Quantum Electronics of Nanostructures

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Lecture 5

Important dates

April, 27, 28 – lectures
May, 8 – seminar
May, 12 – lecture
May 15 – laboratory
May, 18, 19 – lectures
May 22 – laboratory
May, 25, 26 – presentations
May, 29 – seminar
June, 1, 2 – reserved (presentations, revision lecture?)
June, 5 – final exam

April 2020

M	T	W	T	F	S	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30			

May 2020

M	T	W	T	F	S	S
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

June 2020

M	T	W	T	F	S	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30					

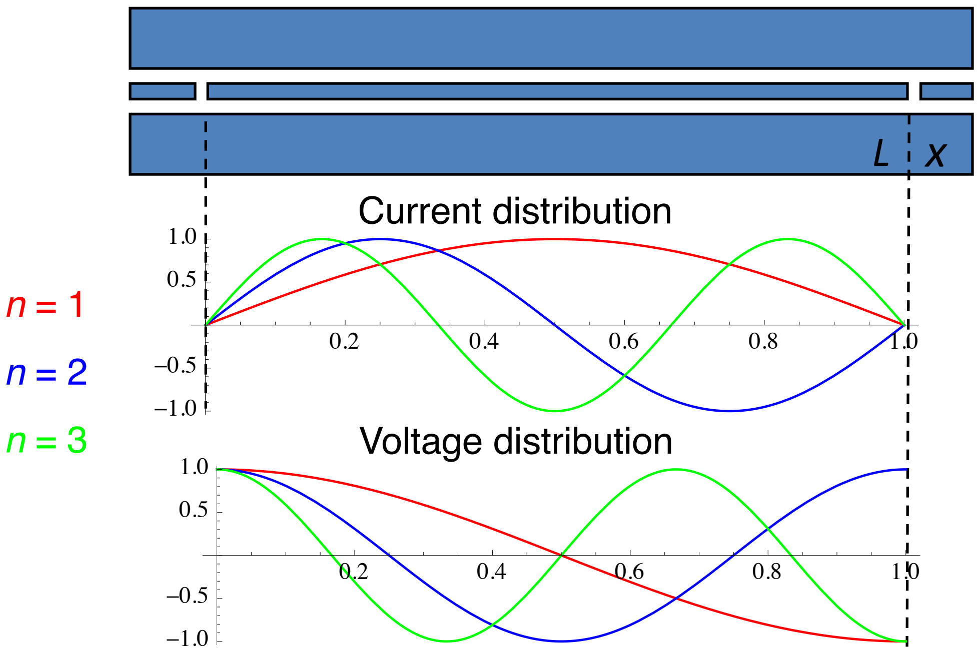
Lecture 5

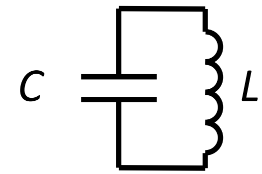
- Two-level system interacting with a harmonic oscillator
- Jaynes-Cummings Hamiltonian
- Energy bands of a two-level system coupled to a resonator
- Energy splitting in two-level system resonantly interacting with a resonator
- Matrix form of the Jaynes-Cummings Hamiltonian
- Measurement setup and experimental observation of qubit-resonator of interaction

Two-level system coupled to a resonator
Cavity Quantum Electrodynamics

Field quantisation in coplanar resonators

$$I_0 \sin\left(\frac{\pi n}{L} x\right) \qquad V = V_0 \cos\left(\frac{\pi n}{L} x\right)$$





$$\hat{V} = \sqrt{\frac{\hbar\omega_0}{2C}}\left(a^{\dagger} + a\right)$$

$$\hat{I} = i\sqrt{\frac{\hbar\omega_0}{2L}}\left(a^{+} - a\right)$$

Voltage and current operators in a coplanar resonator:

$$\boxed{\hat{V} = \sqrt{\frac{\hbar\omega_0}{C}}\left(a + a^{+}\right)\cos\left(\frac{\pi n}{L} x\right)}$$

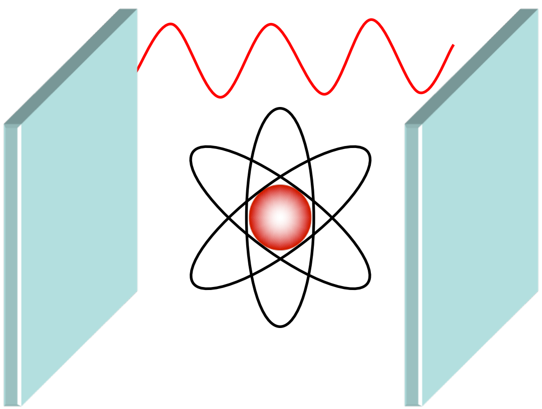
$$\boxed{\hat{I} = i\sqrt{\frac{\hbar\omega_0}{C}}\left(a^{\dagger} - a\right)\sin\left(\frac{\pi n}{L} x\right)}$$

Normalisation condition:

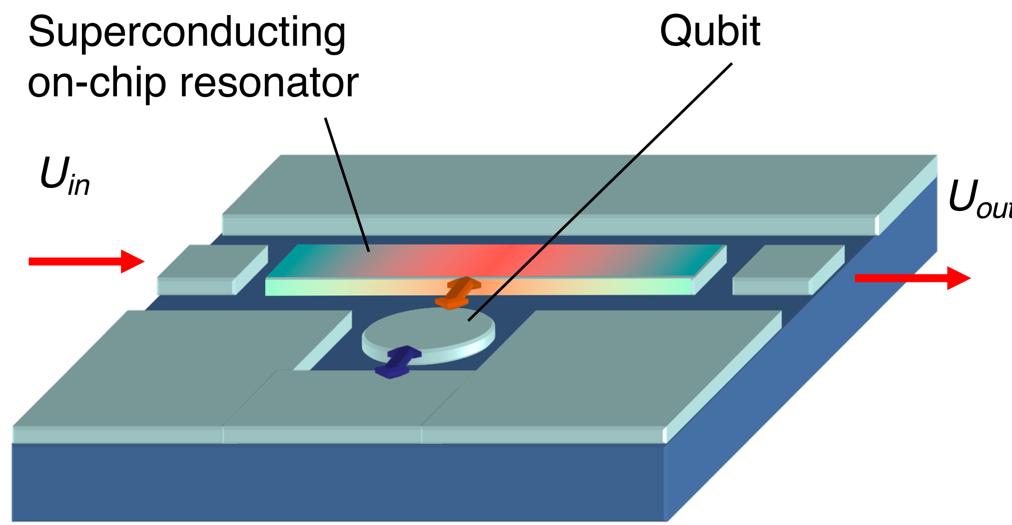
$$\int_0^L \cos^2\left(\frac{\pi n x}{L}\right) dx = \frac{1}{2} \qquad \int_0^L \sin^2\left(\frac{\pi n x}{L}\right) dx = \frac{1}{2}$$

Therefore $\sqrt{2}\cos\left(\frac{\pi x}{L}\right)$ and $\sqrt{2}\sin\left(\frac{\pi x}{L}\right)$ account space distribution of voltage and current

Atom in cavity versus atom in on-chip resonator
Cavity QED versus Circuit QED



Wavelength $\lambda \sim 1\ \mu$m



Wavelength $\lambda \sim 1$ cm

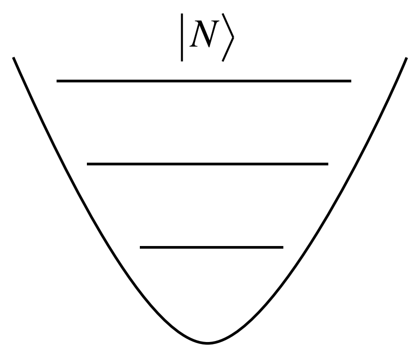
The atom Hamiltonian

$$H_a = -\frac{\Delta E}{2}\sigma_z$$

$$|n_a\rangle$$

The resonator Hamiltonian

$$H_r = \hbar\omega\left(\hat{N} + \frac{1}{2}\right) \qquad H_r' = \hbar\omega\hat{N}$$



Two-level atom creation/annihilation operators

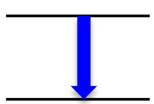
To describe interaction of our atom with external fields we introduce operators of creation/annihilation for the atomic excited state

$$\sigma^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 1|$$

$$\sigma^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0|$$

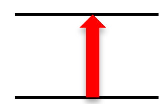
$$\sigma^-|1\rangle = |0\rangle \qquad \sigma^-|0\rangle = 0$$

$$|1\rangle \rightarrow |0\rangle$$



$$\sigma^+|0\rangle = |1\rangle \qquad \sigma^+|1\rangle = 0$$

$$|0\rangle \rightarrow |1\rangle$$



$$\sigma^- = \frac{\sigma_x + i\sigma_y}{2}$$

$$\sigma^+ = \frac{\sigma_x - i\sigma_y}{2}$$

$$\sigma_x = \sigma^+ + \sigma^-$$

Atom-resonator interaction

The system is described by two quantum numbers:

- qubit state n_a: (0, 1)
- photon number state N: $(0,\ 1,\ 2,\ 3,\ldots)$

The system state:

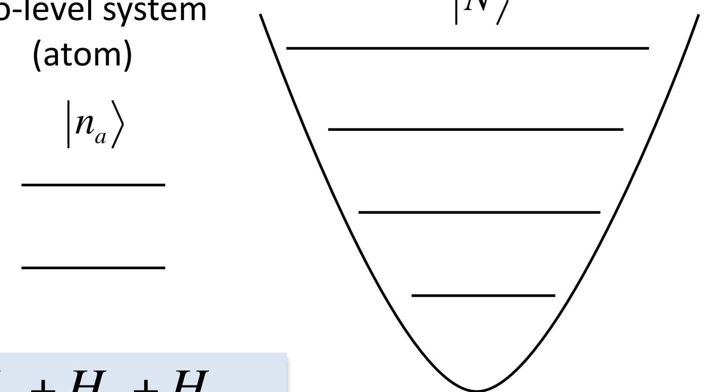
$$\left|n_a N\right\rangle$$

Two-level system (atom)

$\left|n_a\right\rangle$

Harmonic oscillator (resonator)

