

# Quantum Electronics of Nanostructures

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Lecture 6

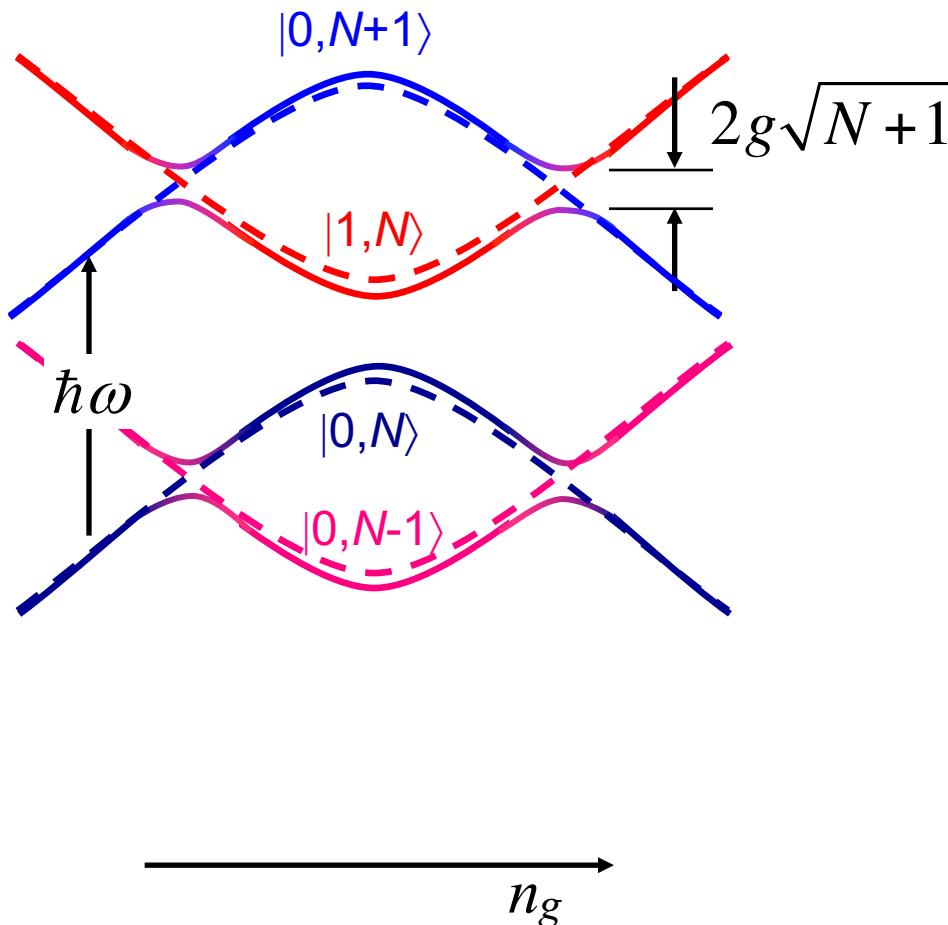
# Lecture 6

- Atom-resonator interaction: second order effects. Ac-Stark and Lamb shifts.
- Unitary transformations
- Time-dependent unitary transformation
- Rotating wave approximation (RWA)
- Quantum state control in qubits

Second order effects  
Dispersive readout

# Second order energy effects

## Shift due to qubit-resonator coupling



# Second order effects

$$H_0 = -\frac{\Delta E}{2} \sigma_z + \hbar \omega_r a^\dagger a \quad H_{\text{int}} = g_0 (a \sigma^+ + a^\dagger \sigma^-)$$

$$H_0 |n, N\rangle = \left[ \left( n - \frac{1}{2} \right) \Delta E \sigma_z + \hbar \omega N \right] |n, N\rangle \Rightarrow E_{n,N} = \left( 1 - \frac{1}{2} \right) \Delta E + \hbar \omega N$$

We are looking for corrections to eigenenergies in the conditions  $|\hbar \omega - \Delta E| \gg g_0$      $\hbar \omega \neq \Delta E$

$$E_{0,N}^{(2)} = \frac{|\langle 0, N | g_0 (a \sigma^+ + a^\dagger \sigma^-) | 1, N-1 \rangle|^2}{E_{0,N} - E_{1,N-1}} = \frac{(g_0 \sqrt{N})^2}{E_{0,N} - E_{1,N-1}} = \frac{g_0^2 N}{\hbar \omega - \Delta E}$$

$$E_{1,N}^{(2)} = \frac{|\langle 1, N | g_0 (a \sigma^+ + a^\dagger \sigma^-) | 0, N+1 \rangle|^2}{E_{1,N} - E_{0,N+1}} = \frac{(g_0 \sqrt{N+1})^2}{E_{1,N} - E_{0,N+1}} = -\frac{g_0^2 (N+1)}{\hbar \omega - \Delta E}$$

$$E'_{0,N} = -\frac{\Delta E}{2} + N \hbar \omega + \frac{g_0^2 N}{\hbar \omega - \Delta E}$$

$$E'_{1,N} = \frac{\Delta E}{2} + N \hbar \omega - \frac{g_0^2 (N+1)}{\hbar \omega - \Delta E}$$

$$E'_{0,N+1} = -\frac{\Delta E}{2} + (N+1) \hbar \omega + \frac{g_0^2 (N+1)}{\hbar \omega - \Delta E}$$

$$E'_{1,N+1} = \frac{\Delta E}{2} + (N+1) \hbar \omega - \frac{g_0^2 (N+2)}{\hbar \omega - \Delta E}$$

Atomic transitions depend on number of photons (Stark shift):

$$E'_{1,N} - E'_{0,N} = \Delta E - \frac{g_0^2 (2N+1)}{\hbar \omega - \Delta E}$$

