Superconducting Quantum Technologies

Oleg Astafiev

Revision lecture

Examination

Friday, June 5

Start: 12:30
End: 15:30 (upload solutions)

Acceptable formats:
- Scan (photo)
- LaTex
- Word

Technical questions can be asked via zoom

Main topics

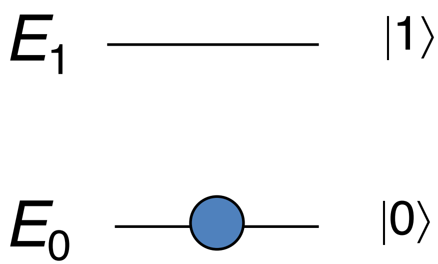
- Quantum bits and control of quantum states
- Practical realisation of quantum systems
- Quantum mechanics of electrical circuits
  Two-level systems in resonators
- Dissipative quantum dynamics
- Measurement circuit

Additional topic to discuss today: Selection rules in qubits

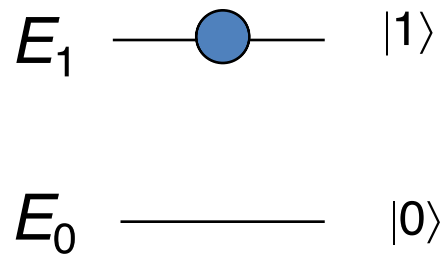
Quantum bits and control of quantum states

Two-level quantum system (qubit)

Ground state: $|0\rangle$



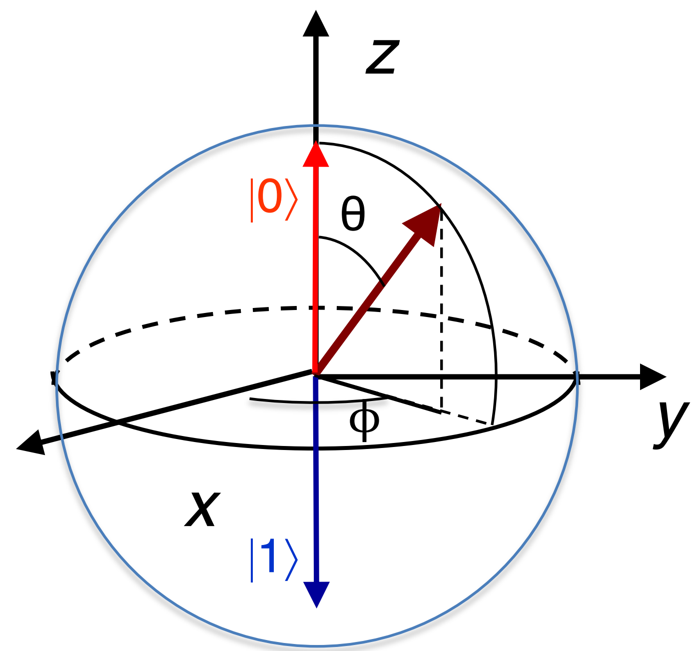
Excited state: $|1\rangle$



Arbitrary state:

$$\Psi = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

$$\Psi = \cos\frac{\theta}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{i\varphi}\sin\frac{\theta}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$



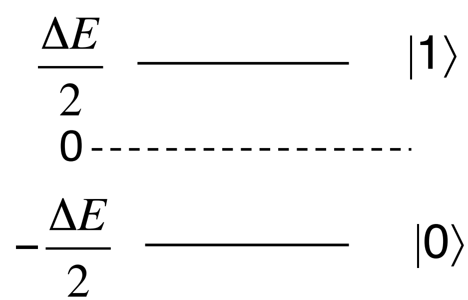
Two-level atom

Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Diagonal form of the two-level Hamiltonian

$$H = \begin{pmatrix} -\frac{\Delta E}{2} & 0 \\ 0 & \frac{\Delta E}{2} \end{pmatrix}$$



$$H = -\frac{\Delta E}{2}\sigma_z$$

Time-dependent unitary transformations

Unitary transformation:

$$\Psi' = U\Psi \qquad \Psi = U^\dagger \Psi'$$

Schrodinger equations:

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \qquad i\hbar \frac{\partial \Psi'}{\partial t} = H'\Psi'$$

Schrodinger equations:

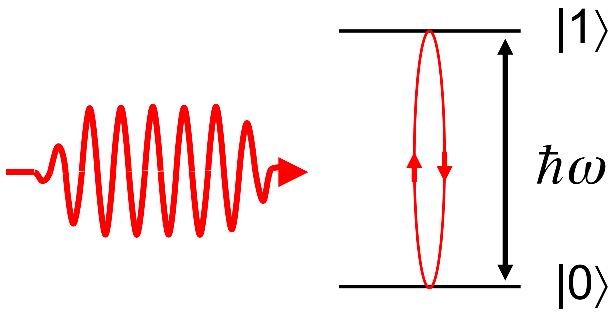
$$i\hbar \frac{\partial U^\dagger \Psi'}{\partial t} = HU^\dagger \Psi' \qquad i\hbar U \frac{\partial U^\dagger}{\partial t}\Psi' + i\hbar UU^\dagger \frac{\partial \Psi'}{\partial t} = UHU^\dagger \Psi'$$

$$i\hbar \frac{\partial \Psi'}{\partial t} = \left(UHU^\dagger - i\hbar U\dot{U}^\dagger\right)\Psi'$$

Time-dependent unitary transformation:

$$\boxed{H' = UHU^\dagger - i\hbar U\dot{U}^\dagger}$$

Transitions in the two-level system under resonant harmonic excitations



Atom driven by an external field:

$$H = -\frac{\hbar\omega}{2}\sigma_z - \hbar\Omega\sigma_x\cos\omega t$$

For example (the charge qubit at the degeneracy point): $\hbar\Omega = VC_k\frac{2e}{C}$

$$U(t) = e^{-i\frac{\omega t}{2}\sigma_z} \qquad H' = UHU^\dagger - i\hbar U\dot{U}^\dagger$$

$$H' = e^{-i\frac{\omega t}{2}\sigma_z} H e^{i\frac{\omega t}{2}\sigma_z} - i\hbar e^{-i\frac{\omega t}{2}\sigma_z}\left(i\frac{\omega}{2}\sigma_z\right)e^{i\frac{\omega t}{2}\sigma_z}$$

$$-\frac{\hbar\omega}{2}\sigma_z - \hbar\Omega\frac{e^{i\omega t}+e^{-i\omega t}}{2}e^{-i\frac{\omega t}{2}\sigma_z}\sigma_x e^{i\frac{\omega t}{2}\sigma_z}$$

$$\frac{\hbar\omega}{2}\sigma_z$$

$$-\frac{\hbar\Omega}{2}\left(e^{i\omega t}+e^{-i\omega t}\right)\begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{i\omega t}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} = -\frac{\hbar\Omega}{2}\left(e^{i\omega t}+e^{-i\omega t}\right)\begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0\end{pmatrix} = -\frac{\hbar\Omega}{2}\begin{pmatrix} 0 & 1+\cancel{e^{-2i\omega t}} \\ 1+\cancel{e^{2i\omega t}} & 0\end{pmatrix}$$

$$\approx -\frac{\hbar\Omega}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} = -\frac{\hbar\Omega}{2}\sigma_x$$

Rotating wave approximation

Driven two-level under external drive

Initial time-dependent Hamiltonian:

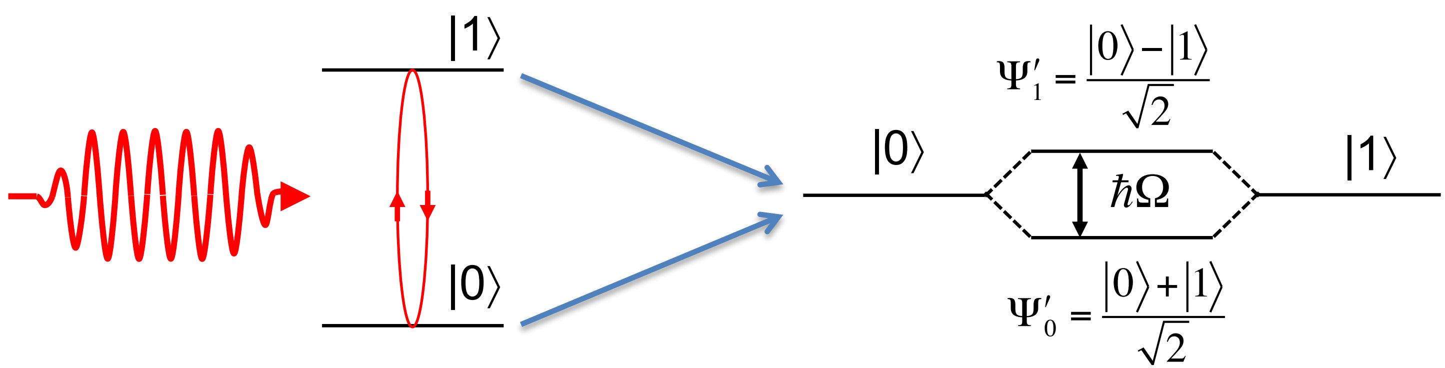
$$H = -\frac{\hbar\omega_a}{2}\sigma_z - \hbar\Omega\sigma_x \cos\omega_a t$$

Time-dependent unitary transformation:

$$U(t) = e^{-i\frac{\omega t}{2}\sigma_z}$$

Transformed Hamiltonian:

$$\boxed{H' \approx -\frac{\hbar\Omega}{2}\sigma_x}$$



Physical meaning of the rotating wave approximation
is coupling of the levels via the radiation

Evolution of the two-level system under the external resonant drive

Initial Hamiltonian and ground state:

$$H = -\frac{\hbar\omega}{2}\sigma_z \qquad \Psi_0 = |0\rangle$$

Resonant drive:

$$H_{\text{int}} = -\hbar\Omega\sigma_x \cos\omega t$$

Transformed Hamiltonian:

$$H' \approx -\frac{\hbar\Omega}{2}\sigma_x$$

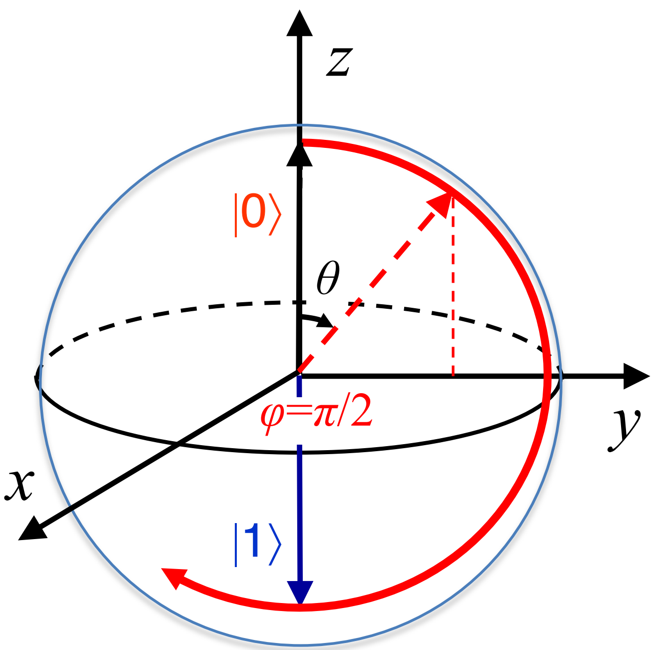
Evolution operator:

$$\boxed{U_{ev} = e^{-i\frac{H'}{\hbar}t} = e^{i\frac{\Omega t}{2}\sigma_x}}$$

$$U_{ev}|0\rangle = \left(\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)\cos\frac{\Omega t}{2} + i\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)\sin\frac{\Omega t}{2}\right)|0\rangle =$$

$$= \cos\frac{\Omega t}{2}|0\rangle + e^{i\pi}\sin\frac{\Omega t}{2}|1\rangle$$

Rotation around *x*-axis:

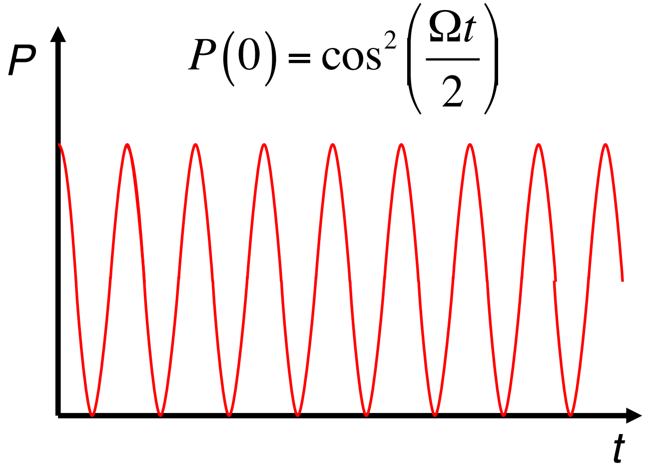


Probability oscillations (without decoherence)

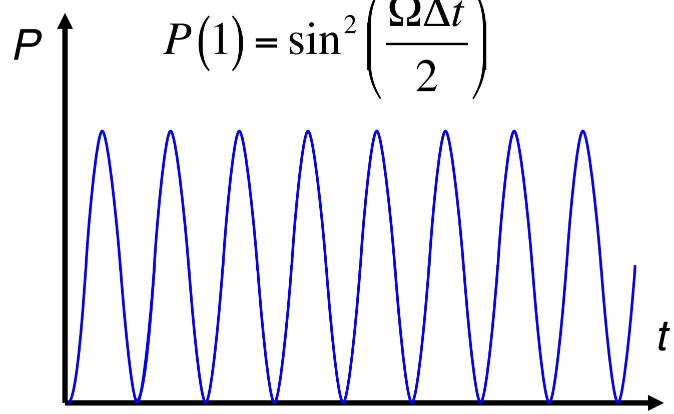
$$H' \approx -\frac{\hbar\Omega}{2}\sigma_x \qquad U_{ev} = e^{i\frac{\Omega t}{2}\sigma_x} = \cos\left(\frac{\Omega t}{2}\right)I + i\sin\left(\frac{\Omega t}{2}\right)\sigma_x \qquad \left|\Psi\right\rangle = U_{ev}\left|0\right\rangle$$

$$U_{ev} = e^{i\frac{\Omega t}{2}\sigma_x} = \cos\left(\frac{\Omega t}{2}\right)\left|0\right\rangle + i\sin\left(\frac{\Omega t}{2}\right)\left|1\right\rangle$$

$$P(0) = \cos^2\left(\frac{\Omega t}{2}\right)$$



$$P(1) = \sin^2\left(\frac{\Omega\Delta t}{2}\right)$$



Frequency of Rabi oscillations: $\frac{\Omega}{2}$

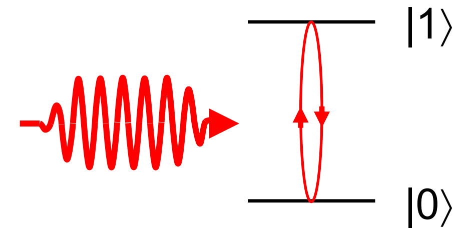
Manipulation with single qubit states

Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Two-level system in the eigenbasis (diagonal Hamiltonian):

$$H = \begin{pmatrix} -\hbar\omega_0/2 & 0 \\ 0 & \hbar\omega_0/2 \end{pmatrix} = -\frac{\hbar\omega_0}{2}\sigma_z = -\frac{\hbar\omega_0}{2}|0\rangle\langle 0| + \frac{\hbar\omega_0}{2}|1\rangle\langle 1|$$

By applying external field we transform the Hamiltonian:



$$i\hbar\frac{\partial\psi(t)}{\partial t} = \hat{H}\psi(t) \qquad \psi(t) = \exp\left(-i\frac{H}{\hbar}t\right)\psi(0)$$

x-rotation:

$$H = -\frac{\hbar\Omega}{2}\sigma_x = -\frac{\hbar\Omega}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \psi(t) = \exp\left(i\frac{\Omega t}{2}\sigma_x\right)\psi(0) \quad R_x(t) = \exp\left(i\frac{\Omega t}{2}\sigma_x\right) = I\cos\left(\frac{\Omega t}{2}\right) + i\sigma_x\sin\left(\frac{\Omega t}{2}\right)$$

y-rotation:

$$H = -\frac{\hbar\Omega}{2}\sigma_y = -\frac{\hbar\Omega}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \psi(t) = \exp\left(i\frac{\Omega t}{2}\sigma_y\right)\psi(0) \quad R_y(t) = \exp\left(i\frac{\Omega t}{2}\sigma_y\right) = I\cos\left(\frac{\Omega t}{2}\right) + i\sigma_y\sin\left(\frac{\Omega t}{2}\right)$$

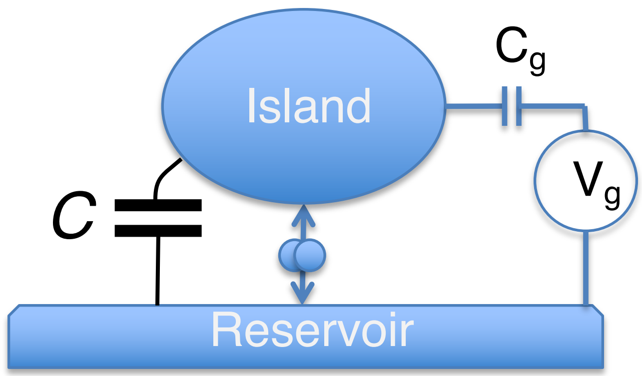
z-rotation:

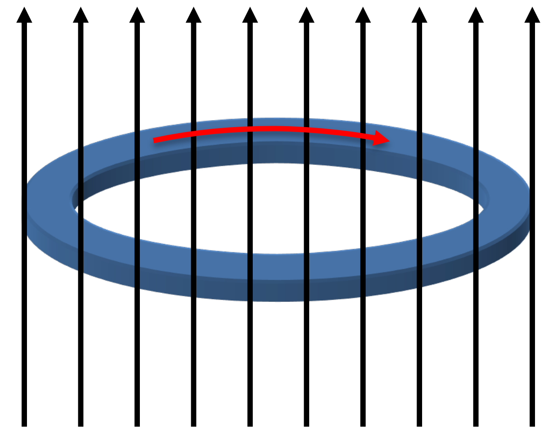
$$H = -\frac{\hbar\omega}{2}\sigma_z = -\frac{\hbar\omega}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \psi(t) = \exp\left(i\frac{\omega t}{2}\sigma_z\right)\psi(0) \quad R_z(t) = \exp\left(i\frac{\omega t}{2}\sigma_z\right) = I\cos\left(\frac{\omega t}{2}\right) + i\sigma_z\sin\left(\frac{\omega t}{2}\right)$$

Physical realization of on-chip quantum systems

Cooper pair box

A superconducting island with capacitance C





- Number of Cooper pairs is: *N*
- We can add or remove charge quantum: 2*e*

$$U = \frac{\left(N2e - C_g V\right)^2}{2C}$$

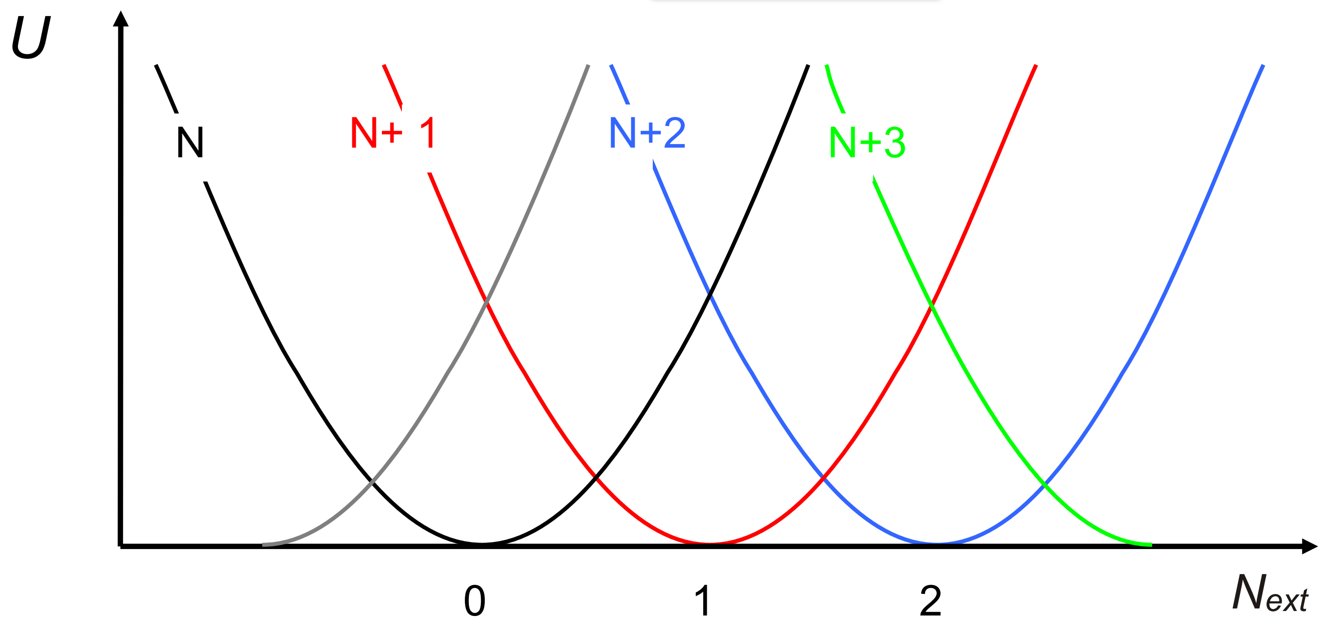
- Number of quantized fluxes: *N*
- We can add or remove flux quantum Φ_0

$$U = \frac{\left(N\Phi_0 - \Phi_{ext}\right)^2}{2L}$$

Electrostatic energies of the Cooper-pair box

$$U = \frac{\left(N2e - C_g V\right)^2}{2C} = \frac{(2e)^2}{2C}\left(N - \frac{C_g V}{2e}\right)^2 = E_C\left(N - N_{ext}\right)^2$$