$\left|N\right\rangle$

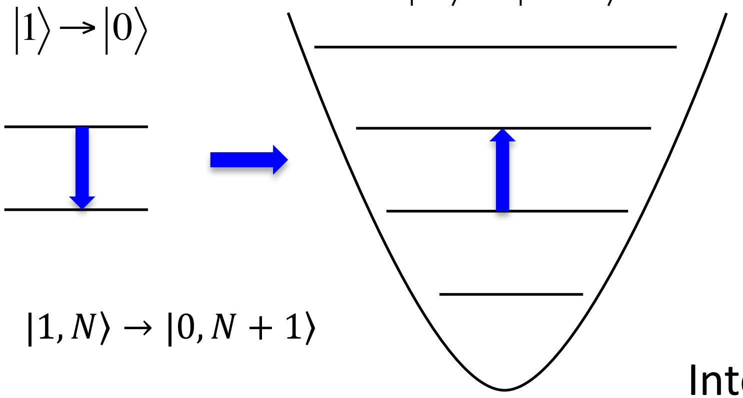


The Hamiltonian of the total system:

$$H_{tot} = H_a + H_r + H_{\text{int}}$$

$\left|1\right\rangle \rightarrow \left|0\right\rangle$

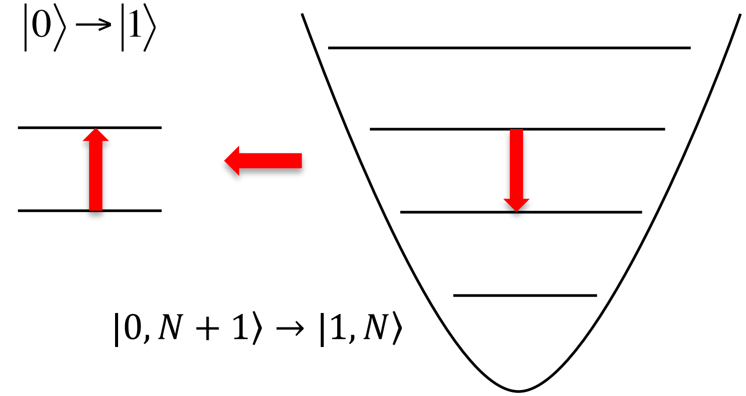
$\left|N\right\rangle \rightarrow \left|N+1\right\rangle$



$|1, N\rangle \rightarrow |0, N+1\rangle$

$\left|0\right\rangle \rightarrow \left|1\right\rangle$

$\left|N+1\right\rangle \rightarrow \left|N\right\rangle$



$|0, N+1\rangle \rightarrow |1, N\rangle$

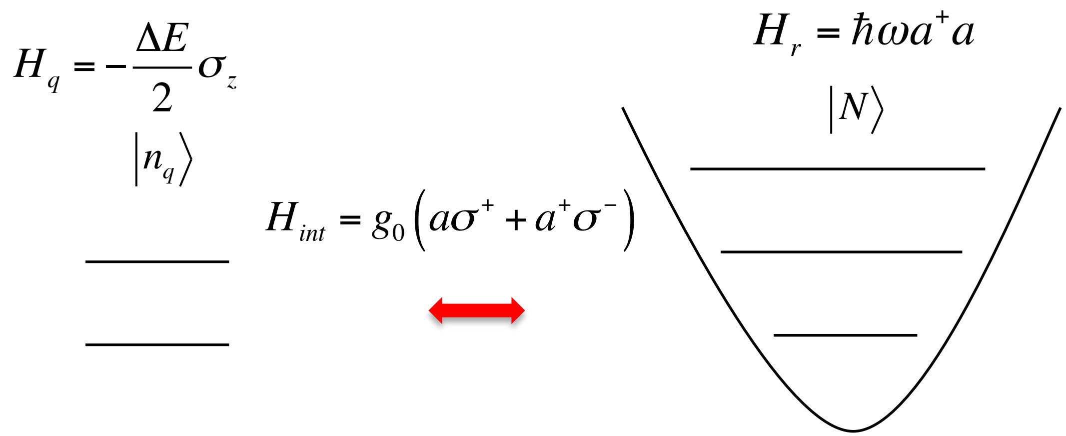
Interaction:

$$|1, N\rangle\langle 0, N+1| + |0, N+1\rangle\langle 1, N|$$

$$a^{+}\sigma^{-} + a^{-}\sigma^{+}$$

Jaynes-Cummings Hamiltonian
and
energy bands of a two-level system coupled to a resonator

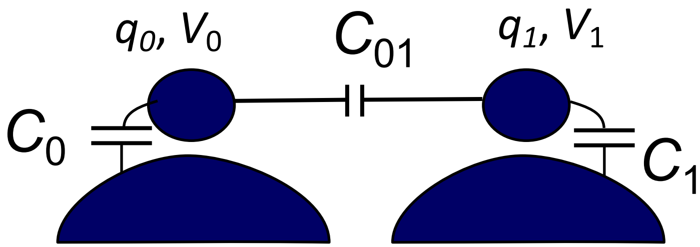
Two-level system interacting with harmonic oscillator
Jaynes-Cummings Hamiltonian



Interaction of the two-level system with the resonator

$$H_{JC} = \underbrace{-\frac{\Delta E}{2}\sigma_z}_{\text{Qubit (atom)}} + \underbrace{\hbar\omega_r a^+ a}_{\text{Oscillator}} + \underbrace{g_0\left(a\sigma^+ + a^+\sigma^-\right)}_{\text{Qubit-resonator interaction}}$$

Electrostatic interaction of two objects (islands)



Capacitance matrix:

$$C = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \qquad \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = C \begin{pmatrix} V_0 \\ V_1 \end{pmatrix}$$

Electrostatic energy:

$$U = \frac{1}{2}\begin{pmatrix} q_0 & q_1 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} V_0 & V_1 \end{pmatrix}\begin{pmatrix} q_0 \\ q_1 \end{pmatrix}$$

$$U = \frac{1}{2}\begin{pmatrix} V_0 & V_1 \end{pmatrix} C \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \qquad U = \frac{1}{2}\begin{pmatrix} q_0 & q_1 \end{pmatrix} C^{-1} \begin{pmatrix} q_0 \\ q_1 \end{pmatrix}$$

Finding elements of the capacitance matrix:

1. $V_1 = 0$: $\begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}\begin{pmatrix} V_0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{00}V_0 \\ a_{10}V_0 \end{pmatrix} \equiv \begin{pmatrix} (C_0 + C_{01})V_0 \\ -C_{01}V_0 \end{pmatrix} \implies a_{00} = C_0 + C_{01}, a_{10} = -C_{01}$

2. $V_0 = 0$: $\begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}\begin{pmatrix} 0 \\ V_1 \end{pmatrix} = \begin{pmatrix} a_{01}V_1 \\ a_{11}V_1 \end{pmatrix} \equiv \begin{pmatrix} -C_{01}V_1 \\ (C_0 + C_{01})V_1 \end{pmatrix} \implies a_{11} = C_1 + C_{01}, a_{01} = -C_{01}$

Capacitance matrix of two coupled islands:

$$C = \begin{pmatrix} C_0 + C_{01} & -C_{01} \\ -C_{01} & C_1 + C_{01} \end{pmatrix}$$

Generalized capacitance matrix:

$$C = \begin{pmatrix} C_0 + C_{01} + C_{02} \ldots & -C_{01} & -C_{02} & \cdots \\ -C_{01} & C_1 + C_{01} + C_{12} \ldots & -C_{12} & \cdots \\ -C_{02} & -C_{12} & C_2 + C_{02} + C_{12} \ldots & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$U = \frac{(C_1 + C_{01})q_0^2}{2(C_0C_1 + C_0C_{01} + C_1C_{01})} + \frac{C_{01}q_0q_1}{(C_0C_1 + C_0C_{01} + C_1C_{01})} + \frac{(C_0 + C_{01})q_1^2}{2(C_0C_1 + C_0C_{01} + C_1C_{01})} \approx \frac{q_0^2}{2C_0} + \frac{C_{01}q_0q_1}{C_0C_1} + \frac{q_1^2}{2C_1}$$

If $C_{01} \ll C_0, C_1$

$$U = \frac{(C_0 + C_{01})V_0^2}{2} - C_{01}V_0V_1 + \frac{(C_1 + C_{01})V_1^2}{2}$$

$$U_{int} = -C_{01}V_0V_1$$

Voltage operator of the charge qubit (degeneracy point)

Potential (voltage) is the energy to remove a unit charge: $V = \frac{\partial E}{\partial q}$

Charge qubit Hamiltonian $\left(E_C \gg E_J\right)$:

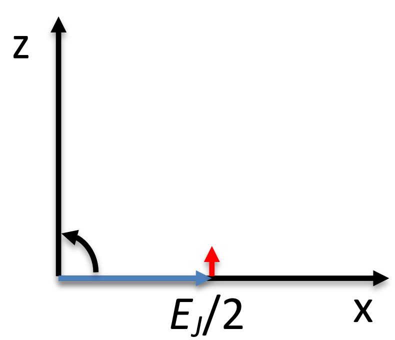
$$H = -\frac{2E_C n}{2}\sigma_z - \frac{E_J}{2}\sigma_x = -\frac{\Delta E}{2}\left(\cos\frac{\theta}{2}\sigma_z + \sin\frac{\theta}{2}\sigma_x\right)$$

where $\Delta E = \sqrt{\varepsilon^2 + {E_J}^2}$ and $\varepsilon = 2E_C n$

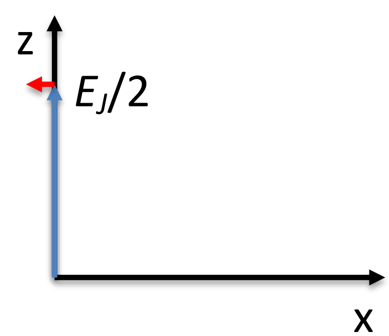
In vicinity of the degeneracy point $(n \ll 1)$, the Hamiltonian can be rewritten as

$$H' = -\frac{E_J}{2}\sigma_z + \frac{2E_C n}{2}\sigma_x$$

0



Basis of diagonalized Hamiltonian



Voltage operator in the charge qubit with a single Cooper pair:

$$\hat{V}_q = \frac{\partial H}{\partial q} = \frac{E_C}{2e}\sigma_x = V_{0q}(\sigma^- + \sigma^+)$$ compare with the resonator field $\hat{V}_r = V_{0r}\left(a + a^\dagger\right)$

