

Superconducting Quantum Technologies

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Lecture 2

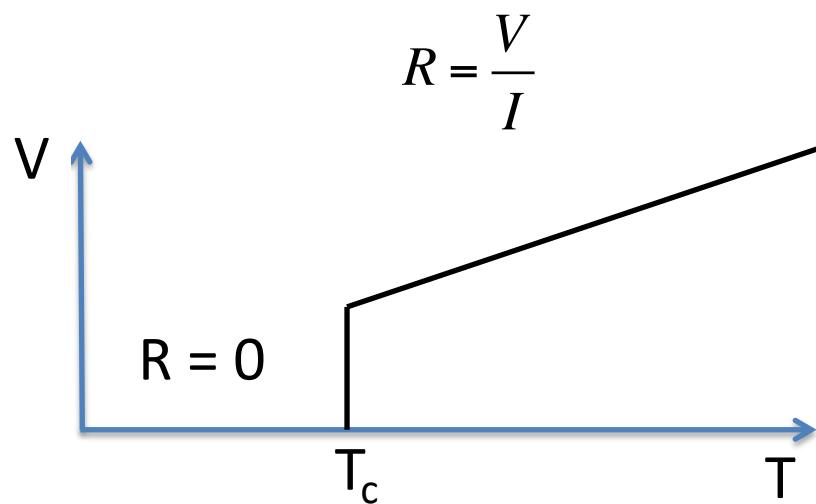
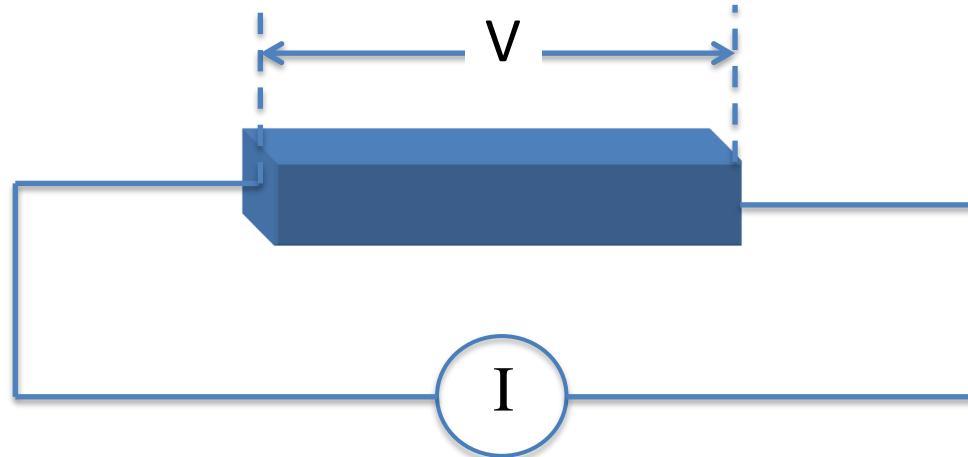
Lecture 2

- Superconductivity. Josephson junction. Josephson energy. Cooper pair tunneling.
- Charging energy.
- The Cooper pair box. Electrostatic energies.
- Quantum mechanics of electric circuits.
- The Hamiltonian of the Cooper pair box.
- The Charge qubit.

Superconductivity
Josephson junctions
Josephson energy

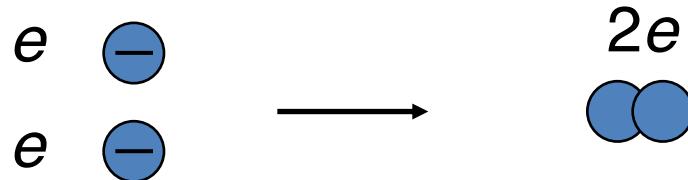
Superconductivity

At $T < T_c$ resistance abruptly vanishes ($R \rightarrow 0$)



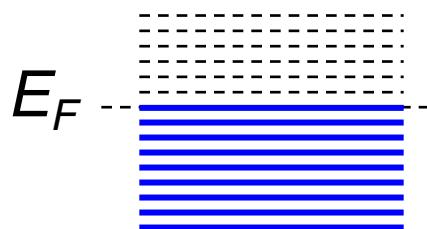
Electron spectrum

Electrons form Cooper pairs due to phonon interaction



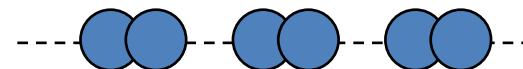
Electron spectrum in
normal metals at $T = 0$:

Conduction band



Valence band

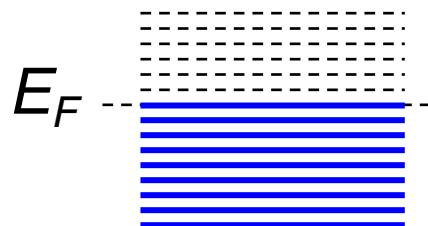
Cooper pairs are Bosons:
All Cooper pairs take the same
energy level, forming condensate



Cooper pairs

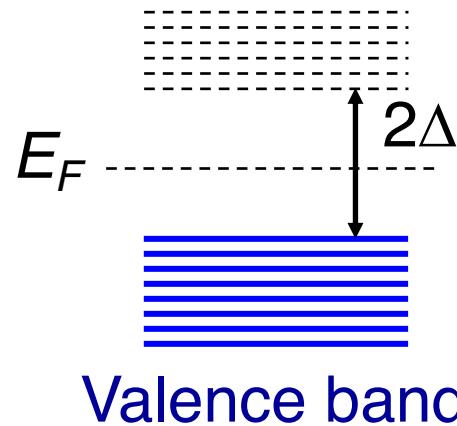
Electron spectrum in
normal metals at $T = 0$:

Conduction band



Valence band

Electron spectrum in
superconductors at $T \ll T_c$:
Conduction band



No normal current flows through the superconductors at $T=0$
Superconducting gap: 2Δ

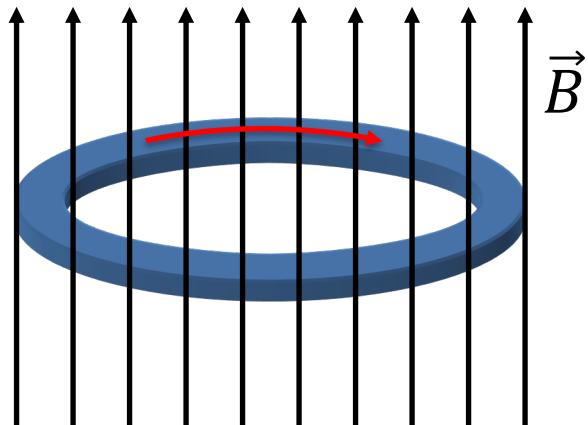
Superconducting phase and flux quantization

