Superconducting Quantum Technologies

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Lecture 3

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- Charge qubits with low anharmonicity
- How to strictly derive quantum mechanical relations for electrical circuits
- Flux qubit: RF-SQUID and double-well potential
- Three-junction flux qubits (classical flux qubit)

Why superconducting quantum systems?

- Low loss. Superconductivity is lossless in DC transport and low dissipative at high frequencies
- Controllable tunneling element. Superconducting Josephson junction is an ideal and well controlled tunneling element
- Fabrication. Methods of fabrication are similar to well established semiconducting circuits
- Scalability. Large integrated circuits with predesigned parameters can be designed and fabricated

Important formulas (from previous lectures)

$$H = T + U$$

$$H = \frac{\hat{Q}^2}{2C} + E_J\left(1 - \cos\hat{\varphi}\right)$$

$$\left[\hat{\varphi}, \hat{N}\right] = i$$

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

$$\hat{N} = -i\frac{\partial}{\partial \varphi}$$

This form of the charge operator can be used, when the charge is not quantized

$$H = \frac{\left(2e\hat{N} - C_g V_g\right)^2}{2C} + E_J\left(1 - \cos\hat{\varphi}\right)$$

$$\boxed{H = E_C\left(\hat{N} - n\right)^2 - E_J\cos\hat{\varphi}}$$

The Charging energy: $\boxed{E_C = \frac{4e^2}{2C}}$

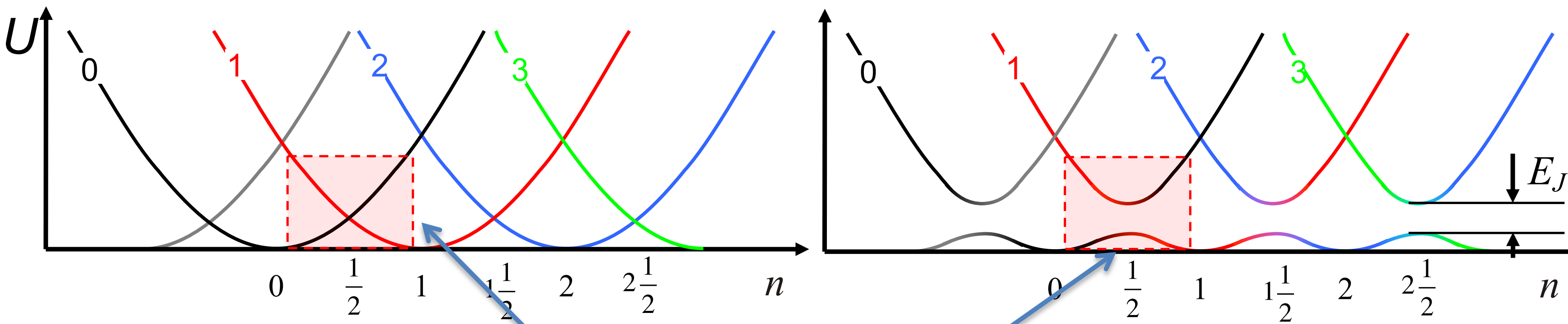
$$n = \frac{C_g V_g}{2e}$$

The Cooper-pair box Hamiltonian in the charge basis (previous lecture)

$$H = E_C (N-n)^2 |N\rangle\langle N| - \frac{1}{2}E_J\left(|N-1\rangle\langle N| + |N-1\rangle\langle N|\right)$$

$$H = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & E_C(-2-n)^2 & -\frac{1}{2}E_J & 0 & 0 & 0 & \cdots \\ \cdots & -\frac{1}{2}E_J & E_C(-1-n)^2 & -\frac{1}{2}E_J & 0 & 0 & \cdots \\ \cdots & 0 & -\frac{1}{2}E_J & E_C n^2 & -\frac{1}{2}E_J & 0 & \cdots \\ \cdots & 0 & 0 & -\frac{1}{2}E_J & E_C(1-n)^2 & -\frac{1}{2}E_J & \cdots \\ \cdots & 0 & 0 & 0 & -\frac{1}{2}E_J & E_C(2-n)^2 & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The superconducting charge qubit (previous lecture)



Two-level approximation

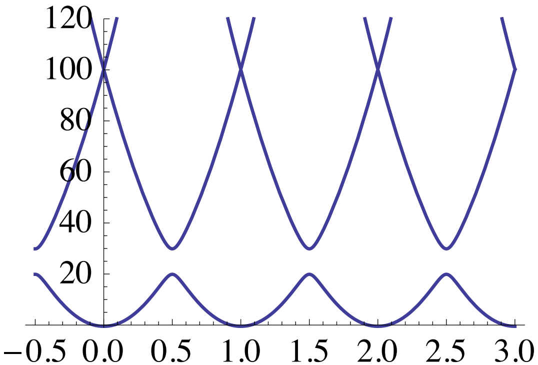
$$
H = \begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & E_C(-2-n)^2 & -\frac{1}{2}E_J & 0 & 0 & 0 & \cdots \\
\cdots & -\frac{1}{2}E_J & E_C(-1-n)^2 & -\frac{1}{2}E_J & 0 & 0 & \cdots \\
\cdots & 0 & -\frac{1}{2}E_J & E_C n^2 & -\frac{1}{2}E_J & 0 & \cdots \\
\cdots & 0 & 0 & -\frac{1}{2}E_J & E_C(1-n)^2 & -\frac{1}{2}E_J & \cdots \\
\cdots & 0 & 0 & 0 & -\frac{1}{2}E_J & E_C(2-n)^2 & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

Charge qubits with low anharmonicity

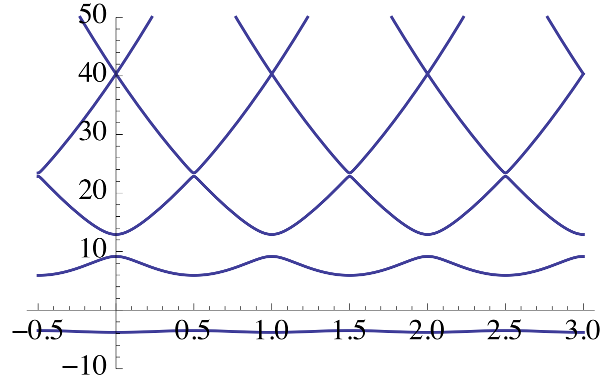
Energy bands

$$H = E_C\left(N - N_{ext}\right)^2 |N\rangle\langle N| - \frac{E_J}{2}\left(|N\rangle\langle N+1| + |N+1\rangle\langle N|\right)$$

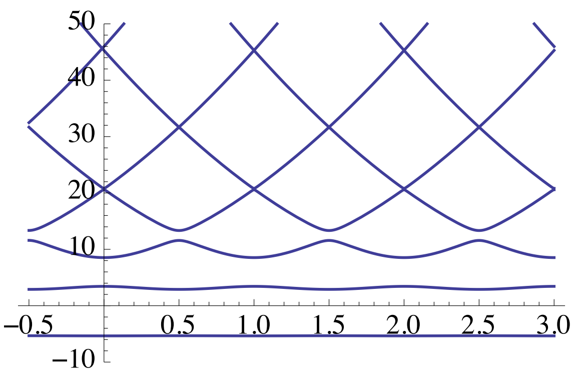
$E_C = 100,\ \ E_J = 10$



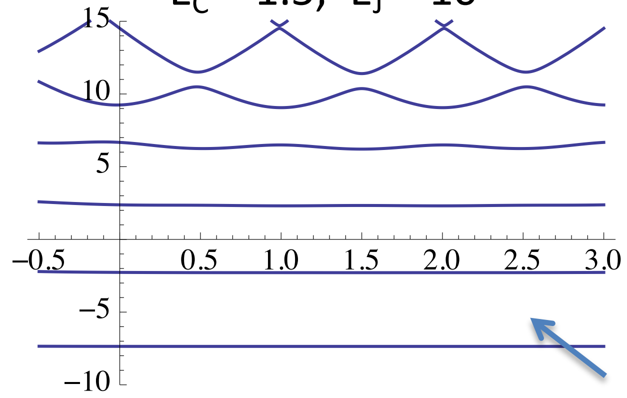
$E_C = 10,\ \ E_J = 10$



$E_C = 5,\ \ E_J = 10$



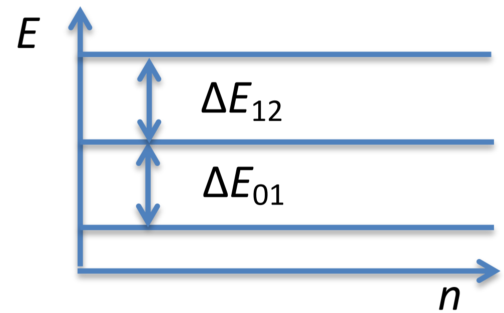
$E_C = 1.5,\ \ E_J = 10$



Low anharmonicity
Almost insensitive to charge

Examples of qubits with weak anharmonicity

$$H = E_C\left(N-n\right)^2 |N\rangle\langle N| - \frac{1}{2}E_J\left(|N-1\rangle\langle N| + |N\rangle\langle N-1|\right)$$

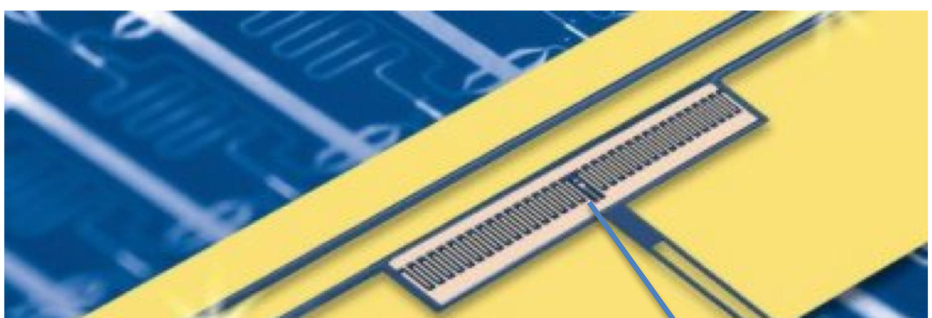


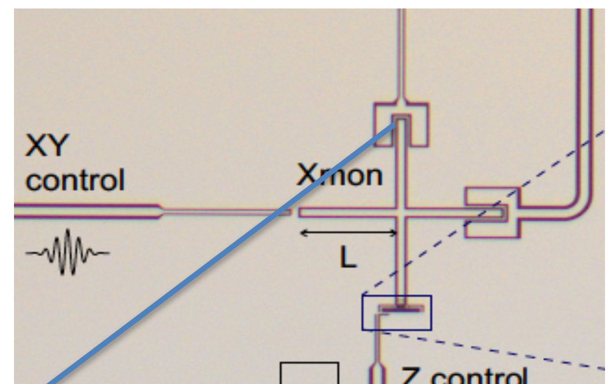
$\Delta E_{01} \neq \Delta E_{12}$

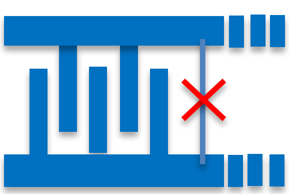
Transmon

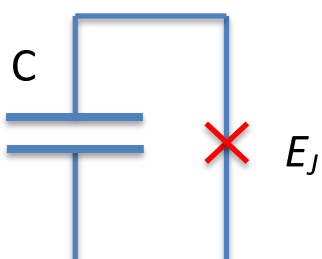
100 – 1000 nm size

X-mon

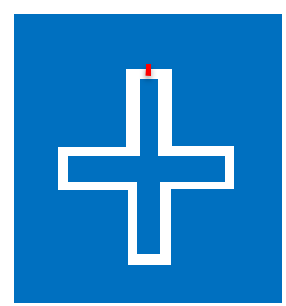


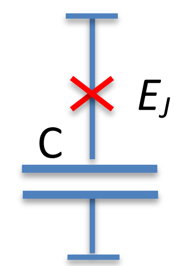






Josephson junctions



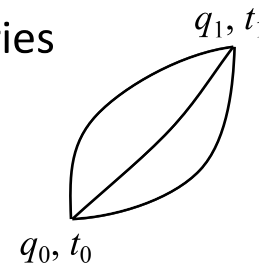


From Lagrangian to Hamiltonian:
Why one can map mechanical systems to electrical

Hamlitonian's principle:

$$\int_{t_0}^{t_1} \delta\mathcal{L}dt = 0$$

Trajectories



Lagrangian: $\mathcal{L} = T - V$ $\qquad \mathcal{L}(q, \dot{q})$

Canonical variables:

$$q \qquad p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

Mechanical system:

$$x \qquad \mathcal{L} = \frac{m\dot{x}^2}{2} - U(x) \qquad p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

Hamiltonian

$$H = \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L}$$

$$H = \dot{x}(m\dot{x}) - \frac{m\dot{x}^2}{2} + U(x) = \frac{m\dot{x}^2}{2} + U(x)$$

In case of quadratic from of kinetic energy: $H = T + U$

How to strictly derive quantum mechanical relations for electrical circuits?