Charging energy: $\boxed{E_C = \frac{(2e)^2}{2C}}$



$\boxed{N + n \rightarrow n}$

The Hamiltonian in the charge basis

Potential energy
(diagonal terms):

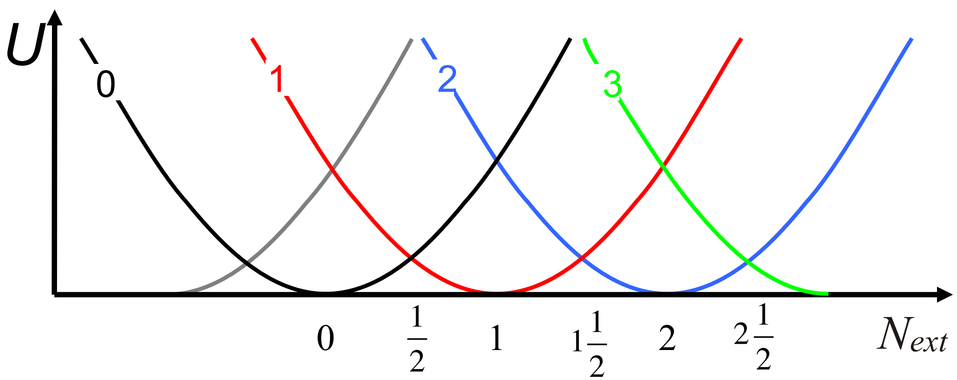
$$U = \frac{\left(2eN - C_g V\right)^2}{2C} |N\rangle\langle N|$$

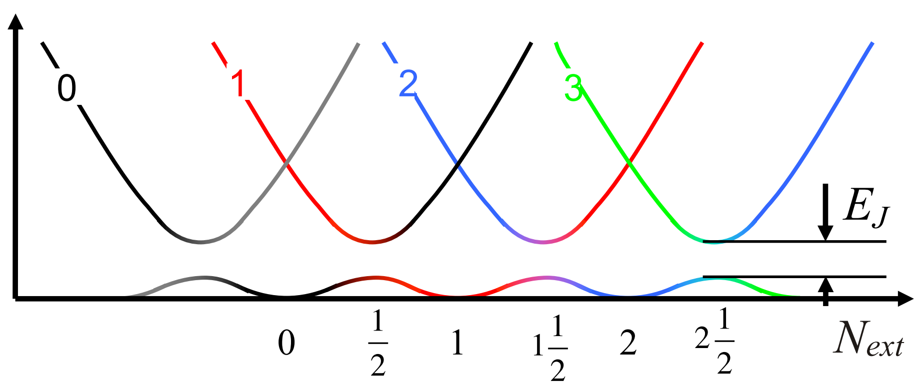
Kinetic energy:

$$T = \frac{1}{2} E_J \left(|N+1\rangle\langle N| + |N-1\rangle\langle N| \right)$$

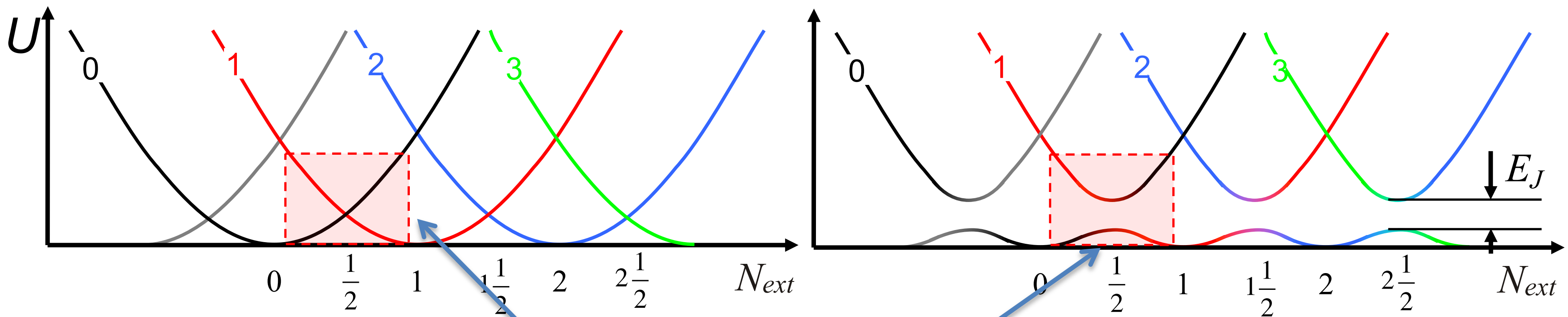
$$\boxed{H = E_C \left(N - N_{ext}\right)^2 |N\rangle\langle N| - \frac{1}{2} E_J \left(|N+1\rangle\langle N| + |N-1\rangle\langle N| \right)}$$

$$U\left(N, N_{ext}\right) = E_C \left(N - N_{ext}\right)^2$$





The superconducting charge qubit



Two-level approximation

$$H=\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} \\
\cdots & E_C\left(-2-N_{ext}\right)^2 & -\frac{1}{2}E_J & 0 & 0 & 0 & \cdots \\
\cdots & -\frac{1}{2}E_J & E_C\left(-1-N_{ext}\right)^2 & -\frac{1}{2}E_J & 0 & 0 & \cdots \\
\cdots & 0 & -\frac{1}{2}E_J & E_C N_{ext}^2 & -\frac{1}{2}E_J & 0 & \cdots \\
\cdots & 0 & 0 & -\frac{1}{2}E_J & E_C\left(1-N_{ext}\right)^2 & -\frac{1}{2}E_J & \cdots \\
\cdots & 0 & 0 & 0 & -\frac{1}{2}E_J & E_C\left(2-N_{ext}\right)^2 & \cdots \\
\cdot^{\cdot^{\cdot}} & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

$$H_q=-\frac{\varepsilon}{2}\sigma_z-\frac{E_J}{2}\sigma_x$$

The charge/flux qubit Hamiltonian

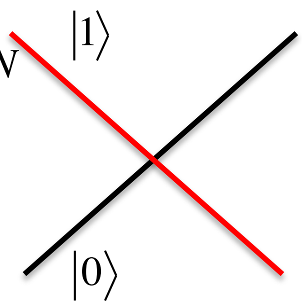
$$H = -\frac{\varepsilon}{2}|0\rangle\langle 0| + \frac{\varepsilon}{2}|1\rangle\langle 1| - \frac{\Delta}{2}\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)$$

Δ - tunneling energy between two states

$$H_q = -\frac{\varepsilon}{2}\sigma_z \qquad\qquad H_q = -\frac{\Delta}{2}\sigma_x$$

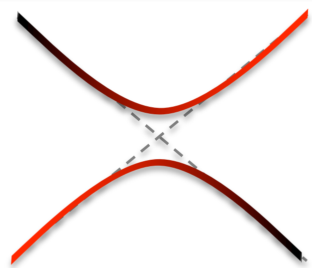
$$-\frac{\varepsilon}{2} = -E_{C,L}\delta N$$

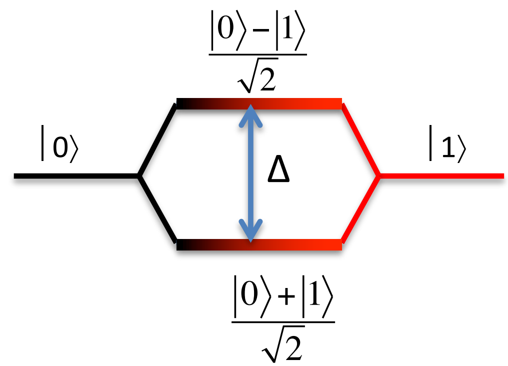
$$\frac{\varepsilon}{2} = E_{C,L}\delta N$$



Qubit Hamiltonian
in the physical basis:

$$\boxed{H_q = -\frac{\varepsilon}{2}\sigma_z - \frac{\Delta}{2}\sigma_x}$$





$$E_0 = -\Delta E \qquad \Psi_0 = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$E_1 = \Delta E \qquad \Psi_1 = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}$$

$$\Delta E = \sqrt{\varepsilon^2 + \Delta^2}$$

$$\tan\theta = \frac{\Delta}{\varepsilon}$$

Quantum mechanics of electrical circuits
Two-level systems in resonators

Charge ⇔ Flux

Capacitance

Inductance

Voltage: V

$V = \dot{\Phi}$

Charge: Q

$Q = CV = C\dot{\Phi}$

$Q = 2eN$

Current: I

$I = \dot{Q}$

Magnetic Flux: Φ

$\Phi = LI = L\dot{Q}$

$\Phi = \frac{\Phi_0}{2\pi}\varphi \qquad \Phi_0 = \frac{h}{2e}$

Quantum mechanics of an electric circuit

Two representations of the electric circuit

Quantum mechanics

$x \qquad p = m\dot{x}$

Commutation relations: $[\hat{x}, \hat{p}] = i\hbar$

Differential form kinetic energy: $\hat{p} = -i\hbar\frac{\partial}{\partial x} \qquad T = \frac{m\dot{x}^2}{2}$

Electric circuit:
charge and charge motion

$Q \qquad \Phi = L\dot{Q}$

$[\hat{Q}, \hat{\Phi}] = i\hbar$

$\hat{\Phi} = -i\hbar\frac{\partial}{\partial Q} \qquad T = \frac{L\dot{Q}^2}{2}$

$Q \leftrightarrow x \qquad \Phi \leftrightarrow p \qquad L \leftrightarrow m$

Electric circuit:
flux and flux motion

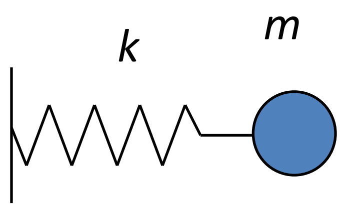
$\Phi \qquad Q = C\dot{\Phi}$

$[\hat{\Phi}, \hat{Q}] = i\hbar$

$\hat{Q} = -i\hbar\frac{\partial}{\partial \Phi} \qquad T = \frac{C\dot{\Phi}^2}{2}$

$\Phi \leftrightarrow x \qquad Q \leftrightarrow p \qquad C \leftrightarrow m$

Classical harmonic oscillators



$$E = \frac{kx^2}{2} + \frac{p^2}{2m}$$

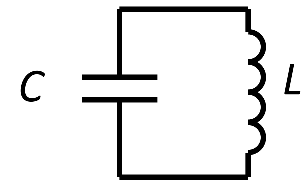
$$E = \frac{kx^2}{2} + \frac{m\dot{x}^2}{2}$$

