Superconducting Quantum Technologies

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Lecture 7

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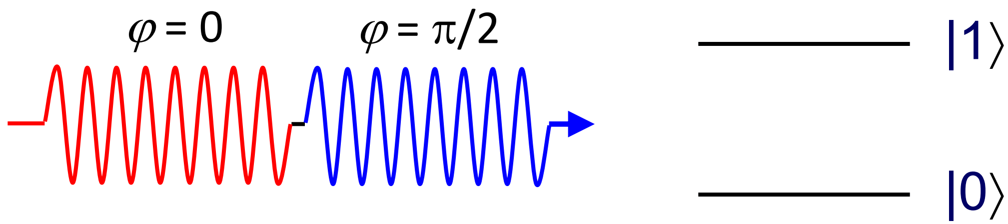
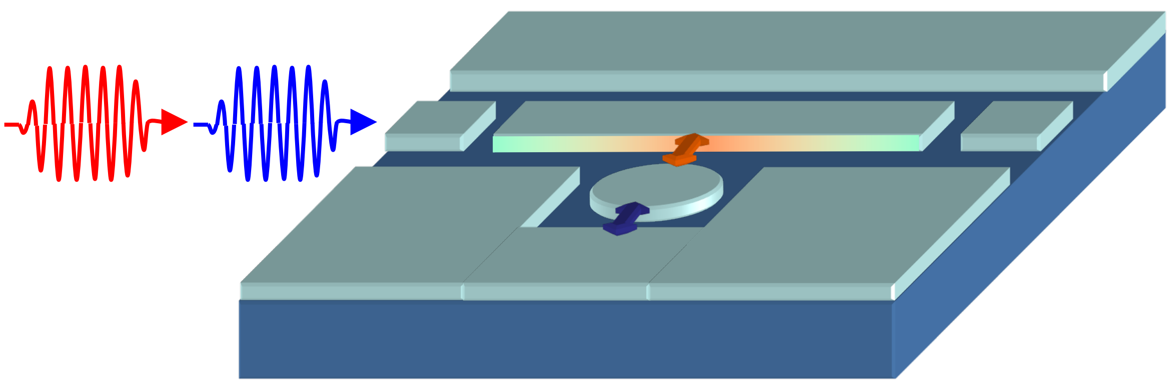
- Experimental quantum state control in two-level systems
- Quantum gates
- Dissipation and decoherence in two-level systems
- Density matrix approach
- Pure and mixed states
- Bloch sphere for mixed states and dissipative dynamics

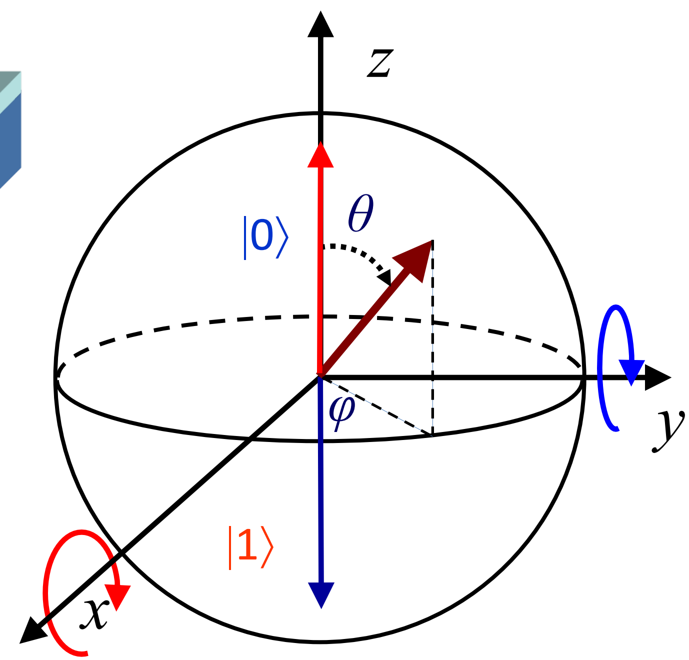
Quantum state control in two-level systems

