Superconducting Quantum Technologies

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Lecture 2

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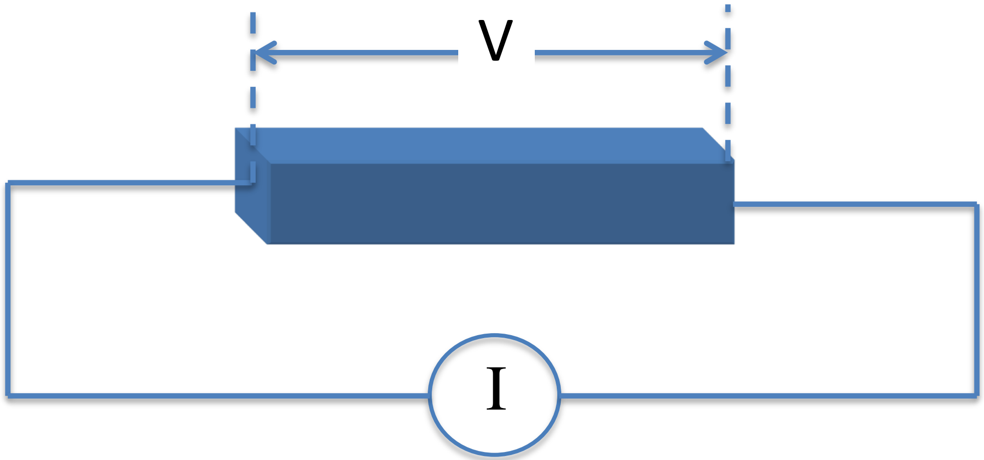
- Superconductivity. Josephson junction. Cooper pair tunneling.
- Charging energy.
- The Cooper pair box. Electrostatic energies.
- Quantum mechanics of electric circuits.
- The Hamiltonian of the Cooper pair box.
- The Charge qubit.

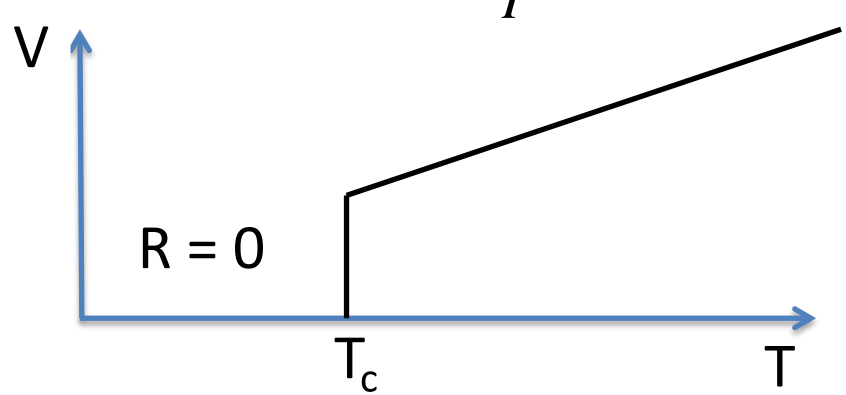
Superconductivity
Josephson junctions

Superconductivity

At T < T_c resistance abruptly vanishes ($R \rightarrow 0$)

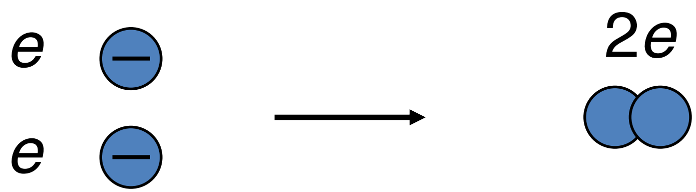


$$R = \frac{V}{I}$$



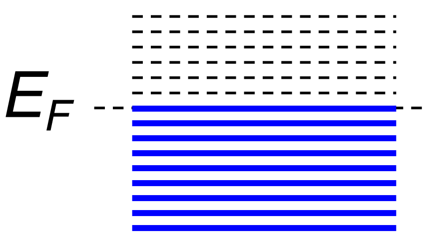
Electron spectrum

Electrons form Cooper pairs due to phonon interaction



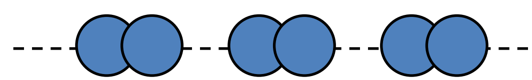
Electron spectrum in normal metals at T = 0:

Conduction band



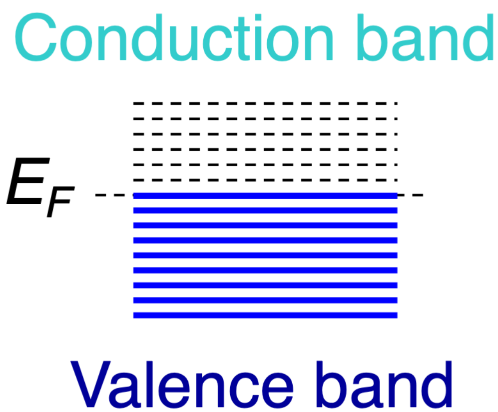
Valence band

Cooper pairs are Bosons:
All Cooper pairs take the same energy level, forming condensate

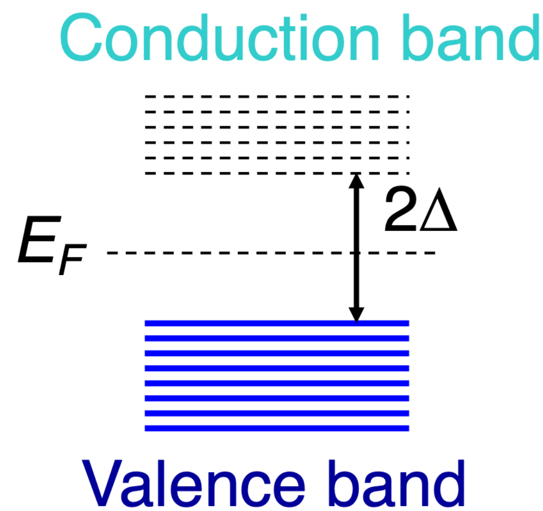


Cooper pairs

Electron spectrum in normal metals at $T = 0$:



Electron spectrum in superconductors at $T \ll T_c$:



No normal current flows through the superconductors at $T=0$

Superconducting gap: 2Δ

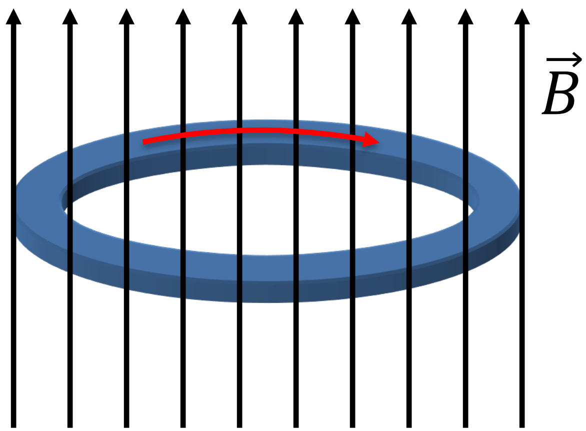
Superconducting phase and flux quantization

