Quantum Electronics of Nanostructures

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Lecture 6

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- Atom-resonator interaction: second order effects. Ac-Stark and Lamb shifts.
- Unitary transformations
- Time-dependent unitary transformation
- Rotating wave approximation (RWA)
- Quantum state control in qubits

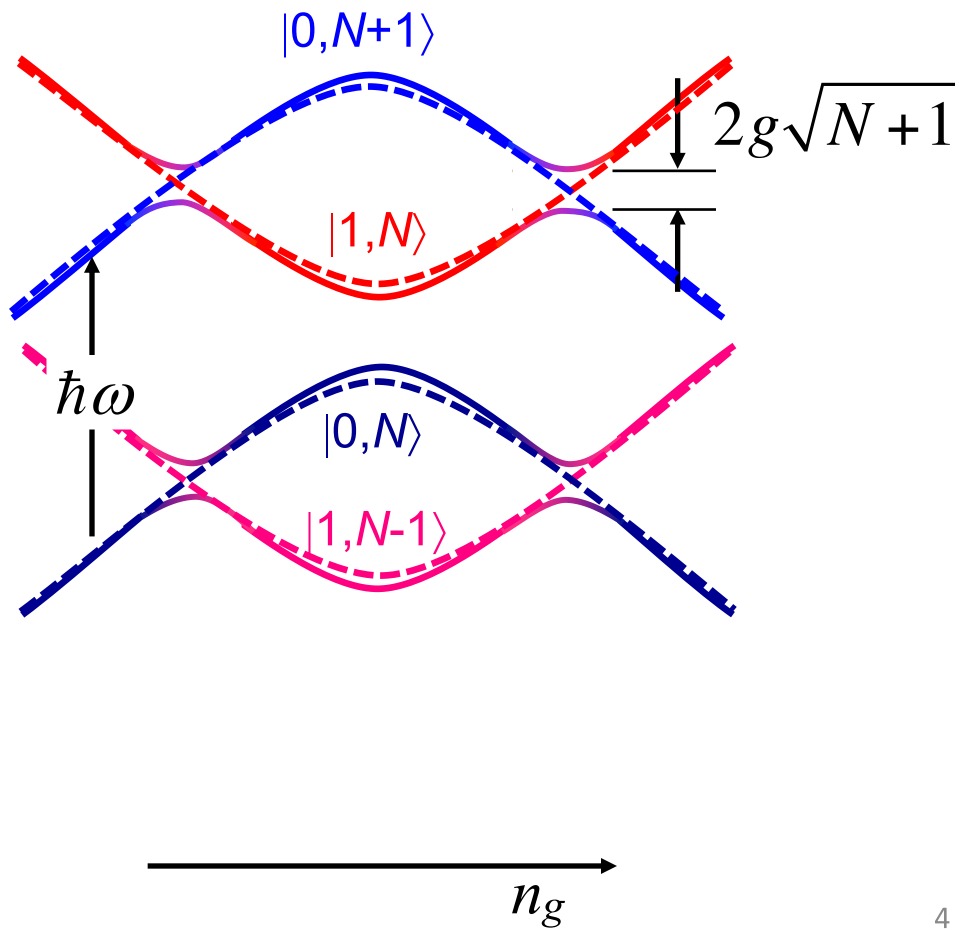
Second order effects
Dispersive readout

Second order energy effects
Shift due to qubit-resonator coupling

We use the following definitions:

Ground state: $|0\rangle$

Excited state: $|1\rangle$



Second order effects

$$H_0 = -\frac{\Delta E}{2}\sigma_z + \hbar\omega_r a^\dagger a \qquad H_{\text{int}} = g_0\left(a\sigma^+ + a^\dagger\sigma^-\right)$$

$$H_0|n,N\rangle = \left[\left(n-\frac{1}{2}\right)\Delta E\sigma_z + \hbar\omega N\right]|n,N\rangle \quad\Rightarrow\quad E_{n,N} = \left(n-\frac{1}{2}\right)\Delta E + \hbar\omega N$$

We are looking for corrections to eigenenergies in the conditions $|\hbar\omega - \Delta E| \gg g_0$ $\qquad \hbar\omega \neq \Delta E$

$$E_{0,N}^{(2)} = \frac{\left|\langle 0,N|g_0(a\sigma^+ + a^\dagger\sigma^-)|1,N-1\rangle\right|^2}{E_{0,N} - E_{1,N-1}} = \frac{\left(g_0\sqrt{N}\right)^2}{E_{0,N} - E_{1,N-1}} = \frac{g_0^2 N}{\hbar\omega - \Delta E}$$

$$E_{1,N}^{(2)} = \frac{\left|\langle 1,N|g_0(a\sigma^+ + a^\dagger\sigma^-)|0,N+1\rangle\right|^2}{E_{1,N} - E_{0,N+1}} = \frac{\left(g_0\sqrt{N+1}\right)^2}{E_{1,N} - E_{0,N+1}} = -\frac{g_0^2 (N+1)}{\hbar\omega - \Delta E}$$

$$E'_{0,N} = -\frac{\Delta E}{2} + N\hbar\omega + \frac{g_0^2 N}{\hbar\omega - \Delta E} \qquad E'_{1,N} = \frac{\Delta E}{2} + N\hbar\omega - \frac{g_0^2 (N+1)}{\hbar\omega - \Delta E}$$

$$E'_{0,N+1} = -\frac{\Delta E}{2} + (N+1)\hbar\omega + \frac{g_0^2 (N+1)}{\hbar\omega - \Delta E} \qquad E'_{1,N+1} = \frac{\Delta E}{2} + (N+1)\hbar\omega - \frac{g_0^2 (N+2)}{\hbar\omega - \Delta E}$$