$$\Psi = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

State manipulation



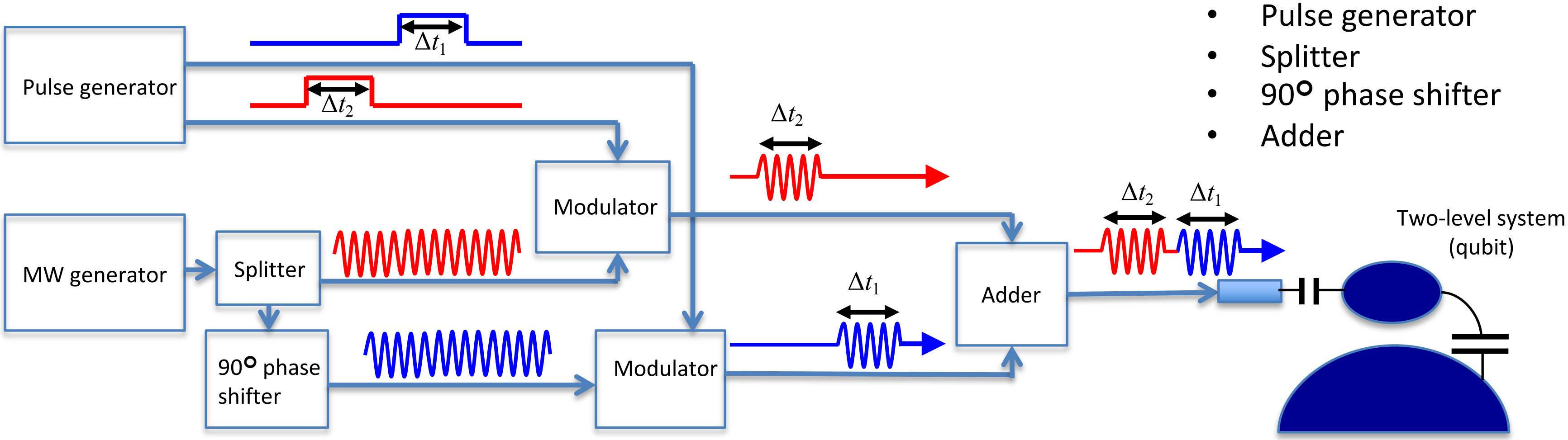




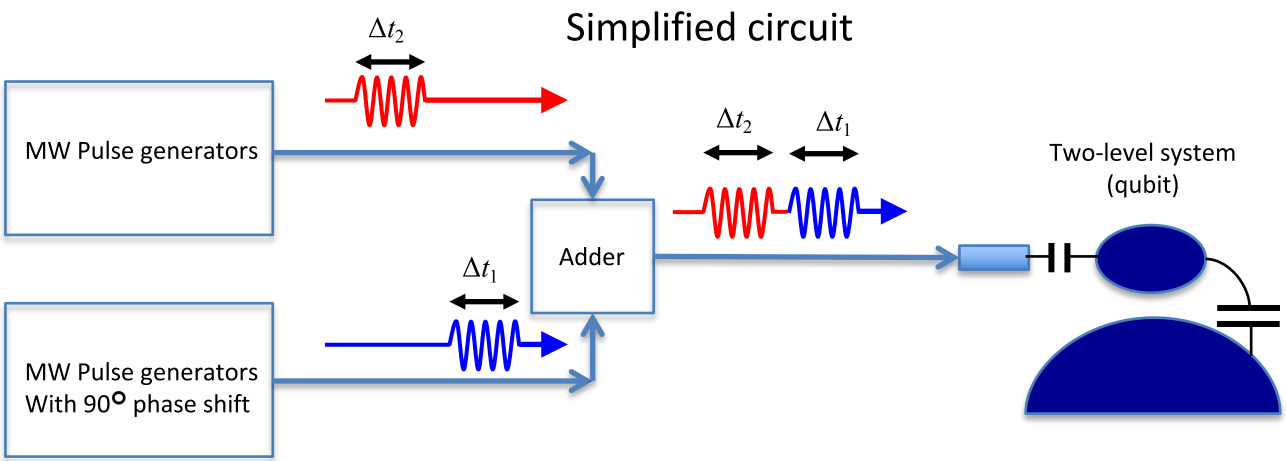
Electrical circuit to prepare arbitrary states

Required devices:

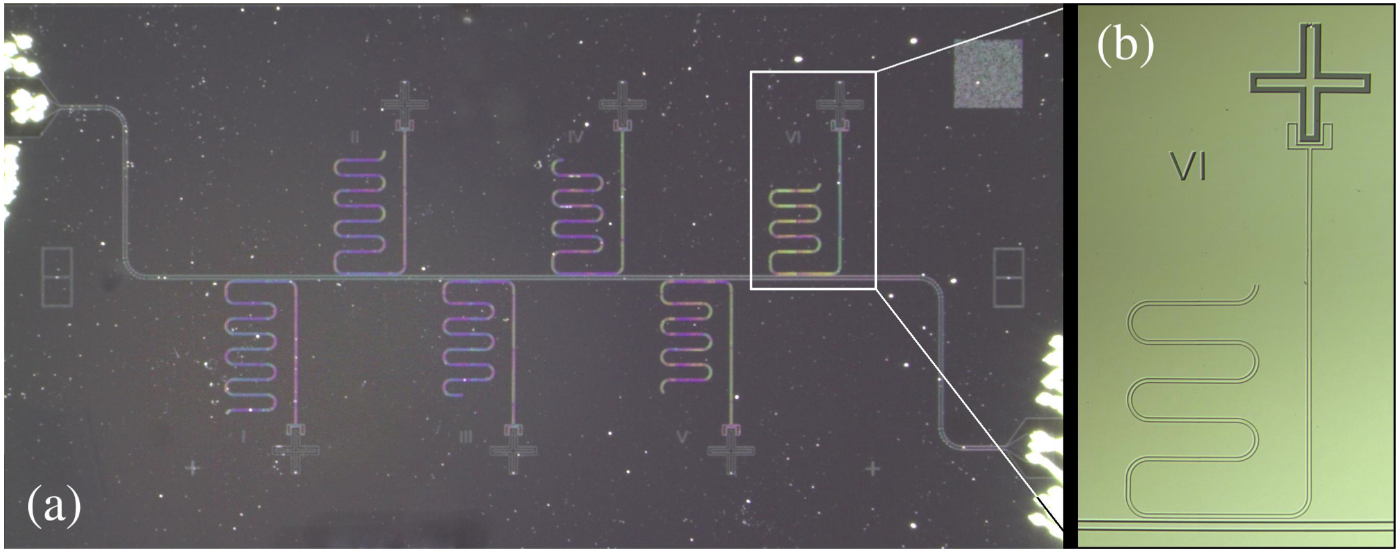
- MW generator
- Pulse generator
- Splitter
- 90° phase shifter
- Adder

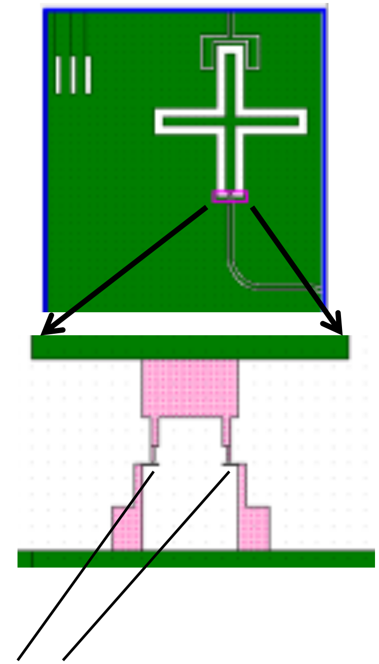


Simplified circuit



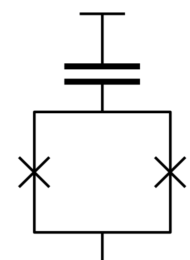
Long-lived low qubits with low unharmonicity
Fabricated and measured at MIPT