Resonator transitions depend on atomic states:

$$E'_{0,N+1} - E'_{0,N} = \hbar \omega + \frac{g_0^2}{\hbar \omega - \Delta E}$$

$$E'_{1,N+1} - E'_{1,N} = \hbar \omega - \frac{g_0^2}{\hbar \omega - \Delta E}$$

# Second order effects (alternative approach)

$$H_{JC} = -\frac{\hbar\omega_q}{2}\sigma_z + \hbar g(a\sigma^+ + a^\dagger\sigma^-) + \hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right)$$

Unitary transformation:

$$U = \exp\left[\frac{g}{\delta\omega}(a\sigma^+ - a^\dagger\sigma^-)\right] \quad \delta\omega = \omega_q - \omega_r \quad \delta\omega \gg g$$

$$H' = UH_{JC}U^\dagger$$

Resonator energy shift

$$H' = \hbar\left(\omega_r + \frac{g^2}{\delta\omega}\sigma_z\right)a^\dagger a + \frac{\hbar}{2}\left(\omega_q + \frac{g^2}{\delta\omega}\right)\sigma_z$$

$$\omega'_r = \omega_r \pm \frac{g^2}{\delta\omega}$$

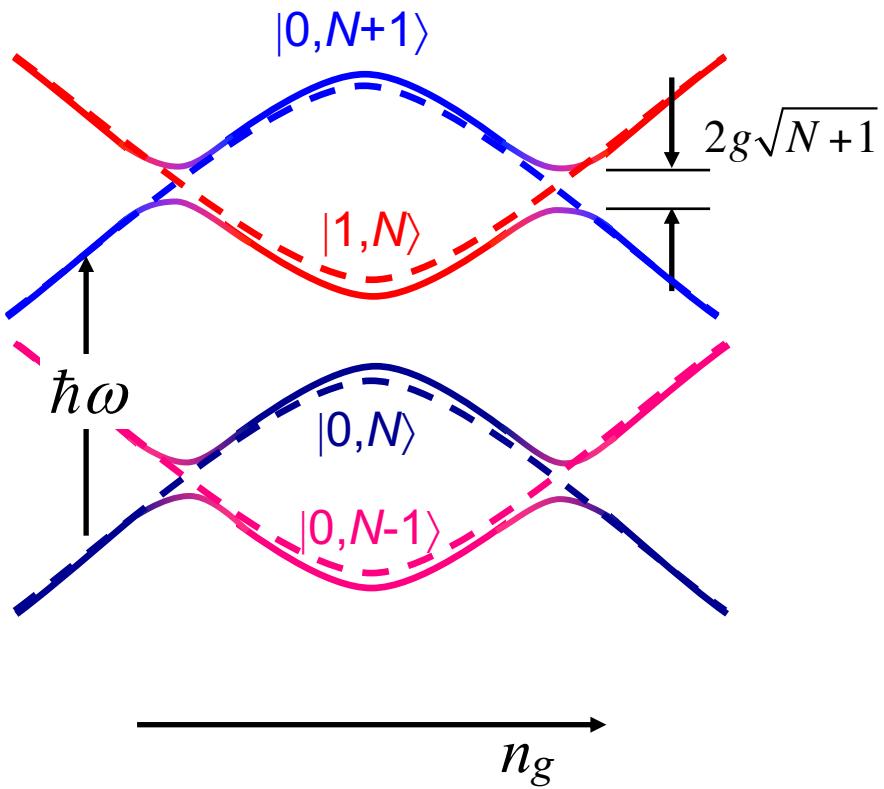
$$H' = \hbar\omega_r a^\dagger a + \frac{\hbar}{2}\left[\omega_q + \frac{g^2}{\delta\omega}(a^\dagger a + 1)\right]\sigma_z$$

Atomic energy shift

$$\omega'_q = \omega_q + \frac{g^2}{\delta\omega}(N + 1)$$

Atom/resonator frequency shift depending on resonator/atom states

# One more way of treatment



$$H'' = -\frac{\varepsilon}{2}\sigma_z - \frac{\Delta}{2}\sigma_x$$

$$\varepsilon = E_{0,N+1} - E_{1,N} = \hbar\omega - \Delta E$$

$$\Delta = 2g\sqrt{N + 1}$$

$$H''_{diag} = -\frac{\sqrt{\varepsilon^2 + \Delta^2}}{2}\sigma_z \approx -\frac{\varepsilon + \frac{\Delta^2}{2\varepsilon}}{2}\sigma_z$$

$$E'_{1,N} = E_{1,N} + \frac{1}{2} \frac{\Delta^2}{2\varepsilon} = E_{1,N} + \frac{g_0^2(N+1)}{\hbar\omega - \Delta E}$$

$$E'_{0,N+1} = E_{0,N+1} - \frac{1}{2} \frac{\Delta^2}{2\varepsilon} = E_{0,N+1} - \frac{g_0^2(N+1)}{\hbar\omega - \Delta E}$$

# AC-Stark shift

$$|0,N\rangle: E_{0,N} = -\frac{\Delta E}{2} + N\hbar(\omega + \Delta\omega)$$

Off-resonant system:

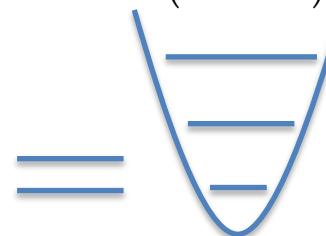
$$\Delta E \neq \hbar\omega$$

$$|1,N\rangle: E_{1,N} = \frac{\Delta E}{2} - \Delta\omega + N\hbar(\omega - \Delta\omega)$$