Atomic transitions depend on number of photons (Stark shift):

$$E'_{1,N} - E'_{0,N} = \Delta E - \frac{g_0^2\left(2N+1\right)}{\hbar\omega - \Delta E}$$

Resonator transitions depend on atomic states:

$$E'_{0,N+1} - E'_{0,N} = \hbar\omega + \frac{g_0^2}{\hbar\omega - \Delta E}$$

$$E'_{1,N+1} - E'_{1,N} = \hbar\omega - \frac{g_0^2}{\hbar\omega - \Delta E}$$

Second order effects (alternative approach)

$$H_{JC} = -\frac{\hbar\omega_q}{2}\sigma_z + \hbar g\left(a\sigma^+ + a^+\sigma^-\right) + \hbar\omega_r\left(a^+a + \frac{1}{2}\right)$$

Unitary transformation:

$$U = \exp\left[\frac{g}{\delta\omega}\left(a\sigma^+ - a^\dagger\sigma^-\right)\right] \qquad \delta\omega = \omega_q - \omega_r \qquad \delta\omega \gg g$$

$$H' = UH_{JC}U^\dagger$$

Resonator energy shift

$$H' = \hbar\left(\omega_r + \frac{g^2}{\delta\omega}\sigma_z\right)a^+a + \frac{\hbar}{2}\left(\omega_q + \frac{g^2}{\delta\omega}\right)\sigma_z$$

$$\omega_r' = \omega_r \pm \frac{g^2}{\delta\omega}$$

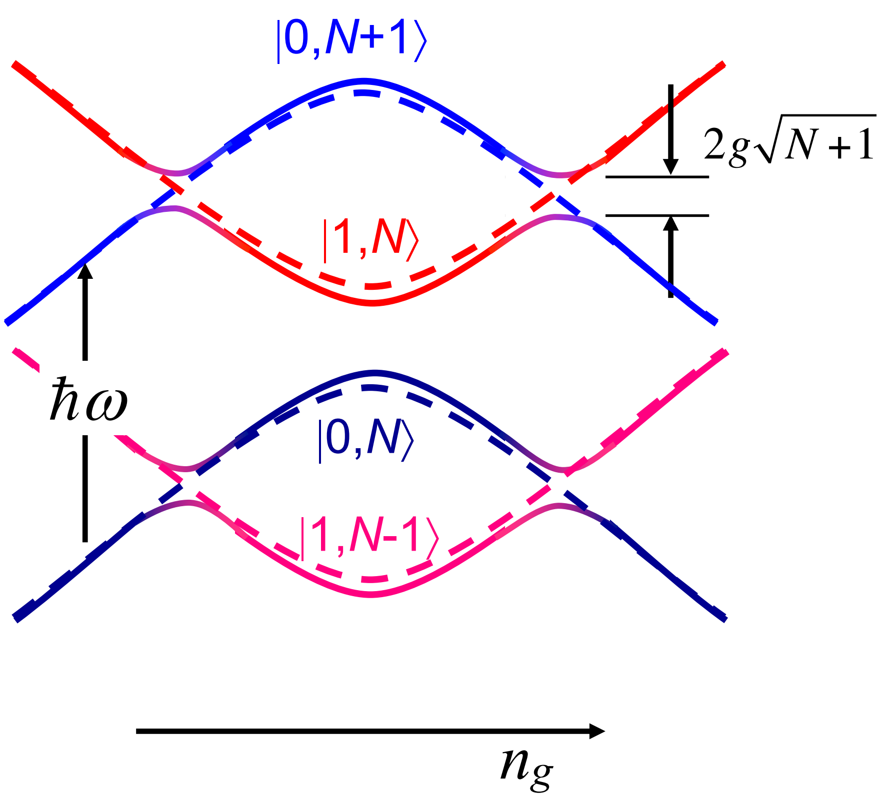
Atomic energy shift

$$H' = \hbar\omega_r a^+a + \frac{\hbar}{2}\left[\omega_q + \frac{g^2}{\delta\omega}\left(a^+a + 1\right)\right]\sigma_z$$

$$\omega_q' = \omega_q + \frac{g^2}{\delta\omega}(N+1)$$

Atom/resonator frequency shift depending on resonator/atom states

One more way of treatment



$$H'' = -\frac{\varepsilon}{2}\sigma_z - \frac{\Delta}{2}\sigma_x$$

$$\varepsilon = E_{0,N+1} - E_{1,N} = \hbar\omega - \Delta E$$

$$\Delta = 2g\sqrt{N+1}$$

$$H''_{diag} = -\frac{\sqrt{\varepsilon^2+\Delta^2}}{2}\sigma_z \approx -\frac{\varepsilon + \frac{\Delta^2}{2\varepsilon}}{2}\sigma_z$$

$$E'_{1,N} = E_{1,N} + \frac{1}{2}\frac{\Delta^2}{2\varepsilon} = E_{1,N} + \frac{g_0^2(N+1)}{\hbar\omega - \Delta E}$$

$$E'_{0,N+1} = E_{0,N+1} - \frac{1}{2}\frac{\Delta^2}{2\varepsilon} = E_{0,N+1} - \frac{g_0^2(N+1)}{\hbar\omega - \Delta E}$$