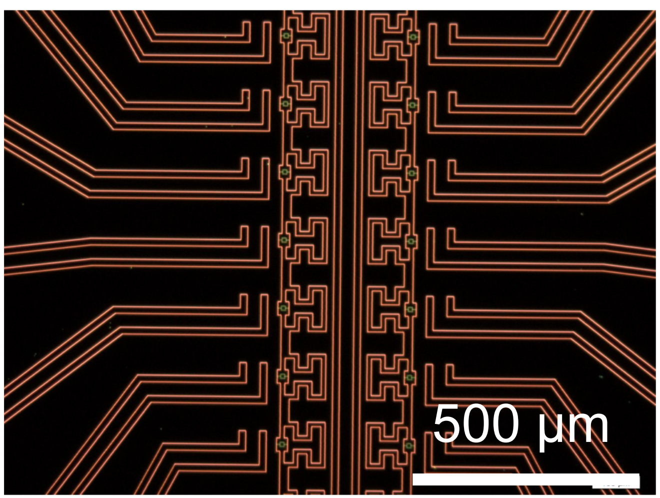




Josephson junctions

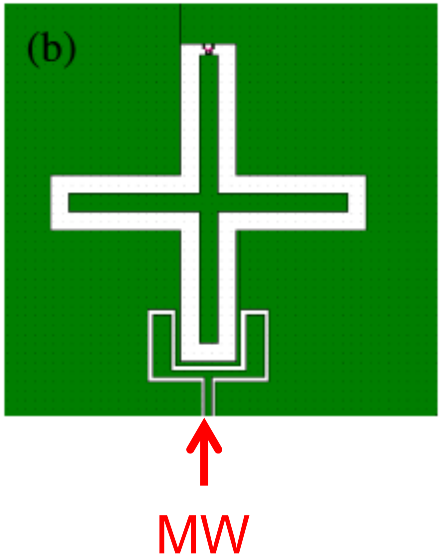
Equivalent electrical circuit

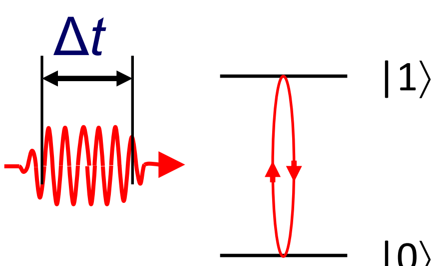


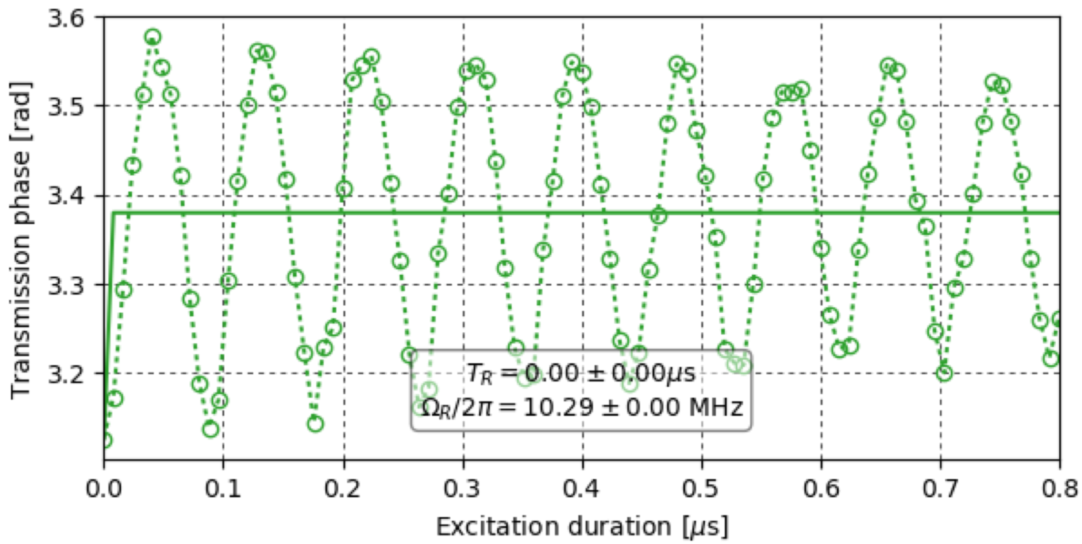


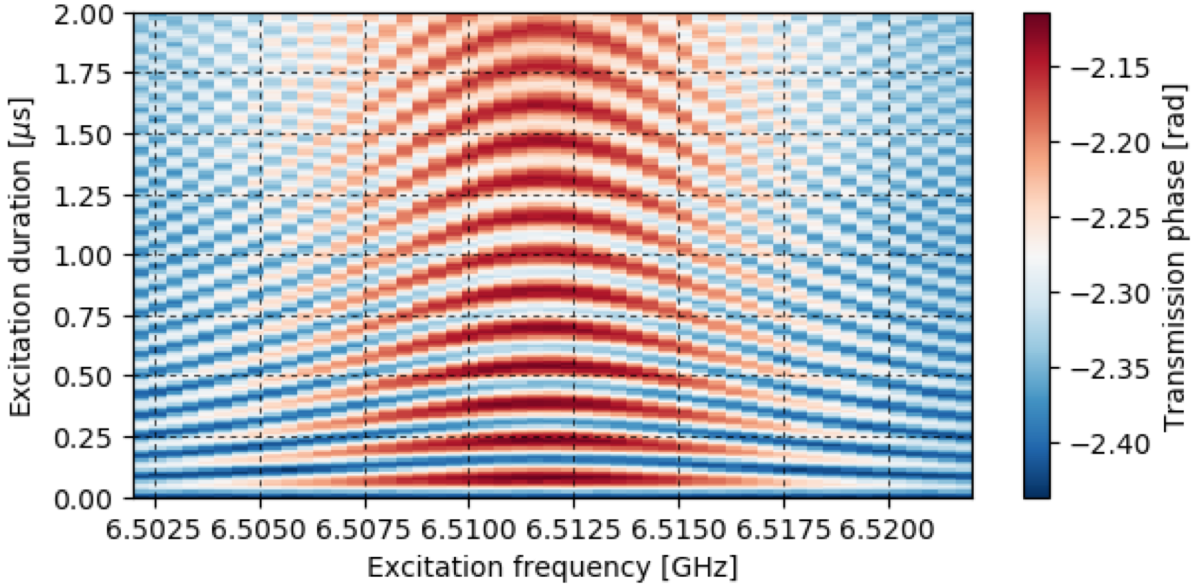
Lifetimes: ~10 µm

Rabi oscillations









Single-qubit operations

$$H = -\frac{\hbar\Omega}{2}\sigma_j \qquad U = \exp\left(i\frac{\Omega t}{2}\sigma_j\right) = I\cos\left(\frac{\Omega t}{2}\right) + i\sigma_j \sin\left(\frac{\Omega t}{2}\right)$$

$$\Omega t = \pi: \qquad U\left(\frac{\pi}{\Omega}\right) = i\sigma_j$$

$$t = \frac{\pi}{\Omega} \quad H = -\frac{\hbar\Omega}{2}\sigma_y \;\Rightarrow\; R = i\sigma_y$$

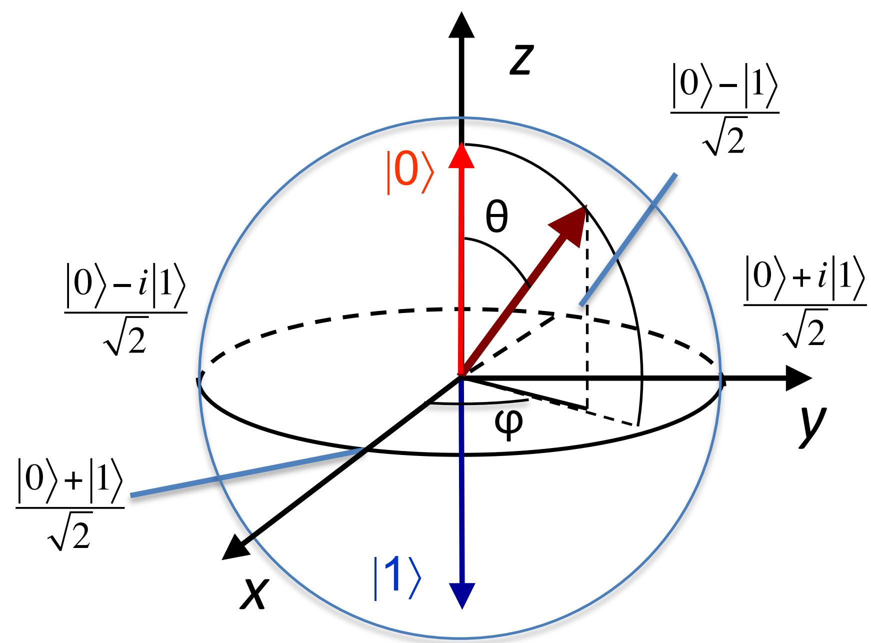
NOT (BIT FLIP) π-rotation around x-axis

$$i\sigma_y|0\rangle = -|1\rangle \quad i\sigma_y|1\rangle = |0\rangle$$

$$t = \frac{\pi}{\Omega} \quad H = -\frac{\hbar\Omega}{2}\sigma_x \;\Rightarrow\; R = i\sigma_x$$

