Quantum Electronics of Nanostructures

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Lecture 7b

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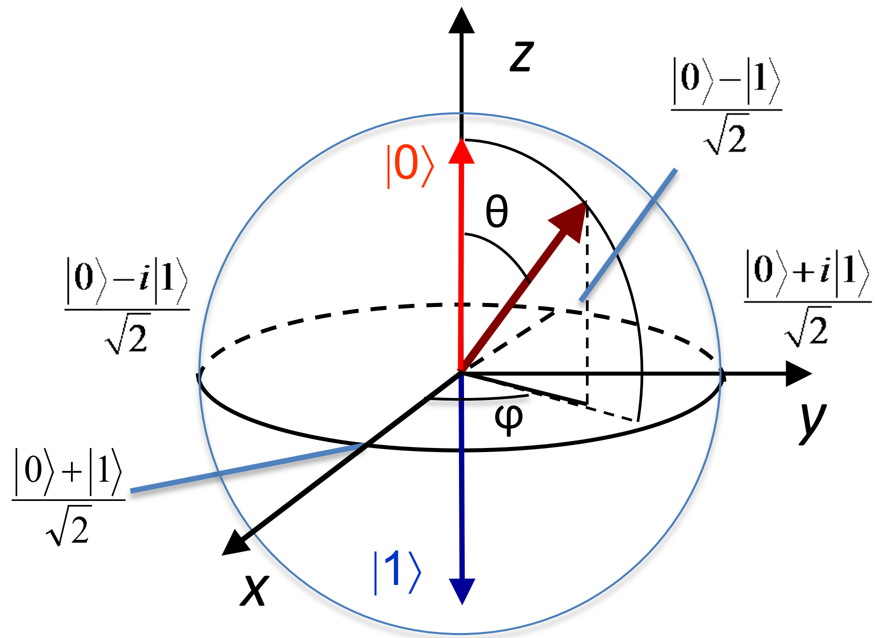
- Dissipation and decoherence in two-level systems
- Density matrix approach
- Pure and mixed states
- Bloch sphere for mixed states and dissipative dynamics
- Relaxation and dephasing

Single-qubit operations

$$H = -\frac{\hbar\Omega}{2}\sigma_j \qquad U = \exp\left(i\frac{\Omega t}{2}\sigma_j\right) = I\cos\left(\frac{\Omega t}{2}\right) + i\sigma_j \sin\left(\frac{\Omega t}{2}\right)$$

$$\Omega t = \pi: \qquad U\left(\frac{\pi}{\Omega}\right) = i\sigma_j$$



$$t = \frac{\pi}{\Omega} \quad H = -\frac{\hbar\Omega}{2}\sigma_y \Rightarrow R = i\sigma_y$$

NOT (BIT FLIP) π-rotation around x-axis

$$i\sigma_y|0\rangle = -|1\rangle \quad i\sigma_y|1\rangle = |0\rangle$$

$$t = \frac{\pi}{\Omega} \quad H = -\frac{\hbar\Omega}{2}\sigma_x \Rightarrow R = i\sigma_x$$

CONJUGATED FLIP π-rotation around y-axis

$$i\sigma_x|0\rangle = i|1\rangle \qquad i\sigma_x|1\rangle = i|0\rangle$$

Master Equation

Schrodinger equation:

$$i\hbar \frac{\partial \left|\Psi\right\rangle}{\partial t} = H\left|\Psi\right\rangle \qquad -i\hbar \frac{\partial \left\langle\Psi\right|}{\partial t} = \left\langle\Psi\right|H$$

$$i\hbar \frac{\partial \left|\Psi\right\rangle\left\langle\Psi\right|}{\partial t} = i\hbar \frac{\partial \left|\Psi\right\rangle}{\partial t}\left\langle\Psi\right| + i\hbar \left|\Psi\right\rangle \frac{\partial \left\langle\Psi\right|}{\partial t} \qquad i\hbar \frac{\partial \left|\Psi\right\rangle\left\langle\Psi\right|}{\partial t} = H\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|H$$

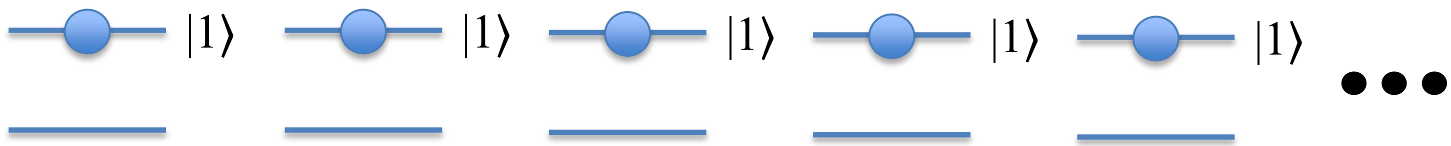
$$\left[H,\rho\right] = H\rho - \rho H$$

$$\boxed{i\hbar \frac{\partial \rho}{\partial t} = \left[H,\rho\right]}$$

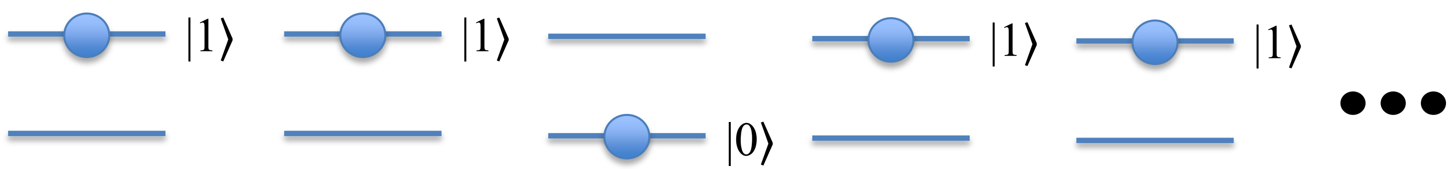
Quantum dynamics of two-level systems with decoherence