Voltage operator of the charge qubit in the general case

$$H = -\frac{2E_C n}{2}\sigma_z - \frac{E_J}{2}\sigma_x = -\frac{\Delta E}{2}\left(\cos\frac{\theta}{2}\sigma_z + \sin\frac{\theta}{2}\sigma_x\right)$$

Voltage operator: $\hat{V} = \frac{\partial H}{\partial q} = -\frac{E_C}{2e}\sigma_z$

The unitary operator $U = I\cos\frac{\theta}{2} - i\,\sigma_y \sin\frac{\theta}{2}$ transforms H to the diagonal form

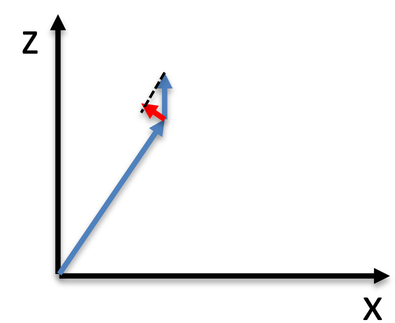
$$H' = UHU^\dagger = -\frac{\Delta E}{2}\sigma_z$$

$$\hat{V}' = U\hat{V}U^\dagger = -\frac{E_C}{2e}U\sigma_z U^\dagger = -\frac{E_C}{2e}(\overbrace{\sigma_z\cos\theta}^{\hat{V}_q^\parallel} + \overbrace{\sigma_x\sin\theta}^{\hat{V}_q^\perp})$$

Off-diagonal matrix elements $\hat{V}_q^\perp$ produce transitions in the two-level system

Qubit dipole voltage operator:

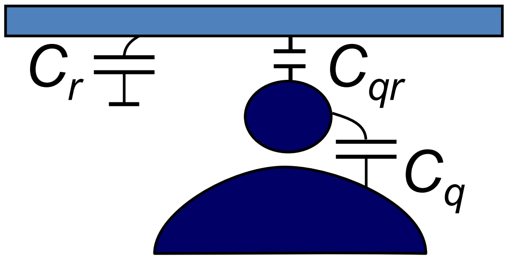
$$\boxed{\hat{V}_q^\perp = V_{q0}\sigma_x} \qquad V_{q0} = -\frac{E_C}{2e}\sin\theta$$



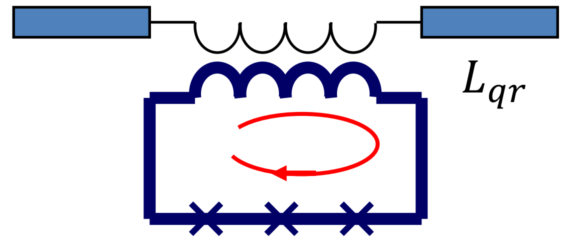
Far away from the degeneracy point $\hat{V}_q^\perp \to 0$

Qubits coupled to a harmonic oscillator

Charge qubit coupled capacitively



Flux qubit coupled inductively



Charge qubit: $H_{int} = \hat{V}_q C_{qr} \hat{V}_r$

Flux qubit: $H_{int} = \hat{I}_q L_{qr} \hat{I}_r$

Off-diagonal matrix elements produce transitions in the two-level system

Qubit dipole voltage operator:

$$\hat{V}_q^{\perp} = V_{q0}\sigma_x$$

$$V_0 = \frac{E_C}{2e}\sin\theta$$

Resonator voltage operator:

$$\hat{V}_r = V_{r0}\left(a^+ + a\right)$$

$$V_{r0} = \sqrt{\frac{\hbar\omega}{C_r}}$$

$$H_{int} = g_0\sigma_x\left(a^+ + a\right)$$

$$g_0 = C_{qr}V_{q0}V_{r0}$$

Interaction Hamiltonian close to resonance

$$\Delta E \approx \hbar\omega$$

$$\boxed{H_{\text{int}} = g_0 \sigma_x \left(a + a^+\right)}$$

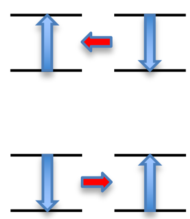
$$\sigma_x = \sigma^+ + \sigma^-$$

$$\boxed{H_{\text{int}} \approx g_0 \left(a + a^+\right)\left(\sigma^+ + \sigma^-\right) = g_0 \left(a\sigma^+ + a^+\sigma^+ + a\sigma^- + a^+\sigma^-\right)}$$

Processes with no energy change

$$\sigma^+ a\left|0N\right\rangle = \sqrt{N}\left|1N\right\rangle$$

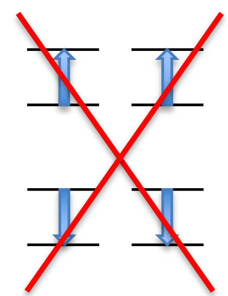
$$\sigma^- a^+\left|1N\right\rangle = \sqrt{N+1}\left|0\left(N+1\right)\right\rangle$$



Processes with energy change by $\pm 2\hbar\omega$

$$\sigma^+ a^+\left|0N\right\rangle = \sqrt{N+1}\left|1\left(N+1\right)\right\rangle$$

$$\sigma^- a\left|1N\right\rangle = \sqrt{N}\left|0\left(N-1\right)\right\rangle$$



$$\boxed{H_{\text{int}} \approx g_0 \left(a\sigma^+ + a^+\sigma^-\right)}$$

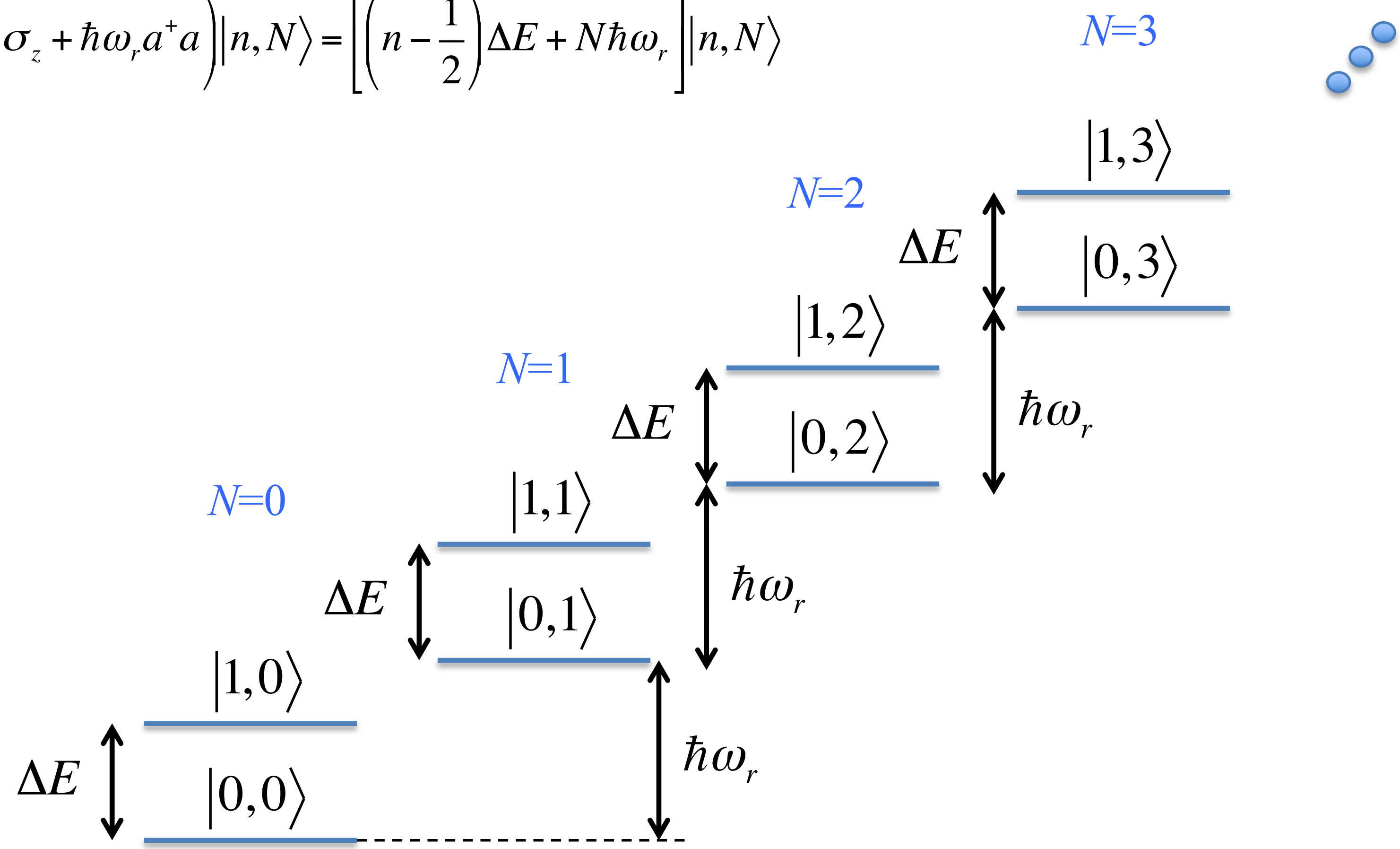
Two-level atom in the resonator (g = 0)