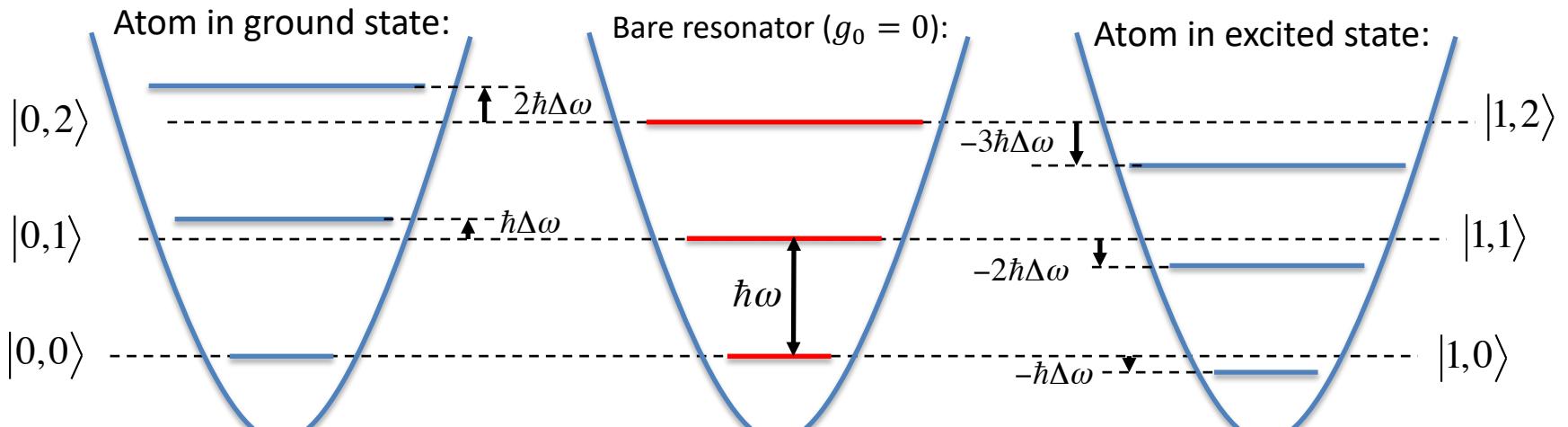
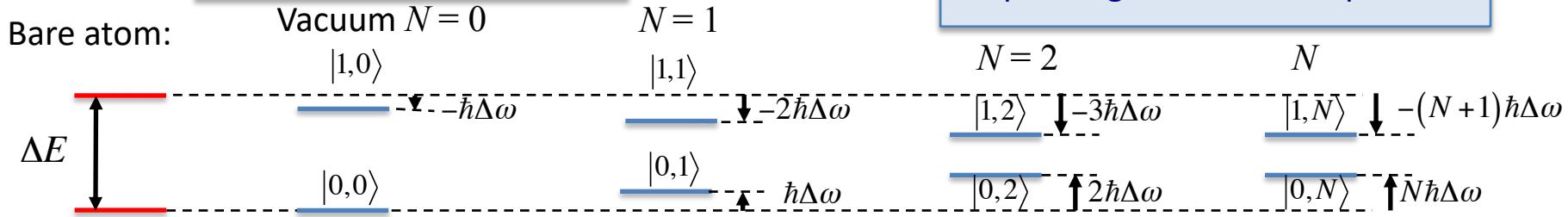
For example:

$$\Delta E \ll \hbar\omega$$

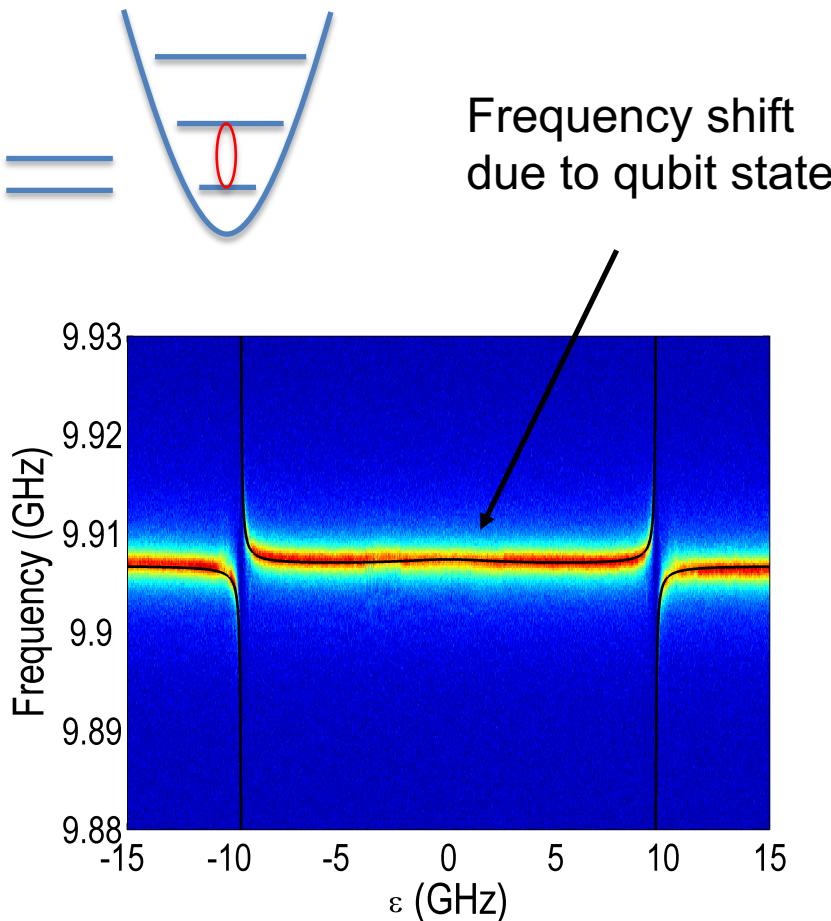
$$\hbar\Delta\omega = \frac{g_0^2}{\hbar\omega - \Delta E}$$



