Superconducting Quantum Technologies

Oleg Astafiev

Lecture 4

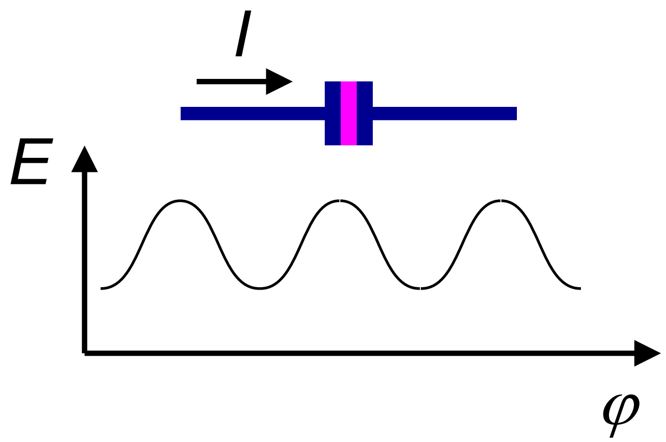
Lecture 4

- Three-junction flux qubits (classical flux qubit)
- Harmonic oscillators
- The number operator and Fock states. Photons in a resonator.
- Coplanar lines and a coplanar resonator
- Filed quantization in coplanar resonators

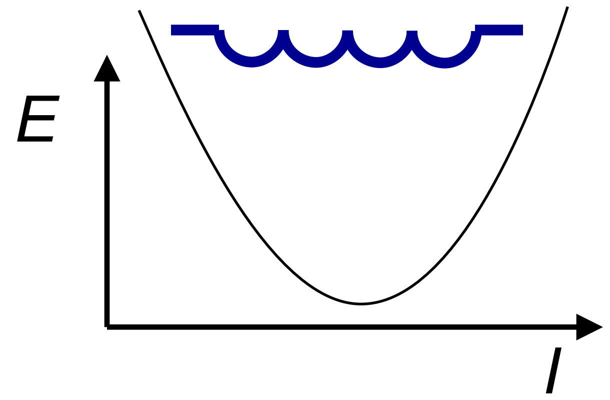
The three-junction flux qubit

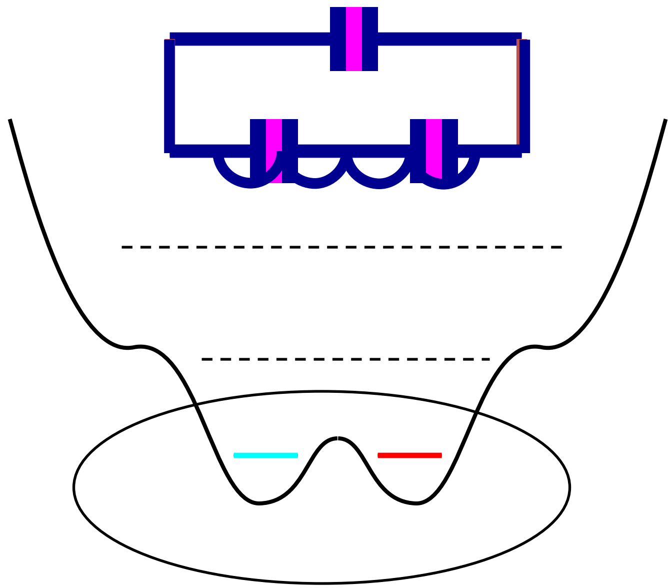
The flux qubit

Josephson junction

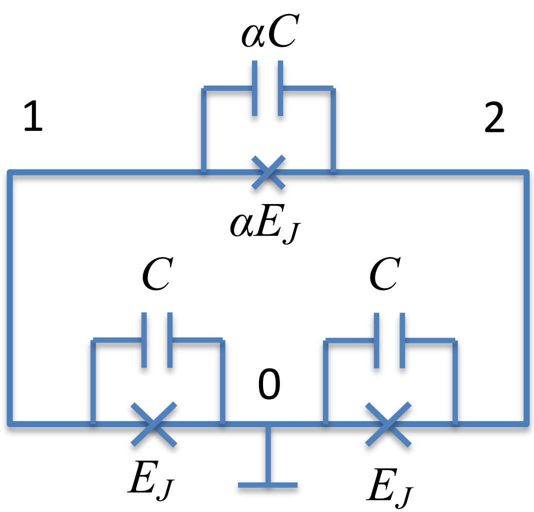


Inductance





The three-junction flux qubit



$$\varphi_{01} + \varphi_{12} + \varphi_{20} = \frac{2\pi}{\Phi_0}\Phi_{ext} = \varphi_{ext}$$

$$U = E_J\left(1 - \cos\varphi_{01}\right) + \alpha E_J\left(1 - \cos\varphi_{12}\right) + E_J\left(1 - \cos\varphi_{20}\right)$$

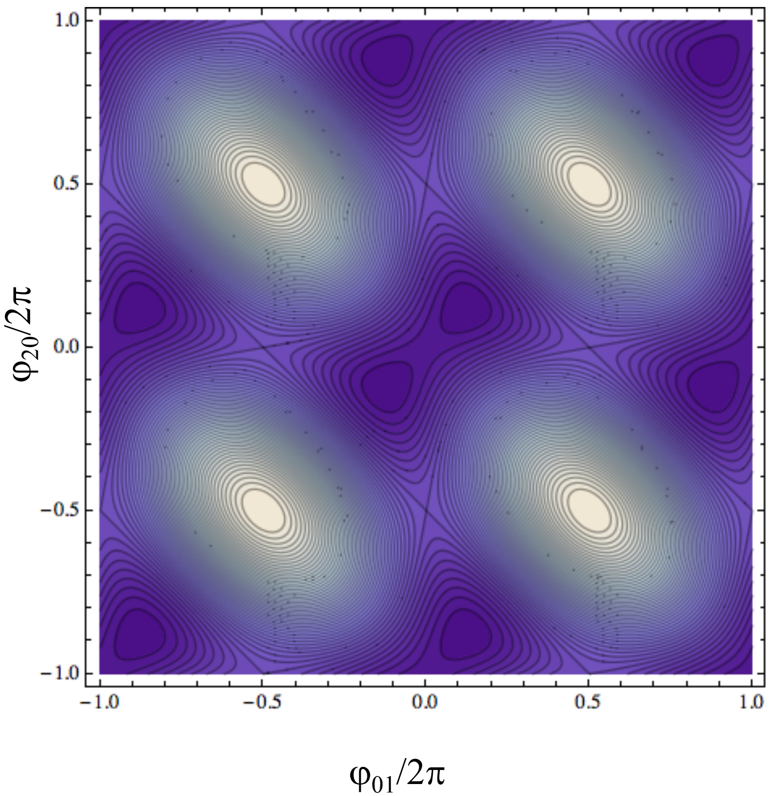
$$U = E_J\left[\left(1 - \cos\varphi_{01}\right) + \alpha\left(1 - \cos\left(\varphi_{ext} - \varphi_{01} - \varphi_{20}\right)\right) + \left(1 - \cos\varphi_{20}\right)\right]$$

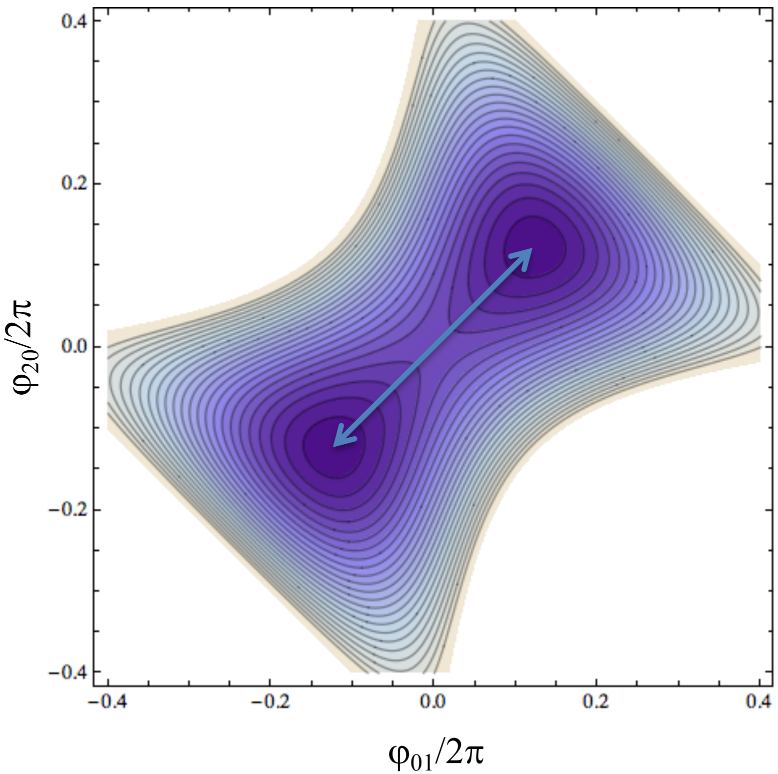
$$U = E_J\left[2 + \alpha - \cos\varphi_{01} - \cos\varphi_{20} - \alpha\cos\left(\varphi_{ext} - \varphi_{01} - \varphi_{20}\right)\right]$$