CONJUGATED FLIP π-rotation around y-axis

$$i\sigma_x|0\rangle = i|1\rangle \qquad i\sigma_x|1\rangle = i|0\rangle$$



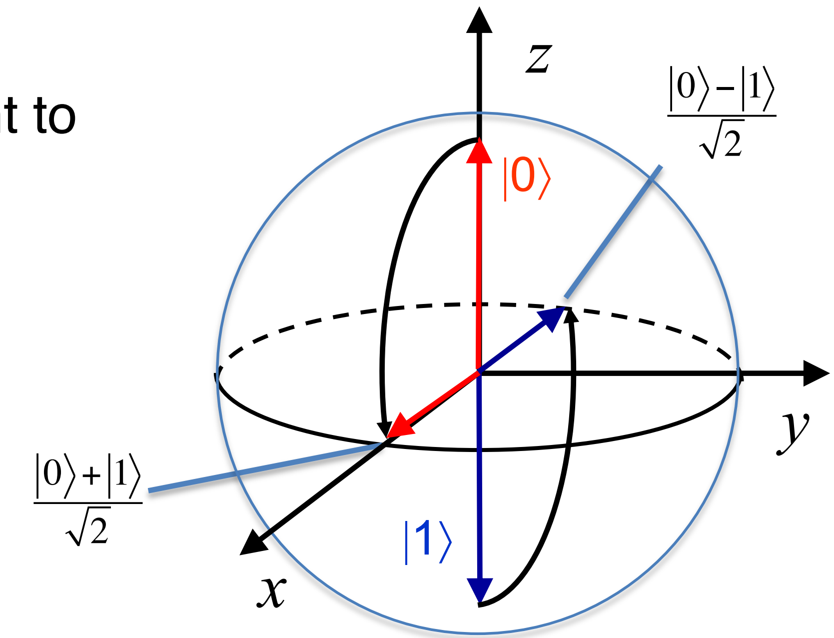
Hadamard gate and superposition of quantum states

$$Hm = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

Hadamard transformation is equivalent to π/2-rotation around y-axis

$$Hm|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$Hm|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



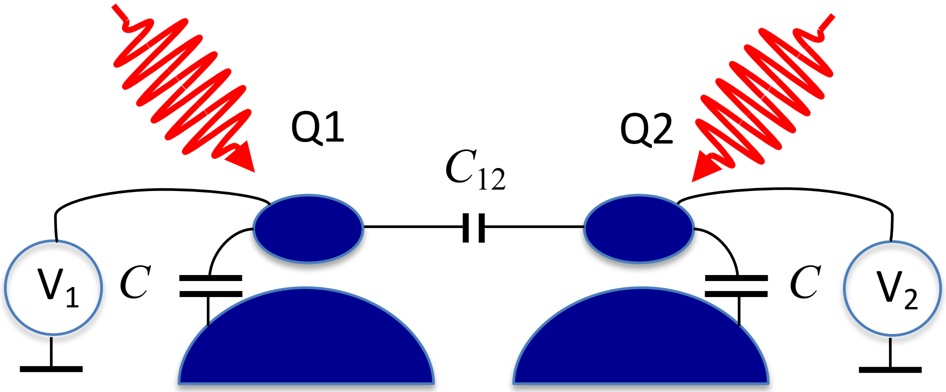
We can check that for

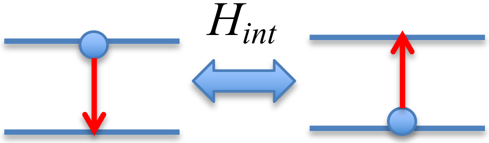
$$H = -\frac{\hbar\Omega}{2}\sigma_y \qquad U(t) = \exp\left[i\frac{\Omega t}{2}\sigma_y\right] = \begin{pmatrix}\cos(\Omega t/2) & \sin(\Omega t/2)\\ -\sin(\Omega t/2) & \cos(\Omega t/2)\end{pmatrix}$$

$$t = \frac{\pi}{2\Omega} \Rightarrow U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} \Rightarrow U|0\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \qquad U|1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$t = \frac{3\pi}{2\Omega} \Rightarrow U = \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ -1 & -1\end{pmatrix} \Rightarrow U|0\rangle = \frac{-|0\rangle - |1\rangle}{\sqrt{2}} \equiv \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad U|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Two-qubit interaction





$$
H = \begin{array}{c}
\begin{array}{cccc} |00\rangle & |10\rangle & |01\rangle & |11\rangle \end{array} \\
\begin{pmatrix}
-\frac{\varepsilon_1}{2}-\frac{\varepsilon_2}{2} & -\frac{\Delta_1}{2} & -\frac{\Delta_2}{2} & 0 \\
-\frac{\Delta_1}{2} & \frac{\varepsilon_1}{2}-\frac{\varepsilon_2}{2} & E_{\text{int}} & -\frac{\Delta_2}{2} \\
-\frac{\Delta_2}{2} & E_{\text{int}} & -\frac{\varepsilon_1}{2}+\frac{\varepsilon_2}{2} & -\frac{\Delta_1}{2} \\
0 & -\frac{\Delta_2}{2} & -\frac{\Delta_1}{2} & \frac{\varepsilon_1}{2}+\frac{\varepsilon_2}{2}
\end{pmatrix}
\end{array}
\begin{array}{c} |00\rangle \\ |10\rangle \\ |01\rangle \\ |11\rangle \end{array}
$$

$$H_1 = -\frac{\varepsilon_1}{2}\sigma_z^{(1)} - \frac{\Delta_1}{2}\sigma_x^{(1)} \qquad H_2 = -\frac{\varepsilon_2}{2}\sigma_z^{(2)} - \frac{\Delta_2}{2}\sigma_x^{(2)} \qquad \varepsilon_1 = E_C n_1 \qquad \varepsilon_2 = E_C n_2$$

$$\sigma_j^{(1)} = \sigma_j \times I$$

$$H_{\text{int}} = E_{\text{int}}\left(\sigma^{(1)-}\sigma^{(2)+} + \sigma^{(1)+}\sigma^{(2)-}\right)$$

$$\sigma_j^{(2)} = I \times \sigma_j$$

$$E_{\text{int}} = 2E_C\frac{C_{12}}{C}$$

CNOT-gate

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Evolution of one of the qubits depends on on the state of the other