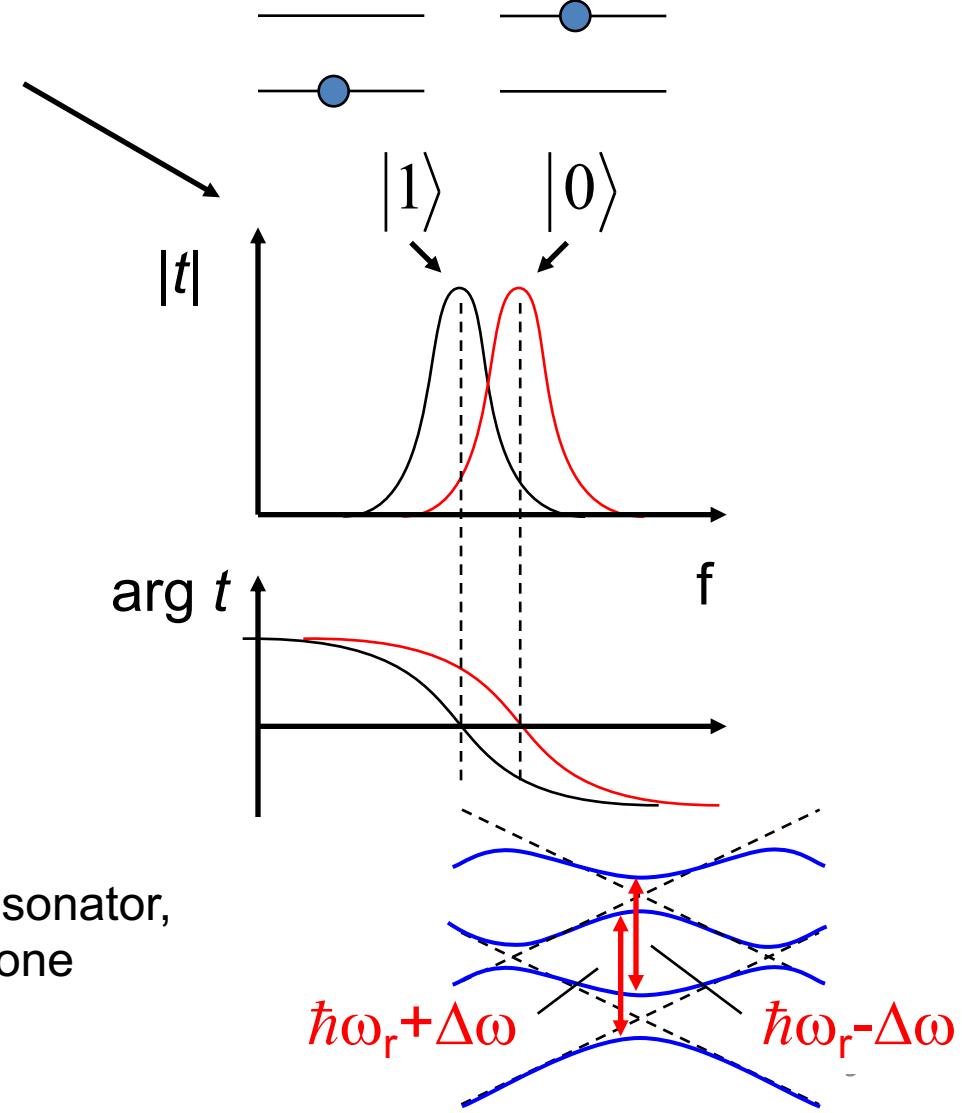
Lamb shift: shift of atomic levels due to vacuum ( $N=0$ )



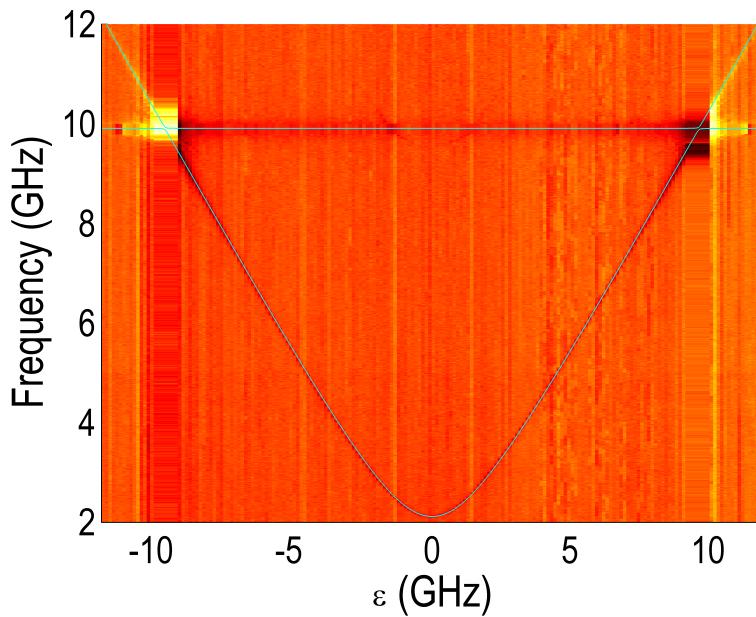
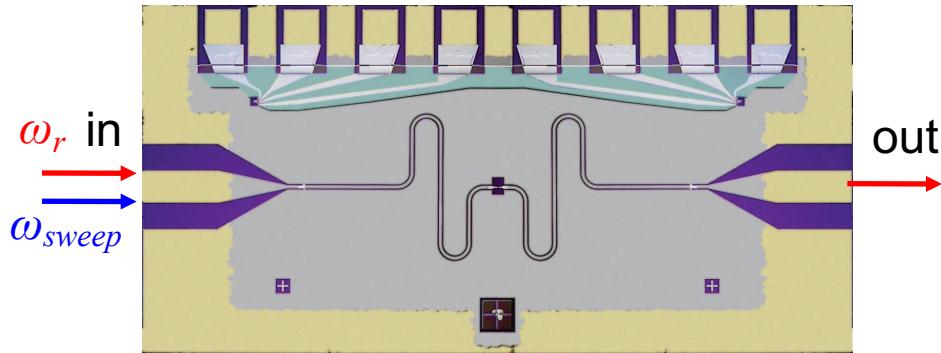
# Principles of dispersive readout



