Quantum Electronics of Nanostructures

Gleb Fedorov

Lecture 7a

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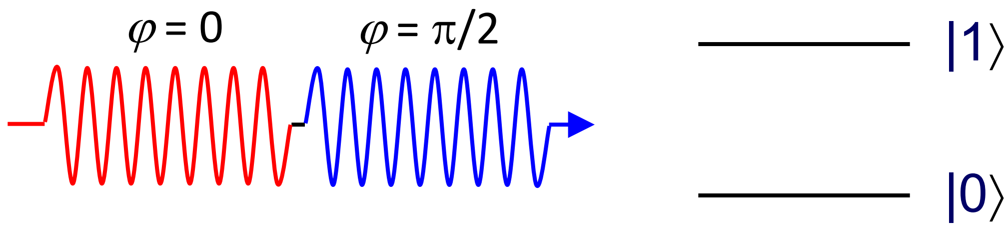
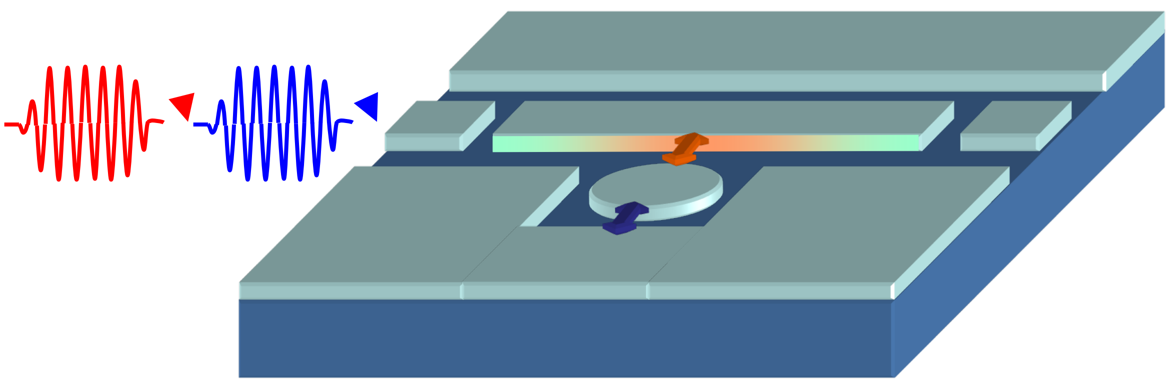
- Experimental quantum state control in two-level systems
- Quantum gates
- Dissipation and decoherence in two-level systems
- Density matrix approach
- Pure and mixed states
- Bloch sphere for mixed states and dissipative dynamics
- Relaxation and dephasing

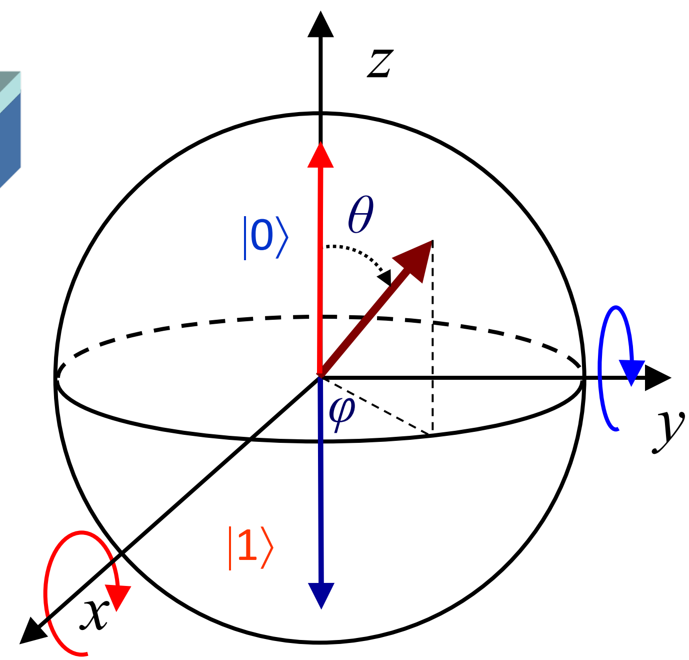
Quantum state control in two-level systems

$$\Psi = \cos\frac{\theta}{2}\left|0\right\rangle + e^{i\varphi}\sin\frac{\theta}{2}\left|1\right\rangle$$

State manipulation



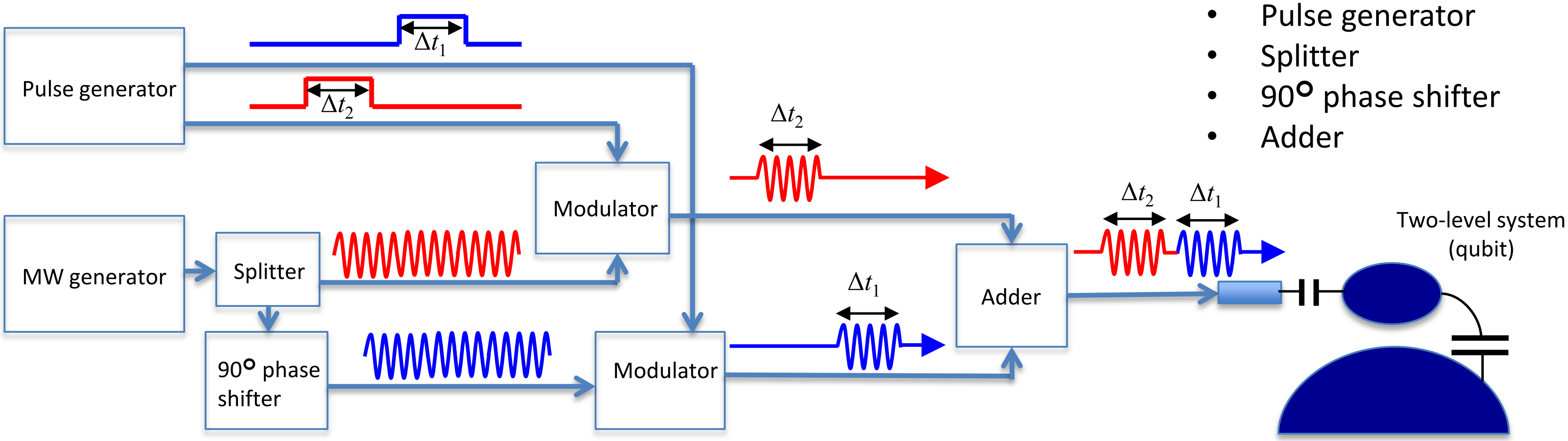




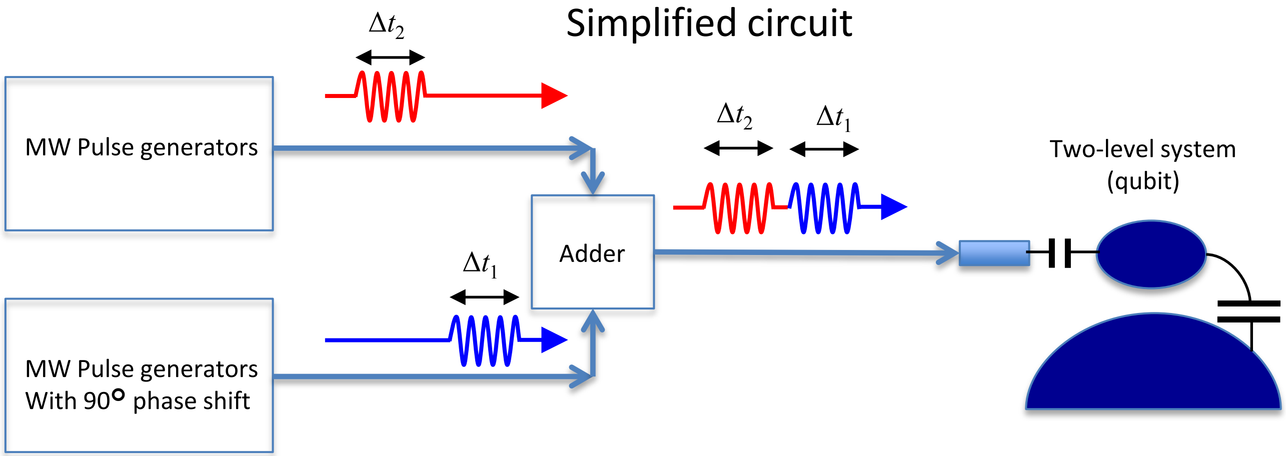
Electrical circuit to prepare arbitrary states

Required devices:

- MW generator
- Pulse generator
- Splitter
- 90° phase shifter
- Adder

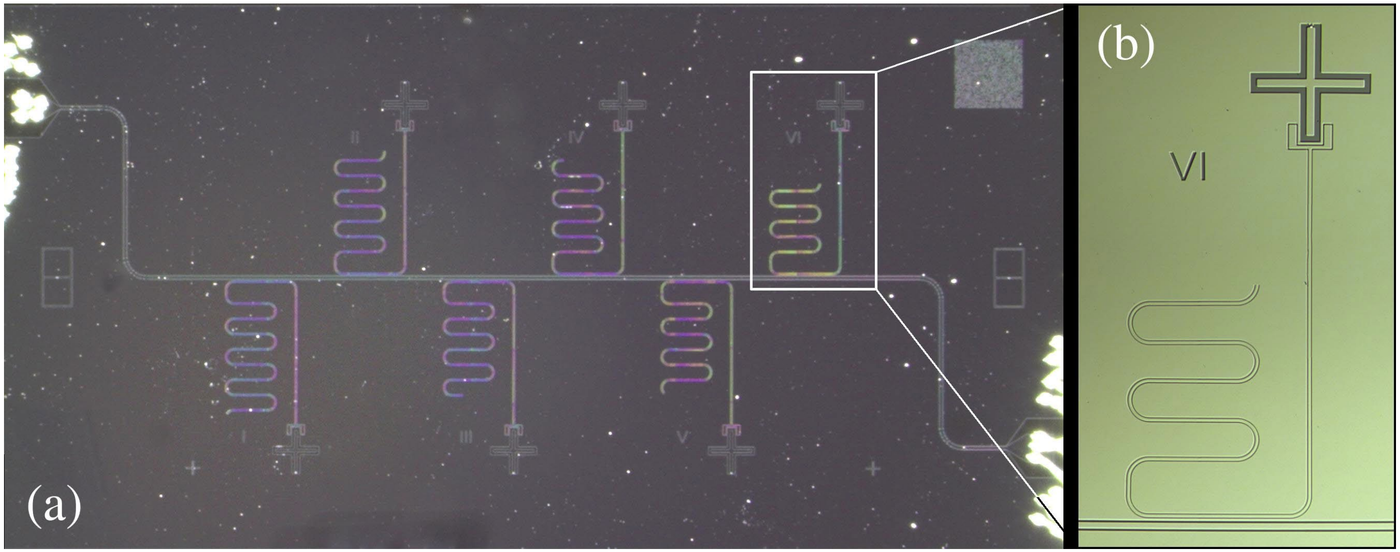


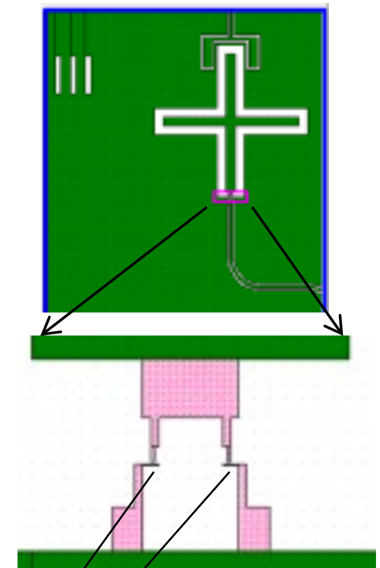
Simplified circuit



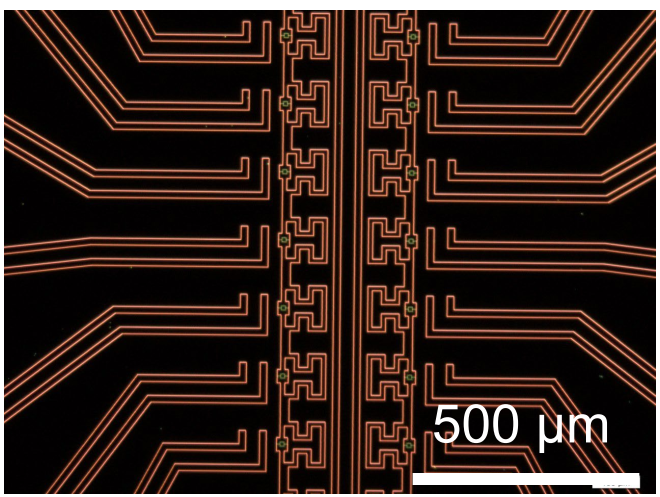
Full experimental realization will be shown later at the seminars

Long-lived qubits with low unharmonicity
Fabricated and measured at MIPT



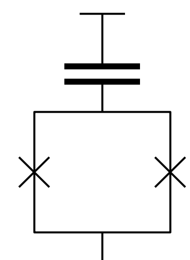


Josephson junctions

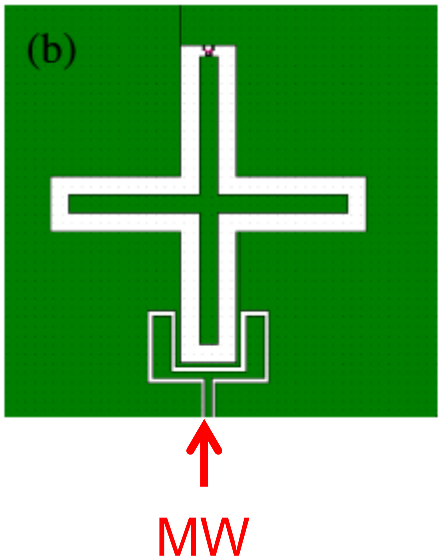


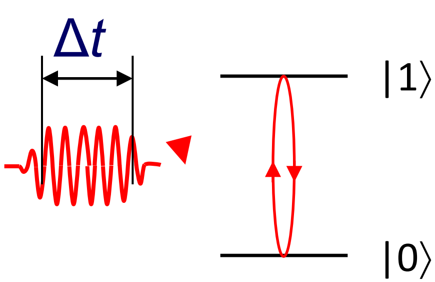
Lifetimes: ~10 μs

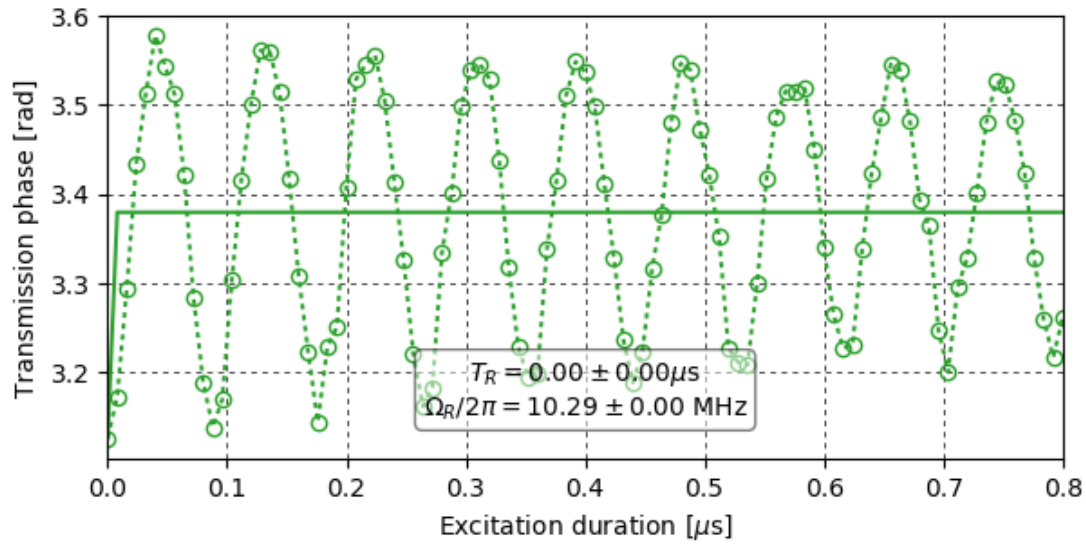
Equivalent electrical circuit

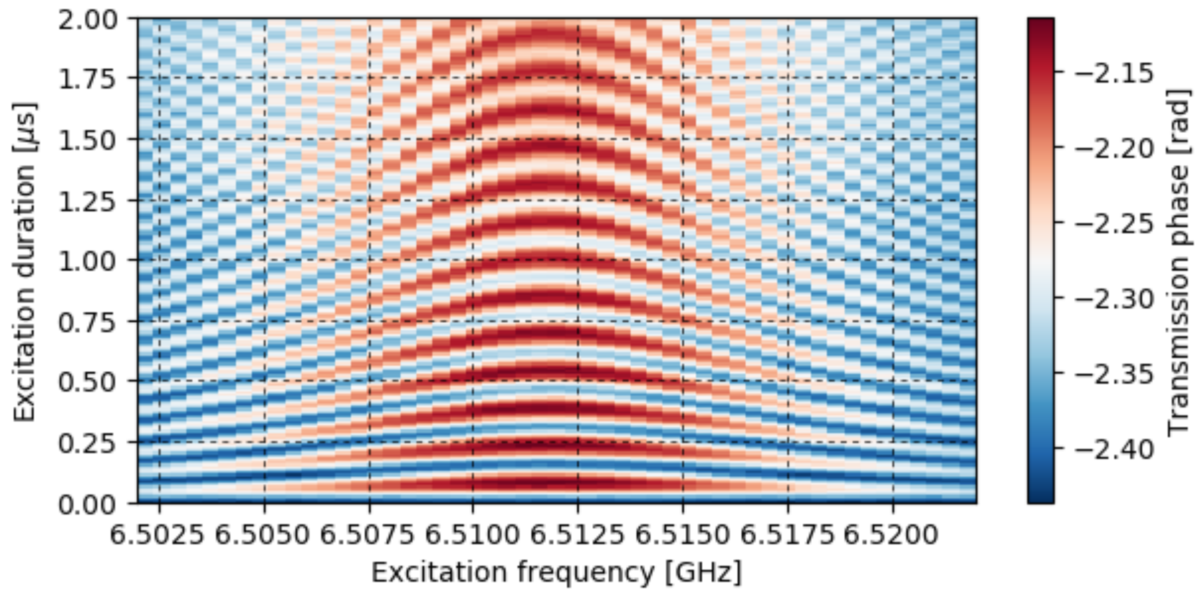


Rabi oscillations









Single-qubit operations

$$H = -\frac{\hbar\Omega}{2}\sigma_j \qquad U = \exp\left(i\frac{\Omega t}{2}\sigma_j\right) = I\cos\left(\frac{\Omega t}{2}\right) + i\sigma_j \sin\left(\frac{\Omega t}{2}\right)$$

$$\Omega t = \pi: \qquad U\left(\frac{\pi}{\Omega}\right) = i\sigma_j$$

$$t = \frac{\pi}{\Omega} \quad H = -\frac{\hbar\Omega}{2}\sigma_y \Rightarrow R = i\sigma_y$$

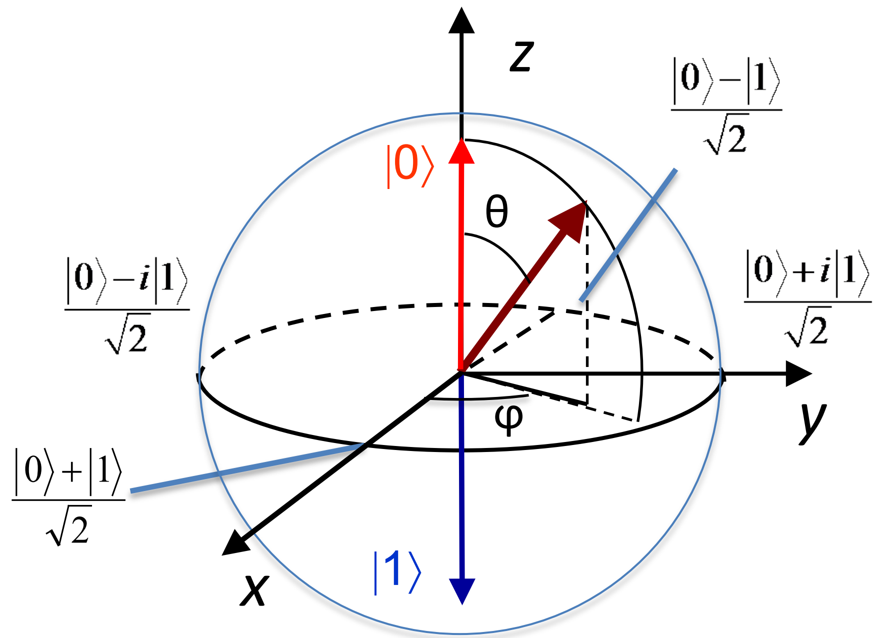
NOT (BIT FLIP) π-rotation around x-axis

$$i\sigma_y|0\rangle = -|1\rangle \quad i\sigma_y|1\rangle = |0\rangle$$

$$t = \frac{\pi}{\Omega} \quad H = -\frac{\hbar\Omega}{2}\sigma_x \Rightarrow R = i\sigma_x$$