Voltage is zero in stationary (dc) transport $V \rightarrow \varphi$

$$\Phi_{ext} = BS$$

$$V = \dot{\Phi}$$

$$\frac{\varphi_{ext}}{2\pi} = \frac{\Phi_{ext}}{\Phi_0}$$



$$\Psi = |\Psi_0| e^{i\varphi(x)}$$

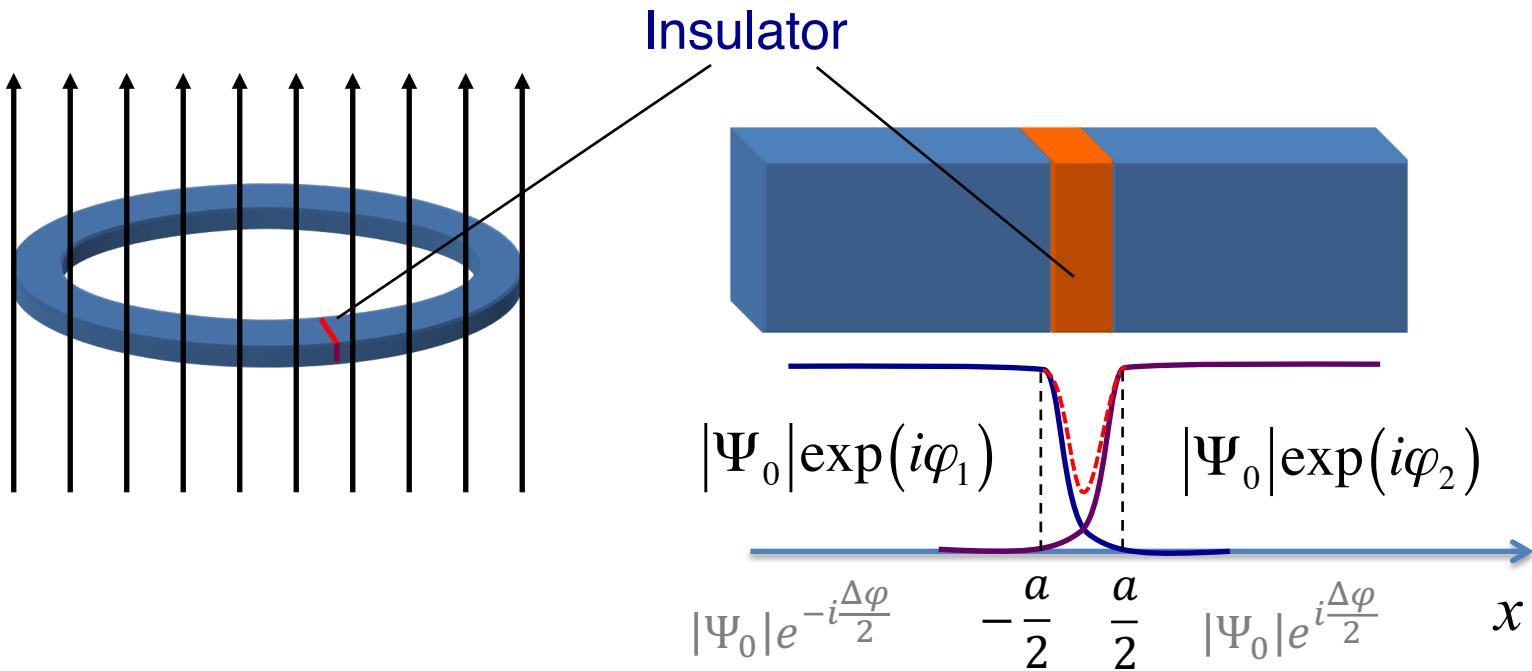
$$\varphi = \varphi_{ext} + 2\pi N$$

$$\Phi = \Phi_{ext} + \Phi_0 N$$

Flux quantum:

$$\Phi_0 = \frac{h}{2e} = 2 \times 10^{-15} \text{ Wb}$$

Josephson junction



$$U\Psi - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi = 0$$

$$\Psi = A \cosh(\kappa x) + B \sinh(\kappa x)$$

$$\kappa = \frac{\sqrt{2mU}}{\hbar}$$

Boundary conditions:

$$\Delta\varphi = \varphi_2 - \varphi_1$$

$$\Psi\left(-\frac{a}{2}\right) = A \cosh\left(\frac{\kappa a}{2}\right) - B \sinh\left(\frac{\kappa a}{2}\right) \equiv |\Psi_0| e^{-i\frac{\Delta\varphi}{2}} \quad \Psi\left(\frac{a}{2}\right) = A \cosh\left(\frac{\kappa a}{2}\right) + B \sinh\left(\frac{\kappa a}{2}\right) \equiv |\Psi_0| e^{i\frac{\Delta\varphi}{2}}$$

$$A \cosh\left(\frac{\kappa a}{2}\right) = |\Psi_0| \cos \frac{\Delta\varphi}{2}$$

$$B \sinh\left(\frac{\kappa a}{2}\right) = |\Psi_0| i \sin \frac{\Delta\varphi}{2}$$

Josephson junction: current-phase relation

$$\Psi = A \cosh(\kappa x) + B \sinh(\kappa x)$$

$$A = |\Psi_0| \cos \frac{\Delta\varphi}{2} \cosh^{-1} \frac{\kappa a}{2}$$

$$B = i|\Psi_0| \sin \frac{\Delta\varphi}{2} \sinh^{-1} \frac{\kappa a}{2}$$

$$\frac{\partial}{\partial x} \Psi = \kappa A \sinh(\kappa x) + \kappa B \cosh(\kappa x)$$

$$\Psi(0) = A \quad \frac{\partial}{\partial x} \Psi(0) = \kappa B$$

At $x=0$: $\Psi = A$ $\frac{\partial \Psi}{\partial x} = \kappa B$ $I = \frac{2e\hbar\kappa}{m} \text{Im}(A^*B) = \frac{2e\hbar\kappa}{m} \sinh^{-1}(\kappa a) \sin(\Delta\varphi)$

$$I = I_c \sin(\varphi_2 - \varphi_1)$$

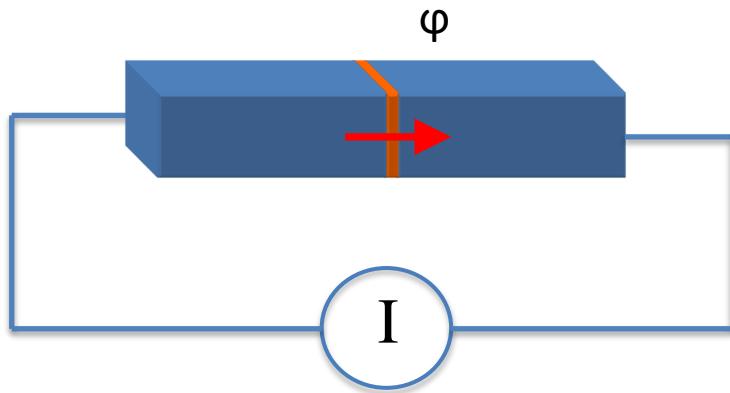
I_c

Supercurrent: $I = -i2e \frac{\hbar}{2m} \left(\Psi^* \frac{\partial}{\partial x} \Psi - \Psi \frac{\partial}{\partial x} \Psi^* \right)$