Voltage is zero in stationary (dc) transport $V \to \varphi$

$$\Phi_{ext} = BS$$

$$\boxed{V = \dot{\Phi}}$$

$$\frac{\varphi_{ext}}{2\pi} = \frac{\Phi_{ext}}{\Phi_0}$$



$$\Psi = |\Psi_0| e^{i\varphi(x)}$$

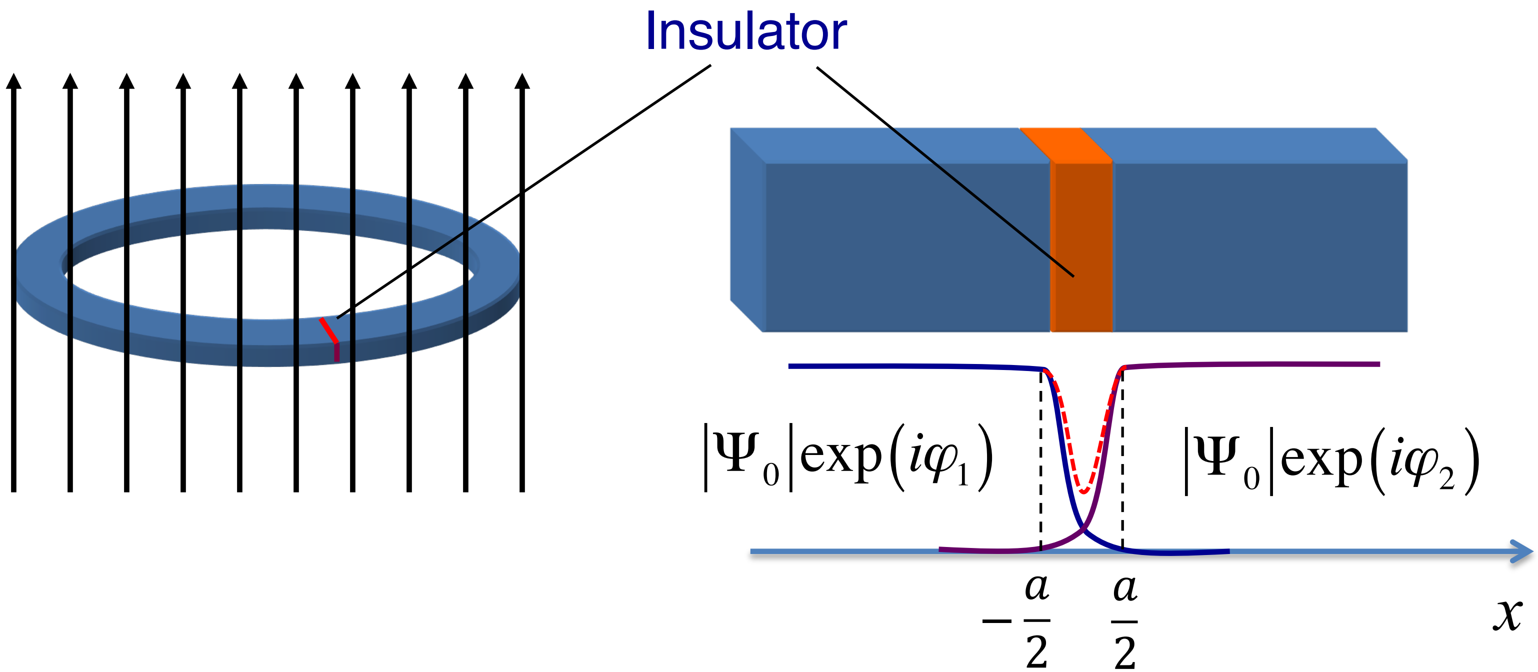
$$\varphi = \varphi_{ext} + 2\pi N$$

$$\Phi = \Phi_{ext} + \Phi_0 \mathrm{N}$$

Flux quantum:

$$\Phi_0 = \frac{h}{2e} = 2 \times 10^{-15}\ \mathrm{Wb}$$

Josephson junction



$$U\Psi - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi = 0 \qquad \Psi = A\cosh(\kappa x) + B\sinh(\kappa x) \qquad \kappa = \frac{\sqrt{2mU}}{\hbar}$$

$$\Psi\left(-\frac{a}{2}\right) = A\cosh\left(\frac{\kappa a}{2}\right) + B\sinh\left(-\frac{\kappa a}{2}\right) \equiv |\Psi_0| e^{i\varphi_1} \qquad \Psi\left(\frac{a}{2}\right) = A\cosh\left(\frac{\kappa a}{2}\right) + B\sinh\left(\frac{\kappa a}{2}\right) \equiv |\Psi_0| e^{i\varphi_2}$$

$$A\cosh\left(\frac{\kappa a}{2}\right) \equiv |\Psi_0| \frac{e^{i\varphi_2} + e^{i\varphi_1}}{2} \qquad B\sinh\left(\frac{\kappa a}{2}\right) \equiv |\Psi_0| \frac{e^{i\varphi_2} - e^{i\varphi_1}}{2}$$

Josephson junction: current-phase relation

$$\Psi = A\cosh(\kappa x) + B\sinh(\kappa x)$$

$$A \equiv |\Psi_0| \frac{e^{i\varphi_2} + e^{i\varphi_1}}{2} \cosh^{-1}\left(\frac{\kappa a}{2}\right)$$

$$B \equiv |\Psi_0| \frac{e^{i\varphi_2} - e^{i\varphi_1}}{2} \sinh^{-1}\left(\frac{\kappa a}{2}\right)$$

Supercurrent: $I = -i2e\frac{\hbar}{2m}\left(\Psi^* \frac{\partial}{\partial x}\Psi - \Psi\frac{\partial}{\partial x}\Psi^*\right)$

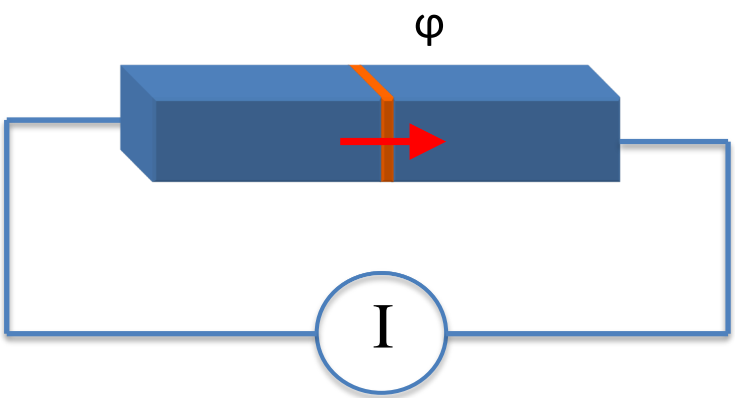
$$I = \frac{2e\hbar}{m}\mathrm{Im}\left(\Psi^*\frac{\partial}{\partial x}\Psi\right)$$

At $x = 0$: $\Psi = A$ $\quad \frac{\partial\Psi}{\partial x} = \kappa B$ $\quad I = \frac{2e\hbar\kappa}{m}\mathrm{Im}\left(A^* B\right) = \frac{2e\hbar\kappa}{m}\sinh^{-1}(\kappa a)e^{i(\varphi_1+\varphi_2)}\sin(\varphi_2 - \varphi_1)$