Josephson potential of the three-junction loop

$$\alpha = 0.7$$

$$U/E_J = \left[2 + \alpha - \cos\varphi_{01} - \cos\varphi_{20} - \alpha\cos\left(\varphi_{ext} - \varphi_{01} - \varphi_{20}\right)\right]$$

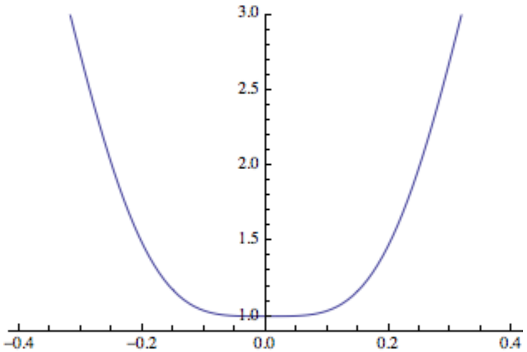




Shape of Josephson potential vs alpha

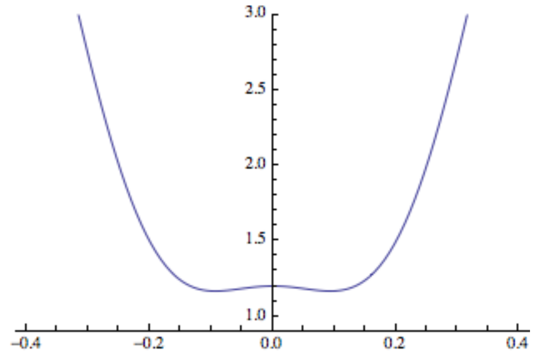
$\varphi_{01}/2\pi = \varphi_{20}/2\pi$

$\alpha = 0.5$



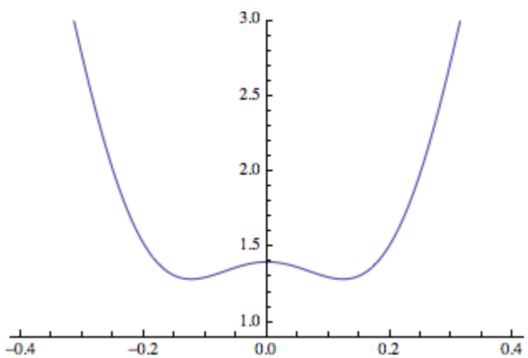
$\varphi_{01}/2\pi$

$\alpha = 0.6$



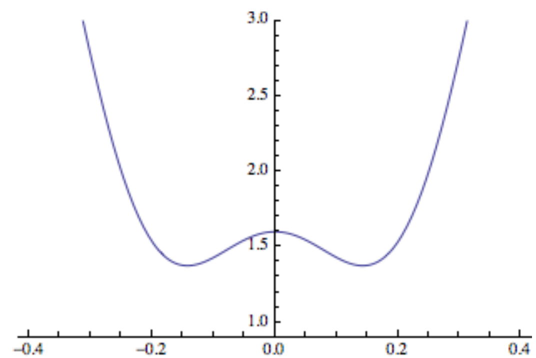
$\varphi_{01}/2\pi$

$\alpha = 0.7$



$\varphi_{01}/2\pi$

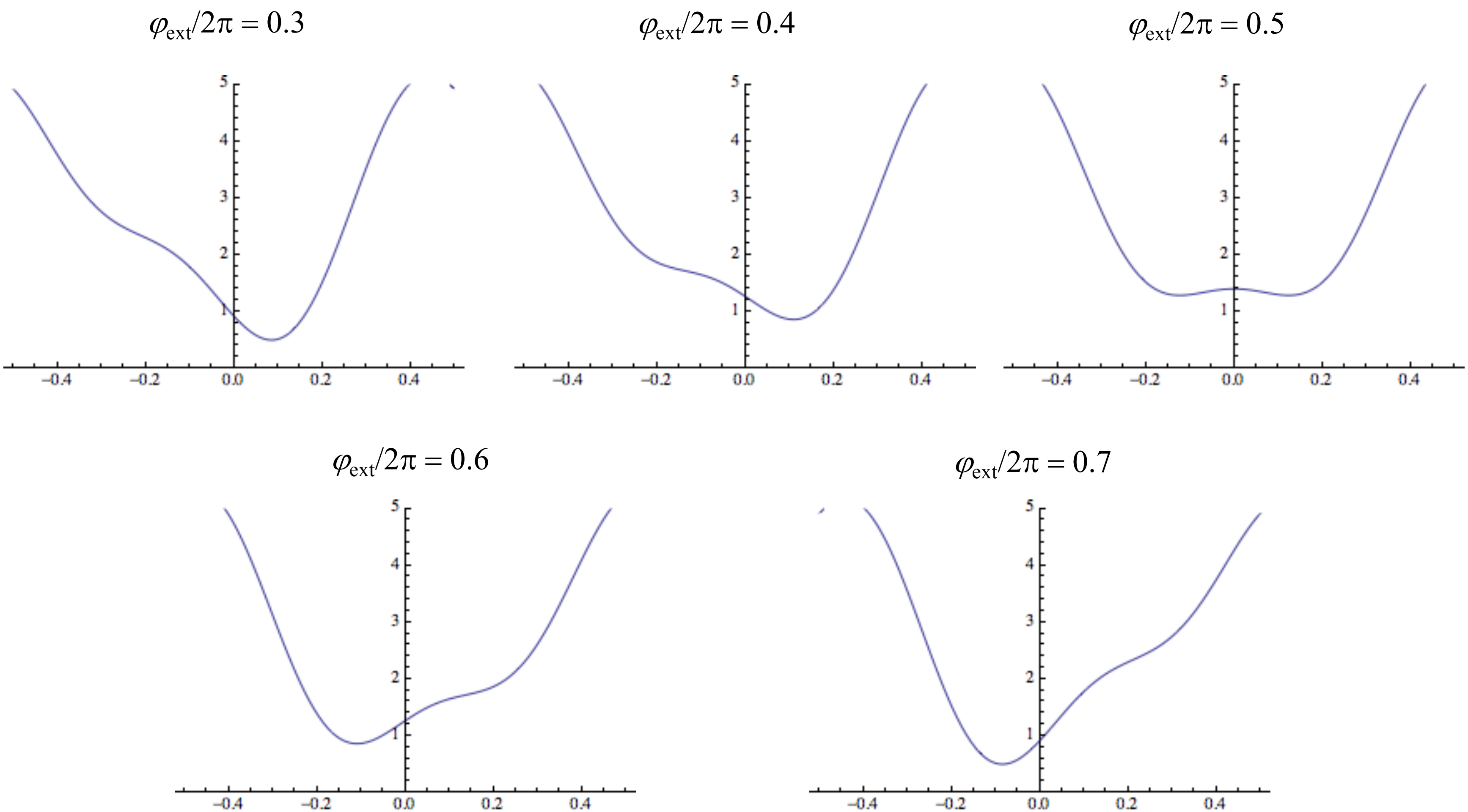
$\alpha = 0.8$



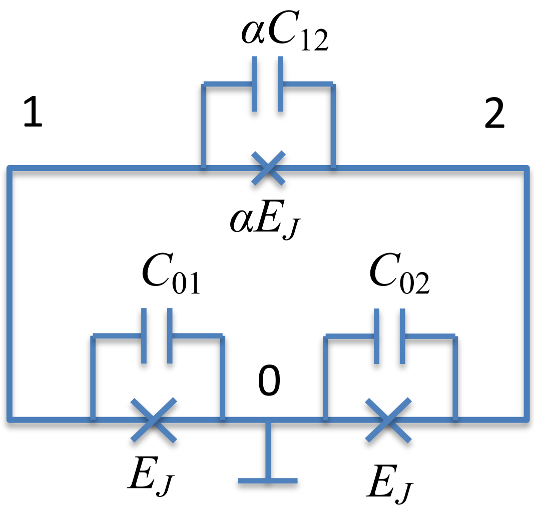
$\varphi_{01}/2\pi$

Josephson potential of the biased flux qubit

$\alpha = 0.7$



Flux biased qubit



Josephson potential:

$$U = E_J\left[2+\alpha-\cos\varphi_{01}-\cos\varphi_{20}-\alpha\cos\left(\varphi_{ext}-\varphi_{01}-\varphi_{20}\right)\right]$$