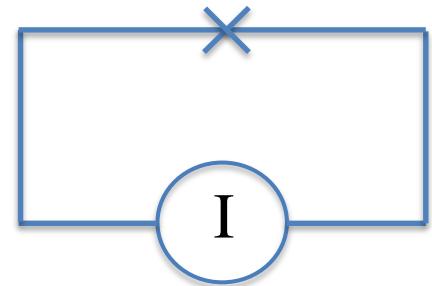
$$I = \frac{2e\hbar}{m} \text{Im} \left(\Psi^* \frac{\partial}{\partial x} \Psi \right)$$

Josephson energy

Current-phase relationship for JJ: $I = I_c \sin \varphi$



$$V = \dot{\Phi} = \frac{\Phi_0}{2\pi} \dot{\varphi}$$

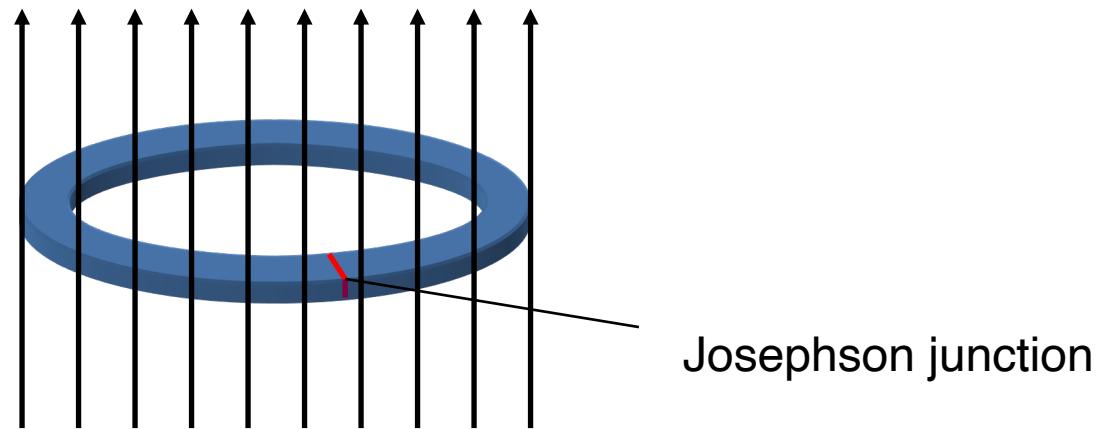
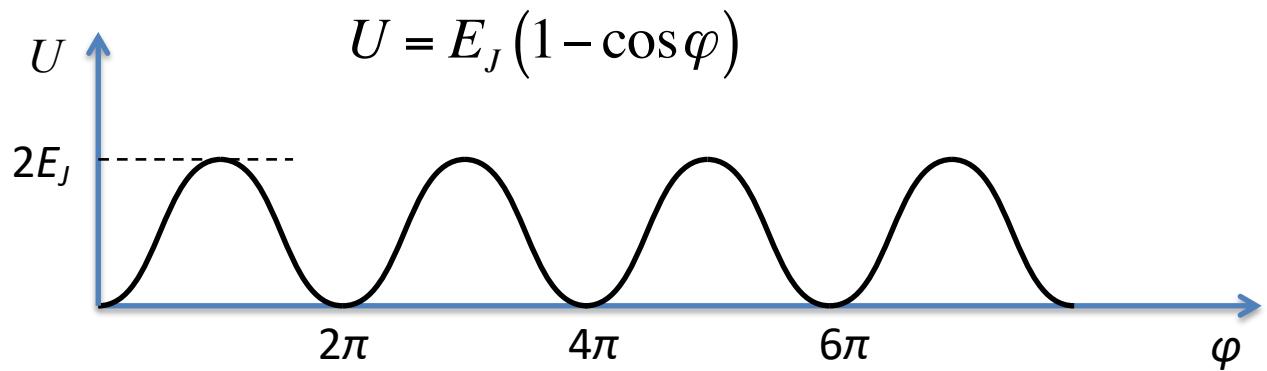


Energy stored in the junction:

$$U = \int_0^t IV dt = \int_0^t I_c \sin \varphi \frac{\Phi_0}{2\pi} \dot{\varphi} dt = I_c \frac{\Phi_0}{2\pi} \int_0^\varphi \sin \varphi d\varphi = \frac{\Phi_0 I_c}{2\pi} (1 - \cos \varphi)$$

Josephson energy: $E_J = I_c \frac{\Phi_0}{2\pi}$

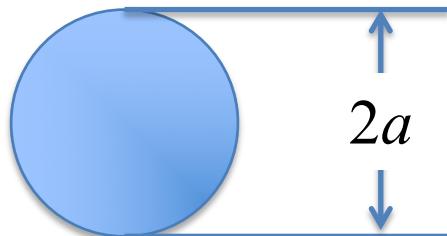
Washboard potential



The system takes one of the minima with phase $\varphi = 2\pi N \Rightarrow N\Phi_0$

The Copper pair box

A sphere in a dielectric



Charging energy (energy of an island with one electron charge):

$$E_c = \frac{e^2}{2C}$$

Thermal fluctuations:

$$k_B T$$

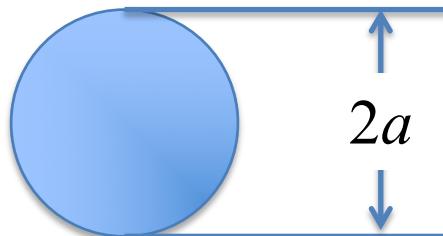
Capacitance of a sphere in

a dielectric ($\epsilon = 10$) (no metallic leads):

$$C = 4\pi\epsilon_0\epsilon a$$

1. Calculate E_c of a sphere with radius $a = 1 \mu\text{m}$ $\epsilon\epsilon_0 = 10^{-10} \text{F/m}$ $e = 1.6 \times 10^{-19} \text{C}$
2. Calculate typical thermal fluctuation at $T = 4\text{K}$ $k_B = 1.38 \times 10^{-23} \text{J/K}$
3. Compare E_c and $k_B T$

A sphere in a dielectric



Charging energy (energy of an island with one electron charge): $E_c = \frac{e^2}{2C}$