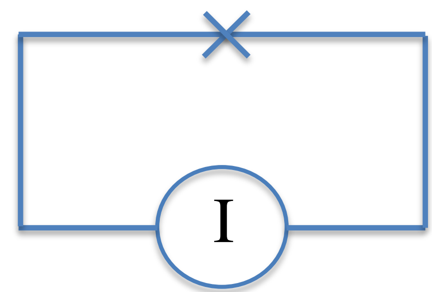
$$\boxed{I = I_c \sin(\varphi_2 - \varphi_1)}$$

Josephson energy

Current-phase relationship for JJ: $I = I_c \sin\varphi$



$$V = \dot{\Phi} = \frac{\Phi_0}{2\pi}\dot{\varphi}$$



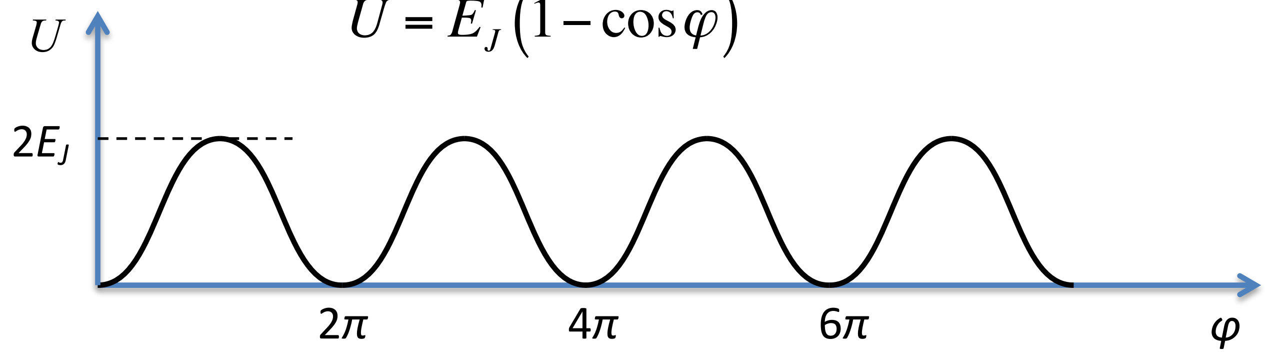
Energy stored in the junction:

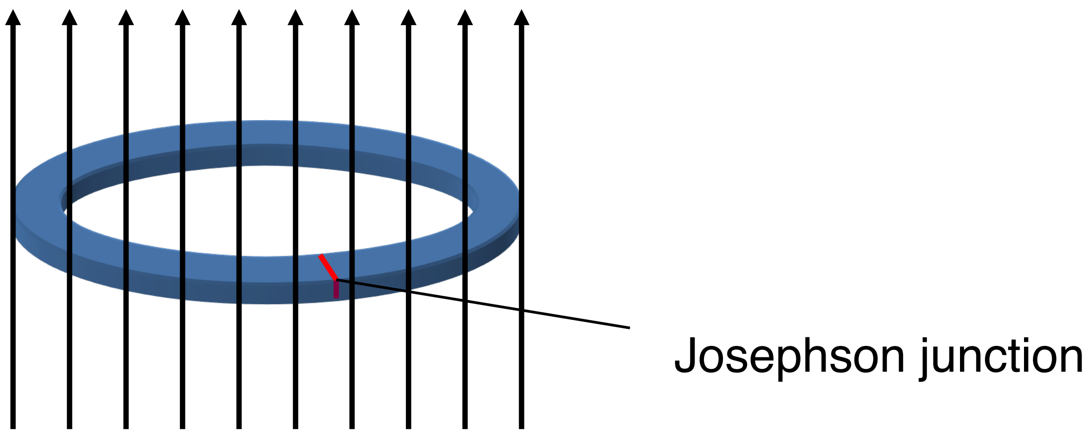
$$U = \int_0^t IV\,dt = \int_0^t I_c \sin\varphi \frac{\Phi_0}{2\pi}\dot{\varphi}\,dt = I_c \frac{\Phi_0}{2\pi}\int_0^{\varphi} \sin\varphi\,d\varphi = \frac{\Phi_0 I_c}{2\pi}\left(1-\cos\varphi\right)$$

Josephson energy: $E_J = I_c \dfrac{\Phi_0}{2\pi}$

Washboard potential

$$U = E_J\left(1-\cos\varphi\right)$$





The system takes one of the minima with phase $\varphi = 2\pi\, N \Rightarrow N\Phi_0$

The Copper pair box

A sphere in a dielectric

Charging energy (energy of an island with one electron charge): $E_c = \frac{e^2}{2C}$

Thermal fluctuations: $k_B T$

Capacitance of a sphere in a dielectric (ε = 10) (no metallic leads): $C = 4\pi\varepsilon_0\varepsilon a$

1. Calculate E_c of a sphere with radius a = 1 μm $\varepsilon\varepsilon_0 = 10^{-10}$F/m $e = 1.6\times10^{-19}$C

2. Calculate typical thermal fluctuation at T = 4K $k_B = 1.38\times10^{-23}$J/K

3. Compare E_c and $k_B T$

A sphere in a dielectric

Charging energy (energy of an island with one electron charge): $E_c = \dfrac{e^2}{2C}$

Thermal fluctuations: $k_B T$

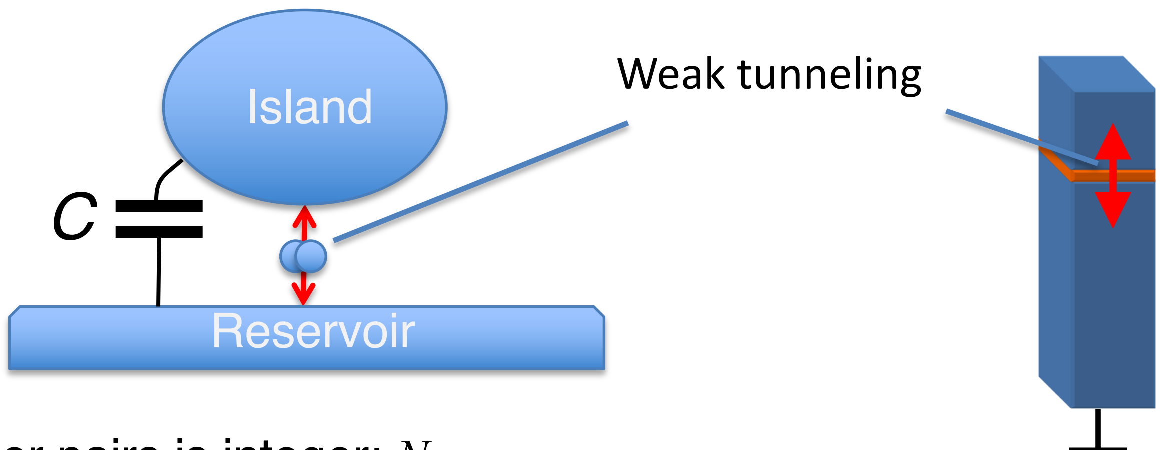
Capacitance of a sphere in a dielectric (ε = 10) (no metallic leads): $C = 4\pi\varepsilon_0\varepsilon a$

