}C^{-1}\vec{n}+E_J\left[2+\alpha-\cos\varphi_{01}-\cos\varphi_{20}-\alpha\cos\left(\varphi_{ext}-\varphi_{01}-\varphi_{20}\right)\right]}$$

$$U(n_1,n_2)=\frac{(2e)^2}{2}\begin{pmatrix} n_1 & n_2 \end{pmatrix}C^{-1}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

$$H=\begin{pmatrix}
\ddots & & & & & & \\
& U(-1,0) & -\frac{\alpha E_J}{2}e^{i\varphi_{ext}} & -\frac{E_J}{2} & 0 & 0 & \\
& -\frac{\alpha E_J}{2}e^{-i\varphi_{ext}} & U(0,-1) & -\frac{E_J}{2} & 0 & 0 & \\
& -\frac{E_J}{2} & -\frac{E_J}{2} & U(0,0) & -\frac{E_J}{2} & -\frac{E_J}{2} & \\
& 0 & 0 & -\frac{E_J}{2} & U(0,1) & -\frac{\alpha E_J}{2}e^{i\varphi_{ext}} & \\
& 0 & 0 & -\frac{E_J}{2} & -\frac{\alpha E_J}{2}e^{-i\varphi_{ext}} & U(1,0) & \\
& & & & & & \ddots
\end{pmatrix}$$