

Quantum Electronics of Nanostructures

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Lecture 5

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- Two-level system interacting with a harmonic oscillator
- Jaynes-Cummings Hamiltonian
- Energy bands of a two-level system coupled to a resonator
- Energy splitting in two-level system resonantly interacting with a resonator
- Matrix form of the Jaynes-Cummings Hamiltonian
- Measurement setup and experimental observation of qubit-resonator interaction

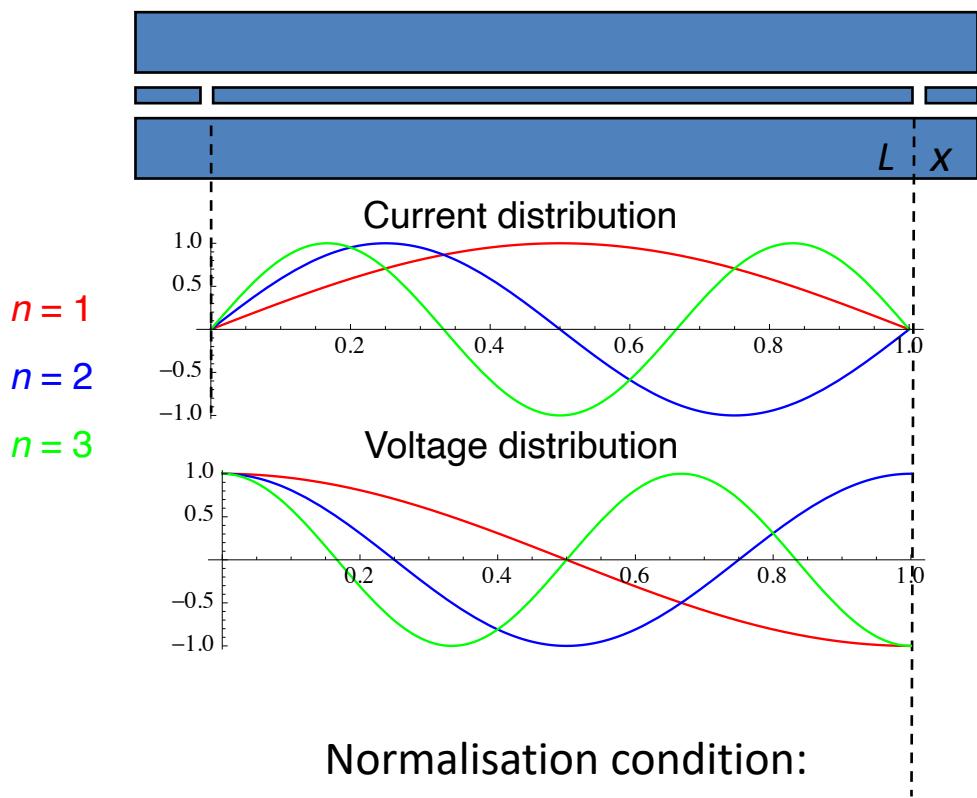
Two-level system coupled to a resonator

Cavity Quantum Electrodynamics

Field quantisation in coplanar resonators

$$I_0 \sin\left(\frac{\pi n}{L}x\right)$$

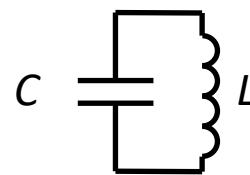
$$V = V_0 \cos\left(\frac{\pi n}{L}x\right)$$



$$\int_0^L \cos^2\left(\frac{\pi n x}{L}\right) dx = \frac{1}{2}$$

$$\int_0^L \sin^2\left(\frac{\pi n x}{L}\right) dx = \frac{1}{2}$$

Therefore $\sqrt{2} \cos\left(\frac{\pi x}{L}\right)$ and $\sqrt{2} \sin\left(\frac{\pi x}{L}\right)$ account space distribution of voltage and current



$$\hat{V} = \sqrt{\frac{\hbar\omega_0}{2C}} (a^\dagger + a)$$

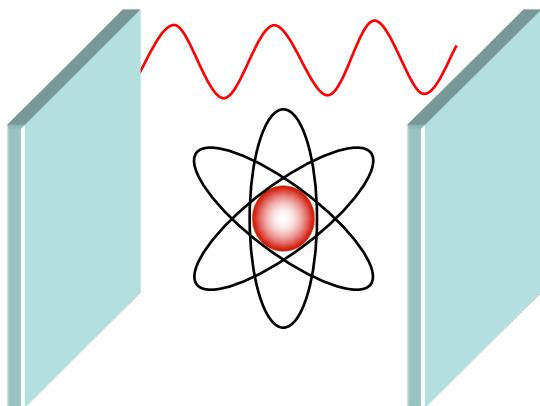
$$\hat{I} = i\sqrt{\frac{\hbar\omega_0}{2L}} (a^\dagger - a)$$

Voltage and current operators in a coplanar resonator:

$$\hat{V} = \sqrt{\frac{\hbar\omega_0}{C}} (a + a^\dagger) \cos\left(\frac{\pi n}{L}x\right)$$

$$\hat{I} = i\sqrt{\frac{\hbar\omega_0}{C}} (a^\dagger - a) \sin\left(\frac{\pi n}{L}x\right)$$

Atom in cavity versus atom in on-chip resonator Cavity QED versus Circuit QED

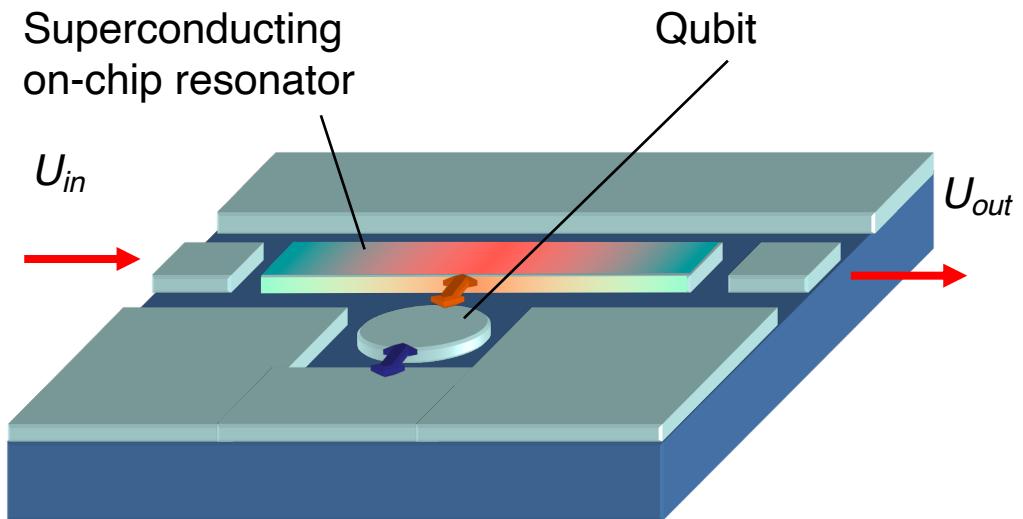
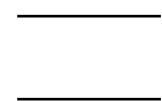


Wavelength $\lambda \sim 1 \mu\text{m}$

The atom Hamiltonian

$$H_a = -\frac{\Delta E}{2} \sigma_z$$

$$|n_a\rangle$$

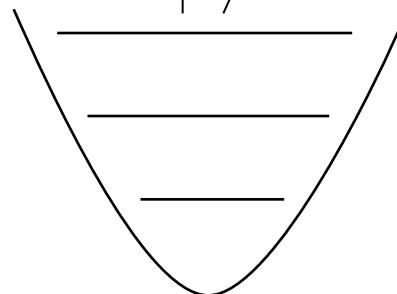


Wavelength $\lambda \sim 1 \text{ cm}$

The resonator Hamiltonian

$$H_r = \hbar\omega \left(\hat{N} + \frac{1}{2} \right)$$

$$|N\rangle$$



Two-level atom creation/annihilation operators

To describe interaction of our atom with external fields we introduce operators of creation/annihilation for the atomic excited state

$$\sigma^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 1|$$

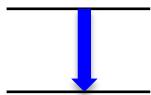
$$\sigma^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0|$$

$$\sigma^- |1\rangle = |0\rangle \quad \sigma^- |0\rangle = 0$$

$$\sigma^+ |0\rangle = |1\rangle \quad \sigma^+ |1\rangle = 0$$

$$|1\rangle \rightarrow |0\rangle$$

$$|0\rangle \rightarrow |1\rangle$$



$$\sigma^- = \frac{\sigma_x + i\sigma_y}{2}$$

$$\sigma^+ = \frac{\sigma_x - i\sigma_y}{2}$$

$$\sigma_x = \sigma^+ + \sigma^-$$

Atom-resonator interaction

The system is described by two quantum numbers:

- qubit state n_a : (0, 1)
- photon number state N : (0, 1, 2, 3, ...)