We measure transmission through the resonator, while the qubit is excited by the second tone



# Dispersive readout: Two-tone spectroscopy measurements



- The plot shows change of the phase of the transmitted signal
- As predicted between two anti-crossings we observe a deep

# Unitary transformations

# Transformation of Hamiltonians (time independent)

Unitary transformation:

$$\Psi' = U\Psi \quad \Psi = U^\dagger\Psi'$$

Schrodinger equations:

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \quad i\hbar \frac{\partial \Psi'}{\partial t} = H'\Psi'$$

We are looking for  $H'$

Schrodinger equations:

$$i\hbar \frac{\partial U^\dagger \Psi'}{\partial t} = HU^\dagger \Psi'$$

$$UU^\dagger i\hbar \frac{\partial \Psi'}{\partial t} = UHU^\dagger \Psi'$$

$$i\hbar \frac{\partial \Psi'}{\partial t} = UHU^\dagger \Psi'$$

$$H' = UHU^\dagger$$

# Unitary operators

Operator  $U$       Hermitian conjugate:  $U^\dagger = (U^*)^T$

Unitary operator:  $U^\dagger U = I$

Example of unitary operators:

Pauli matrices:  $\sigma_x, \sigma_y, \sigma_z$        $(\sigma_x)^\dagger \sigma_x = I$        $(\sigma_y)^\dagger \sigma_y = I$        $(\sigma_z)^\dagger \sigma_z = I$

Rotation operator:  $U = \exp(i\alpha\sigma_y) = \cos(\alpha\sigma_y) + i\sin(\alpha\sigma_y)$

$$\cos(\alpha\sigma_y) = \sum_{k=0}^{\infty} \frac{(\alpha\sigma_y)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{(2k)!} I = I \cos \alpha$$

$$\sin(\alpha\sigma_y) = \sum_{k=0}^{\infty} \frac{(\alpha\sigma_y)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{\alpha^{2k+1}}{(2k+1)!} \sigma_y = \sigma_y \sin \alpha$$

$$U = I \cos \alpha + i\sigma_y \sin \alpha$$

$$U = \exp(i\alpha\sigma_j) \rightarrow$$

$$U = I \cos \alpha + i\sigma_j \sin \alpha$$

# Examples of the unitary operator

$$U = \exp(i\alpha\sigma_y) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$U = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \quad U^\dagger = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$U^\dagger U = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Operators  $U = e^{i\alpha\sigma_j}$  are unitary

$$U = I \cos\alpha + i\sigma_j \sin\alpha \quad U^\dagger = I \cos\alpha - i\sigma_j \sin\alpha$$

$$U^\dagger U = (I \cos\alpha - i\sigma_j \sin\alpha)(I \cos\alpha + i\sigma_j \sin\alpha) = I \cos^2\alpha - ii\sigma_j \sigma_j \sin^2\alpha = I$$

+I

# Properties of the rotation operator

$$U_y = e^{i\alpha\sigma_y}$$

Commutation properties:

$$U_y \sigma_z = e^{i\alpha\sigma_y} \sigma_z = (I \cos \alpha + i \sigma_y \sin \alpha) \sigma_z = \sigma_z (I \cos \alpha - i \sigma_y \sin \alpha) = \sigma_z U_y^\dagger$$

$$U_y \sigma_x = e^{i\alpha\sigma_y} \sigma_x = (I \cos \alpha + i \sigma_y \sin \alpha) \sigma_x = \sigma_x (I \cos \alpha - i \sigma_y \sin \alpha) = \sigma_x U_y^\dagger$$

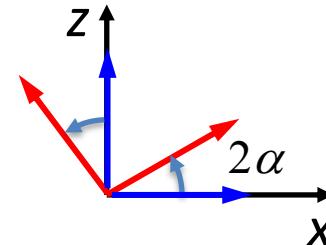
$$U_y \sigma_y = \sigma_y e^{i\alpha\sigma_y} = e^{i\alpha\sigma_y} \sigma_y = \sigma_y U_y$$

$$U_y \sigma_z U_y^\dagger = e^{i\alpha\sigma_y} \sigma_z e^{-i\alpha\sigma_y} = (I \cos 2\alpha + i \sigma_y \sin 2\alpha) \sigma_z = \sigma_z \cos 2\alpha - \sigma_x \sin 2\alpha$$

$$U_y \sigma_x U_y^\dagger = e^{i\alpha\sigma_y} \sigma_x e^{-i\alpha\sigma_y} = (I \cos 2\alpha + i \sigma_y \sin 2\alpha) \sigma_x = \sigma_x \cos 2\alpha + \sigma_z \sin 2\alpha$$

$$U_y \sigma_y U_y^\dagger = e^{i\alpha\sigma_y} \sigma_y e^{-i\alpha\sigma_y} = \sigma_y$$

$U_y = e^{i\alpha\sigma_y}$  is counter-clockwise rotation around  $y$ -axis by  $2\alpha$



# Diagonalization of the qubit Hamiltonian

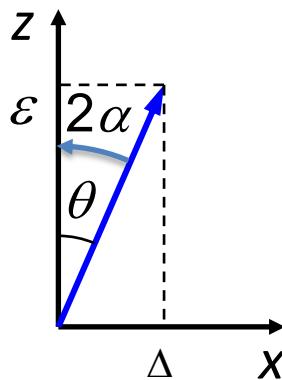
Unitary transformation:  $\Psi' = U\Psi$        $\Psi = U^\dagger \Psi'$

$$H' = UHU^\dagger$$

$$H = -\frac{\varepsilon}{2}\sigma_z - \frac{\Delta}{2}\sigma_x \quad H = -\frac{\Delta E}{2}(\sigma_z \sin \theta + \sigma_x \cos \theta) \quad \Delta E = \sqrt{\varepsilon^2 + \Delta^2}$$