AC-Stark shift

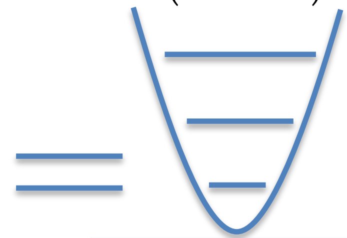
$$|0,N\rangle: \; E_{0,N} = -\frac{\Delta E}{2} + N\hbar(\omega + \Delta\omega) \qquad |1,N\rangle: \; E_{1,N} = \frac{\Delta E}{2} - \Delta\omega + N\hbar(\omega - \Delta\omega) \qquad \hbar\Delta\omega = \frac{g_0^2}{\hbar\omega - \Delta E}$$

Off-resonant system:

$\Delta E \neq \hbar\omega$

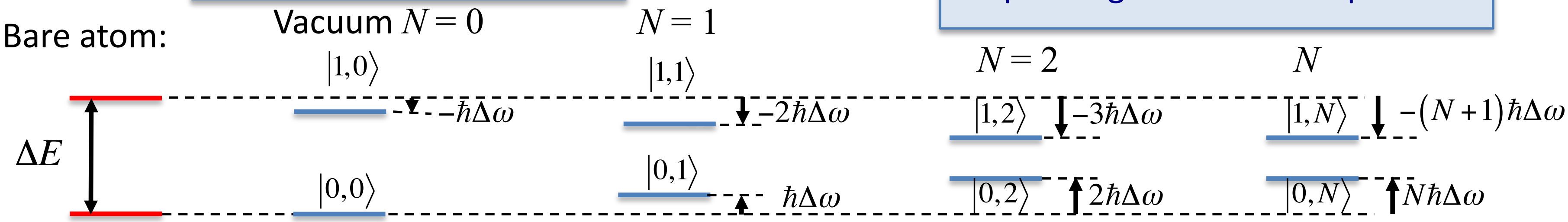
For example:

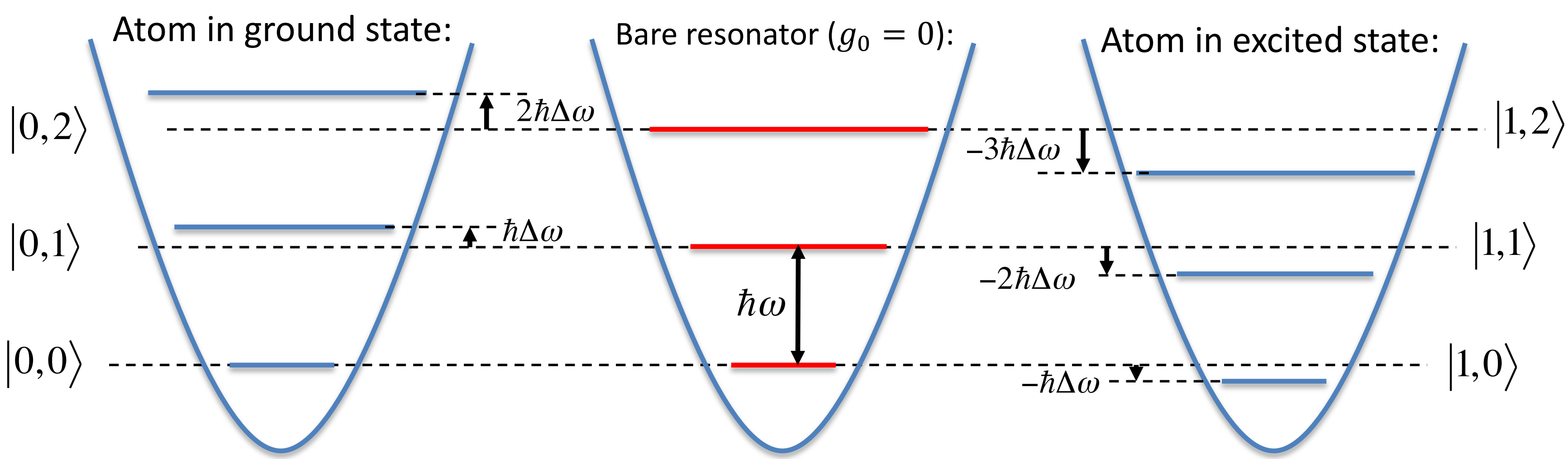
$\Delta E << \hbar\omega$



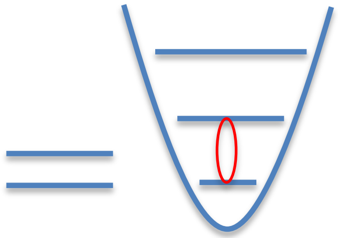
Lamb shift: shift of atomic levels due to vacuum (*N*=0)