The system state:

$$|n_a N\rangle$$

Two-level system
(atom)

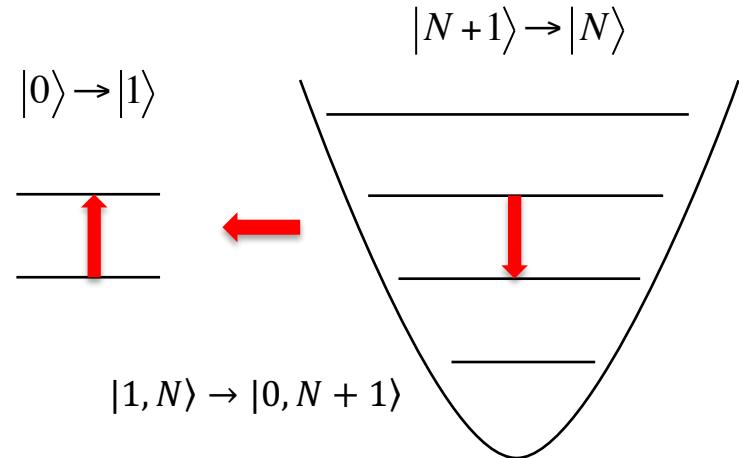
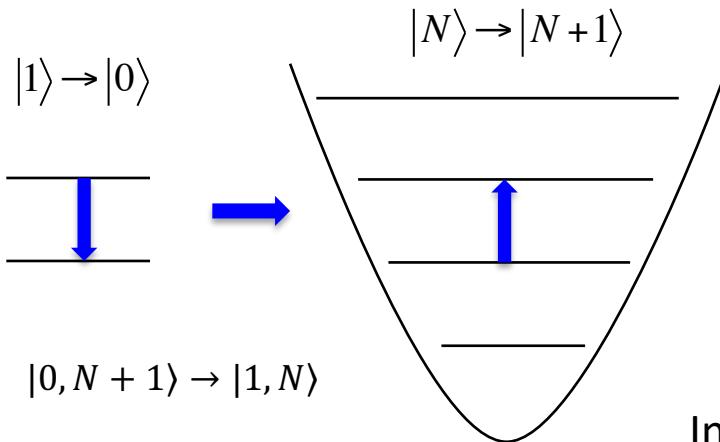
$$\begin{array}{c} |n_a\rangle \\ \hline \\ \hline \end{array}$$

Harmonic oscillator
(resonator)

$$|N\rangle$$

The Hamiltonian of the total system:

$$H_{tot} = H_a + H_r + H_{int}$$



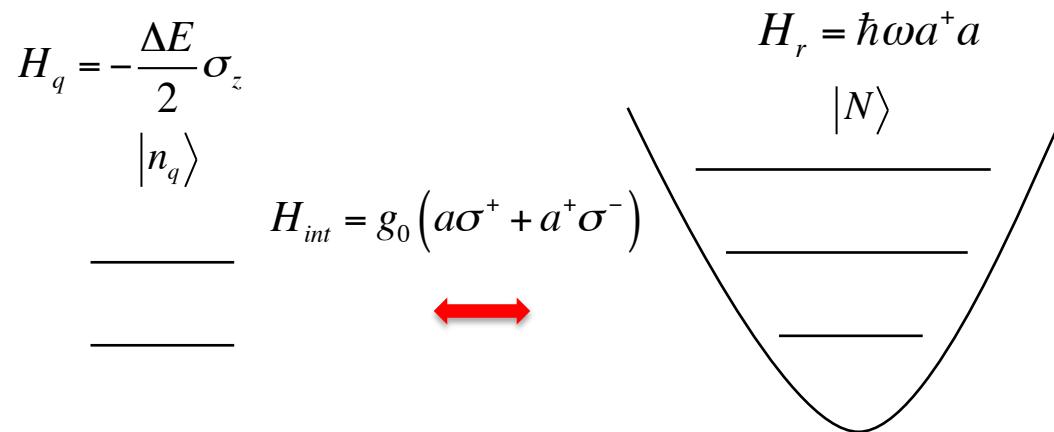
Interaction:

$$|0, N+1\rangle\langle 1, N| + |1, N\rangle\langle 0, N+1|$$

$$a^+ \sigma^- + a^- \sigma^+$$

Jaynes-Cummings Hamiltonian
and
energy bands of a two-level system coupled to a resonator

Two-level system interacting with harmonic oscillator Jaynes-Cummings Hamiltonian



Interaction of the two-level system with the resonator

$$H_{JC} = -\frac{\Delta E}{2} \sigma_z + \hbar\omega_r a^+ a + g_0(a\sigma^+ + a^+\sigma^-)$$

Qubit (atom) Oscillator Qubit-resonator interaction

Voltage operator of the charge qubit (degeneracy point)

Potential (voltage) is the energy to remove a unit charge:

$$V = \frac{\partial E}{\partial q}$$

Charge qubit Hamiltonian ($E_C \gg E_J$):

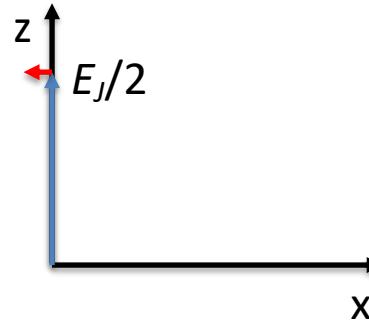
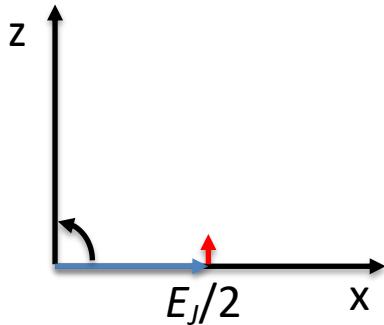
$$H = -\frac{2E_C n}{2}\sigma_z - \frac{E_J}{2}\sigma_x = -\frac{\Delta E}{2}\left(\cos\frac{\theta}{2}\sigma_z + \sin\frac{\theta}{2}\sigma_x\right)$$

where $\Delta E = \sqrt{\varepsilon^2 + E_J^2}$ and $\varepsilon = 2E_C n$

In vicinity of the degeneracy point ($n \ll 1$), the Hamiltonian can be rewritten as

$$H' = -\frac{E_J}{2}\sigma_z + \frac{2E_C n}{2}\sigma_x \xrightarrow{0}$$

Basis of diagonalized Hamiltonian



Voltage operator in the charge qubit with a single Cooper pair:

$$\hat{V}_q = \frac{\partial H}{\partial q} = \frac{E_C}{2e}\sigma_x = V_{0q}(\sigma^- + \sigma^+) \quad \text{compare with the resonator field } \hat{V}_r = V_{0r}(a + a^\dagger)$$

Voltage operator of the charge qubit in the general case

$$H = -\frac{2E_C n}{2}\sigma_z - \frac{E_J}{2}\sigma_x = -\frac{\Delta E}{2}\left(\cos\frac{\theta}{2}\sigma_z + \sin\frac{\theta}{2}\sigma_x\right)$$

Voltage operator: $\hat{V} = \frac{\partial H}{\partial q} = -\frac{E_C}{2e}\sigma_z$

The unitary operator $U = I \cos\frac{\theta}{2} - i\sigma_y \sin\frac{\theta}{2}$ transforms H to the diagonal form

$$H' = UHU^\dagger = -\frac{\Delta E}{2}\sigma_z$$

$$\hat{V}' = U\hat{V}U^\dagger = -\frac{E_C}{2e}U\sigma_zU^\dagger = -\frac{E_C}{2e}(\underbrace{\sigma_z \cos\theta}_{\hat{V}_q^{\parallel}} + \underbrace{\sigma_x \sin\theta}_{\hat{V}_q^{\perp}})$$

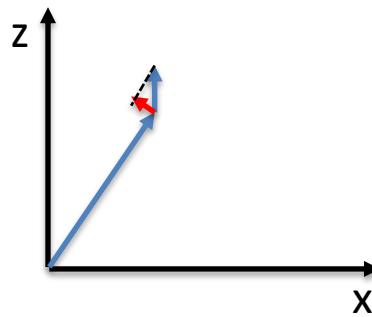
Off-diagonal matrix elements \hat{V}_q^{\perp} produce transitions in the two-level system

Qubit dipole voltage operator:

$$\hat{V}_q^{\perp} = V_{q0}\sigma_x$$

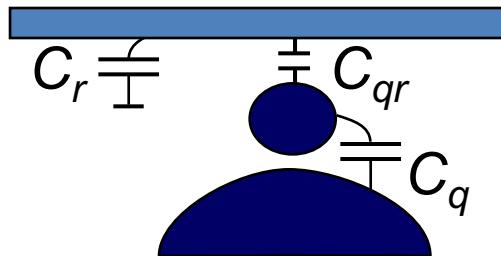
$$V_{q0} = -\frac{E_C}{2e}\sin\theta$$

Far away from the degeneracy point $\hat{V}_q^{\perp} \rightarrow 0$

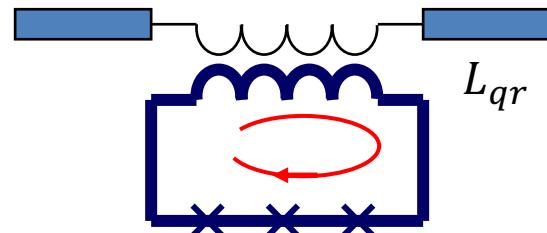


Qubits coupled to a harmonic oscillator

Charge qubit coupled capacitively



Flux qubit coupled inductively



Charge qubit: $H_{int} = \hat{V}_q C_{qr} \hat{V}_r$

Flux qubit: $H_{int} = \hat{I}_q L_{qr} \hat{I}_r$

Off-diagonal matrix elements produce transitions in the two-level system

Qubit dipole voltage operator:

$$\hat{V}_q^\perp = V_{q0} \sigma_x$$

$$V_0 = \frac{E_C}{2e} \sin \theta$$

Resonator voltage operator:

$$\hat{V}_r = V_{r0} (a^\dagger + a)$$

$$H_{int} = g_0 \sigma_x (a^\dagger + a)$$

$$V_{r0} = \sqrt{\frac{\hbar \omega}{C_r}}$$

$$g_0 = C_{qr} V_{q0} V_{r0}$$

Interaction Hamiltonian close to resonance

$$\Delta E \approx \hbar\omega$$

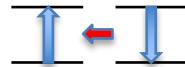
$$H_{\text{int}} = g_0 \sigma_x (a + a^\dagger)$$

$$\sigma_x = \sigma^+ + \sigma^-$$

$$H_{\text{int}} \approx g_0 (a + a^\dagger) (\sigma^+ + \sigma^-) = g_0 (a\sigma^+ + a^\dagger\sigma^+ + a\sigma^- + a^\dagger\sigma^-)$$