Thermal fluctuations: $k_B T$

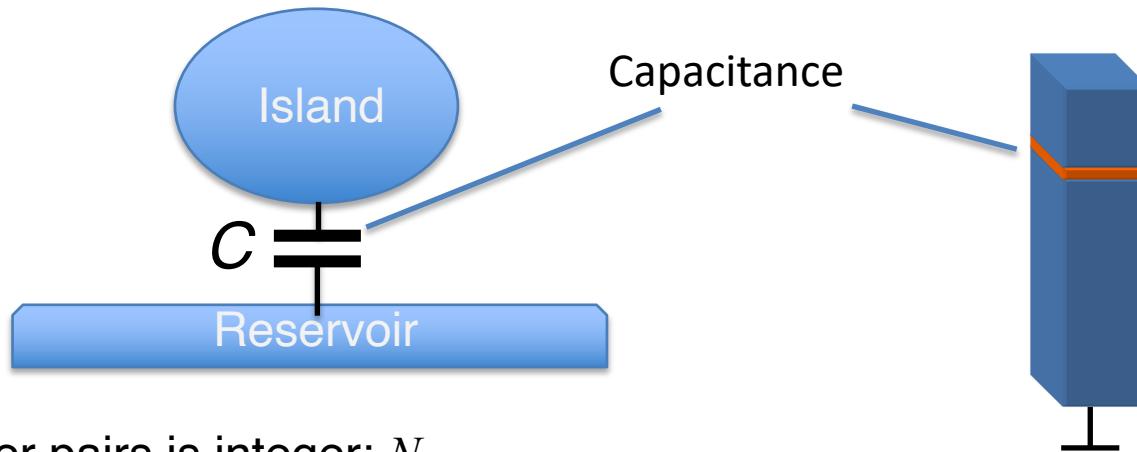
Capacitance of a sphere in a dielectric ($\epsilon = 10$) (no metallic leads): $C = 4\pi\epsilon_0\epsilon a$

Size	Temperature	Capacitance	Charging Energy	Thermal fluctuations	E_c / kT
1 mm	300 K	$1.1 \times 10^{-12} \text{ F}$	$2.3 \times 10^{-26} \text{ J}$	$4.1 \times 10^{-21} \text{ J}$	5×10^{-6}
1 μm	1 K	$1.1 \times 10^{-15} \text{ F}$	$2.3 \times 10^{-23} \text{ J}$	$1.4 \times 10^{-23} \text{ J}$	1.6
100 nm	0.1 K	$1.1 \times 10^{-16} \text{ F}$	$2.3 \times 10^{-22} \text{ J}$	$1.4 \times 10^{-24} \text{ J}$	160

To reach the quantum regime ($E_c \gg k_B T$) with quantized charges nanostructures (nanofabrication) and millikelvin temperatures are required

Cooper pair box

A superconducting island with capacitance C



Number of Cooper pairs is integer: N

We can add or remove a Cooper pair $2e$

Total energy of the island:

$$U = \frac{Q^2}{2C} = \frac{(N2e)^2}{2C}$$

$$9E_C \quad |3\rangle$$

N is a number of excess Cooper pairs

Charging energy:

$$U = \frac{(2e)^2}{2C} N^2 = E_C N^2$$

$$E_C = \frac{(2e)^2}{2C}$$

$$4E_C \quad |2\rangle$$

$$\begin{array}{c} E_C \\ 0 \end{array} \quad \begin{array}{c} |1\rangle \\ |0\rangle \end{array}$$

Number operator for particles/charges

$$\hat{N} = \sum_0^{\infty} N|N\rangle\langle N|$$

Compact form:

$$\hat{N} = N|N\rangle\langle N|$$

Matrix form:

$$\hat{N} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

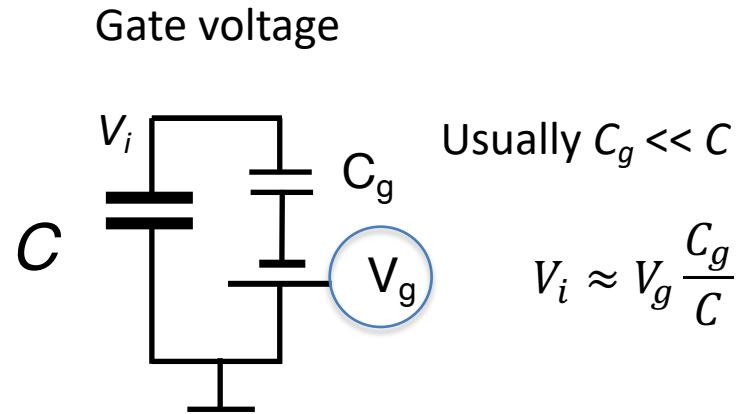
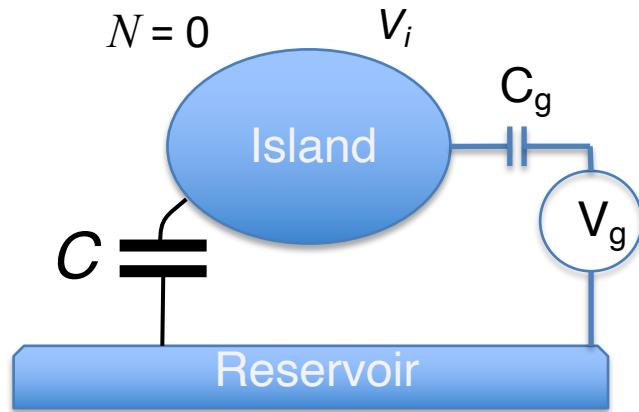
Operator of electrostatic energy
in the charge basis:

$$\hat{U} = E_c \hat{N}^2 = E_c N^2 |N\rangle\langle N|$$

$$H = \hat{U}$$

$$\hat{U} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & E_c & 0 & 0 & \dots \\ 0 & 0 & 4E_c & 0 & \dots \\ 0 & 0 & 0 & 9E_c & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Cooper pair box



Electrostatic Energy:

$$U \approx \frac{CV_i^2}{2} = \frac{(C_g V_g)^2}{2C} = \frac{(2e)^2}{2C} \left(\frac{C_g V_g}{2e} \right)^2 = E_c n^2$$

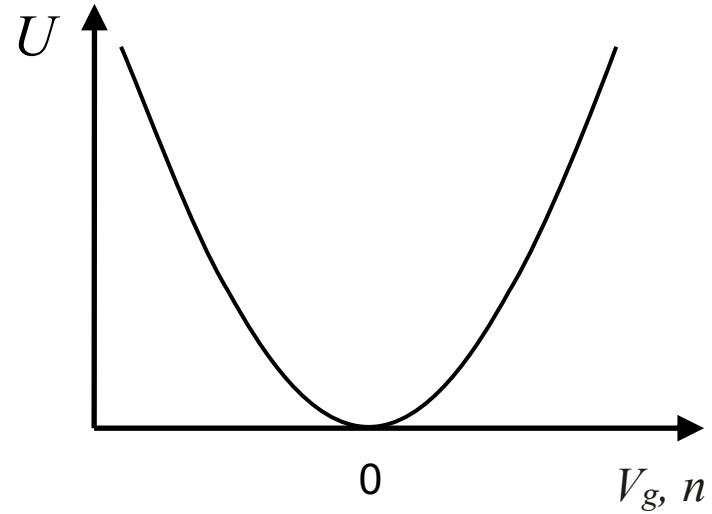
Cooper pair
charging energy

$$E_c = \frac{(2e)^2}{2C}$$

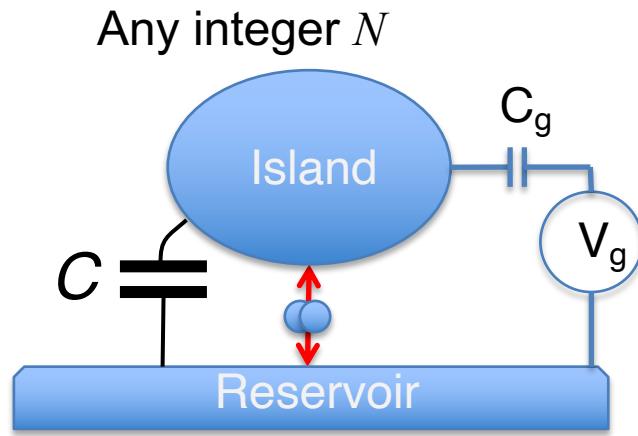
Gate induced charge:

$$n = \frac{C_g V_g}{2e}$$

$$U = E_c n^2$$



Cooper pair box



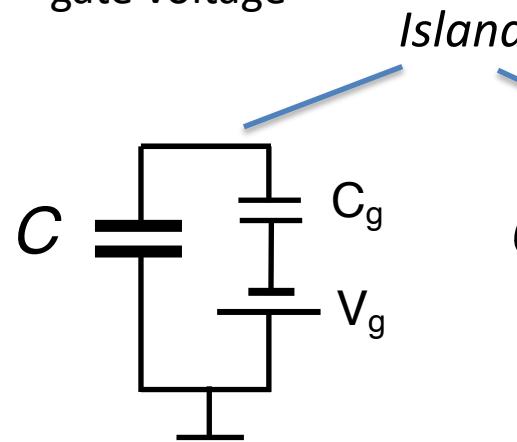
Electrostatic Energy:

$$U = \frac{C(V_{ig} + V_{iq})^2}{2} = E_c(n - N)^2$$

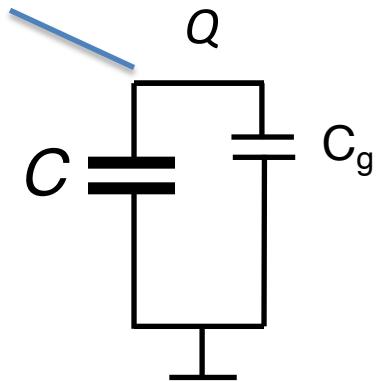
$$n = \frac{C_g V_g}{2e}$$

$$E_c = \frac{(2e)^2}{2C_{tot}}$$

Effect of
gate voltage



Effect of
island charge



$$V_{iq} = \frac{V_g C_g}{C} = \frac{Q_g}{C} = \frac{2en}{C}$$

$$V_{iQ} = \frac{Q}{C} = \frac{-2eN}{C}$$

$$V_i = V_{ig} + V_{iq} = \frac{2e}{C}(n - N)$$

$$U = E_c(N - n)^2$$

n is continuous variable $\sim V_g$

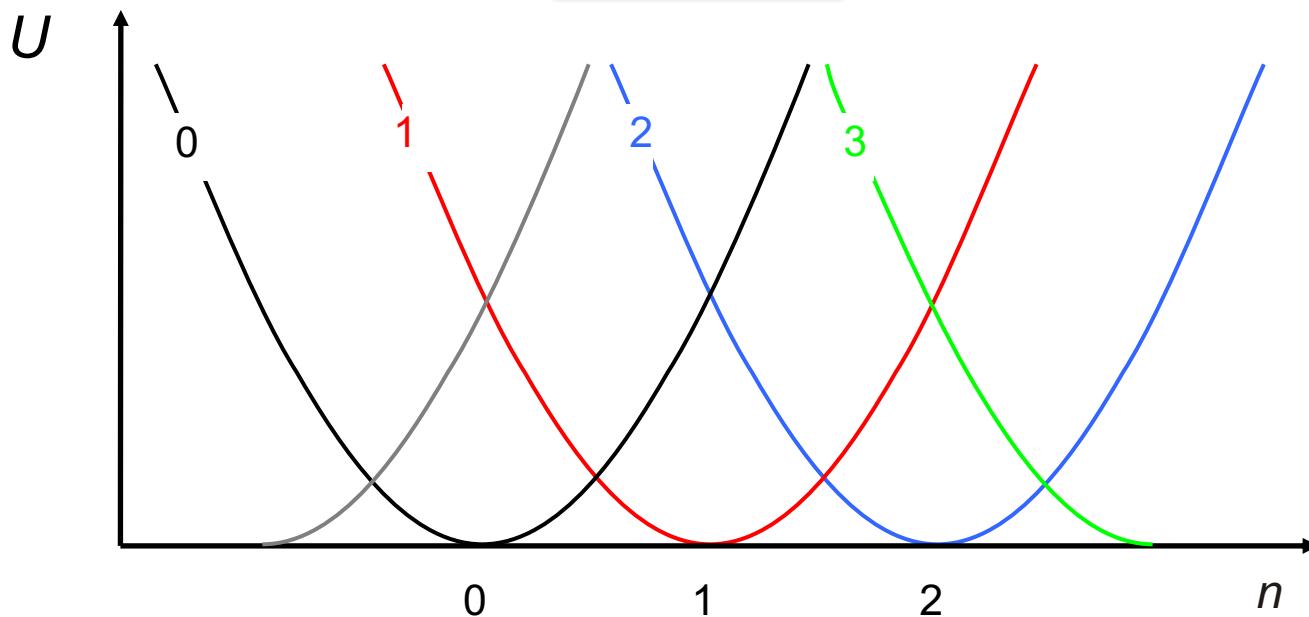
Electrostatic energies of the Cooper-pair box

$$U = E_C (N - n)^2$$

Charging energy:

$$E_C = \frac{(2e)^2}{2C}$$

Gate induced charge: $n = \frac{C_g V}{2e}$
continuous variable,
controlled by the gate voltage



A Hamiltonian of an isolated island (without tunneling)

$$U = E_C (N - n)^2$$

$$H = E_c (\hat{N} - n)^2$$

Without tunneling energy kinetic energy $T = 0$ and the Hamiltonian

$$H = U$$

In the charge basis the Hamiltonian has diagonal form $H = E_C (N - n)^2 |N\rangle\langle N|$

$$H = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \dots & E_C (-2-n)^2 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & E_C (-1-n)^2 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & E_C n^2 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & E_C (1-n)^2 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & E_C (2-n)^2 & \dots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Quantum mechanics of an electric circuit