Charge:

$$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

Potential:

$$\vec{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

Capacitance matrix:

$$C = \begin{pmatrix} C_{01}+C_{12} & -C_{12} \\ -C_{12} & C_{02}+C_{12} \end{pmatrix}$$

$$\vec{n} = \frac{C\vec{V}}{2e} \qquad \vec{V} = 2eC^{-1}\vec{n}$$

Electrostatic energy: $T = \dfrac{(2e)^2}{2}\vec{n}C^{-1}\vec{n}$

The Hamiltonian:

$$\boxed{H = \frac{(2e)^2}{2}\vec{n}C^{-1}\vec{n} + E_J\left[2+\alpha-\cos\varphi_{01}-\cos\varphi_{20}-\alpha\cos\left(\varphi_{ext}-\varphi_{01}-\varphi_{20}\right)\right]}$$

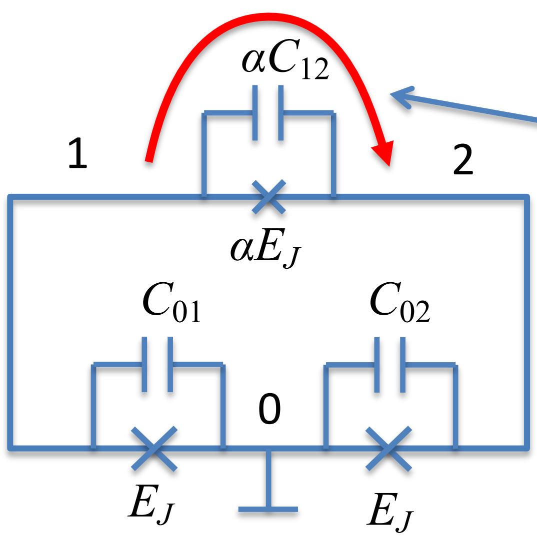
Phase operators

$$\cos\hat{\varphi} = \frac{e^{i\hat{\varphi}} + e^{-i\hat{\varphi}}}{2} \qquad e^{i\hat{\varphi}} = |N\rangle\langle N-1| \qquad e^{-i\hat{\varphi}} = |N-1\rangle\langle N|$$

$$\boxed{\cos\hat{\varphi}_{01} = \frac{1}{2}\left(|N_1\rangle\langle N_1-1| + |N_1-1\rangle\langle N_1|\right)} \qquad \boxed{\cos\hat{\varphi}_{20} = \frac{1}{2}\left(|N_2\rangle\langle N_2-1| + |N_2-1\rangle\langle N_2|\right)}$$

$$\cos\left(\varphi_{ext} - \hat{\varphi}_{01} - \hat{\varphi}_{20}\right) = \frac{e^{i(\varphi_{ext}-\hat{\varphi}_{01}-\hat{\varphi}_{20})} + e^{-i(\varphi_{ext}-\hat{\varphi}_{01}-\hat{\varphi}_{20})}}{2}$$

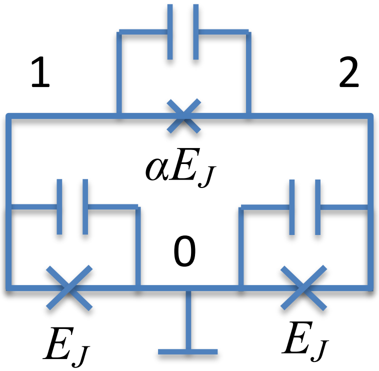
$$e^{i(\varphi_{ext}-\hat{\varphi}_{01}-\hat{\varphi}_{20})} = e^{i\varphi_{ext}} e^{-i\hat{\varphi}_{01}} e^{-i\hat{\varphi}_{20}} = e^{i\varphi_{ext}} |N_1-1\rangle\langle N_1||N_2\rangle\langle N_2-1| = e^{i\varphi_{ext}} \boxed{|N_1-1,N_2\rangle\langle N_1,N_2-1|}$$



Physical meaning: tunneling of a charge quantum (Cooper pair) from island 1 to island 2

$$\boxed{\cos\left(\varphi_{ext} - \hat{\varphi}_{01} - \hat{\varphi}_{20}\right) = \frac{1}{2}\left(e^{i\varphi_{ext}} |N_1-1,N_2\rangle\langle N_1,N_2-1| + e^{-i\varphi_{ext}} |N_1,N_2-1\rangle\langle N_1-1,N_2|\right)}$$

The flux qubit Hamiltonian in the charge basis

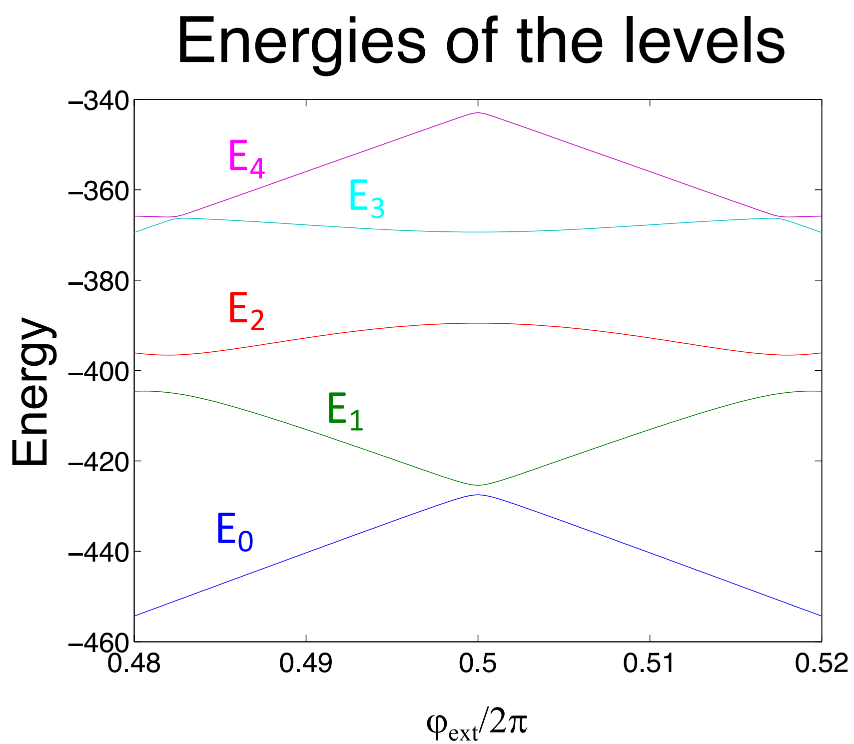


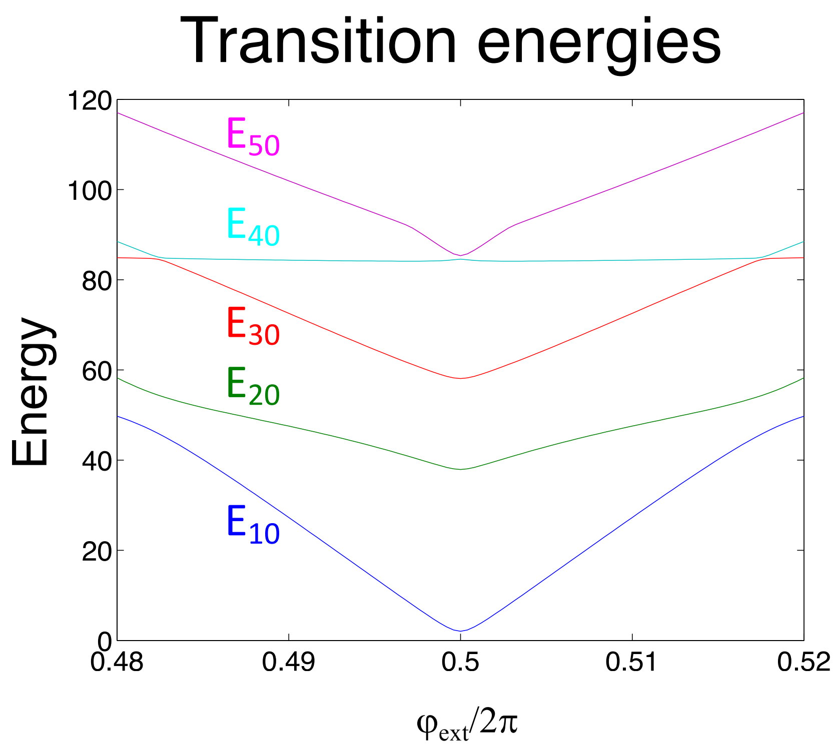
$$\boxed{H = \frac{(2e)^2}{2}\vec{n}C^{-1}\vec{n} + E_J\left[2+\alpha-\cos\varphi_{01}-\cos\varphi_{20}-\alpha\cos\left(\varphi_{ext}-\varphi_{01}-\varphi_{20}\right)\right]}$$

$$U(n_1,n_2) = \frac{(2e)^2}{2}\begin{pmatrix} n_1 & n_2 \end{pmatrix} C^{-1}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