Pure and Mixed states

Pure state: wavefunction is $|1\rangle$; Density matrix: $\rho = |1\rangle\langle 1|$



Mixed state: the state which can not be described by a wavefunction => probabilities
Density matrix: $\rho = 0.8|1\rangle\langle 1| + 0.2|0\rangle\langle 0|$



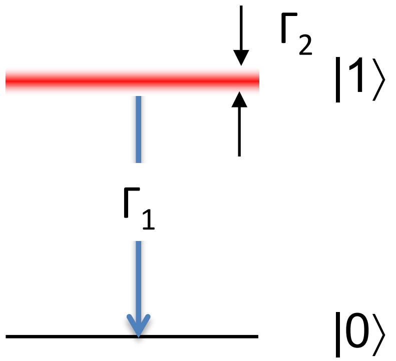
Phase fluctuations:

$$\left|\Psi_1\right\rangle = \frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}} \quad \rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad \left|\Psi_2\right\rangle = \frac{\left|0\right\rangle + e^{i\varphi}\left|1\right\rangle}{\sqrt{2}} \quad \rho = \begin{pmatrix} \frac{1}{2} & \frac{e^{i\varphi}}{2} \\ \frac{e^{-i\varphi}}{2} & \frac{1}{2} \end{pmatrix} \qquad \left|\Psi_3\right\rangle = \frac{\left|0\right\rangle + e^{-i\varphi}\left|1\right\rangle}{\sqrt{2}} \quad \rho = \begin{pmatrix} \frac{1}{2} & \frac{e^{-i\varphi}}{2} \\ \frac{e^{i\varphi}}{2} & \frac{1}{2} \end{pmatrix} \quad \bullet\bullet\bullet$$

$$\rho_{00} = \rho_{11} = \frac{1}{2} \qquad \rho_{01} = \frac{1}{N}\left(\frac{1}{2} + \frac{e^{i\phi}}{2} + \frac{e^{-i\phi}}{2} + \ldots\right) \approx \frac{1}{2}\frac{1}{N}\left(1 + \left(1 + i\phi + \frac{(i\phi)^2}{2!}\right) + \left(1 - i\phi + \frac{(-i\phi)^2}{2!}\right) + \ldots\right) = \frac{1 - \frac{\phi^2}{2}}{2}$$

Mixed state: $$\left|\rho_{01}\right| = \left|\rho_{10}\right| < \sqrt{\left|\rho_{00}\right|\left|\rho_{11}\right|} = \frac{1}{2}$$

The Master Equation and the Lindblad term



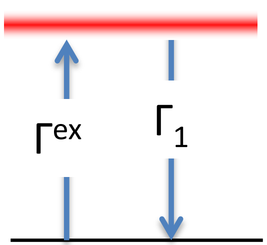
Relaxation and dephasing: Γ_1, Γ_2

The Lindblad term

$$\boxed{\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar}[H,\rho] + L}$$

$$\begin{pmatrix} \dot{\rho}_{00} & \dot{\rho}_{01} \\ \dot{\rho}_{10} & \dot{\rho}_{11} \end{pmatrix}$$

$$\boxed{L = \begin{pmatrix} \Gamma_1 \rho_{11} & -\Gamma_2 \rho_{01} \\ -\Gamma_2 \rho_{10} & -\Gamma_1 \rho_{11} \end{pmatrix}}$$

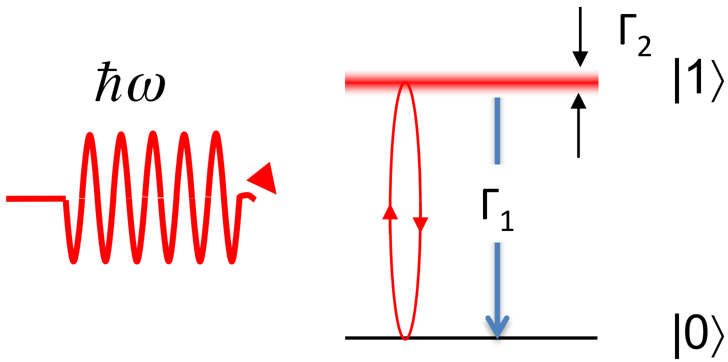


Accounting excitation (e.g. T > 0)

$$\boxed{L = \begin{pmatrix} \Gamma_1 \rho_{11} - \Gamma^{ex} \rho_{00} & -\Gamma_2 \rho_{01} \\ -\Gamma_2 \rho_{10} & -\Gamma_1 \rho_{11} + \Gamma^{ex} \rho_{00} \end{pmatrix}}$$

Driven two-level system with incoherent processes

The general form of the Hamiltonian driven by a wave with an arbitrary phase shift