Canonical variables:

$$q \qquad p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \qquad [\hat{q}, \hat{p}] = -i\hbar$$

Mechanical system

$$x \qquad p = \frac{\partial \mathcal{L}}{\partial \dot{x}} \qquad [\hat{x}, \hat{p}] = -i\hbar$$

Electric circuit (charge-flux):

$$Q \qquad \Phi = \frac{\partial \mathcal{L}}{\partial \dot{Q}} \qquad [\hat{Q}, \hat{\Phi}] = -i\hbar$$

$$H = \dot{Q}\frac{\partial \mathcal{L}}{\partial \dot{Q}} - \mathcal{L}$$

$$\mathrm{T} = \frac{L\dot{Q}^2}{2} + E_J\left(1 - \cos\left(\frac{2\pi L\dot{Q}}{\Phi_0}\right)\right) \qquad U = \frac{CQ^2}{2}$$

Electric circuit (flux-charge):

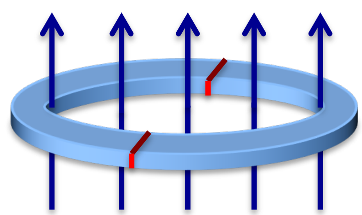
$$\Phi \qquad Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi} \qquad [\hat{\Phi}, \hat{Q}] = -i\hbar \qquad T = \frac{C\dot{\Phi}^2}{2} \qquad U = E_J\left(1 - \cos\left(\frac{2\pi\Phi}{\Phi_0}\right)\right)$$

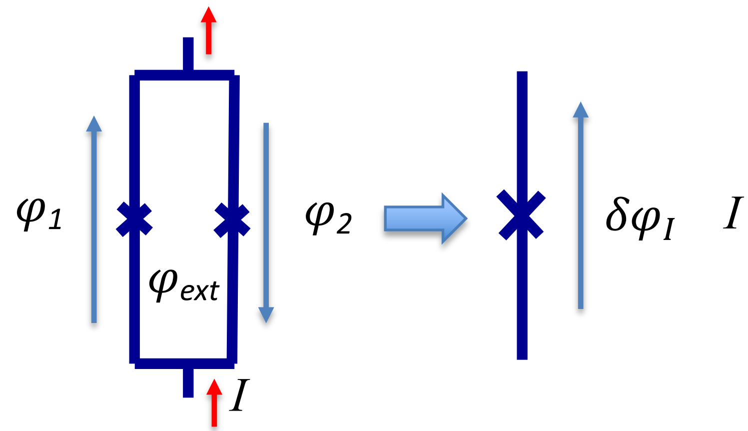
$$H = \dot{\Phi}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}} - \mathcal{L} \qquad H = \frac{C\dot{\Phi}^2}{2} + E_J\left(1 - \cos\left(\frac{2\pi\Phi}{\Phi_0}\right)\right) \qquad \mathcal{L} = \frac{C\dot{\Phi}^2}{2} - E_J\left(1 - \cos\left(\frac{2\pi\Phi}{\Phi_0}\right)\right)$$

$$H = \frac{\hat{Q}^2}{2C} + E_J\left(1 - \cos\hat{\varphi}\right)$$

Flux qubits based on RF-SQUID

Superconducting Quantum Interference Device (SQUID)



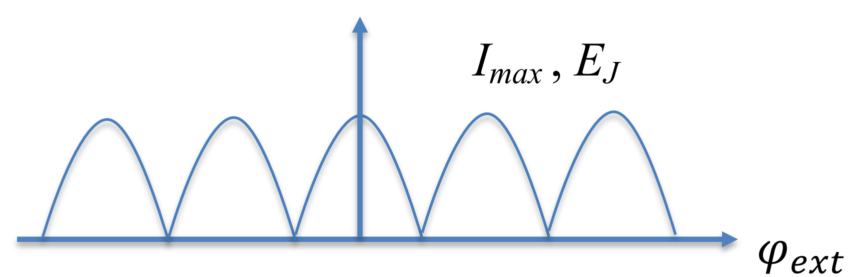


$$\varphi_1 + \varphi_2 = \varphi_{ext} \quad \Rightarrow \quad \varphi_1 = \varphi_2 = \frac{\varphi_{ext}}{2}$$

$$I = I_c \sin\left(\frac{\varphi_{ext}}{2} + \delta\varphi\right) + I_c \sin\left(-\frac{\varphi_{ext}}{2} + \delta\varphi\right) = 2I_c \cos\left(\frac{\varphi_{ext}}{2}\right) \sin\delta\varphi$$

$$I_{max} = 2I_c \left|\cos\left(\frac{\varphi_{ext}}{2}\right)\right|$$

$$E_J = 2E_J \left|\cos\left(\frac{\varphi_{ext}}{2}\right)\right|$$



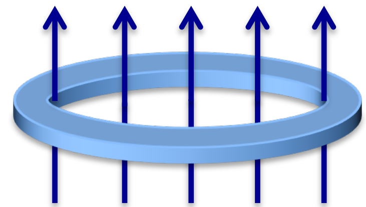
Alternative explanation: interference of two waves with amplitudes $1 + e^{i\varphi_{ext}}$

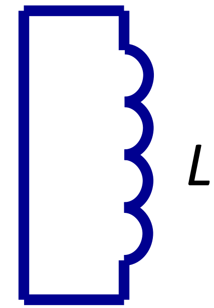
Absolute value of such a wave is $\left|1 + e^{i\varphi_{ext}}\right| = 2\cos\frac{\varphi_{ext}}{2}$

Energy of inductance

Energy of inductance:

$$E = \frac{LI^2}{2} = \frac{\Phi^2}{2L} = \frac{\Phi_0^2}{2L}\left(\frac{\Phi}{\Phi_0}\right)^2$$





$$E = E_L f^2$$

Magnetic energy:

$$E_L = \frac{\Phi_0^2}{2L}$$

Normalized flux:

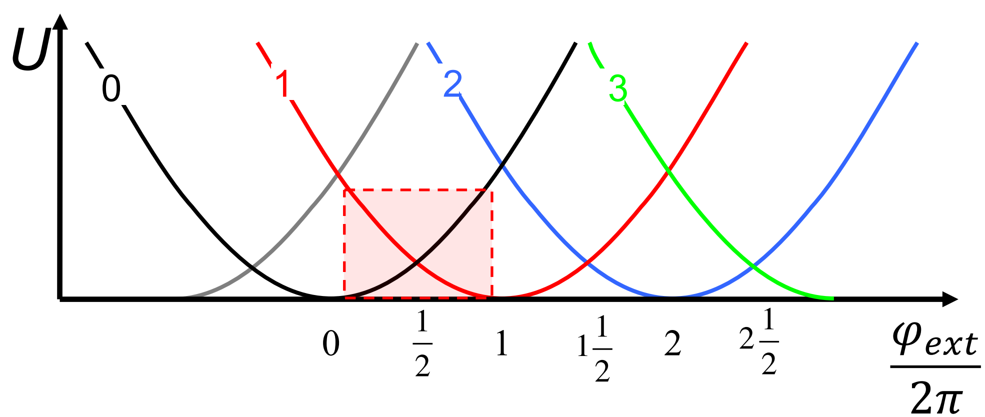
$$f = \frac{\Phi}{\Phi_0} = \frac{\varphi}{2\pi}$$

It can be convenient to redefine the magnetic energy

$$E'_L = \frac{\Phi_0^2}{2L(2\pi)^2}$$

$$E = E'_L \varphi^2$$

$$E = E'_L(\varphi_{ext} - 2\pi N)^2$$



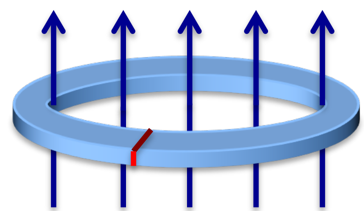
The flux qubit: RF-SQUID

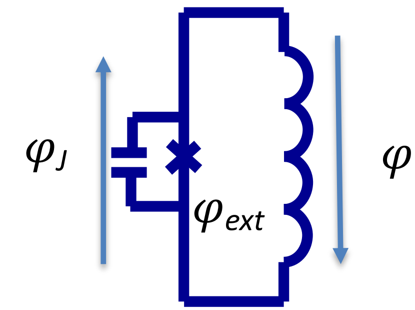
RF-SQUID: One junction is replaced by an inductance (e.g. finite length wire)

$$T = \frac{Q^2}{2C} = E_C \hat{N}^2$$

$$\hat{N}_J = -i\frac{\partial}{\partial \varphi_J}$$

$$H = -E_C \frac{\partial^2}{\partial \varphi_J^2} - E_J \cos\varphi_J + E_L\left(\varphi_{ext} - \varphi_J\right)^2 \qquad E_L = \left(\frac{\Phi_0}{2\pi}\right)^2 \frac{1}{2L}$$

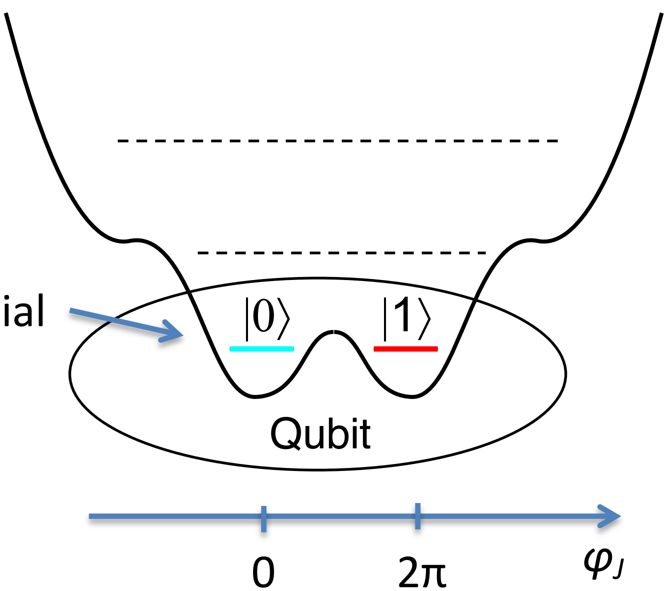




$$\varphi_J + \varphi = \varphi_{ext}$$

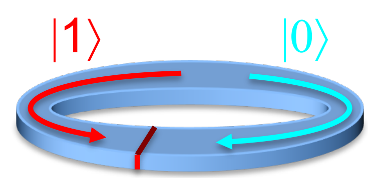
Compare with $H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x)$ $\qquad x \Leftrightarrow \varphi \qquad m \Leftrightarrow C$