Size	Temperature	Capacitance	Charging Energy	Thermal fluctuations	Ec /kT
1 mm	300 K	1.1×10^{-12} F	2.3×10^{-26} J	4.1×10^{-21} J	5×10^{-6}
1 µm	1 K	1.1×10^{-15} F	2.3×10^{-23} J	1.4×10^{-23} J	1.6
100 nm	0.1 K	1.1×10^{-16} F	2.3×10^{-22} J	1.4×10^{-24} J	160

To reach the quantum regime ($E_C >> k_B T$) with quantized charges nanostructures (nanofabriaction) and millikelvin temperatures are required

Cooper pair box

A superconducting island with capacitance *C*



Number of Cooper pairs is integer: N
We can add or remove a Cooper pair $2e$

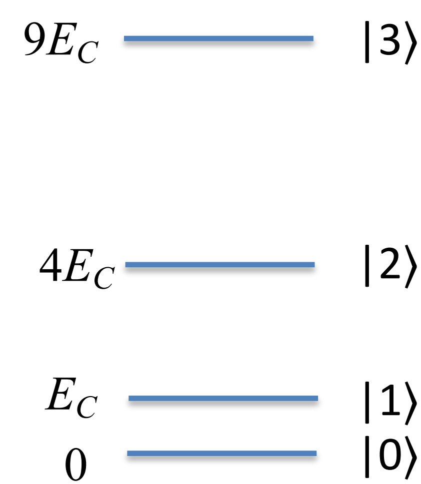
Total energy of the island: $$U = \frac{Q^2}{2C} = \frac{(N2e)^2}{2C}$$

N is a number of excess Cooper pairs

$$U = \frac{(2e)^2}{2C} N^2 = \boxed{E_C N^2}$$

Charging energy:

$$\boxed{E_C = \frac{(2e)^2}{2C}}$$



Number operator for particles/charges

$$\hat{N} = \sum_{0}^{\infty} N|N\rangle\langle N|$$

Compact form:

$$\hat{N} = N|N\rangle\langle N|$$

Matrix form:

$$\hat{N} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

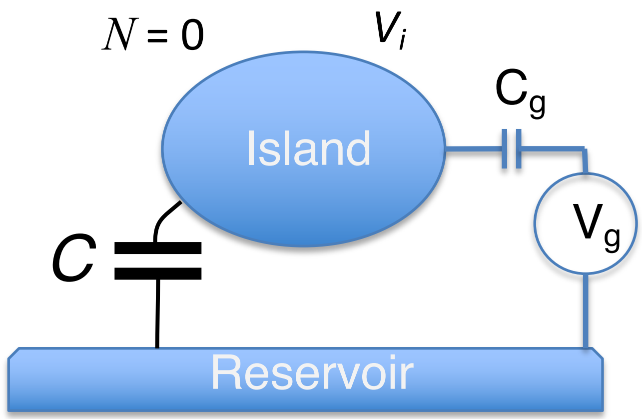
Operator of electrostatic energy
in the charge basis:

$$\hat{U} = E_c \hat{N}^2 = E_c N^2 |N\rangle\langle N|$$

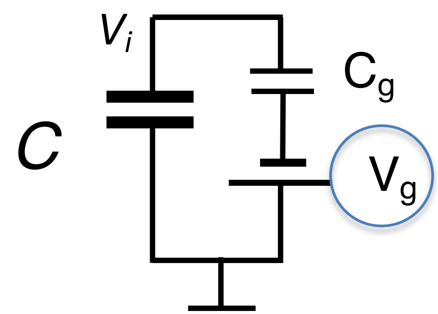
$$H = \hat{U}$$

$$\hat{U} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & E_c & 0 & 0 & \cdots \\ 0 & 0 & 4E_c & 0 & \cdots \\ 0 & 0 & 0 & 9E_c & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Cooper pair box



Gate voltage



Usually $C_g << C$

$$V_i \approx V_g \frac{C_g}{C}$$

Electrostatic Energy:

$$U \approx \frac{CV_i^2}{2} = \frac{\left(C_g V_g\right)^2}{2C} = \frac{(2e)^2}{2C}\left(\frac{C_g V_g}{2e}\right)^2 = E_c n^2$$

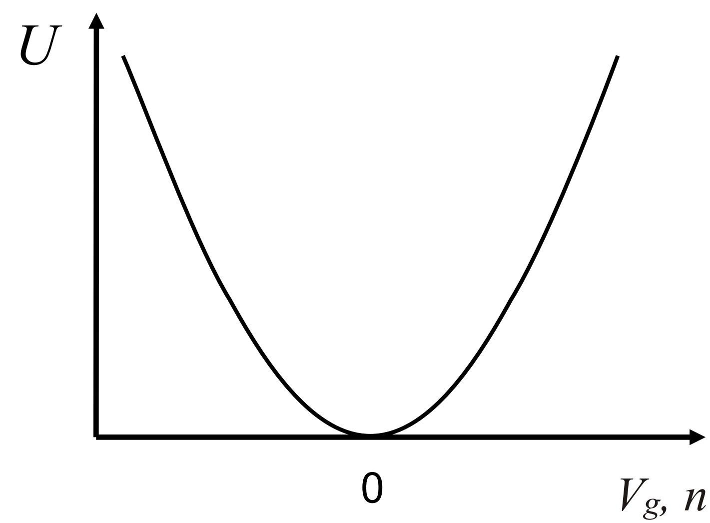
Cooper pair charging energy

$$E_c = \frac{(2e)^2}{2C}$$

Gate induced charge:

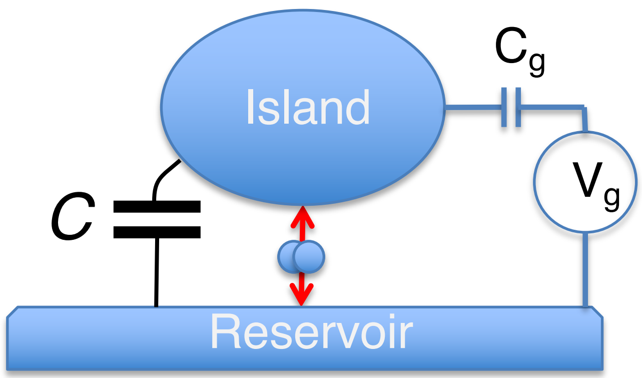
$$n = \frac{C_g V_g}{2e}$$

$$\boxed{U = E_C n^2}$$



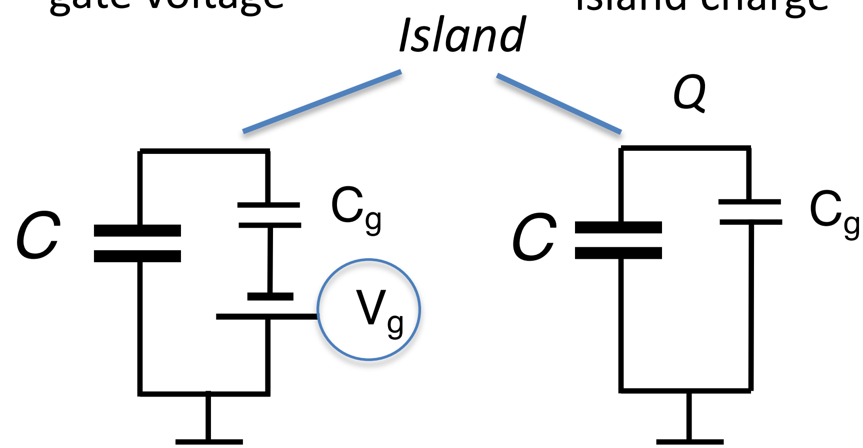
Cooper pair box

Any integer N



Effect of gate voltage

Effect of island charge



Electrostatic Energy:

$$U = \frac{C\left(V_{ig} + V_{iq}\right)^2}{2} = E_c(n - N)^2$$

$$V_{iq} = \frac{V_g C_g}{C} = \frac{Q_g}{C} = \frac{2en}{C} \qquad V_{iQ} = \frac{Q}{C} = \frac{-2eN}{C}$$

$$n = \frac{C_g V_g}{2e} \qquad E_C = \frac{(2e)^2}{2C_{tot}}$$

$$V_i = V_{ig} + V_{iq} = \frac{2e}{C}(n - N)$$

$$\boxed{U = E_C\left(N - n\right)^2}$$

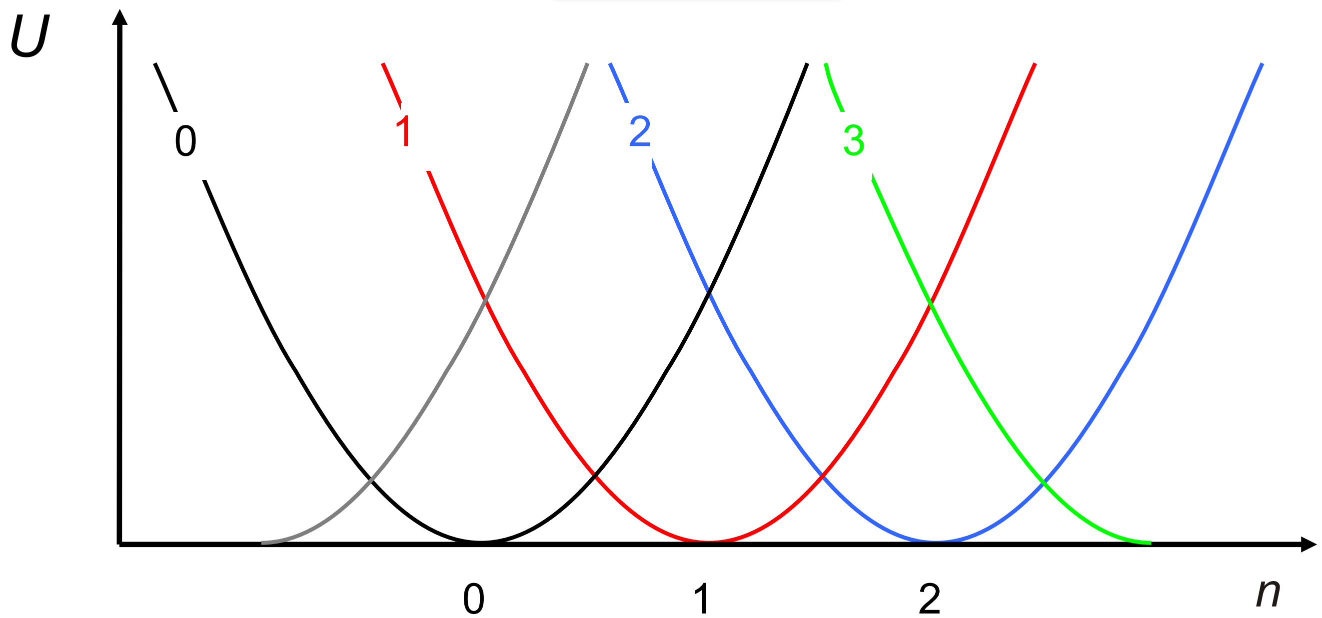
n is continuous variable $\sim V_g$

Electrostatic energies of the Cooper-pair box

$$U = E_C \left(N - n\right)^2$$

Charging energy: $\boxed{E_C = \dfrac{(2e)^2}{2C}}$

Gate induced charge: $n = \dfrac{C_g V}{2e}$
continuous variable,
controlled by the gate voltage



A Hamiltonian of an isolated island (without tunneling)

$$U = E_C\left(N-n\right)^2$$

$$H = E_c\left(\widehat{N}-n\right)^2$$

Without tunneling energy kinetic energy $T = 0$ and the Hamiltonian $\boxed{H = U}$

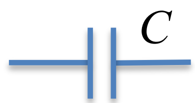
In the charge basis the Hamiltonian has diagonal form $H = E_C\left(N-n\right)^2 |N\rangle\langle N|$

$$H = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & E_C\left(-2-n\right)^2 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & E_C\left(-1-n\right)^2 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & E_C n^2 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & E_C\left(1-n\right)^2 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & E_C\left(2-n\right)^2 & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Quantum mechanics of an electric circuit

Charge ⇔ Flux

Capacitance



Voltage: V

$$V = \dot{\Phi}$$

Charge: Q

$$Q = CV = C\dot{\Phi}$$

$$Q = 2eN$$

Inductance

Current: I

$$I = \dot{Q}$$

Magnetic Flux: Φ

$$\Phi = LI = L\dot{Q}$$

$$\Phi = \frac{\Phi_0}{2\pi}\varphi \qquad \Phi_0 = \frac{h}{2e}$$

Quantum mechanics of an electric circuit

Quantum mechanics

$$x \qquad p = m\dot{x}$$

Commutation relations: $\left[\hat{x}, \hat{p}\right] = i\hbar$

Kinetic energy: $T = \dfrac{m\dot{x}^2}{2}$

Two representations of the electric circuit

Electric circuit: charge and charge motion

$$Q \qquad \Phi = L\dot{Q}$$

$$\left[\hat{Q}, \hat{\Phi}\right] = i\hbar$$

$$T = \frac{L\dot{Q}^2}{2}$$

$$Q \leftrightarrow x \quad \Phi \leftrightarrow p \quad L \leftrightarrow m$$

Electric circuit: flux and flux motion

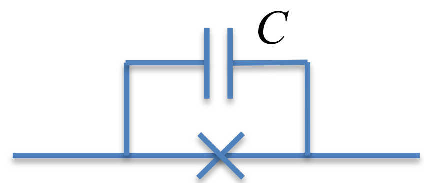
$$\Phi \qquad Q = C\dot{\Phi}$$

$$\left[\hat{\Phi}, \hat{Q}\right] = i\hbar$$

$$T = \frac{C\dot{\Phi}^2}{2}$$

$$\Phi \leftrightarrow x \quad Q \leftrightarrow p \quad C \leftrightarrow m$$

The Cooper-pair box: Josephson junction with capacitance



It is convenient to use the second approach: Electric circuit with flux and flux motion

$$\Phi = \frac{\Phi_0}{2\pi}\varphi \qquad Q = 2eN \qquad \left[\hat{\Phi},\hat{Q}\right] = i\hbar \;\rightarrow\; \left[\frac{\Phi_0}{2\pi}\hat{\varphi}, 2e\hat{N}\right] = i\hbar \;\rightarrow\; \boxed{\left[\hat{\varphi},\hat{N}\right] = i}$$