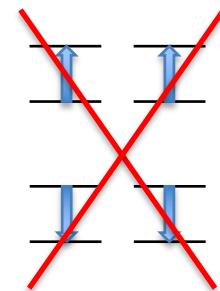
Processes with no energy change

$$\sigma^+ a |0N\rangle = \sqrt{N} |1N\rangle$$

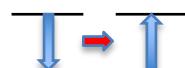


Processes with energy change by $\pm 2\hbar\omega$

$$\sigma^+ a^\dagger |0N\rangle = \sqrt{N+1} |1(N+1)\rangle$$



$$\sigma^- a^\dagger |1N\rangle = \sqrt{N+1} |0(N+1)\rangle$$



$$\sigma^- a |1N\rangle = \sqrt{N} |0(N-1)\rangle$$



$$H_{\text{int}} \approx g_0 (a\sigma^+ + a^\dagger\sigma^-)$$

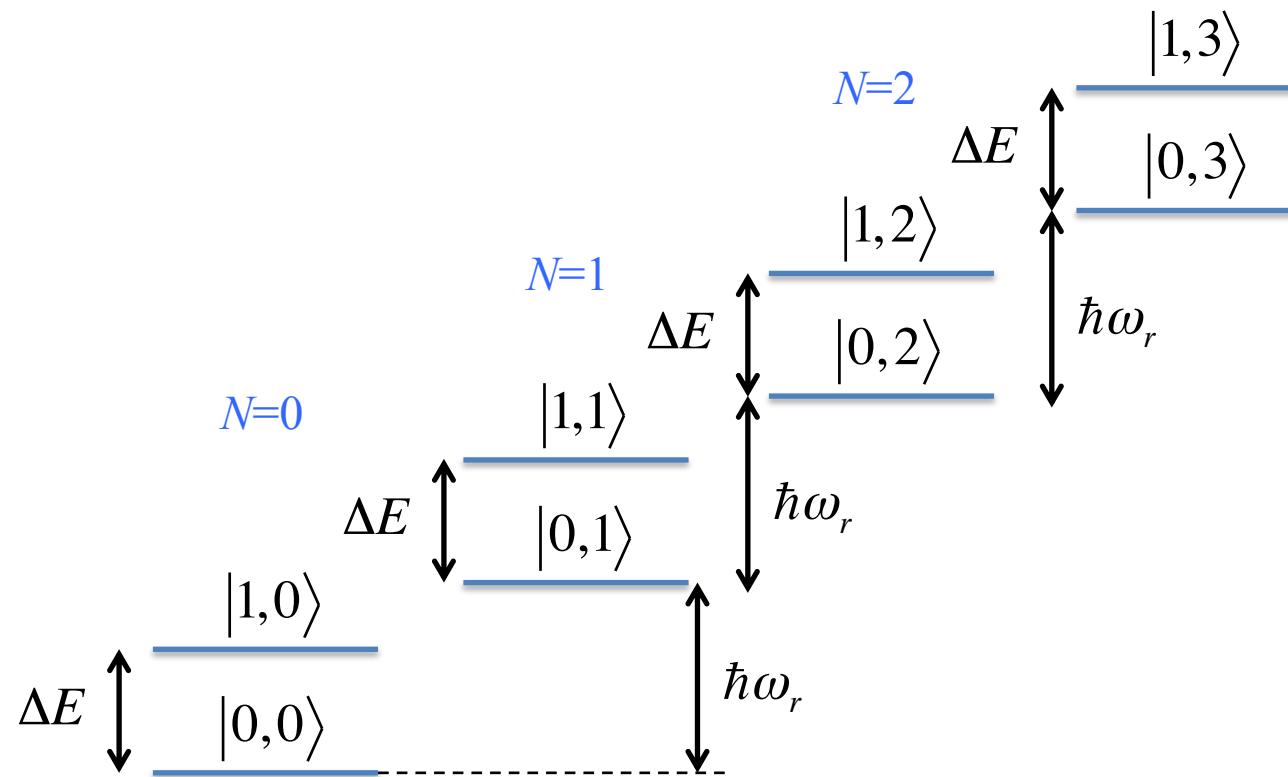
Two-level atom in a resonator ($g = 0$)

$$|n\rangle \otimes |N\rangle = |n, N\rangle$$

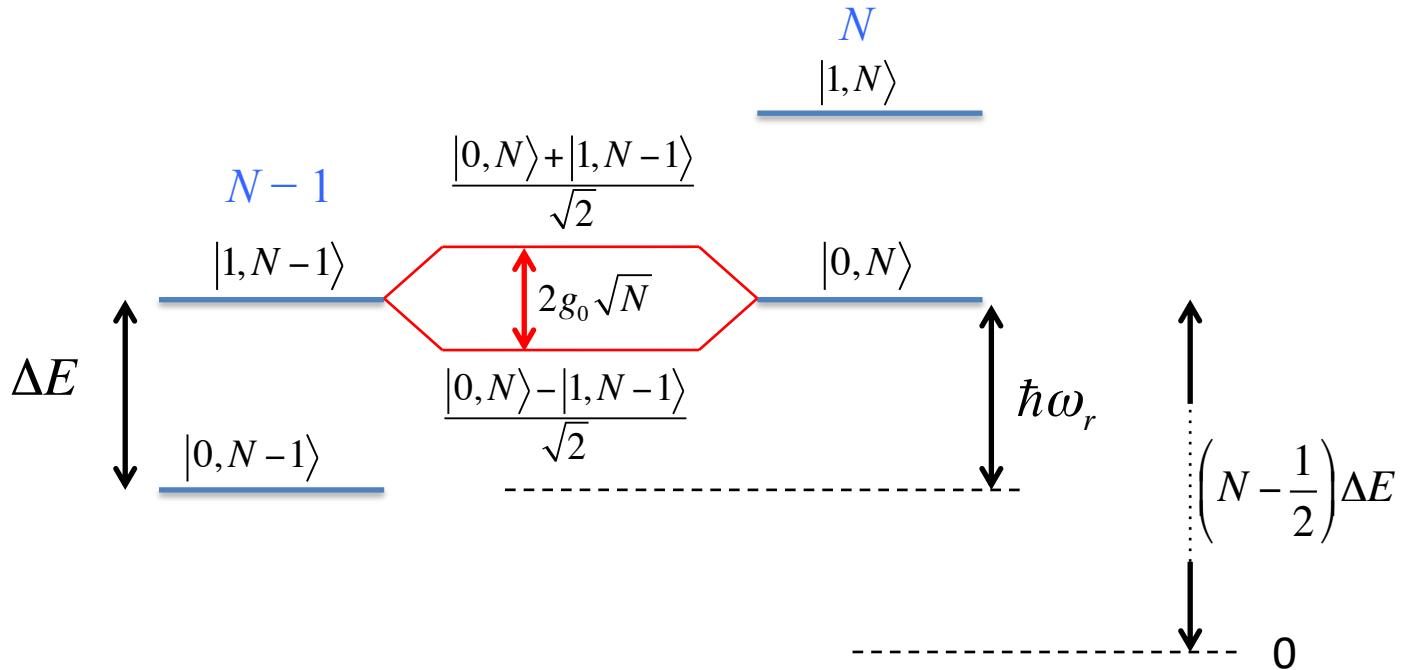
n is the atomic state (0 or 1)
 N is the number of photons in the resonator

$|n, N\rangle$ are eigenstates of $H_0 = -\frac{\Delta E}{2}\sigma_z + \hbar\omega_r a^\dagger a$

$$\left(-\frac{\Delta E}{2}\sigma_z + \hbar\omega_r a^\dagger a\right)|n, N\rangle = \left[\left(n - \frac{1}{2}\right)\Delta E + N\hbar\omega_r\right]|n, N\rangle$$



Atom-resonator resonance ($\Delta E = \hbar\omega$)



$$H_0 = -\frac{\Delta E}{2} \sigma_z + \hbar\omega a^\dagger a$$

$$H_{int} = g_0(a^\dagger \sigma^- + a \sigma^+)$$

$$H_0 |1, N-1\rangle = \left[\frac{\Delta E}{2} + (N-1)\hbar\omega \right] |1, N-1\rangle = \left(N - \frac{1}{2} \right) \Delta E |1, N-1\rangle$$

$$H_0 |0, N\rangle = \left[-\frac{\Delta E}{2} + N\hbar\omega \right] |0, N\rangle = \left(N - \frac{1}{2} \right) \Delta E |0, N\rangle$$