Double well potential



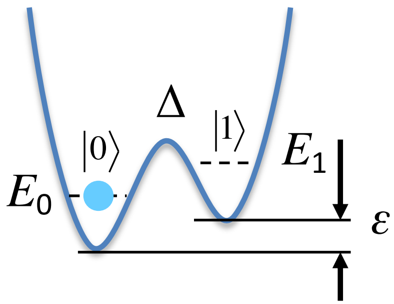
$$H|0\rangle = E_0|0\rangle$$

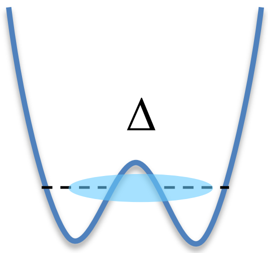
$$H|1\rangle = E_1|1\rangle$$



Simplified Hamiltonian

$$H = -E_C \frac{\partial^2}{\partial \varphi_J^2} - E_J \cos\varphi_J + E_L \left(\varphi_{ext} - \varphi_J\right)^2$$





$$H_0 \approx -\frac{\varepsilon}{2}\sigma_z \quad \Psi_0 \approx |0\rangle \qquad\qquad H_1 = -\frac{\Delta}{2}\sigma_x$$

Δ is the tunneling energy between two wells

Degeneracy point: $E_L \varphi_{ext}^2 = E_L \left(2\pi - \varphi_{ext}\right)^2 \qquad \varphi_{ext} = \pi$

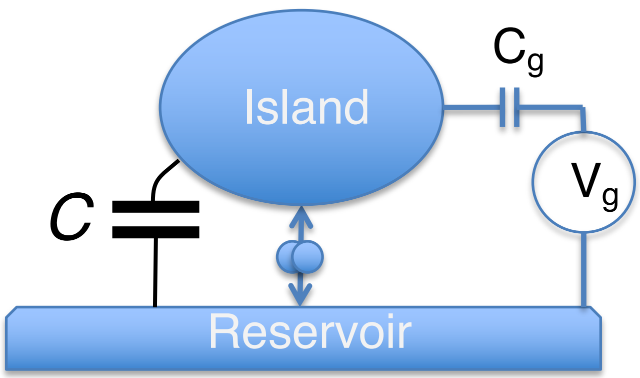
$\varepsilon = -2E_L \delta\varphi$, where $\delta\varphi = \varphi_{ext} - \pi$

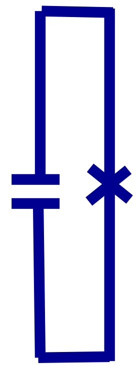
$$\boxed{H \approx -\frac{\varepsilon}{2}\sigma_z - \frac{\Delta}{2}\sigma_x}$$

The charge vs flux qubit

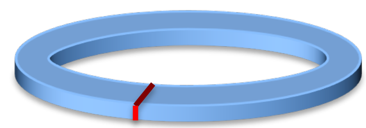
$$H \approx -\frac{\varepsilon}{2}\sigma_z - \frac{\Delta}{2}\sigma_x$$

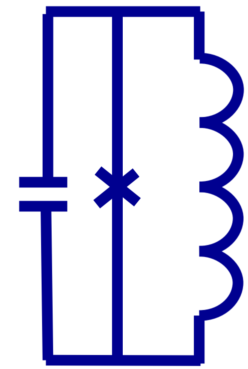
The charge qubit





The flux qubit





Charge states (0, 2e): $|0\rangle$, $|1\rangle$

Cooper pair tunneling in/out

Charging energy: E_C

Tunneling energy: E_J

Controlled by V

$\varepsilon = -2E_C\delta N$

Flux states (0, 2π): $|0\rangle$, $|1\rangle$

Flux tunneling tunneling in/out

Magnetic energy: E_L

Tunneling energy: Δ

Controlled by B

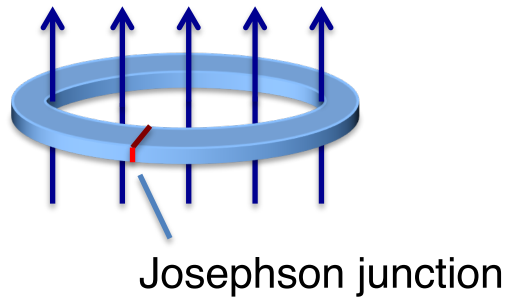
$\varepsilon = -2E_L\delta\varphi$

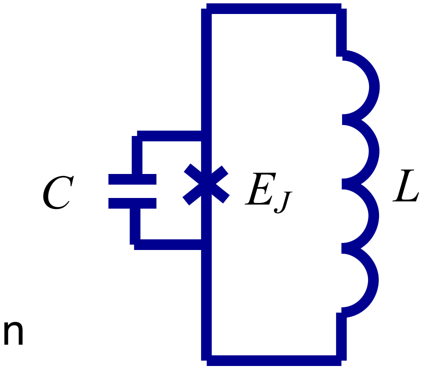
The flux qubit: RF-SQUID

$$U = -E_J \cos\varphi_J + \frac{\Phi_L^2}{2L}$$

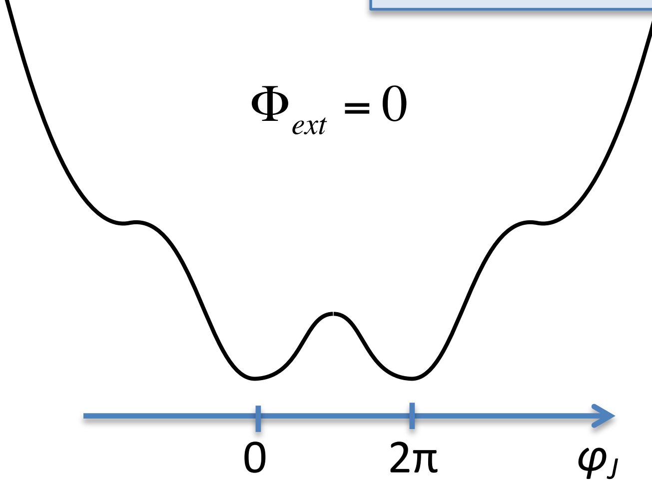
$$\varphi_L = \frac{2\pi\Phi_L}{\Phi_0} \qquad \varphi_J = \frac{2\pi\Phi_J}{\Phi_0}$$

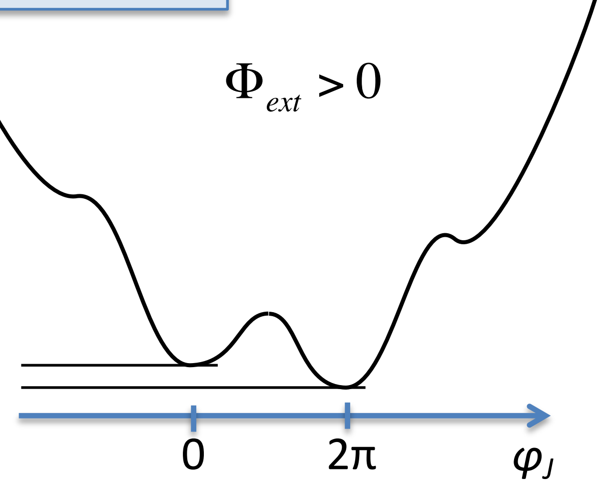
$$\Phi_{ext} - \Phi_J - \Phi_L = 0 \qquad\qquad \varphi_J + \varphi_L = \varphi_{ext}$$





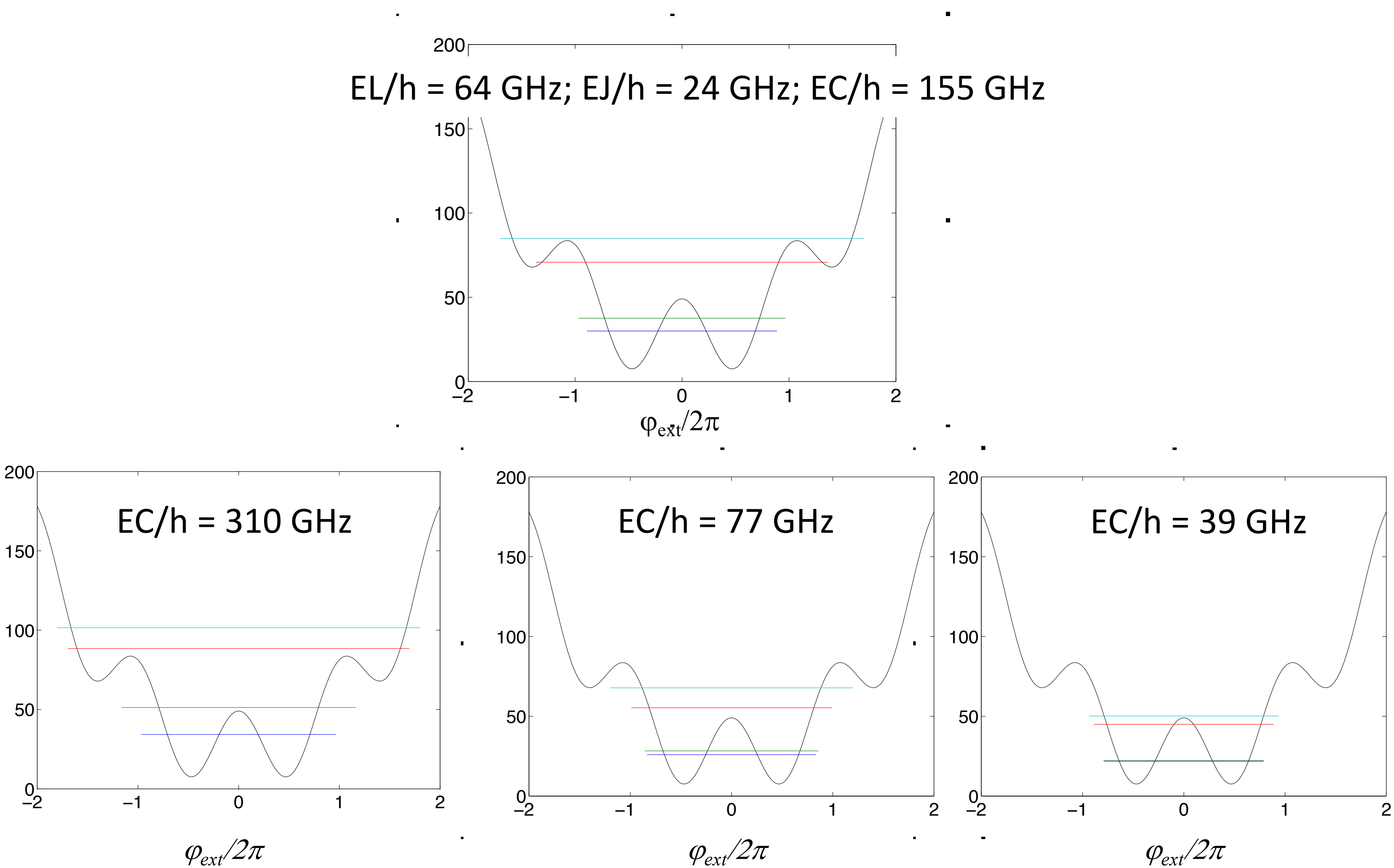
$$\boxed{U = -E_J \cos\varphi_J + \frac{\Phi_0}{2\pi}\frac{\left(\varphi_{ext} - \varphi_J\right)^2}{2L}}$$