Coordinate: x

Momentum: $p = m\dot{x}$

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad E = \frac{m}{2}\left(\omega_0^2 x^2 + \dot{x}^2\right)$$

$$x = Ae^{i\omega t} + Be^{-i\omega t}$$



$$E = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

$$E = \frac{Q^2}{2C} + \frac{L\dot{Q}^2}{2}$$

Q

$p' = L\dot{Q} \rightarrow \Phi$

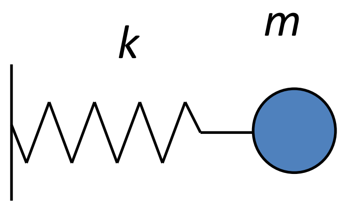
$$E = \frac{C}{2}\left(\omega_0^2 Q^2 + \dot{Q}^2\right) \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = Ae^{i\omega t} + Be^{-i\omega t}$$

The equations are transformed from one to another with the substitutions

$$x \rightarrow Q \qquad p \rightarrow \Phi \qquad k \rightarrow \frac{1}{C} \qquad m \rightarrow L$$

Classical harmonic oscillators (alternative approach)



$$E = \frac{kx^2}{2} + \frac{p^2}{2m}$$

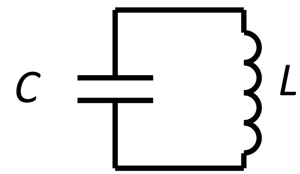
$$E = \frac{kx^2}{2} + \frac{m\dot{x}^2}{2}$$

Coordinate: x

Momentum: $p = m\dot{x}$

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad E = \frac{m}{2}\left(\omega_0^2 x^2 + \dot{x}^2\right)$$

$$x = Ae^{i\omega t} + Be^{-i\omega t}$$



$$E = \frac{\Phi^2}{2L} + \frac{Q^2}{2C}$$

$$E = \frac{\Phi^2}{2L} + \frac{C\dot{\Phi}^2}{2}$$

Φ

$p' = C\dot{\Phi} \rightarrow Q$

$$E = \frac{L}{2}\left(\omega_0^2 \Phi^2 + \dot{\Phi}^2\right) \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

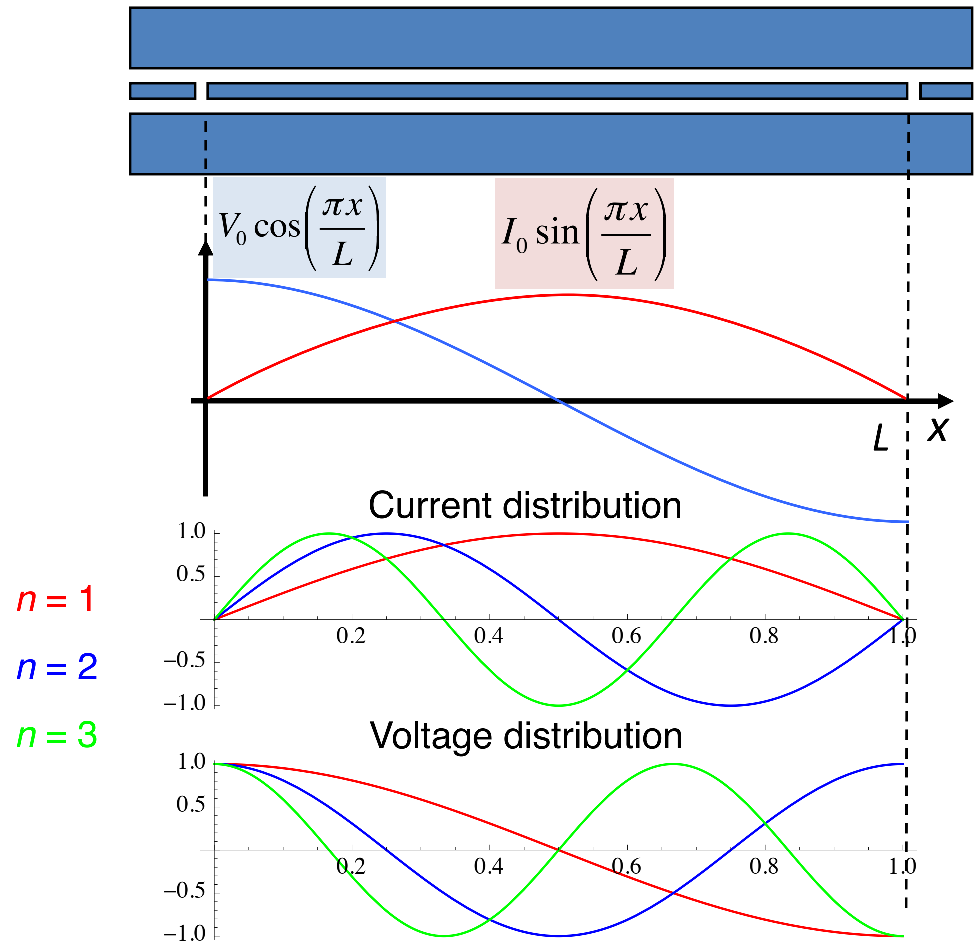
$$\Phi = Ae^{i\omega t} + Be^{-i\omega t}$$

The equations are identical with the following substitutions

$$x \rightarrow \Phi \qquad p \rightarrow Q \qquad k \rightarrow \frac{1}{L} \qquad m \rightarrow C$$

Field distribution in coplanar resonators

$$I_0 \sin\left(\frac{\pi n}{L} x\right) \qquad V = V_0 \cos\left(\frac{\pi n}{L} x\right)$$



Quantum mechanics of coplanar resonators

Classical field

$$V = V_0 \cos\left(\frac{\pi n}{L} x\right)$$

$$I = I_0 \sin\left(\frac{\pi n}{L} x\right)$$

Quantum field

$$\hat{V} = \sqrt{\frac{\hbar\omega}{2C}}\left(a + a^\dagger\right)\cos\left(\frac{\pi n}{L} x\right)$$

$$\hat{I} = i\sqrt{\frac{\hbar\omega}{2L}}\left(a^\dagger - a\right)\sin\left(\frac{\pi n}{L} x\right)$$

$$H = \int_0^L \left(l\frac{\hat{I}^2(x)}{2} + c\frac{\hat{V}^2(x)}{2}\right)dx = \frac{\hbar\omega}{L_r}\frac{lL}{4}\left(a + a^\dagger\right)^2 - \frac{\hbar\omega}{C_r}\frac{cL}{4}\left(a - a^\dagger\right)^2 = \frac{\hbar\omega}{2}\left(2a^\dagger a + 1\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

Integration over space gives the usual form of the Hamiltonian

$$H = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

Interaction Hamiltonian

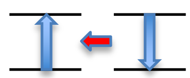
$$\boxed{H_{\text{int}} = g_0 \sigma_x \left(a + a^+\right)}$$

$$\sigma_x = \sigma^+ + \sigma^-$$

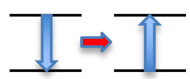
$$\boxed{H_{\text{int}} \approx g_0 \left(a + a^+\right)\left(\sigma^+ + \sigma^-\right) = g_0 \left(a\sigma^+ + a^+\sigma^+ + a\sigma^- + a^+\sigma^-\right)}$$

Processes with no energy change

$$\sigma^+ a \left|0N\right\rangle = \sqrt{N-1}\left|1(N-1)\right\rangle$$



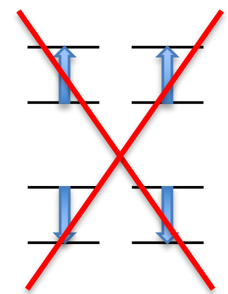
$$\sigma^- a^+ \left|1N\right\rangle = \sqrt{N}\left|0(N+1)\right\rangle$$



Processes with energy change by $\pm 2\hbar\omega$

$$\sigma^+ a^+ \left|0N\right\rangle = \sqrt{N}\left|1(N+1)\right\rangle$$

$$\sigma^- a \left|1N\right\rangle = \sqrt{N-1}\left|0(N-1)\right\rangle$$



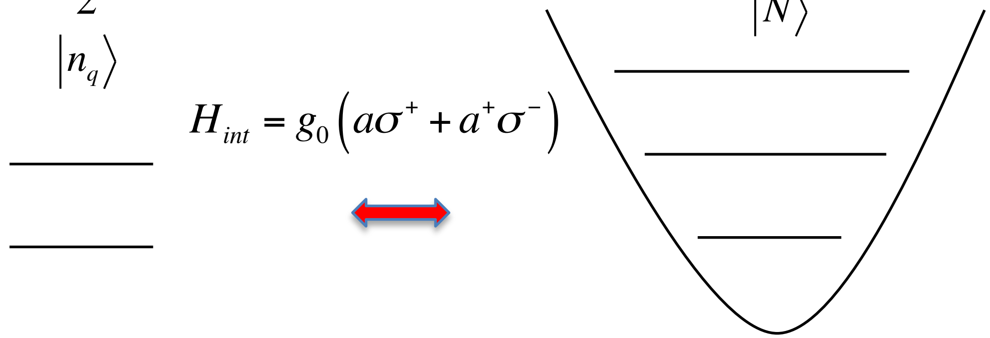
$$\boxed{H_{\text{int}} \approx g_0 \left(a\sigma^+ + a^+\sigma^-\right)}$$

Two-level system interacting with harmonic oscillator Jaynes-Cummings Hamiltonian

$$H_q = -\frac{\Delta E}{2}\sigma_z \qquad H_r = \hbar\omega a^+ a$$

$$|n_q\rangle \qquad |N\rangle$$

$$H_{int} = g_0\left(a\sigma^+ + a^+\sigma^-\right)$$



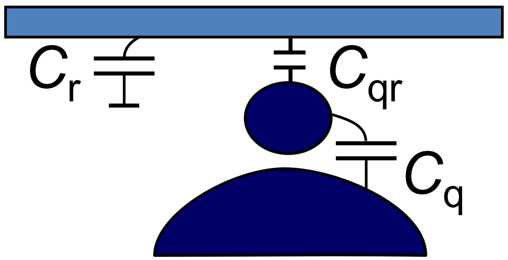
Interaction of the two-level system with the resonator

$$H_{JC} = -\frac{\Delta E}{2}\sigma_z + \hbar\omega_r a^+ a + g_0\left(a\sigma^+ + a^+\sigma^-\right)$$

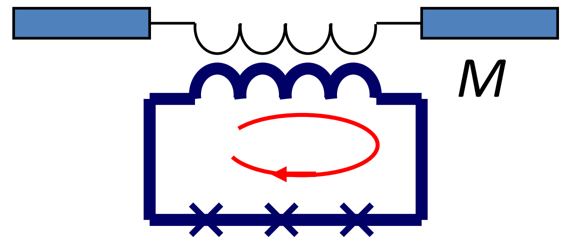
Qubit (atom) Oscillator Qubit-resonator interaction

Qubits coupled to a harmonic oscillator

Charge qubit coupled capacitively



Flux qubit coupled inductively



Charge qubit: $H_{int} = \hat{V}_q C_{qr} \hat{V}_r$

Flux qubit: $H_{int} = \hat{I}_q M \hat{I}_r$

Off-diagonal matrix elements produce transitions in the two-level system

Qubit dipole voltage operator:

$$\boxed{\hat{V}_q^{\perp} = V_{q0}\sigma_x}$$

$$V_0 = \frac{E_C}{2e}\sin\theta$$

Resonator voltage operator:

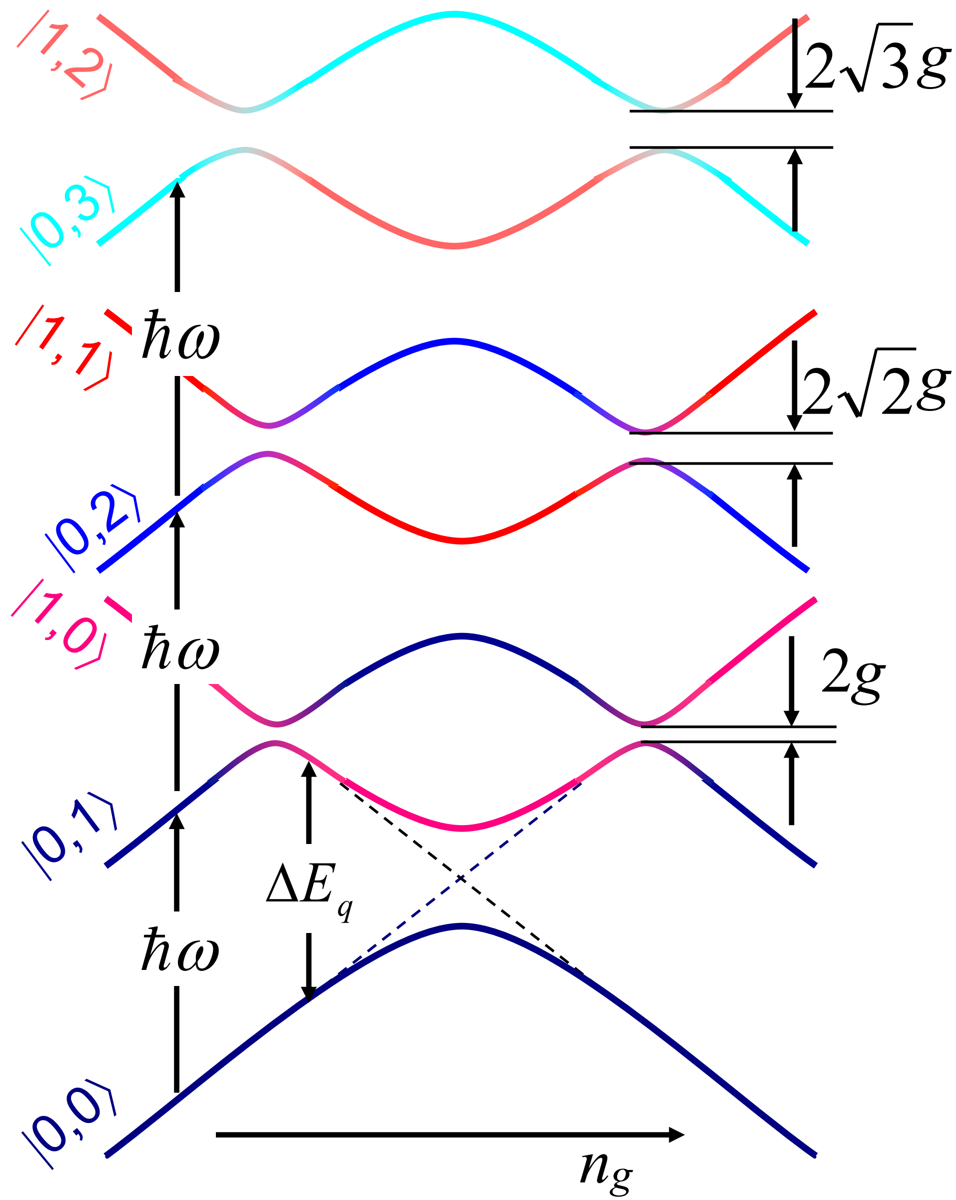
$$\boxed{\hat{V}_r = V_{r0}\left(a + a^+\right)}$$

$$V_{r0} = \sqrt{\frac{\hbar\omega}{C_r}}$$

$$\boxed{H_{int} = g_0\sigma_x\left(a + a^+\right)}$$

$$g_0 = C_{qr} V_{q0} V_{r0}$$

Qubit-resonator energy diagram



Dissipative quantum dynamics

Incoherent processes: Energy dissipation

Probabilities to find the two-level system in the excited and ground states: P_1 and P_0

$$dP_1 = -P_1 \frac{dt}{T_1} \qquad P_1(t) = P_1(0)e^{-\frac{t}{T}} \qquad P_0(t) = 1 - P_1(0)e^{-\frac{t}{T_1}}$$

$$P_0 + P_1 = 1$$

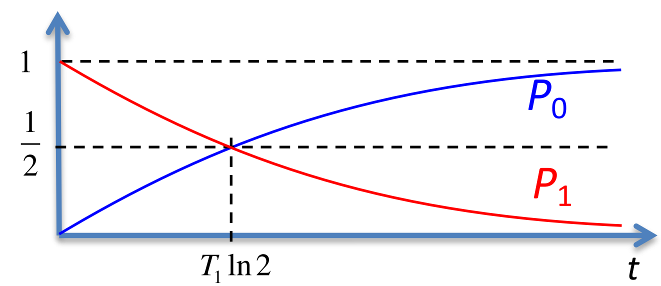
T_1 is the energy relaxation time

Mixed state (is not described by wavefunctions):

$$\rho_{11}(0) = 1 \qquad \rho(0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \rho(T_1 \ln 2) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \qquad \rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

The wavefunction giving the same probabilities:

$$\Psi = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow \rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$



$$\Psi = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad \rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

ρ_{01} and ρ_{10} - coherence

Expectation values

$$\langle \hat{O} \rangle = \langle \Psi | \hat{O} | \Psi \rangle = \text{Trace}\left(|\Psi\rangle\langle\Psi| \hat{O} \right) = \text{Trace}\left(\rho \hat{O} \right)$$

Pure states (described by wavefunctions):

$$|\Psi\rangle\langle\Psi| = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \begin{pmatrix} a_0^* & a_1^* \end{pmatrix} = \begin{pmatrix} a_0 a_0^* & a_0 a_1^* \\ a_1 a_0^* & a_1 a_1^* \end{pmatrix}$$

Pure state:

$$|\rho_{01}|^2 = |\rho_{00}\rho_{11}| \qquad \rho_{00} + \rho_{11} = 1$$

Examples:

$$|0\rangle\langle 0| = \frac{\sigma_z + 1}{2} \qquad \frac{\langle \sigma_z \rangle + 1}{2} = \frac{\rho_{00} - \rho_{11} + 1}{2} = \rho_{00}$$

$$|1\rangle\langle 1| = \frac{1 - \sigma_z}{2} \qquad \frac{1 - \langle \sigma_z \rangle}{2} = \frac{1 - \rho_{00} + \rho_{11}}{2} = \rho_{11}$$

$$\sigma_x \qquad \langle \sigma_x \rangle = \rho_{01} + \rho_{10}$$

$$\sigma_y \qquad \langle \sigma_y \rangle = i\rho_{01} - i\rho_{10}$$

$$\langle \sigma_z \rangle = \text{tr}\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = \text{tr} \begin{pmatrix} \rho_{00} & -\rho_{01} \\ \rho_{10} & -\rho_{11} \end{pmatrix} = \rho_{00} - \rho_{11}$$

$$\langle \sigma_x \rangle = \text{tr}\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \text{tr} \begin{pmatrix} \rho_{01} & \rho_{00} \\ \rho_{11} & \rho_{10} \end{pmatrix} = \rho_{01} + \rho_{10}$$

$$\langle \sigma_y \rangle = \text{tr}\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] = \text{tr} \begin{pmatrix} i\rho_{01} & -i\rho_{00} \\ i\rho_{11} & -i\rho_{10} \end{pmatrix} = i\rho_{01} - i\rho_{10}$$

Master Equation

Schrodinger equation:

$$i\hbar\frac{\partial|\Psi\rangle}{\partial t}=H|\Psi\rangle \qquad -i\hbar\frac{\partial\langle\Psi|}{\partial t}=\langle\Psi|H$$

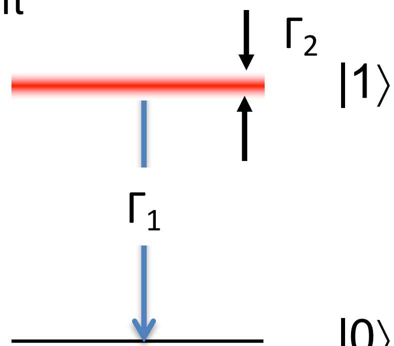
$$i\hbar\frac{\partial|\Psi\rangle\langle\Psi|}{\partial t}=i\hbar\frac{\partial|\Psi\rangle}{\partial t}\langle\Psi|+i\hbar|\Psi\rangle\frac{\partial\langle\Psi|}{\partial t}$$

$$i\hbar\frac{\partial|\Psi\rangle\langle\Psi|}{\partial t}=H|\Psi\rangle\langle\Psi|-|\Psi\rangle\langle\Psi|H$$

$$\boxed{i\hbar\frac{\partial\rho}{\partial t}=[H,\rho]} \qquad [H,\rho]=H\rho-\rho H$$

The Master Equation and the Lindblad operator

The general form of the Hamiltonian driven by a wave with an arbitrary phase shift



Relaxation and dephasing: Γ_1, Γ_2

Master equation:

$$\frac{\partial \rho}{\partial t} = -\frac{1}{i\hbar}[H,\rho] + L$$

The Lindblad term for two-level system at $T = 0$:

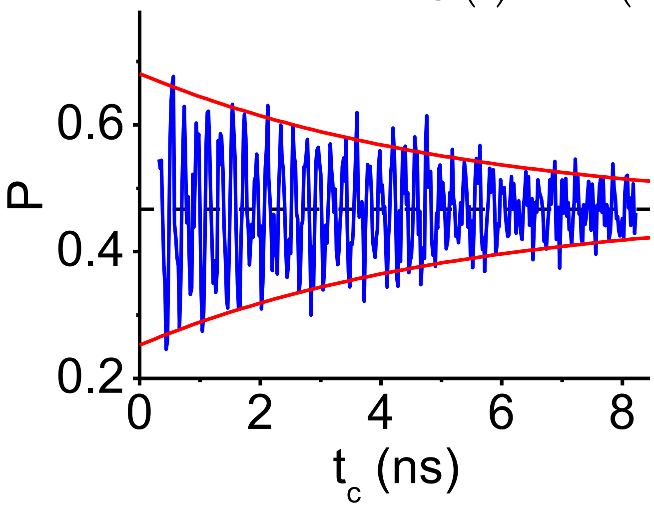
$$L = \begin{pmatrix} \Gamma_1 \rho_{11} & -\Gamma_2 \rho_{01} \\ -\Gamma_2 \rho_{10} & -\Gamma_1 \rho_{11} \end{pmatrix}$$