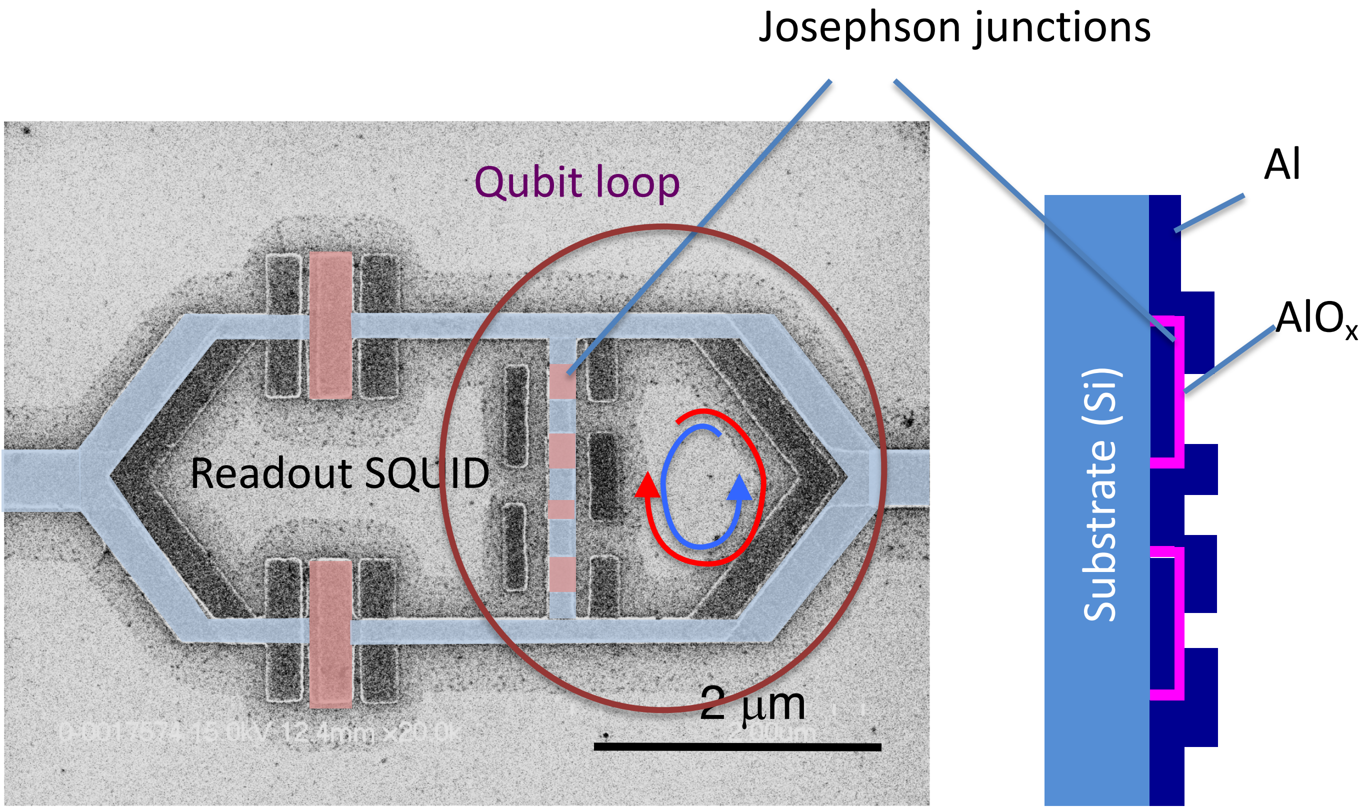
$$H = \begin{pmatrix}
\ddots & & & & & \\
& U(-1,0) & -\frac{\alpha E_J}{2}e^{i\varphi_{ext}} & -\frac{E_J}{2} & 0 & 0 \\
& -\frac{\alpha E_J}{2}e^{-i\varphi_{ext}} & U(0,-1) & -\frac{E_J}{2} & 0 & 0 \\
& -\frac{E_J}{2} & -\frac{E_J}{2} & U(0,0) & -\frac{E_J}{2} & -\frac{E_J}{2} \\
& 0 & 0 & -\frac{E_J}{2} & U(0,1) & -\frac{\alpha E_J}{2}e^{i\varphi_{ext}} \\
& 0 & 0 & -\frac{E_J}{2} & -\frac{\alpha E_J}{2}e^{-i\varphi_{ext}} & U(1,0) \\
& & & & & \ddots
\end{pmatrix}$$