CONJUGATED FLIP π-rotation around y-axis

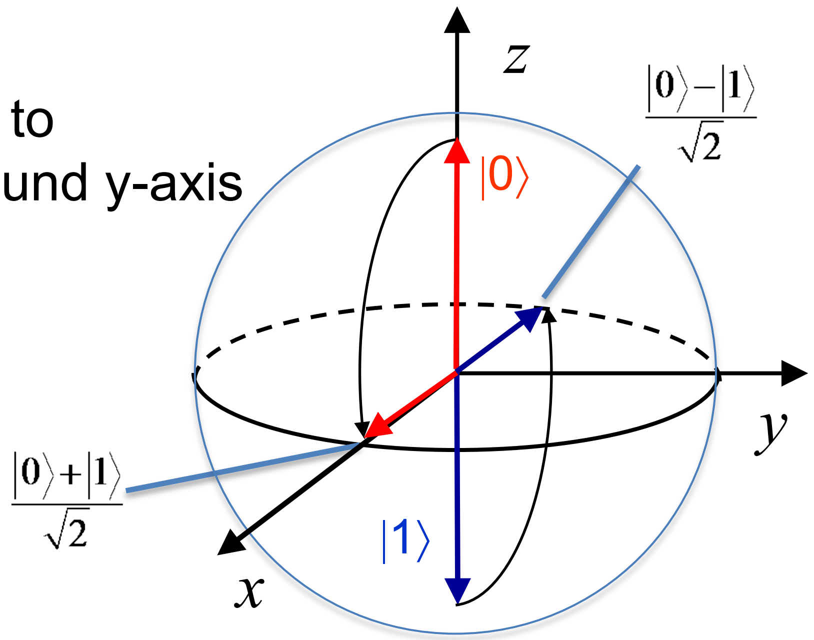
$$i\sigma_x|0\rangle = i|1\rangle \qquad i\sigma_x|1\rangle = i|0\rangle$$



Hadamard gate and superposition of quantum states

$$Hm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Hadamard transformation is equivalent to
π-rotation around z *and* π/2-rotation around y-axis



$$Hm|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$Hm|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

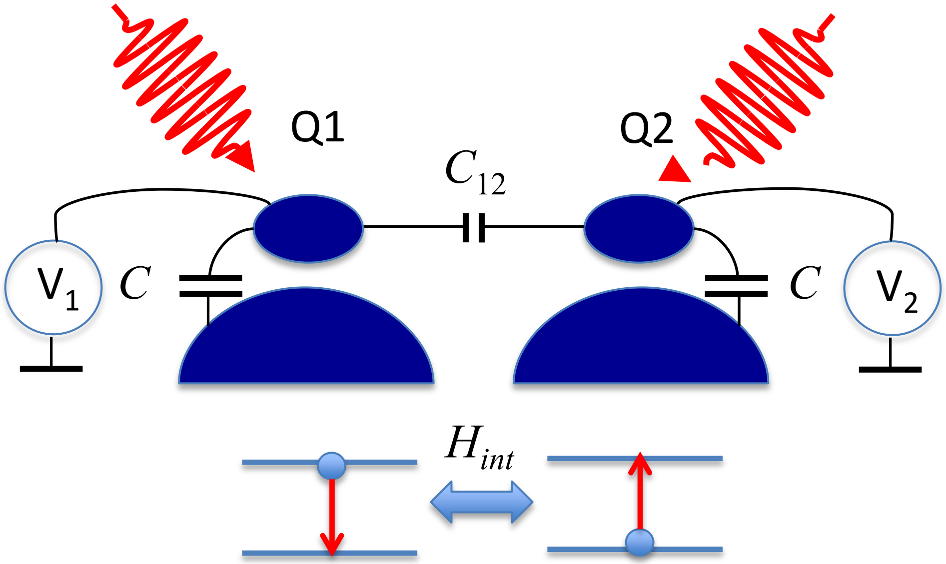
For z-eigenstates just a y-rotation:

$$H = -\frac{\hbar\Omega}{2}\sigma_y \qquad U(t) = \exp\left[i\frac{\Omega t}{2}\sigma_y\right] = \begin{pmatrix} \cos(\Omega t/2) & \sin(\Omega t/2) \\ -\sin(\Omega t/2) & \cos(\Omega t/2) \end{pmatrix}$$

$$t = \frac{\pi}{2\Omega} \Rightarrow U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow U|0\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \qquad U|1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$t = \frac{3\pi}{2\Omega} \Rightarrow U = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \Rightarrow U|0\rangle = \frac{-|0\rangle - |1\rangle}{\sqrt{2}} \equiv \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad U|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Two-qubit interaction



$$H=\begin{array}{c} \\ \left(\begin{array}{cccc} -\frac{\varepsilon_1}{2}-\frac{\varepsilon_2}{2} & -\frac{\Delta_1}{2} & -\frac{\Delta_2}{2} & 0 \\ -\frac{\Delta_1}{2} & \frac{\varepsilon_1}{2}-\frac{\varepsilon_2}{2} & E_{\text{int}} & -\frac{\Delta_2}{2} \\ -\frac{\Delta_2}{2} & E_{\text{int}} & -\frac{\varepsilon_1}{2}+\frac{\varepsilon_2}{2} & -\frac{\Delta_1}{2} \\ 0 & -\frac{\Delta_2}{2} & -\frac{\Delta_1}{2} & \frac{\varepsilon_1}{2}+\frac{\varepsilon_2}{2} \end{array}\right) \end{array} \begin{array}{c} |00\rangle \\ |10\rangle \\ |01\rangle \\ |11\rangle \end{array}$$

with columns labelled $|00\rangle \quad |10\rangle \quad |01\rangle \quad |11\rangle$

$$H_1=-\frac{\varepsilon_1}{2}\sigma_z^{(1)}-\frac{\Delta_1}{2}\sigma_x^{(1)} \qquad H_2=-\frac{\varepsilon_2}{2}\sigma_z^{(2)}-\frac{\Delta_2}{2}\sigma_x^{(2)}$$

$$\varepsilon_1=E_C n_1 \qquad \varepsilon_2=E_C n_2$$

$$\sigma_j^{(1)}=\sigma_j\times I$$

$$\sigma_j^{(2)}=I\times\sigma_j$$

$$H_{\text{int}}=E_{\text{int}}\left(\sigma^{(1)-}\sigma^{(2)+}+\sigma^{(1)+}\sigma^{(2)-}\right)$$

$$E_{\text{int}}=2E_C\frac{C_{12}}{C}$$

CNOT-gate

$$U=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Evolution of one of the qubits depends on on the state of the other

Lecture on experimental realization of two-qubit gates by will be given on the 4th of May

Quantum parallelism

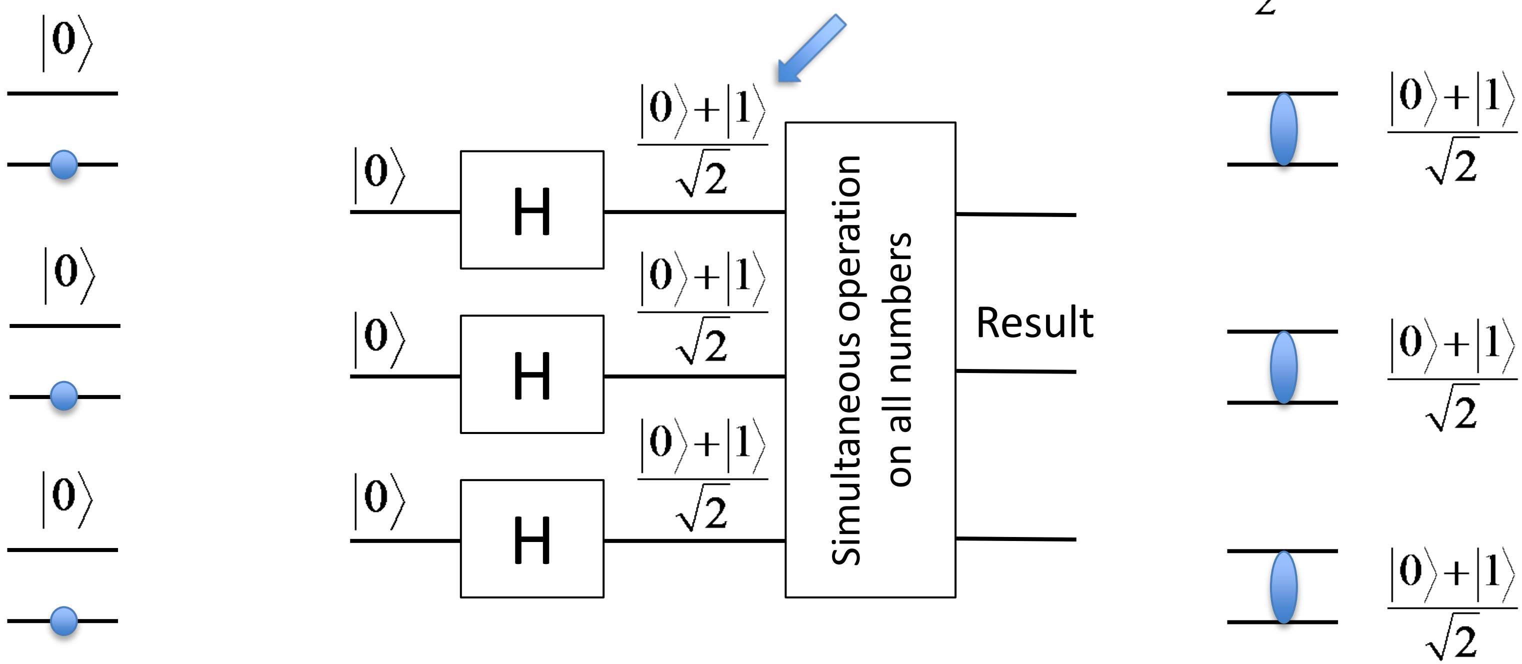
Initial state

$$|000\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle$$

State after Hadamard transformation

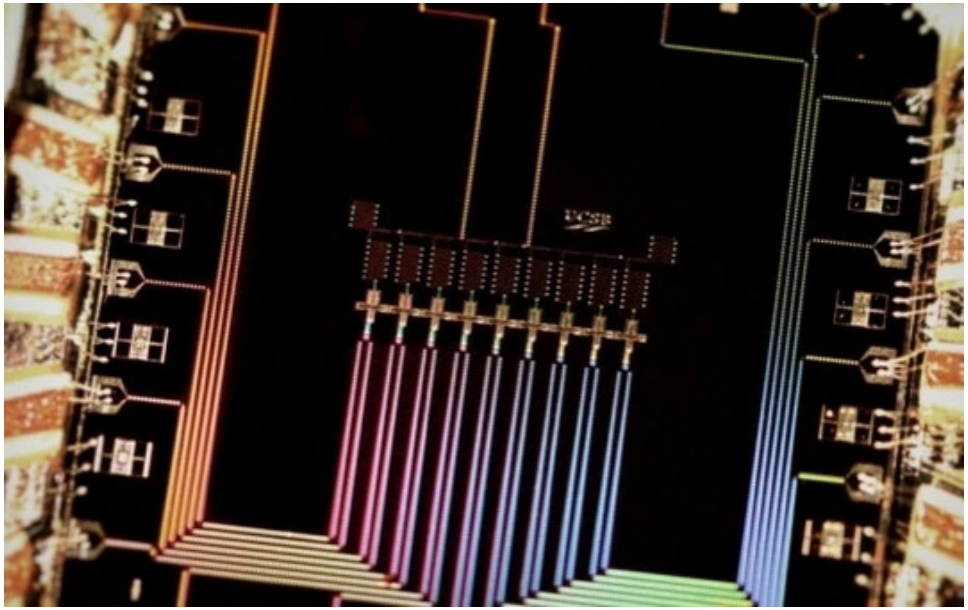
$$|000\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} =$$

