

# Quantum Electronics of Nanostructures

Oleg Astafiev

Lecture 7

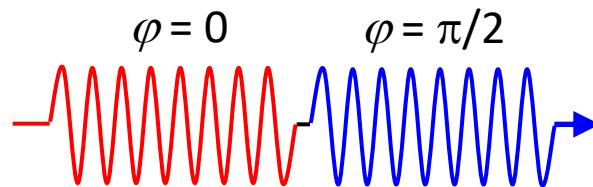
# Lecture 7

- Experimental quantum state control in two-level systems
- Quantum gates
- Dissipation and decoherence in two-level systems
- Density matrix approach
- Pure and mixed states
- Bloch sphere for mixed states and dissipative dynamics
- Relaxation and dephasing

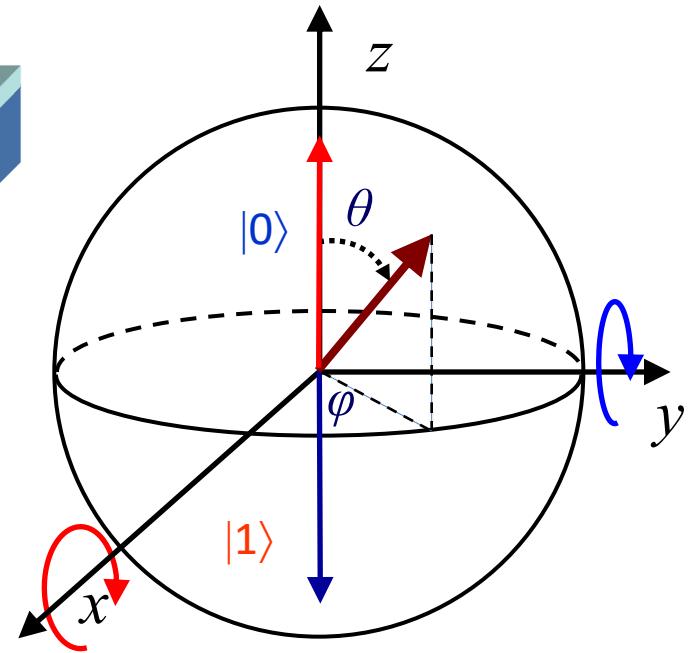
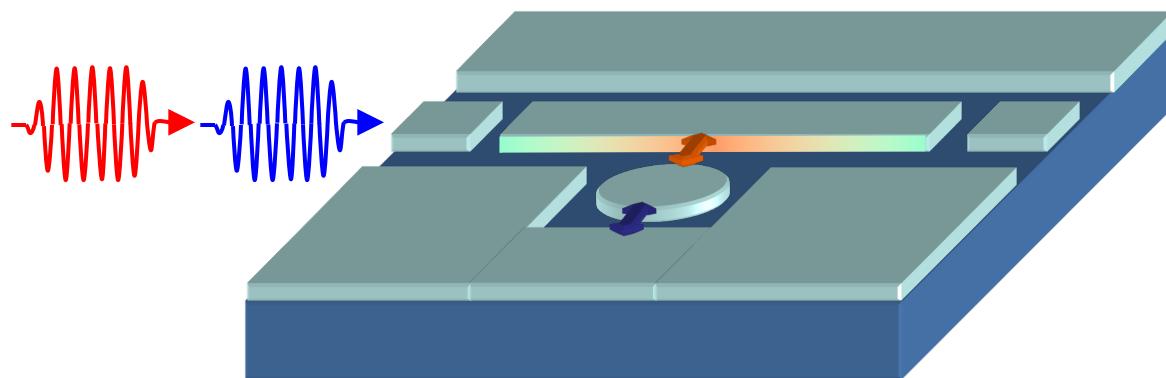
# Quantum state control in two-level systems

$$\Psi = \cos \frac{\theta}{2} |0\rangle + e^{i\varphi} \sin \frac{\theta}{2} |1\rangle$$

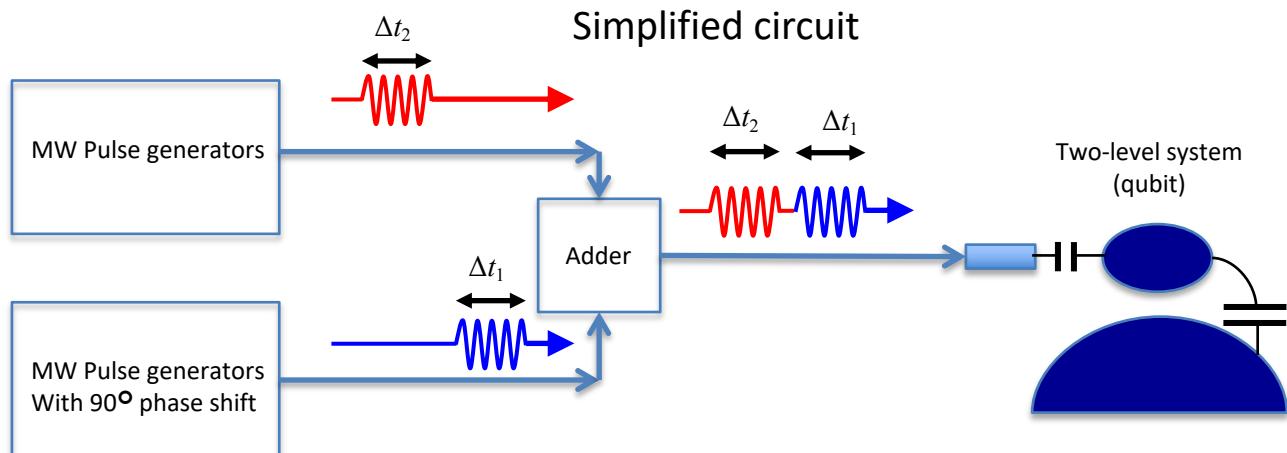
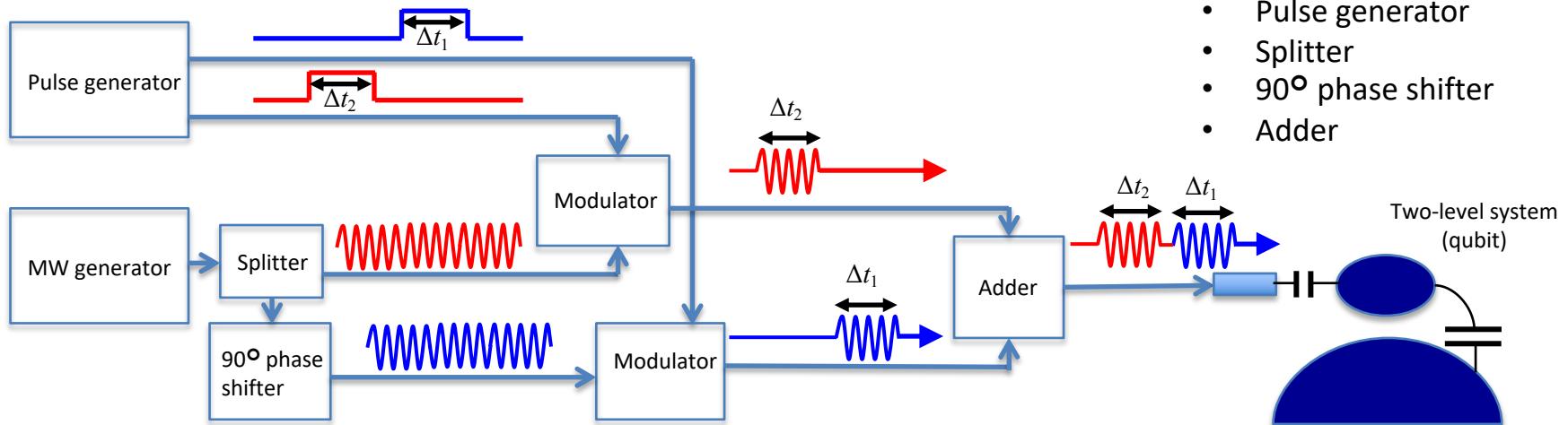
State manipulation



$|1\rangle$   
 $|0\rangle$



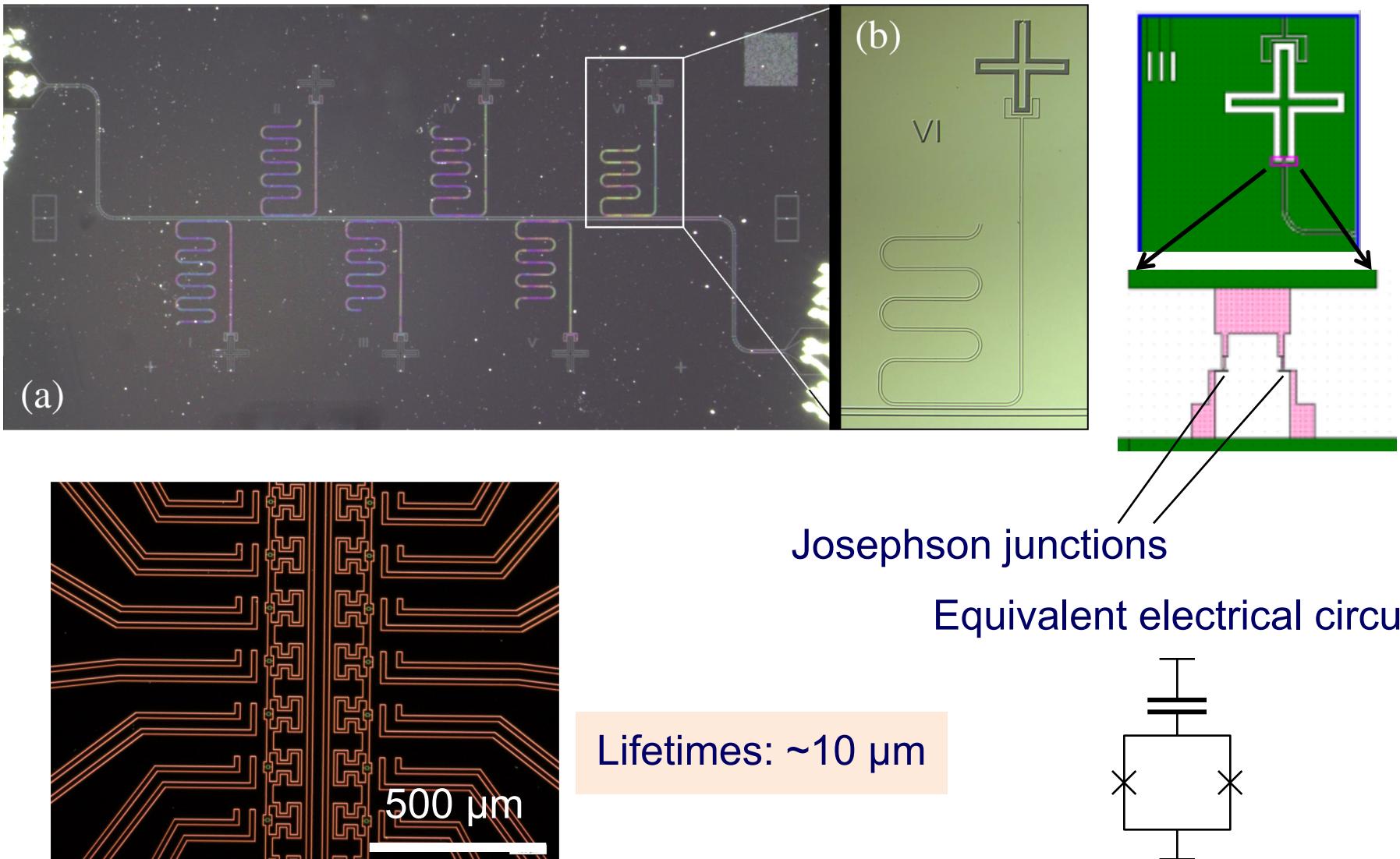
# Electrical circuit to prepare arbitrary states