Stark shift: shift of atomic levels depending on number of photons

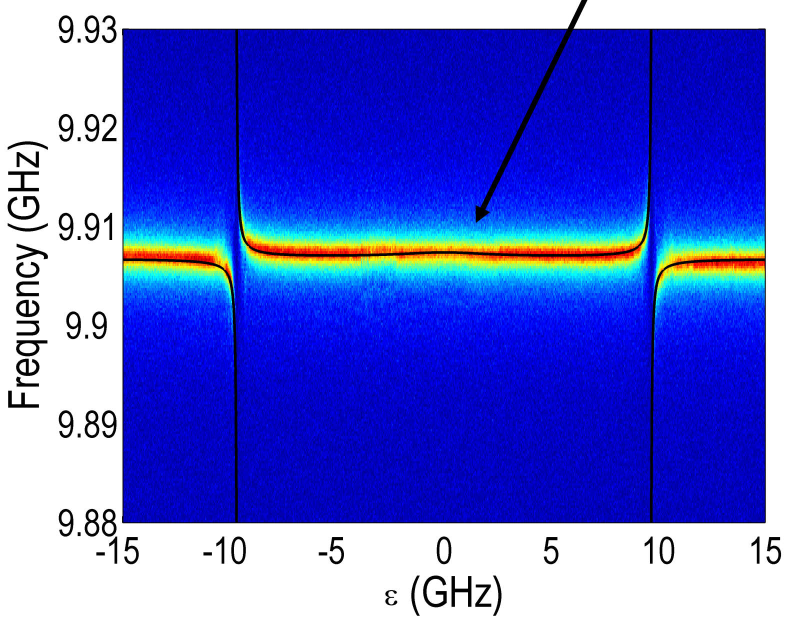


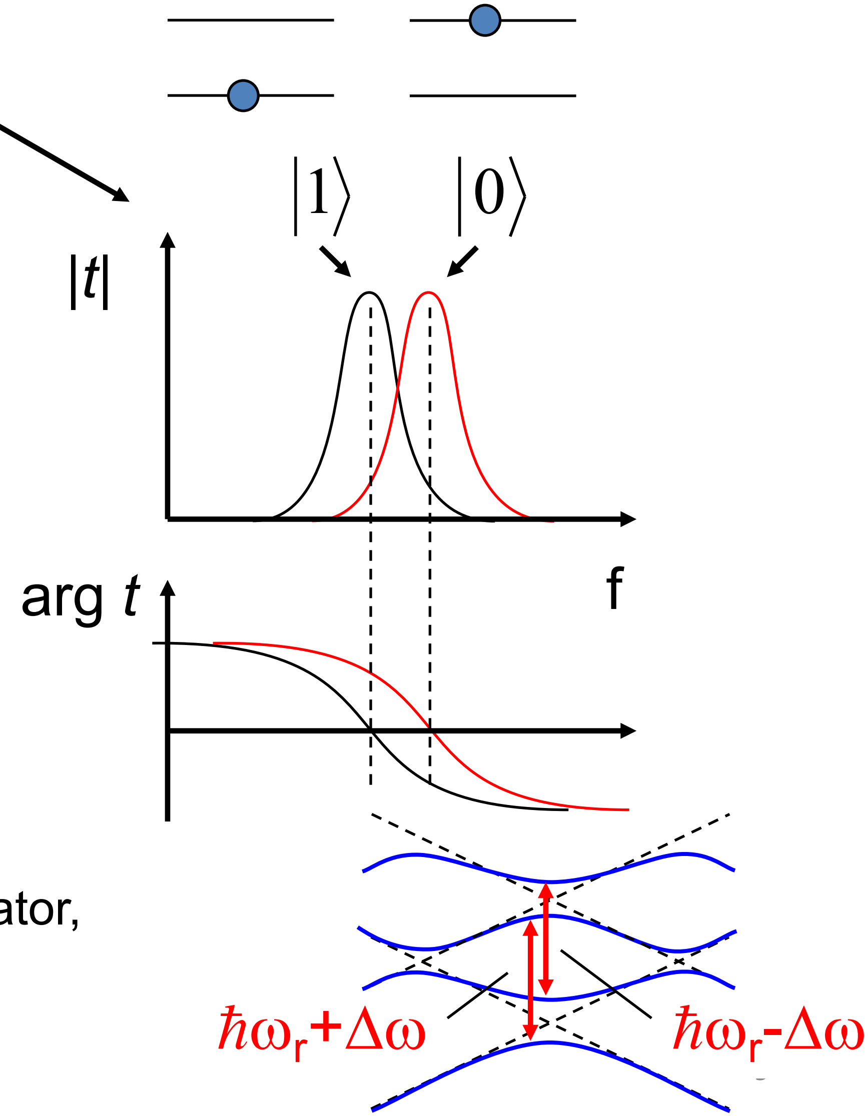


Principles of dispersive readout



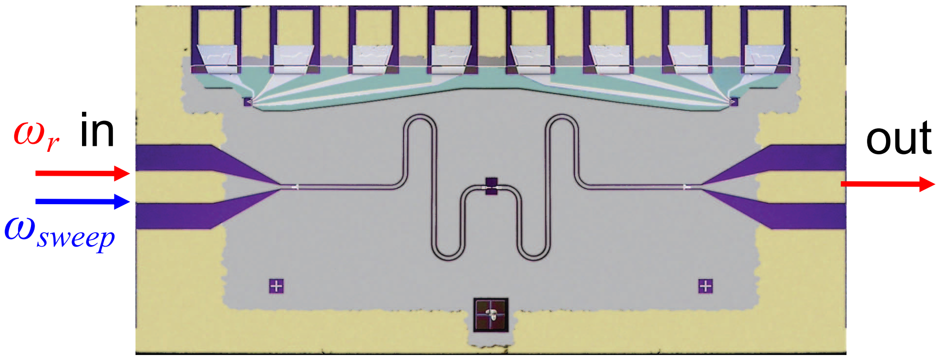
Frequency shift
due to qubit state

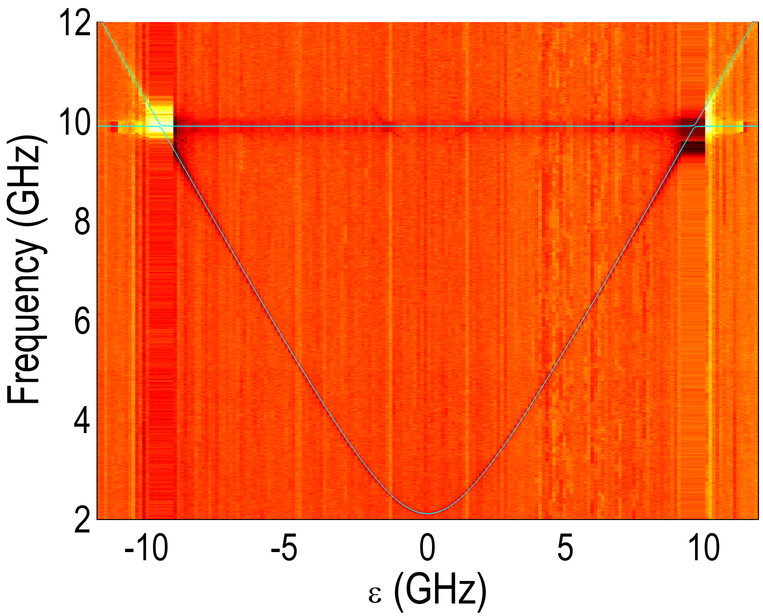




We measure transmission through the resonator, while the qubit is excited by the second tone

Dispersive readout:
Two-tone spectroscopy measurements





- The plot shows change of the phase of the transmitted signal
- As predicted between two anti-crossings we observe a deep

Unitary transformations

Transformation of Hamiltonians (time independent)

Unitary transformation:

$$\Psi' = U\Psi \qquad \Psi = U^{\dagger}\Psi'$$

Schrodinger equations:

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi \qquad i\hbar\frac{\partial\Psi'}{\partial t} = H'\Psi'$$

We are looking for H'

Schrodinger equations:

$$i\hbar\frac{\partial U^{\dagger}\Psi'}{\partial t} = HU^{\dagger}\Psi' \qquad UU^{\dagger}i\hbar\frac{\partial\Psi'}{\partial t} = UHU^{\dagger}\Psi' \qquad i\hbar\frac{\partial\Psi'}{\partial t} = UHU^{\dagger}\Psi'$$

$$\boxed{H' = UHU^{\dagger}}$$

Unitary operators

Operator U Hermitian conjugate: $U^{\dagger} = \left(U^{*}\right)^{T}$

Unitary operator: $U^{\dagger}U = I$

Example of unitary operators:

Pauli matrices: σ_x, σ_y, σ_z $\quad (\sigma_x)^{\dagger}\sigma_x = I \quad (\sigma_y)^{\dagger}\sigma_y = I \quad (\sigma_z)^{\dagger}\sigma_z = I$

Rotation operator: $U = \exp\left(i\alpha\sigma_y\right) = \cos\left(\alpha\sigma_y\right) + i\sin\left(\alpha\sigma_y\right)$

$$\cos\left(\alpha\sigma_y\right) = \sum_{k=0}^{\infty}\frac{\left(\alpha\sigma_y\right)^{2k}}{(2k)!} = \sum_{k=0}^{\infty}\frac{\alpha^{2k}}{(2k)!}I = I\cos\alpha$$

$$\sin\left(\alpha\sigma_y\right) = \sum_{k=0}^{\infty}\frac{\left(\alpha\sigma_y\right)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}\frac{\alpha^{2k+1}}{(2k+1)!}\sigma_y = \sigma_y\sin\alpha$$

$$U = I\cos\alpha + i\sigma_y\sin\alpha$$

$$U = \exp\left(i\alpha\sigma_j\right) \Rightarrow \boxed{U = I\cos\alpha + i\sigma_j\sin\alpha}$$

Examples of the unitary operator

$$U = \exp\left(i\alpha\sigma_y\right) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$U = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \qquad U^\dagger = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$U^\dagger U = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Operators $U = e^{i\alpha\sigma_j}$ are unitary

$$U = I\cos\alpha + i\sigma_j \sin\alpha \qquad U^\dagger = I\cos\alpha - i\sigma_j \sin\alpha$$

$$U^\dagger U = \left(I\cos\alpha - i\sigma_j\sin\alpha\right)\left(I\cos\alpha + i\sigma_j\sin\alpha\right) = I\cos^2\alpha \underbrace{- ii\sigma_j\sigma_j}_{+I} \sin^2\alpha = I$$