Charge \leftrightarrow Flux

Capacitance



Voltage: V

$$V = \dot{\Phi}$$

Charge: Q

$$Q = CV = C\dot{\Phi}$$

$$Q = 2eN$$

Inductance



Current: I

$$I = \dot{Q}$$

Magnetic Flux: Φ

$$\Phi = LI = L\dot{Q}$$

$$\Phi = \frac{\Phi_0}{2\pi}\varphi$$

$$\Phi_0 = \frac{h}{2e}$$

Quantum mechanics of an electric circuit

Two representations of the electric circuit

Quantum mechanics

$$x$$

$$p = m\dot{x}$$

Commutation relations: $[\hat{x}, \hat{p}] = i\hbar$

Kinetic energy:

$$T = \frac{m\dot{x}^2}{2}$$

Electric circuit:
charge and charge motion

$$Q \quad \Phi = L\dot{Q}$$

$$[\hat{Q}, \hat{\Phi}] = i\hbar$$

$$T = \frac{L\dot{Q}^2}{2}$$

$$Q \leftrightarrow x \quad \Phi \leftrightarrow p \quad L \leftrightarrow m$$

Electric circuit:
flux and flux motion

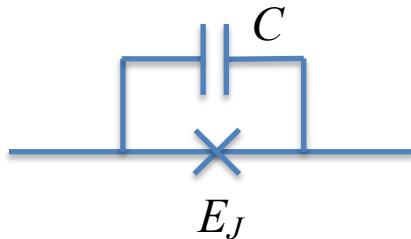
$$\Phi \quad Q = C\dot{\Phi}$$

$$[\hat{\Phi}, \hat{Q}] = i\hbar$$

$$T = \frac{C\dot{\Phi}^2}{2}$$

$$\Phi \leftrightarrow x \quad Q \leftrightarrow p \quad C \leftrightarrow m$$

The Cooper-pair box: Josephson junction with capacitance



It is convenient to use the second approach: Electric circuit with flux and flux motion

$$\Phi = \frac{\Phi_0}{2\pi} \varphi \quad Q = 2eN \quad [\hat{\Phi}, \hat{Q}] = i\hbar \rightarrow \left[\frac{\Phi_0}{2\pi} \hat{\varphi}, 2e\hat{N} \right] = i\hbar \rightarrow [\hat{\varphi}, \hat{N}] = i$$

Potential energy:

$$U = E_J(1 - \cos \varphi)$$

Kinetic energy:

$$T = \frac{C\dot{\Phi}^2}{2} = \frac{Q^2}{2C}$$

$$H = U + T$$

$$H = E_J(1 - \cos \varphi) + \frac{Q^2}{2C} \quad H = E_J(1 - \cos \varphi) + \frac{4e^2}{2C}(N - n)^2$$

$$H = E_J(1 - \cos \hat{\varphi}) + E_C(\hat{N} - n)^2$$

It has a diagonal form in a charge basis $|N\rangle$ as it was shown earlier

Important formulas

$$H = T + U$$

$$H = \frac{Q^2}{2C} + E_J(1 - \cos \varphi)$$

$$[\hat{\varphi}, \hat{N}] = i$$

$$p = -i\hbar \frac{\partial}{\partial x}$$

$$\hat{N} = -i \frac{\partial}{\partial \varphi}$$

This form of the charge operator can be used, when the charge is not quantised

$$H = \frac{(2e\hat{N} - C_g V)^2}{2C} + E_J(1 - \cos \hat{\varphi})$$

$$H = E_C (\hat{N} - n)^2 - E_J \cos \hat{\varphi}$$

The Charging energy:

$$E_C = \frac{4e^2}{2C}$$

$$n = \frac{C_g V}{2e}$$

Operator of cosine of phase in the charge basis

$$\cos \hat{\varphi}$$

We first derive commutation relations for $\hat{\varphi}^k$ and \hat{N} :

$$\left[\hat{\varphi}, \hat{N} \right] = i \quad \rightarrow \quad \hat{\varphi} \hat{N} - \hat{N} \hat{\varphi} = i$$

$$\varphi^k N = \varphi^{k-1}(N\varphi + i) = \varphi^{k-1}N\varphi + i\varphi^{k-1} + \varphi^{k-2}(N\varphi + i)\varphi + i\varphi^{k-1} = \varphi^{k-2}N\varphi^2 + 2i\varphi^{k-1} = \\ \equiv \dots \equiv N\varphi^k + ki\varphi^{k-1} \quad \xrightarrow{\hspace{1cm}} \quad [\varphi^k, N] = ki\varphi^{k-1}$$

$$e^{i\varphi}N = \sum_{k=0}^{\infty} \frac{(i\varphi)^k}{k!} N = \sum_{k=0}^{\infty} N \frac{(i\varphi)^k}{k!} + i \sum_{k=1}^{\infty} \frac{ik(i\varphi)^{k-1}}{k!} = N \sum_{k=0}^{\infty} \frac{(i\varphi)^k}{k!} - \sum_{k=1}^{\infty} \frac{(i\varphi)^{k-1}}{(k-1)!} = Ne^{i\varphi} - e^{i\varphi}$$

$$e^{i\hat{\varphi}} \hat{N} = (\hat{N} - 1) e^{i\hat{\varphi}}$$

$$e^{i\hat{\varphi}} = |N\rangle\langle N -$$

$$\hat{N} = N|N\rangle\langle N|$$

$$(\hat{N}-1)e^{i\hat{\phi}} = (N-1)|N\rangle\langle N|N\rangle\langle N-1| = (N-1)|N\rangle\langle N-1|$$

$$e^{i\hat{\varphi}} \hat{N} = |N\rangle\langle N-1|(N-1)|N-1\rangle\langle N-1| = (N-1)|N\rangle\langle N-1|$$

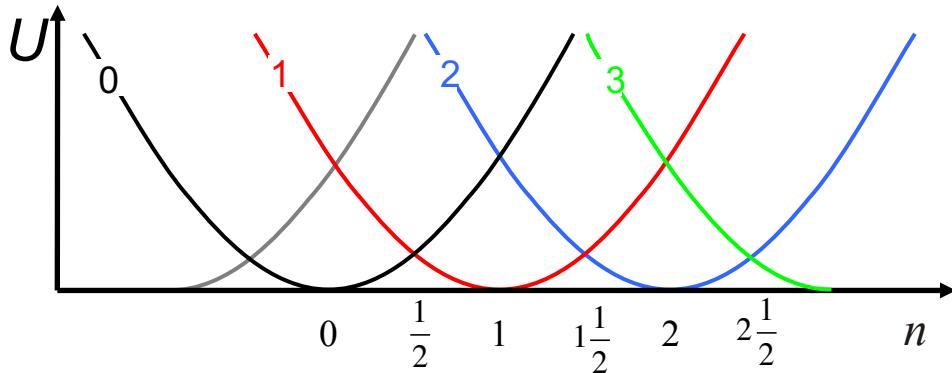
The Cooper-pair box Hamiltonian in the charge basis

$$H = E_C(N-n)^2 |N\rangle\langle N| - \frac{1}{2}E_J(|N-1\rangle\langle N| + |N\rangle\langle N-1|)$$

$$H = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & E_C(-2-n)^2 & -\frac{1}{2}E_J & 0 & 0 & 0 & \cdots \\ \cdots & -\frac{1}{2}E_J & E_C(-1-n)^2 & -\frac{1}{2}E_J & 0 & 0 & \cdots \\ \cdots & 0 & -\frac{1}{2}E_J & E_C n^2 & -\frac{1}{2}E_J & 0 & \cdots \\ \cdots & 0 & 0 & -\frac{1}{2}E_J & E_C(1-n)^2 & -\frac{1}{2}E_J & \cdots \\ \cdots & 0 & 0 & 0 & -\frac{1}{2}E_J & E_C(2-n)^2 & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Tunneling energy