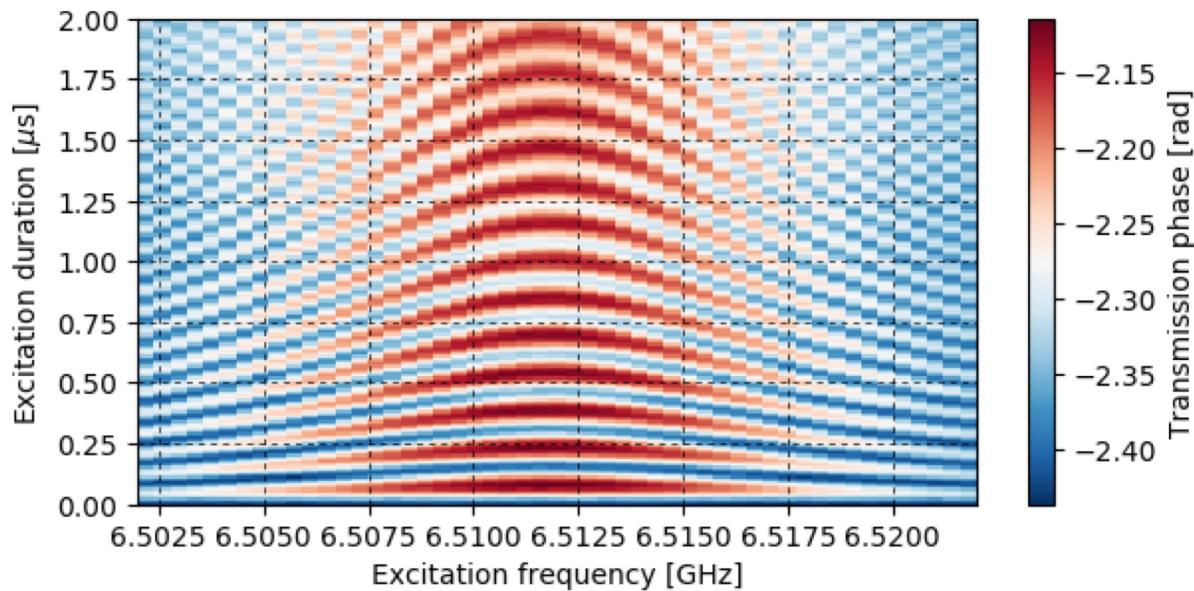
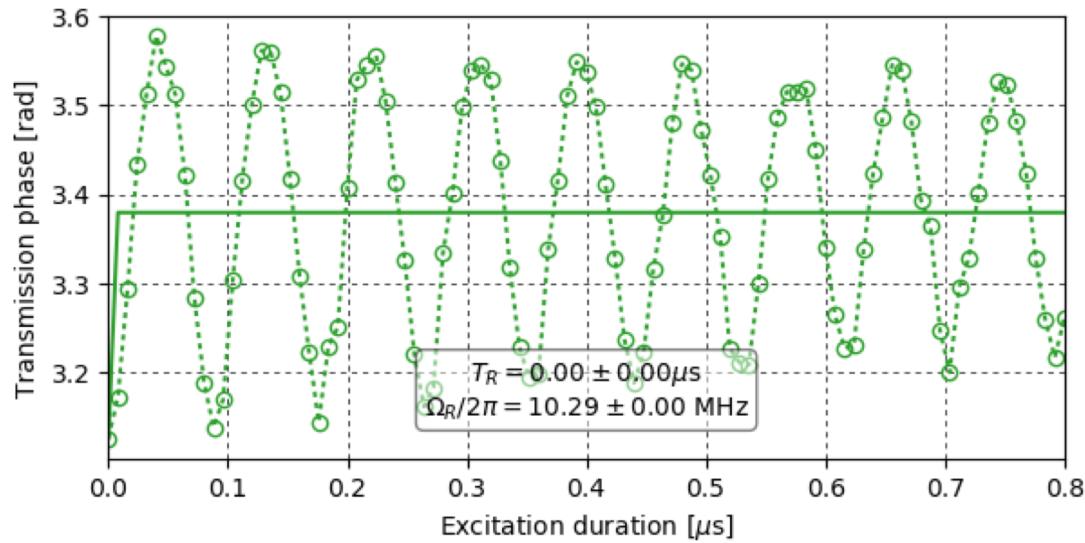
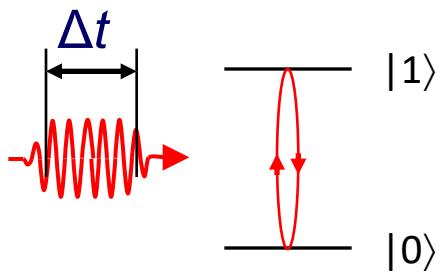
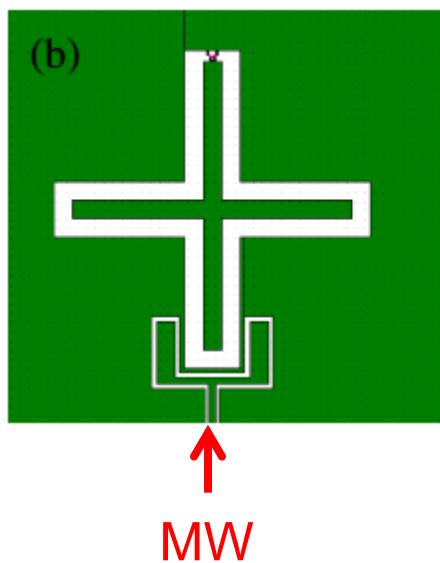
Experimental realization will be shown on 15<sup>th</sup> and 22<sup>nd</sup> May by Shamil Kadyrmetov (PhD student, MIPT)

# Long-lived low qubits with low unharmonicity

## Fabricated and measured at MIPT



# Rabi oscillations



# Single-qubit operations

$$H = -\frac{\hbar\Omega}{2}\sigma_j \quad U = \exp\left(i\frac{\Omega t}{2}\sigma_j\right) = I \cos\left(\frac{\Omega t}{2}\right) + i\sigma_j \sin\left(\frac{\Omega t}{2}\right)$$

$$\Omega t = \pi: \quad U\left(\frac{\pi}{\Omega}\right) = i\sigma_j$$

$$t = \frac{\pi}{\Omega} \quad H = -\frac{\hbar\Omega}{2}\sigma_y \quad \xrightarrow{\text{blue arrow}} \quad R = i\sigma_y$$

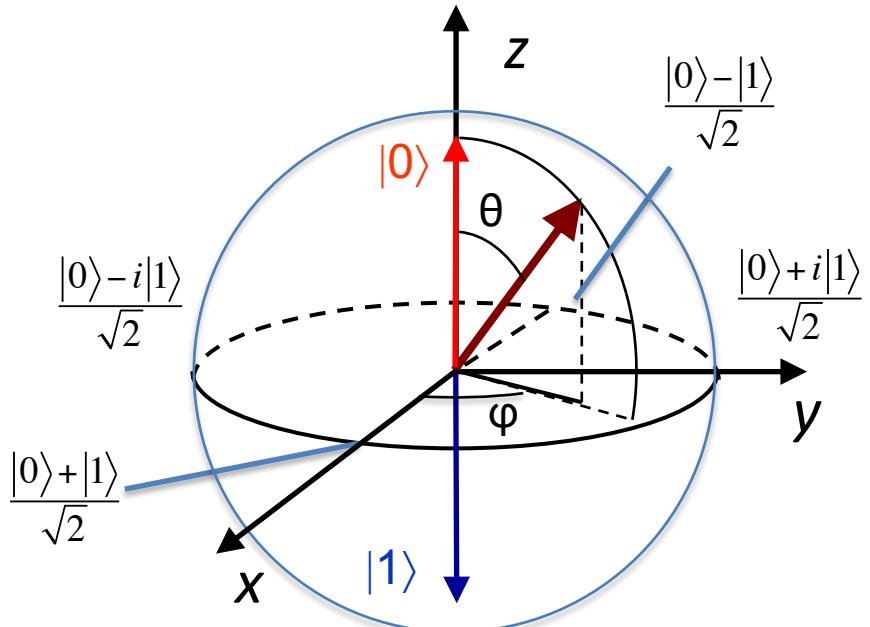
NOT (BIT FLIP)  $\pi$ -rotation around x-axis

$$i\sigma_y|0\rangle = -|1\rangle \quad i\sigma_y|1\rangle = |0\rangle$$

$$t = \frac{\pi}{\Omega} \quad H = -\frac{\hbar\Omega}{2}\sigma_x \quad \xrightarrow{\text{blue arrow}} \quad R = i\sigma_x$$