$$|n\rangle \otimes |N\rangle = |n, N\rangle$$

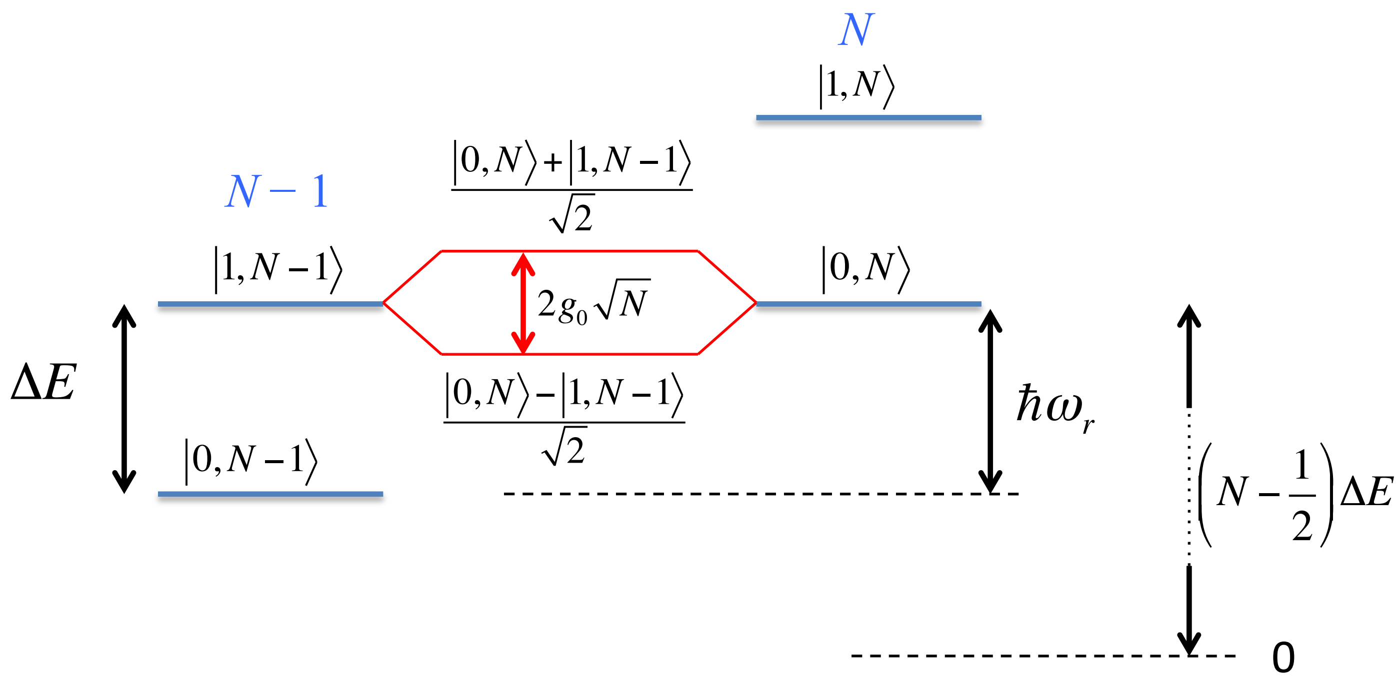
n is the atomic state (0 or 1)
N is the number of photons in the resonator

$|n, N\rangle$ are eigenstates of $H_0 = -\frac{\Delta E}{2}\sigma_z + \hbar\omega_r a^+ a$

$$\left(-\frac{\Delta E}{2}\sigma_z + \hbar\omega_r a^+ a\right)|n, N\rangle = \left[\left(n - \frac{1}{2}\right)\Delta E + N\hbar\omega_r\right]|n, N\rangle$$



Atom-resonator resonance ($\Delta E = \hbar\omega$)



$$H_0 = -\frac{\Delta E}{2}\sigma_z + \hbar\omega a^\dagger a \qquad H_{int} = g_0\left(a^\dagger\sigma^- + a\sigma^+\right)$$

$$H_0|1,N-1\rangle = \left[\frac{\Delta E}{2} + (N-1)\hbar\omega\right]|1,N-1\rangle = \left(N-\frac{1}{2}\right)\Delta E|1,N-1\rangle$$

$$H_0|0,N\rangle = \left[-\frac{\Delta E}{2} + N\hbar\omega\right]|0,N\rangle = \left(N-\frac{1}{2}\right)\Delta E|0,N\rangle$$

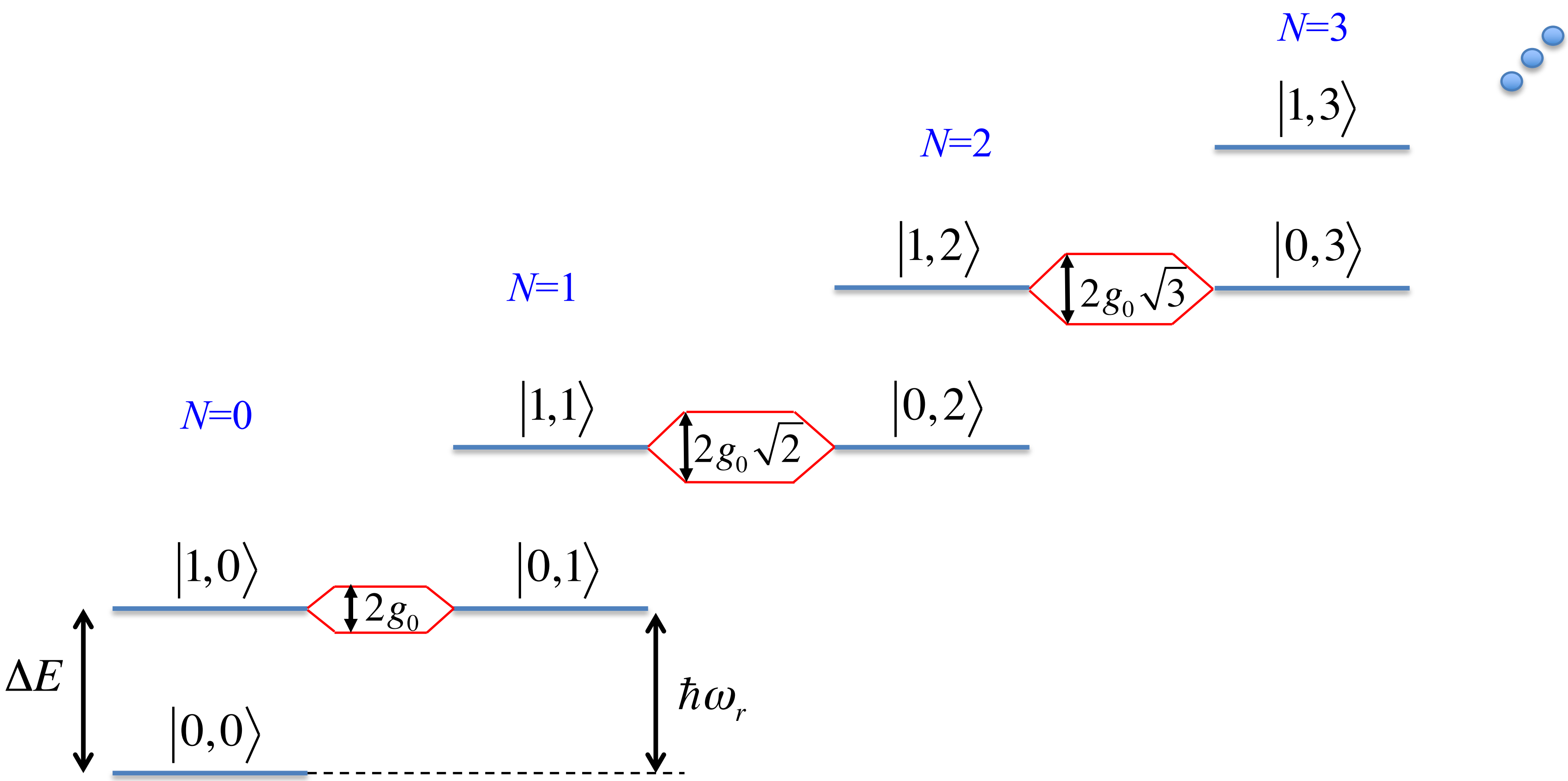
$$H_{int}\frac{|0,N\rangle \pm |1,N-1\rangle}{\sqrt{2}} = g_0\left(a^\dagger\sigma^- + a\sigma^+\right)\frac{|0,N\rangle \pm |1,N-1\rangle}{\sqrt{2}} = g_0\sqrt{N}\frac{|1,N-1\rangle \pm |0,N\rangle}{\sqrt{2}} = \pm g_0\sqrt{N}\frac{|0,N\rangle \pm |1,N-1\rangle}{\sqrt{2}}$$

$$H_{JC}\frac{|0,N\rangle \pm |1,N-1\rangle}{\sqrt{2}} = \left[\left(N-\frac{1}{2}\right)\Delta E \pm g_0\sqrt{N}\right]\frac{|0,N\rangle \pm |1,N-1\rangle}{\sqrt{2}}$$

Two-level atom in the resonator ($g \neq 0$)

$$|n\rangle \otimes |N\rangle = |n, N\rangle$$

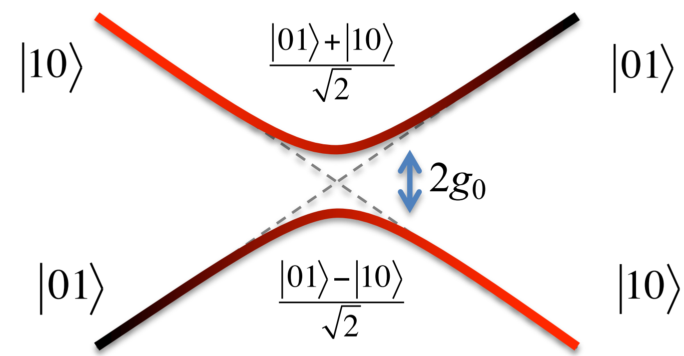
n is the atomic state (0 or 1)
N is the number of photons in the resonator



Single-photon interaction

$$H_{JC} = -\frac{\Delta E}{2}\sigma_z + \hbar\omega_r a^+ a + g_0\left(a\sigma^+ + a^\dagger\sigma^-\right) \qquad \left|n_q N\right\rangle$$

$$H = \begin{array}{c} \\ \end{array}\begin{array}{cccccc} & |00\rangle & |10\rangle & |01\rangle & |11\rangle & \\ & \left(\begin{array}{c} -\frac{\Delta E}{2} \\ 0 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ \frac{\Delta E}{2} \\ g_0 \\ 0 \end{array} & \begin{array}{c} 0 \\ g_0 \\ -\frac{\Delta E}{2}+\hbar\omega_r \\ 0 \end{array} & \left.\begin{array}{c} 0 \\ 0 \\ 0 \\ \frac{\Delta E}{2}+\hbar\omega_r \end{array}\right) & \begin{array}{c} |00\rangle \\ |10\rangle \\ |01\rangle \\ |11\rangle \end{array} \end{array}$$