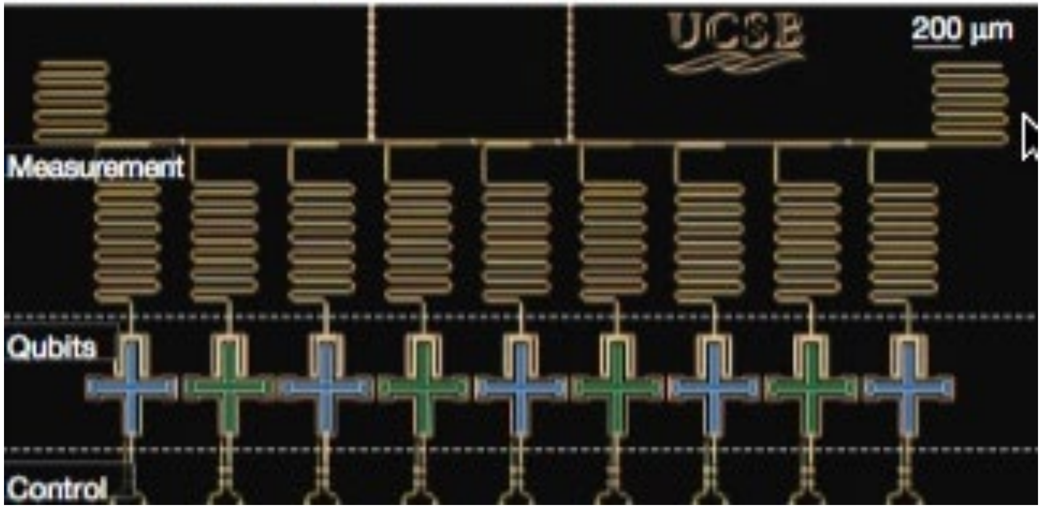
$$= \frac{|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle}{2^{3/2}}$$



The operation can be produced simultaneously on all possible combinations of numbers

Quantum processor





Dissipative dynamics

Density matrix

Density matrix: $\rho = |\Psi\rangle\langle\Psi|$

A diagonal element ρ_{kk} defines a probability of finding the system in the state $|k\rangle$

Examples:

$$\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\Psi\rangle\langle\Psi| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Psi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |\Psi\rangle\langle\Psi| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |\Psi\rangle\langle\Psi| = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\Psi = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} \qquad |\Psi\rangle\langle\Psi| = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}$$

Expectation values

$$\langle \hat{O} \rangle = \langle \Psi | \hat{O} | \Psi \rangle = \mathrm{Trace}\left(|\Psi\rangle\langle\Psi| \hat{O} \right) = \mathrm{Trace}\left(\rho \hat{O}\right)$$

$$|\Psi\rangle = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \qquad |\Psi\rangle\langle\Psi| = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}\begin{pmatrix} a_0^* & a_1^* \end{pmatrix} = \begin{pmatrix} a_0 a_0^* & a_0 a_1^* \\ a_1 a_0^* & a_1 a_1^* \end{pmatrix} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \qquad \rho_{11} + \rho_{22} = 1$$

Pure states (described by wavefunctions): $|\rho_{01}|^2 = \rho_{00}\rho_{11}$

Examples:

$$|0\rangle\langle 0| = \frac{\sigma_z + 1}{2} \qquad \frac{\langle\sigma_z\rangle + 1}{2} = \frac{\rho_{00} - \rho_{11} + 1}{2} = \rho_{00}$$

$$\langle\sigma_z\rangle = \mathrm{tr}\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \mathrm{tr}\begin{pmatrix} \rho_{00} & -\rho_{01} \\ \rho_{10} & -\rho_{11} \end{pmatrix} = \rho_{00} - \rho_{11}$$

$$|1\rangle\langle 1| = \frac{1 - \sigma_z}{2} \qquad \frac{1 - \langle\sigma_z\rangle}{2} = \frac{1 - \rho_{00} + \rho_{11}}{2} = \rho_{11}$$

$$\langle\sigma_x\rangle = \mathrm{tr}\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \mathrm{tr}\begin{pmatrix} \rho_{01} & \rho_{00} \\ \rho_{11} & \rho_{10} \end{pmatrix} = \rho_{01} + \rho_{10}$$

$$\sigma_x \qquad \langle\sigma_x\rangle = \rho_{01} + \rho_{10}$$

$$\sigma_y \qquad \langle\sigma_y\rangle = i\rho_{01} - i\rho_{10}$$

$$\langle\sigma_y\rangle = \mathrm{tr}\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = \mathrm{tr}\begin{pmatrix} i\rho_{01} & -i\rho_{00} \\ i\rho_{11} & -i\rho_{10} \end{pmatrix} = i\rho_{01} - i\rho_{10}$$

$$\sigma^+ = \frac{\sigma_x - i\sigma_y}{2} \qquad \langle\sigma^+\rangle = \frac{\langle\sigma_x\rangle - i\langle\sigma_y\rangle}{2} = \frac{\rho_{01} + \rho_{10} + \rho_{01} - \rho_{10}}{2} = \rho_{01}$$

$$\sigma^- = \frac{\sigma_x + i\sigma_y}{2} \qquad \langle\sigma^-\rangle = \frac{\langle\sigma_x\rangle + i\langle\sigma_y\rangle}{2} = \frac{\rho_{01} + \rho_{10} - \rho_{01} + \rho_{10}}{2} = \rho_{10}$$

Master Equation

Schrodinger equation:

$$i\hbar\frac{\partial|\Psi\rangle}{\partial t}=H|\Psi\rangle \qquad -i\hbar\frac{\partial\langle\Psi|}{\partial t}=\langle\Psi|H$$

$$i\hbar\frac{\partial|\Psi\rangle\langle\Psi|}{\partial t}=i\hbar\frac{\partial|\Psi\rangle}{\partial t}\langle\Psi|+i\hbar|\Psi\rangle\frac{\partial\langle\Psi|}{\partial t} \qquad i\hbar\frac{\partial|\Psi\rangle\langle\Psi|}{\partial t}=H|\Psi\rangle\langle\Psi|-|\Psi\rangle\langle\Psi|H$$

$$\boxed{i\hbar\frac{\partial\rho}{\partial t}=[H,\rho]} \qquad [H,\rho]=H\rho-\rho H$$

Two-level system diagonal Hamiltonian: $H=-\frac{\hbar\omega_a}{2}\sigma_z$

$$i\hbar\begin{pmatrix}\dot\rho_{00} & \dot\rho_{01}\\ \dot\rho_{10} & \dot\rho_{11}\end{pmatrix}=-\frac{\hbar\omega_a}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix}+\frac{\hbar\omega_a}{2}\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \begin{pmatrix}\dot\rho_{00} & \dot\rho_{01}\\ \dot\rho_{10} & \dot\rho_{11}\end{pmatrix}=\frac{-i\omega_a}{2}\begin{pmatrix}0 & -2\rho_{01}\\ 2\rho_{10} & 0\end{pmatrix}$$

$$\rho_{00}(t)=\rho_{00}(0) \qquad \rho_{01}(t)=\rho_{01}(0)e^{i\omega_a t}$$

$$\rho_{10}(t)=\rho_{10}(0)e^{-i\omega_a t} \qquad \rho_{11}(t)=\rho_{11}(0)$$

$$H=-\frac{\hbar\Omega}{2}\sigma_x \qquad i\hbar\begin{pmatrix}\dot\rho_{00} & \dot\rho_{01}\\ \dot\rho_{10} & \dot\rho_{11}\end{pmatrix}=-\frac{\hbar\Omega}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix}+\frac{\hbar\Omega}{2}\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \begin{pmatrix}\dot\rho_{00} & \dot\rho_{01}\\ \dot\rho_{10} & \dot\rho_{11}\end{pmatrix}=\frac{i\Omega}{2}\begin{pmatrix}\rho_{10}-\rho_{01} & \rho_{11}-\rho_{00}\\ \rho_{00}-\rho_{11} & \rho_{01}-\rho_{10}\end{pmatrix}$$

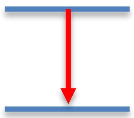
$$\dot\rho_{11}-\dot\rho_{00}=i\Omega(\rho_{01}-\rho_{10}) \qquad \ddot\rho_{11}-\ddot\rho_{00}=-\Omega^2(\rho_{11}-\rho_{00})$$

$$\dot\rho_{01}-\dot\rho_{10}=i\Omega(\rho_{11}-\rho_{00}) \qquad \rho_{11}(t)=A\sin\Omega t+B\cos\Omega t$$

Incoherent processes: Energy dissipation

Probabilities to find the two-level system in the excited and ground states: P_1 and P_0

Excited two-level system



$$\Gamma = \frac{1}{T}$$

$$dP_1 = -P_1 \frac{dt}{T}$$

$$P_0 + P_1 = 1$$

$$\dot{P}_1 = -\frac{P_1}{T} = -\Gamma P_1$$

$$P_1(t) = P_1(0) e^{-\frac{t}{T}} \qquad P_0(t) = 1 - P_1(0) e^{-\frac{t}{T_1}}$$

T_1 is the energy relaxation time

$$\rho_{11}(0) = 1 \qquad \rho(0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$$