Relaxation and dephasing: Γ_1, Γ_2

Dephasing: $\Gamma_2 = \frac{\Gamma_1}{2} + \Gamma_\varphi$

Pure dephasing

$$H = -\frac{\hbar\omega}{2}\sigma_z + \hbar\Omega\sigma_x\cos(\omega t - \varphi) \qquad U = e^{-i\frac{\omega t}{2}\sigma_z}$$

$$\boxed{H' = \frac{\hbar\Omega}{2}\left(\sigma_x\cos\varphi + \sigma_y\sin\varphi\right)}$$

$$\frac{\partial\rho}{\partial t} = -\frac{i}{\hbar}[H',\rho] + L \qquad \rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \qquad L = \begin{pmatrix} \Gamma_1\rho_{11} & -\Gamma_2\rho_{01} \\ -\Gamma_2\rho_{10} & -\Gamma_1\rho_{11} \end{pmatrix}$$

$$L = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -\Gamma_1\rho_{00} & -\Gamma_2\rho_{01} \\ -\Gamma_2\rho_{10} & -\Gamma_1\rho_{11} \end{pmatrix}$$

Driven two-level system with incoherent processes

The RWA Hamiltonian:

$$H = \frac{\hbar\Omega}{2}\left(\sigma_x \cos\varphi + \sigma_y \sin\varphi\right)$$

The Master Equation:

$$\frac{\partial\rho}{\partial t} = -\frac{i}{\hbar}[H,\rho] + L$$

The Lindblad term:

$$L = \begin{pmatrix} \Gamma_1\rho_{11} & -\Gamma_2\rho_{01} \\ -\Gamma_2\rho_{10} & -\Gamma_1\rho_{11} \end{pmatrix}$$

Three independent equations for $\rho_{00},\ \rho_{01},\ \rho_{10}$

$$\rho_{00} + \rho_{11} = 1$$

$$\begin{pmatrix} \dot\rho_{00} & \dot\rho_{01} \\ \dot\rho_{10} & \dot\rho_{11} \end{pmatrix} = -\frac{i}{\hbar}(H\rho - \rho H) + L$$

$$\frac{\partial\langle\sigma_j\rangle}{\partial t} = \mathrm{tr}\left[\sigma_j \frac{\partial\rho}{\partial t}\right]$$

$$\frac{\partial\langle\sigma_j\rangle}{\partial t} = \mathrm{tr}\left[-\frac{i}{\hbar}\sigma_j(H\rho - \rho H) + \sigma_j L\right] = -\frac{i}{\hbar}\mathrm{tr}\left[\sigma_j H\rho - H\sigma_j\rho\right] + \mathrm{tr}\left[\sigma_j L\right] = \frac{i}{\hbar}\left\langle\left[H,\sigma_j\right]\right\rangle + \mathrm{tr}\left[\sigma_j L\right]$$

$$\frac{\partial\langle\sigma_x\rangle}{\partial t} = -i\frac{\Omega}{2}\left(\langle[\sigma_x,\sigma_x]\rangle\cos\varphi + \langle[\sigma_y,\sigma_x]\rangle\sin\varphi\right) + \mathrm{tr}[\sigma_x L]$$

$$\frac{\partial\langle\sigma_y\rangle}{\partial t} = -i\frac{\Omega}{2}\left(\langle[\sigma_x,\sigma_y]\rangle\cos\varphi + \langle[\sigma_y,\sigma_y]\rangle\sin\varphi\right) + \mathrm{tr}[\sigma_y L]$$

$$\frac{\partial\langle\sigma_z\rangle}{\partial t} = -i\frac{\Omega}{2}\left(\langle[\sigma_x,\sigma_z]\rangle\cos\varphi + \langle[\sigma_y,\sigma_z]\rangle\sin\varphi\right) + \mathrm{tr}[\sigma_z L]$$

$$\frac{\partial\langle\sigma_x\rangle}{\partial t} = -\Omega\langle\sigma_z\rangle\sin\varphi - \Gamma_2\langle\sigma_x\rangle$$

$$\frac{\partial\langle\sigma_y\rangle}{\partial t} = \Omega\langle\sigma_z\rangle\cos\varphi - \Gamma_2\langle\sigma_y\rangle$$

$$\frac{\partial\langle\sigma_z\rangle}{\partial t} = \Omega\left(-\langle\sigma_y\rangle\cos\varphi + \langle\sigma_x\rangle\sin\varphi\right) - \Gamma_1\langle\sigma_z\rangle + \Gamma_1$$

$$\underbrace{\frac{\partial}{\partial t}\begin{pmatrix}\langle\sigma_x\rangle \\ \langle\sigma_y\rangle \\ \langle\sigma_z\rangle\end{pmatrix}}_{\frac{\partial\vec\sigma}{\partial t}} = \underbrace{\begin{pmatrix} -\Gamma_2 & 0 & -\Omega\sin\varphi \\ 0 & -\Gamma_2 & \Omega\cos\varphi \\ \Omega\sin\varphi & -\Omega\cos\varphi & -\Gamma_1 \end{pmatrix}}_{B}\underbrace{\begin{pmatrix}\langle\sigma_x\rangle \\ \langle\sigma_y\rangle \\ \langle\sigma_z\rangle\end{pmatrix}}_{\vec\sigma} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ \Gamma_1 \end{pmatrix}}_{\vec b}$$