$$U(N, N_{ext}) = E_C (N - N_{ext})^2$$

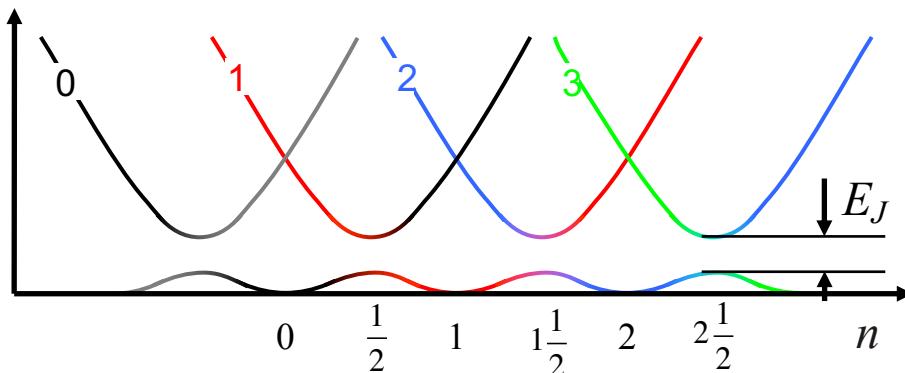


$$H = E_C (N - n)^2 |N\rangle\langle N|$$

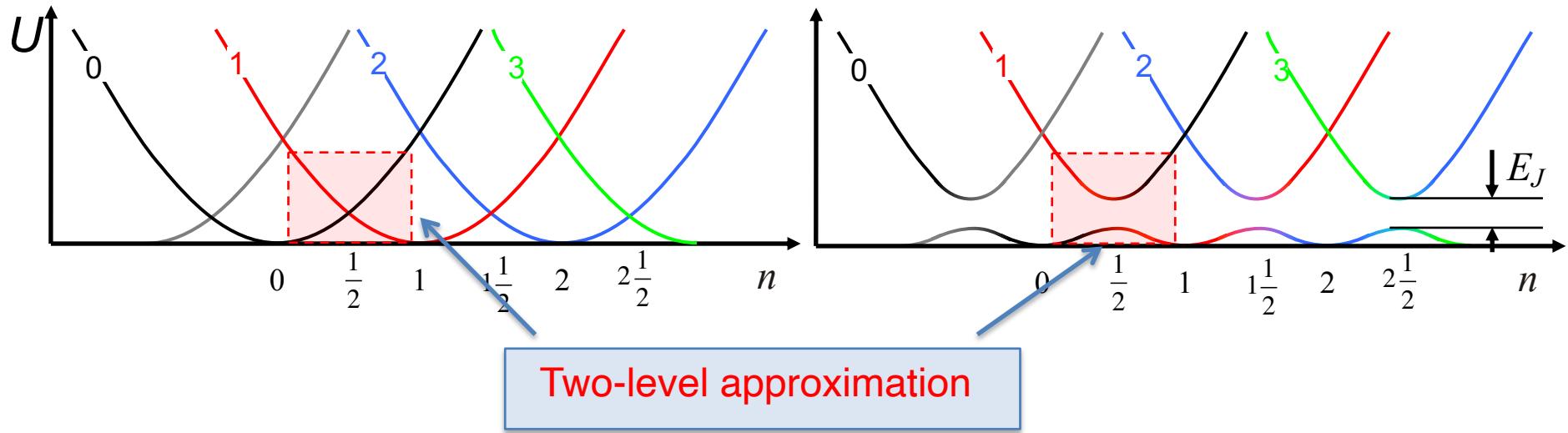
Degeneracy point: $U(N, n) = U(N + 1, n)$

$$E_C (N - n)^2 = E_C (N + 1 - n)^2 \quad 2(N - n) + 1 = 0 \quad n = N + \frac{1}{2}$$

$$H = E_C (N - n)^2 |N\rangle\langle N| - \frac{1}{2} E_J (|N\rangle\langle N-1| + |N-1\rangle\langle N|)$$



The superconducting charge qubit



$$H = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & E_C(-2-n)^2 & -\frac{1}{2}E_J & 0 & 0 & 0 & \cdots \\ \cdots & -\frac{1}{2}E_J & E_C(-1-n)^2 & -\frac{1}{2}E_J & 0 & 0 & \cdots \\ \cdots & 0 & -\frac{1}{2}E_J & E_Cn^2 & -\frac{1}{2}E_J & 0 & \cdots \\ \cdots & 0 & 0 & -\frac{1}{2}E_J & E_C(1-n)^2 & -\frac{1}{2}E_J & \cdots \\ \cdots & 0 & 0 & 0 & -\frac{1}{2}E_J & E_C(2-n)^2 & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The superconducting charge qubit Hamiltonian (symmetric form)

$$H_q = \begin{pmatrix} E_C n^2 & -\frac{1}{2} E_J \\ -\frac{1}{2} E_J & E_C (1-n)^2 \end{pmatrix}$$

$$H_q = E_C n^2 |0\rangle\langle 0| + E_C (1-n)^2 |1\rangle\langle 1| - \frac{E_J}{2} (|0\rangle\langle 1| + |1\rangle\langle 0|)$$

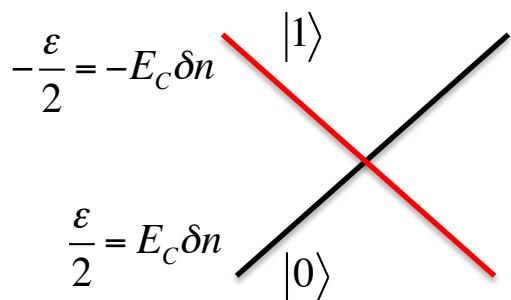
$\varepsilon = E_C (1-n)^2 - E_C n^2 = -2E_C \delta n$, where $\delta n = n - \frac{1}{2}$ is the normalized charge measured from the degeneracy point

$$H_q = -\frac{\varepsilon}{2} |0\rangle\langle 0| + \frac{\varepsilon}{2} |1\rangle\langle 1| - \frac{E_J}{2} (|0\rangle\langle 1| + |1\rangle\langle 0|)$$