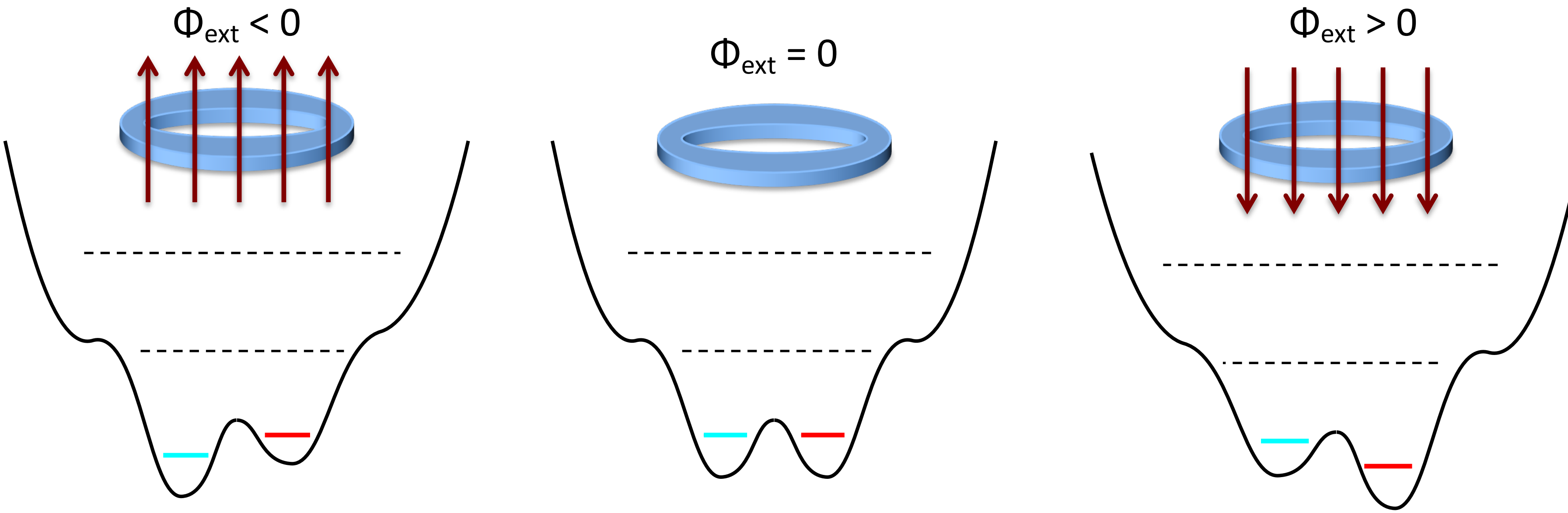
Two preferable states: $\varphi_J = 0$ and $\varphi_J = 2\pi$

The RF-SQUID flux qubit: particle mass dependence



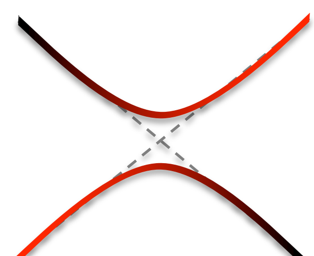
Level position and tunneling energy depend on the particle mass

The flux qubit

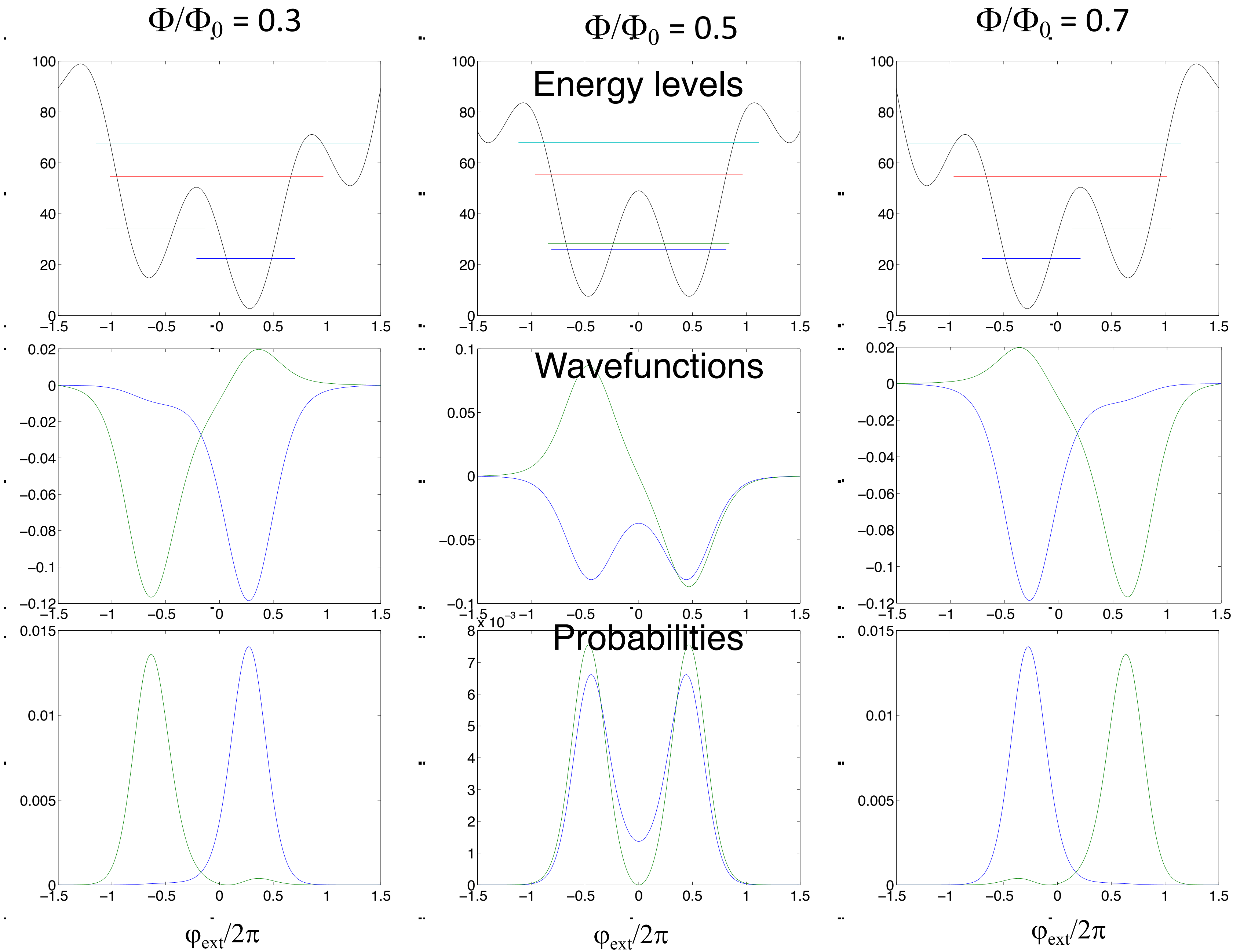


$$H = \begin{pmatrix} -E_L n_{ext} & -\Delta/2 \\ -\Delta/2 & E_L n_{ext} \end{pmatrix} = -\frac{\varepsilon}{2}\sigma_z - \frac{\Delta}{2}\sigma_x$$

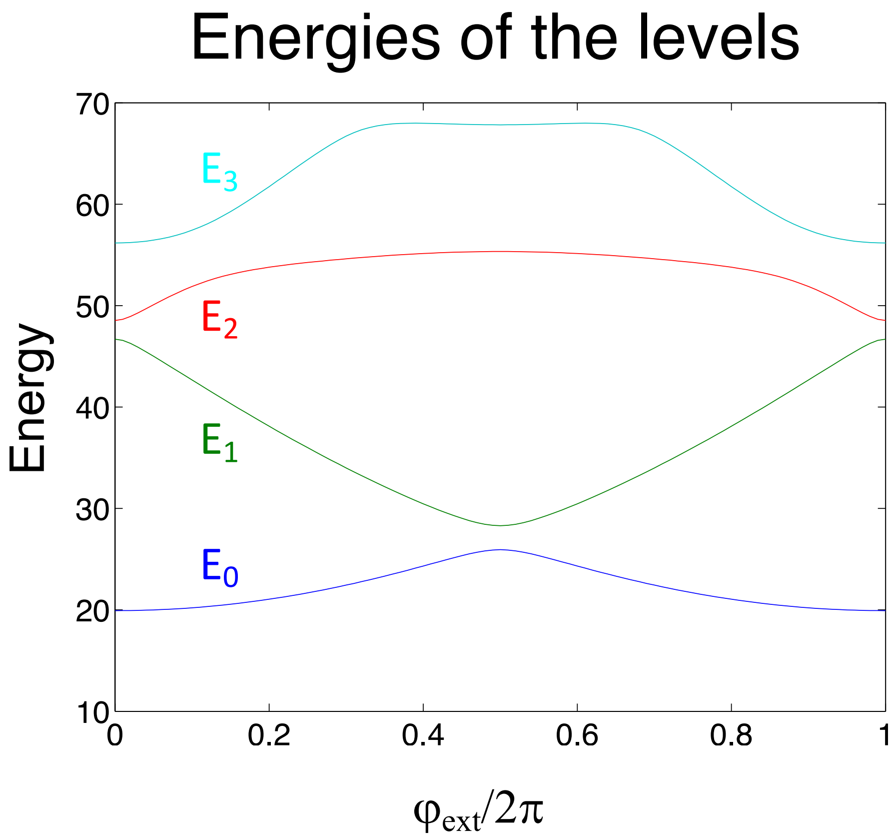
$$\varepsilon = 2E_L n_{ext}$$

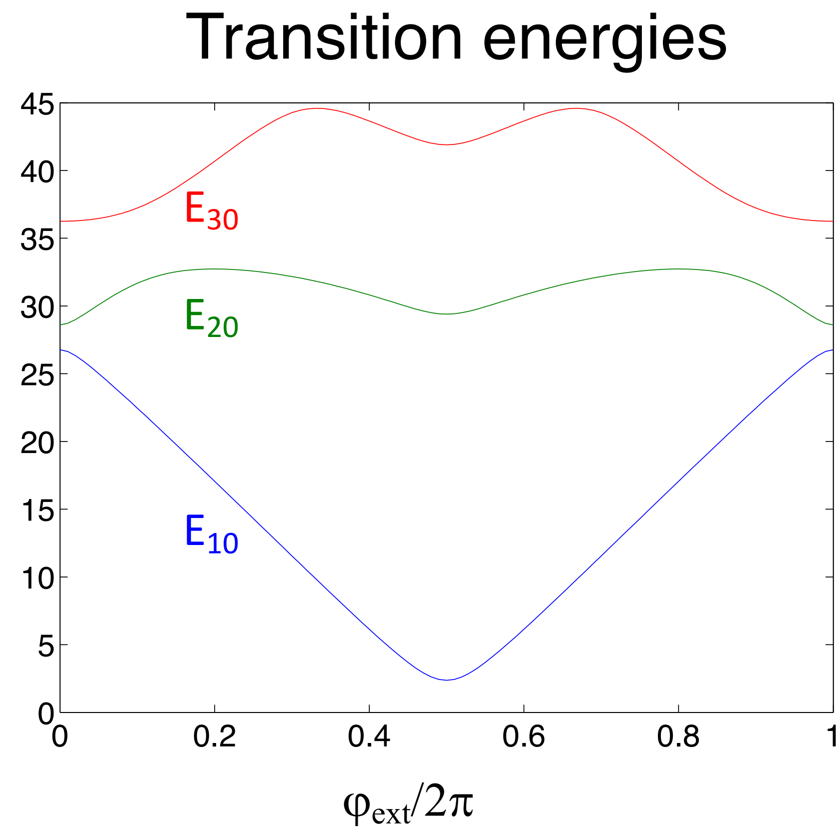


EL/h = 64 GHz; EJ/h = 24 GHz; EC/h = 77 Ghz;