Dynamics of the two level system is described by:

$$\frac{d\vec\sigma}{dt} = B\vec\sigma + \vec b$$

Bloch 'ball' for dissipative spin dynamics

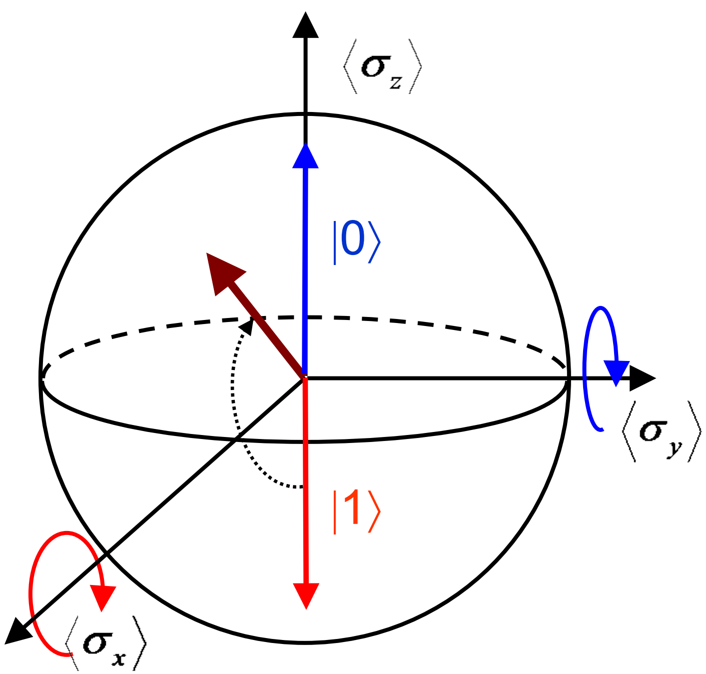
$$\frac{\partial}{\partial t}\begin{pmatrix}\langle\sigma_x\rangle\\ \langle\sigma_y\rangle\\ \langle\sigma_z\rangle\end{pmatrix}=\begin{pmatrix}-\Gamma_2 & 0 & -\Omega\sin\varphi\\ 0 & -\Gamma_2 & \Omega\cos\varphi\\ \Omega\sin\varphi & -\Omega\cos\varphi & -\Gamma_1\end{pmatrix}\begin{pmatrix}\langle\sigma_x\rangle\\ \langle\sigma_y\rangle\\ \langle\sigma_z\rangle\end{pmatrix}+\begin{pmatrix}0\\ 0\\ \Gamma_1\end{pmatrix}$$

$$\frac{d\vec{\sigma}}{dt}=B\vec{\sigma}+\vec{b}$$

Vector length can be less than one
(alternative criterion of mixed states)

$$\vec{\sigma}(t)=e^{Bt}\vec{\sigma}(0)+B^{-1}\left(e^{Bt}-1\right)\vec{b}$$

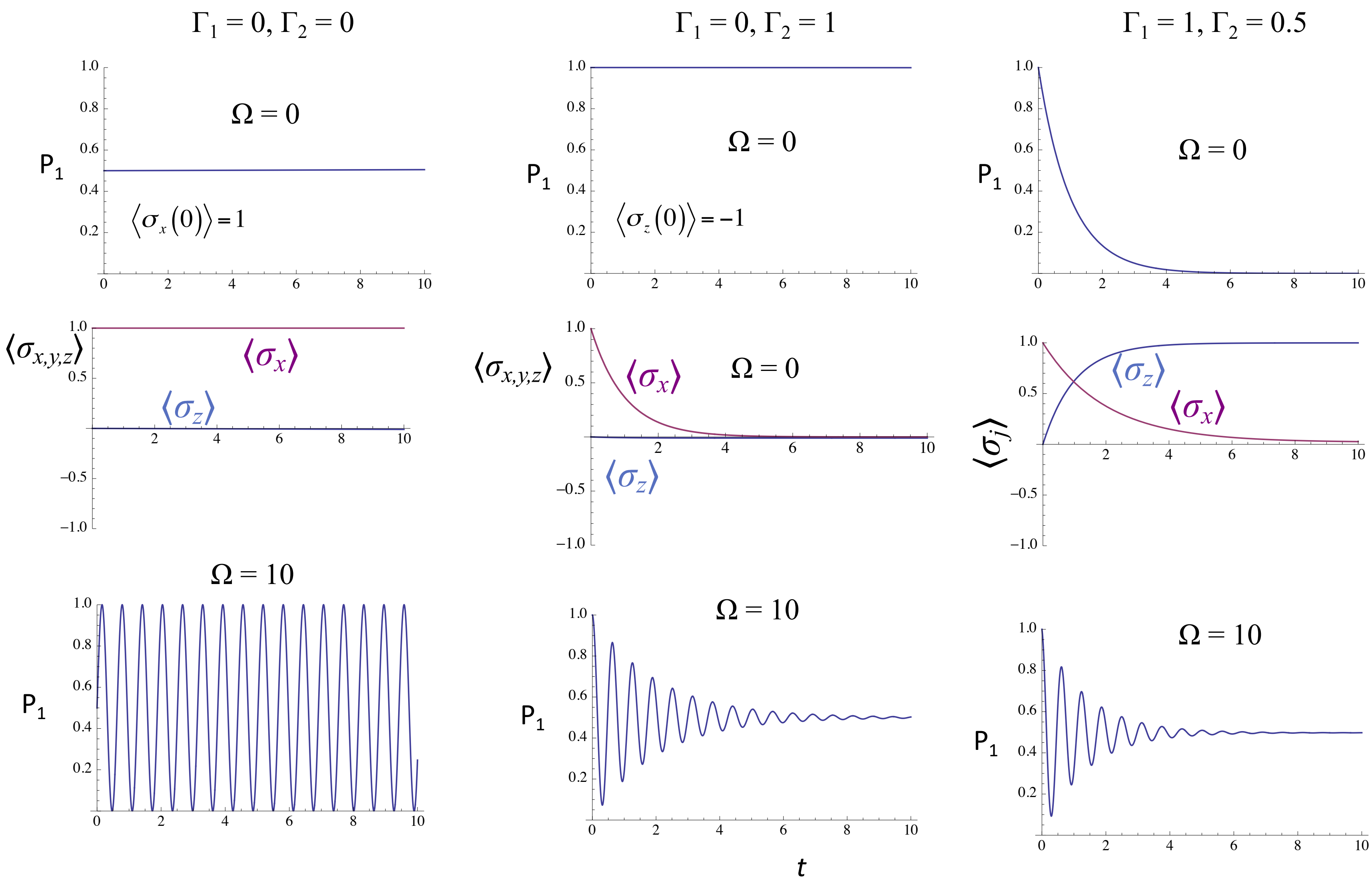
$$\vec{\sigma}(t)=e^{Bt}\left[\vec{\sigma}(0)+B^{-1}\vec{b}\right]-B^{-1}\vec{b}$$