CONJUGATED FLIP  $\pi$ -rotation around y-axis

$$i\sigma_x|0\rangle = i|1\rangle \quad i\sigma_x|1\rangle = i|0\rangle$$



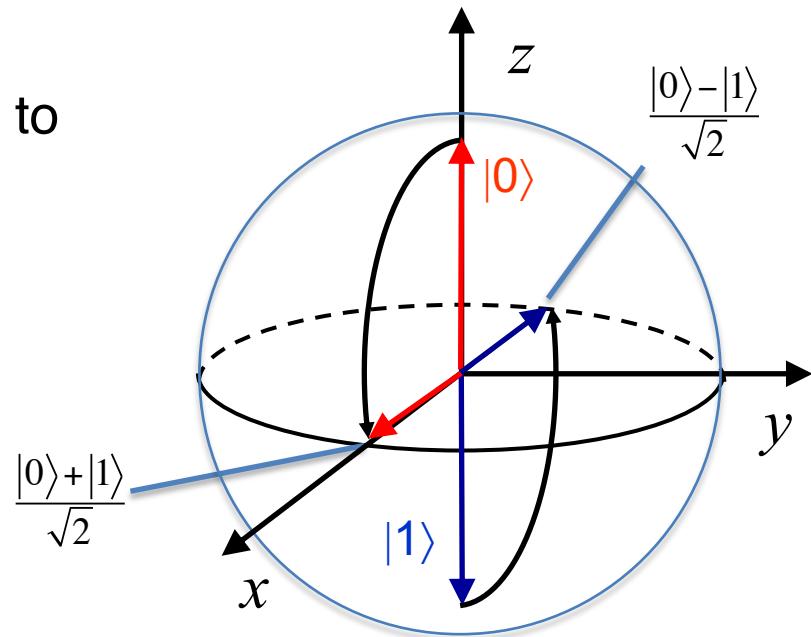
# Hadamard gate and superposition of quantum states

$$Hm = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Hadamard transformation is equivalent to  $\pi/2$ -rotation around y-axis

$$Hm|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$Hm|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



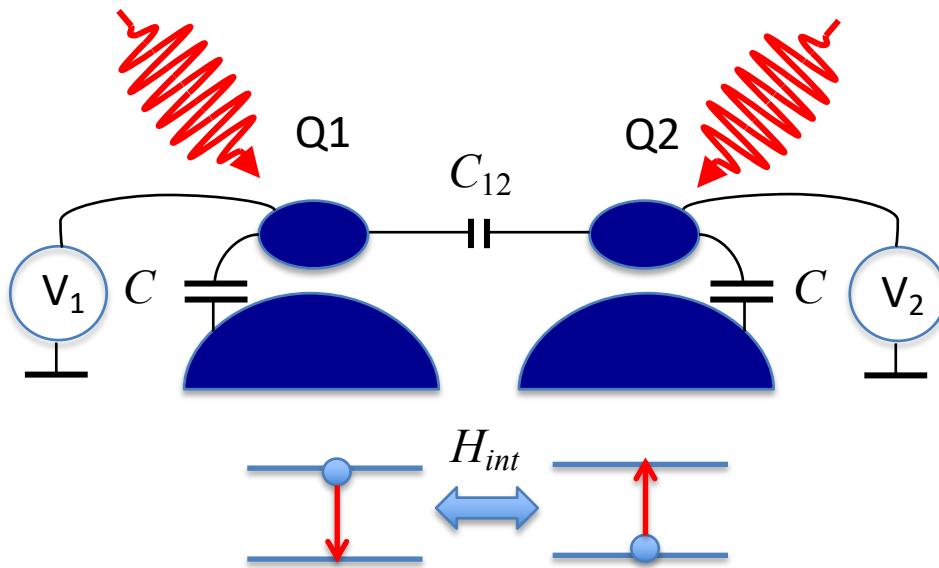
We can check that for

$$H = -\frac{\hbar\Omega}{2}\sigma_y \quad U(t) = \exp\left[i\frac{\Omega t}{2}\sigma_y\right] = \begin{pmatrix} \cos(\Omega t/2) & \sin(\Omega t/2) \\ -\sin(\Omega t/2) & \cos(\Omega t/2) \end{pmatrix}$$

$$t = \frac{\pi}{2\Omega} \quad \rightarrow \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \rightarrow \quad U|0\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \quad U|1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$t = \frac{3\pi}{2\Omega} \quad \rightarrow \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \quad \rightarrow \quad U|0\rangle = \frac{-|0\rangle - |1\rangle}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad U|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

# Two-qubit interaction



$$H = \begin{pmatrix} |00\rangle & |10\rangle & |01\rangle & |11\rangle \\ |00\rangle & -\frac{\varepsilon_1 - \varepsilon_2}{2} & -\frac{\Delta_1}{2} & -\frac{\Delta_2}{2} & 0 \\ |10\rangle & -\frac{\Delta_1}{2} & \frac{\varepsilon_1 - \varepsilon_2}{2} & E_{\text{int}} & -\frac{\Delta_2}{2} \\ |01\rangle & -\frac{\Delta_2}{2} & E_{\text{int}} & -\frac{\varepsilon_1 + \varepsilon_2}{2} & -\frac{\Delta_1}{2} \\ |11\rangle & 0 & -\frac{\Delta_2}{2} & -\frac{\Delta_1}{2} & \frac{\varepsilon_1 + \varepsilon_2}{2} \end{pmatrix} |00\rangle$$

$$H_1 = -\frac{\varepsilon_1}{2} \sigma_z^{(1)} - \frac{\Delta_1}{2} \sigma_x^{(1)}$$

$$H_2 = -\frac{\varepsilon_2}{2} \sigma_z^{(2)} - \frac{\Delta_2}{2} \sigma_x^{(2)}$$

$$\varepsilon_1 = E_C n_1$$

$$\varepsilon_2 = E_C n_2$$

$$\sigma_j^{(1)} = \sigma_j \times I$$

$$H_{\text{int}} = E_{\text{int}} \left( \sigma^{(1)-} \sigma^{(2)+} + \sigma^{(1)+} \sigma^{(2)-} \right)$$

$$\sigma_j^{(2)} = I \times \sigma_j$$

$$E_{\text{int}} = 2E_C \frac{C_{12}}{C}$$

CNOT-gate

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Evolution of one of the qubits depends on the state of the other

18<sup>th</sup> May: Lecture on experimental realization of the two-qubit gate by PhD students MIPT Alexey Dmitriev and Gleb Fedorov

# Quantum parallelism

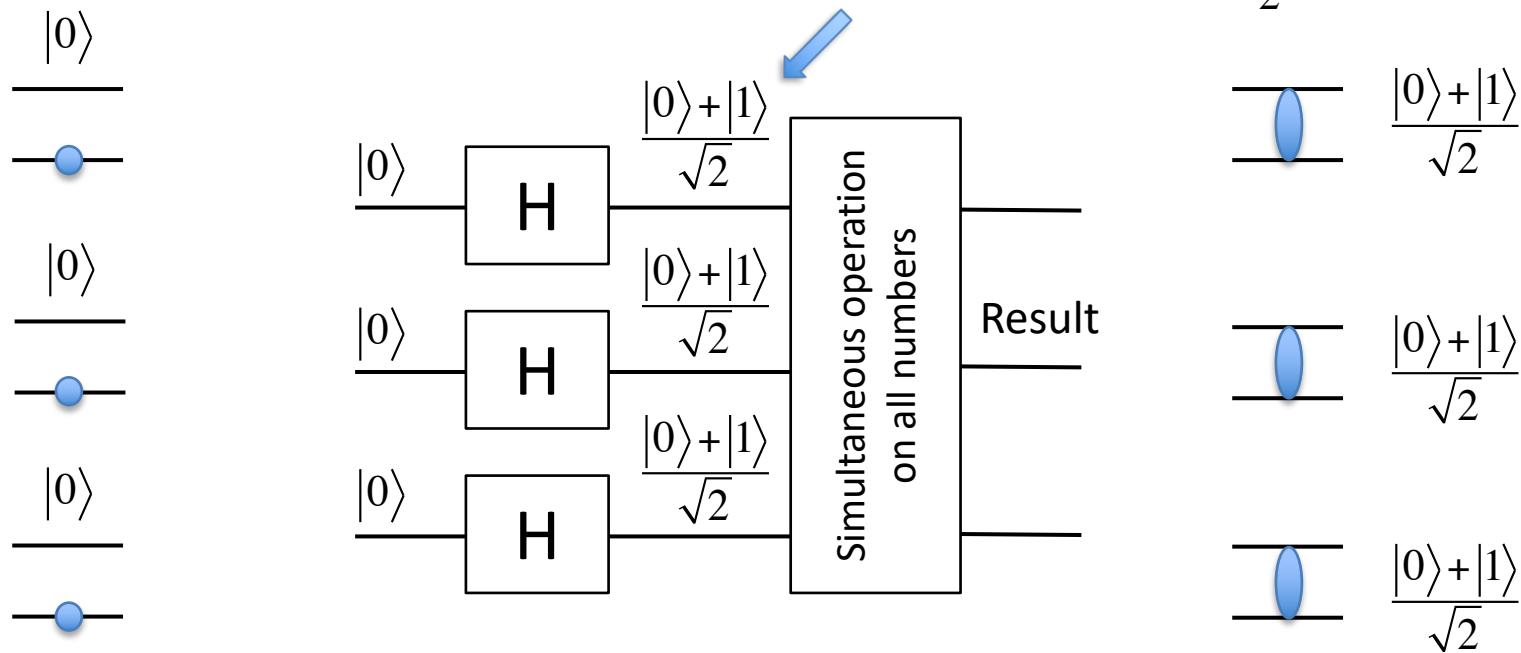
Initial state

$$|000\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle$$

State after Hadamard transformation

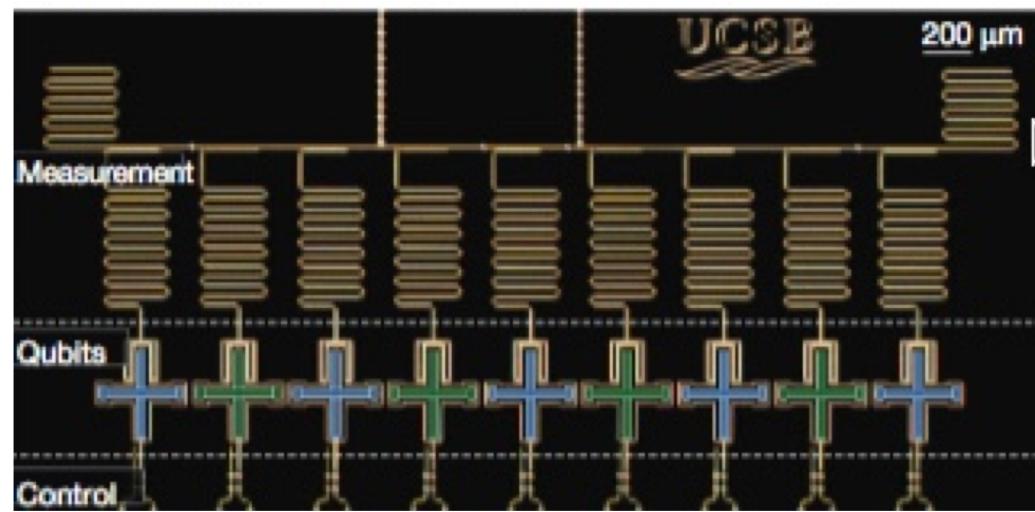
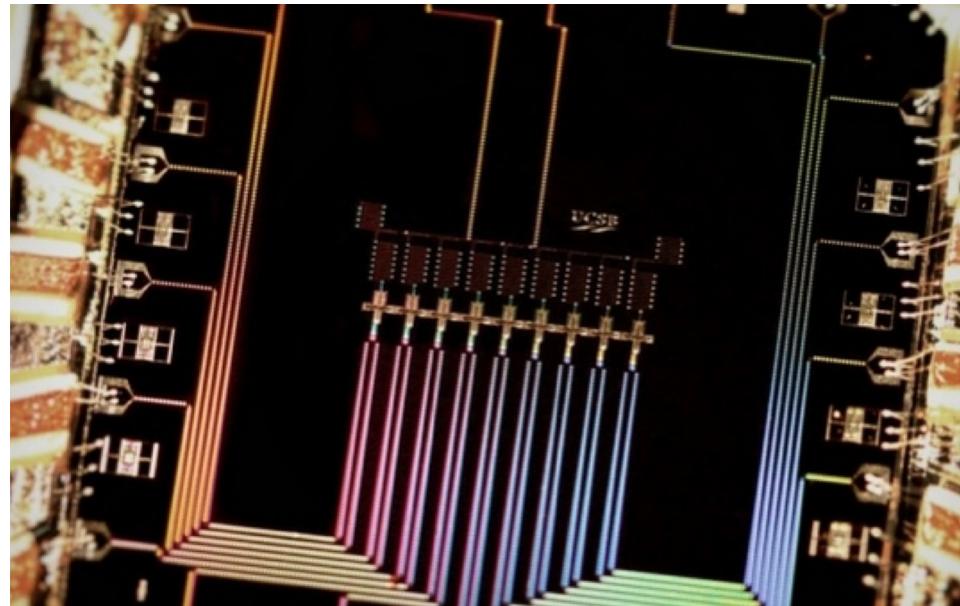
$$|000\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} =$$