Potential energy:

$$U = E_J\left(1-\cos\varphi\right)$$

Kinetic energy:

$$T = \frac{C\dot{\Phi}^2}{2} \equiv \frac{Q^2}{2C}$$

$$H = U + T$$

$$H = E_J\left(1-\cos\varphi\right) + \frac{Q^2}{2C} \qquad H = E_J\left(1-\cos\varphi\right) + \frac{4e^2}{2C}\left(N-n\right)^2$$

$$\boxed{H = E_J\left(1-\cos\hat{\varphi}\right) + E_C\left(\hat{N}-n\right)^2}$$

It has a diagonal form in a charge basis $|N\rangle$ as it was shown earlier

Important formulas

$$H = T + U$$

$$H = \frac{Q^2}{2C} + E_J\left(1 - \cos\varphi\right)$$

$$\left[\hat{\varphi}, \hat{N}\right] = i$$

$$p = -i\hbar\frac{\partial}{\partial x}$$

$$\hat{N} = -i\frac{\partial}{\partial \varphi}$$

This form of the charge operator can be used, when the charge is not quantised

$$H = \frac{\left(2e\hat{N} - C_g V\right)^2}{2C} + E_J\left(1 - \cos\hat{\varphi}\right)$$

$$\boxed{H = E_C\left(\hat{N} - n\right)^2 - E_J\cos\hat{\varphi}}$$

The Charging energy: $\boxed{E_C = \frac{4e^2}{2C}}$

$$n = \frac{C_g V}{2e}$$

Operator of cosine of phase in the charge basis

$$\boxed{\cos\hat{\varphi}}$$

We first derive commutation relations for $\hat{\varphi}^k$ and $\hat{N}$:

$$\left[\hat{\varphi},\hat{N}\right]=i \quad \rightarrow \quad \hat{\varphi}\hat{N}-\hat{N}\hat{\varphi}=i$$

$$\varphi^k N=\varphi^{k-1}(N\varphi+i)=\varphi^{k-1}N\varphi+i\varphi^{k-1}+\varphi^{k-2}(N\varphi+i)\varphi+i\varphi^{k-1}=\varphi^{k-2}N\varphi^2+2i\varphi^{k-1}=$$

$$=\cdots=N\varphi^k+ki\varphi^{k-1} \quad \Rightarrow \quad [\varphi^k,N]=ki\varphi^{k-1}$$

$$e^{i\varphi}N=\sum_{k=0}^{\infty}\frac{(i\varphi)^k}{k!}N=\sum_{k=0}^{\infty}N\frac{(i\varphi)^k}{k!}+i\sum_{k=1}^{\infty}\frac{ik(i\varphi)^{k-1}}{k!}=N\sum_{k=0}^{\infty}\frac{(i\varphi)^k}{k!}-\sum_{k=1}^{\infty}\frac{(i\varphi)^{k-1}}{(k-1)!}=Ne^{i\varphi}-e^{i\varphi}$$

$$e^{i\hat{\varphi}}\hat{N}=\left(\hat{N}-1\right)e^{i\hat{\varphi}} \qquad \boxed{e^{i\hat{\varphi}}=|N\rangle\langle N-1|} \qquad \boxed{\hat{N}=N|N\rangle\langle N|}$$

$$\left(\hat{N}-1\right)e^{i\hat{\varphi}}=(N-1)|N\rangle\langle N|N\rangle\langle N-1|=(N-1)|N\rangle\langle N-1|$$

$$e^{i\hat{\varphi}}\hat{N}=|N\rangle\langle N-1|(N-1)|N-1\rangle\langle N-1|=(N-1)|N\rangle\langle N-1|$$

$$\boxed{\cos\hat{\varphi}=\frac{e^{i\hat{\varphi}}+e^{-i\hat{\varphi}}}{2}=\frac{1}{2}\left(|N\rangle\langle N-1|+|N-1\rangle\langle N|\right)}$$

$$\cos\hat{\varphi}=\frac{1}{2}\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 1 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

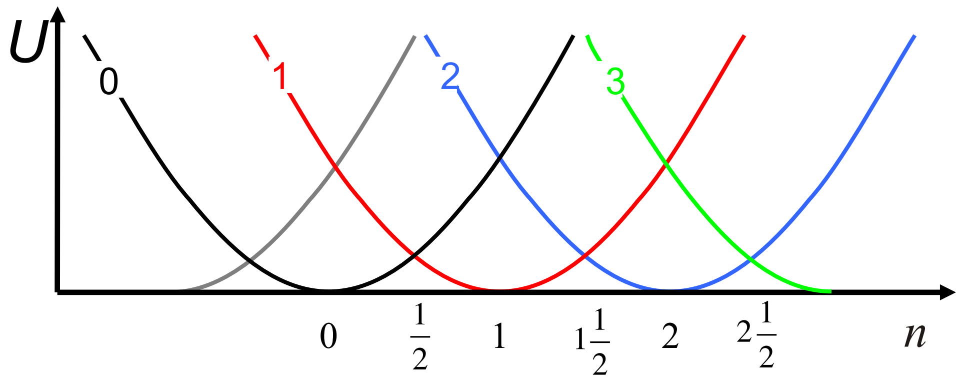
The Cooper-pair box Hamiltonian in the charge basis

$$H = E_C\left(N-n\right)^2|N\rangle\langle N| - \frac{1}{2}E_J\left(|N-1\rangle\langle N| + |N\rangle\langle N-1|\right)$$

$$H = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & E_C\left(-2-n\right)^2 & -\frac{1}{2}E_J & 0 & 0 & 0 & \cdots \\ \cdots & -\frac{1}{2}E_J & E_C\left(-1-n\right)^2 & -\frac{1}{2}E_J & 0 & 0 & \cdots \\ \cdots & 0 & -\frac{1}{2}E_J & E_C n^2 & -\frac{1}{2}E_J & 0 & \cdots \\ \cdots & 0 & 0 & -\frac{1}{2}E_J & E_C\left(1-n\right)^2 & -\frac{1}{2}E_J & \cdots \\ \cdots & 0 & 0 & 0 & -\frac{1}{2}E_J & E_C\left(2-n\right)^2 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Tunneling energy