Energy bands in the flux quantum system





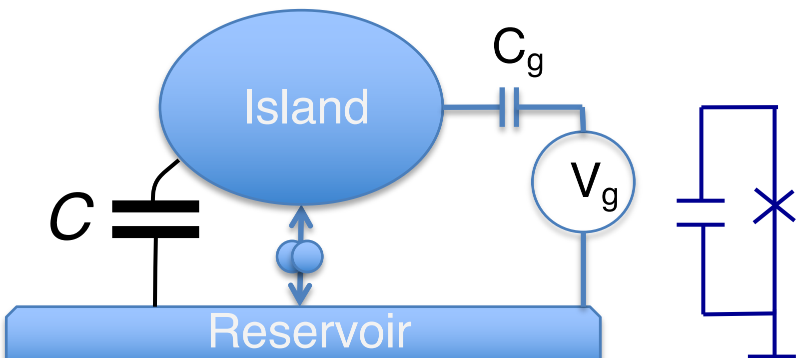
The four-junction flux qubit



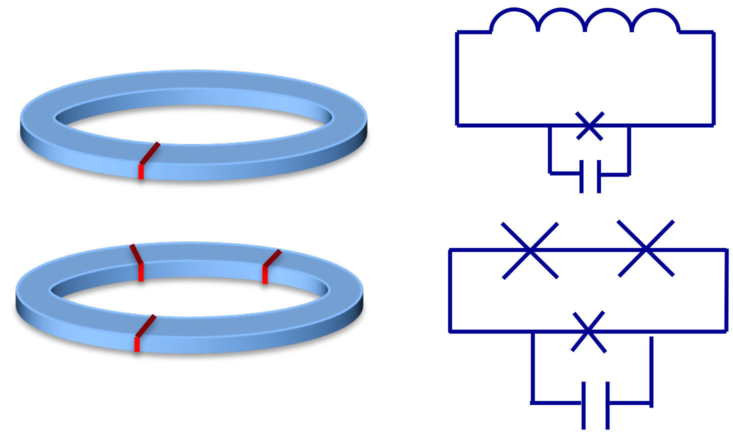
The charge vs flux qubits

$$H \approx -\frac{\varepsilon}{2}\sigma_z - \frac{\Delta}{2}\sigma_x$$

The charge qubit



The flux qubit



Any type of superconducting quantum systems utilize charge and flux quantization

Charge states (0, 2e): $|0\rangle$, $|1\rangle$

Cooper pair tunneling in/out

Charging energy: E_C

Tunneling energy: E_J

Controlled by V

$\varepsilon = -2E_C \delta N$

Flux states (0, 2π): $|0\rangle$, $|1\rangle$

Flux tunneling tunneling in/out

Magnetic energy: E_L

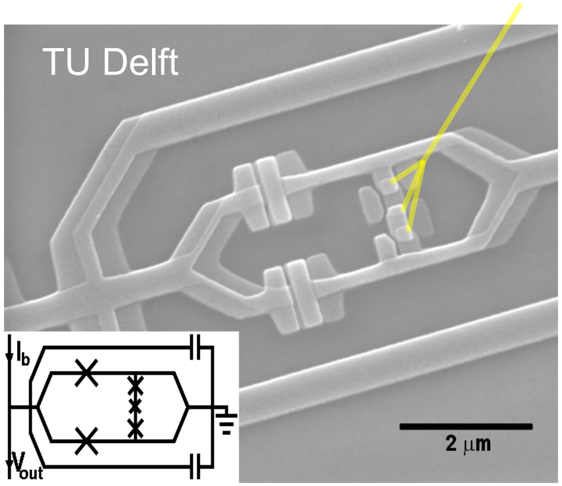
Tunneling energy: Δ

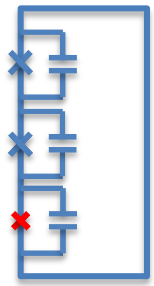
Controlled by B

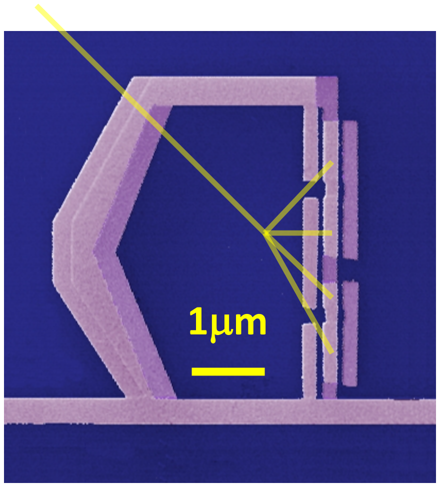
$\varepsilon = -2E_L \delta\varphi$

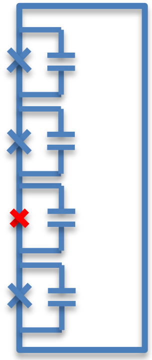
Josephson junction flux qubit

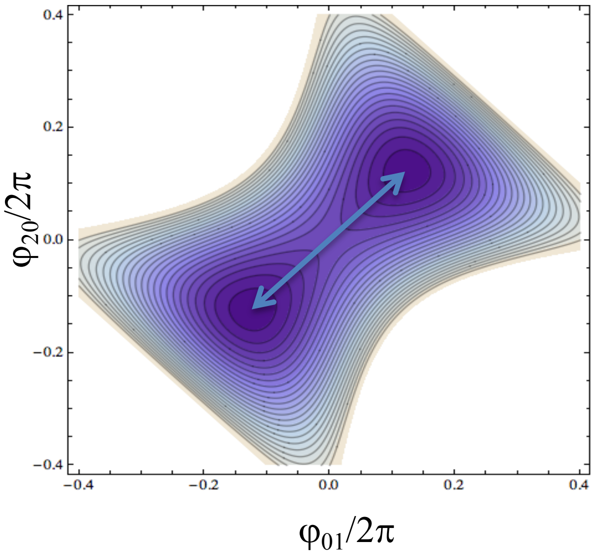
Josephson junctions

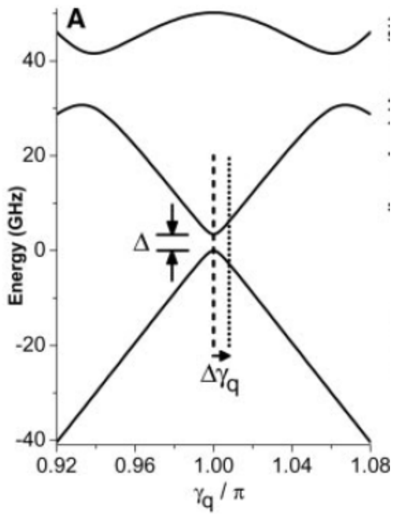








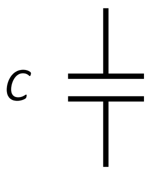




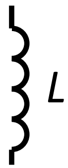
Quantum mechanics of electrical circuits.
Capacitance, inductance, resonators

Electric circuit

Capacitor



Inductor



Voltage, current: V, I

Charge, magnetic flux: $Q = CV \qquad \Phi = LI$

$$I = \dot{Q} \qquad V = \dot{\Phi}$$

Potential energy

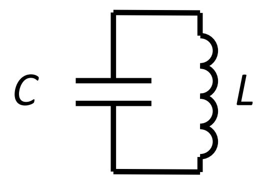
$$U = \frac{CV^2}{2}$$

$$U = \frac{Q^2}{2C}$$

Kinetic energy

$$T = \frac{LI^2}{2}$$

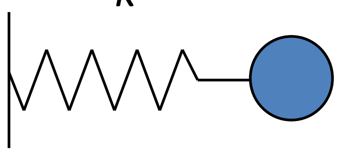
$$T = \frac{\Phi^2}{2L}$$



Total energy of the system

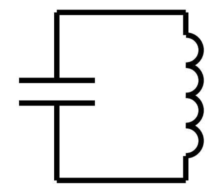
$$\boxed{E = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}} \qquad \boxed{E = \frac{CV^2}{2} + \frac{LI^2}{2}}$$

Classical harmonic oscillators



$$E = \frac{kx^2}{2} + \frac{p^2}{2m}$$

$$E = \frac{kx^2}{2} + \frac{m\dot{x}^2}{2}$$



$$E = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

$$E = \frac{Q^2}{2C} + \frac{L\dot{Q}^2}{2}$$

Coordinate: x

Momentum: $p = m\dot{x}$

Q

$p' = L\dot{Q} \rightarrow \Phi$

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad E = \frac{m}{2}\left(\omega_0^2 x^2 + \dot{x}^2\right)$$

$$E = \frac{L}{2}\left(\omega_0^2 \Phi^2 + \dot{\Phi}^2\right) \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

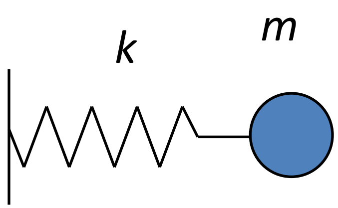
$$x = Ae^{i\omega t} + Be^{-i\omega t}$$

$$Q = Ae^{i\omega t} + Be^{-i\omega t}$$

The equations are transformed from one to another with the substitutions

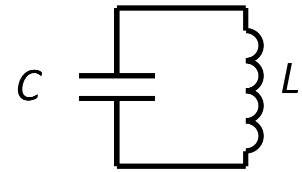
$$x \rightarrow Q \qquad p \rightarrow \Phi \qquad k \rightarrow \frac{1}{C} \qquad m \rightarrow L$$

Classical harmonic oscillators (alternative approach)



$$E = \frac{kx^2}{2} + \frac{p^2}{2m}$$

$$E = \frac{kx^2}{2} + \frac{m\dot{x}^2}{2}$$



$$E = \frac{\Phi^2}{2L} + \frac{Q^2}{2C}$$

$$E = \frac{\Phi^2}{2L} + \frac{C\dot{\Phi}^2}{2}$$

Coordinate: x $\qquad\Phi$

Momentum: $p = m\dot{x}$ $\qquad p' = C\dot{\Phi} \rightarrow Q$

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad E = \frac{m}{2}\left(\omega_0^2 x^2 + \dot{x}^2\right)$$

$$E = \frac{C}{2}\left(\omega_0^2 Q^2 + \dot{Q}^2\right) \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$x = Ae^{i\omega t} + Be^{-i\omega t} \qquad Q = Ae^{i\omega t} + Be^{-i\omega t}$$

The equations are identical with the following substitutions

$$\boxed{x \rightarrow \Phi \qquad p \rightarrow Q \qquad k \rightarrow \frac{1}{L} \qquad m \rightarrow C}$$

Quantum mechanics of harmonic oscillators

$$H = \frac{\hbar\omega_0}{2}\left(\frac{\hat{x}^2}{x_0^2} + \frac{\hat{p}^2}{p_0^2}\right) \qquad H = \frac{\hbar\omega_0}{2}\left(\frac{\hat{Q}^2}{Q_0^2} + \frac{\hat{\Phi}^2}{\phi_0^2}\right)$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega_0}} \qquad p_0 = \frac{\hbar}{x_0} \qquad Q_0 = \sqrt{\frac{\hbar}{L\omega_0}} = \sqrt{\hbar C\omega_0} \qquad \phi_0 = \frac{\hbar}{Q_0} = \sqrt{\hbar L\omega_0}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Because $CV = Q$ $\quad LI = \Phi$ we can rewrite the Hamiltonian of LC-resonator in the form

$$H = \frac{\hbar\omega_0}{2}\left(\frac{\hat{V}^2}{V_0^2} + \frac{\hat{I}^2}{I_0^2}\right) \qquad V_0 = \frac{Q_0}{C} = \sqrt{\frac{\hbar C\omega_0}{C^2}} = \boxed{\sqrt{\frac{\hbar\omega_0}{C}}} \qquad I_0 = \frac{\phi_0}{L} = \sqrt{\frac{\hbar L\omega_0}{L^2}} = \boxed{\sqrt{\frac{\hbar\omega_0}{L}}}$$

Hint to remember: $\frac{CV_0^2}{2} = \frac{LI_0^2}{2} = \frac{\hbar\omega_0}{2}$

Position operator: $\hat{x}$

Momentum operator: $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

Commutation relations: $\left[\hat{p}, \hat{x}\right] = -i\hbar$

Voltage operator: $\hat{V} = \hat{Q}/C$

Current operator: $\hat{I} = \frac{\hat{\Phi}}{L} = -i\frac{\hbar}{L}\frac{\partial}{\partial Q} = -i\frac{\hbar}{LC}\frac{\partial}{\partial V}$

$$\left[L\hat{I}, C\hat{V}\right] = -i\hbar$$

$$y = \frac{x}{x_0} \qquad y = \frac{Q}{Q_0} \text{ or } y = \frac{V}{V_0}$$

$$\boxed{\hat{H} = \frac{\hbar\omega_0}{2}\left(y^2 - \frac{\partial^2}{\partial y^2}\right)}$$

Quantum mechanics of harmonic oscillators

$$H = \frac{\hbar\omega_0}{2}\left(\frac{\hat{x}^2}{x_0^2} + \frac{\hat{p}^2}{p_0^2}\right)$$

$$H = \frac{\hbar\omega_0}{2}\left(\frac{\hat{V}^2}{V_0^2} + \frac{\hat{I}^2}{I_0^2}\right)$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega_0}} \qquad p_0 = \frac{\hbar}{x_0}$$

$$V_0 = \sqrt{\frac{\hbar\omega_0}{C}} \qquad I_0 = \sqrt{\frac{\hbar\omega_0}{L}}$$