Transformation from physical basis to eigenbasis:

$$U = R_y = e^{i\frac{\theta}{2}\sigma_y}$$



$$H' = UHU^\dagger = R_y R_y H = e^{i\theta\sigma_y} H = -\frac{\Delta E}{2} \sigma_z$$

$$-\Delta E : \Psi'_0 = |0\rangle \quad +\Delta E : \Psi'_1 = |1\rangle$$

$$\Psi_0 = U^\dagger \Psi'_0 = e^{-i\frac{\theta}{2}\sigma_y} |0\rangle = \left( I \cos \frac{\theta}{2} - i\sigma_y \sin \frac{\theta}{2} \right) |0\rangle = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix}$$

$$\Psi_1 = U^\dagger \Psi'_1 = e^{-i\frac{\theta}{2}\sigma_y} |1\rangle = \left( I \cos \frac{\theta}{2} - i\sigma_y \sin \frac{\theta}{2} \right) |1\rangle = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix}$$

# Time dependent unitary transformations and manipulations with qubit

# Time-dependent unitary transformations

Unitary transformation:

$$\Psi' = U\Psi \quad \Psi = U^\dagger\Psi'$$

Schrodinger equations:

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \quad i\hbar \frac{\partial \Psi'}{\partial t} = H'\Psi'$$

Schrodinger equations:

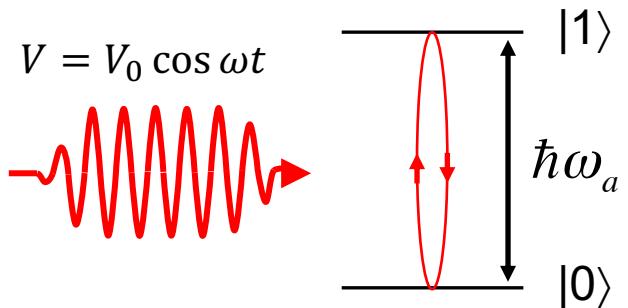
$$i\hbar \frac{\partial U^\dagger \Psi'}{\partial t} = HU^\dagger \Psi' \quad i\hbar U \frac{\partial U^\dagger}{\partial t} \Psi' + i\hbar UU^\dagger \frac{\partial \Psi'}{\partial t} = UHU^\dagger \Psi'$$

$$i\hbar \frac{\partial \Psi'}{\partial t} = (UHU^\dagger - i\hbar U\dot{U}^\dagger) \Psi'$$

Time-dependent unitary transformation:

$$H' = UHU^\dagger - i\hbar U\dot{U}^\dagger$$

# Transitions in the two-level system under resonant harmonic excitations



Atom driven by an external MW field:

$$H = -\frac{\hbar\omega_a}{2}\sigma_z + \hbar\Omega\sigma_x \cos \omega t$$

$$\hbar\Omega = V_0 C_k \frac{2e}{C_q}$$

For example (the charge qubit at the degeneracy point):

$$U(t) = e^{-i\frac{\omega t}{2}\sigma_z}$$

$$H' = UHU^\dagger - i\hbar U\dot{U}^\dagger$$

$$H' = e^{-i\frac{\omega t}{2}\sigma_z} H e^{i\frac{\omega t}{2}\sigma_z} - i\hbar e^{-i\frac{\omega t}{2}\sigma_z} \left( i\frac{\omega}{2}\sigma_z \right) e^{i\frac{\omega t}{2}\sigma_z}$$

$$-\frac{\hbar\omega_a}{2}\sigma_z + \hbar\Omega \frac{e^{i\omega t} + e^{-i\omega t}}{2} e^{-i\frac{\omega t}{2}\sigma_z} \sigma_x e^{i\frac{\omega t}{2}\sigma_z}$$

$$\frac{\hbar\omega}{2}\sigma_z$$

$$\frac{\hbar\Omega}{2}(e^{i\omega t} + e^{-i\omega t}) \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{i\omega t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar\Omega}{2}(e^{i\omega t} + e^{-i\omega t}) \begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix} =$$

$$= \frac{\hbar\Omega}{2} \begin{pmatrix} 0 & 1 \\ 1 + e^{2i\omega t} & 0 \end{pmatrix} \approx \frac{\hbar\Omega}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar\Omega}{2}\sigma_x$$

Rotating wave approximation

# Driven two-level system

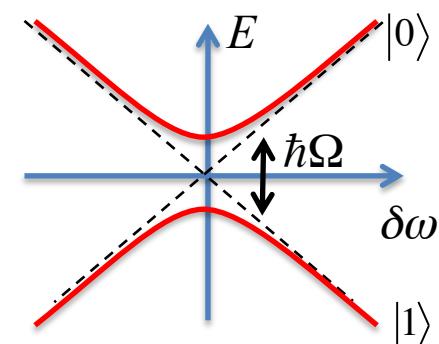
Initial time-dependent Hamiltonian: Time-dependent unitary transformation:

$$H = -\frac{\hbar\omega_a}{2}\sigma_z + \frac{\hbar\Omega}{2}\sigma_x \cos\omega_a t$$