$$= \frac{|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle}{2^{3/2}}$$



The operation can be produced simultaneously  
on all possible combination of numbers

# Quantum processor



# Dissipative dynamics

# Density matrix

Density matrix:  $\rho = |\Psi\rangle\langle\Psi|$

A diagonal element  $\rho_{kk}$  defines a probability of finding the system in the state  $|k\rangle$

Examples:

$$\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\Psi\rangle\langle\Psi| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Psi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad |\Psi\rangle\langle\Psi| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |\Psi\rangle\langle\Psi| = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\Psi = \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\varphi} \sin \frac{\theta}{2} \end{pmatrix} \quad |\Psi\rangle\langle\Psi| = \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\varphi} \sin \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} & e^{-i\varphi} \sin \frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos^2 \frac{\theta}{2} & e^{-i\varphi} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \\ e^{i\varphi} \sin \frac{\theta}{2} \cos \frac{\theta}{2} & \sin^2 \frac{\theta}{2} \end{pmatrix}$$

# Expectation values

$$\langle \hat{O} \rangle = \langle \Psi | \hat{O} | \Psi \rangle = \text{Trace}(\Psi \langle \Psi | \hat{O}) = \text{Trace}(\rho \hat{O})$$

$$|\Psi\rangle = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \quad |\Psi\rangle\langle\Psi| = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \begin{pmatrix} a_0^* & a_1^* \end{pmatrix} = \begin{pmatrix} a_0 a_0^* & a_0 a_1^* \\ a_1 a_0^* & a_1 a_1^* \end{pmatrix} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \quad \rho_{11} + \rho_{22} = 1$$

Pure states (described by wavefunctions):  $|\rho_{01}|^2 = \rho_{00}\rho_{11}$

Examples:

$$|0\rangle\langle 0| = \frac{\sigma_z + 1}{2} \quad \frac{\langle \sigma_z \rangle + 1}{2} = \frac{\rho_{00} - \rho_{11} + 1}{2} = \rho_{00} \quad \langle \sigma_z \rangle = \text{tr} \left[ \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = \text{tr} \begin{pmatrix} \rho_{00} & -\rho_{01} \\ \rho_{10} & -\rho_{11} \end{pmatrix} = \rho_{00} - \rho_{11}$$

$$|1\rangle\langle 1| = \frac{1 - \sigma_z}{2} \quad \frac{1 - \langle \sigma_z \rangle}{2} = \frac{1 - \rho_{00} + \rho_{11}}{2} = \rho_{11}$$

$$\sigma_x \quad \langle \sigma_x \rangle = \rho_{01} + \rho_{10}$$

$$\langle \sigma_x \rangle = \text{tr} \left[ \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \text{tr} \begin{pmatrix} \rho_{01} & \rho_{00} \\ \rho_{11} & \rho_{10} \end{pmatrix} = \rho_{01} + \rho_{10}$$

$$\sigma_y \quad \langle \sigma_y \rangle = i\rho_{01} - i\rho_{10}$$

$$\langle \sigma_y \rangle = \text{tr} \left[ \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] = \text{tr} \begin{pmatrix} i\rho_{01} & -i\rho_{00} \\ i\rho_{11} & -i\rho_{10} \end{pmatrix} = i\rho_{01} - i\rho_{10}$$

$$\sigma^+ = \frac{\sigma_x - i\sigma_y}{2} \quad \langle \sigma^+ \rangle = \frac{\langle \sigma_x \rangle - i\langle \sigma_y \rangle}{2} = \frac{\rho_{01} + \rho_{10} + \rho_{01} - \rho_{10}}{2} = \rho_{01}$$

$$\sigma^- = \frac{\sigma_x + i\sigma_y}{2} \quad \langle \sigma^- \rangle = \frac{\langle \sigma_x \rangle + i\langle \sigma_y \rangle}{2} = \frac{\rho_{01} + \rho_{10} - \rho_{01} + \rho_{10}}{2} = \rho_{10}$$

# Master Equation

Schrodinger equation:

$$i\hbar \frac{\partial |\Psi\rangle}{\partial t} = H|\Psi\rangle \quad -i\hbar \frac{\partial \langle \Psi|}{\partial t} = \langle \Psi|H$$

$$i\hbar \frac{\partial |\Psi\rangle\langle \Psi|}{\partial t} = i\hbar \frac{\partial |\Psi\rangle}{\partial t}\langle \Psi| + i\hbar |\Psi\rangle \frac{\partial \langle \Psi|}{\partial t} \quad i\hbar \frac{\partial |\Psi\rangle\langle \Psi|}{\partial t} = H|\Psi\rangle\langle \Psi| - |\Psi\rangle\langle \Psi|H$$

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho]$$

$$[H, \rho] = H\rho - \rho H$$

Two-level system diagonal Hamiltonian:  $H = -\frac{\hbar\omega_a}{2}\sigma_z$

$$i\hbar \begin{pmatrix} \dot{\rho}_{00} & \dot{\rho}_{01} \\ \dot{\rho}_{10} & \dot{\rho}_{11} \end{pmatrix} = -\frac{\hbar\omega_a}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} + \frac{\hbar\omega_a}{2} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} \dot{\rho}_{00} & \dot{\rho}_{01} \\ \dot{\rho}_{10} & \dot{\rho}_{11} \end{pmatrix} = \frac{-i\omega_a}{2} \begin{pmatrix} 0 & -2\rho_{01} \\ 2\rho_{10} & 0 \end{pmatrix}$$

$$\rho_{00}(t) = \rho_{00}(0) \quad \rho_{01}(t) = \rho_{01}(0)e^{i\omega_a t}$$

$$\rho_{10}(t) = \rho_{10}(0)e^{-i\omega_a t} \quad \rho_{11}(t) = \rho_{11}(0)$$

$$H = -\frac{\hbar\Omega}{2}\sigma_x \quad i\hbar \begin{pmatrix} \dot{\rho}_{00} & \dot{\rho}_{01} \\ \dot{\rho}_{10} & \dot{\rho}_{11} \end{pmatrix} = -\frac{\hbar\Omega}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} + \frac{\hbar\Omega}{2} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} \dot{\rho}_{00} & \dot{\rho}_{01} \\ \dot{\rho}_{10} & \dot{\rho}_{11} \end{pmatrix} = \frac{i\Omega}{2} \begin{pmatrix} \rho_{10} - \rho_{01} & \rho_{11} - \rho_{00} \\ \rho_{00} - \rho_{11} & \rho_{01} - \rho_{10} \end{pmatrix}$$

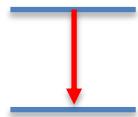
$$\dot{\rho}_{11} - \dot{\rho}_{00} = i\Omega(\rho_{01} - \rho_{10}) \quad \ddot{\rho}_{11} - \ddot{\rho}_{00} = -\Omega^2(\rho_{11} - \rho_{00})$$

$$\dot{\rho}_{01} - \dot{\rho}_{10} = i\Omega(\rho_{11} - \rho_{00}) \quad \rho_{11}(t) = A \sin \Omega t + B \cos \Omega t$$

# Incoherent processes: Energy dissipation

Probabilities to find the two-level system in the excited and ground states:  $P_1$  and  $P_0$

Excited two-level system



$$\Gamma = \frac{1}{T}$$

$$dP_1 = -P_1 \frac{dt}{T}$$

$$P_0 + P_1 = 1$$

$$\dot{P}_1 = -\frac{P_1}{T} = -\Gamma P_1$$

$$P_1(t) = P_1(0) e^{-\frac{t}{T}}$$

$$P_0(t) = 1 - P_1(0) e^{-\frac{t}{T}}$$

$T_1$  is the energy relaxation time

$$\rho_{11}(0) = 1$$

$$\rho(0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$



Mixed state (is not described by wavefunctions):

$$\rho(T_1 \ln 2) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad \rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

The wavefunction giving the same probabilities:

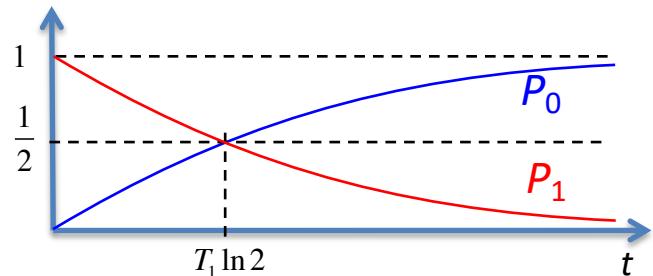
$$\Psi = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$



$$\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Psi = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$



$\rho_{01}$  and  $\rho_{10}$  - coherence

# Incoherent processes: Dissipation and dephasing

The Master Equation including incoherent processes:

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} [H, \rho] + L$$

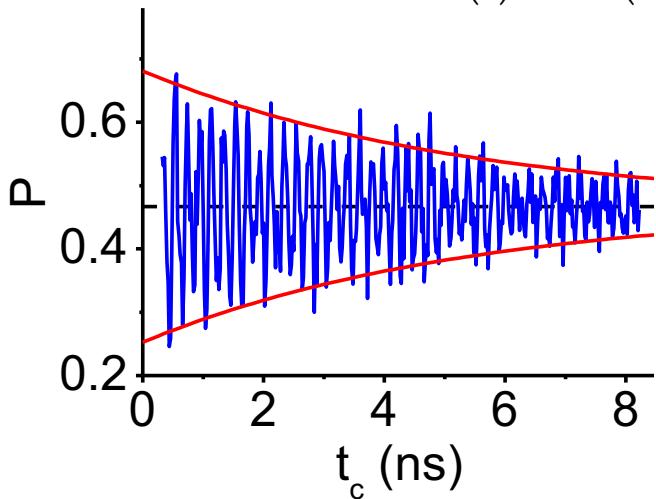
$$\begin{aligned} dP_1 &= -P_1 \frac{dt}{T_1} & \frac{d\rho_{11}}{dt} &= -\rho_{11} \Gamma_1 \\ dP_0 &= P_1 \frac{dt}{T_1} & \frac{d\rho_{00}}{dt} &= +\rho_{11} \Gamma_1 \end{aligned} \quad \Gamma_1 = \frac{1}{T_1} \quad L = \begin{pmatrix} \Gamma_1 \rho_{11} & -\Gamma_2 \rho_{01} \\ -\Gamma_2 \rho_{10} & -\Gamma_1 \rho_{11} \end{pmatrix}$$

$\Gamma_1$  is the energy relaxation

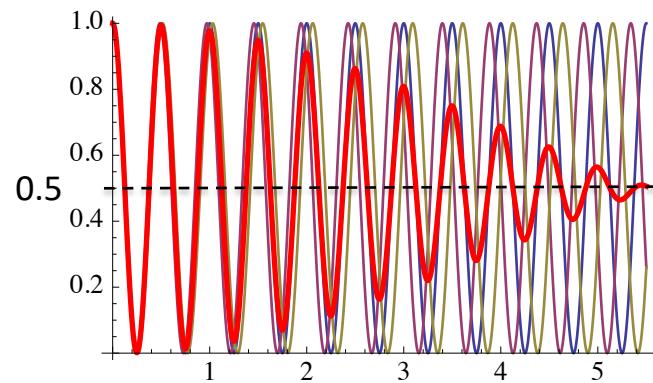
$\Gamma_2$  is dephasing (decay of the off-diagonal elements)

# Dephasing

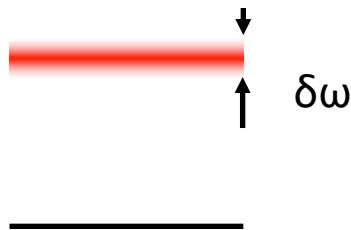
$$\psi(t) = \cos(\Omega t/2)|0\rangle + e^{-i\frac{\pi}{2}} \sin(\Omega t/2)|1\rangle$$



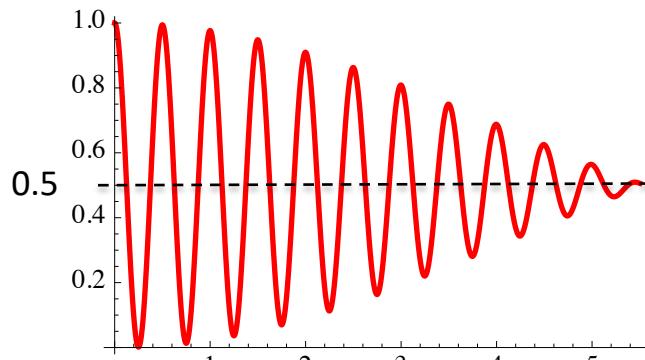
$$P_1 = \cos^2 \frac{\Omega_1 t}{2} \quad P_2 = \cos^2 \frac{\Omega_2 t}{2} \quad P_3 = \cos^2 \frac{\Omega_3 t}{2}$$



Dephasing:  
energy/frequency  $\Rightarrow$  phase fluctuation

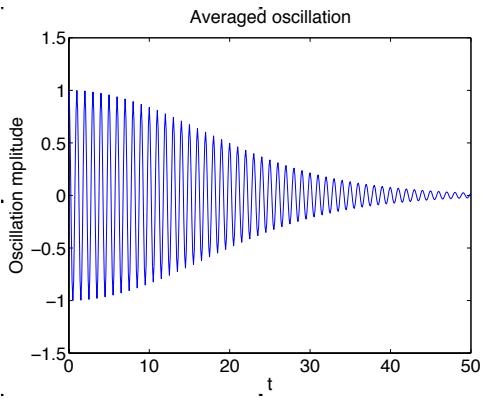


$$P(0) = \frac{1}{3} \left( \cos^2 \frac{\Omega_1 t}{2} + \cos^2 \frac{\Omega_2 t}{2} + \cos^2 \frac{\Omega_3 t}{2} \right)$$

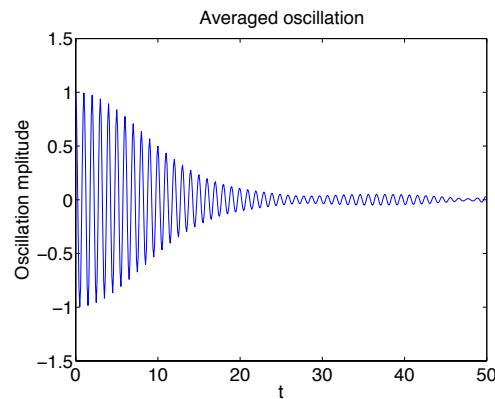


# Dephasing due to energy level fluctuations

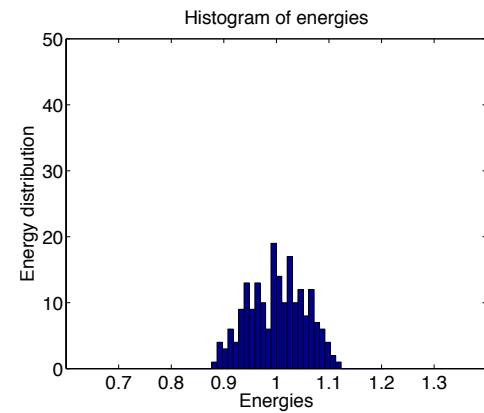
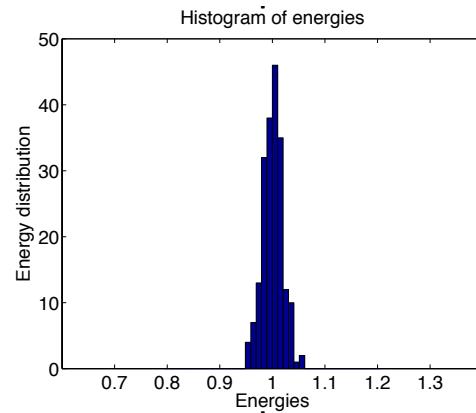
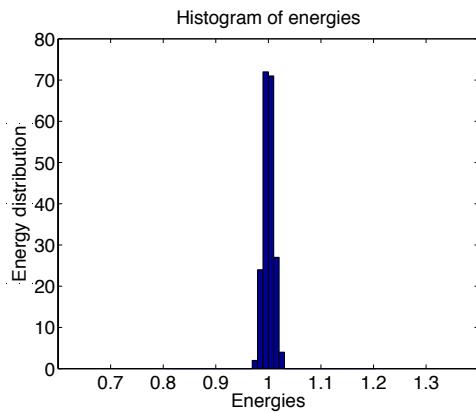
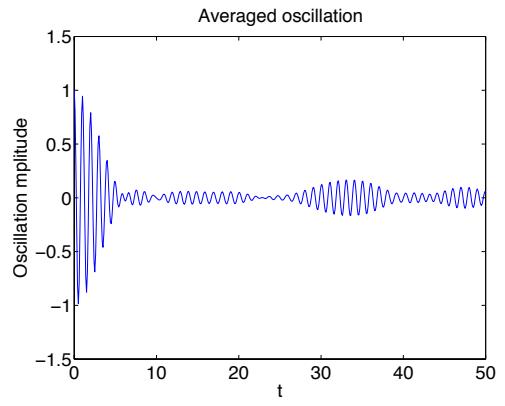
$\langle f \rangle = 1, \sigma_f = 0.01$



$\langle f \rangle = 1, \sigma_f = 0.02$



$\langle f \rangle = 1, \sigma_f = 0.05$



# Expectation values of Pauli operators

Density matrix of the two-level system has three independent real variables:  $\rho_{00}$ ,  $\text{Re}(\rho_{01})$ ,  $\text{Im}(\rho_{01})$

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \quad \rho_{00} + \rho_{11} = 1 \quad \rho_{01} = \rho_{10}^*$$

$$\langle \sigma_x \rangle = \text{tr} \left[ \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \text{tr} \begin{pmatrix} \rho_{01} & \rho_{00} \\ \rho_{11} & \rho_{10} \end{pmatrix} = \rho_{01} + \rho_{10} = 2 \text{Re}[\rho_{01}]$$

$$\langle \sigma_y \rangle = \text{tr} \left[ \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] = \text{tr} \begin{pmatrix} i\rho_{01} & -i\rho_{00} \\ i\rho_{11} & -i\rho_{10} \end{pmatrix} = i\rho_{01} - i\rho_{10} = -2 \text{Im}[\rho_{01}]$$

$$\langle \sigma_z \rangle = \text{tr} \left[ \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = \text{tr} \begin{pmatrix} \rho_{00} & -\rho_{01} \\ -\rho_{11} & -\rho_{11} \end{pmatrix} = \rho_{00} - \rho_{11} = \rho_{00} - (1 - \rho_{00}) = 2\rho_{00} - 1$$

$$\rho_{00} = \frac{1 + \langle \sigma_z \rangle}{2} \quad \rho_{01} = \frac{\langle \sigma_x \rangle - i\langle \sigma_y \rangle}{2} = \langle \sigma^+ \rangle \quad \rho_{10} = \frac{\langle \sigma_x \rangle + i\langle \sigma_y \rangle}{2} = \langle \sigma^- \rangle \quad \rho_{11} = \frac{1 - \langle \sigma_z \rangle}{2}$$

The dynamics can be alternatively represented via expectation values of the three Pauli matrices

# Bloch Sphere for dissipative spin dynamics

$$|\psi\rangle\langle\psi| = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2} \\ e^{i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{1}{2}e^{-i\varphi}\sin\theta \\ \frac{1}{2}e^{i\varphi}\sin\theta & \sin^2\frac{\theta}{2} \end{pmatrix}$$

$$\rho_{00} + \rho_{11} = 1$$

$$\langle\sigma_z\rangle = 1 \Rightarrow \rho_{00} - \rho_{11} = 1 \Rightarrow \rho_{00} = 1$$

$$|\psi\rangle = |0\rangle$$

$$\langle\sigma_z\rangle = -1 \Rightarrow \rho_{00} - \rho_{11} = -1 \Rightarrow \rho_{11} = 1$$

$$|\psi\rangle = |1\rangle$$

$$\langle\sigma_x\rangle = 1 \Rightarrow \rho_{01} + \rho_{10} = 1 \Rightarrow \rho_{01} = \rho_{10} = \frac{1}{2}$$

$$\theta = \frac{\pi}{2} \quad \varphi = 0 \quad |\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