Stationary Master Equation $\left(\frac{\partial \rho}{\partial t} = 0\right)$:

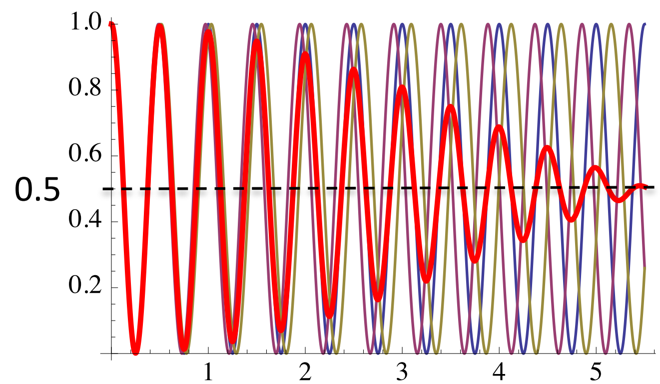
$$-\frac{1}{i\hbar}[H,\rho] + L = 0$$

Dephasing

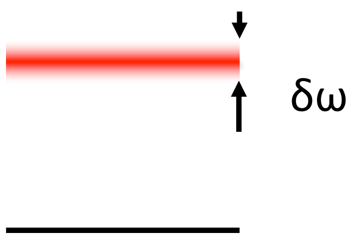
$$\psi(t) = \cos(\Omega t/2)|0\rangle + e^{-i\frac{\pi}{2}} \sin(\Omega t/2)|1\rangle$$



$$\varphi = \omega t + \delta\varphi$$



Dephasing:
energy/frequency => phase fluctuation



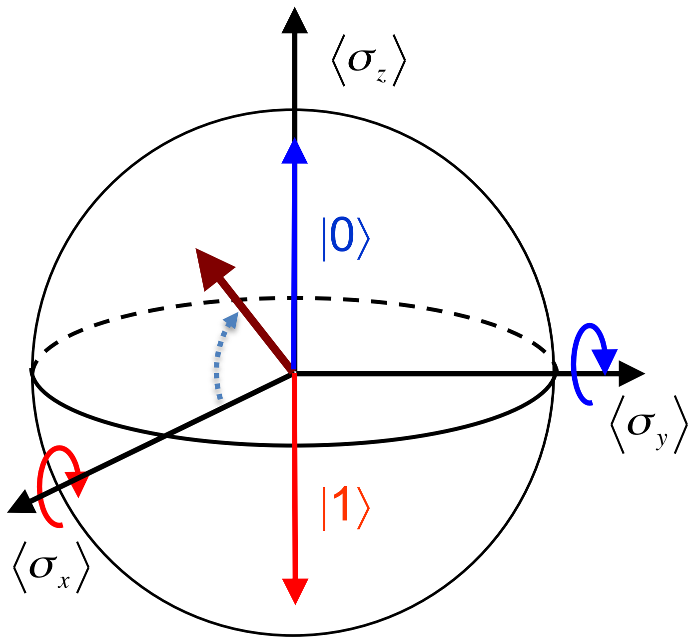
$$\Gamma_1 = \frac{2\pi}{\hbar^2} \left|\frac{\partial H}{\partial X}\right|^2 S_X(\omega)$$

Relaxation of the charge qubit due to potential fluctuations

$$\Gamma_1 = \frac{2\pi\mu^2}{\hbar^2} S_V(\omega)$$

Bloch Sphere for dissipative spin dynamics

The vector can be less than one
(alternative criteria for mixed states)

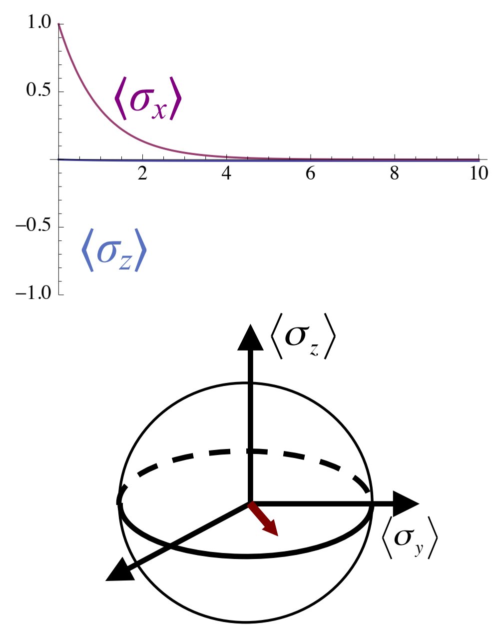


Dynamics of the two-level system is exactly same as the dynamics of spin 1/2

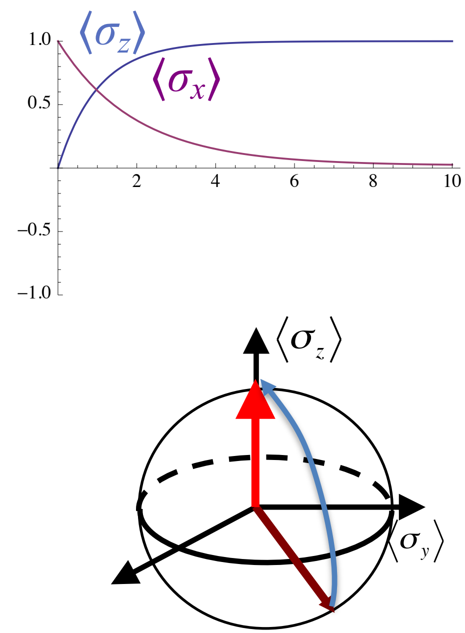
Relaxation results in decay of $\langle \sigma_z \rangle$

Dephasing results in decay of $\langle \sigma_x \rangle$, $\langle \sigma_y \rangle$

Pure dephasing



Relaxation



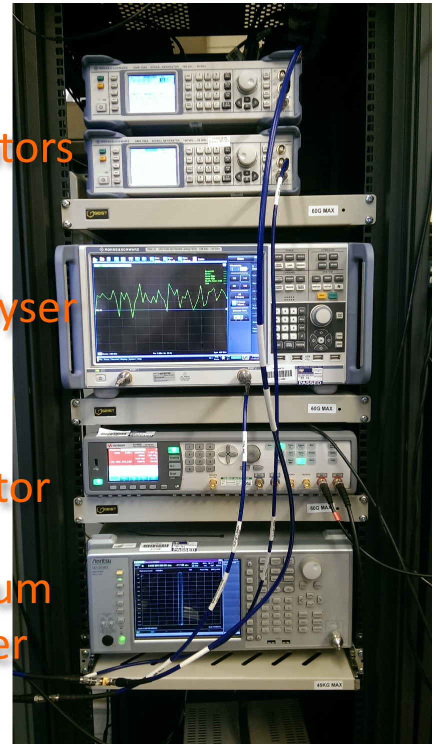
Measurement circuits

Dilution refrigerator and measurement equipmemt

Dilution refrigerator

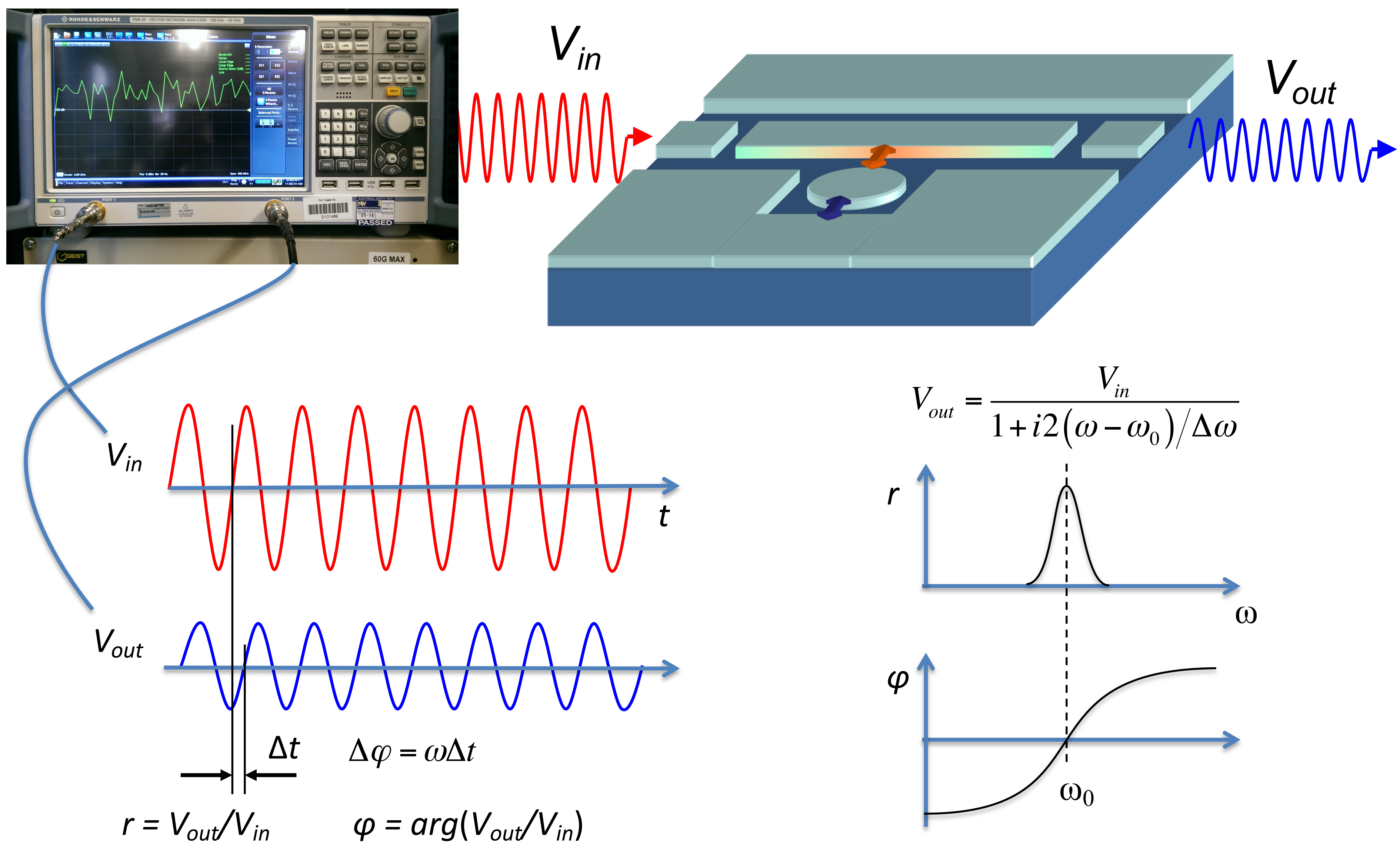
Internal view

Measurement MW equipment



The lowest temperature is 10 mK

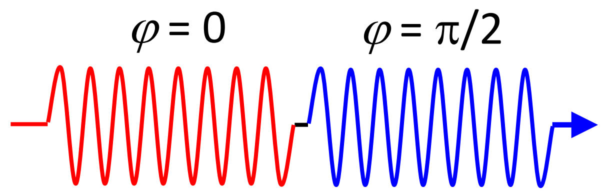
Phase-sensitive detection of transmitted signal by a network analyzer