Quantum parallelism

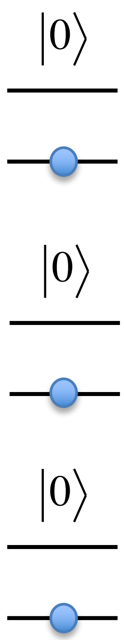
Initial state

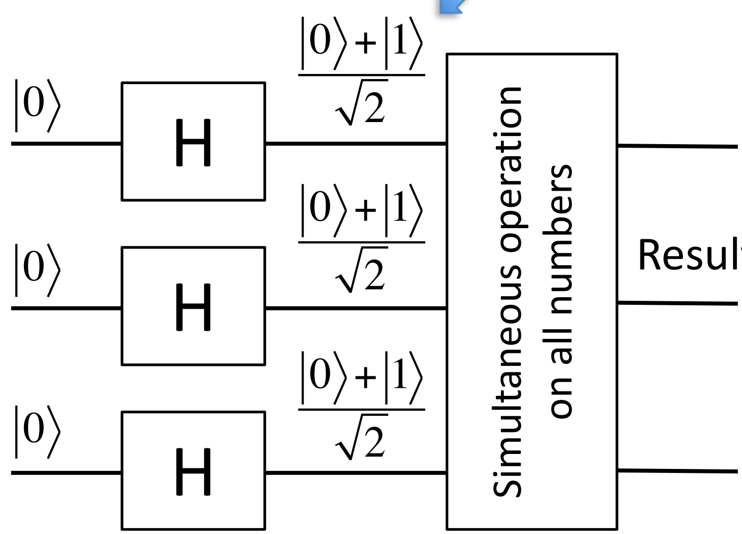
$$|000\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle$$

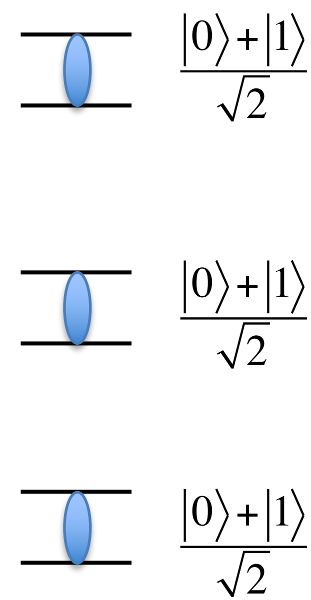
State after Hadamard transformation

$$|000\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} =$$

$$= \frac{|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle}{2^{3/2}}$$

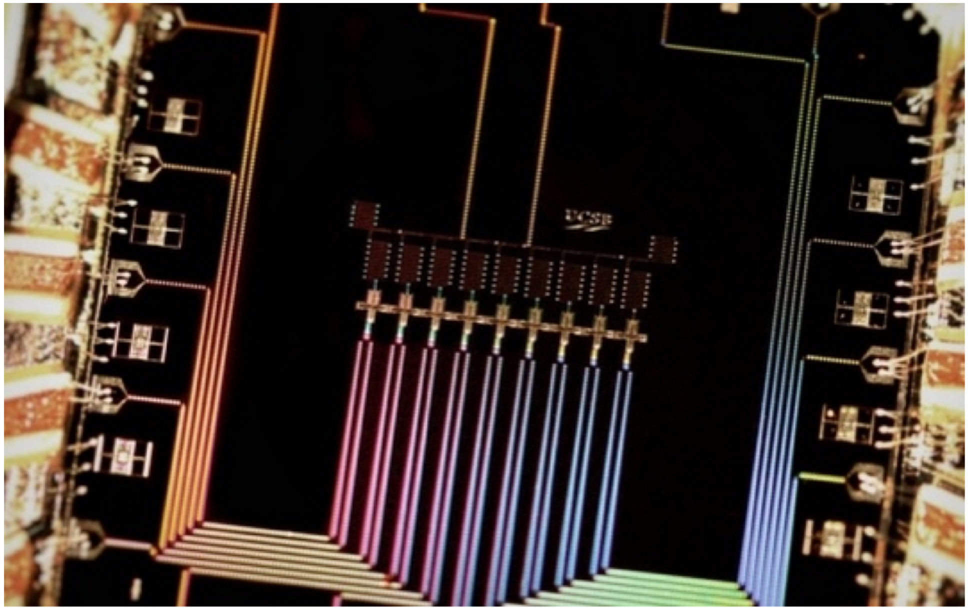


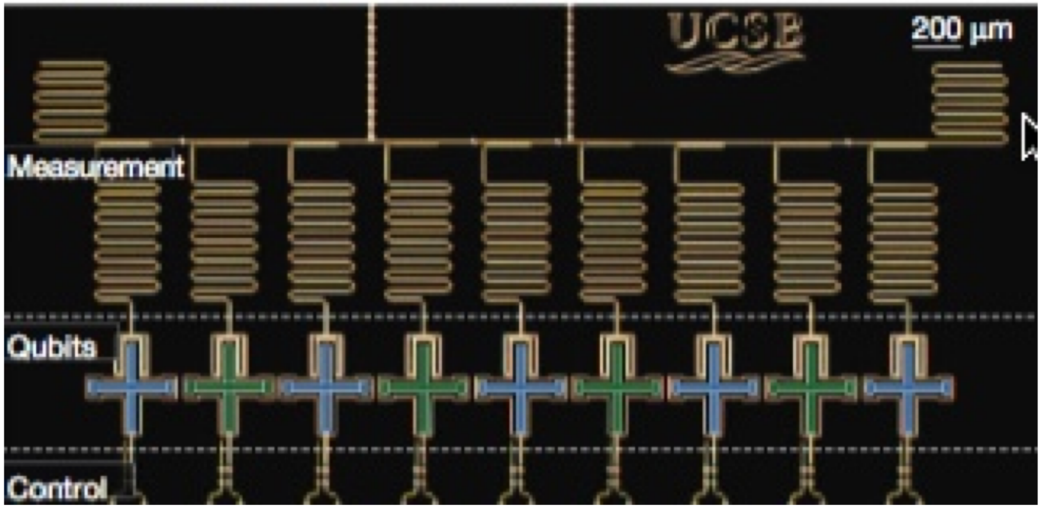




The operation can be produced simultaneously
on all possible combination of numbers

Quantum processor





Dissipative dynamics

Density matrix

Density matrix: $\rho = |\Psi\rangle\langle\Psi|$

A diagonal element ρ_{kk} defines a probability of finding the system in the state $|k\rangle$

Examples:

$$\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\Psi\rangle\langle\Psi| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Psi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |\Psi\rangle\langle\Psi| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |\Psi\rangle\langle\Psi| = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\Psi = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} \qquad |\Psi\rangle\langle\Psi| = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}$$

Expectation values of operators

$$\langle \hat{O} \rangle = \langle \Psi | \hat{O} | \Psi \rangle = \mathrm{Trace}\left(|\Psi\rangle\langle\Psi| \hat{O} \right) = \mathrm{Trace}\left(\rho \hat{O} \right)$$

$$|\Psi\rangle = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \qquad |\Psi\rangle\langle\Psi| = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \begin{pmatrix} a_0^* & a_1^* \end{pmatrix} = \begin{pmatrix} a_0 a_0^* & a_0 a_1^* \\ a_1 a_0^* & a_1 a_1^* \end{pmatrix} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \qquad \rho_{11} + \rho_{22} = 1$$

$$|\Psi\rangle\langle\Psi| = \rho_{00}|0\rangle\langle 0| + \rho_{01}|0\rangle\langle 1| + \rho_{10}|1\rangle\langle 0| + + \rho_{11}|1\rangle\langle 1|$$