$$U\left(N, N_{ext}\right) = E_C\left(N - N_{ext}\right)^2$$

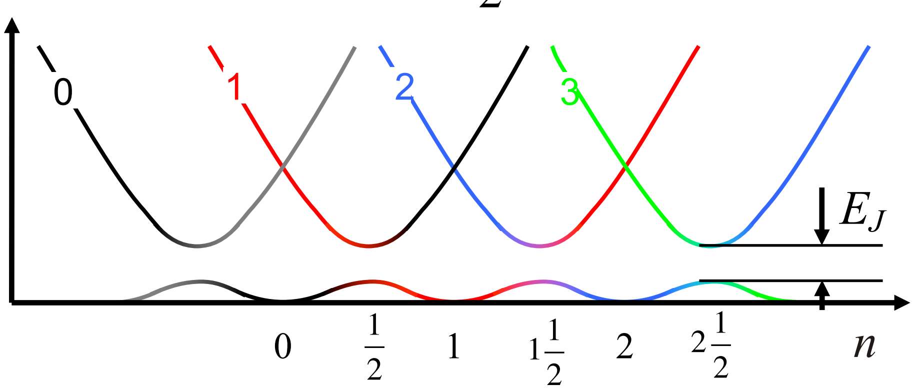


$$H = E_C\left(N - n\right)^2 \left|N\right\rangle\left\langle N\right|$$

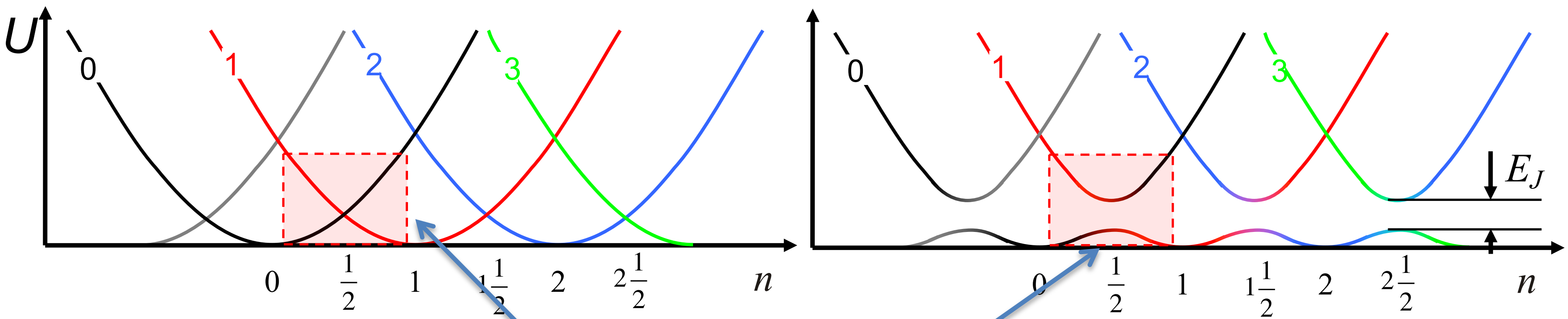
Degeneracy point: $U\left(N, n\right) = U\left(N+1, n\right)$

$$E_C\left(N - n\right)^2 = E_C\left(N + 1 - n\right)^2 \qquad 2\left(N - n\right) + 1 = 0 \qquad n = N + \frac{1}{2}$$

$$H = E_C\left(N - n\right)^2 \left|N\right\rangle\left\langle N\right| - \frac{1}{2}E_J\left(\left|N\right\rangle\left\langle N-1\right| + \left|N-1\right\rangle\left\langle N\right|\right)$$



The superconducting charge qubit



Two-level approximation

$$H=\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & E_C(-2-n)^2 & -\frac{1}{2}E_J & 0 & 0 & 0 & \cdots \\
\cdots & -\frac{1}{2}E_J & E_C(-1-n)^2 & -\frac{1}{2}E_J & 0 & 0 & \cdots \\
\cdots & 0 & -\frac{1}{2}E_J & E_C n^2 & -\frac{1}{2}E_J & 0 & \cdots \\
\cdots & 0 & 0 & -\frac{1}{2}E_J & E_C(1-n)^2 & -\frac{1}{2}E_J & \cdots \\
\cdots & 0 & 0 & 0 & -\frac{1}{2}E_J & E_C(2-n)^2 & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

The superconducting charge qubit Hamiltonian (symmetric form)

$$H_q = \begin{pmatrix} E_C n^2 & -\frac{1}{2}E_J \\ -\frac{1}{2}E_J & E_C(1-n)^2 \end{pmatrix}$$

$$H_q = E_C n^2 |0\rangle\langle 0| + E_C(1-n)^2 |1\rangle\langle 1| - \frac{E_J}{2}\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)$$

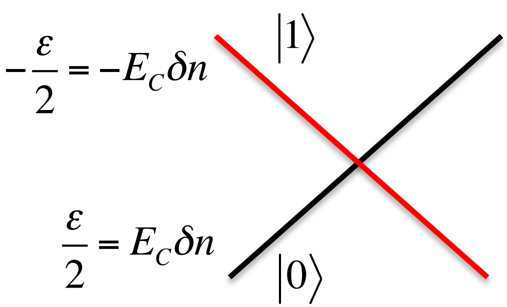
$\varepsilon = E_C(1-n)^2 - E_C n^2 = \boxed{-2E_C\delta n}$, where $\delta n = n - \frac{1}{2}$ is the normalized charge measured from the degeneracy point

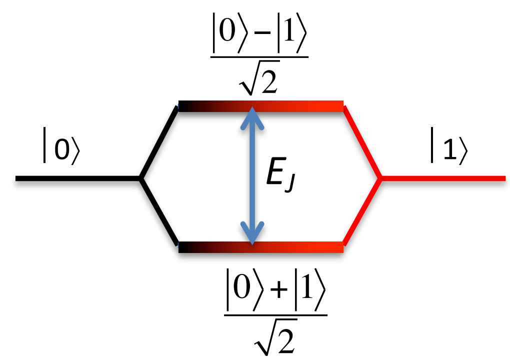
$$H_q = -\frac{\varepsilon}{2}|0\rangle\langle 0| + \frac{\varepsilon}{2}|1\rangle\langle 1| - \frac{E_J}{2}\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)$$

$$H_q = -\frac{\varepsilon}{2}\sigma_z - \frac{E_J}{2}\sigma_x$$

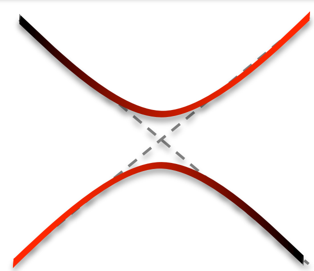
$$H_q = -\frac{\varepsilon}{2}\sigma_z$$