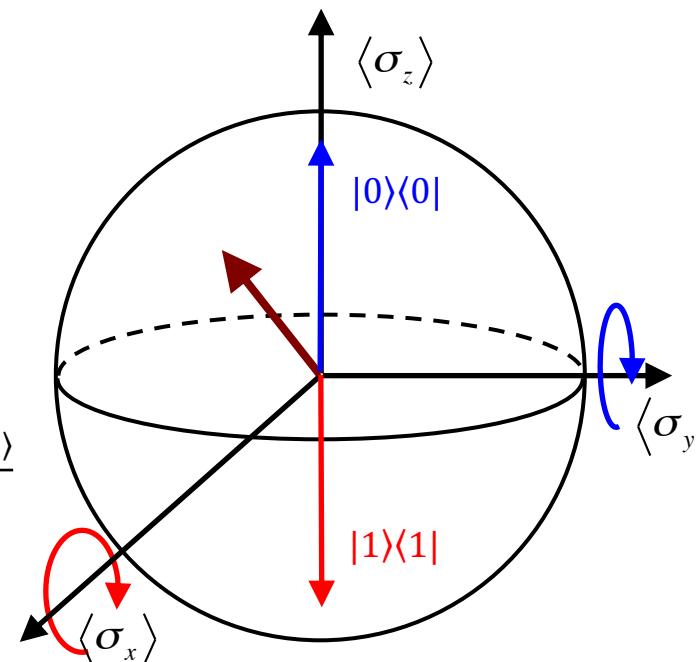
$$\langle\sigma_x\rangle = -1 \Rightarrow \rho_{01} + \rho_{10} = -1 \Rightarrow \rho_{01} = \rho_{10} = -\frac{1}{2}$$

$$\theta = \frac{\pi}{2} \quad \varphi = \pi \quad |\psi\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$\langle\sigma_y\rangle = 1 \Rightarrow i\rho_{01} - i\rho_{10} = 1 \Rightarrow \rho_{01} = -\rho_{01} = \frac{1}{2i}$$

$$\theta = \frac{\pi}{2} \quad \varphi = \frac{\pi}{2} \quad |\psi\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$

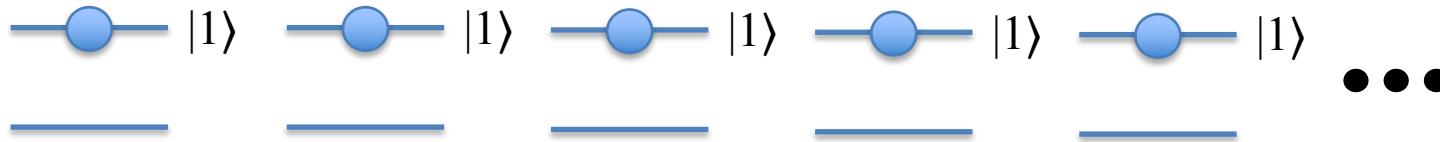
$$\langle\sigma_y\rangle = -1 \Rightarrow i\rho_{01} - i\rho_{10} = -1 \Rightarrow \rho_{01} = -\rho_{01} = -\frac{1}{2i} \quad \theta = \frac{\pi}{2} \quad \varphi = -\frac{\pi}{2} \quad |\psi\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$$



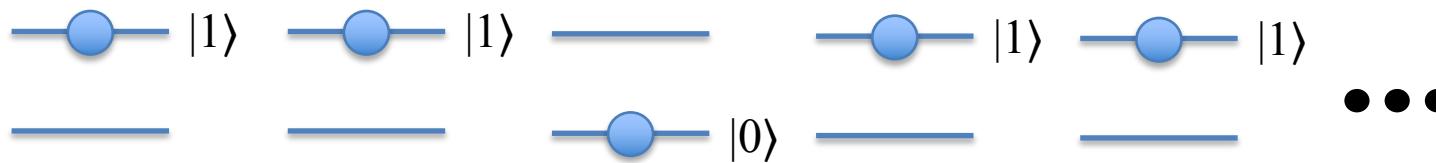
# Quantum dynamics of two-level systems with decoherence

# Pure and Mixed states

Pure state: wavefunction is  $|1\rangle$ ; Density matrix:  $\rho = |1\rangle\langle 1|$



Mixed state: the state which can not be described by a wavefunction => probabilities  
 Density matrix:  $\rho = 0.8|1\rangle\langle 1| + 0.2|0\rangle\langle 0|$



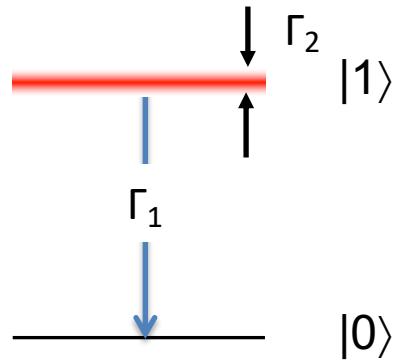
Phase fluctuations:

$$|\Psi_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad |\Psi_2\rangle = \frac{|0\rangle + e^{i\varphi}|1\rangle}{\sqrt{2}} \quad \rho = \begin{pmatrix} \frac{1}{2} & \frac{e^{i\varphi}}{2} \\ \frac{e^{-i\varphi}}{2} & \frac{1}{2} \end{pmatrix} \quad |\Psi_3\rangle = \frac{|0\rangle + e^{-i\varphi}|1\rangle}{\sqrt{2}} \quad \rho = \begin{pmatrix} \frac{1}{2} & \frac{e^{-i\varphi}}{2} \\ \frac{e^{i\varphi}}{2} & \frac{1}{2} \end{pmatrix} \quad \dots$$

$$\rho_{00} = \rho_{11} = \frac{1}{2} \quad \rho_{01} = \frac{1}{N} \left( \frac{1}{2} + \frac{e^{i\varphi}}{2} + \frac{e^{-i\varphi}}{2} + \dots \right) \approx \frac{1}{2} \frac{1}{N} \left( 1 + \left( 1 + i\varphi + \frac{(i\varphi)^2}{2!} \right) + \left( 1 - i\varphi + \frac{(-i\varphi)^2}{2!} \right) + \dots \right) = \frac{1 - \frac{\varphi^2}{3}}{2}$$

Mixed state:  $|\rho_{01}| = |\rho_{10}| < \sqrt{|\rho_{00}||\rho_{11}|} = \frac{1}{2}$

# The Master Equation and the Lindblad term



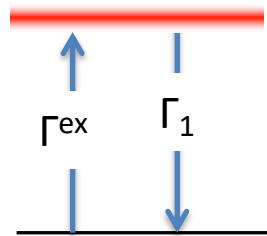
Relaxation and dephasing:  $\Gamma_1, \Gamma_2$

The Lindblad term

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar} [H, \rho] + L$$

$$\begin{pmatrix} \dot{\rho}_{00} & \dot{\rho}_{01} \\ \dot{\rho}_{10} & \dot{\rho}_{11} \end{pmatrix}$$

$$L = \begin{pmatrix} \Gamma_1 \rho_{11} & -\Gamma_2 \rho_{01} \\ -\Gamma_2 \rho_{10} & -\Gamma_1 \rho_{11} \end{pmatrix}$$

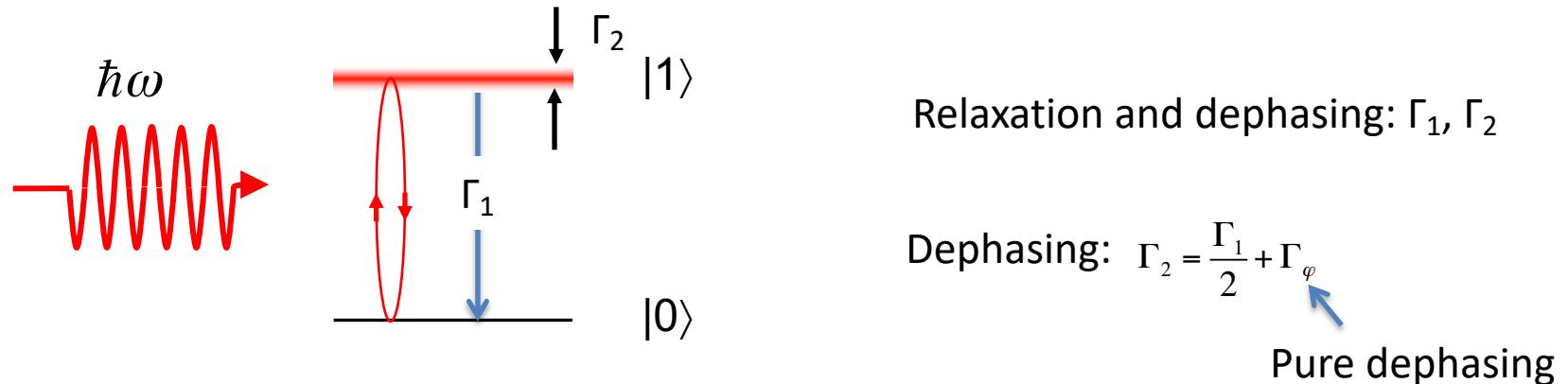


Accounting excitation (e.g.  $T > 0$ )

$$L = \begin{pmatrix} \Gamma_1 \rho_{11} - \Gamma^{ex} \rho_{00} & -\Gamma_2 \rho_{01} \\ -\Gamma_2 \rho_{10} & -\Gamma_1 \rho_{11} + \Gamma^{ex} \rho_{00} \end{pmatrix}$$

# Driven two-level system with incoherent processes

The general form of the Hamiltonian driven by a wave with an arbitrary phase shift



$$H = -\frac{\hbar\omega}{2}\sigma_z + \hbar\Omega\sigma_x \cos(\omega t - \varphi) \quad U = e^{-i\frac{\omega t}{2}\sigma_z}$$

$$H' = \frac{\hbar\Omega}{2}(\sigma_x \cos\varphi + \sigma_y \sin\varphi)$$

$$\frac{\partial\rho}{\partial t} = -\frac{i}{\hbar}[H', \rho] + L \quad \rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \quad L = \begin{pmatrix} \Gamma_1\rho_{11} & -\Gamma_2\rho_{01} \\ -\Gamma_2\rho_{10} & -\Gamma_1\rho_{11} \end{pmatrix}$$

$$L = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -\Gamma_1\rho_{00} & -\Gamma_2\rho_{01} \\ -\Gamma_2\rho_{10} & -\Gamma_1\rho_{11} \end{pmatrix}$$