Pure states (described by wavefunctions): $|\rho_{01}|^2 = \rho_{00}\rho_{11}$

Examples:

$$|0\rangle\langle 0| = \frac{\sigma_z + 1}{2} \qquad \frac{\langle \sigma_z \rangle + 1}{2} = \frac{\rho_{00} - \rho_{11} + 1}{2} = \rho_{00}$$

$$\langle \sigma_z \rangle = \mathrm{tr}\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = \mathrm{tr} \begin{pmatrix} \rho_{00} & -\rho_{01} \\ \rho_{10} & -\rho_{11} \end{pmatrix} = \rho_{00} - \rho_{11}$$

$$|1\rangle\langle 1| = \frac{1 - \sigma_z}{2} \qquad \frac{1 - \langle \sigma_z \rangle}{2} = \frac{1 - \rho_{00} + \rho_{11}}{2} = \rho_{11}$$

$$\langle \sigma_x \rangle = \mathrm{tr}\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \mathrm{tr} \begin{pmatrix} \rho_{01} & \rho_{00} \\ \rho_{11} & \rho_{10} \end{pmatrix} = \rho_{01} + \rho_{10}$$

$$\sigma_x \qquad \langle \sigma_x \rangle = \rho_{01} + \rho_{10}$$

$$\sigma_y \qquad \langle \sigma_y \rangle = i\rho_{01} - i\rho_{10}$$

$$\langle \sigma_y \rangle = \mathrm{tr}\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] = \mathrm{tr} \begin{pmatrix} i\rho_{01} & -i\rho_{00} \\ i\rho_{11} & -i\rho_{10} \end{pmatrix} = i\rho_{01} - i\rho_{10}$$

$$\sigma^+ = \frac{\sigma_x - i\sigma_y}{2} \qquad \langle \sigma^+ \rangle = \frac{\langle \sigma_x \rangle - i\langle \sigma_y \rangle}{2} = \frac{\rho_{01} + \rho_{10} + \rho_{01} - \rho_{10}}{2} = \rho_{01}$$

$$\sigma^- = \frac{\sigma_x + i\sigma_y}{2} \qquad \langle \sigma^- \rangle = \frac{\langle \sigma_x \rangle + i\langle \sigma_y \rangle}{2} = \frac{\rho_{01} + \rho_{10} - \rho_{01} + \rho_{10}}{2} = \rho_{10}$$

Master Equation

Schrodinger equation:

$$i\hbar\frac{\partial|\Psi\rangle}{\partial t} = H|\Psi\rangle \qquad -i\hbar\frac{\partial\langle\Psi|}{\partial t} = \langle\Psi|H$$

$$i\hbar\frac{\partial|\Psi\rangle\langle\Psi|}{\partial t} = i\hbar\frac{\partial|\Psi\rangle}{\partial t}\langle\Psi| + i\hbar|\Psi\rangle\frac{\partial\langle\Psi|}{\partial t} \qquad i\hbar\frac{\partial|\Psi\rangle\langle\Psi|}{\partial t} = H|\Psi\rangle\langle\Psi| - |\Psi\rangle\langle\Psi|H$$

$$\boxed{i\hbar\frac{\partial\rho}{\partial t} = [H,\rho]} \qquad [H,\rho] = H\rho - \rho H$$

Two-level system diagonal Hamiltonian: $H = -\frac{\hbar\omega_a}{2}\sigma_z$

$$i\hbar\begin{pmatrix}\dot\rho_{00} & \dot\rho_{01}\\ \dot\rho_{10} & \dot\rho_{11}\end{pmatrix} = -\frac{\hbar\omega_a}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix} + \frac{\hbar\omega_a}{2}\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \begin{pmatrix}\dot\rho_{00} & \dot\rho_{01}\\ \dot\rho_{10} & \dot\rho_{11}\end{pmatrix} = \frac{-i\omega_a}{2}\begin{pmatrix}0 & -2\rho_{01}\\ 2\rho_{10} & 0\end{pmatrix}$$

$$\rho_{00}(t) = \rho_{00}(0) \qquad \rho_{01}(t) = \rho_{01}(0)e^{i\omega_a t}$$

$$\rho_{10}(t) = \rho_{10}(0)e^{-i\omega_a t} \qquad \rho_{11}(t) = \rho_{11}(0)$$

$$H = -\frac{\hbar\Omega}{2}\sigma_x \qquad i\hbar\begin{pmatrix}\dot\rho_{00} & \dot\rho_{01}\\ \dot\rho_{10} & \dot\rho_{11}\end{pmatrix} = -\frac{\hbar\Omega}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix} + \frac{\hbar\Omega}{2}\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \begin{pmatrix}\dot\rho_{00} & \dot\rho_{01}\\ \dot\rho_{10} & \dot\rho_{11}\end{pmatrix} = \frac{i\Omega}{2}\begin{pmatrix}\rho_{10}-\rho_{01} & \rho_{11}-\rho_{00}\\ \rho_{00}-\rho_{11} & \rho_{01}-\rho_{10}\end{pmatrix}$$

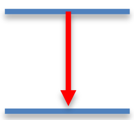
$$\dot\rho_{11} - \dot\rho_{00} = i\Omega(\rho_{01} - \rho_{10}) \qquad \ddot\rho_{11} - \ddot\rho_{00} = -\Omega^2(\rho_{11} - \rho_{00})$$

$$\dot\rho_{01} - \dot\rho_{10} = i\Omega(\rho_{11} - \rho_{00}) \qquad \rho_{11}(t) = A\sin\Omega t + B\cos\Omega t$$

Incoherent processes: Energy dissipation

Probabilities to find the two-level system in the excited and ground states: P_1 and P_0

Excited two-level system



$$\Gamma = \frac{1}{T}$$

$$dP_1 = -P_1 \frac{dt}{T}$$

$$P_0 + P_1 = 1$$

$$\dot{P}_1 = -\frac{P_1}{T} = -\Gamma P_1$$

$$P_1(t) = P_1(0) e^{-\frac{t}{T}} \qquad P_0(t) = 1 - P_1(0) e^{-\frac{t}{T_1}}$$

T_1 is the energy relaxation time

$$\rho_{11}(0) = 1 \qquad \rho(0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$$

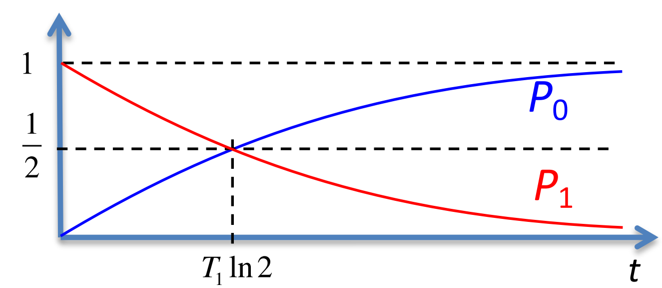
Mixed state (is not described by wavefunctions):

$$\rho(T_1 \ln 2) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \qquad \rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