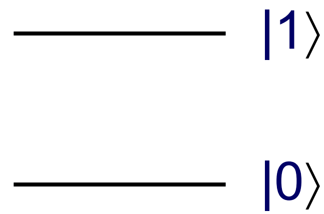


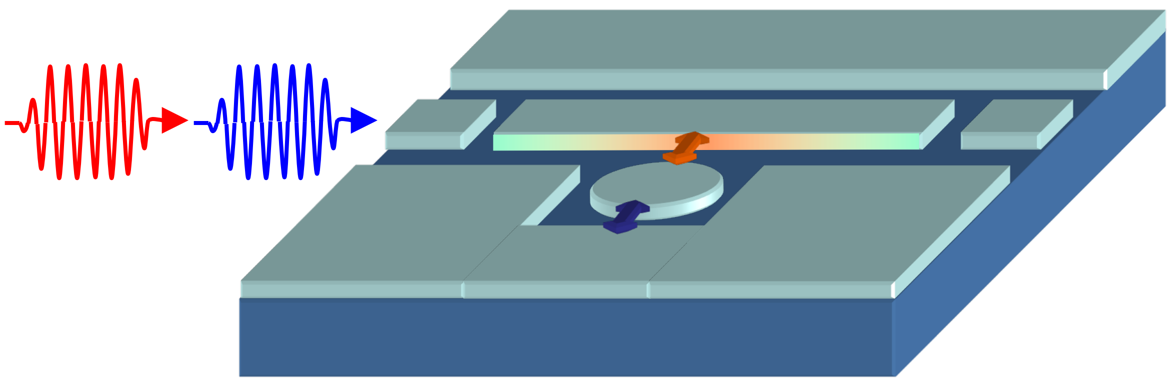
Quantum state control in two-level systems

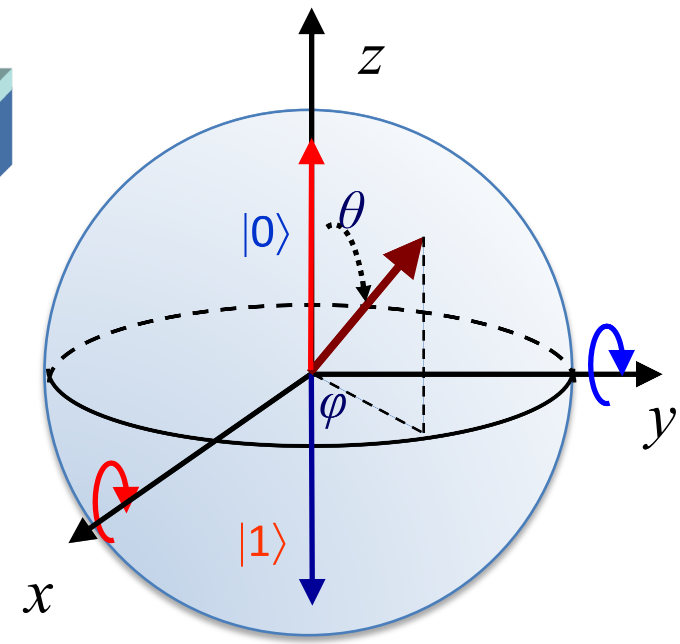
$$\Psi = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

State manipulation





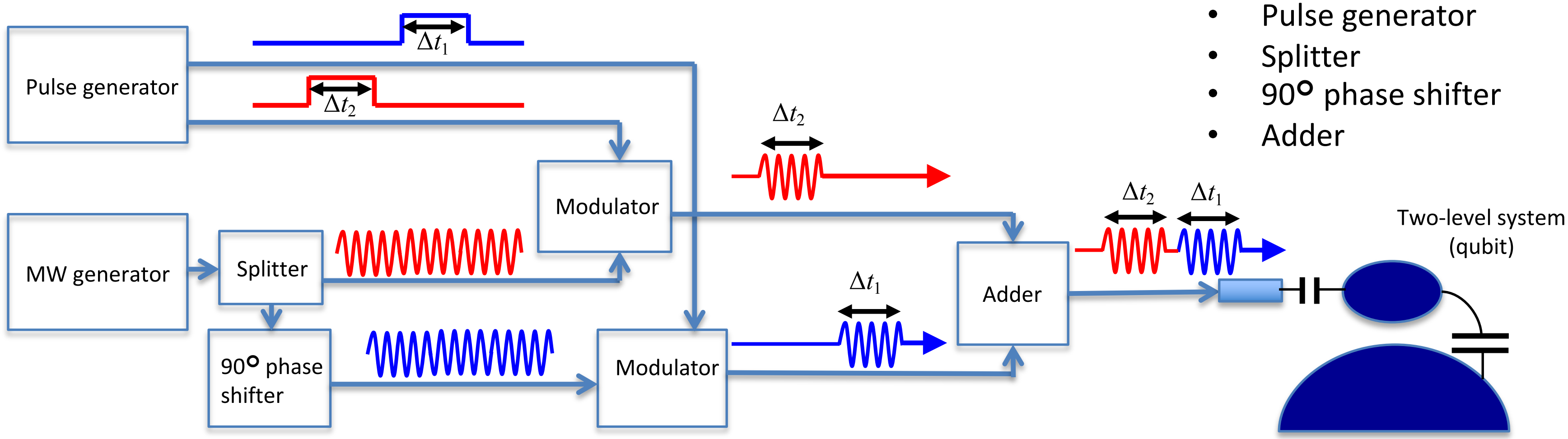




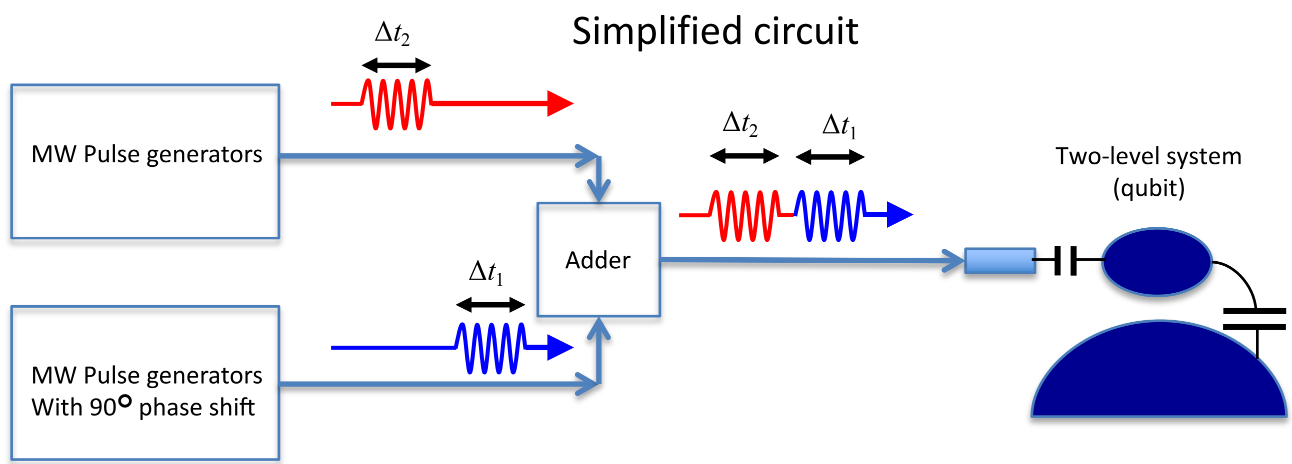
Electrical circuit to prepare arbitrary states

Required devices:

- MW generator
- Pulse generator
- Splitter
- 90° phase shifter
- Adder



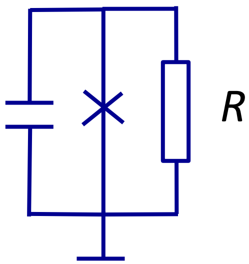
Simplified circuit



Selection rules

Relaxation due to ohmic environment

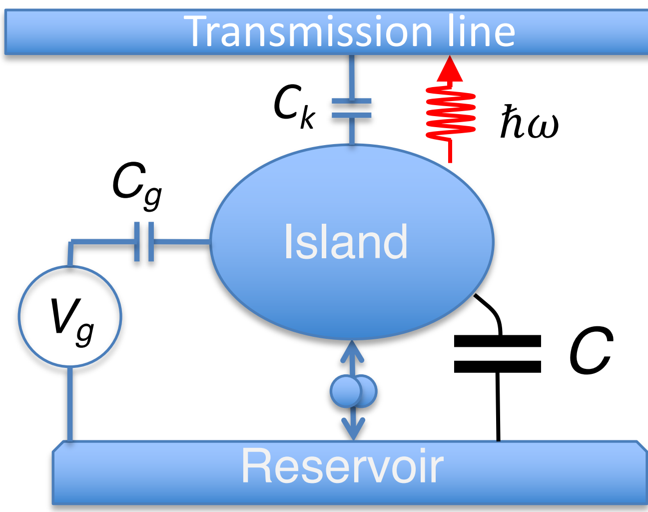
Absorber

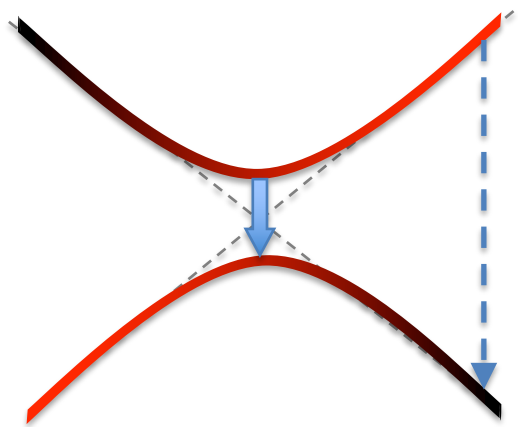


$$S_V(\omega) = \frac{\hbar\omega R}{\pi} \qquad R \to \mathrm{Re}[Z] \qquad \mu_q = C_\kappa \frac{E_C}{2e}\sin\theta$$

$$\Gamma_{10} = \frac{2\pi\mu_q^2}{\hbar^2} S_V(\omega) = \frac{2\hbar\omega\mu_q^2 R}{\hbar^2} \qquad \Gamma_1 = \frac{2\mu_q^2\omega}{\hbar}\mathrm{Re}[Z]$$

Charge qubit: $\Gamma_1 = \frac{2\omega Z_0}{2\hbar}\left(\frac{2eC_\kappa}{C_q}\sin\theta\right)^2$