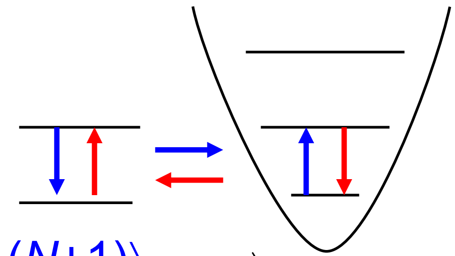
$$\boxed{\Delta E = \hbar\omega_r}$$

$$H = \begin{pmatrix} \frac{\Delta E}{2} & g_0 \\ g_0 & -\frac{\Delta E}{2}+\hbar\omega_r \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{\Delta E-\hbar\omega_r}{2} & g_0 \\ g_0 & \frac{-\Delta E+\hbar\omega_r}{2} \end{pmatrix} = \frac{\Delta E-\hbar\omega_r}{2}\sigma_z + \frac{2g_0}{2}\sigma_x$$

Compare with the qubit Hamiltonian in the physical basis: $H = -\frac{\varepsilon}{2}\sigma_z - \frac{\Delta}{2}\sigma_x$

Nearly resonant interaction of atom-harmonic oscillator (dressed states)

$$H_{JC} = -\frac{\Delta E}{2}\sigma_z + g\left(a\sigma^+ + a^\dagger\sigma^-\right) + \hbar\omega_r a^+ a$$

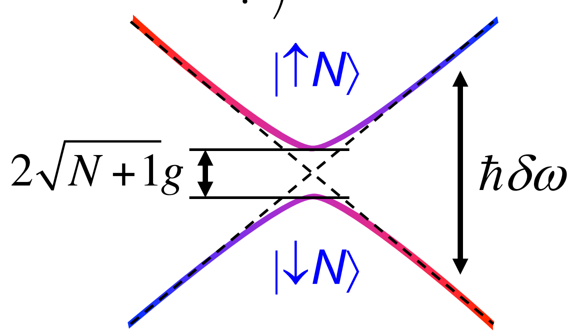


$$H = \begin{array}{c|cccc|c} & |0N\rangle & |1N\rangle & |0(N+1)\rangle & |1(N+1)\rangle & \\ \hline & \ddots & & & & \\ & -\frac{\Delta E}{2}+\hbar N\omega_r & 0 & 0 & 0 & |0N\rangle \\ & 0 & \frac{\Delta E}{2}+\hbar N\omega_r & \sqrt{N+1}g & 0 & |1N\rangle \\ & 0 & \sqrt{N+1}g & -\frac{\Delta E}{2}+\hbar(N+1)\omega_r & 0 & |0(N+1)\rangle \\ & 0 & 0 & 0 & \frac{\Delta E}{2}+\hbar(N+1)\omega_r & |1(N+1)\rangle \\ & & & & \ddots & \end{array}$$

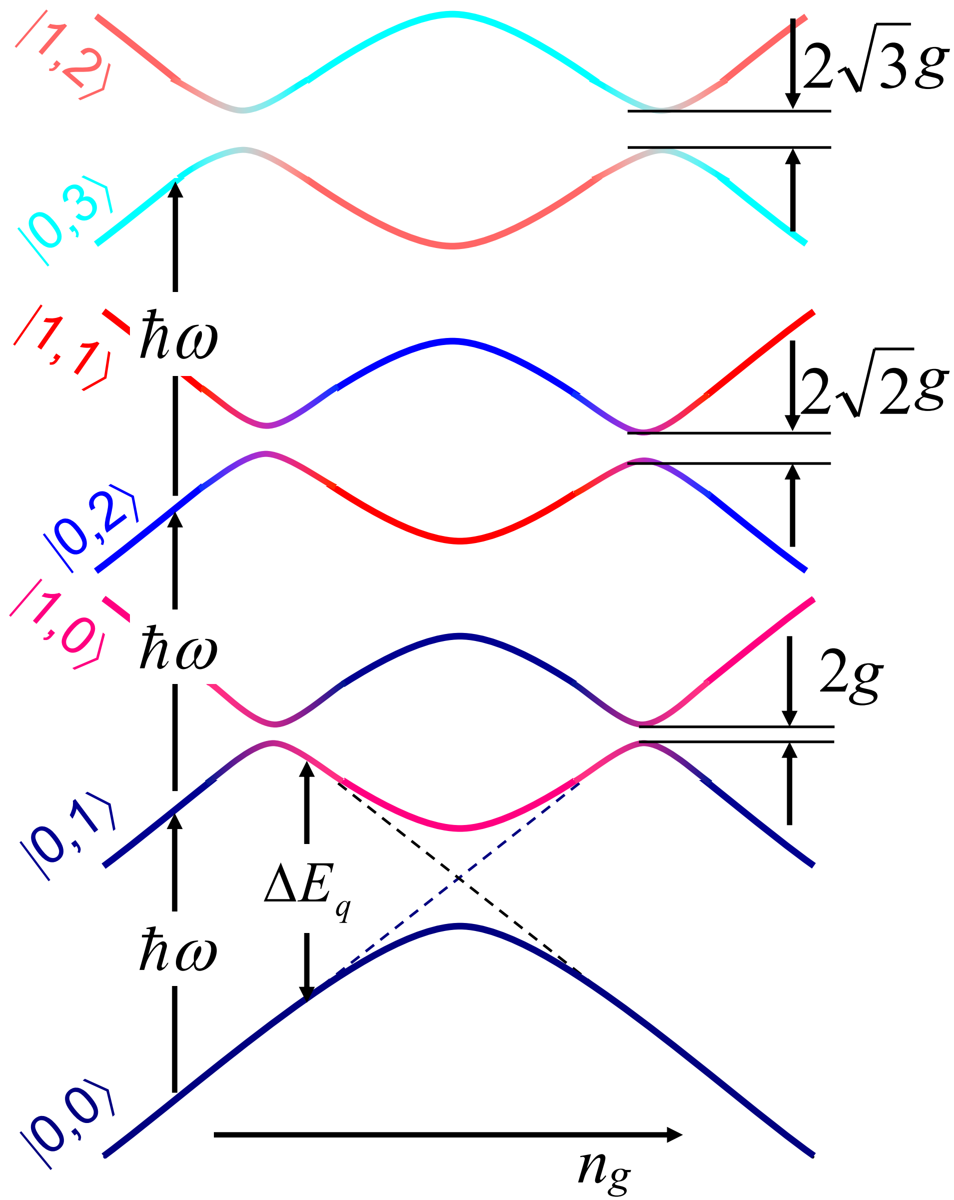
$$H' = \begin{pmatrix} -\hbar\frac{\delta\omega}{2} & \sqrt{N+1}g \\ \sqrt{N+1}g & \hbar\frac{\delta\omega}{2} \end{pmatrix}$$

$$H' = -\frac{\hbar\delta\omega}{2}\sigma_z + 2g\sqrt{N+1}\sigma_x$$

$$\delta\omega = \omega_r - \omega_a$$



Qubit-resonator energy diagram



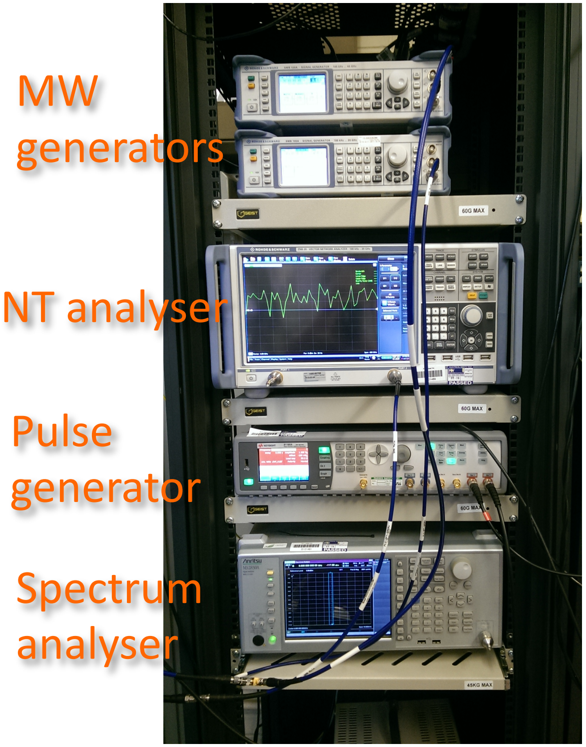
Measurement set-up
for microwave characterization of
superconducting quantum systems