# Driven two-level system with incoherent processes

The RWA Hamiltonian:

$$H = \frac{\hbar\Omega}{2}(\sigma_x \cos\varphi + \sigma_y \sin\varphi)$$

The Master Equation:

$$\frac{\partial\rho}{\partial t} = -\frac{i}{\hbar}[H, \rho] + L$$

The Lindblad term:

$$L = \begin{pmatrix} \Gamma_1 \rho_{11} & -\Gamma_2 \rho_{01} \\ -\Gamma_2 \rho_{10} & -\Gamma_1 \rho_{11} \end{pmatrix}$$

Three independent equations for

$$\rho_{00} + \rho_{11} = 1$$

$$\rho_{00}, \rho_{01}, \rho_{10}$$

$$\frac{\partial\langle\sigma_j\rangle}{\partial t} = \text{tr}\left[\sigma_j \frac{\partial\rho}{\partial t}\right]$$

$$\frac{\partial\langle\sigma_j\rangle}{\partial t} = \text{tr}\left[-\frac{i}{\hbar}\sigma_j(H\rho - \rho H) + \sigma_j L\right] = -\frac{i}{\hbar}\text{tr}\left[\sigma_j H\rho - H\sigma_j \rho\right] + \text{tr}\left[\sigma_j L\right] = \frac{i}{\hbar}\langle[H, \sigma_j]\rangle + \text{tr}\left[\sigma_j L\right]$$

$$\frac{\partial\langle\sigma_x\rangle}{\partial t} = -i\frac{\Omega}{2}(\langle[\sigma_x, \sigma_x]\rangle \cos\varphi + \langle[\sigma_y, \sigma_x]\rangle \sin\varphi) + \text{tr}[\sigma_x L]$$

$$\frac{\partial\langle\sigma_y\rangle}{\partial t} = -i\frac{\Omega}{2}(\langle[\sigma_x, \sigma_y]\rangle \cos\varphi + \langle[\sigma_y, \sigma_y]\rangle \sin\varphi) + \text{tr}[\sigma_y L]$$

$$\frac{\partial\langle\sigma_z\rangle}{\partial t} = -i\frac{\Omega}{2}(\langle[\sigma_x, \sigma_z]\rangle \cos\varphi + \langle[\sigma_y, \sigma_z]\rangle \sin\varphi) + \text{tr}[\sigma_z L]$$

$$\begin{pmatrix} \dot{\rho}_{00} & \dot{\rho}_{01} \\ \dot{\rho}_{10} & \dot{\rho}_{11} \end{pmatrix} = -\frac{i}{\hbar}(H\rho - \rho H) + L$$

$$\frac{\partial\langle\sigma_x\rangle}{\partial t} = -\Omega\langle\sigma_z\rangle \sin\varphi - \Gamma_2\langle\sigma_x\rangle$$

$$\frac{\partial\langle\sigma_y\rangle}{\partial t} = \Omega\langle\sigma_z\rangle \cos\varphi - \Gamma_2\langle\sigma_y\rangle$$

$$\frac{\partial\langle\sigma_z\rangle}{\partial t} = \Omega(-\langle\sigma_y\rangle \cos\varphi + \langle\sigma_x\rangle \sin\varphi) - \Gamma_1\langle\sigma_z\rangle + \Gamma_1$$

$$\frac{\partial \vec{\sigma}}{\partial t} = \underbrace{\begin{pmatrix} \langle\sigma_x\rangle \\ \langle\sigma_y\rangle \\ \langle\sigma_z\rangle \end{pmatrix}}_{\vec{\sigma}} \underbrace{\begin{pmatrix} -\Gamma_2 & 0 & -\Omega \sin\varphi \\ 0 & -\Gamma_2 & \Omega \cos\varphi \\ \Omega \sin\varphi & -\Omega \cos\varphi & -\Gamma_1 \end{pmatrix}}_B \underbrace{\begin{pmatrix} \langle\sigma_x\rangle \\ \langle\sigma_y\rangle \\ \langle\sigma_z\rangle \end{pmatrix}}_{\vec{\sigma}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ \Gamma_1 \end{pmatrix}}_{\vec{b}}$$

Dynamics of the two level system is described by:

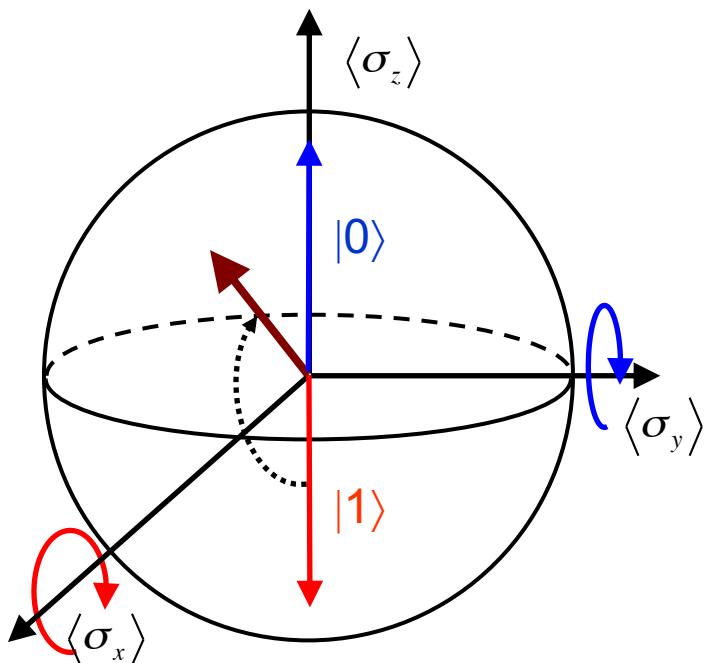
$$\frac{d\vec{\sigma}}{dt} = B\vec{\sigma} + \vec{b}$$

# Bloch Sphere for dissipative spin dynamics

$$\frac{\partial}{\partial t} \begin{pmatrix} \langle \sigma_x \rangle \\ \langle \sigma_y \rangle \\ \langle \sigma_z \rangle \end{pmatrix} = \begin{pmatrix} -\Gamma_2 & 0 & -\Omega \sin \varphi \\ 0 & -\Gamma_2 & \Omega \cos \varphi \\ \Omega \sin \varphi & -\Omega \cos \varphi & -\Gamma_1 \end{pmatrix} \begin{pmatrix} \langle \sigma_x \rangle \\ \langle \sigma_y \rangle \\ \langle \sigma_z \rangle \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Gamma_1 \end{pmatrix}$$

$$\frac{d\vec{\sigma}}{dt} = B\vec{\sigma} + \vec{b}$$

The vector can be less than one  
(alternative criteria of mixed states)



$$\vec{\sigma}(t) = e^{Bt} \vec{\sigma}(0) + B^{-1} (e^{Bt} - 1) \vec{b}$$

$$\vec{\sigma}(t) = e^{Bt} [\vec{\sigma}(0) + B^{-1} \vec{b}] - B^{-1} \vec{b}$$

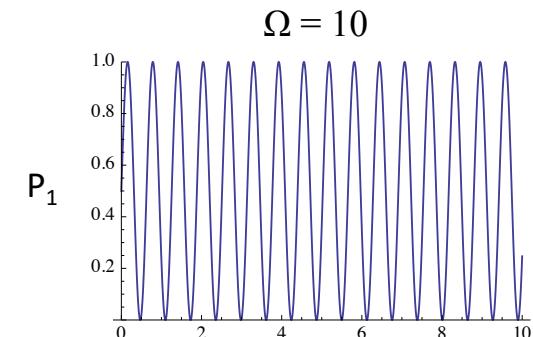
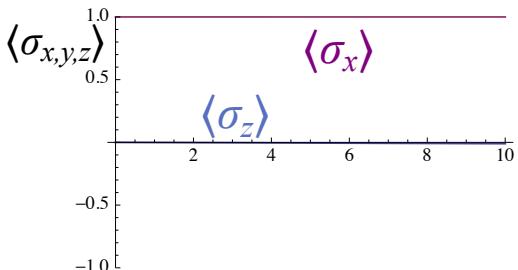
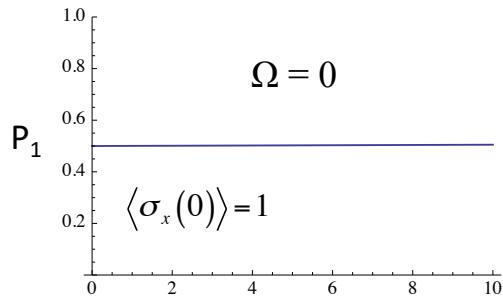
Dynamics of the two-level system is exactly same as the dynamics of spin 1/2

# Rabi oscillations: Oscillations in a two-level system under external drive.

## Oscillations of atomic states.

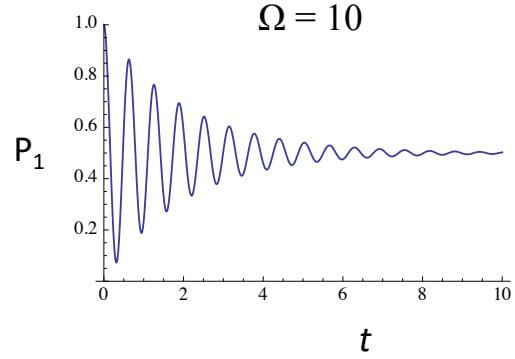
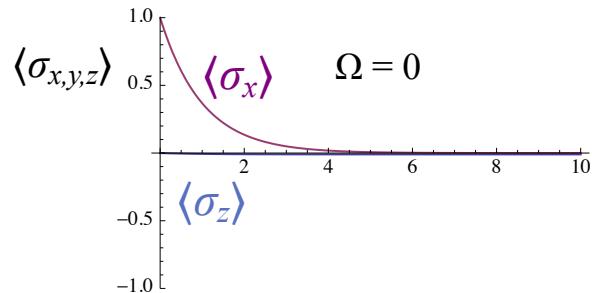
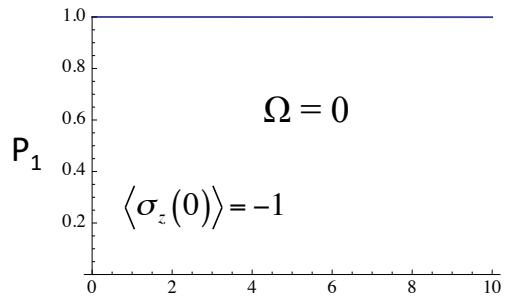
No decoherence