Energies versus external flux (phase)





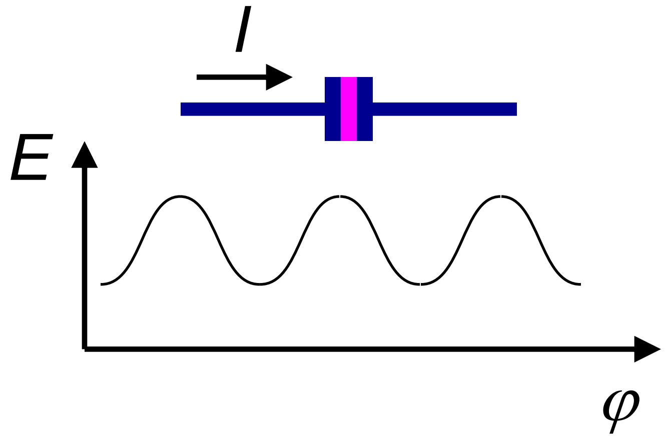
Two-level approximation:

$$H \approx -\frac{\varepsilon}{2}\sigma_z - \frac{\Delta}{2}\sigma_x$$

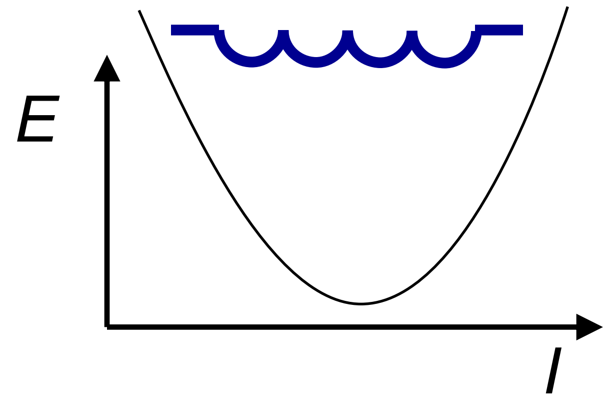
The flux qubit

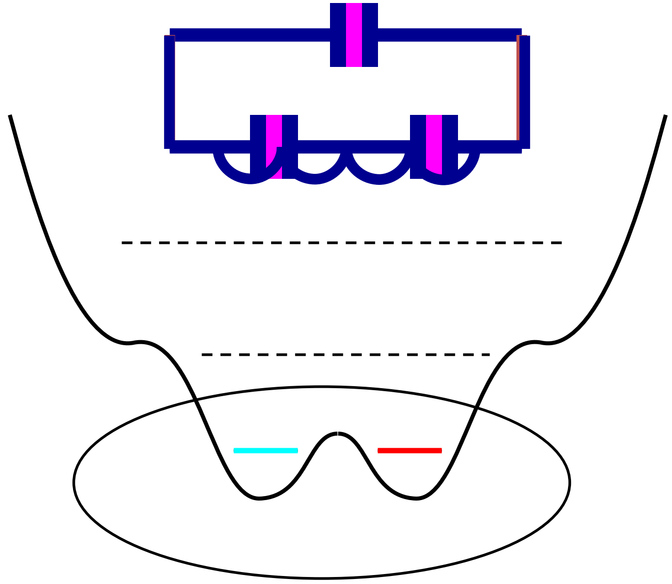
The flux qubit

Josephson junction

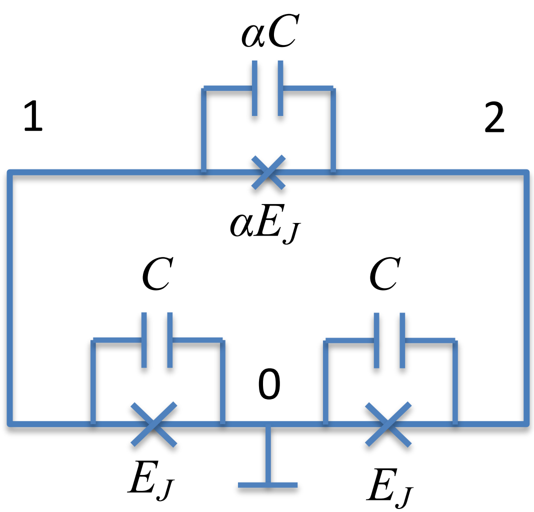


Inductance





The three-junction flux qubit



$$\varphi_{01}+\varphi_{12}+\varphi_{20}=\frac{2\pi}{\Phi_0}\Phi_{ext}=\varphi_{ext}$$

$$U=E_J\left(1-\cos\varphi_{01}\right)+\alpha E_J\left(1-\cos\varphi_{12}\right)+E_J\left(1-\cos\varphi_{20}\right)$$

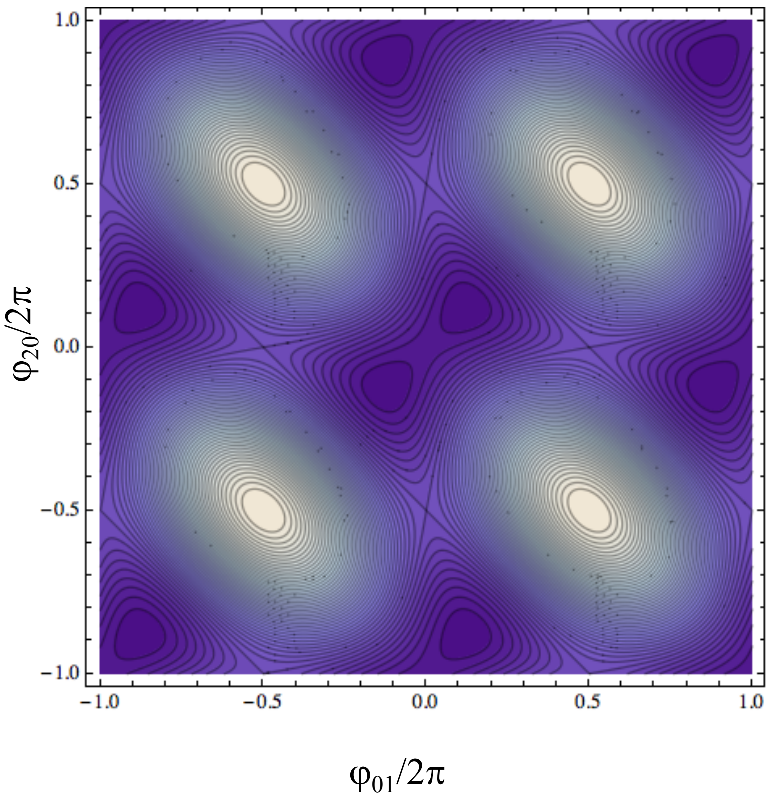
$$U=E_J\left[\left(1-\cos\varphi_{01}\right)+\alpha\left(1-\cos\left(\varphi_{ext}-\varphi_{01}-\varphi_{20}\right)\right)+\left(1-\cos\varphi_{20}\right)\right]$$

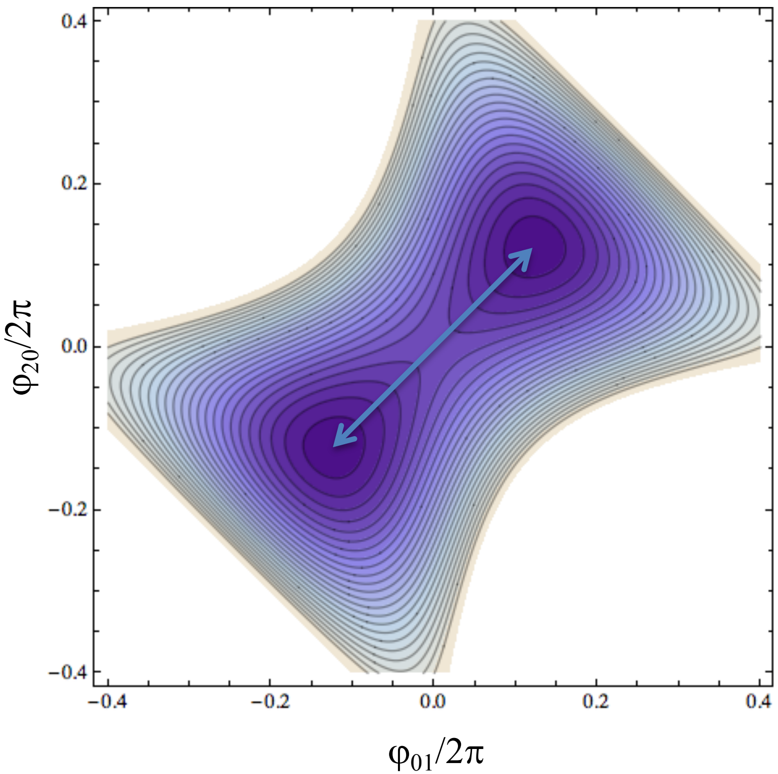
$$U=E_J\left[2+\alpha-\cos\varphi_{01}-\cos\varphi_{20}-\alpha\cos\left(\varphi_{ext}-\varphi_{01}-\varphi_{20}\right)\right]$$

Josephson potential of the three-junction loop

$$\alpha = 0.7$$

$$U/E_J = \left[2 + \alpha - \cos\varphi_{01} - \cos\varphi_{20} - \alpha\cos\left(\varphi_{ext} - \varphi_{01} - \varphi_{20}\right)\right]$$