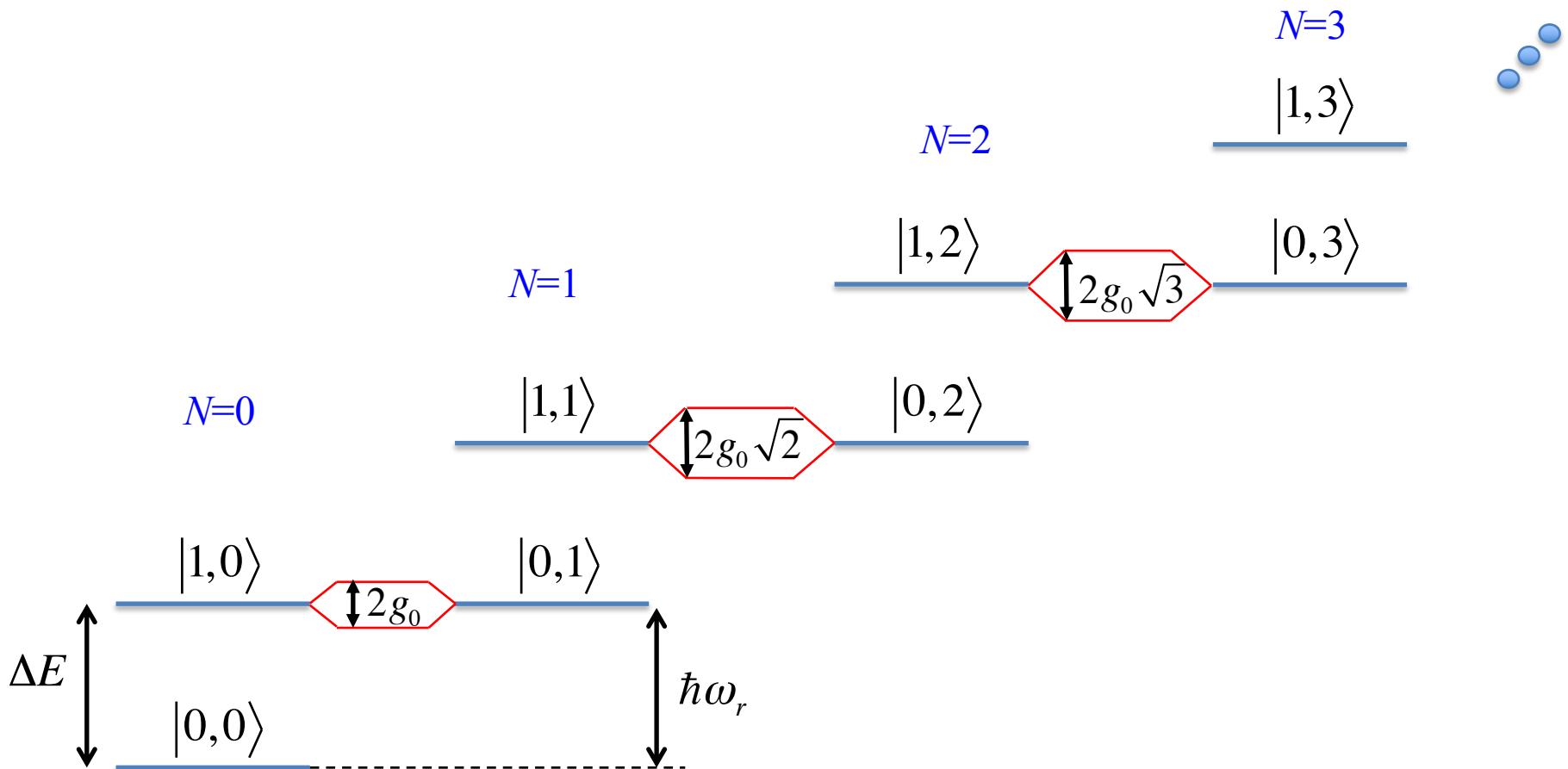
$$H_{int} \frac{|0, N\rangle \pm |1, N-1\rangle}{\sqrt{2}} = g_0(a^\dagger \sigma^- + a \sigma^+) \frac{|0, N\rangle \pm |1, N-1\rangle}{\sqrt{2}} = g_0 \sqrt{N} \frac{|1, N-1\rangle \pm |0, N\rangle}{\sqrt{2}} = \pm g_0 \sqrt{N} \frac{|0, N\rangle \pm |1, N-1\rangle}{\sqrt{2}}$$

$$H_{JC} \frac{|0, N\rangle \pm |1, N-1\rangle}{\sqrt{2}} = \left[\left(N - \frac{1}{2} \right) \Delta E \pm g_0 \sqrt{N} \right] \frac{|0, N\rangle \pm |1, N-1\rangle}{\sqrt{2}}$$

Two-level atom in the resonator ($g \neq 0$)

$$|n\rangle \otimes |N\rangle = |n, N\rangle$$

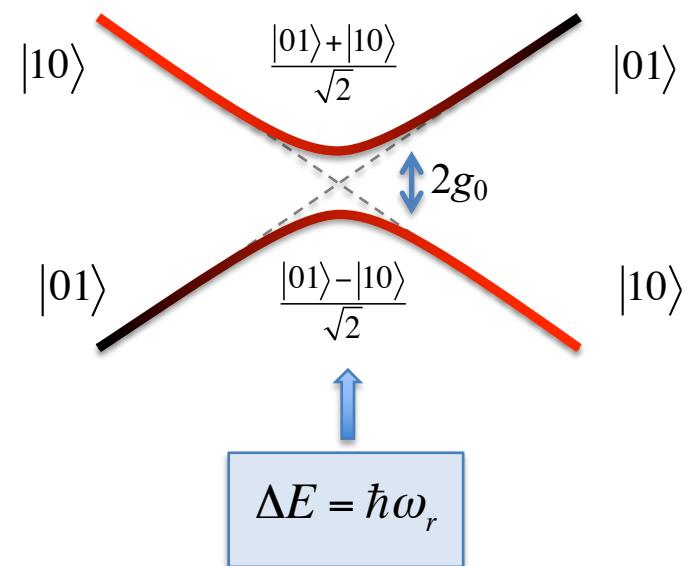
n is the atomic state (0 or 1)
 N is the number of photons in the resonator



Single-photon interaction

$$H_{JC} = -\frac{\Delta E}{2}\sigma_z + \hbar\omega_r a^\dagger a + g_0(a\sigma^+ + a^\dagger\sigma^-) \quad |n_q N\rangle$$

$$H = \begin{pmatrix} |00\rangle & |10\rangle & |01\rangle & |11\rangle \\ -\frac{\Delta E}{2} & 0 & 0 & 0 \\ 0 & \frac{\Delta E}{2} & g_0 & 0 \\ g_0 & -\frac{\Delta E}{2} + \hbar\omega_r & 0 & 0 \\ 0 & 0 & 0 & \frac{\Delta E}{2} + \hbar\omega_r \end{pmatrix} \begin{pmatrix} |00\rangle \\ |10\rangle \\ |01\rangle \\ |11\rangle \end{pmatrix}$$

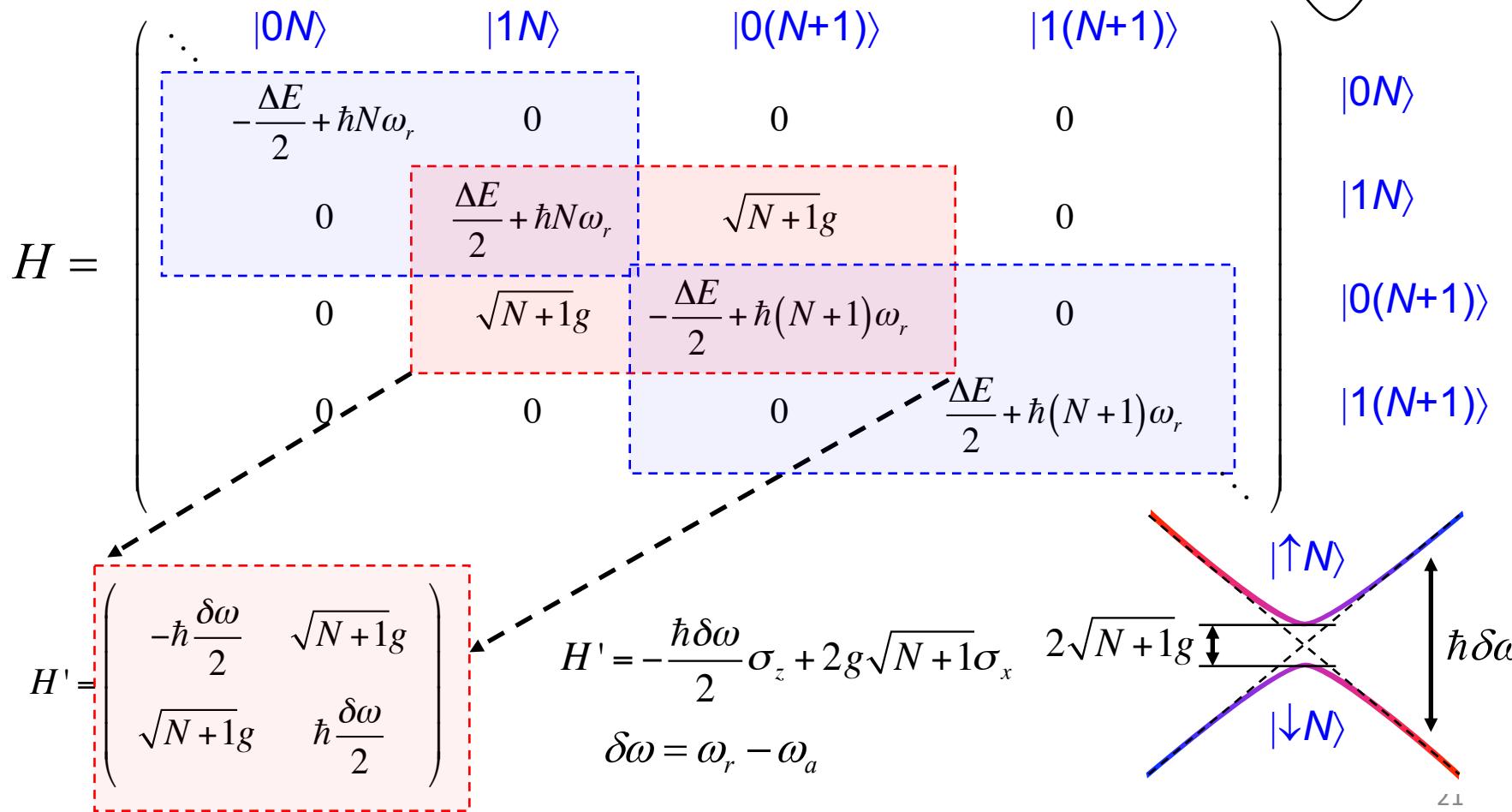


$$H = \begin{pmatrix} \frac{\Delta E}{2} & g_0 \\ g_0 & -\frac{\Delta E}{2} + \hbar\omega_r \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{\Delta E - \hbar\omega_r}{2} & g_0 \\ g_0 & \frac{-\Delta E + \hbar\omega_r}{2} \end{pmatrix} = \frac{\Delta E - \hbar\omega_r}{2}\sigma_z + \frac{2g_0}{2}\sigma_x$$