$$\Gamma_1 = 0, \Gamma_2 = 0$$



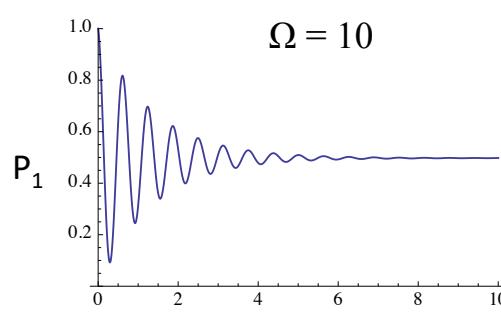
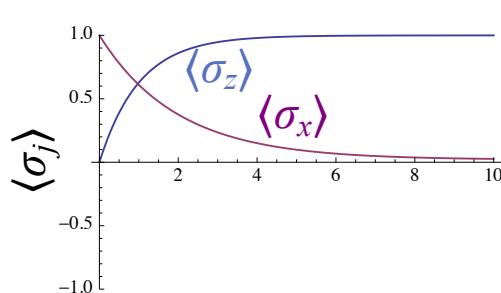
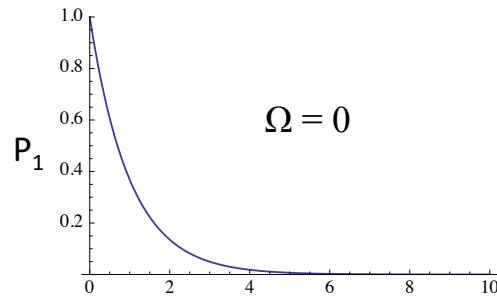
Pure dephasing

$$\Gamma_1 = 0, \Gamma_2 = 1$$



Relaxation

$$\Gamma_1 = 1, \Gamma_2 = 0.5$$



No decoherence  $\Gamma_1 = 0, \Gamma_2 = 0$

$$\Psi = \cos \frac{\Omega t}{2} |0\rangle + \sin \frac{\Omega t}{2} |1\rangle$$

$$\rho_{00} = \sin^2 \frac{\Omega t}{2} = \frac{1 + \cos \Omega t}{2}$$

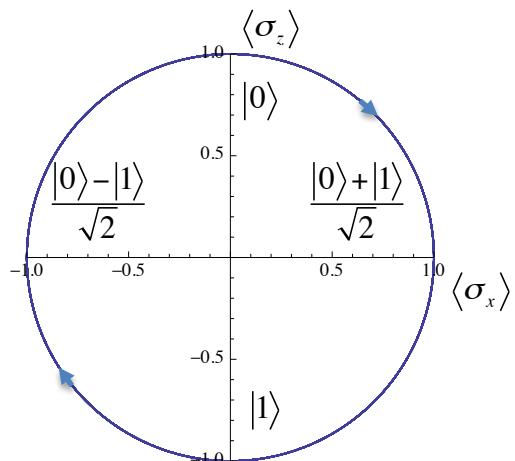
$$\rho_{01} = \cos \frac{\Omega t}{2} \sin \frac{\Omega t}{2} = \frac{\sin \Omega t}{2}$$

$$\rho_{10} = \cos \frac{\Omega t}{2} \sin \frac{\Omega t}{2} = \frac{\sin \Omega t}{2}$$

$$\langle \sigma_x \rangle = \rho_{01} + \rho_{10} = \sin \Omega t$$

$$\langle \sigma_y \rangle = i\rho_{01} - i\rho_{10} = 0$$

$$\langle \sigma_z \rangle = 2\rho_{00} - 1 = \cos \Omega t$$



With decoherence

$$\rho_{00} = 1$$

$$\langle \sigma_x \rangle \approx e^{-\gamma t} \sin \Omega t$$

$$\langle \sigma_y \rangle = 0$$

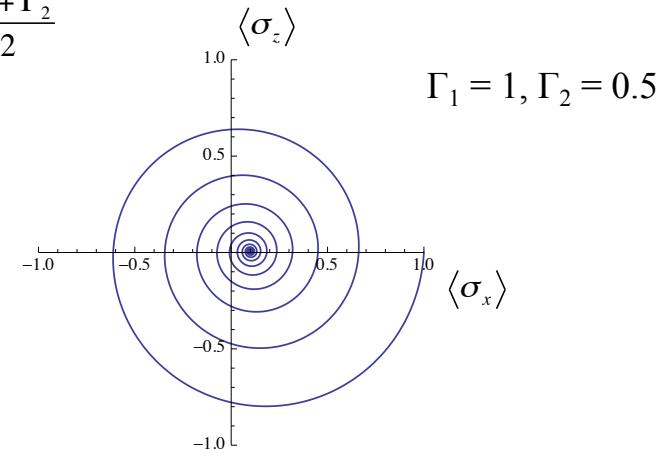
$$\langle \sigma_z \rangle \approx e^{-\gamma t} \cos \Omega t$$

$$\rho_{00} = \frac{\langle \sigma_z + 1 \rangle}{2} = \frac{1 + e^{-\gamma t} \cos \Omega t}{2}$$

$$\rho_{01} = \frac{e^{-\gamma t} \sin \Omega t}{2}$$

$$\rho_{10} \approx \frac{e^{-\gamma t} \sin \Omega t}{2}$$

$$\gamma = \frac{\Gamma_1 + \Gamma_2}{2}$$



# Stationary Master Equation

$$\frac{\partial \rho}{\partial t} = 0$$

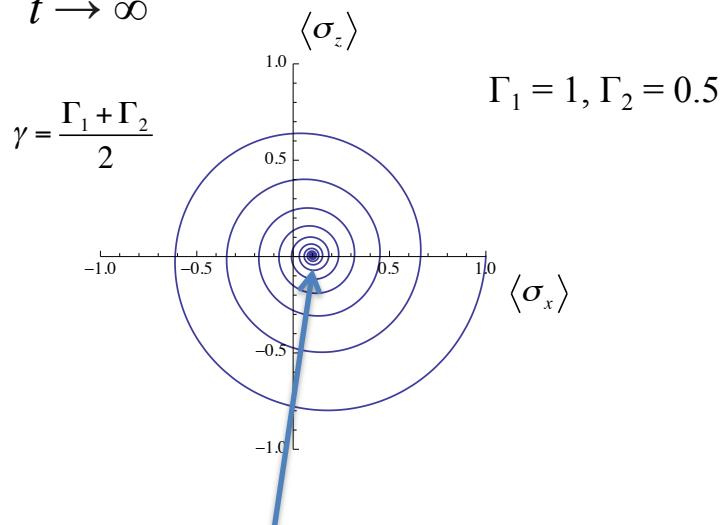
$$-\frac{i}{\hbar} [H, \rho] + L = 0$$

$$\frac{\partial \vec{\sigma}}{\partial t} = 0$$

$$B\vec{\sigma} + \vec{b} = 0$$

$$\vec{\sigma} = -B^{-1}\vec{b}$$

$t \rightarrow \infty$



$$\langle \sigma_x \rangle = \frac{\Gamma_1 \Omega}{\Gamma_1 \Gamma_2 + \Omega^2} \quad \xrightarrow{\Omega \rightarrow \infty} \frac{\Gamma_1}{\Omega}$$

$$\langle \sigma_y \rangle = 0$$

$$\langle \sigma_z \rangle = \frac{\Gamma_1 \Gamma_2}{\Gamma_1 \Gamma_2 + \Omega^2}$$

Stationary conditions

# Driven two-level system with detuning

$$H = -\frac{\hbar\omega_0}{2}\sigma_z + \frac{\hbar\Omega}{2}\sigma_x \cos(\omega t + \varphi) \quad U = e^{-i\frac{\omega t}{2}\sigma_z}$$

$$H' = -\frac{\hbar\delta\omega}{2}\sigma_z + \frac{\hbar\Omega}{2}(\sigma_x \cos\varphi + \sigma_y \sin\varphi)$$

$$\delta\omega = \omega_0 - \omega$$

$$\frac{\partial \langle \sigma_x \rangle}{\partial t} = -i \frac{\delta\omega}{2} \langle [\sigma_z, \sigma_x] \rangle + \dots$$

$$\frac{\partial \langle \sigma_y \rangle}{\partial t} = -i \frac{\delta\omega}{2} \langle [\sigma_z, \sigma_y] \rangle + \dots$$

$$\frac{\partial \langle \sigma_z \rangle}{\partial t} = -i \frac{\delta\omega}{2} \langle [\sigma_z, \sigma_z] \rangle + \dots$$

$$\frac{\partial \langle \sigma_x \rangle}{\partial t} = -\delta\omega \langle \sigma_y \rangle - \Omega \langle \sigma_z \rangle \sin\varphi - \Gamma_2 \langle \sigma_x \rangle$$

$$\frac{\partial \langle \sigma_y \rangle}{\partial t} = \delta\omega \langle \sigma_x \rangle + \Omega \langle \sigma_z \rangle \cos\varphi - \Gamma_2 \langle \sigma_y \rangle$$

$$\frac{\partial \langle \sigma_z \rangle}{\partial t} = \Omega (\langle \sigma_x \rangle \cos\varphi - \langle \sigma_y \rangle \sin\varphi) - \Gamma_1 \langle \sigma_z \rangle + \Gamma_1$$

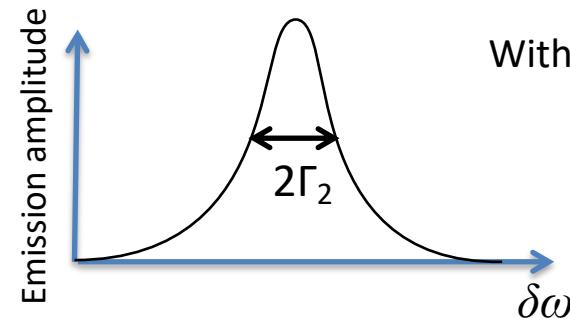
$$\frac{\partial}{\partial t} \begin{pmatrix} \langle \sigma_x \rangle \\ \langle \sigma_y \rangle \\ \langle \sigma_z \rangle \end{pmatrix} = \begin{pmatrix} -\Gamma_2 & -\delta\omega & -\Omega \sin\varphi \\ \delta\omega & -\Gamma_2 & \Omega \cos\varphi \\ \Omega \sin\varphi & -\Omega \cos\varphi & -\Gamma_1 \end{pmatrix} \begin{pmatrix} \langle \sigma_x \rangle \\ \langle \sigma_y \rangle \\ \langle \sigma_z \rangle \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Gamma_1 \end{pmatrix}$$

Stationary solution:

$$B\vec{\sigma} + \vec{b} = 0 \quad \vec{\sigma} = -B^{-1}\vec{b}$$

In the limit of weak drive:

$$V_{emit} = V_0 \langle \sigma^+ \rangle \propto \frac{1}{\Gamma_2 + i\delta\omega}$$



Without dephasing:

$$\Gamma_2 = \frac{\Gamma_1}{2}$$