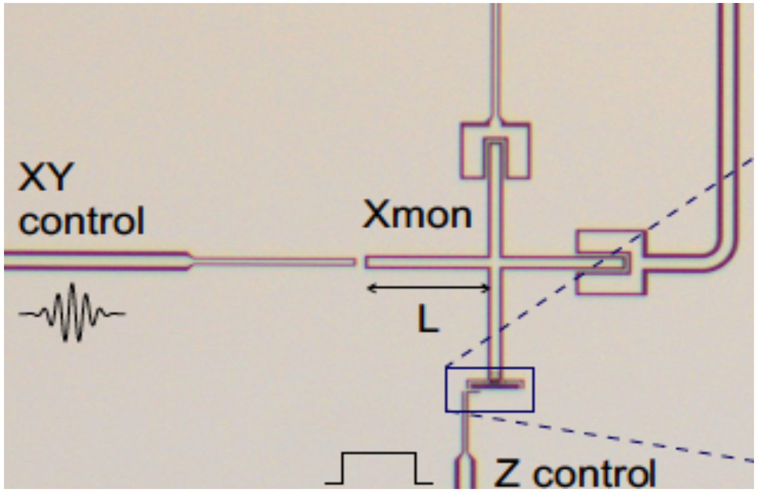




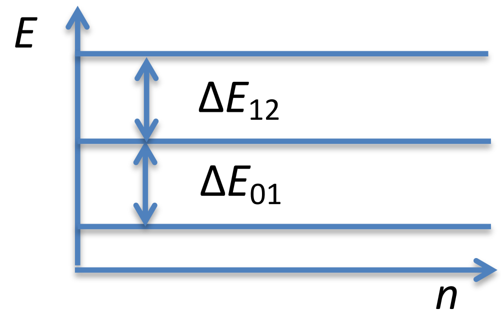
$$\Gamma_1 \sim \frac{E_J^2}{\omega}$$

At T = 0, relaxation is a result of high frequency quantum noise

Charge qubit with low charging energy (high capacitance)



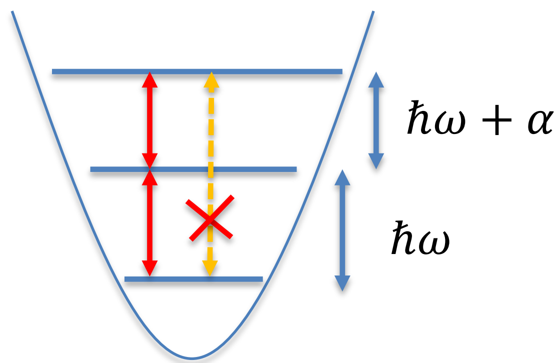
$$H = E_C\left(N-n\right)^2 |N\rangle\langle N| - \frac{1}{2}E_J\left(|N-1\rangle\langle N| + |N\rangle\langle N-1|\right)$$



$\Delta E_{01} \neq \Delta E_{12}$

$E_C \ll E_J$

Non-linear resonator



$$H = \hbar\omega_q a^\dagger a + \frac{1}{2}\alpha a^\dagger a\left(a^\dagger a - 1\right)$$

$N = 0: 0$

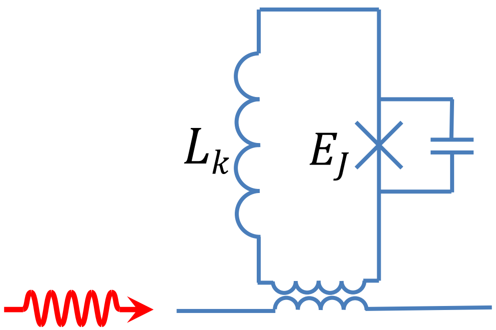
$N = 1: \hbar\omega_q$

$N = 1: 2\hbar\omega_q + \alpha$

$N = 1: 3\hbar\omega_q + 3\alpha$

Transitions between neighboring states are only allowed

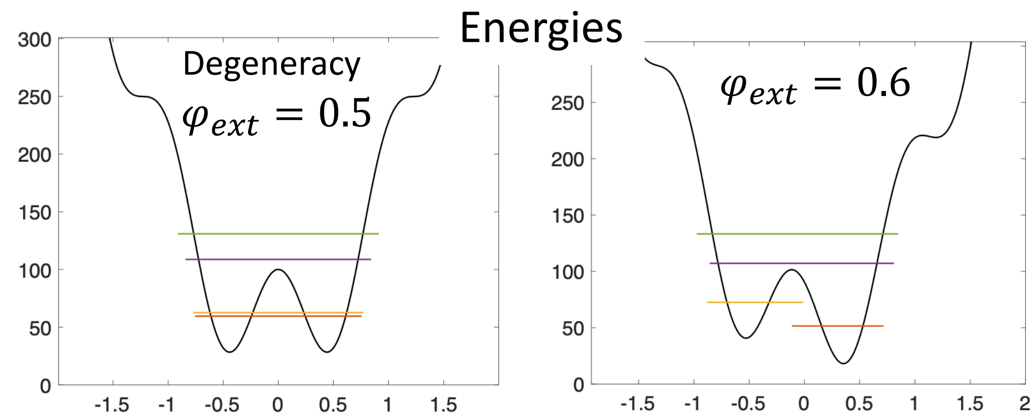
Selection rules in RF-SQUID



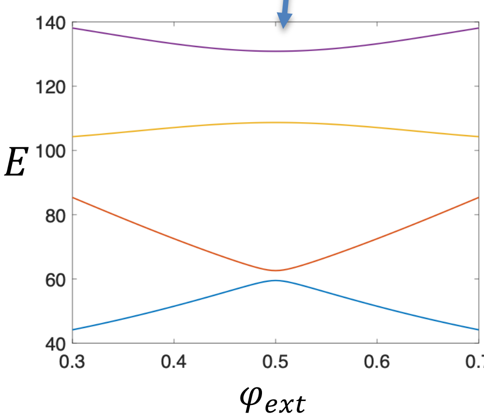
$$H_{int} = M\hat{\varphi}\,\varphi_{ext}$$

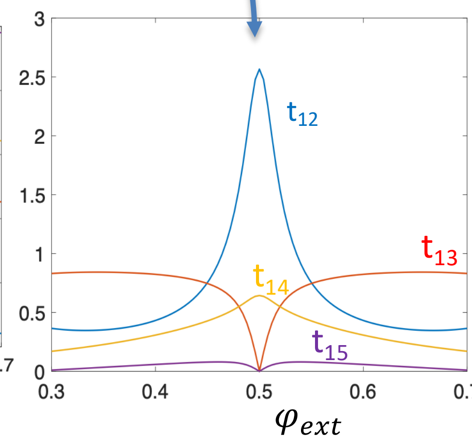
$$t_{nm} = \langle n|\hat{\varphi}|m\rangle$$

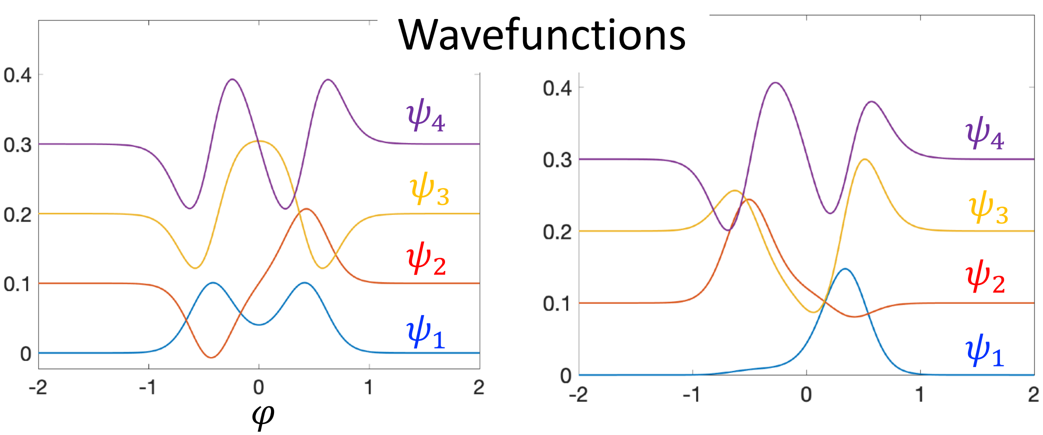
$$E_L = 256$$
$$E_J = 50$$
$$E_C = 100$$



Degeneracy point



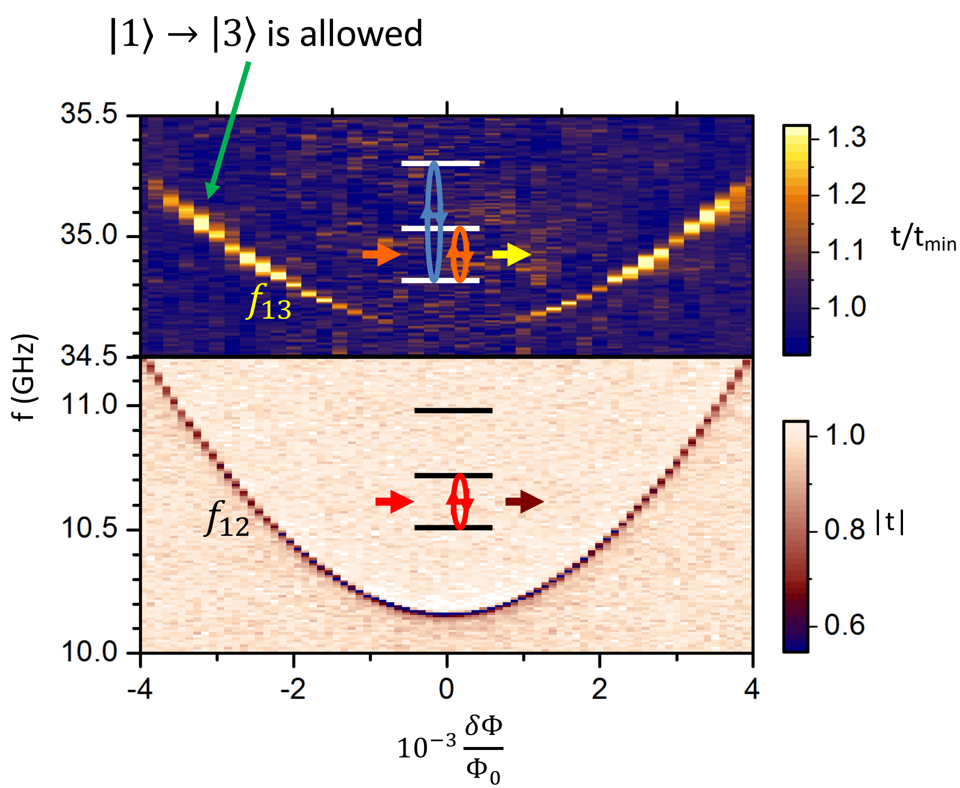


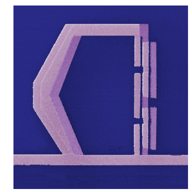


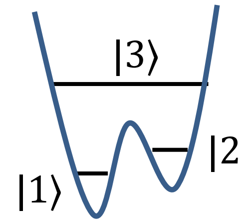
- ➢ Transition between symmetric and antisymmetric states (even-odd) are allowed
- ➢ Selection rules are working at the degeneracy point only

Transmission spectroscopy of three-level atom (flux qubit)

~ 2008 – 2009







Transmission coefficient
at different driving conditions