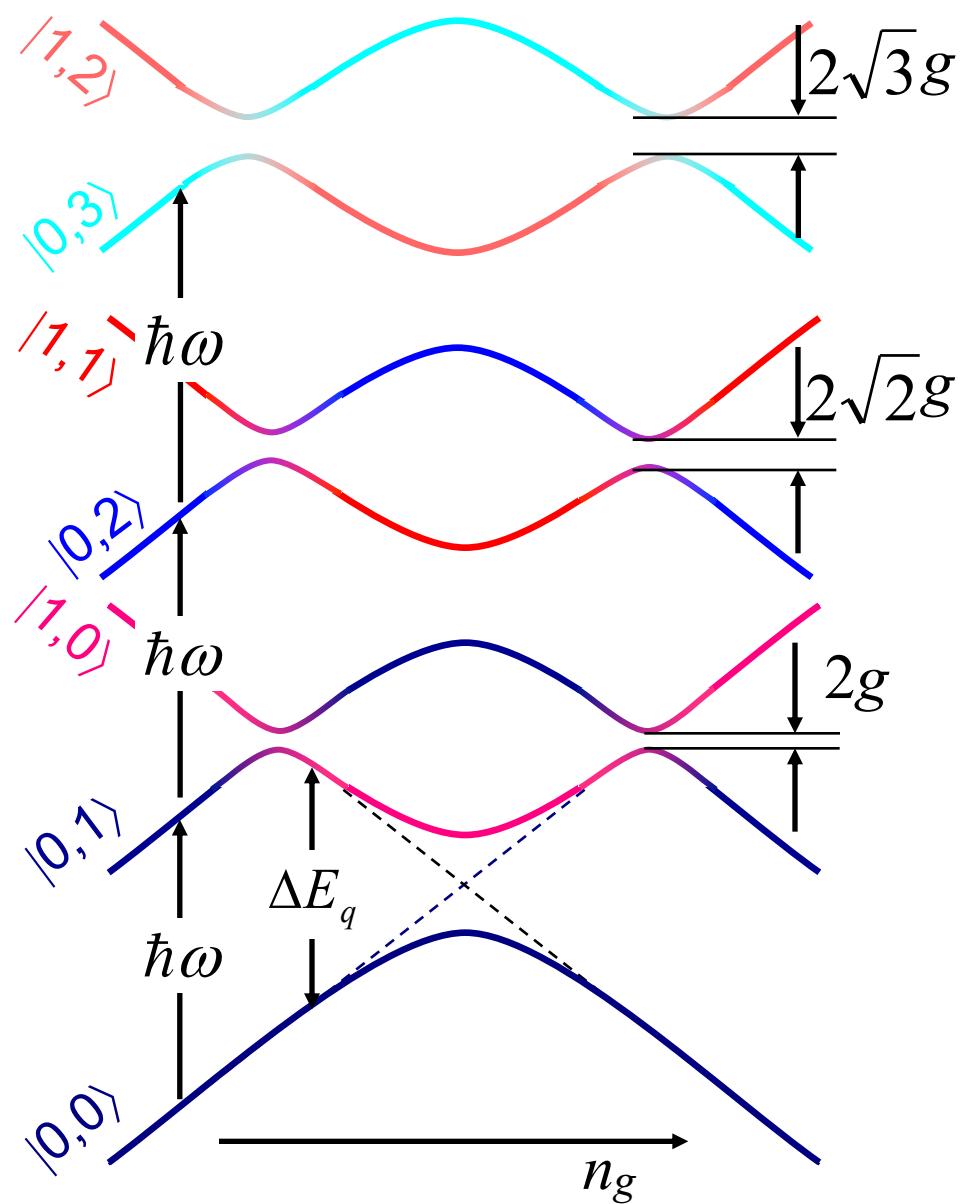
Compare with the qubit Hamiltonian in the physical basis: $H = -\frac{\epsilon}{2}\sigma_z - \frac{\Delta}{2}\sigma_x$

Nearly resonant interaction of atom-harmonic oscillator (dressed states)

$$H_{JC} = -\frac{\Delta E}{2}\sigma_z + g(a\sigma^+ + a^\dagger\sigma^-) + \hbar\omega_r a^\dagger a$$



Qubit-resonator energy diagram



Measurement set-up for microwave characterization of superconducting quantum systems