Dynamics of the two-level system is equivalent to the dynamics of spin 1/2

Rabi oscillations: Oscillations in a two-level system under external drive. Oscillations of atomic states.

No decoherence | Pure dephasing | Relaxation



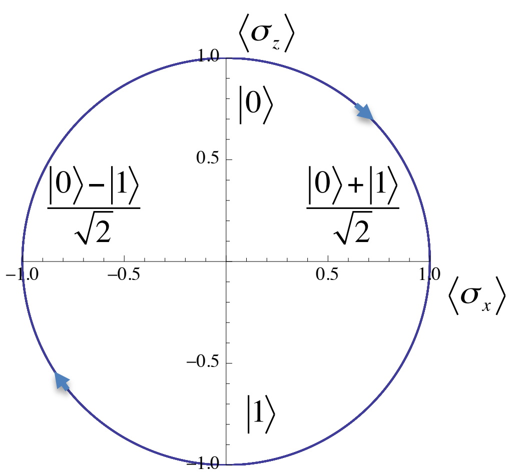
No decoherence $\Gamma_1 = 0, \Gamma_2 = 0$

$$\Psi = \cos\frac{\Omega t}{2}|0\rangle + \sin\frac{\Omega t}{2}|1\rangle$$

$$\rho_{00} = \sin^2\frac{\Omega t}{2} = \frac{1+\cos\Omega t}{2} \qquad \langle\sigma_x\rangle = \rho_{01} + \rho_{10} = \sin\Omega t$$

$$\rho_{01} = \cos\frac{\Omega t}{2}\sin\frac{\Omega t}{2} = \frac{\sin\Omega t}{2} \qquad \langle\sigma_y\rangle = i\rho_{01} - i\rho_{10} = 0$$

$$\rho_{10} = \cos\frac{\Omega t}{2}\sin\frac{\Omega t}{2} = \frac{\sin\Omega t}{2} \qquad \langle\sigma_z\rangle = 2\rho_{00} - 1 = \cos\Omega t$$



With decoherence

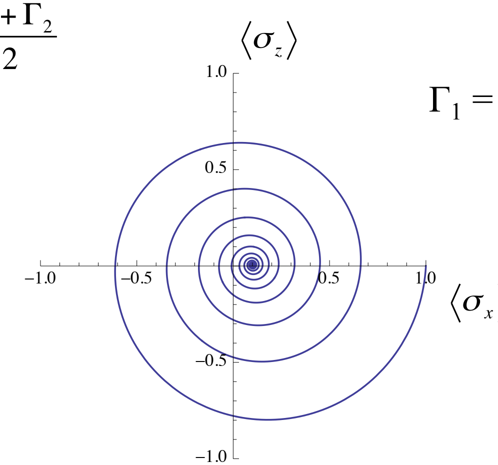
$$\rho_{00} = 1$$

$$\langle\sigma_x\rangle \approx e^{-\gamma t}\sin\Omega t \qquad \rho_{00} = \frac{\langle\sigma_z + 1\rangle}{2} = \frac{1 + e^{-\gamma t}\cos\Omega t}{2}$$

$$\langle\sigma_y\rangle = 0 \qquad \rho_{01} = \frac{e^{-\gamma t}\sin\Omega t}{2}$$

$$\langle\sigma_z\rangle \approx e^{-\gamma t}\cos\Omega t \qquad \rho_{10} \approx \frac{e^{-\gamma t}\sin\Omega t}{2}$$