Dilution refrigerator and measurement equipmemt

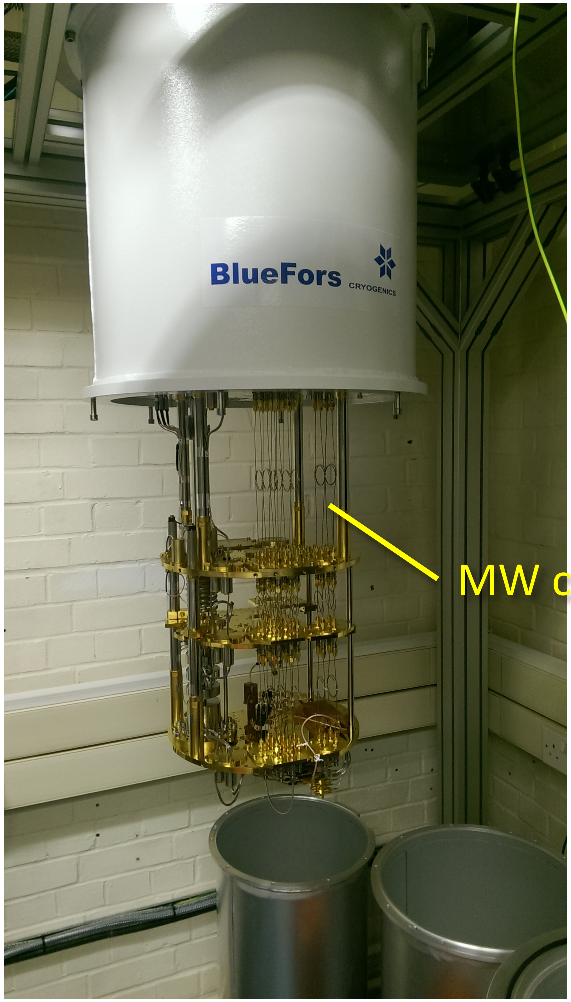
Dilution refrigerator

Internal view

Measurement MW equipment

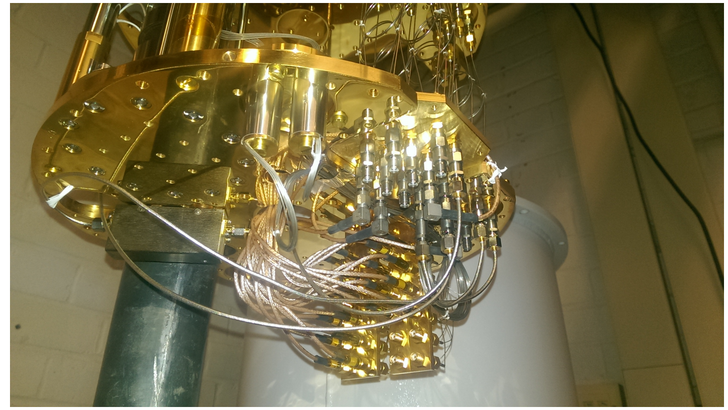


The lowest temperature is 10 mK

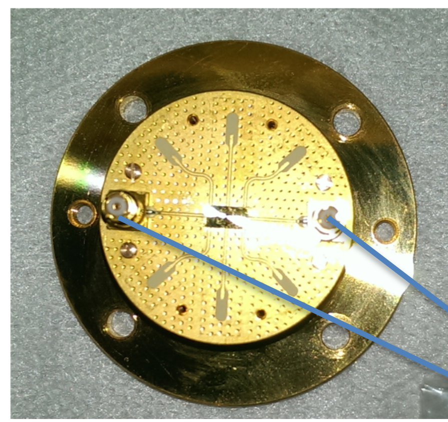


MW cables

Bottom view of a cold plate

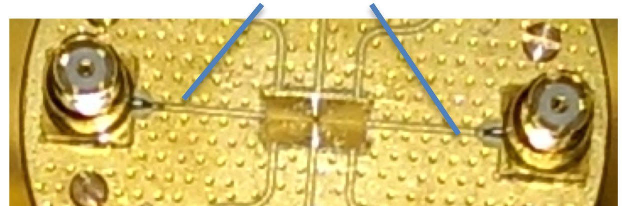


Sample holder

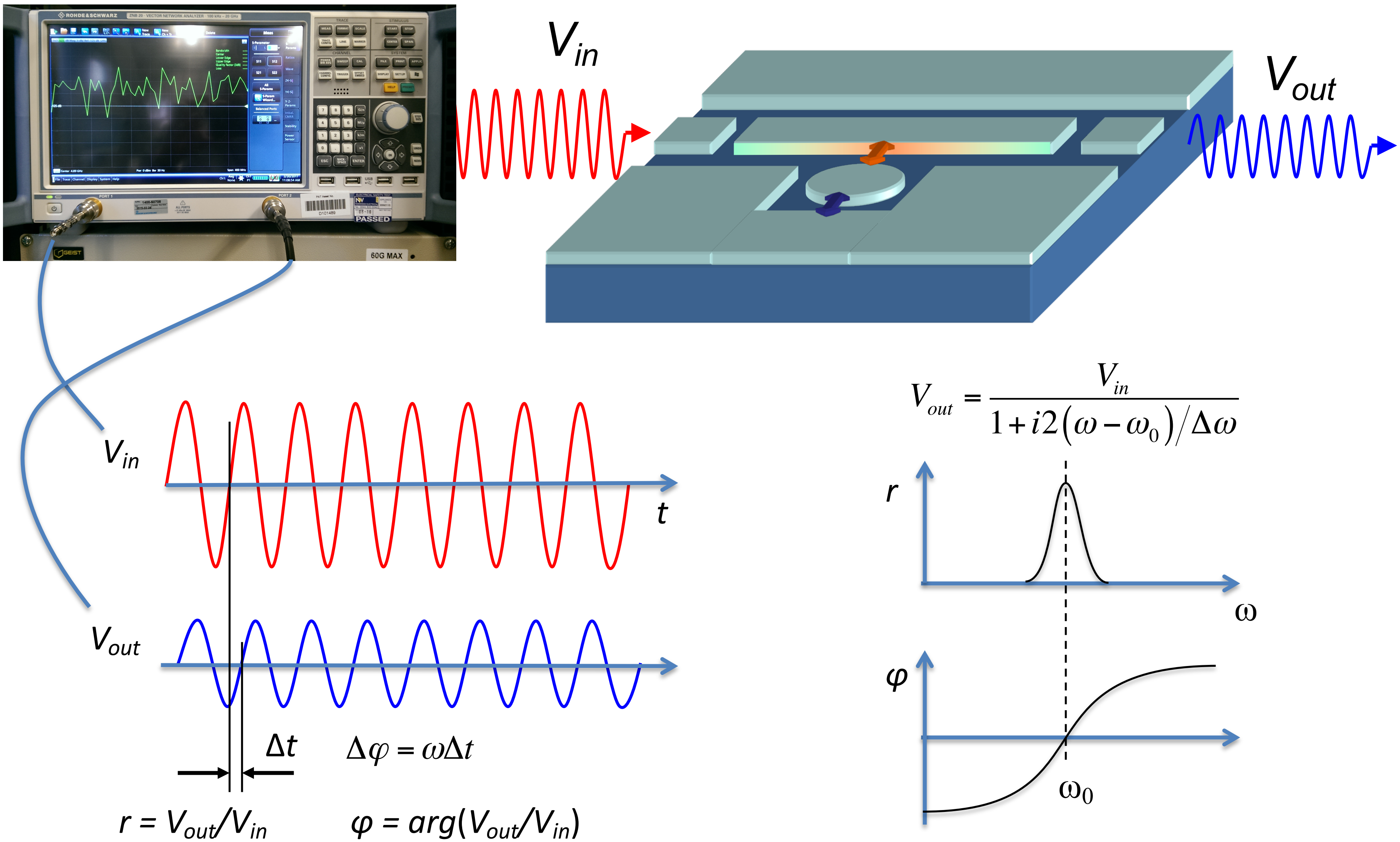


MW connectors

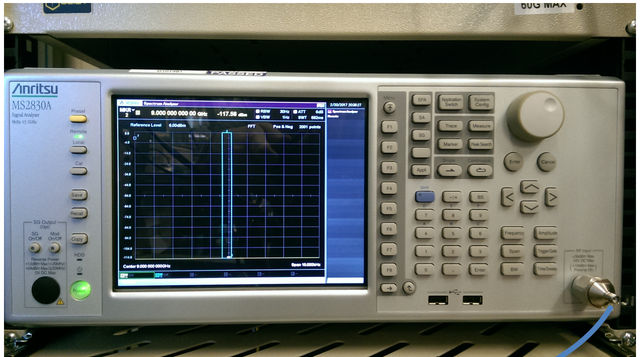
MW lines

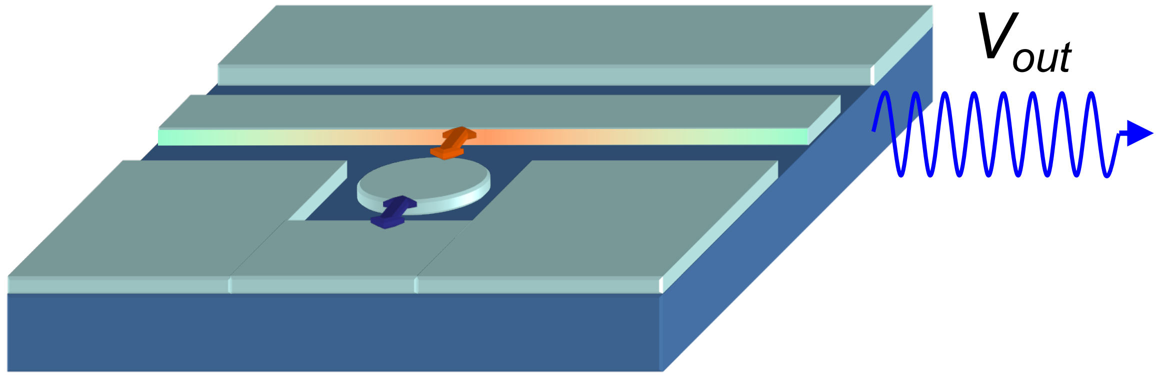


Phase-sensitive detection of transmitted signal by a network analyzer



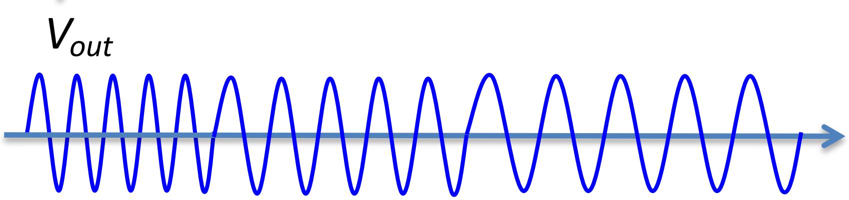
Spectrum detection of emission by a spectrum analyzer