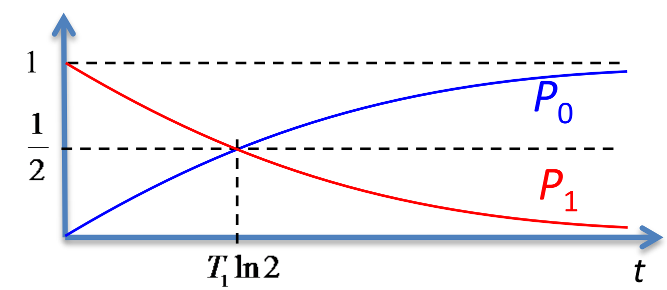
Mixed state (is not described by wavefunctions):

$$\rho(T_1 \ln 2) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \qquad \rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

The wavefunction giving the same probabilities:

$$\Psi = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow \rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Psi = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad \rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$



ρ_{01} and ρ_{10} - coherence

Incoherent processes: Dissipation and dephasing

The Master Equation including incoherent processes:

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}\left[H,\rho\right] + L$$

$$dP_1 = -P_1 \frac{dt}{T_1} \qquad \frac{d\rho_{11}}{dt} = -\rho_{11}\Gamma_1$$

$$dP_0 = P_1 \frac{dt}{T_1} \qquad \frac{d\rho_{00}}{dt} = +\rho_{11}\Gamma_1$$

$$\Gamma_1 = \frac{1}{T_1}$$

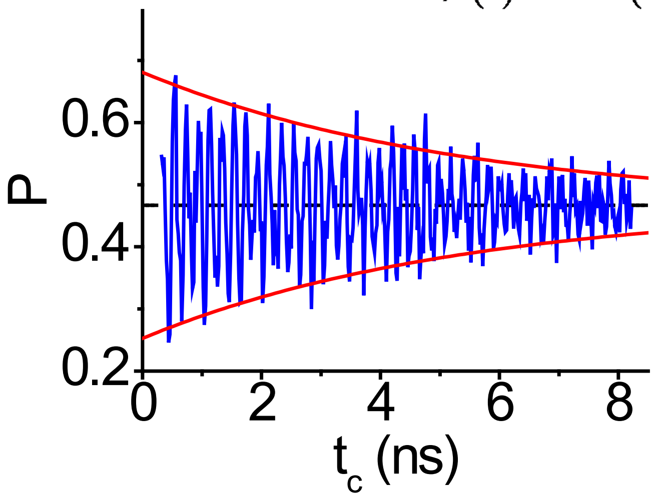
$$L = \begin{pmatrix} \Gamma_1 \rho_{11} & -\Gamma_2 \rho_{01} \\ -\Gamma_2 \rho_{10} & -\Gamma_1 \rho_{11} \end{pmatrix}$$

Γ_1 is the energy relaxation

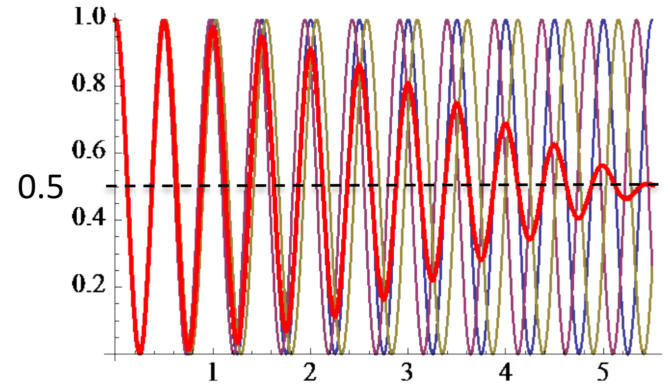
Γ_2 is dephasing (decay of the off-diagonal elements)

Dephasing

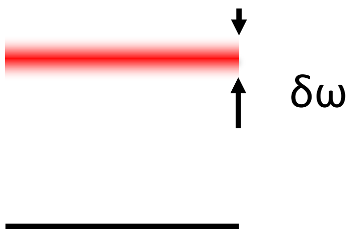
$$\psi(t) = \cos(\Omega t/2)|0\rangle + e^{-i\frac{\pi}{2}}\sin(\Omega t/2)|1\rangle$$



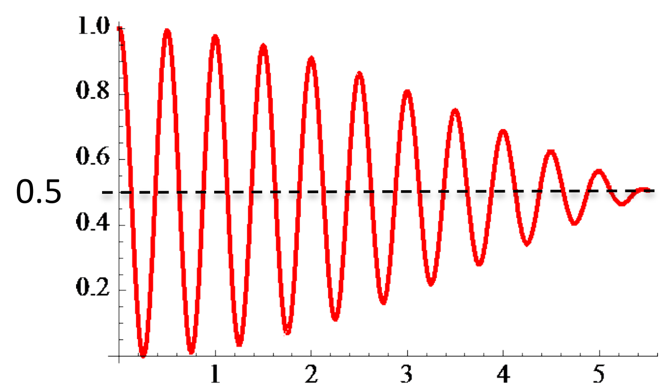
$$P_1 = \cos^2\frac{\Omega_1 t}{2} \qquad P_2 = \cos^2\frac{\Omega_2 t}{2} \qquad P_3 = \cos^2\frac{\Omega_3 t}{2}$$



Dephasing:
energy/frequency => phase fluctuation

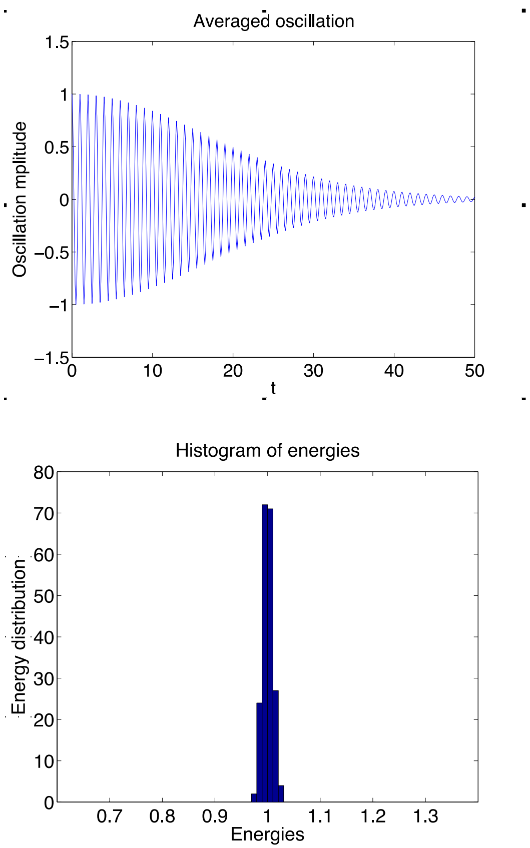


$$P(0) = \frac{1}{3}\left(\cos^2\frac{\Omega_1 t}{2} + \cos^2\frac{\Omega_2 t}{2} + \cos^2\frac{\Omega_3 t}{2}\right)$$

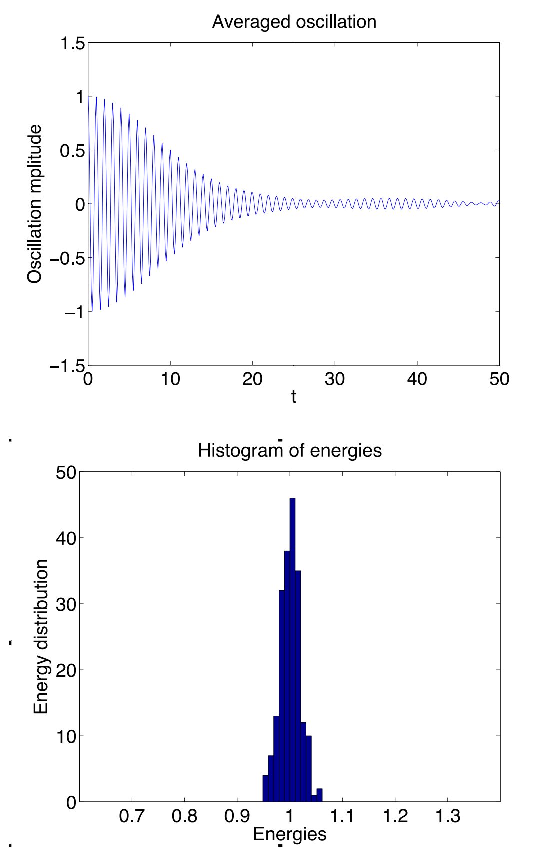


Dephasing due to energy level fluctuations

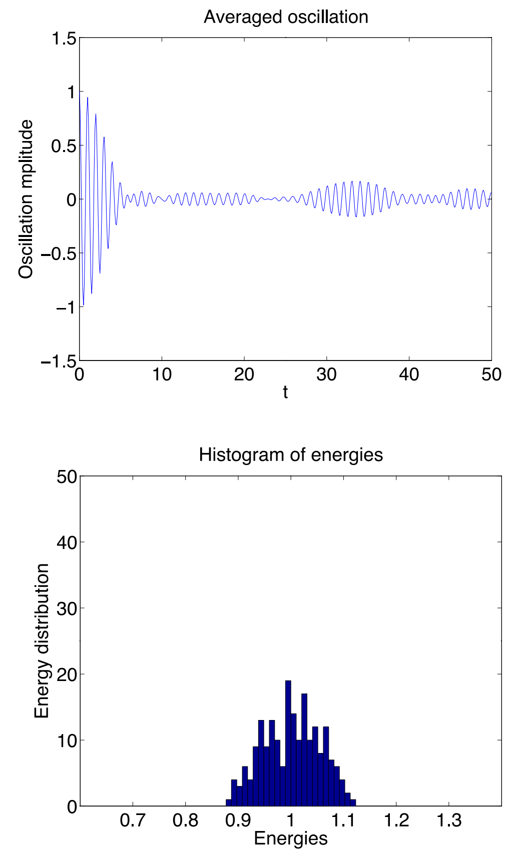
$\langle f \rangle = 1, \sigma_f = 0.01$



$\langle f \rangle = 1, \sigma_f = 0.02$



$\langle f \rangle = 1, \sigma_f = 0.05$



Expectation values of Pauli operators

Density matrix of the two-level system has three independent real variables: ρ_{00}, Re(ρ_{01}), Im(ρ_{01})