Properties of the rotation operator

$$U_y = e^{i\alpha\sigma_y}$$

Commutation properties:

$$U_y\sigma_z = e^{i\alpha\sigma_y}\sigma_z = \left(I\cos\alpha + i\sigma_y\sin\alpha\right)\sigma_z = \sigma_z\left(I\cos\alpha - i\sigma_y\sin\alpha\right) = \sigma_z U_y^\dagger$$

$$U_y\sigma_x = e^{i\alpha\sigma_y}\sigma_x = \left(I\cos\alpha + i\sigma_y\sin\alpha\right)\sigma_x = \sigma_x\left(I\cos\alpha - i\sigma_y\sin\alpha\right) = \sigma_x U_y^\dagger$$

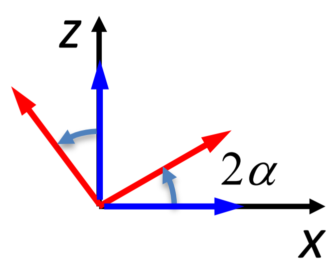
$$U_y\sigma_y = \sigma_y e^{i\alpha\sigma_y} = e^{i\alpha\sigma_y}\sigma_y = \sigma_y U_y$$

$$U_y\sigma_z U_y^\dagger = e^{i\alpha\sigma_y}\sigma_z e^{-i\alpha\sigma_y} = \left(I\cos 2\alpha + i\,\sigma_y\sin 2\alpha\right)\sigma_z = \sigma_z\cos 2\alpha - \sigma_x\sin 2\alpha$$

$$U_y\sigma_x U_y^\dagger = e^{i\alpha\sigma_y}\sigma_x e^{-i\alpha\sigma_y} = \left(I\cos 2\alpha + i\,\sigma_y\sin 2\alpha\right)\sigma_x = \sigma_x\cos 2\alpha + \sigma_z\sin 2\alpha$$

$$U_y\sigma_y U_y^\dagger = e^{i\alpha\sigma_y}\sigma_y e^{-i\alpha\sigma_y} = \sigma_y$$

$U_y = e^{i\alpha\sigma_y}$ is counter-clockwise rotation around y-axis by 2α

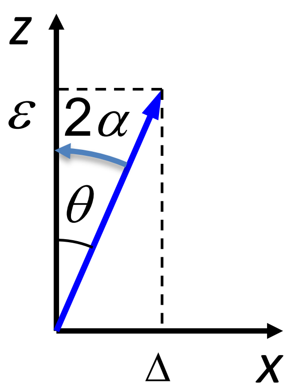


Diagonalization of the qubit Hamiltonian

Unitary transformation: $\Psi' = U\Psi$ $\Psi = U^\dagger \Psi'$ $\boxed{H' = UHU^\dagger}$

$$H = -\frac{\varepsilon}{2}\sigma_z - \frac{\Delta}{2}\sigma_x \qquad H = -\frac{\Delta E}{2}\left(\sigma_z \sin\theta + \sigma_x \cos\theta\right) \qquad \Delta E = \sqrt{\varepsilon^2 + \Delta^2}$$

Transformation from physical basis to eigenbasis: $\boxed{U = R_y = e^{i\frac{\theta}{2}\sigma_y}}$



$$H' = UHU^\dagger = R_y R_y H = e^{i\theta\sigma_y} H = -\frac{\Delta E}{2}\sigma_z$$

$$-\Delta E: \ \Psi_0' = |0\rangle \qquad +\Delta E: \ \Psi_1' = |1\rangle$$

$$\Psi_0 = U^\dagger \Psi_0' = e^{-i\frac{\theta}{2}\sigma_y}|0\rangle = \left(I\cos\frac{\theta}{2} - i\sigma_y \sin\frac{\theta}{2}\right)|0\rangle = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$\Psi_1 = U^\dagger \Psi_1' = e^{-i\frac{\theta}{2}\sigma_y}|1\rangle = \left(I\cos\frac{\theta}{2} - i\sigma_y \sin\frac{\theta}{2}\right)|1\rangle = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

Time dependent unitary transformations
and
manipulations with qubit

Time-dependent unitary transformations

Unitary transformation:

$$\Psi' = U\Psi \qquad \Psi = U^\dagger \Psi'$$

Schrodinger equations:

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \qquad i\hbar \frac{\partial \Psi'}{\partial t} = H'\Psi'$$

Schrodinger equations:

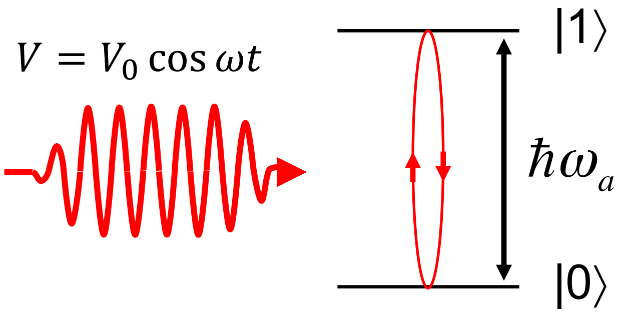
$$i\hbar \frac{\partial U^\dagger \Psi'}{\partial t} = HU^\dagger \Psi' \qquad i\hbar U \frac{\partial U^\dagger}{\partial t}\Psi' + i\hbar U U^\dagger \frac{\partial \Psi'}{\partial t} = UHU^\dagger \Psi'$$

$$i\hbar \frac{\partial \Psi'}{\partial t} = \left(UHU^\dagger - i\hbar U\dot{U}^\dagger\right)\Psi'$$