Dilution refrigerator and measurement equipment

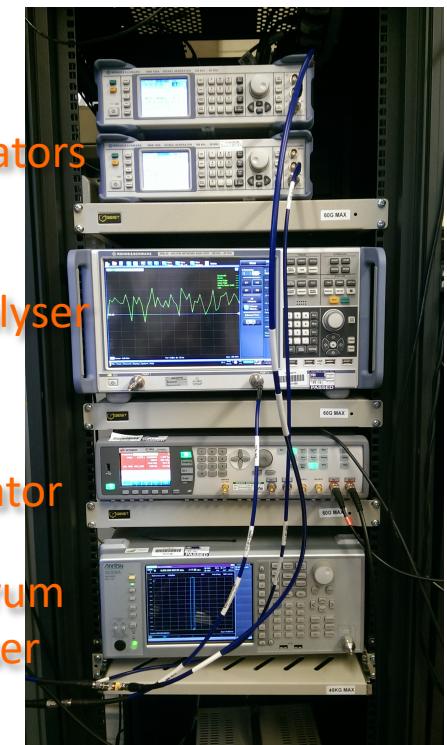
Dilution refrigerator



Internal view

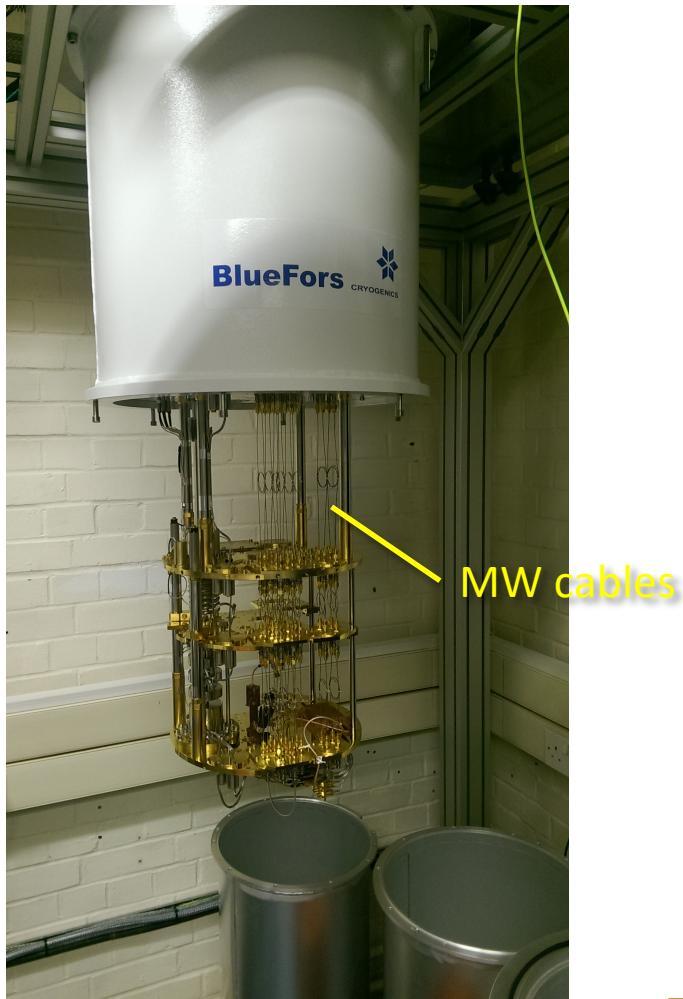


Measurement MW equipment

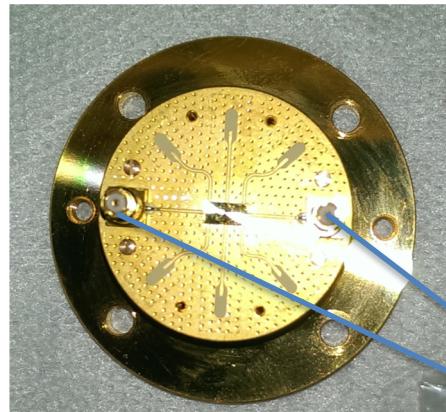


The lowest temperature is 10 mK

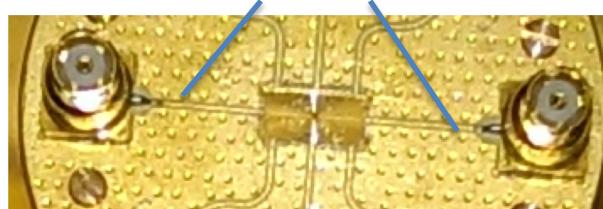
Bottom view of a cold plate



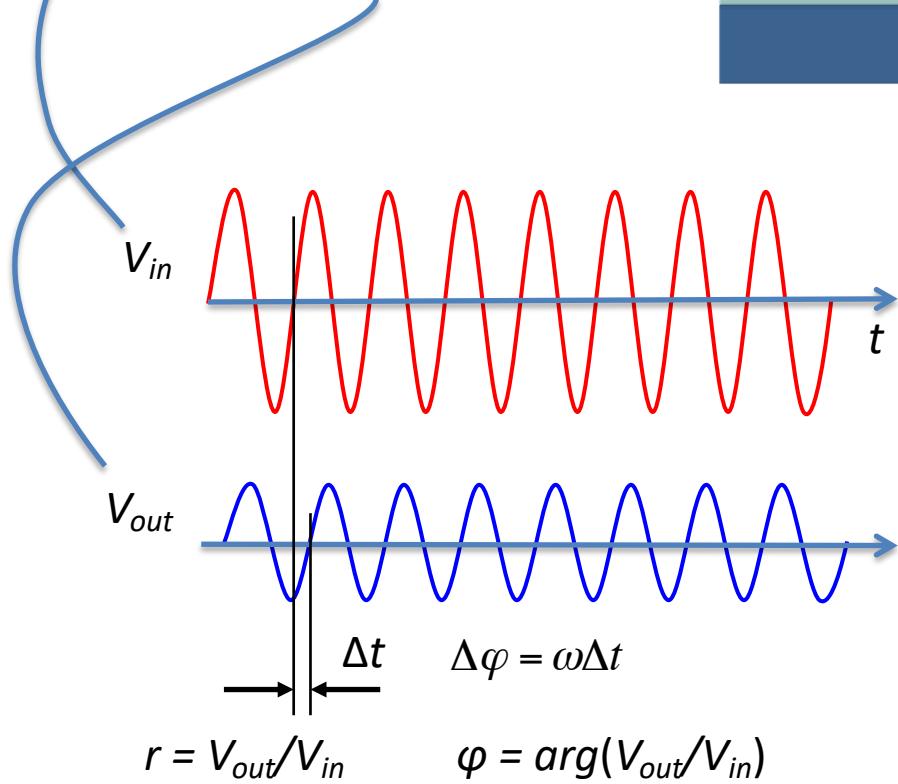
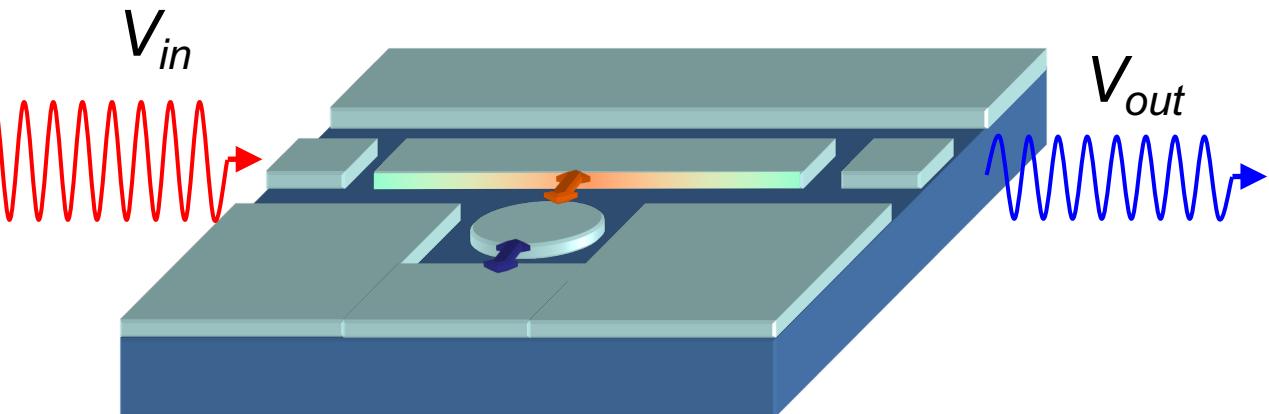
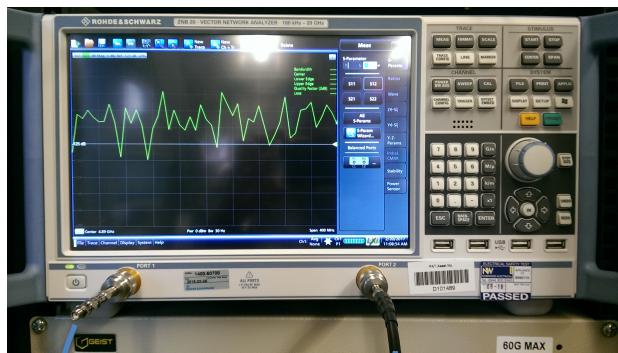
Sample holder



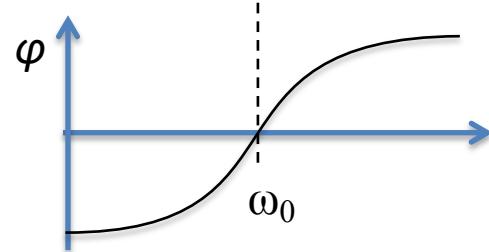
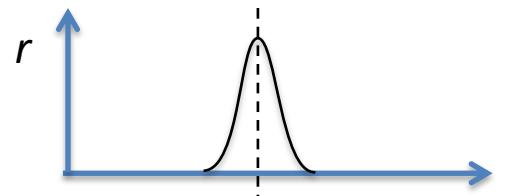
MW lines



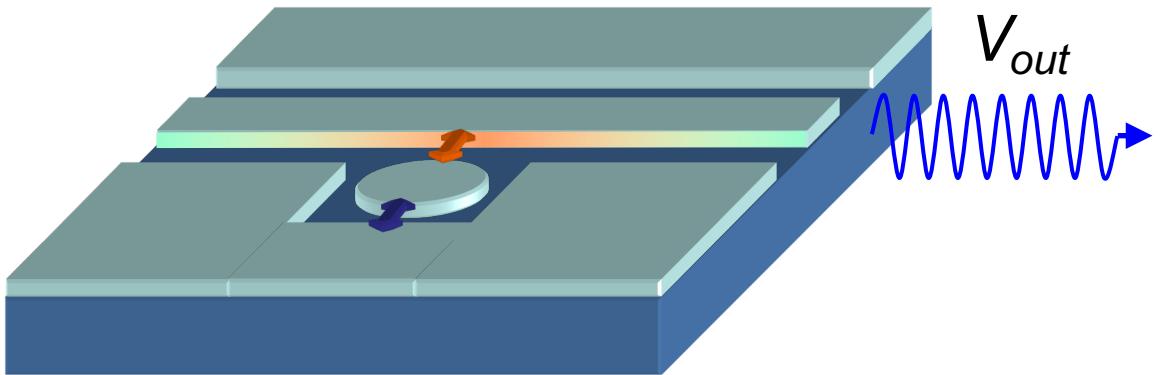
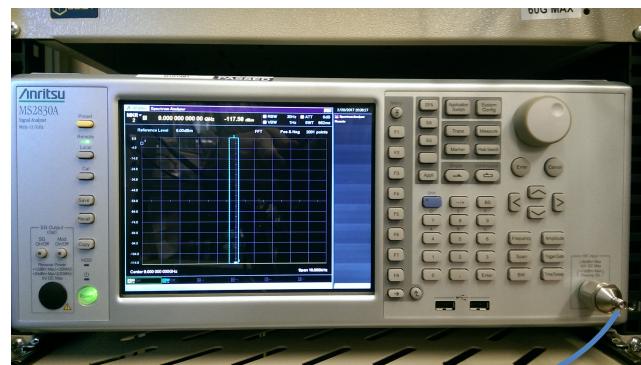
Phase-sensitive detection of transmitted signal by a network analyzer



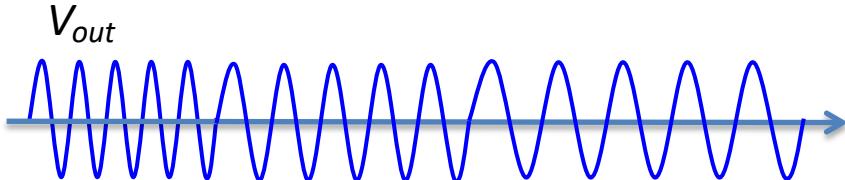
$$V_{out} = \frac{V_{in}}{1 + i2(\omega - \omega_0)/\Delta\omega}$$