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \qquad \rho_{00} + \rho_{11} = 1 \qquad \rho_{01} = \rho_{10}^{*}$$

$$\langle \sigma_x \rangle = \mathrm{tr}\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \mathrm{tr}\begin{pmatrix} \rho_{01} & \rho_{00} \\ \rho_{11} & \rho_{10} \end{pmatrix} = \rho_{01} + \rho_{10} = 2\,\mathrm{Re}[\rho_{01}]$$

$$\langle \sigma_y \rangle = \mathrm{tr}\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = \mathrm{tr}\begin{pmatrix} i\rho_{01} & -i\rho_{00} \\ i\rho_{11} & -i\rho_{10} \end{pmatrix} = i\rho_{01} - i\rho_{10} = -2\,\mathrm{Im}[\rho_{01}]$$

$$\langle \sigma_z \rangle = \mathrm{tr}\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \mathrm{tr}\begin{pmatrix} \rho_{00} & -\rho_{01} \\ -\rho_{11} & -\rho_{11} \end{pmatrix} = \rho_{00} - \rho_{11} = \rho_{00} - (1 - \rho_{00}) = 2\rho_{00} - 1$$

$$\rho_{00} = \frac{1 + \langle \sigma_z \rangle}{2} \qquad \rho_{01} = \frac{\langle \sigma_x \rangle - i\langle \sigma_y \rangle}{2} = \langle \sigma^+ \rangle \qquad \rho_{10} = \frac{\langle \sigma_x \rangle + i\langle \sigma_y \rangle}{2} = \langle \sigma^- \rangle \qquad \rho_{11} = \frac{1 - \langle \sigma_z \rangle}{2}$$

The dynamics can be alternatively represented via expectation values of the three Pauli matrices

Bloch Sphere for dissipative spin dynamics

$$|\psi\rangle\langle\psi| = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{1}{2}e^{-i\varphi}\sin\theta \\ \frac{1}{2}e^{i\varphi}\sin\theta & \sin^2\frac{\theta}{2} \end{pmatrix}$$

$$\rho_{00} + \rho_{11} = 1$$

$$\langle\sigma_z\rangle = 1 \Rightarrow \quad \rho_{00} - \rho_{11} = 1 \Rightarrow \rho_{00} = 1 \qquad |\psi\rangle = |0\rangle$$

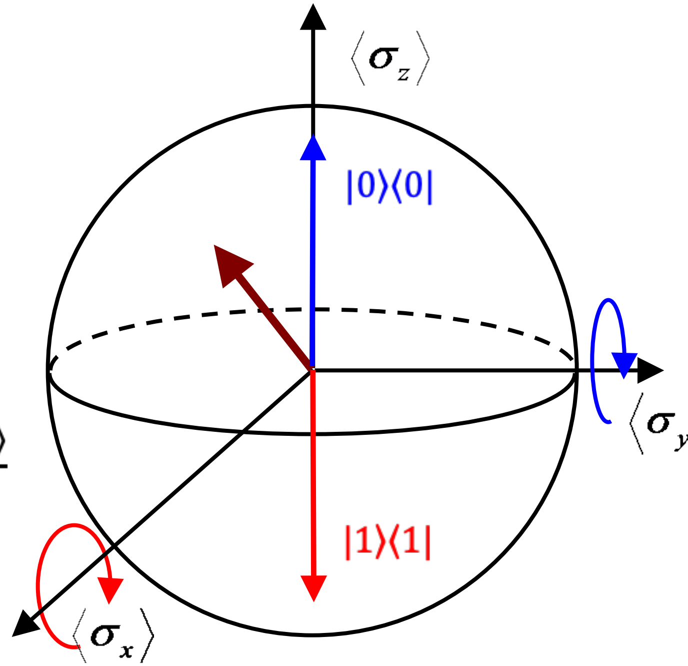
$$\langle\sigma_z\rangle = -1 \Rightarrow \rho_{00} - \rho_{11} = -1 \Rightarrow \rho_{11} = 1 \qquad |\psi\rangle = |1\rangle$$

$$\langle\sigma_x\rangle = 1 \Rightarrow \quad \rho_{01} + \rho_{10} = 1 \Rightarrow \rho_{01} = \rho_{10} = \tfrac{1}{2} \qquad \theta = \frac{\pi}{2} \quad \varphi = 0 \quad |\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$\langle\sigma_x\rangle = -1 \Rightarrow \quad \rho_{01} + \rho_{10} = -1 \Rightarrow \rho_{01} = \rho_{10} = -\tfrac{1}{2} \qquad \theta = \frac{\pi}{2} \quad \varphi = \pi \quad |\psi\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$\langle\sigma_y\rangle = 1 \Rightarrow \quad i\rho_{01} - i\rho_{10} = 1 \Rightarrow \rho_{01} = -\rho_{01} = \tfrac{1}{2i} \qquad \theta = \frac{\pi}{2} \quad \varphi = \frac{\pi}{2} \quad |\psi\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$

$$\langle\sigma_y\rangle = -1 \Rightarrow \quad i\rho_{01} - i\rho_{10} = -1 \Rightarrow \rho_{01} = -\rho_{01} = -\tfrac{1}{2i} \qquad \theta = \frac{\pi}{2} \quad \varphi = -\frac{\pi}{2} \quad |\psi\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$$

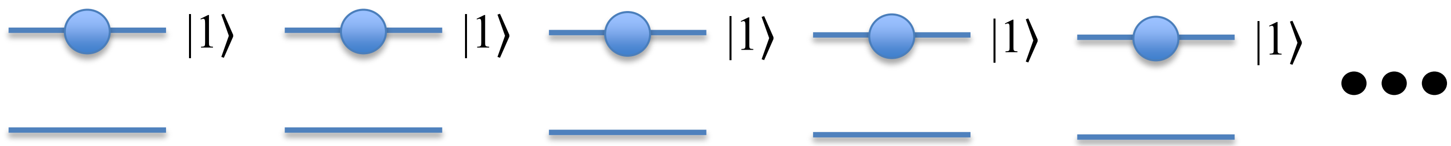


For non-pure states, the vector tip may go inside the sphere (Bloch 'ball')

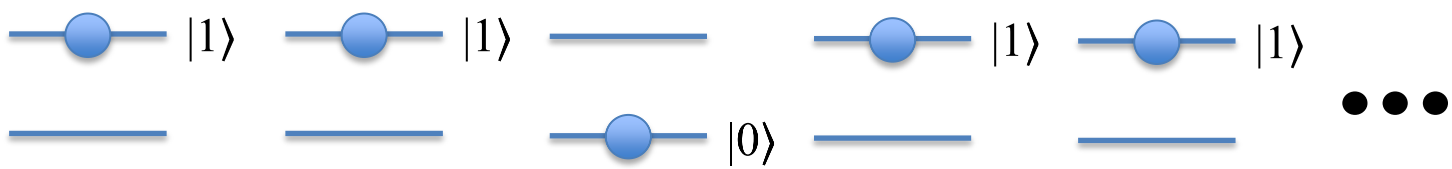
Quantum dynamics of two-level systems with decoherence

Pure and Mixed states

Pure state: wavefunction is $|1\rangle$; Density matrix: $\rho = |1\rangle\langle 1|$



Mixed state: the state which can not be described by a wavefunction => probabilities
Density matrix: $\rho = 0.8|1\rangle\langle 1| + 0.2|0\rangle\langle 0|$



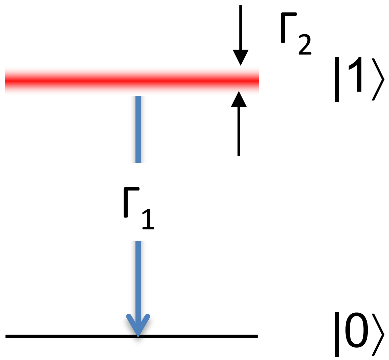
Phase fluctuations:

$$\left|\Psi_1\right\rangle = \frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}} \quad \rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad \left|\Psi_2\right\rangle = \frac{\left|0\right\rangle + e^{i\varphi}\left|1\right\rangle}{\sqrt{2}} \quad \rho = \begin{pmatrix} \frac{1}{2} & \frac{e^{i\varphi}}{2} \\ \frac{e^{-i\varphi}}{2} & \frac{1}{2} \end{pmatrix} \qquad \left|\Psi_3\right\rangle = \frac{\left|0\right\rangle + e^{-i\varphi}\left|1\right\rangle}{\sqrt{2}} \quad \rho = \begin{pmatrix} \frac{1}{2} & \frac{e^{-i\varphi}}{2} \\ \frac{e^{i\varphi}}{2} & \frac{1}{2} \end{pmatrix} \quad \bullet\bullet\bullet$$

$$\rho_{00} = \rho_{11} = \frac{1}{2} \qquad \rho_{01} = \frac{1}{N}\left(\frac{1}{2} + \frac{e^{i\varphi}}{2} + \frac{e^{-i\varphi}}{2} + \ldots\right) \approx \frac{1}{2}\frac{1}{N}\left(1 + \left(1 + i\varphi + \frac{(i\varphi)^2}{2!}\right) + \left(1 - i\varphi + \frac{(-i\varphi)^2}{2!}\right) + \ldots\right) = \frac{1 - \frac{\varphi^2}{3}}{2}$$

Mixed state: $\left|\rho_{01}\right| = \left|\rho_{10}\right| < \sqrt{\left|\rho_{00}\right|\left|\rho_{11}\right|} = \frac{1}{2}$

The Master Equation and the Lindblad term



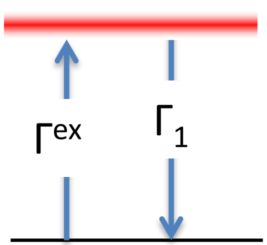
Relaxation and dephasing: Γ_1, Γ_2

$$\boxed{\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar}\left[H,\rho\right] + L}$$

$$\begin{pmatrix} \dot{\rho}_{00} & \dot{\rho}_{01} \\ \dot{\rho}_{10} & \dot{\rho}_{11} \end{pmatrix}$$

The Lindblad term

$$\boxed{L = \begin{pmatrix} \Gamma_1 \rho_{11} & -\Gamma_2 \rho_{01} \\ -\Gamma_2 \rho_{10} & -\Gamma_1 \rho_{11} \end{pmatrix}}$$

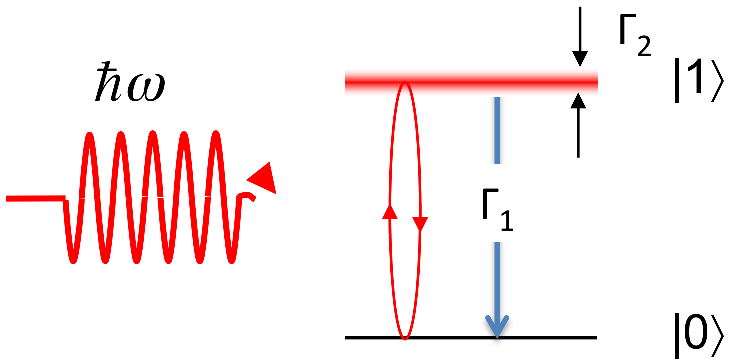


Accounting excitation (e.g. T > 0)

$$\boxed{L = \begin{pmatrix} \Gamma_1 \rho_{11} - \Gamma^{ex} \rho_{00} & -\Gamma_2 \rho_{01} \\ -\Gamma_2 \rho_{10} & -\Gamma_1 \rho_{11} + \Gamma^{ex} \rho_{00} \end{pmatrix}}$$