Position operator: $\hat{x}$

Momentum operator: $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

Commutation relations: $\left[\hat{p}, \hat{x}\right] = -i\hbar$

Voltage operator: $\hat{V}$

Current operator: $\hat{I} = -i\frac{\hbar}{LC}\frac{\partial}{\partial V}$

$\left[L\hat{I}, C\hat{V}\right] = -i\hbar$

$$y = \frac{x}{x_0} \qquad y = \frac{V}{V_0} \qquad \hat{H} = \frac{\hbar\omega_0}{2}\left(y^2 - \frac{\partial^2}{\partial y^2}\right)$$

Quantum mechanics of harmonic oscillators

$$H = \frac{\hbar\omega_0}{2}\left(\frac{\hat{x}^2}{x_0^2} + \frac{\hat{p}^2}{p_0^2}\right)$$

$$H = \frac{\hbar\omega_0}{2}\left(\frac{\hat{Q}^2}{Q_0^2} + \frac{\hat{\Phi}^2}{\phi_0^2}\right) \quad \text{or} \quad H = \frac{\hbar\omega_0}{2}\left(\frac{\hat{V}^2}{V_0^2} + \frac{\hat{I}^2}{I_0^2}\right)$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega_0}} \qquad p_0 = \frac{\hbar}{x_0}$$

$$Q_0 = \sqrt{\hbar\omega_0 C} \qquad \phi_0 = \sqrt{\hbar\omega_0 L} \qquad V_0 = \sqrt{\frac{\hbar\omega_0}{C}} \qquad I_0 = \sqrt{\frac{\hbar\omega_0}{L}}$$

Normalized position operator: x/x_0

Momentum operator: $\hat{p}/p_0 = -i\dfrac{\partial}{\partial(x/x_0)}$

Commutation relations: $\left[\dfrac{\hat{x}}{x_0}, \dfrac{\hat{p}}{p_0}\right] = i$

Charge (voltage) operator: $Q/Q_0 \quad V/V_0$

Flux (current) operator: $\hat{\Phi}/\phi_0 = -i\dfrac{\partial}{\partial(Q/Q_0)}$

$$\left[\frac{\hat{Q}}{Q_0}, \frac{\hat{\Phi}}{\phi_0}\right] = \left[\frac{\hat{V}}{V_0}, \frac{\hat{I}}{I_0}\right] = i \qquad \hat{I}/I_0 = -i\frac{\partial}{\partial(V/V_0)}$$

$$y = \frac{x}{x_0} \qquad y = \frac{Q}{Q_0} \qquad y = \frac{V}{V_0}$$

General form of the Hamiltonian

$$\hat{H} = \frac{\hbar\omega_0}{2}\left(y^2 - \frac{\partial^2}{\partial y^2}\right)$$

Number operator

$$H = \frac{\hbar\omega_0}{2}\left(y^2 - \frac{\partial^2}{\partial y^2}\right)$$

$$H\psi_n(y) = E_n\psi_n(y)$$

Hermit polynomials

$$\psi_n(y) = \frac{1}{\pi^{\frac{1}{4}}\sqrt{2^n n!}} e^{-\frac{y^2}{2}} H_n(y)$$

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n}\left(e^{-y^2}\right)$$

$$H_0(y) = 1$$

$$H_1(y) = 2y$$

$$H_2(y) = 4y^2 - 2$$

$$H_3(y) = 8y^3 - 12y$$

$$E_n = \hbar\omega_0\left(n + \frac{1}{2}\right) \qquad H = \frac{\hbar\omega_0}{2}\left(y^2 - \frac{\partial^2}{\partial y^2}\right)$$

Number operator

$$\hat{N} = \frac{1}{2}\left(y^2 - \frac{\partial^2}{\partial y^2} - 1\right) \qquad \hat{N}\psi_n = \frac{1}{2}\left(y^2 - \frac{\partial^2}{\partial y^2} - 1\right)\psi_n = n\psi_n$$

$$\hat{N}|n\rangle = n \qquad H = \hbar\omega_0\left(\hat{N} + \frac{1}{2}\right) \qquad \hat{N} = n|n\rangle\langle n|$$

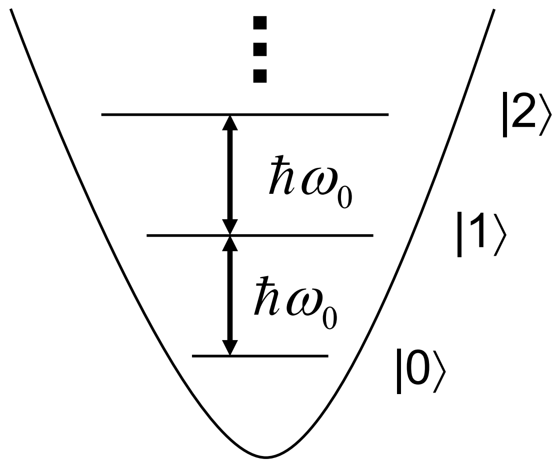
Number operator and Fock states

Harmonic oscillator Hamiltonian

$$\hat{H} = \hbar\omega_0\left(\hat{N} + \frac{1}{2}\right)$$

$$\hat{H}\Psi_n = E_n\Psi_n \qquad E_n = \hbar\omega_0\left(n + \frac{1}{2}\right)$$

Photons



Photon number operator

$$\hat{N} = \sum_{n=0}^{\infty} n|n\rangle\langle n| = \begin{pmatrix} 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 2 & 0 & \\ 0 & 0 & 0 & 3 & \\ & & & & \ddots \end{pmatrix}$$

Fock states are photon number states

$$\Psi_n = |n\rangle$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix} \quad \cdots$$

a and $a^{\dagger}$ operators

$$H = \frac{\hbar\omega_0}{2}\left(y^2 - \frac{\partial^2}{\partial y^2}\right) \qquad [\xi,\eta] = \frac{1}{\hbar}[x,\hat{p}] = i \qquad \xi = y \qquad \eta = -i\frac{\partial}{\partial \xi}$$

$$a = \frac{1}{\sqrt{2}}\left(y + \frac{\partial}{\partial y}\right) \qquad a^{\dagger} = \frac{1}{\sqrt{2}}\left(y - \frac{\partial}{\partial y}\right)$$

$y = \frac{Q}{Q_0} = \frac{V}{V_0}$ or $y = \frac{\Phi}{\Phi_0} = \frac{I}{I_0}$

$$a = \frac{1}{\sqrt{2}}(\xi + i\eta) \qquad a^{\dagger} = \frac{1}{\sqrt{2}}(\xi - i\eta)$$

$$aa^{\dagger} = \frac{1}{2}\left(\xi^2 + \eta^2 \overbrace{-i[\xi,\eta]}^{+1}\right)$$

$$a^{\dagger}a = \frac{1}{2}\left(\xi^2 + \eta^2 \underbrace{+i[\xi,\eta]}_{-1}\right)$$

$$H = \frac{\hbar\omega_0}{2}\left(a^{\dagger}a + aa^{\dagger}\right) \qquad \left[a,a^{\dagger}\right] = 1 \qquad \hat{H} = \hbar\omega_0\left(a^{\dagger}a + \frac{1}{2}\right)$$

The solution of the equation $\hat{H}\psi_n = E_n\psi_n$ is $E_n = \hbar\omega_0\left(n + \frac{1}{2}\right)$

$$\hat{N} = \frac{1}{2}\left(y^2 - \frac{\partial^2}{\partial y^2} - 1\right) \qquad \hat{H} = \hbar\omega_0\left(\hat{N} + \frac{1}{2}\right) \qquad \hat{N} = a^{\dagger}a$$

Derivation of explicit forms of operators a and $a^\dagger$

$$\hat{N}a|n\rangle = a^\dagger a a|n\rangle = \left(aa^\dagger - 1\right)a|n\rangle = \left(aa^\dagger a - a\right)|n\rangle = a\left(a^\dagger a - 1\right)|n\rangle = a\left(\hat{N} - 1\right)|n\rangle = a\left(n-1\right)|n\rangle$$

Therefore $\quad a = c_n|n-1\rangle\langle n|$

$$\hat{N}a^\dagger|n\rangle = a^\dagger a a^\dagger|n\rangle = a^\dagger\left(a^\dagger a + 1\right)|n\rangle = a^\dagger\left(n+1\right)|n\rangle = a\left(a^\dagger a - 1\right)|n\rangle = \left(n+1\right)a^\dagger|n\rangle$$

Therefore $\quad a^\dagger = d_n|n+1\rangle\langle n|$

We verify:

$$\hat{N}a|n\rangle = c_n\left(n-1\right)|n-1\rangle\langle n-1||n-1\rangle\langle n| = c_n\left(n-1\right)|n-1\rangle\langle n| = \left(n-1\right)a|n\rangle$$

$$\hat{N}a^\dagger|n\rangle = d_n\left(n+1\right)|n+1\rangle\langle n+1||n+1\rangle\langle n| = d_n\left(n+1\right)|n+1\rangle\langle n| = \left(n+1\right)a^\dagger|n\rangle$$