Spectrum detection of emission by a spectrum analyzer

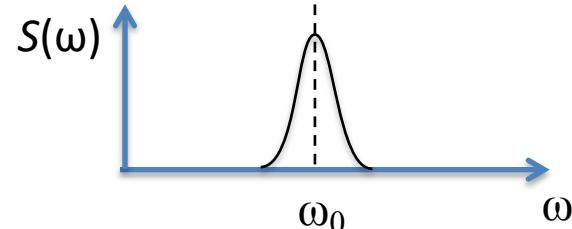


The output signal is uncorrelated in phase with excitation

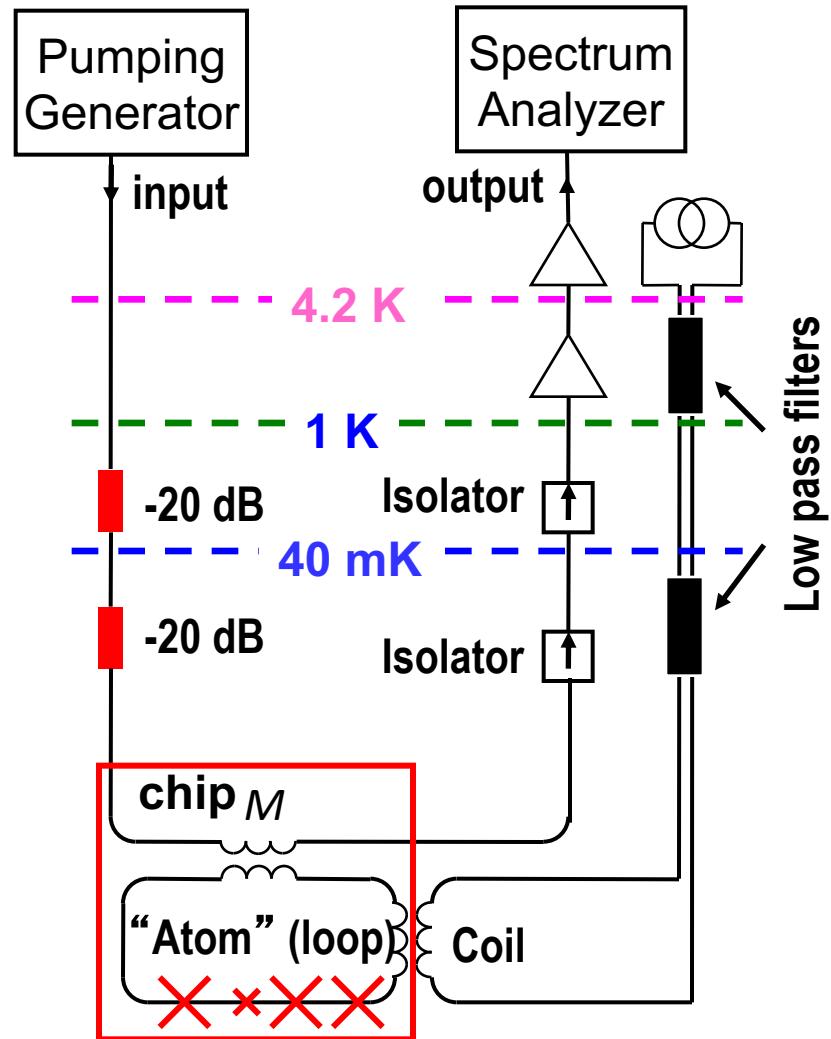
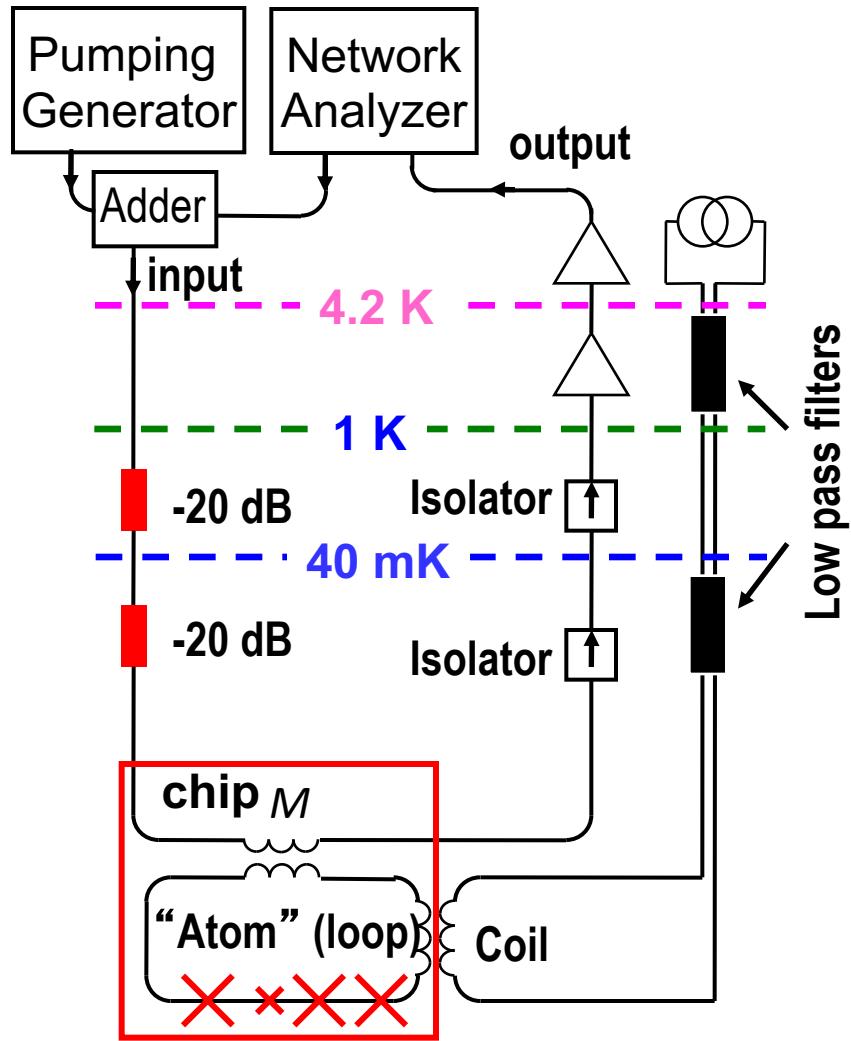


Spectral power density: $S(\omega)$

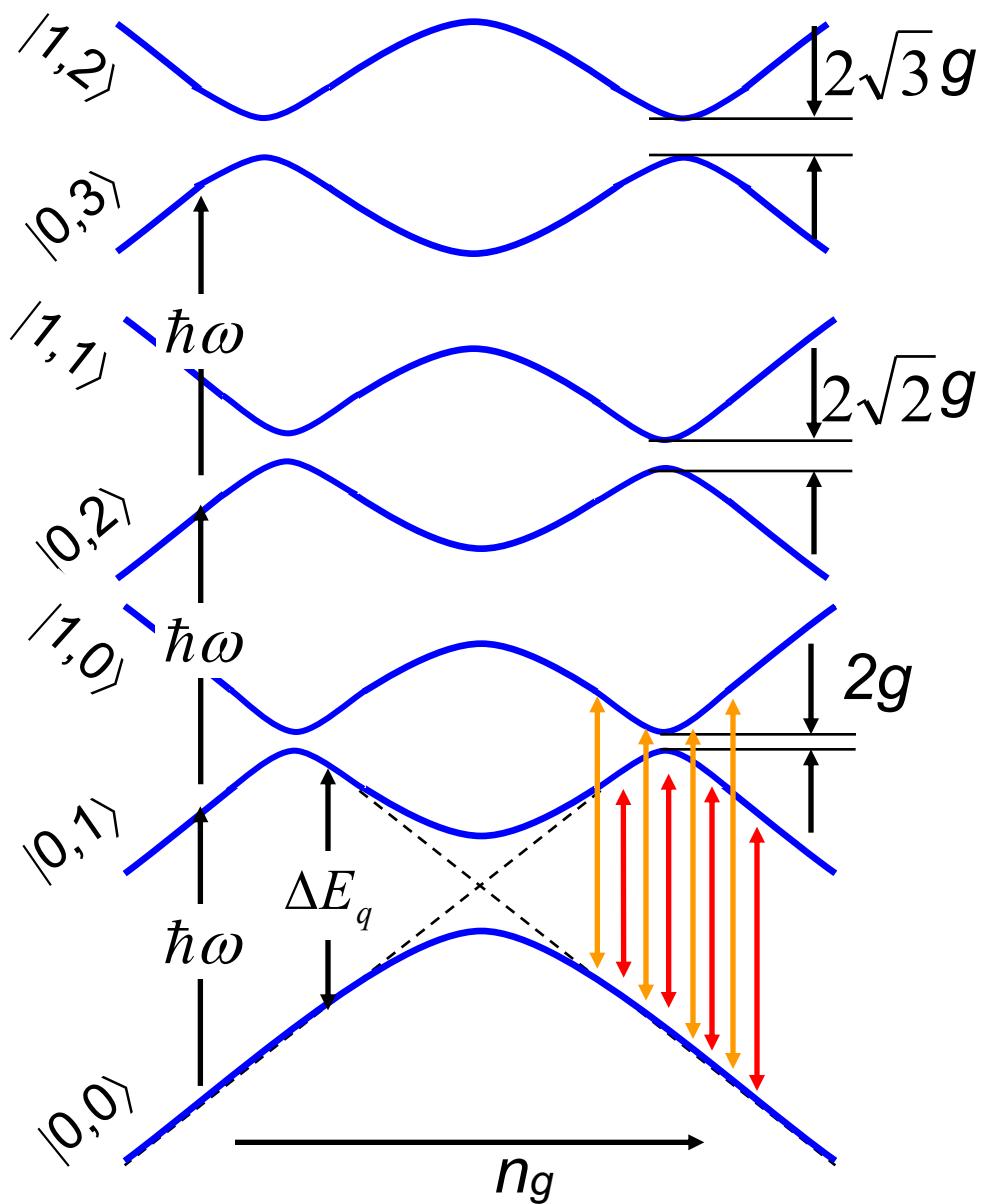
$$S_{out}(\omega) = \frac{S_0}{1 + [2(\omega - \omega_0)/\Delta\omega]^2}$$



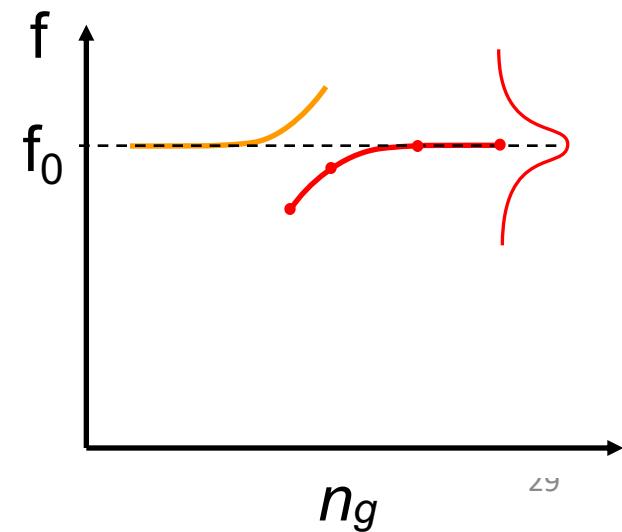
Measurement circuits for Superconducting quantum systems