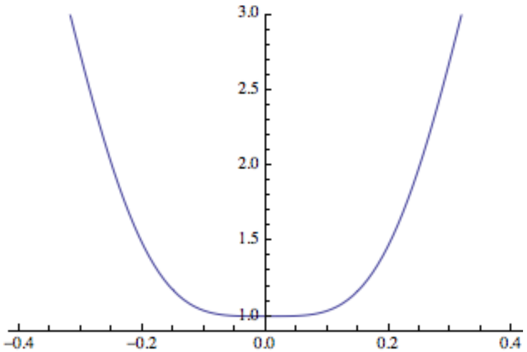




Shape of Josephson potential vs alpha

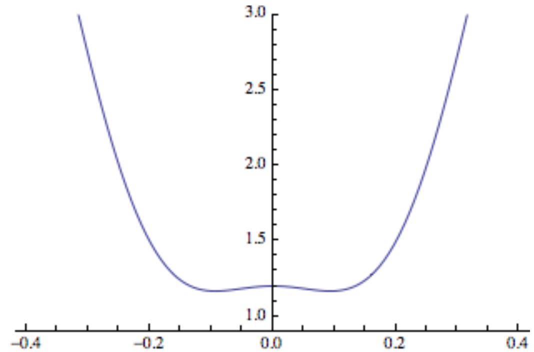
$\varphi_{01}/2\pi = \varphi_{20}/2\pi$

$\alpha = 0.5$



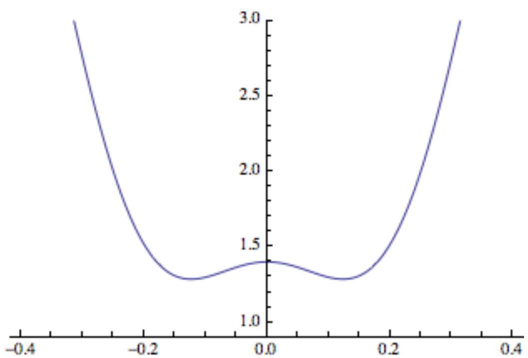
$\varphi_{01}/2\pi$

$\alpha = 0.6$



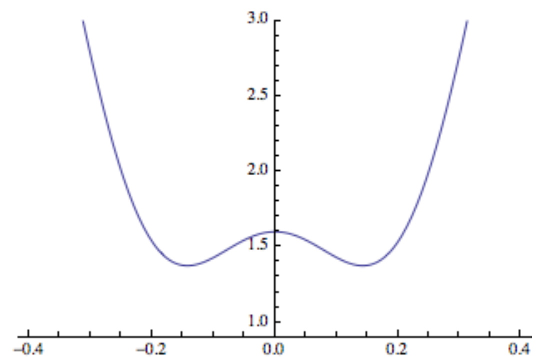
$\varphi_{01}/2\pi$

$\alpha = 0.7$



$\varphi_{01}/2\pi$

$\alpha = 0.8$

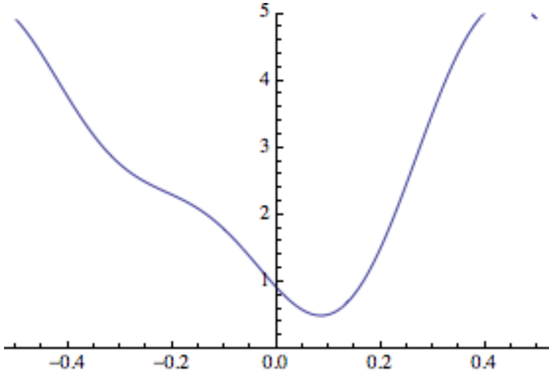


$\varphi_{01}/2\pi$

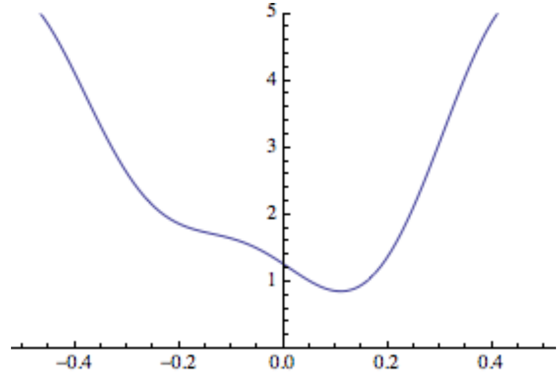
Josephson potential of the biased flux qubit

$$\alpha = 0.7$$

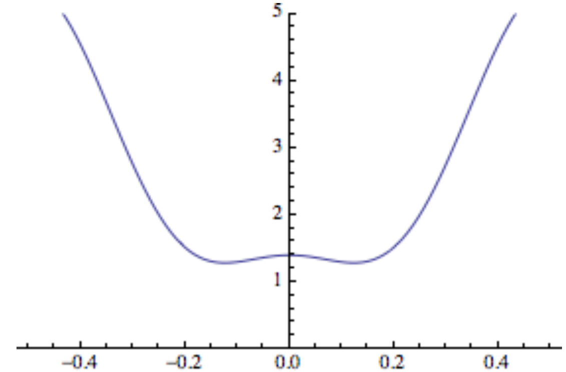
$\varphi_{ext}/2\pi = 0.3$



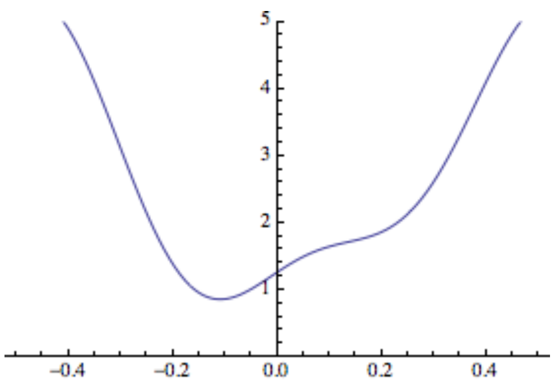
$\varphi_{ext}/2\pi = 0.4$



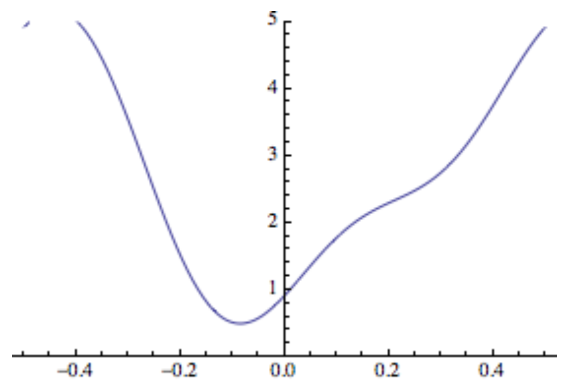
$\varphi_{ext}/2\pi = 0.5$



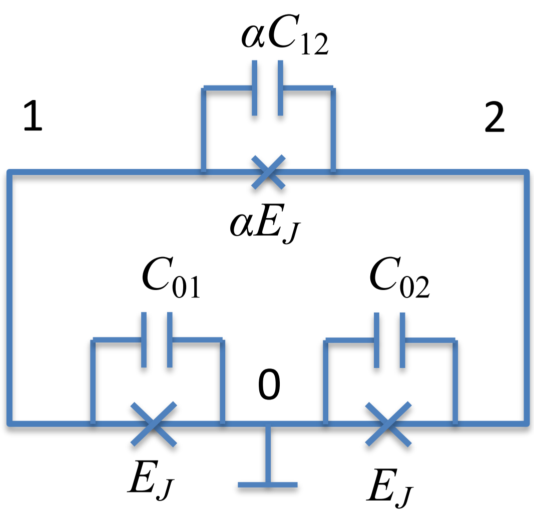
$\varphi_{ext}/2\pi = 0.6$



$\varphi_{ext}/2\pi = 0.7$



Biased flux qubit



Josephson potential:

$$U = E_J\left[2+\alpha-\cos\varphi_{01}-\cos\varphi_{20}-\alpha\cos\left(\varphi_{ext}-\varphi_{01}-\varphi_{20}\right)\right]$$

Charge:

$$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

Potential:

$$\vec{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

Capacitance matrix:

$$C = \begin{pmatrix} C_{01}+C_{12} & -C_{12} \\ -C_{12} & C_{02}+C_{12} \end{pmatrix}$$

$$\vec{n} = \frac{C\vec{V}}{2e} \qquad \vec{V} = 2eC^{-1}\vec{n}$$

Electrostatic energy: $T = \dfrac{(2e)^2}{2}\vec{n}C^{-1}\vec{n}$

The Hamiltonian:

$$\boxed{H = \frac{(2e)^2}{2}\vec{n}C^{-1}\vec{n} + E_J\left[2+\alpha-\cos\varphi_{01}-\cos\varphi_{20}-\alpha\cos\left(\varphi_{ext}-\varphi_{01}-\varphi_{20}\right)\right]}$$

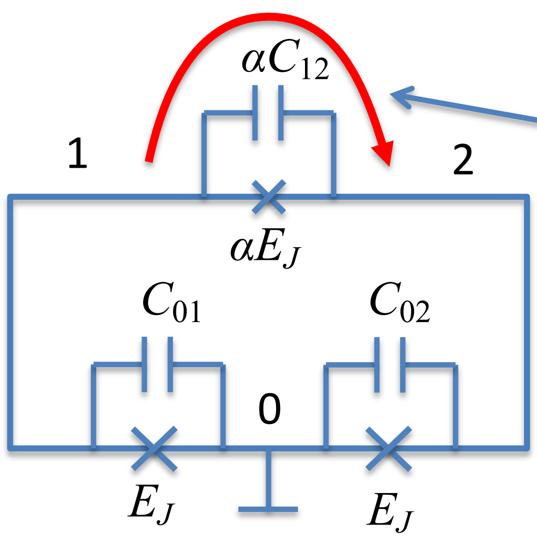
Phase operators

$$\cos\hat{\varphi} = \frac{e^{i\hat{\varphi}} + e^{-i\hat{\varphi}}}{2} \qquad e^{i\hat{\varphi}} = |N\rangle\langle N-1| \qquad e^{-i\hat{\varphi}} = |N-1\rangle\langle N|$$

$$\boxed{\cos\hat{\varphi}_{01} = \frac{1}{2}\left(|N_1\rangle\langle N_1-1| + |N_1-1\rangle\langle N_1|\right)} \qquad \boxed{\cos\hat{\varphi}_{20} = \frac{1}{2}\left(|N_2\rangle\langle N_2-1| + |N_2-1\rangle\langle N_2|\right)}$$

$$\cos\left(\varphi_{ext} - \hat{\varphi}_{01} - \hat{\varphi}_{20}\right) = \frac{e^{i(\varphi_{ext}-\hat{\varphi}_{01}-\hat{\varphi}_{20})} + e^{-i(\varphi_{ext}-\hat{\varphi}_{01}-\hat{\varphi}_{20})}}{2}$$

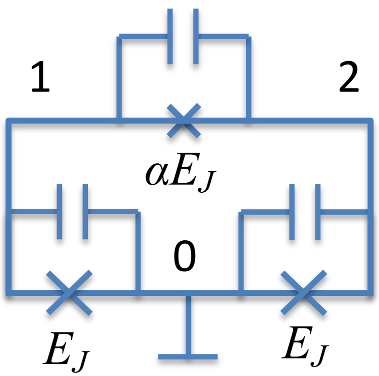
$$e^{i(\varphi_{ext}-\hat{\varphi}_{01}-\hat{\varphi}_{20})} = e^{i\varphi_{ext}} e^{-i\hat{\varphi}_{01}} e^{-i\hat{\varphi}_{20}} = e^{i\varphi_{ext}} |N_1-1\rangle\langle N_1||N_2\rangle\langle N_2-1| = e^{i\varphi_{ext}} \boxed{|N_1-1, N_2\rangle\langle N_1, N_2-1|}$$



Physical meaning: tunneling of a charge quantum (Cooper pair) from island 1 to island 2

$$\boxed{\cos\left(\varphi_{ext} - \hat{\varphi}_{01} - \hat{\varphi}_{20}\right) = \frac{1}{2}\left(e^{i\varphi_{ext}} |N_1-1, N_2\rangle\langle N_1, N_2-1| + e^{-i\varphi_{ext}} |N_1, N_2-1\rangle\langle N_1-1, N_2|\right)}$$