Time-dependent unitary transformation:

$$\boxed{H' = UHU^\dagger - i\hbar U\dot{U}^\dagger}$$

Transitions in the two-level system under resonant harmonic excitations



Atom driven by an external MW field:

$$\boxed{H = -\frac{\hbar\omega_a}{2}\sigma_z + \hbar\Omega\sigma_x \cos \omega t} \qquad \hbar\Omega = V_0 C_k \frac{2e}{C_q}$$

For example (the charge qubit at the degeneracy point):

$$\boxed{U(t) = e^{-i\frac{\omega t}{2}\sigma_z}}$$

$$H' = UHU^{\dagger} - i\hbar U\dot{U}^{\dagger}$$

$$H' = e^{-i\frac{\omega t}{2}\sigma_z} H e^{i\frac{\omega t}{2}\sigma_z} - i\hbar e^{-i\frac{\omega t}{2}\sigma_z}\left(i\frac{\omega}{2}\sigma_z\right)e^{i\frac{\omega t}{2}\sigma_z}$$

$$-\frac{\hbar\omega_a}{2}\sigma_z + \underbrace{\hbar\Omega\frac{e^{i\omega t}+e^{-i\omega t}}{2}e^{-i\frac{\omega t}{2}\sigma_z}\sigma_x e^{i\frac{\omega t}{2}\sigma_z}}$$

$$\frac{\hbar\omega}{2}\sigma_z$$

$$\frac{\hbar\Omega}{2}\left(e^{i\omega t}+e^{-i\omega t}\right)\begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{i\omega t}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} = \frac{\hbar\Omega}{2}\left(e^{i\omega t}+e^{-i\omega t}\right)\begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0\end{pmatrix} =$$

$$= \frac{\hbar\Omega}{2}\begin{pmatrix} 0 & 1+\cancel{e^{-2i\omega t}} \\ 1+\cancel{e^{2i\omega t}} & 0\end{pmatrix} \approx \frac{\hbar\Omega}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} = \frac{\hbar\Omega}{2}\sigma_x$$

Rotating wave approximation

Driven two-level system

Initial time-dependent Hamiltonian:

$$H = -\frac{\hbar\omega_a}{2}\sigma_z + \frac{\hbar\Omega}{2}\sigma_x \cos\omega_a t$$

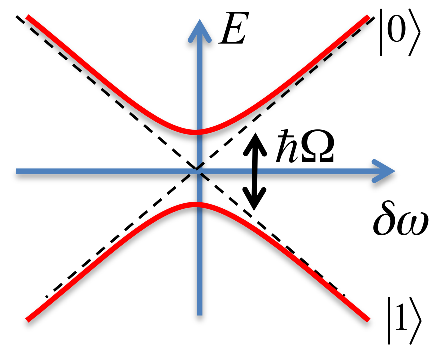
Time-dependent unitary transformation:

$$U(t) = e^{-i\frac{\omega t}{2}\sigma_z}$$

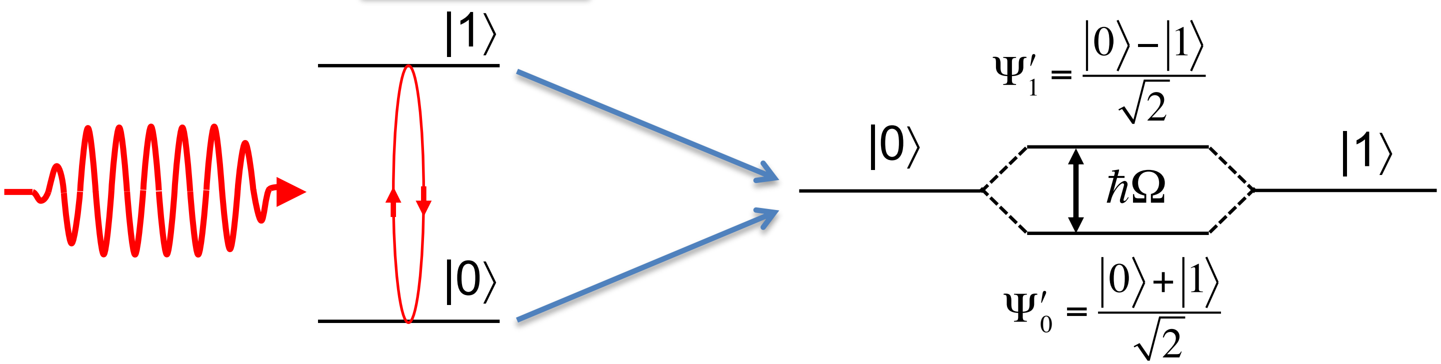
Transformed Hamiltonian:

$$\boxed{H = \frac{\hbar\delta\omega}{2}\sigma_z + \frac{\hbar\Omega}{2}\sigma_x}$$