$$H_q = -\frac{E_J}{2}\sigma_x$$

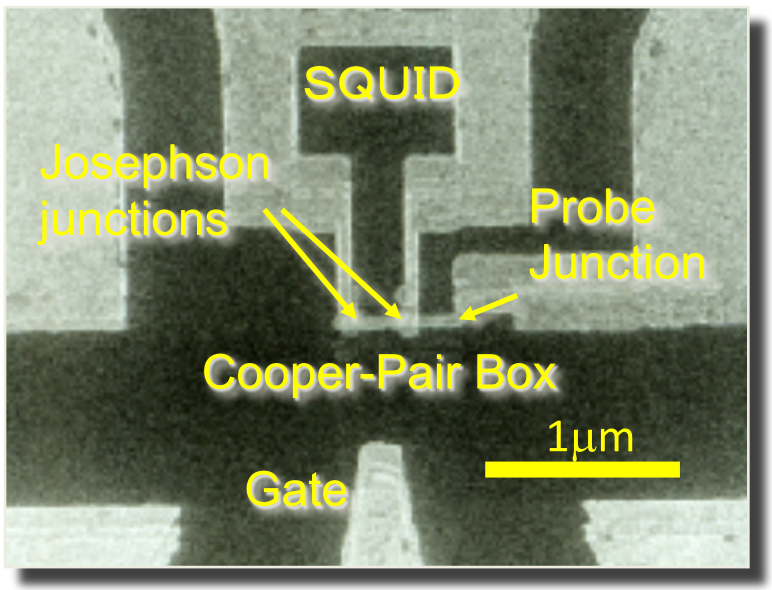


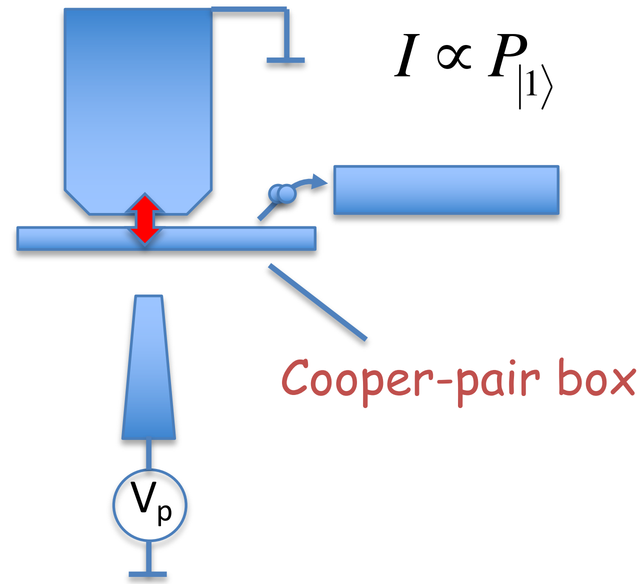


$$\boxed{H_q = -\frac{\varepsilon}{2}\sigma_z - \frac{E_J}{2}\sigma_x}$$

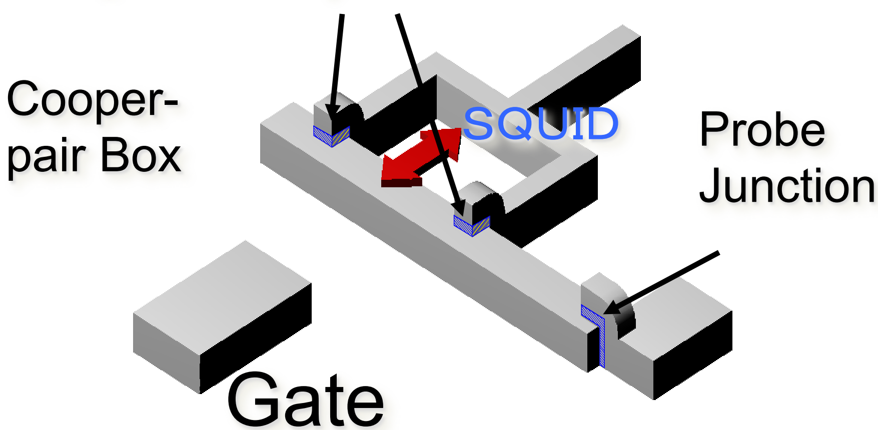


First qubit demonstration

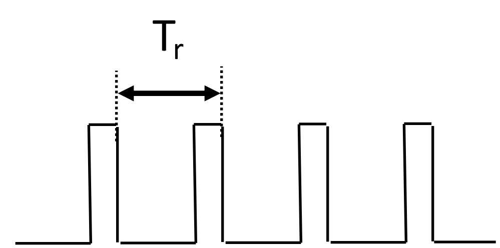




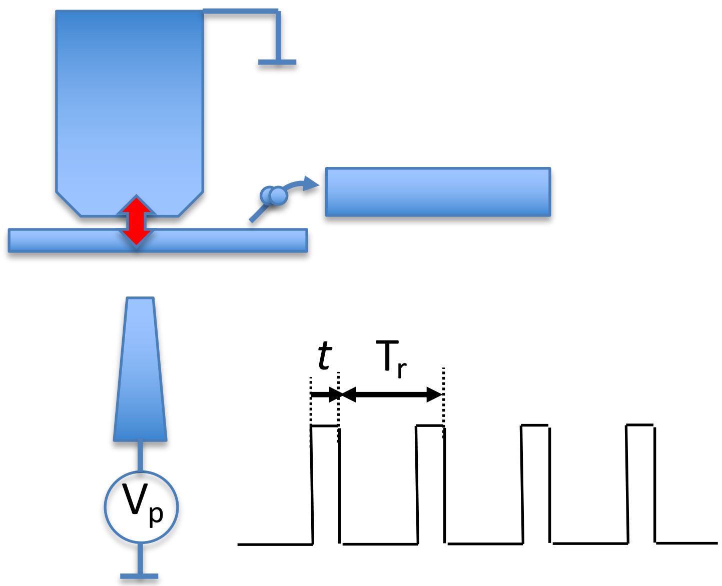
Al/AlO$_x$/Al tunnel junctions



Control-pulse sequence



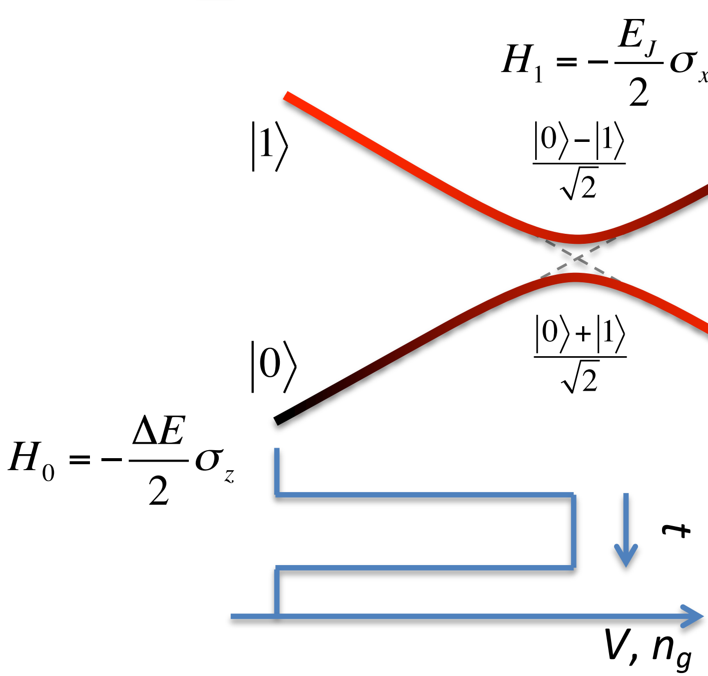
First qubit demonstration



$$\Psi(t) = \exp\left(-i\frac{H}{\hbar}t\right)\Psi(0)$$

$$\Psi(t) = \exp\left(i\frac{E_J}{2\hbar}\sigma_x t\right)|0\rangle$$

$$\Psi(t) = \cos\left(\frac{E_J}{2\hbar}t\right)|0\rangle + i\sin\left(\frac{E_J}{2\hbar}t\right)|1\rangle$$



$$I \propto P_{|1\rangle}$$