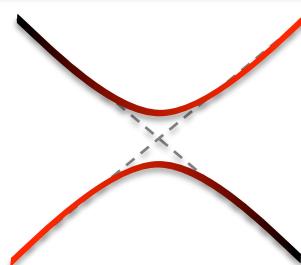
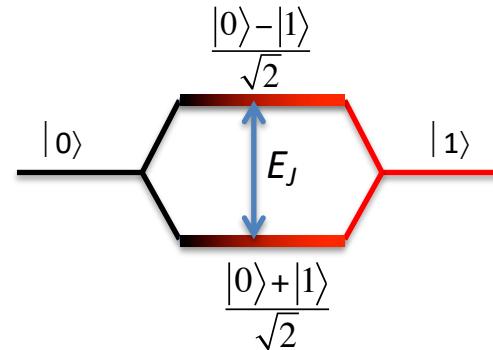
$$H_q = -\frac{\varepsilon}{2} \sigma_z$$

$$H_q = -\frac{\varepsilon}{2} \sigma_z - \frac{E_J}{2} \sigma_x$$

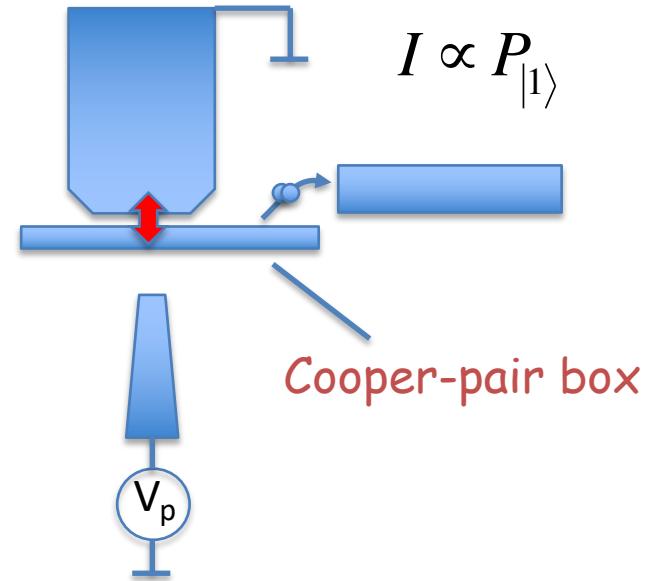
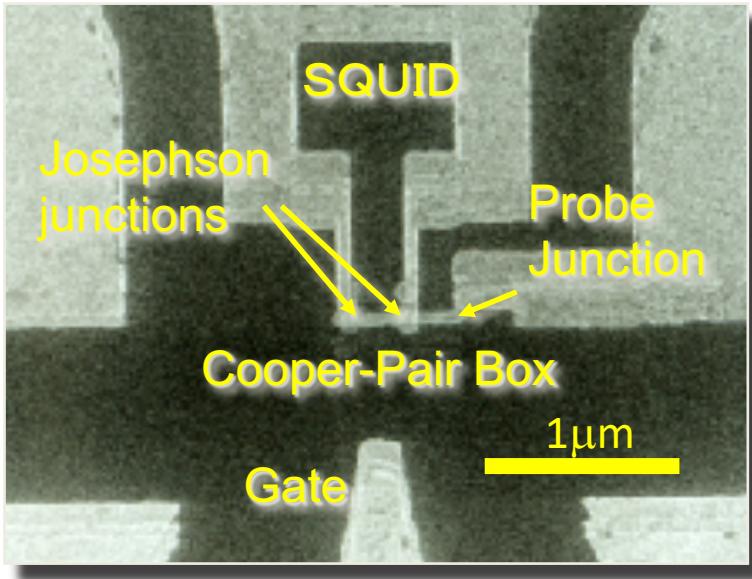
$$H_q = -\frac{E_J}{2} \sigma_x$$



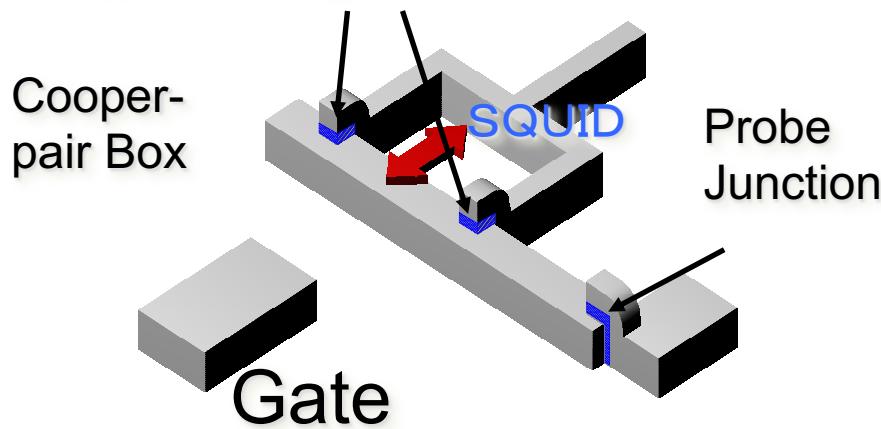
$$H_q = -\frac{\varepsilon}{2} \sigma_z - \frac{E_J}{2} \sigma_x$$



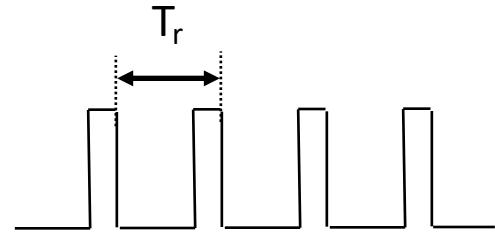
First qubit demonstration



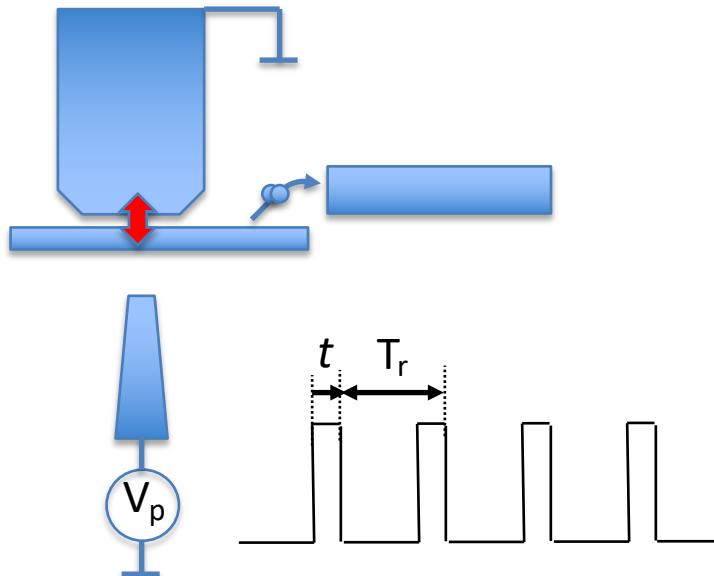
Al/AlO_x/Al tunnel junctions



Control-pulse sequence



First qubit demonstration



$$\Psi(t) = \exp\left(-i\frac{H}{\hbar}t\right)\Psi(0)$$

$$\Psi(t) = \exp\left(i\frac{E_J}{2\hbar}\sigma_x t\right)|0\rangle$$

$$\Psi(t) = \cos\left(\frac{E_J}{2\hbar}t\right)|0\rangle + i\sin\left(\frac{E_J}{2\hbar}t\right)|1\rangle$$