$$U(t) = e^{-i\frac{\omega t}{2}\sigma_z}$$

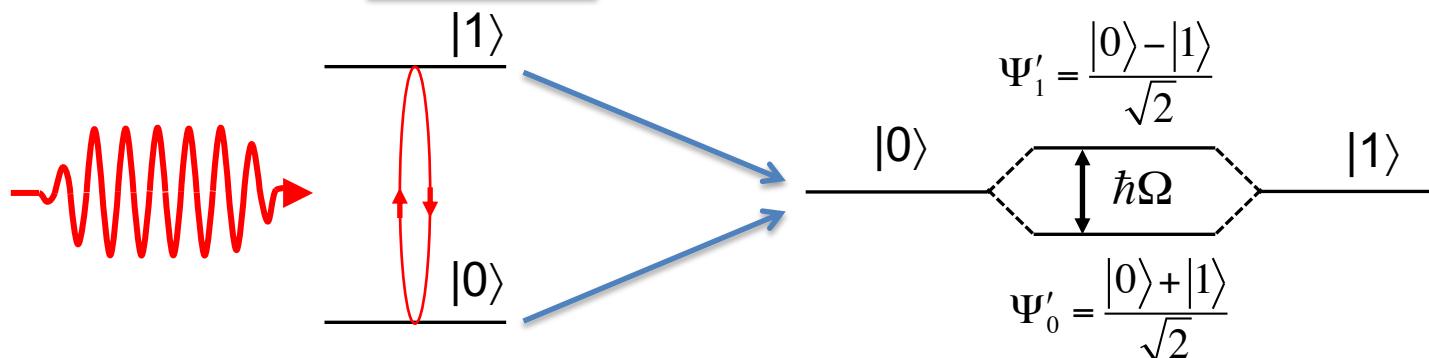
## Transformed Hamiltonian:

$$H = \frac{\hbar\delta\omega}{2}\sigma_z + \frac{\hbar\Omega}{2}\sigma_x \quad \delta\omega = \omega - \omega_a$$



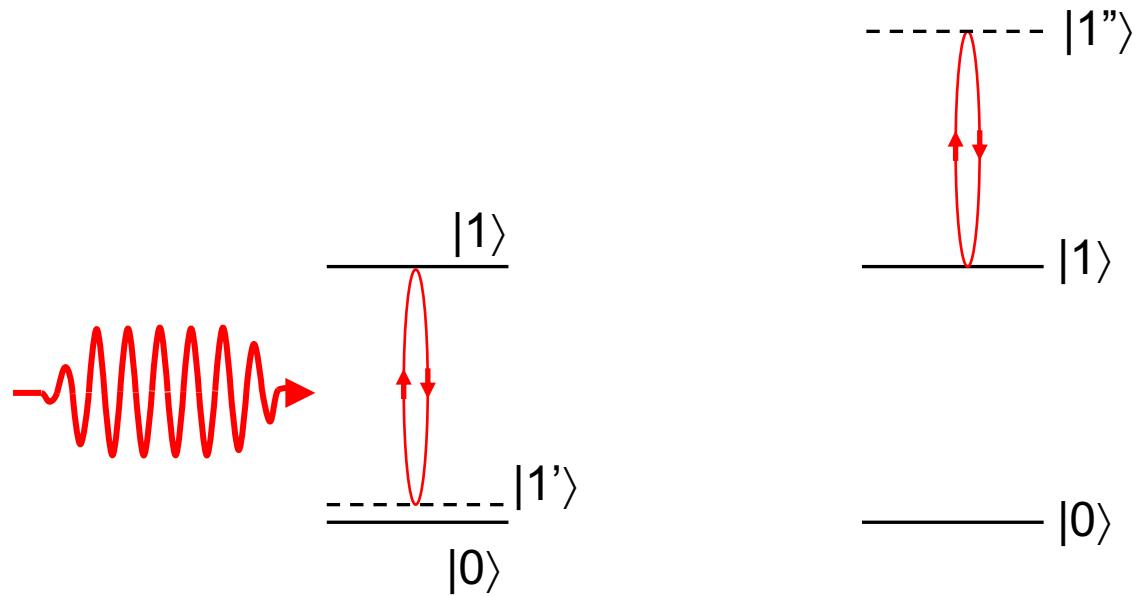
If  $\omega = \omega_a$  :

$$H = \frac{\hbar\Omega}{2} \sigma_x$$



Physical meaning of the rotating wave approximation  
is coupling of the levels via the radiation

# Physical interpretation of the fast rotating terms



# Evolution of the two-level system under the external resonant drive

Initial Hamiltonian and ground state:

$$H = -\frac{\hbar\omega}{2}\sigma_z \quad \Psi_0 = |0\rangle$$

Resonant drive:

$$H_{int} = \hbar\Omega\sigma_x \cos\omega_a t$$

Transformed Hamiltonian:

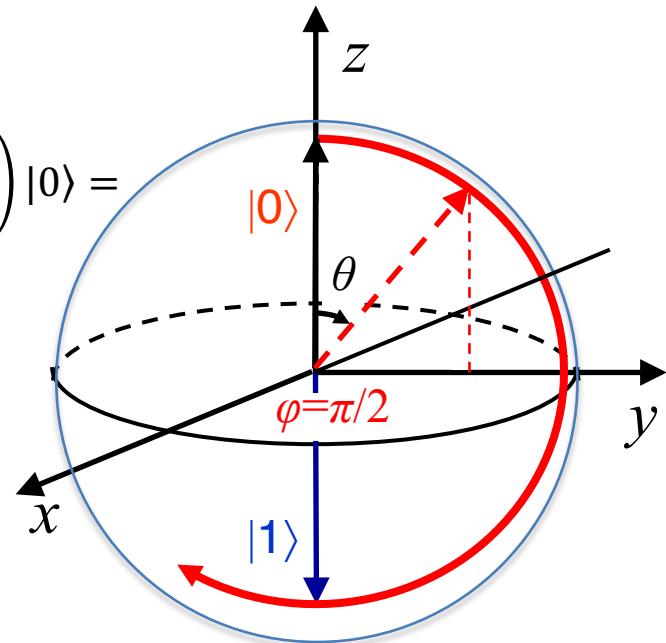
$$H' \approx -\frac{\hbar\Omega}{2}\sigma_x$$