From $a^\dagger a = \hat{N}$ we find

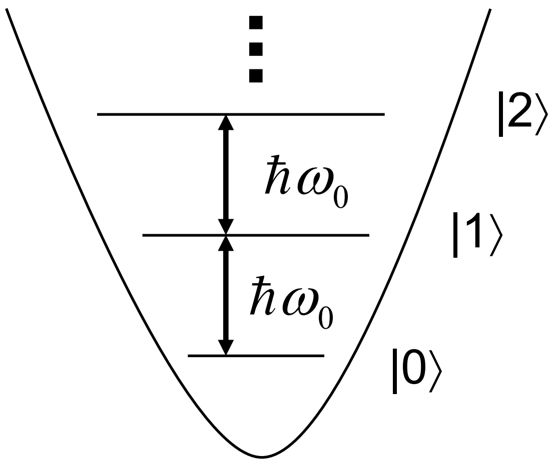
$$a^\dagger a|n\rangle = d_{n-1}|n\rangle\langle n-1|c_n|n-1\rangle\langle n||n\rangle = d_{n-1}c_n|n\rangle = n|n\rangle$$ therefore $d_{n-1}c_n = n$

From $a^\dagger = \left(a\right)^* = c_n|n\rangle\langle n-1| \equiv d_{n-1}|n\rangle\langle n-1|$ we find $d_{n-1} = c_n = \sqrt{n}$

Finally $\quad \boxed{a = \sqrt{n}|n-1\rangle\langle n| \qquad a^\dagger = \sqrt{n+1}|n+1\rangle\langle n|}$

Explicit form of a-operators

Harmonic oscillator



$\hat{a}$ - creation operator

$\hat{a}^{+}$ - annihilation operator

$$a^{+} = \sum_{n=0}^{\infty} \sqrt{n+1}\,|n+1\rangle\langle n| \qquad a = \sum_{n=0}^{\infty} \sqrt{n}\,|n-1\rangle\langle n|$$

$$a^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 & \\ 0 & 0 & \sqrt{2} & 0 & \\ 0 & 0 & 0 & \sqrt{3} & \\ 0 & 0 & 0 & 0 & \\ & & & & \ddots \end{pmatrix} \qquad a^{+} = \begin{pmatrix} 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & \\ 0 & \sqrt{2} & 0 & 0 & \\ 0 & 0 & \sqrt{3} & 0 & \\ & & & & \ddots \end{pmatrix}$$

$$H = \hbar\omega_0\left(\sum_{n=0}^{\infty} n|n\rangle\langle n| + \frac{1}{2}\right) = \hbar\omega_0\left(\hat{N} + \frac{1}{2}\right)$$

Photon number operator:

$$\hat{N} = a^{+}a = \sum_{n=0}^{\infty} n|n\rangle\langle n|$$

Quantized fields in LC-resonator

$$H = \frac{C\hat{V}^2}{2} + \frac{L\hat{I}^2}{2} \qquad H = \frac{\hbar\omega_0}{2}\left(\frac{\hat{V}^2}{V_0^2} + \frac{\hat{I}^2}{I_0^2}\right) \qquad V_0 = \sqrt{\frac{\hbar\omega_0}{C}} \qquad I_0 = \sqrt{\frac{\hbar\omega_0}{L}}$$

Voltage operator: $\hat{V}$

Current operator: $\hat{I} = -i\frac{\hbar}{LC}\frac{\partial}{\partial V}$

$$a = \frac{1}{\sqrt{2}}\left(\frac{\hat{V}}{V_0} + i\frac{\hat{I}}{I_0}\right) \qquad a^+ = \frac{1}{\sqrt{2}}\left(\frac{\hat{V}}{V_0} - i\frac{\hat{I}}{I_0}\right)$$

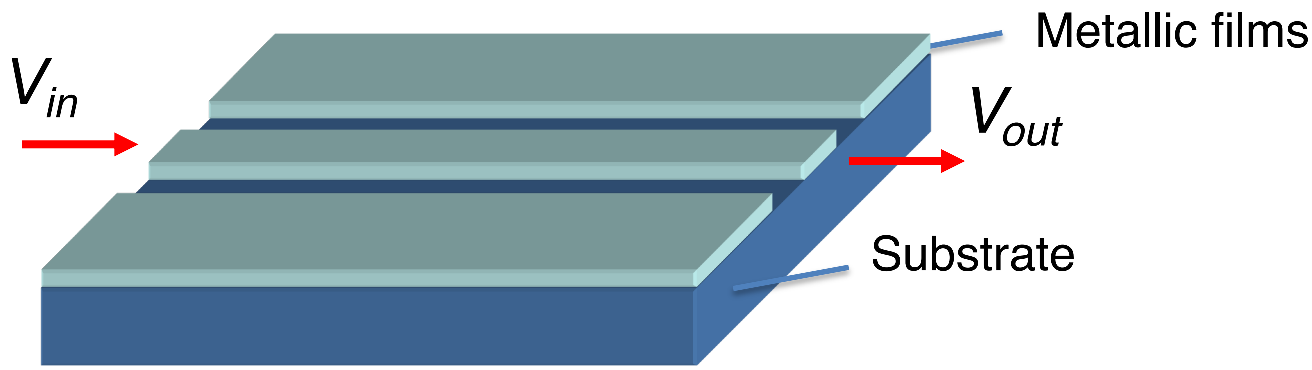
$$\boxed{\hat{V} = \sqrt{\frac{\hbar\omega_0}{2C}}\left(a + a^+\right)} \qquad \boxed{\hat{I} = i\sqrt{\frac{\hbar\omega_0}{2L}}\left(a^\dagger - a\right)}$$

$$\boxed{H = \hbar\omega_0\left(a^+a + \frac{1}{2}\right)}$$

$$\boxed{\hat{N} = a^+a \qquad a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle}$$

Coplanar waveguide transmission lines

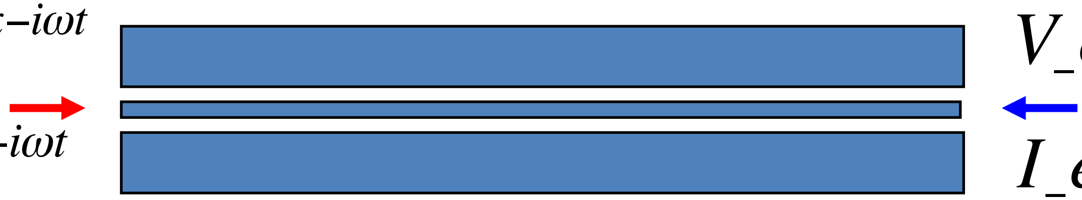
Coplanar transmission line



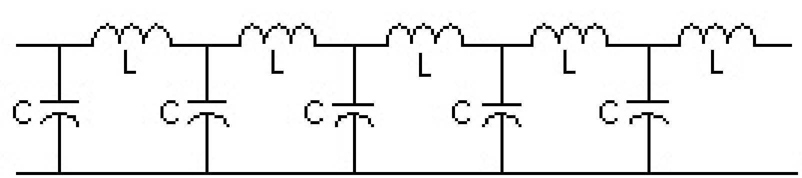
Right propagating wave

Left propagating wave (e.g. reelected wave)

$$I_R = \frac{V_R}{Z} \qquad V_+ e^{ikx-i\omega t} \qquad I_+ e^{ikx-i\omega t}$$



$$V_- e^{-ikx-i\omega t} \qquad I_- e^{-ikx-i\omega t} \qquad I_L = -\frac{V_L}{Z}$$



l – inductance per length

c – capacitance per length

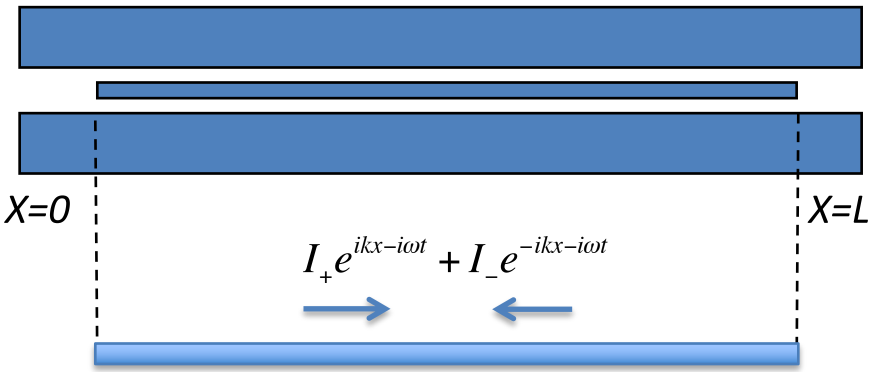
Telegrapher's equations:

$$\frac{\partial V}{\partial x} = -l \frac{\partial I}{\partial t} \qquad ikV_0 e^{ikx-i\omega t} = li\omega I_0 e^{ikx-i\omega t} \qquad Z = \frac{V_0}{I_0} = \frac{\omega l}{k}$$

$$\frac{\partial I}{\partial x} = -c \frac{\partial V}{\partial t} \qquad ikI_0 e^{ikx-i\omega t} = \omega i c V_0 e^{ikx-i\omega t} \qquad Z = \frac{V_0}{I_0} = \frac{k}{\omega c}$$

$$Z = \sqrt{\frac{l}{c}}$$

Quantum mechanics of coplanar resonators



$$I_+ e^{ikx-i\omega t} + I_- e^{-ikx-i\omega t}$$

$$x = 0: \quad I_+ + I_- = 0 \qquad\qquad x = L: \quad I_+ e^{ikL} + I_- e^{-ikL} = 0$$