$$\gamma = \frac{\Gamma_1 + \Gamma_2}{2}$$



$\Gamma_1 = 1, \Gamma_2 = 0.5$

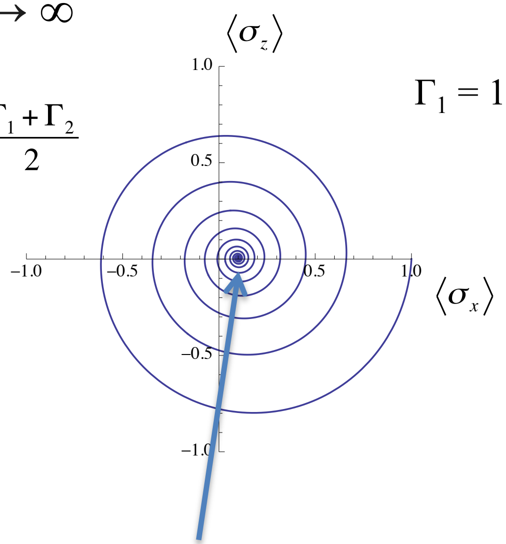
Stationary Master Equation

$$\frac{\partial \rho}{\partial t} = 0 \qquad -\frac{i}{\hbar}\left[H,\rho\right] + L = 0$$

$$\frac{\partial \vec{\sigma}}{\partial t} = 0 \qquad B\vec{\sigma} + \vec{b} = 0 \qquad \vec{\sigma} = -B^{-1}\vec{b}$$

$$t \longrightarrow \infty$$

$$\gamma = \frac{\Gamma_1 + \Gamma_2}{2}$$



$\Gamma_1 = 1, \Gamma_2 = 0.5$

$$\langle \sigma_x \rangle = \frac{\Gamma_1 \Omega}{\Gamma_1 \Gamma_2 + \Omega^2} \xrightarrow[\Omega \to \infty]{} \frac{\Gamma_1}{\Omega}$$

$$\langle \sigma_y \rangle = 0$$

$$\langle \sigma_z \rangle = \frac{\Gamma_1 \Gamma_2}{\Gamma_1 \Gamma_2 + \Omega^2}$$

Stationary conditions

Driven two-level system with detuning

$$H = -\frac{\hbar\omega_0}{2}\sigma_z + \frac{\hbar\Omega}{2}\sigma_x\cos(\omega t+\varphi) \qquad U = e^{-i\frac{\omega t}{2}\sigma_z}$$

$$\boxed{H' = -\frac{\hbar\delta\omega}{2}\sigma_z + \frac{\hbar\Omega}{2}\left(\sigma_x\cos\varphi + \sigma_y\sin\varphi\right)}$$

$$\delta\omega = \omega_0 - \omega$$

$$\frac{\partial\langle\sigma_x\rangle}{\partial t} = -i\frac{\delta\omega}{2}\langle[\sigma_z,\sigma_x]\rangle + \dots$$

$$\frac{\partial\langle\sigma_y\rangle}{\partial t} = -i\frac{\delta\omega}{2}\langle[\sigma_z,\sigma_y]\rangle + \dots$$

$$\frac{\partial\langle\sigma_z\rangle}{\partial t} = -i\frac{\delta\omega}{2}\langle[\sigma_z,\sigma_z]\rangle + \dots$$

$$\frac{\partial\langle\sigma_x\rangle}{\partial t} = -\delta\omega\langle\sigma_y\rangle - \Omega\langle\sigma_z\rangle\sin\varphi - \Gamma_2\langle\sigma_x\rangle$$

$$\frac{\partial\langle\sigma_y\rangle}{\partial t} = \delta\omega\langle\sigma_x\rangle + \Omega\langle\sigma_z\rangle\cos\varphi - \Gamma_2\langle\sigma_y\rangle$$

$$\frac{\partial\langle\sigma_z\rangle}{\partial t} = \Omega\left(\langle\sigma_x\rangle\cos\varphi - \langle\sigma_y\rangle\sin\varphi\right) - \Gamma_1\langle\sigma_z\rangle + \Gamma_1$$

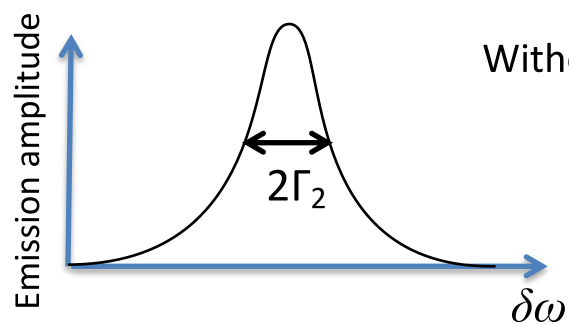
$$\frac{\partial}{\partial t}\begin{pmatrix}\langle\sigma_x\rangle\\ \langle\sigma_y\rangle\\ \langle\sigma_z\rangle\end{pmatrix} = \begin{pmatrix}-\Gamma_2 & -\delta\omega & -\Omega\sin\varphi\\ \delta\omega & -\Gamma_2 & \Omega\cos\varphi\\ \Omega\sin\varphi & -\Omega\cos\varphi & -\Gamma_1\end{pmatrix}\begin{pmatrix}\langle\sigma_x\rangle\\ \langle\sigma_y\rangle\\ \langle\sigma_z\rangle\end{pmatrix} + \begin{pmatrix}0\\0\\ \Gamma_1\end{pmatrix}$$

Stationary solution:

$$B\vec{\sigma} + \vec{b} = 0 \qquad \vec{\sigma} = -B^{-1}\vec{b}$$

In the limit of weak drive:

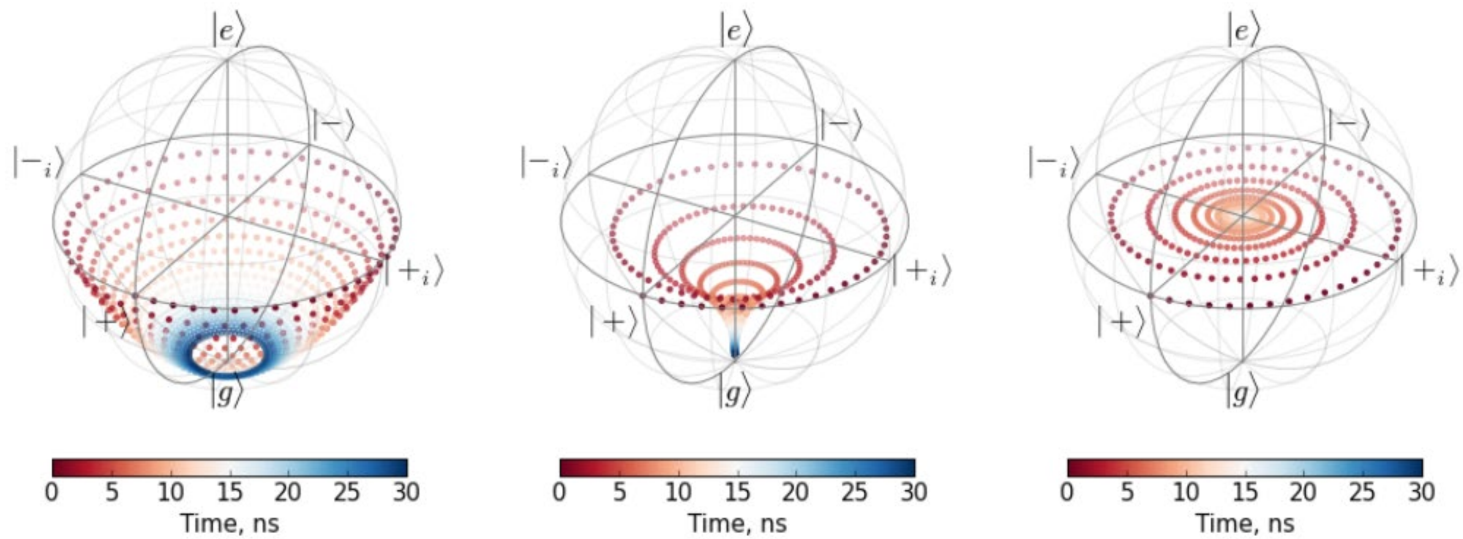
$$V_{emit} = V_0\langle\sigma^+\rangle \propto \frac{1}{\Gamma_2 + i\delta\omega}$$



Without dephasing:

$$\Gamma_2 = \frac{\Gamma_1}{2}$$

Relaxation and dephasing in the lab frame

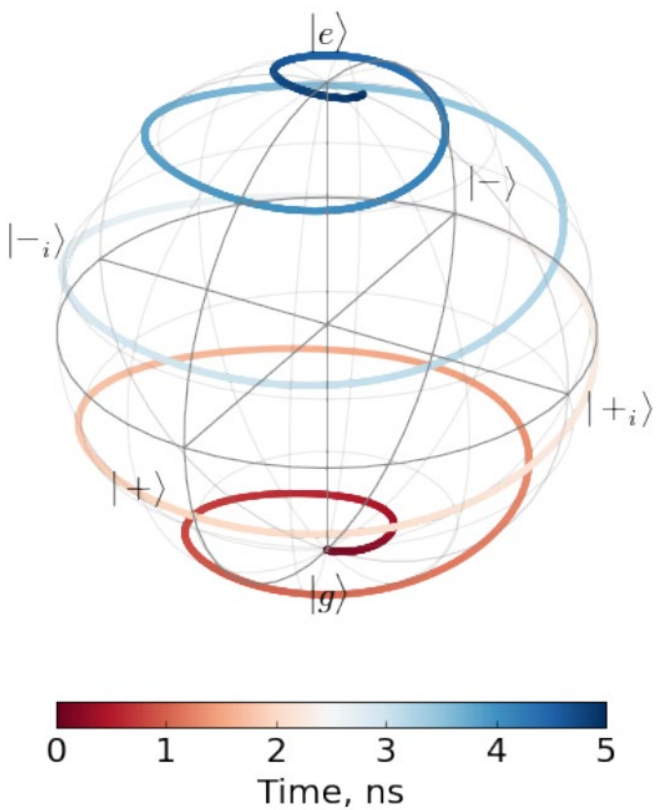


$\gamma = 0.1, \;\; \gamma_\phi = 0$	$\gamma = \gamma_\phi = 0.1$	$\gamma = 0, \;\; \gamma_\phi = 0.1$
Relaxation	Both	Pure dephasing