Driven two-level system with incoherent processes

The general form of the Hamiltonian driven by a wave with an arbitrary phase shift



Relaxation and dephasing: Γ_1, Γ_2

Dephasing: $\Gamma_2 = \frac{\Gamma_1}{2} + \Gamma_\varphi$

Pure dephasing

$$H = -\frac{\hbar\omega}{2}\sigma_z + \hbar\Omega\sigma_x\cos(\omega t - \varphi) \qquad U = e^{-i\frac{\omega t}{2}\sigma_z}$$

$$\boxed{H' = \frac{\hbar\Omega}{2}\left(\sigma_x\cos\varphi + \sigma_y\sin\varphi\right)}$$

$$\frac{\partial\rho}{\partial t} = -\frac{i}{\hbar}\left[H',\rho\right] + L \qquad \rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \qquad L = \begin{pmatrix} \Gamma_1\rho_{11} & -\Gamma_2\rho_{01} \\ -\Gamma_2\rho_{10} & -\Gamma_1\rho_{11} \end{pmatrix}$$

$$L = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -\Gamma_1\rho_{00} & -\Gamma_2\rho_{01} \\ -\Gamma_2\rho_{10} & -\Gamma_1\rho_{11} \end{pmatrix}$$

Driven two-level system with incoherent processes

The RWA Hamiltonian:

$$\boxed{H=\frac{\hbar\Omega}{2}\left(\sigma_x\cos\varphi+\sigma_y\sin\varphi\right)}$$

The Master Equation:

$$\boxed{\frac{\partial\rho}{\partial t}=-\frac{i}{\hbar}\left[H,\rho\right]+L}$$

The Lindblad term:

$$L=\begin{pmatrix}\Gamma_1\rho_{11} & -\Gamma_2\rho_{01}\\ -\Gamma_2\rho_{10} & -\Gamma_1\rho_{11}\end{pmatrix}$$

Three independent equations for $\rho_{00},\ \rho_{01},\ \rho_{10}$

$$\rho_{00}+\rho_{11}=1$$

$$\begin{pmatrix}\dot\rho_{00} & \dot\rho_{01}\\ \dot\rho_{10} & \dot\rho_{11}\end{pmatrix}=-\frac{i}{\hbar}\left(H\rho-\rho H\right)+L$$

$$\frac{\partial\langle\sigma_j\rangle}{\partial t}=\mathrm{tr}\left[\sigma_j\frac{\partial\rho}{\partial t}\right]$$

$$\frac{\partial\langle\sigma_j\rangle}{\partial t}=\mathrm{tr}\left[-\frac{i}{\hbar}\sigma_j\left(H\rho-\rho H\right)+\sigma_jL\right]=-\frac{i}{\hbar}\mathrm{tr}\left[\sigma_jH\rho-H\sigma_j\rho\right]+\mathrm{tr}\left[\sigma_jL\right]=\frac{i}{\hbar}\left\langle\left[H,\sigma_j\right]\right\rangle+\mathrm{tr}\left[\sigma_jL\right]$$

$$\frac{\partial\langle\sigma_x\rangle}{\partial t}=-i\frac{\Omega}{2}\left(\langle[\sigma_x,\sigma_x]\rangle\cos\varphi+\langle[\sigma_y,\sigma_x]\rangle\sin\varphi\right)+\mathrm{tr}[\sigma_xL]$$

$$\frac{\partial\langle\sigma_y\rangle}{\partial t}=-i\frac{\Omega}{2}\left(\langle[\sigma_x,\sigma_y]\rangle\cos\varphi+\langle[\sigma_y,\sigma_y]\rangle\sin\varphi\right)+\mathrm{tr}[\sigma_yL]$$

$$\frac{\partial\langle\sigma_z\rangle}{\partial t}=-i\frac{\Omega}{2}\left(\langle[\sigma_x,\sigma_z]\rangle\cos\varphi+\langle[\sigma_y,\sigma_z]\rangle\sin\varphi\right)+\mathrm{tr}[\sigma_zL]$$

$$\frac{\partial\langle\sigma_x\rangle}{\partial t}=-\Omega\langle\sigma_z\rangle\sin\varphi-\Gamma_2\langle\sigma_x\rangle$$

$$\frac{\partial\langle\sigma_y\rangle}{\partial t}=\Omega\langle\sigma_z\rangle\cos\varphi-\Gamma_2\langle\sigma_y\rangle$$

$$\frac{\partial\langle\sigma_z\rangle}{\partial t}=\Omega\left(-\langle\sigma_y\rangle\cos\varphi+\langle\sigma_x\rangle\sin\varphi\right)-\Gamma_1\langle\sigma_z\rangle+\Gamma_1$$

$$\underbrace{\frac{\partial}{\partial t}\begin{pmatrix}\langle\sigma_x\rangle\\ \langle\sigma_y\rangle\\ \langle\sigma_z\rangle\end{pmatrix}}_{\frac{\partial\vec\sigma}{\partial t}}=\underbrace{\begin{pmatrix}-\Gamma_2 & 0 & -\Omega\sin\varphi\\ 0 & -\Gamma_2 & \Omega\cos\varphi\\ \Omega\sin\varphi & -\Omega\cos\varphi & -\Gamma_1\end{pmatrix}}_{B}\underbrace{\begin{pmatrix}\langle\sigma_x\rangle\\ \langle\sigma_y\rangle\\ \langle\sigma_z\rangle\end{pmatrix}}_{\vec\sigma}+\underbrace{\begin{pmatrix}0\\ 0\\ \Gamma_1\end{pmatrix}}_{\vec b}$$

Dynamics of the two level system is described by:

$$\boxed{\frac{d\vec\sigma}{dt}=B\vec\sigma+\vec b}$$

Bloch Sphere for dissipative spin dynamics

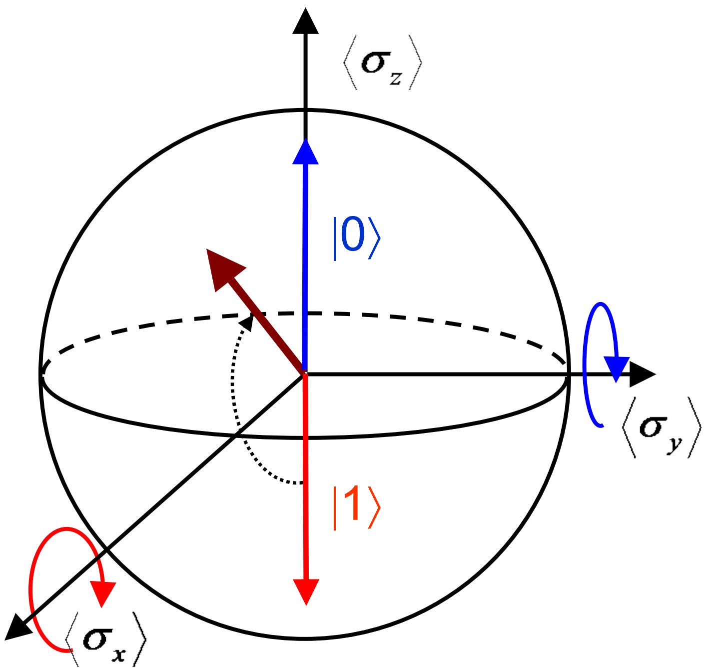
$$\frac{\partial}{\partial t}\begin{pmatrix}\langle\sigma_x\rangle\\ \langle\sigma_y\rangle\\ \langle\sigma_z\rangle\end{pmatrix}=\begin{pmatrix}-\Gamma_2 & 0 & -\Omega\sin\varphi\\ 0 & -\Gamma_2 & \Omega\cos\varphi\\ \Omega\sin\varphi & -\Omega\cos\varphi & -\Gamma_1\end{pmatrix}\begin{pmatrix}\langle\sigma_x\rangle\\ \langle\sigma_y\rangle\\ \langle\sigma_z\rangle\end{pmatrix}+\begin{pmatrix}0\\ 0\\ \Gamma_1\end{pmatrix}$$

$$\frac{d\vec{\sigma}}{dt}=B\vec{\sigma}+\vec{b}$$

The vector can be less than one
(alternative criteria of mixed states)

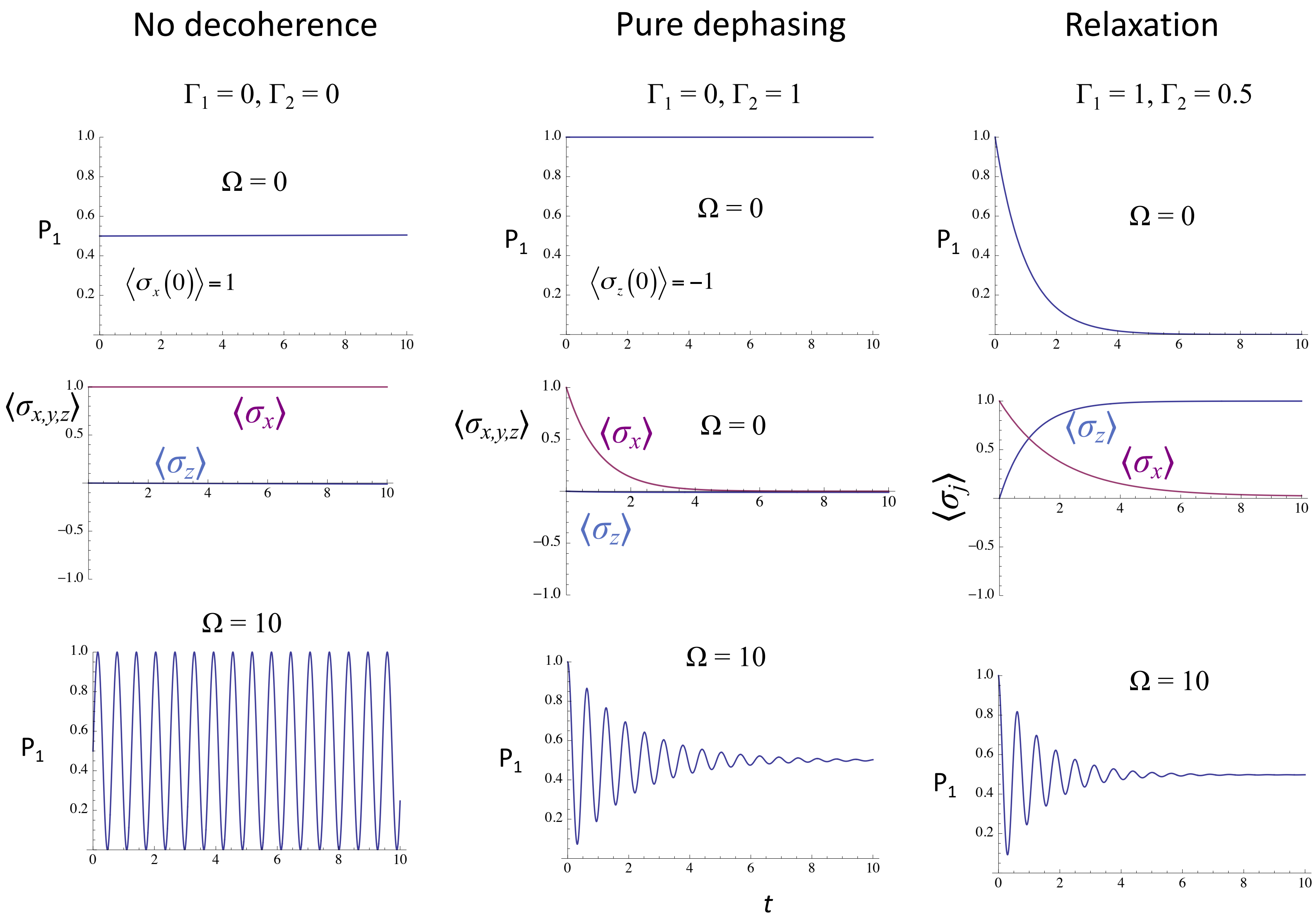
$$\vec{\sigma}(t)=e^{Bt}\vec{\sigma}(0)+B^{-1}\left(e^{Bt}-1\right)\vec{b}$$