The flux qubit Hamiltonian in the charge basis

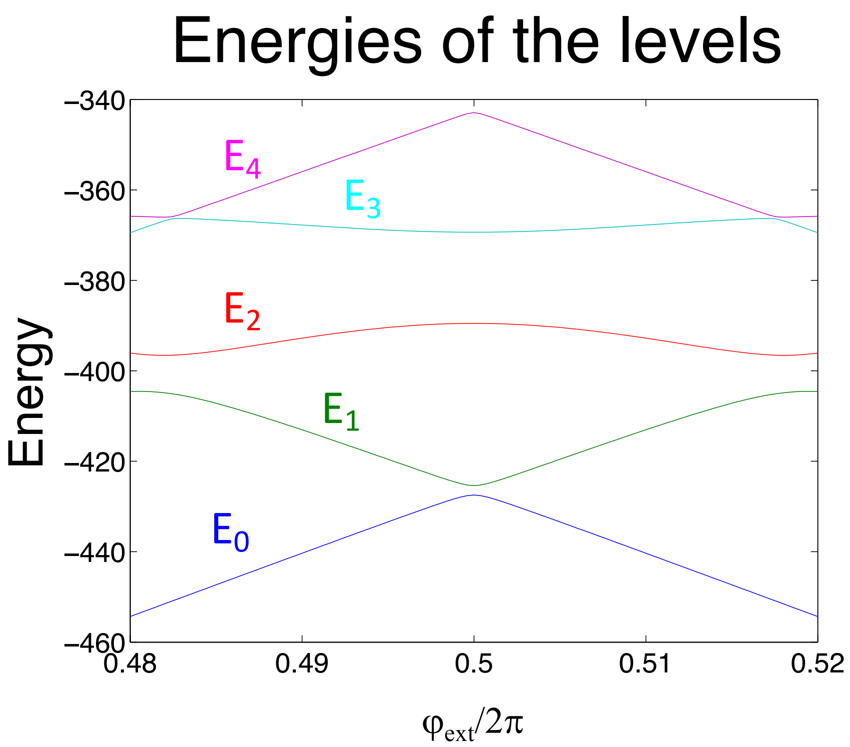


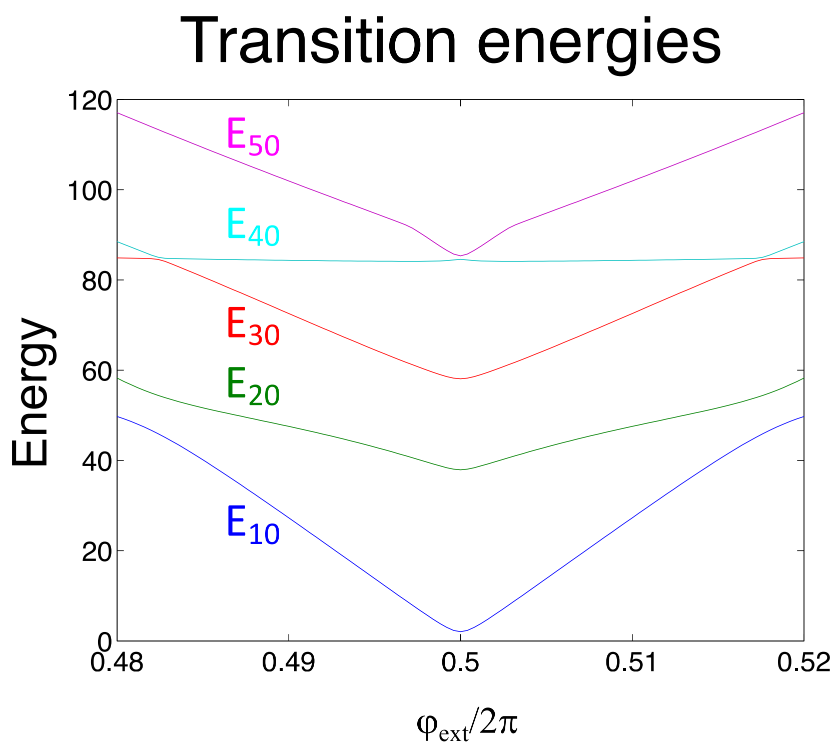
$$H = \frac{(2e)^2}{2}\vec{n}C^{-1}\vec{n} + E_J\left[2+\alpha-\cos\varphi_{01}-\cos\varphi_{20}-\alpha\cos\left(\varphi_{ext}-\varphi_{01}-\varphi_{20}\right)\right]$$

$$U(n_1,n_2) = \frac{(2e)^2}{2}\begin{pmatrix} n_1 & n_2 \end{pmatrix} C^{-1}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

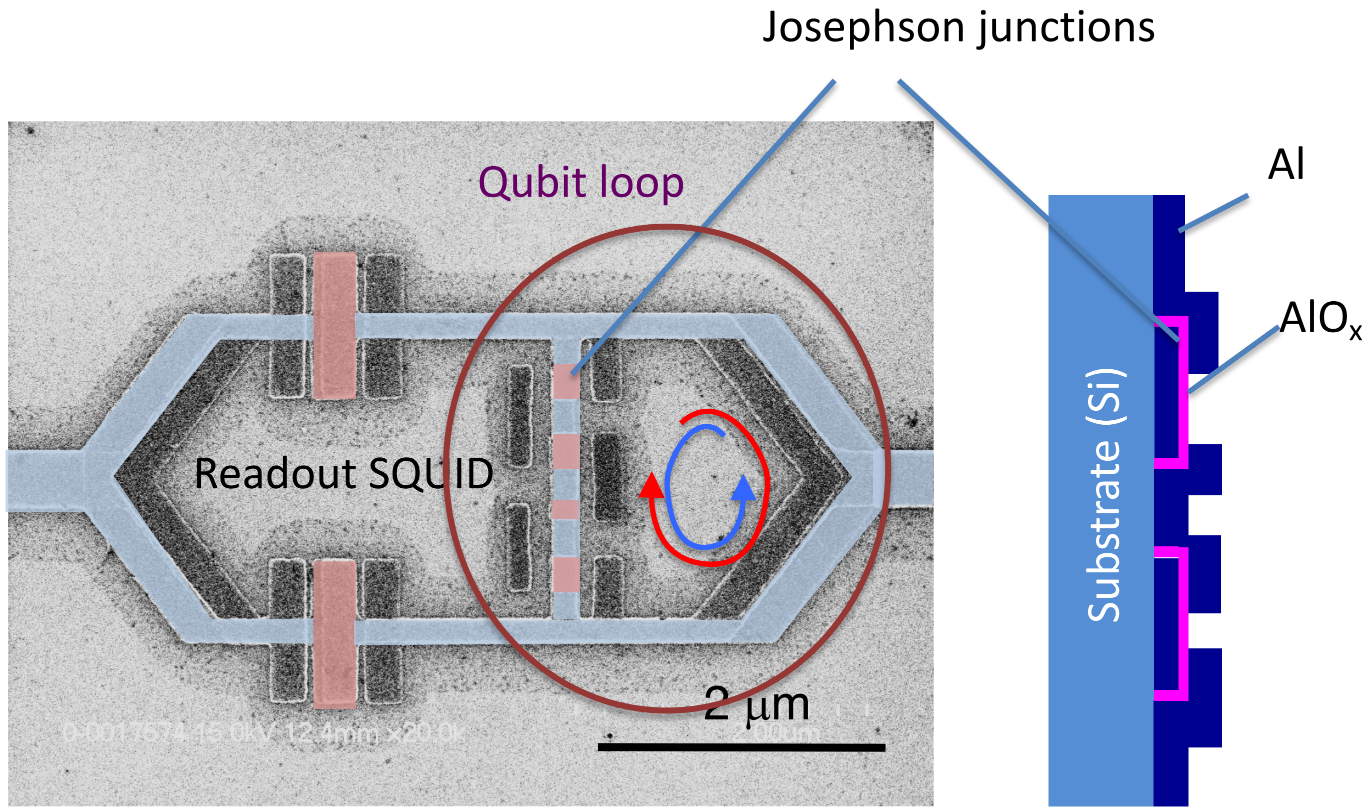
$$H = \begin{pmatrix} \ddots & & & & & & \\ & U(-1,0) & -\frac{\alpha E_J}{2}e^{i\varphi_{ext}} & -\frac{E_J}{2} & 0 & 0 & \\ & -\frac{\alpha E_J}{2}e^{-i\varphi_{ext}} & U(0,-1) & -\frac{E_J}{2} & 0 & 0 & \\ & -\frac{E_J}{2} & -\frac{E_J}{2} & U(0,0) & -\frac{E_J}{2} & -\frac{E_J}{2} & \\ & 0 & 0 & -\frac{E_J}{2} & U(0,1) & -\frac{\alpha E_J}{2}e^{i\varphi_{ext}} & \\ & 0 & 0 & -\frac{E_J}{2} & -\frac{\alpha E_J}{2}e^{-i\varphi_{ext}} & U(1,0) & \\ & & & & & & \ddots \end{pmatrix}$$