The wavefunction giving the same probabilities:

$$\Psi = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow \rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Psi = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad \rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$



ρ_{01} and ρ_{10} - coherence

Incoherent processes: Dissipation and dephasing

The Master Equation including incoherent processes:

$$\boxed{\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}\left[H,\rho\right] + L}$$

$$dP_1 = -P_1 \frac{dt}{T_1} \qquad \frac{d\rho_{11}}{dt} = -\rho_{11}\Gamma_1$$

$$dP_0 = P_1 \frac{dt}{T_1} \qquad \frac{d\rho_{00}}{dt} = +\rho_{11}\Gamma_1$$

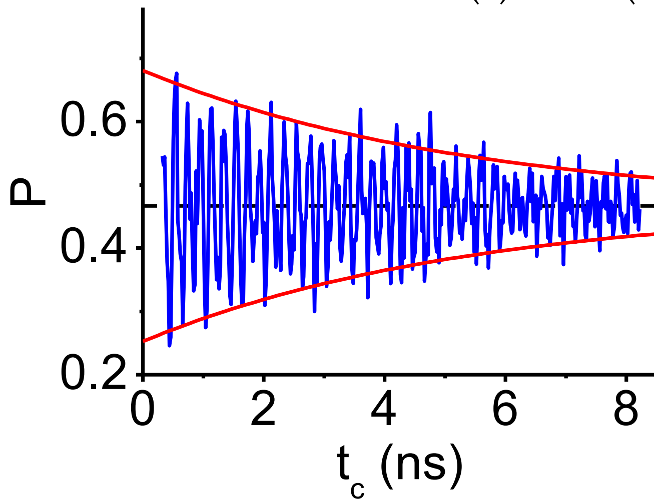
$$\Gamma_1 = \frac{1}{T_1} \qquad L = \begin{pmatrix} \Gamma_1\rho_{11} & -\Gamma_2\rho_{01} \\ -\Gamma_2\rho_{10} & -\Gamma_1\rho_{11} \end{pmatrix}$$

Γ_1 is the energy relaxation

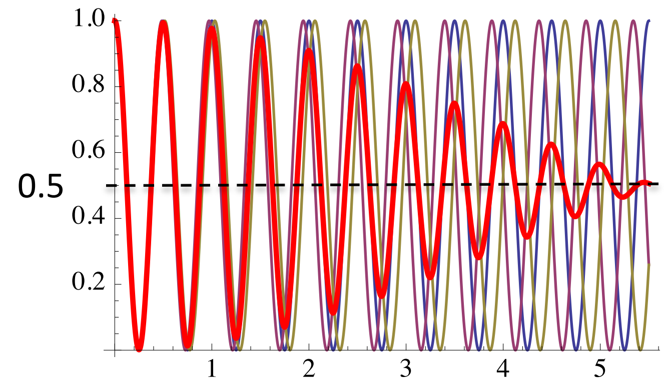
Γ_2 is dephasing (decay of the off-diagonal elements)

Dephasing

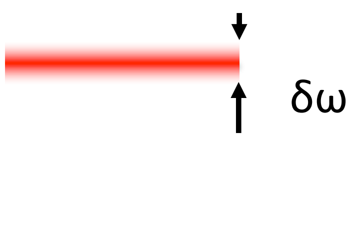
$$\psi(t)=\cos(\Omega t/2)|0\rangle+e^{-i\frac{\pi}{2}}\sin(\Omega t/2)|1\rangle$$



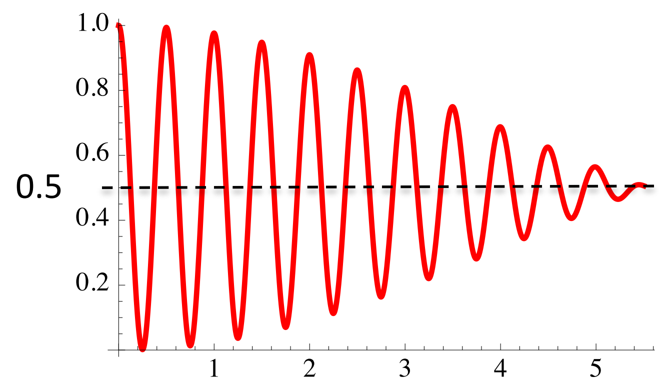
$$P_1=\cos^2\frac{\Omega_1 t}{2}\qquad P_2=\cos^2\frac{\Omega_2 t}{2}\qquad P_3=\cos^2\frac{\Omega_3 t}{2}$$