$$\vec{\sigma}(t)=e^{Bt}\left[\vec{\sigma}(0)+B^{-1}\vec{b}\right]-B^{-1}\vec{b}$$



Dynamics of the two-level system is exactly same as the dynamics of spin 1/2

Rabi oscillations: Oscillations in a two-level system under external drive. Oscillations of atomic states.



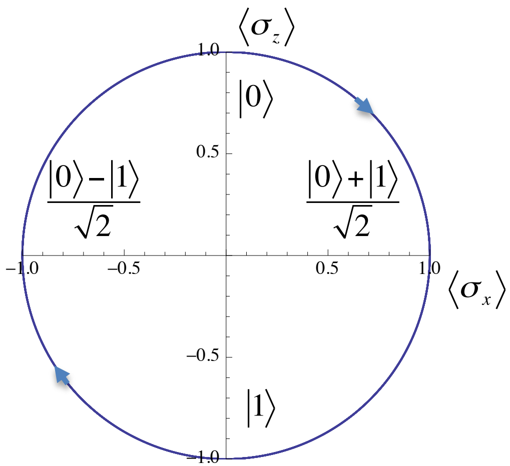
No decoherence $\Gamma_1 = 0, \Gamma_2 = 0$

$$\Psi = \cos\frac{\Omega t}{2}|0\rangle + \sin\frac{\Omega t}{2}|1\rangle$$

$$\rho_{00} = \sin^2\frac{\Omega t}{2} = \frac{1+\cos\Omega t}{2} \qquad \langle\sigma_x\rangle = \rho_{01}+\rho_{10} = \sin\Omega t$$

$$\rho_{01} = \cos\frac{\Omega t}{2}\sin\frac{\Omega t}{2} = \frac{\sin\Omega t}{2} \qquad \langle\sigma_y\rangle = i\rho_{01} - i\rho_{10} = 0$$

$$\rho_{10} = \cos\frac{\Omega t}{2}\sin\frac{\Omega t}{2} = \frac{\sin\Omega t}{2} \qquad \langle\sigma_z\rangle = 2\rho_{00} - 1 = \cos\Omega t$$



With decoherence

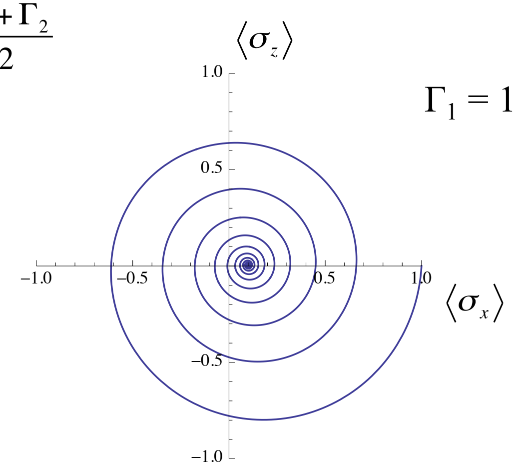
$$\rho_{00} = 1$$

$$\langle\sigma_x\rangle \approx e^{-\gamma t}\sin\Omega t \qquad \rho_{00} = \frac{\langle\sigma_z + 1\rangle}{2} = \frac{1+e^{-\gamma t}\cos\Omega t}{2}$$

$$\langle\sigma_y\rangle = 0 \qquad \rho_{01} = \frac{e^{-\gamma t}\sin\Omega t}{2}$$

$$\langle\sigma_z\rangle \approx e^{-\gamma t}\cos\Omega t \qquad \rho_{10} \approx \frac{e^{-\gamma t}\sin\Omega t}{2}$$

$$\gamma = \frac{\Gamma_1+\Gamma_2}{2}$$



$\Gamma_1 = 1, \Gamma_2 = 0.5$

Stationary Master Equation

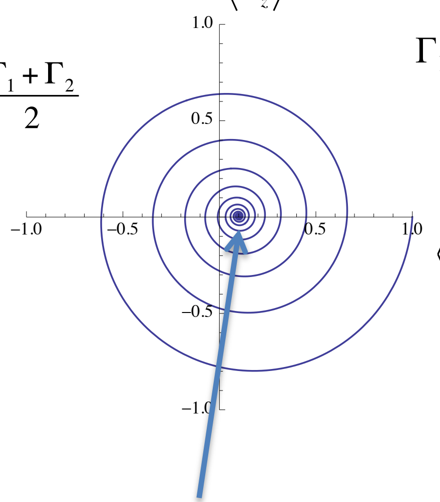
$$\frac{\partial \rho}{\partial t} = 0 \qquad -\frac{i}{\hbar}[H,\rho] + L = 0$$

$$\frac{\partial \vec{\sigma}}{\partial t} = 0 \qquad B\vec{\sigma} + \vec{b} = 0 \qquad \vec{\sigma} = -B^{-1}\vec{b}$$

$$t \longrightarrow \infty$$

$$\gamma = \frac{\Gamma_1 + \Gamma_2}{2}$$

$\Gamma_1 = 1, \Gamma_2 = 0.5$



Stationary conditions

$$\langle \sigma_x \rangle = \frac{\Gamma_1 \Omega}{\Gamma_1 \Gamma_2 + \Omega^2} \xrightarrow[\Omega \to \infty]{} \frac{\Gamma_1}{\Omega}$$

$$\langle \sigma_y \rangle = 0$$

$$\langle \sigma_z \rangle = \frac{\Gamma_1 \Gamma_2}{\Gamma_1 \Gamma_2 + \Omega^2}$$

Driven two-level system with detuning

$$H=-\frac{\hbar\omega_0}{2}\sigma_z+\frac{\hbar\Omega}{2}\sigma_x\cos(\omega t+\varphi) \qquad U=e^{-i\frac{\omega t}{2}\sigma_z}$$

$$\boxed{H'=-\frac{\hbar\delta\omega}{2}\sigma_z+\frac{\hbar\Omega}{2}\left(\sigma_x\cos\varphi+\sigma_y\sin\varphi\right)}$$

$$\delta\omega=\omega_0-\omega$$

$$\frac{\partial\langle\sigma_x\rangle}{\partial t}=-i\frac{\delta\omega}{2}\langle[\sigma_z,\sigma_x]\rangle+\ldots$$
$$\frac{\partial\langle\sigma_y\rangle}{\partial t}=-i\frac{\delta\omega}{2}\langle[\sigma_z,\sigma_y]\rangle+\ldots$$
$$\frac{\partial\langle\sigma_z\rangle}{\partial t}=-i\frac{\delta\omega}{2}\langle[\sigma_z,\sigma_z]\rangle+\ldots$$

$$\frac{\partial\langle\sigma_x\rangle}{\partial t}=-\delta\omega\langle\sigma_y\rangle-\Omega\langle\sigma_z\rangle\sin\varphi-\Gamma_2\langle\sigma_x\rangle$$
$$\frac{\partial\langle\sigma_y\rangle}{\partial t}=\delta\omega\langle\sigma_x\rangle+\Omega\langle\sigma_z\rangle\cos\varphi-\Gamma_2\langle\sigma_y\rangle$$
$$\frac{\partial\langle\sigma_z\rangle}{\partial t}=\Omega\left(\langle\sigma_x\rangle\cos\varphi-\langle\sigma_y\rangle\sin\varphi\right)-\Gamma_1\langle\sigma_z\rangle+\Gamma_1$$

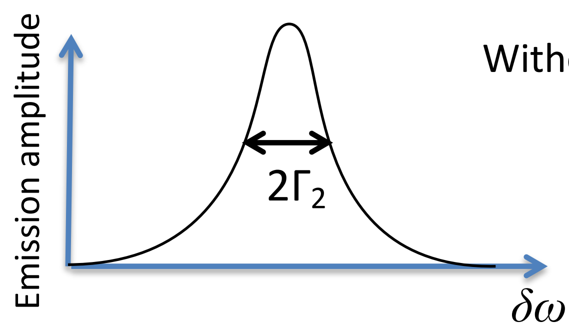
$$\frac{\partial}{\partial t}\begin{pmatrix}\langle\sigma_x\rangle\\ \langle\sigma_y\rangle\\ \langle\sigma_z\rangle\end{pmatrix}=\begin{pmatrix}-\Gamma_2 & -\delta\omega & -\Omega\sin\varphi\\ \delta\omega & -\Gamma_2 & \Omega\cos\varphi\\ \Omega\sin\varphi & -\Omega\cos\varphi & -\Gamma_1\end{pmatrix}\begin{pmatrix}\langle\sigma_x\rangle\\ \langle\sigma_y\rangle\\ \langle\sigma_z\rangle\end{pmatrix}+\begin{pmatrix}0\\0\\ \Gamma_1\end{pmatrix}$$

Stationary solution:

$$B\vec{\sigma}+\vec{b}=0 \qquad \vec{\sigma}=-B^{-1}\vec{b}$$

In the limit of weak drive:

$$V_{emit}=V_0\langle\sigma^+\rangle\propto\frac{1}{\Gamma_2+i\delta\omega}$$



Without dephasing:

$$\Gamma_2=\frac{\Gamma_1}{2}$$