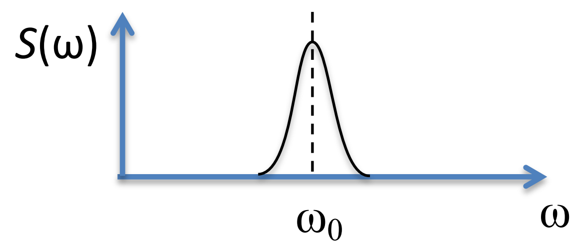




The output signal is uncorrelated in phase with excitation

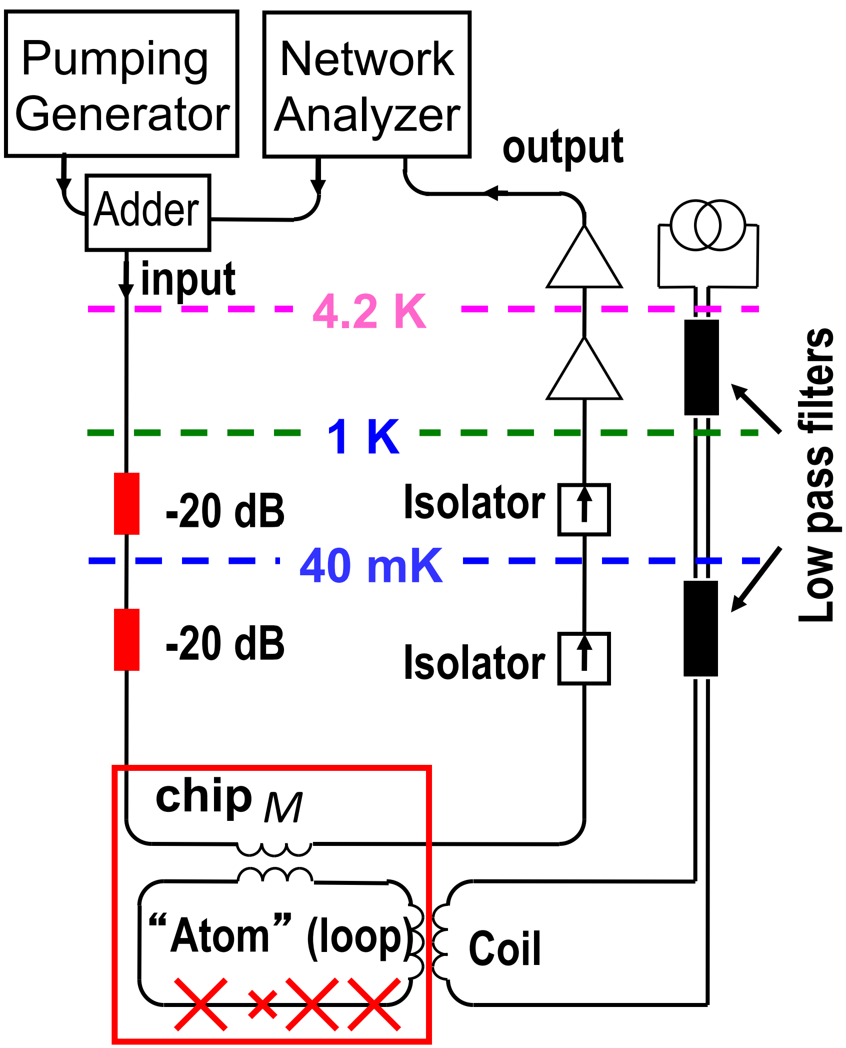
$$S_{out}(\omega) = \frac{S_0}{1+\left[2(\omega-\omega_0)/\Delta\omega\right]^2}$$