Dephasing:
energy/frequency => phase fluctuation

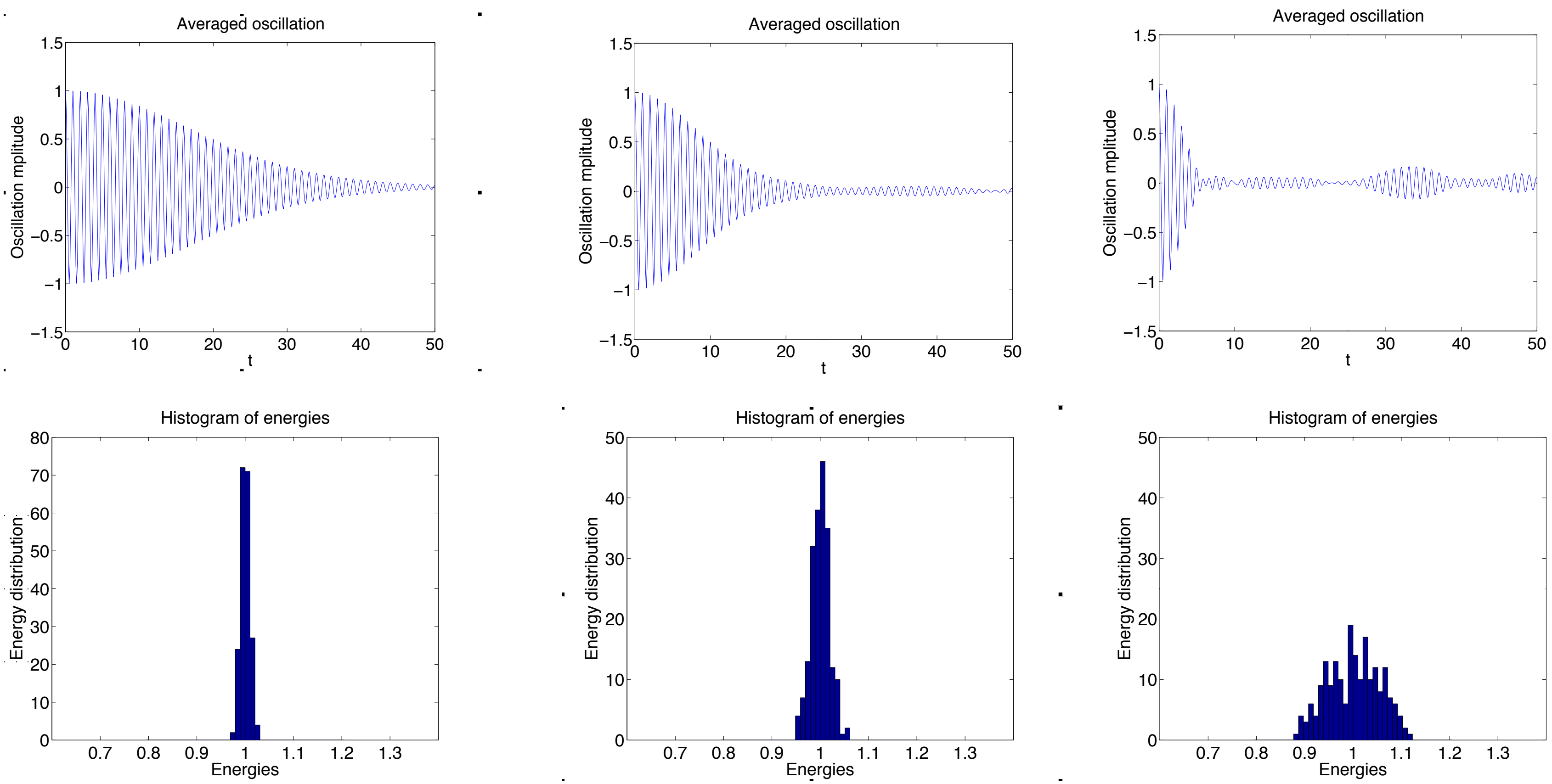


$$P(0)=\frac{1}{3}\left(\cos^2\frac{\Omega_1 t}{2}+\cos^2\frac{\Omega_2 t}{2}+\cos^2\frac{\Omega_3 t}{2}\right)$$



Dephasing due to energy level fluctuations

$\langle f \rangle = 1, \sigma_f = 0.01$ $\langle f \rangle = 1, \sigma_f = 0.02$ $\langle f \rangle = 1, \sigma_f = 0.05$



Expectation values of Pauli operators

Density matrix of the two-level system has three independent real variables: ρ_{00}, Re(ρ_{01}), Im(ρ_{01})

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \qquad \rho_{00} + \rho_{11} = 1 \qquad \rho_{01} = \rho_{10}^{*}$$

$$\langle \sigma_x \rangle = \mathrm{tr}\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \mathrm{tr} \begin{pmatrix} \rho_{01} & \rho_{00} \\ \rho_{11} & \rho_{10} \end{pmatrix} = \rho_{01} + \rho_{10} = 2\,\mathrm{Re}\left[\rho_{01}\right]$$

$$\langle \sigma_y \rangle = \mathrm{tr}\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] = \mathrm{tr} \begin{pmatrix} i\rho_{01} & -i\rho_{00} \\ i\rho_{11} & -i\rho_{10} \end{pmatrix} = i\rho_{01} - i\rho_{10} = -2\,\mathrm{Im}\left[\rho_{01}\right]$$

$$\langle \sigma_z \rangle = \mathrm{tr}\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = \mathrm{tr} \begin{pmatrix} \rho_{00} & -\rho_{01} \\ -\rho_{11} & -\rho_{11} \end{pmatrix} = \rho_{00} - \rho_{11} = \rho_{00} - (1 - \rho_{00}) = 2\rho_{00} - 1$$

$$\rho_{00} = \frac{1 + \langle \sigma_z \rangle}{2} \qquad \rho_{01} = \frac{\langle \sigma_x \rangle - i\langle \sigma_y \rangle}{2} = \langle \sigma^+ \rangle \qquad \rho_{10} = \frac{\langle \sigma_x \rangle + i\langle \sigma_y \rangle}{2} = \langle \sigma^- \rangle \qquad \rho_{11} = \frac{1 - \langle \sigma_z \rangle}{2}$$

The dynamics can be alternatively represented via expectation values of the three Pauli matrices

Bloch Sphere for dissipative spin dynamics

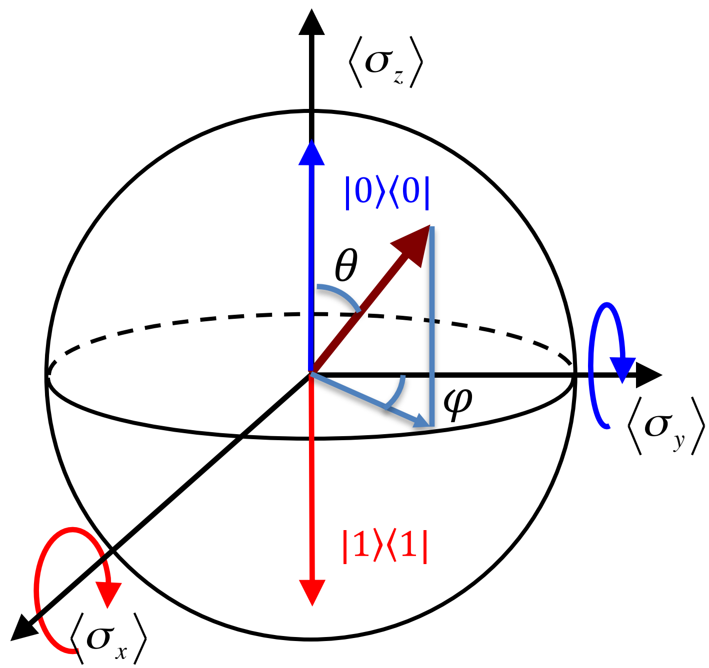
$$|\psi\rangle\langle\psi| = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{1}{2}e^{-i\varphi}\sin\theta \\ \frac{1}{2}e^{i\varphi}\sin\theta & \sin^2\frac{\theta}{2} \end{pmatrix}$$

$$\rho_{00} + \rho_{11} = 1$$

$$\langle\sigma_x\rangle = \rho_{01} + \rho_{10} = \frac{1}{2}e^{-i\varphi}\sin\theta + \frac{1}{2}e^{i\varphi}\sin\theta = \cos\varphi\sin\theta$$

$$\langle\sigma_y\rangle = i\rho_{01} - i\rho_{10} = \frac{i}{2}e^{-i\varphi}\sin\theta + \frac{i}{2}e^{i\varphi}\sin\theta = \sin\varphi\sin\theta$$

$$\langle\sigma_z\rangle = \rho_{00} - \rho_{11} = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta$$



Polar coordinates:

$$x = \cos\varphi\sin\theta$$

$$y = \sin\varphi\sin\theta$$

$$z = \cos\theta$$

More generally:

$$x \to \langle\sigma_x\rangle \quad y \to \langle\sigma_y\rangle \quad z \to \langle\sigma_z\rangle$$