Energy bands in the flux quantum system





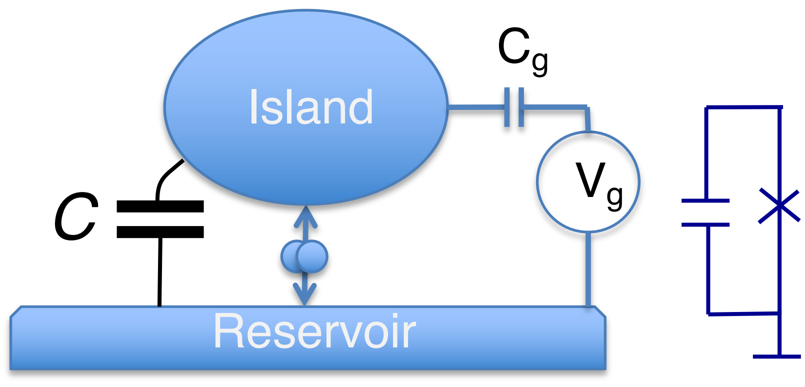
The four-junction flux qubit



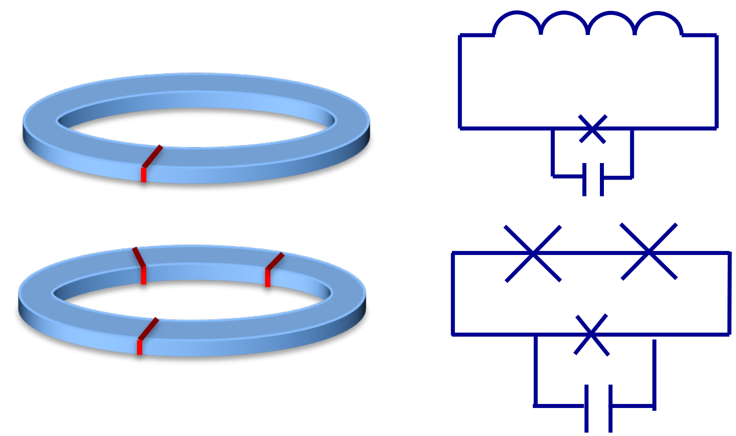
The charge vs flux qubits

$$H \approx -\frac{\varepsilon}{2}\sigma_z - \frac{\Delta}{2}\sigma_x$$

The charge qubit



The flux qubit



Any type of superconducting quantum systems utilize charge and flux quantization

Charge states (0, 2e): $|0\rangle$, $|1\rangle$

Cooper pair tunneling in/out

Charging energy: E_C

Tunneling energy: E_J

Controlled by V

$\varepsilon = -2E_C \delta N$

Flux states (0, 2π): $|0\rangle$, $|1\rangle$

Flux tunneling tunneling in/out

Magnetic energy: E_L

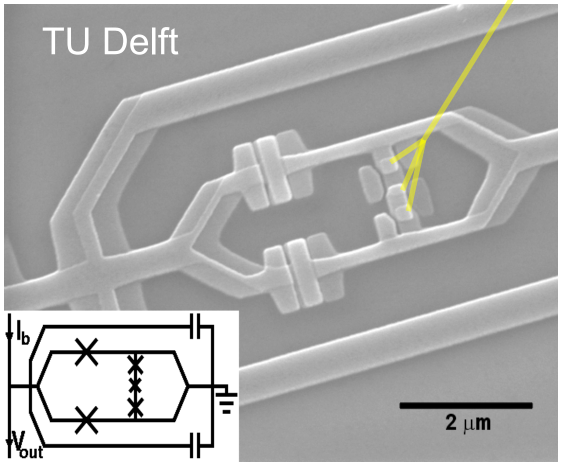
Tunneling energy: Δ

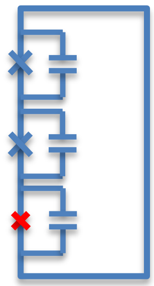
Controlled by B

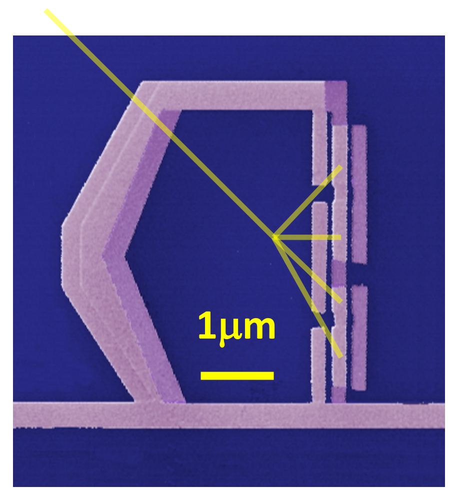
$\varepsilon = -2E_L \delta\varphi$

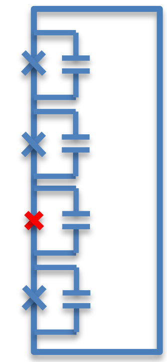
Josephson junction flux qubit

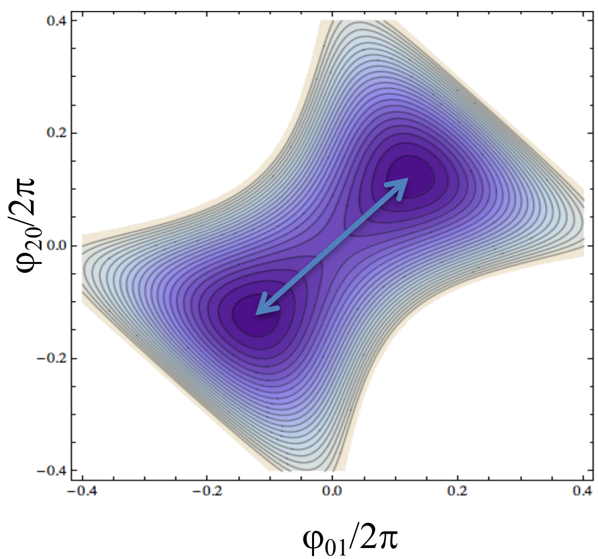
Josephson junctions

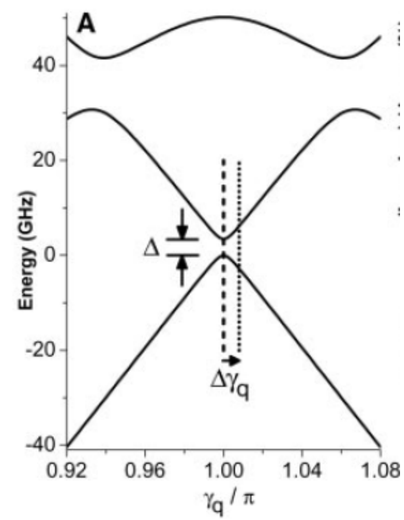








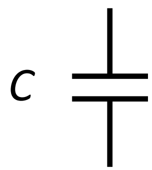




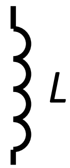
Quantum mechanics of electrical circuits. Capacitance, inductance, resonators

Electric circuit

Capacitor



Inductor



Voltage, current: V, I

Charge, magnetic flux: $Q = CV \quad \Phi = LI$

$$I = \dot{Q} \qquad V = \dot{\Phi}$$

Potential energy

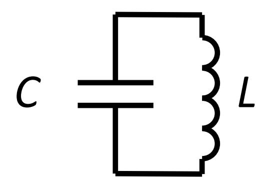
$$U = \frac{CV^2}{2}$$

$$U = \frac{Q^2}{2C}$$

Kinetic energy

$$T = \frac{LI^2}{2}$$

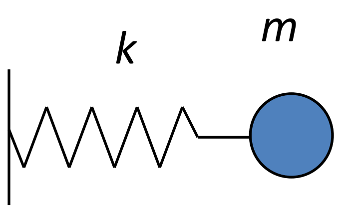
$$T = \frac{\Phi^2}{2L}$$



Total energy of the system

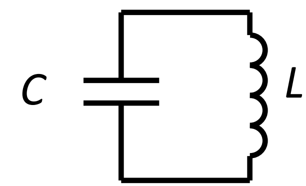
$$\boxed{E = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}} \qquad \boxed{E = \frac{CV^2}{2} + \frac{LI^2}{2}}$$

Classical harmonic oscillators



$$E = \frac{kx^2}{2} + \frac{p^2}{2m}$$

$$E = \frac{kx^2}{2} + \frac{m\dot{x}^2}{2}$$



$$E = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

$$E = \frac{Q^2}{2C} + \frac{L\dot{Q}^2}{2}$$

Coordinate: x

Momentum: $p = m\dot{x}$

Q

$p' = L\dot{Q} \rightarrow \Phi$

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad E = \frac{m}{2}\left(\omega_0^2 x^2 + \dot{x}^2\right)$$

$$E = \frac{L}{2}\left(\omega_0^2 \Phi^2 + \dot{\Phi}^2\right) \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

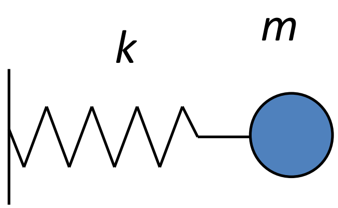
$$x = Ae^{i\omega t} + Be^{-i\omega t}$$

$$Q = Ae^{i\omega t} + Be^{-i\omega t}$$

The equations are transformed from one to another with the substitutions

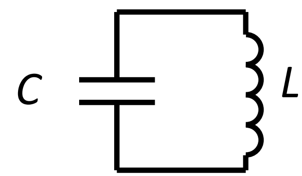
$$x \rightarrow Q \qquad p \rightarrow \Phi \qquad k \rightarrow \frac{1}{C} \qquad m \rightarrow L$$

Classical harmonic oscillators (alternative approach)



$$E = \frac{kx^2}{2} + \frac{p^2}{2m}$$

$$E = \frac{kx^2}{2} + \frac{m\dot{x}^2}{2}$$



$$E = \frac{\Phi^2}{2L} + \frac{Q^2}{2C}$$

$$E = \frac{\Phi^2}{2L} + \frac{C\dot{\Phi}^2}{2}$$

Coordinate: x $\qquad$ Φ

Momentum: $p = m\dot{x}$ $\qquad$ $p' = C\dot{\Phi} \to Q$

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad E = \frac{m}{2}\left(\omega_0^2 x^2 + \dot{x}^2\right)$$

$$E = \frac{C}{2}\left(\omega_0^2 Q^2 + \dot{Q}^2\right) \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$x = Ae^{i\omega t} + Be^{-i\omega t}$$

$$Q = Ae^{i\omega t} + Be^{-i\omega t}$$

The equations are identical with the following substitutions

$$\boxed{x \to \Phi \qquad p \to Q \qquad k \to \frac{1}{L} \qquad m \to C}$$

Quantum mechanics of harmonic oscillators

$$H = \frac{\hbar\omega_0}{2}\left(\frac{\hat{x}^2}{x_0^2} + \frac{\hat{p}^2}{p_0^2}\right) \qquad H = \frac{\hbar\omega_0}{2}\left(\frac{\hat{Q}^2}{Q_0^2} + \frac{\hat{\Phi}^2}{\phi_0^2}\right)$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega_0}} \qquad p_0 = \frac{\hbar}{x_0} \qquad Q_0 = \sqrt{\frac{\hbar}{L\omega_0}} = \sqrt{\hbar C\omega_0} \qquad \phi_0 = \frac{\hbar}{Q_0} = \sqrt{\hbar L\omega_0}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Because $CV = Q \quad LI = \Phi$ we can rewrite the Hamiltonian of LC-resonator in the form

$$H = \frac{\hbar\omega_0}{2}\left(\frac{\hat{V}^2}{V_0^2} + \frac{\hat{I}^2}{I_0^2}\right) \qquad V_0 = \frac{Q_0}{C} = \sqrt{\frac{\hbar C\omega_0}{C^2}} = \boxed{\sqrt{\frac{\hbar\omega_0}{C}}} \qquad I_0 = \frac{\phi_0}{L} = \sqrt{\frac{\hbar L\omega_0}{L^2}} = \boxed{\sqrt{\frac{\hbar\omega_0}{L}}}$$

Hint to remember: $\frac{CV_0^2}{2} = \frac{LI_0^2}{2} = \frac{\hbar\omega_0}{2}$

Position operator: $\hat{x}$

Momentum operator: $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

Commutation relations: $\left[\hat{p}, \hat{x}\right] = -i\hbar$

Voltage operator: $\hat{V} = \hat{Q}/C$

Current operator: $\hat{I} = \frac{\hat{\Phi}}{L} = -i\frac{\hbar}{L}\frac{\partial}{\partial Q} = -i\frac{\hbar}{LC}\frac{\partial}{\partial V}$

$$\left[L\hat{I}, C\hat{V}\right] = -i\hbar$$

$$y = \frac{x}{x_0} \qquad y = \frac{Q}{Q_0} \text{ or } y = \frac{V}{V_0}$$

$$\boxed{\hat{H} = \frac{\hbar\omega_0}{2}\left(y^2 - \frac{\partial^2}{\partial y^2}\right)}$$

Quantum mechanics of harmonic oscillators

$$H = \frac{\hbar\omega_0}{2}\left(\frac{\hat{x}^2}{x_0^2} + \frac{\hat{p}^2}{p_0^2}\right)$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega_0}} \qquad p_0 = \frac{\hbar}{x_0}$$

$$H = \frac{\hbar\omega_0}{2}\left(\frac{\hat{V}^2}{V_0^2} + \frac{\hat{I}^2}{I_0^2}\right)$$

$$V_0 = \sqrt{\frac{\hbar\omega_0}{C}} \qquad I_0 = \sqrt{\frac{\hbar\omega_0}{L}}$$

Position operator: $\hat{x}$

Momentum operator: $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

Commutation relations: $\left[\hat{p}, \hat{x}\right] = -i\hbar$

Voltage operator: $\hat{V}$

Current operator: $\hat{I} = -i\frac{\hbar}{LC}\frac{\partial}{\partial V}$

$\left[L\hat{I}, C\hat{V}\right] = -i\hbar$

$$y = \frac{x}{x_0} \qquad y = \frac{V}{V_0}$$

$$\hat{H} = \frac{\hbar\omega_0}{2}\left(y^2 - \frac{\partial^2}{\partial y^2}\right)$$