Evolution operator:

$$U_{ev} = e^{-i\frac{H'}{\hbar}t} = e^{-i\frac{\Omega t}{\hbar}\sigma_x}$$

$$\begin{aligned} U_{ev}|0\rangle &= \left( (|0\rangle\langle 0| + |1\rangle\langle 1|) \cos\frac{\Omega t}{2} - i(|0\rangle\langle 1| + |1\rangle\langle 0|) \sin\frac{\Omega t}{2} \right) |0\rangle = \\ &= \cos\frac{\Omega t}{2} |0\rangle + e^{-i\pi/2} \sin\frac{\Omega t}{2} |1\rangle \end{aligned}$$

Rotation around  $x$ -axis:



# Evolution of the two-level system under the external resonant drive with a phase shift

Resonant drive:

$$H_{int} = \hbar\Omega\sigma_x \cos\left(\omega t - \frac{\pi}{2}\right) = \hbar\Omega\sigma_x \sin\omega t$$

$$\frac{\hbar\Omega}{2}(-ie^{i\omega t} + ie^{-i\omega t}) \begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix} = \frac{\hbar\Omega}{2} \begin{pmatrix} 0 & -i + ie^{-2i\omega t} \\ i - ie^{2i\omega t} & 0 \end{pmatrix} \approx \frac{\hbar\Omega}{2}\sigma_y$$

This is the same as  $H_{int} = \hbar\Omega\sigma_y \cos\omega t$

$$UH_{int}U^\dagger = \frac{\hbar\Omega}{2}(e^{i\omega t} + e^{-i\omega t})e^{-i\omega t\sigma_z}\sigma_y = \frac{\hbar\Omega}{2}(e^{i\omega t} + e^{-i\omega t}) \begin{pmatrix} 0 & -ie^{-i\omega t} \\ ie^{i\omega t} & 0 \end{pmatrix} = \frac{\hbar\Omega}{2} \begin{pmatrix} 0 & -i + ie^{-2i\omega t} \\ i - ie^{2i\omega t} & 0 \end{pmatrix} \approx \frac{\hbar\Omega}{2}\sigma_y$$

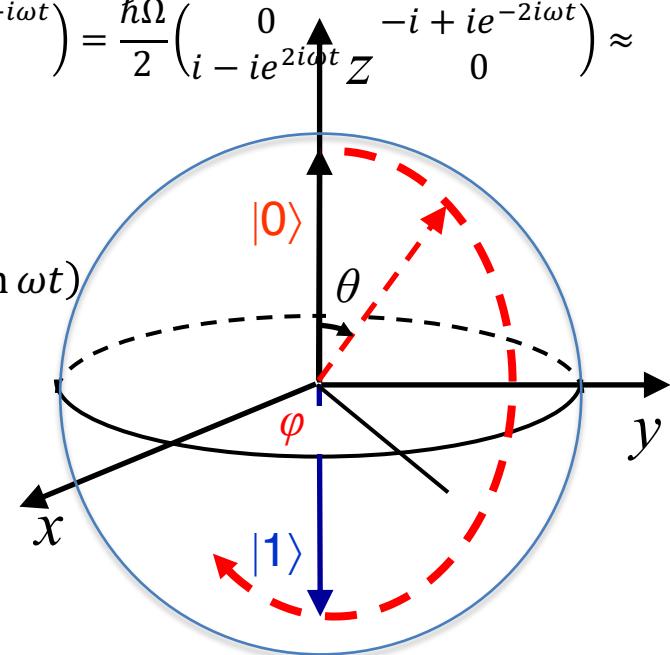
An arbitrary phase shift in the incident wave

$$H_{int} = \hbar\Omega\sigma_x \cos(\omega t - \varphi) = \hbar\Omega\sigma_x(\cos\varphi \cos\omega t + \sin\varphi \sin\omega t)$$

is equivalent to

$$H'_{int} = \frac{\hbar\Omega}{2}(\sigma_x \cos\varphi + \sigma_y \sin\varphi)$$

It describes rotation around axis in  $x$ - $y$  plane at angle  $\varphi$



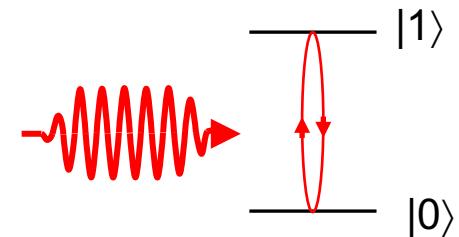
# Manipulation with single qubit states

Two-level system in the eigenbasis (diagonal Hamiltonian):

$$H = \begin{pmatrix} -\hbar\omega_0/2 & 0 \\ 0 & \hbar\omega_0/2 \end{pmatrix} = -\frac{\hbar\omega_0}{2}\sigma_z = -\frac{\hbar\omega_0}{2}|0\rangle\langle 0| + \frac{\hbar\omega_0}{2}|1\rangle\langle 1|$$

By applying external field we transform the Hamiltonian:

$$i\hbar \frac{\partial \psi(t)}{\partial t} = \hat{H}\psi(t) \quad \psi(t) = \exp\left(-i\frac{H}{\hbar}t\right)\psi(0)$$



x-rotation:

$$H = \frac{\hbar\Omega}{2}\sigma_x \quad \psi(t) = \exp\left(-i\frac{\Omega t}{2}\sigma_x\right)\psi(0) \quad R_x(t) = \exp\left(-i\frac{\Omega t}{2}\sigma_x\right)$$

y-rotation:

$$H = \frac{\hbar\Omega}{2}\sigma_y \quad \psi(t) = \exp\left(-i\frac{\Omega t}{2}\sigma_y\right)\psi(0) \quad R_y(t) = \exp\left(-i\frac{\Omega t}{2}\sigma_y\right)$$

By applying two sequential pulses with  $\frac{\pi}{2}$  shifted phases, we can prepare any arbitrary state