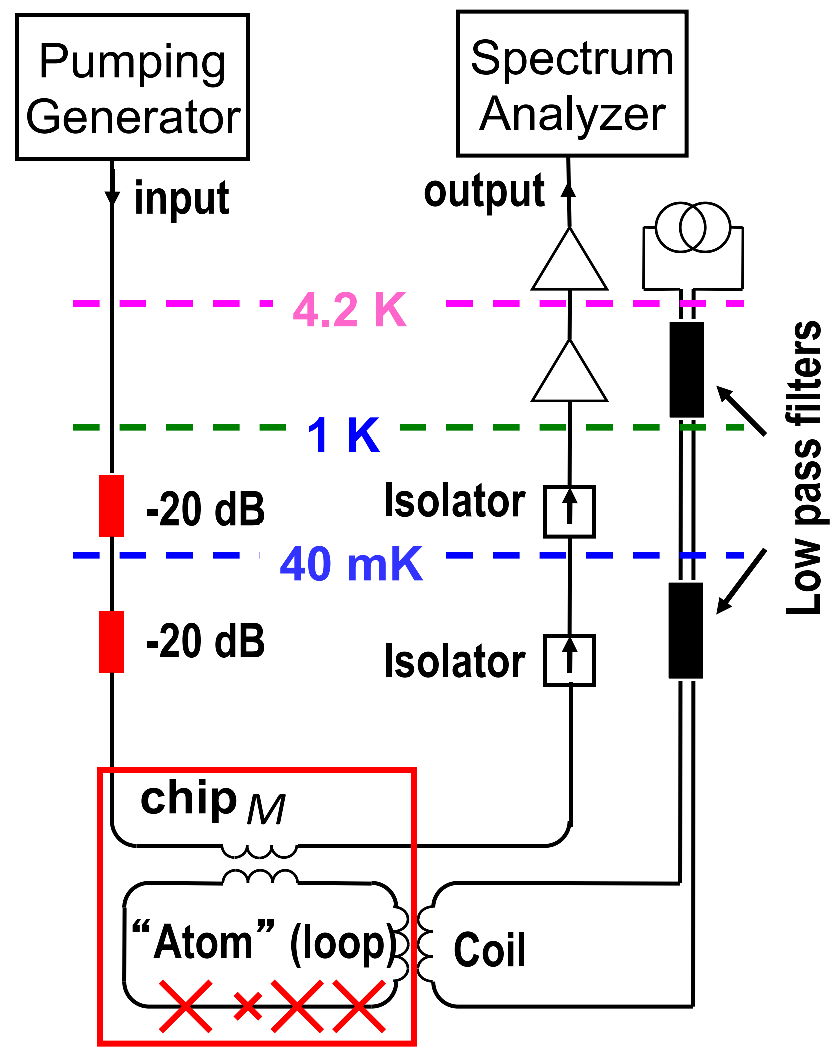




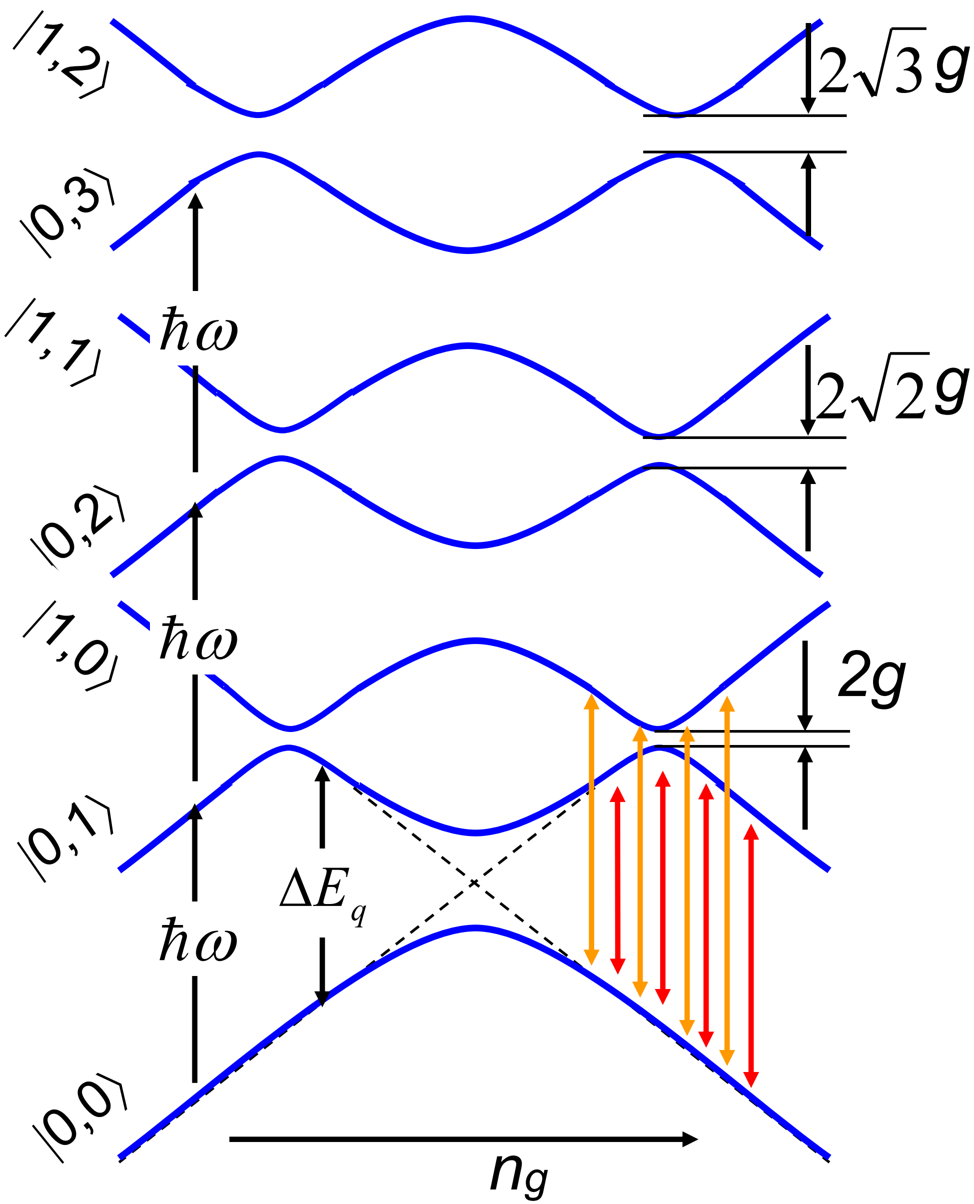
Spectral power density: $S(\omega)$

Measurement circuits for Superconducting quantum systems

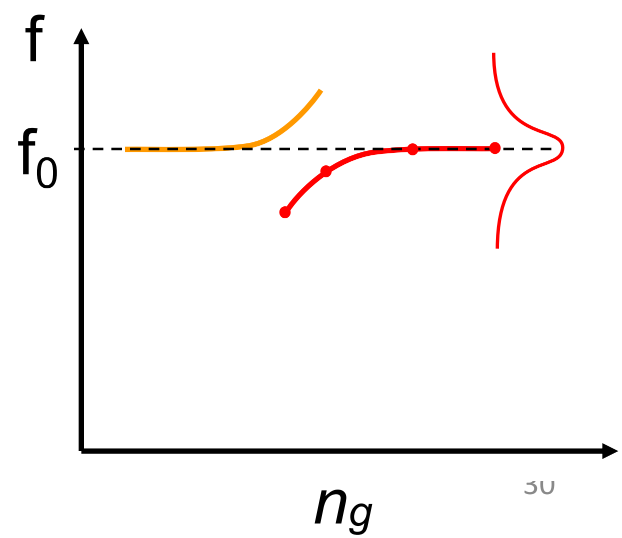




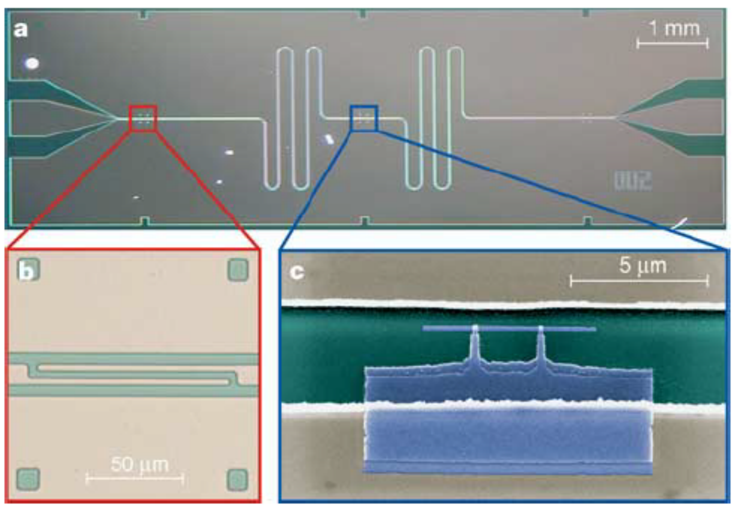
Microwave (optical) probing

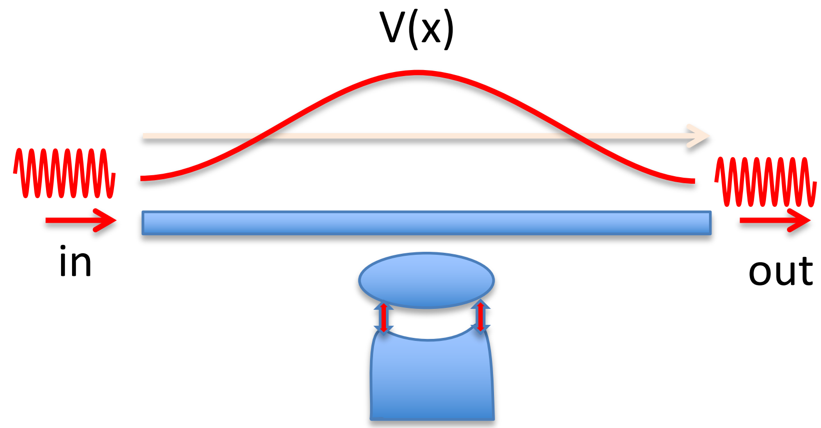


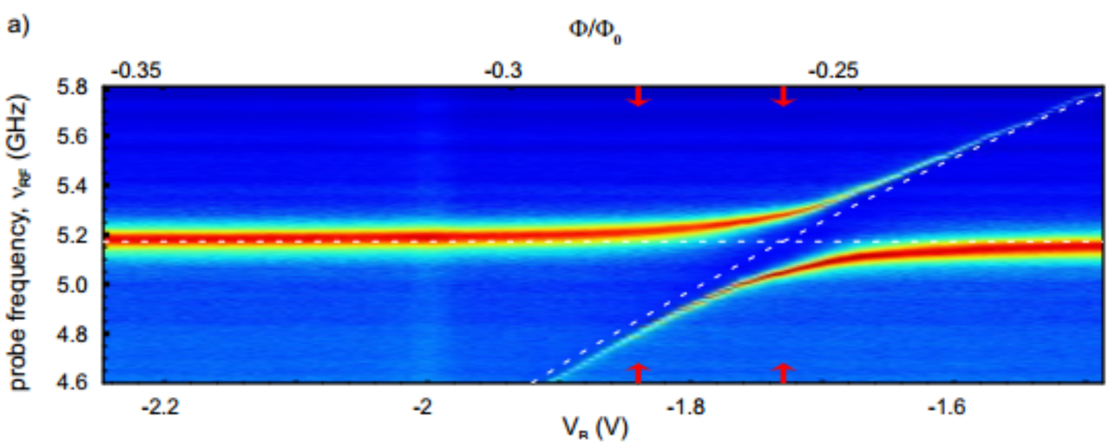
Spectroscopy:
Transitions in the oscillator:
$|n,0\rangle \rightarrow |n,1\rangle$

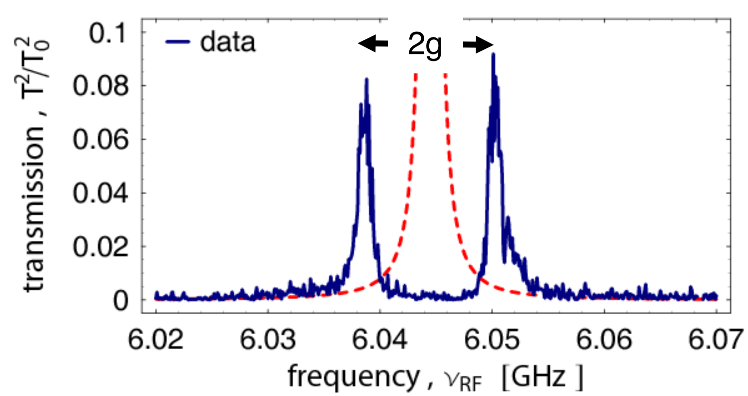


Qubit coupled to a resonator

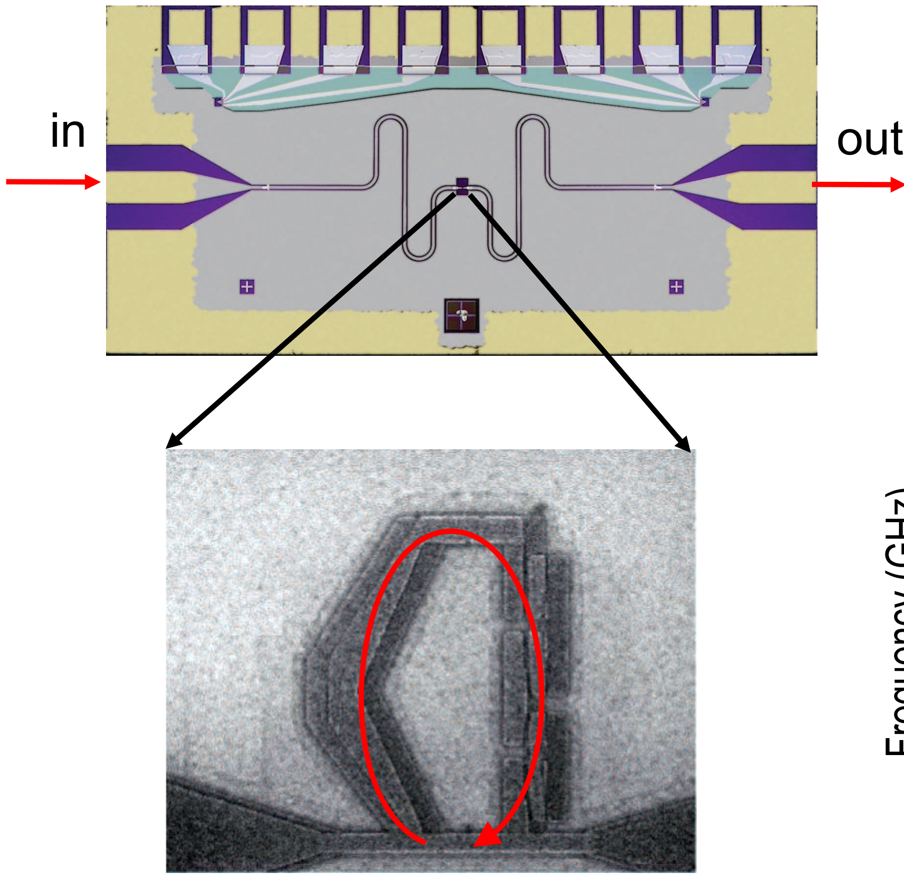


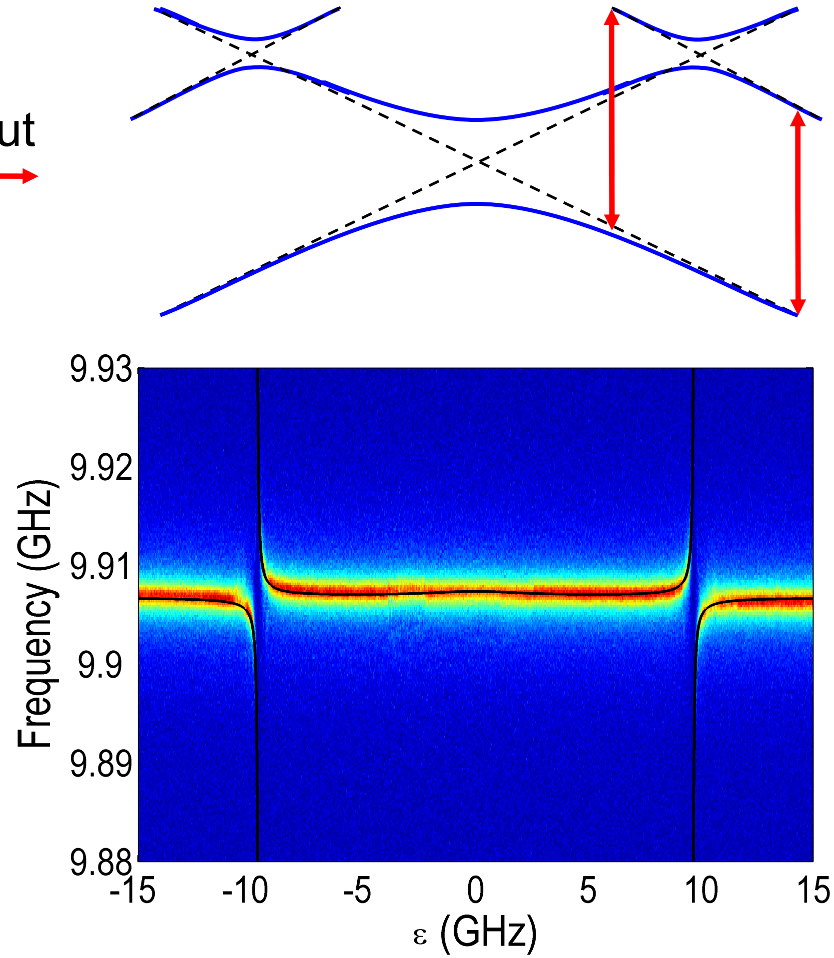






Flux qubit coupled to the coplanar waveguide resonator





We demonstrated coupling of the flux qubit to the resonator