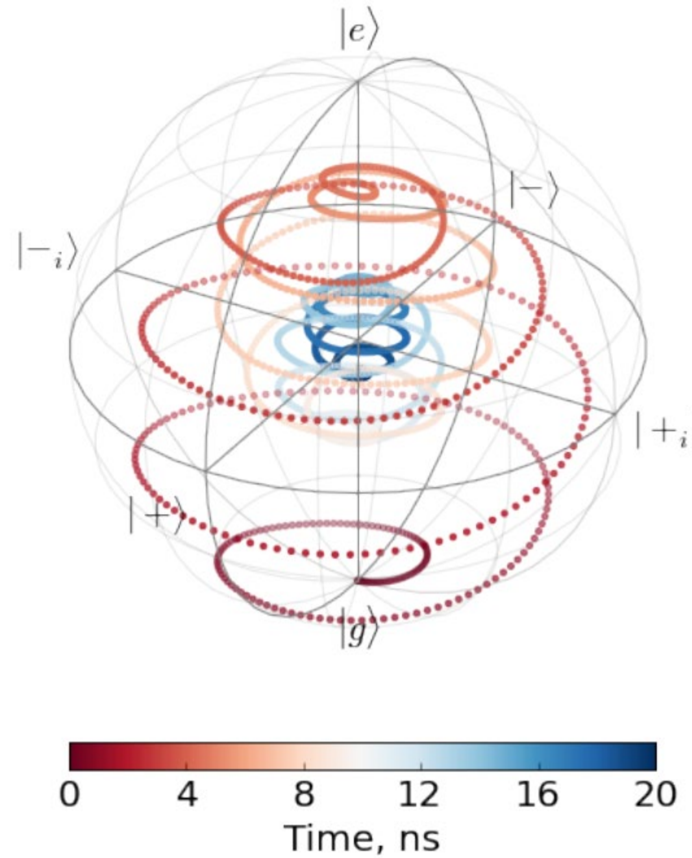
Driven-dissipative qubit in the lab frame

$$\xcancel{(\hat{\sigma}^{+} + \hat{\sigma}^{-})(e^{i\omega t} + e^{-i\omega t}) \to (\hat{\sigma}^{-} e^{-i\omega t} + \hat{\sigma}^{+} e^{i\omega t})}$$

No RWA!



Fast pi-pulse, no dissipation

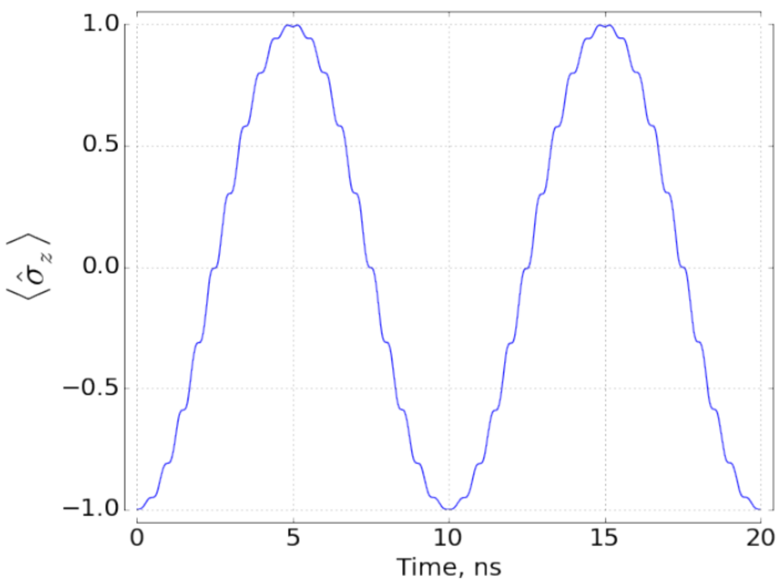


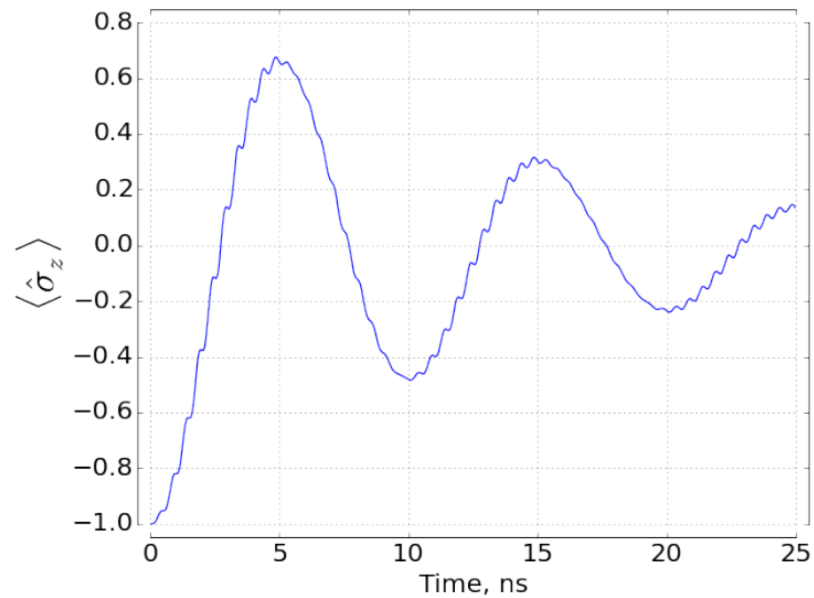
Fast driving, with dissipation

Driven-dissipative qubit in the lab frame

$$\cancel{(\hat{\sigma}^+ + \hat{\sigma}^-)(e^{i\omega t} + e^{-i\omega t}) \rightarrow (\hat{\sigma}^- e^{-i\omega t} + \hat{\sigma}^+ e^{i\omega t})}$$

No RWA!





Fast pi-pulse, no dissipation

Fast driving, with dissipation