Boundary conditions:
No current at the ends

$$I_- = -I_+ \qquad\qquad I_- = -I_+ e^{i2kL}$$

$$e^{i2kL} = 1 \Rightarrow kL = \pi n \Rightarrow k = \frac{\pi n}{L}$$

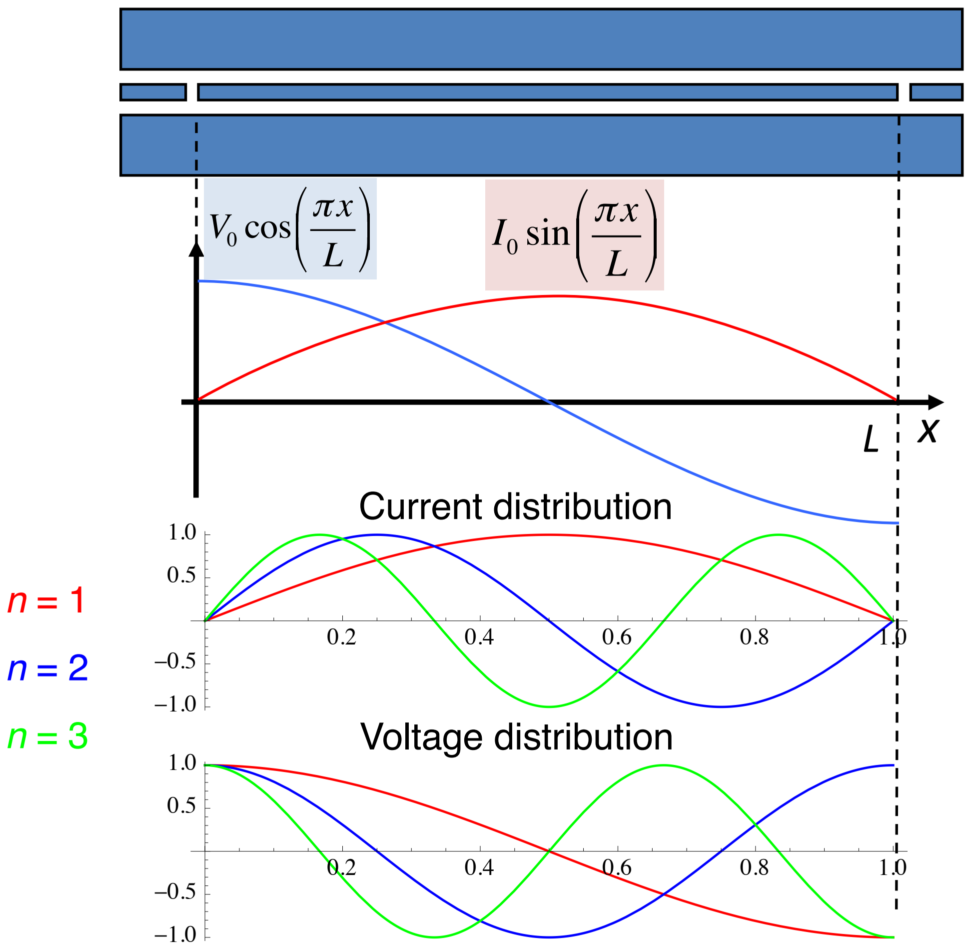
$$I_+ e^{i\frac{\pi n}{L}x - i\omega t} - I_+ e^{-i\frac{\pi n}{L}x - i\omega t} = \boxed{I_0 \sin\left(\frac{\pi n}{L}x\right) e^{-i\omega t}} \qquad n = 1,2,3\ldots$$

$$\frac{\partial I}{\partial x} = -c\frac{\partial V}{\partial t} \Rightarrow \frac{\partial I}{\partial x} = i\omega c V \Rightarrow \boxed{V = V_0 \cos\left(\frac{\pi n}{L}x\right) e^{-i\omega t - i\frac{\pi}{2}}}$$

$$V_0 = Z I_0$$

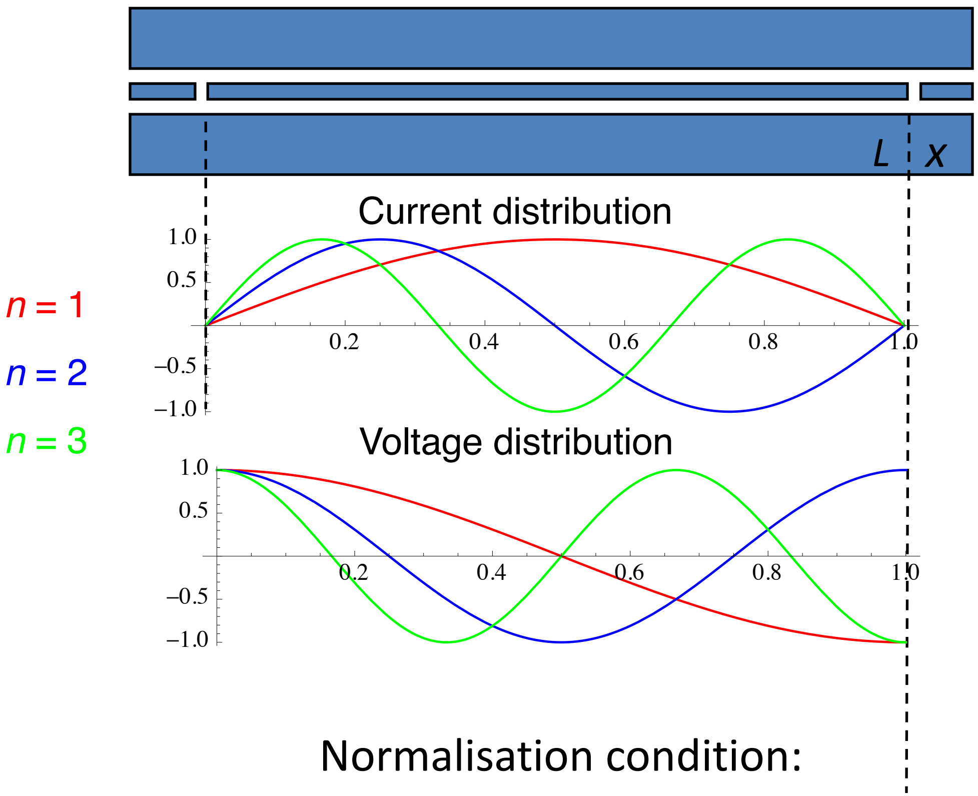
Field distribution

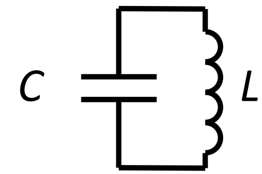
$$I_0 \sin\left(\frac{\pi n}{L} x\right) \qquad V = V_0 \cos\left(\frac{\pi n}{L} x\right)$$



Field quantisation in coplanar resonators

$$I_0 \sin\left(\frac{\pi n}{L}x\right) \qquad V = V_0 \cos\left(\frac{\pi n}{L}x\right)$$





$$\hat{V} = \sqrt{\frac{\hbar\omega_0}{2C}}\left(a^\dagger + a\right)$$

$$\hat{I} = i\sqrt{\frac{\hbar\omega_0}{2L}}\left(a^+ - a\right)$$

Voltage and current operators in a coplanar resonator:

$$\boxed{\hat{V} = \sqrt{\frac{\hbar\omega_0}{C}}\left(a + a^+\right)\cos\left(\frac{\pi n}{L}x\right)}$$

$$\boxed{\hat{I} = i\sqrt{\frac{\hbar\omega_0}{C}}\left(a^\dagger - a\right)\sin\left(\frac{\pi n}{L}x\right)}$$

Normalisation condition:

$$\int_0^L \cos^2\left(\frac{\pi n x}{L}\right)dx = \frac{1}{2} \qquad \int_0^L \sin^2\left(\frac{\pi n x}{L}\right)dx = \frac{1}{2}$$

Therefore $\sqrt{2}\cos\left(\frac{\pi x}{L}\right)$ and $\sqrt{2}\sin\left(\frac{\pi x}{L}\right)$ account space distribution of voltage and current

Coplanar waveguide resonator on silicon substrate