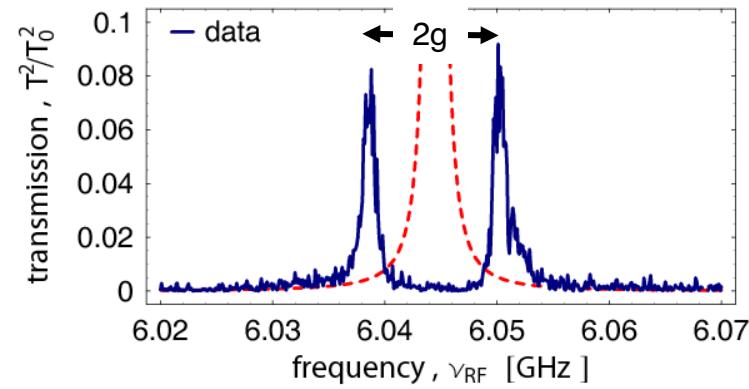
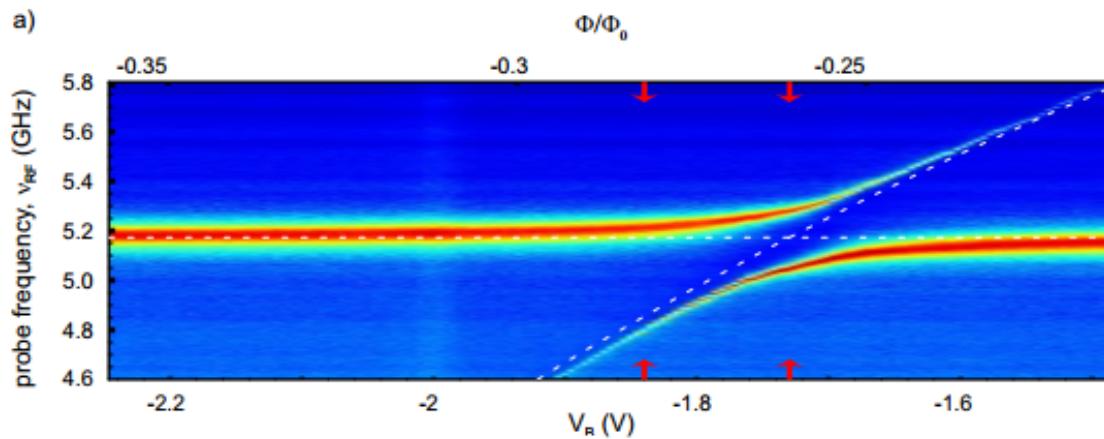
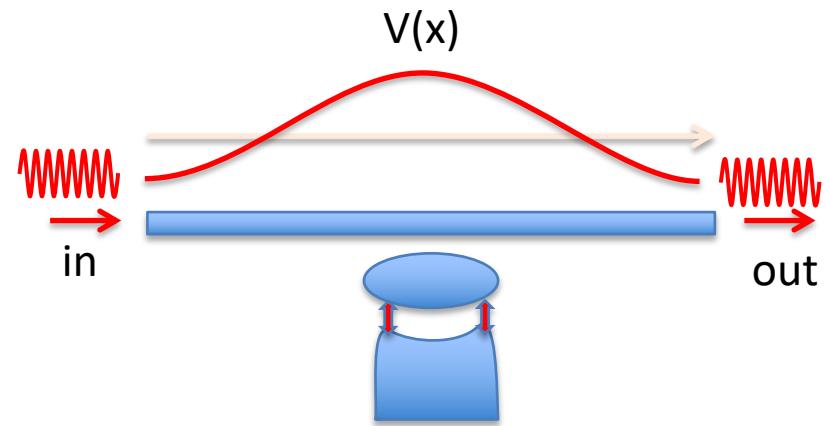
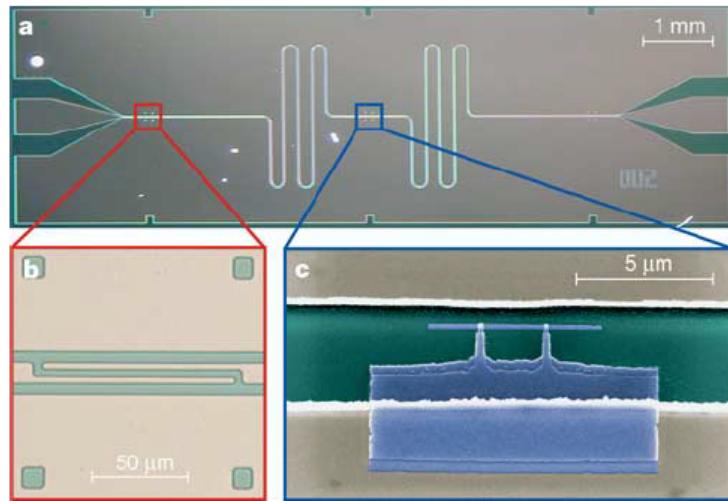
Microwave (optical) probing



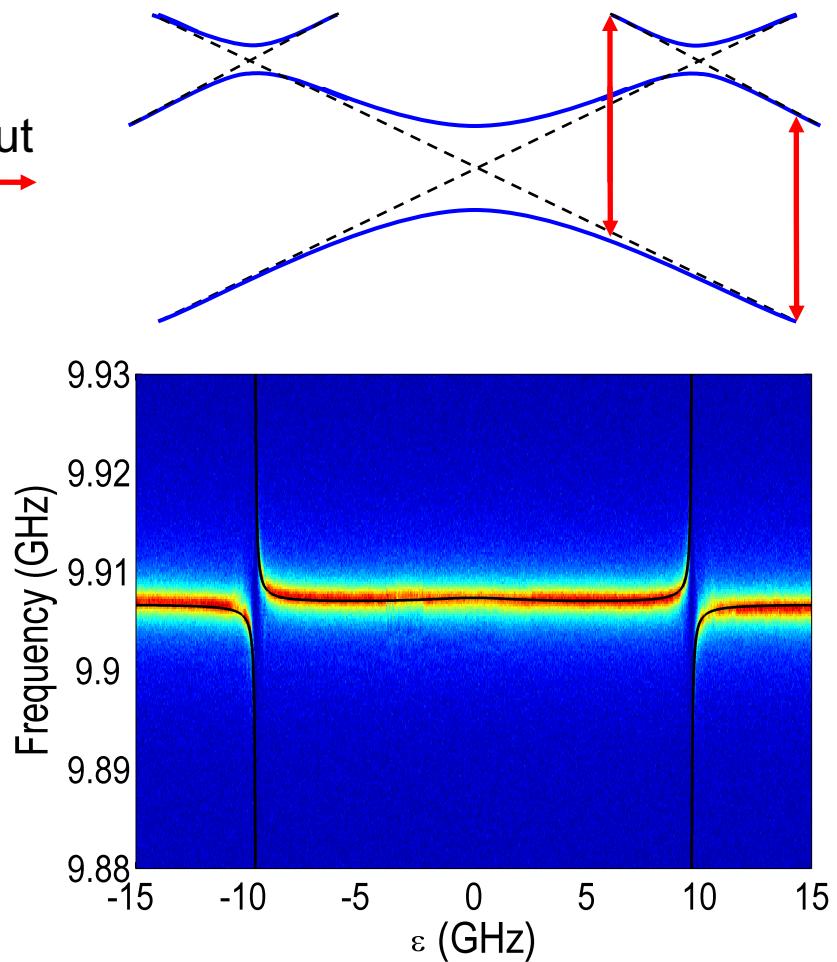
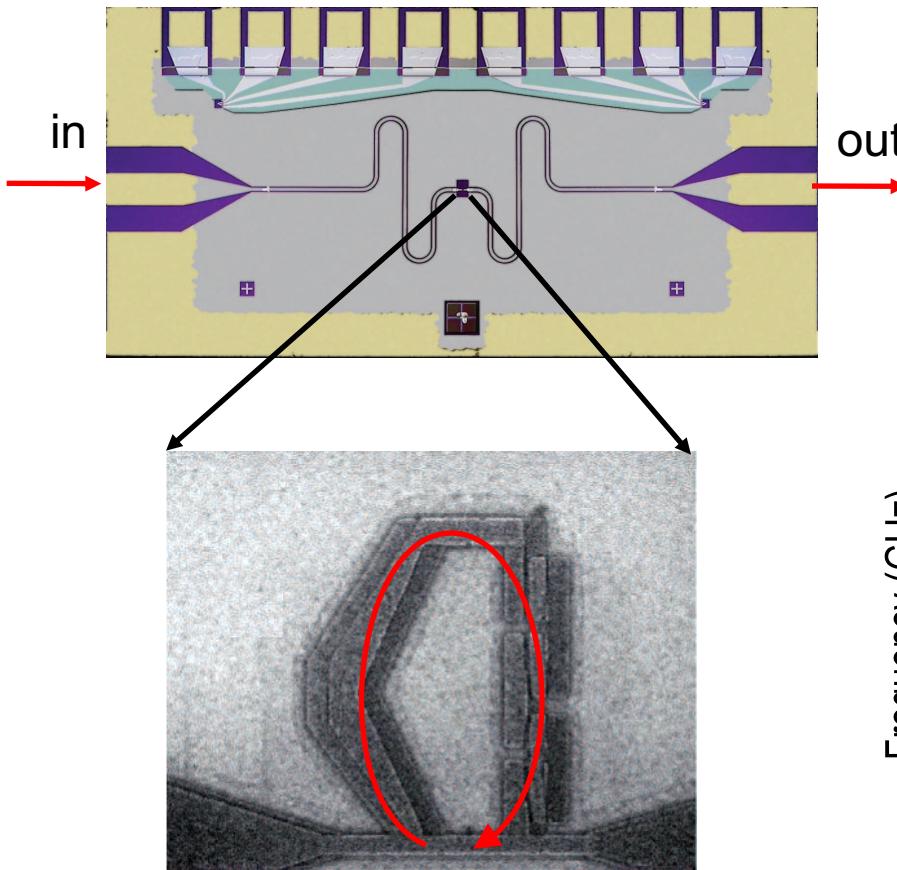
Spectroscopy:
Transitions in the
oscillator:
 $|n,0\rangle \rightarrow |n,1\rangle$



Qubit coupled to a resonator



Flux qubit coupled to the coplanar waveguide resonator



We demonstrated coupling of the flux qubit to the resonator