$$\delta\omega = \omega - \omega_a$$

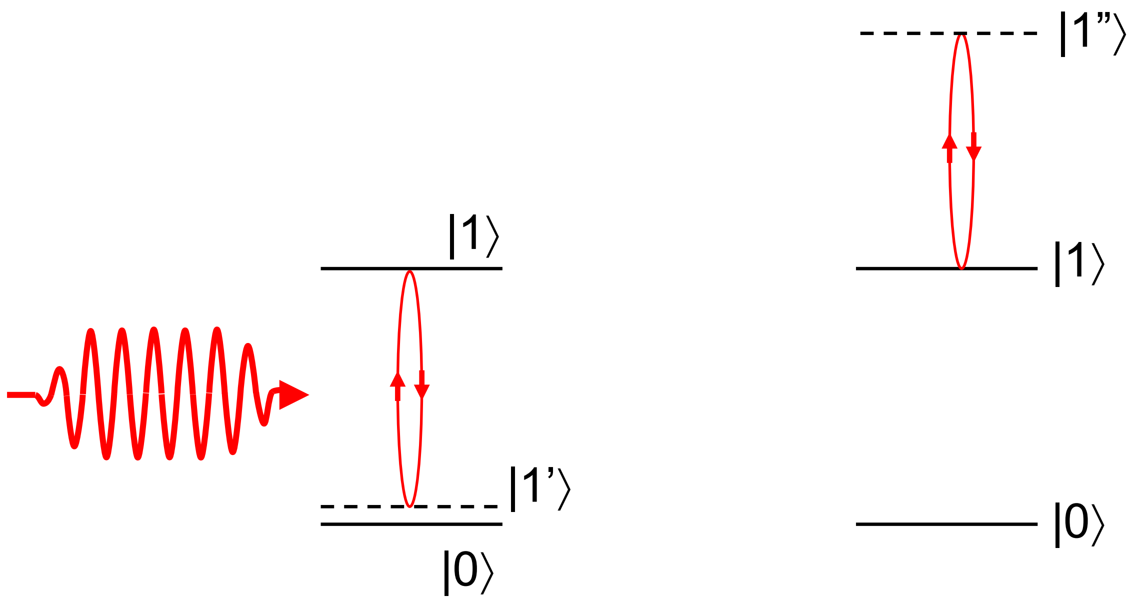


If $\omega = \omega_a$: $\boxed{H = \frac{\hbar\Omega}{2}\sigma_x}$



Physical meaning of the rotating wave approximation is coupling of the levels via the radiation

Physical interpretation of the fast rotating terms



Evolution of the two-level system under the external resonant drive

Initial Hamiltonian and ground state:

$$H = -\frac{\hbar\omega}{2}\sigma_z \qquad \Psi_0 = |0\rangle$$

Resonant drive:

$$H_{int} = \hbar\Omega\sigma_x \cos\omega_a t$$

Transformed Hamiltonian:

$$H' \approx -\frac{\hbar\Omega}{2}\sigma_x$$

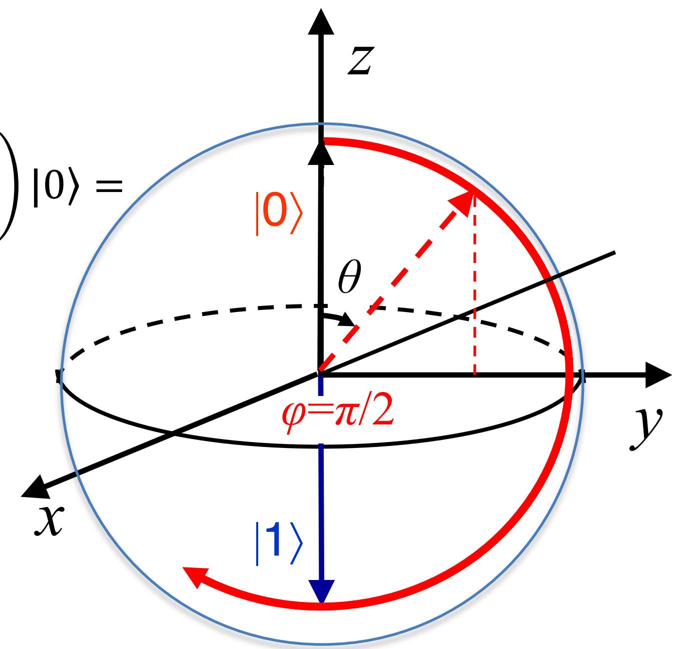
Evolution operator:

$$U_{ev} = e^{-i\frac{H'}{\hbar}t} = e^{-i\frac{\Omega t}{\hbar}\sigma_x}$$

$$U_{ev}|0\rangle = \left((|0\rangle\langle 0| + |1\rangle\langle 1|)\cos\frac{\Omega t}{2} - i(|0\rangle\langle 1| + |1\rangle\langle 0|)\sin\frac{\Omega t}{2} \right)|0\rangle =$$

$$= \cos\frac{\Omega t}{2}|0\rangle + e^{-i\pi/2}\sin\frac{\Omega t}{2}|1\rangle$$

Rotation around *x*-axis:



Evolution of the two-level system under the external resonant drive with a phase shift

Resonant drive:

$$H_{int} = \hbar\Omega\sigma_x \cos\left(\omega t - \frac{\pi}{2}\right) = \hbar\Omega\sigma_x \sin\omega t$$

$$\frac{\hbar\Omega}{2}\left(-ie^{i\omega t} + ie^{-i\omega t}\right)\begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix} = \frac{\hbar\Omega}{2}\begin{pmatrix} 0 & -i + ie^{-2i\omega t} \\ i - ie^{2i\omega t} & 0 \end{pmatrix} \approx \frac{\hbar\Omega}{2}\sigma_y$$

This is the same as $H_{int} = \hbar\Omega\sigma_y \cos\omega t$

$$UH_{int}U^\dagger = \frac{\hbar\Omega}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)e^{-i\omega t\sigma_z}\sigma_y = \frac{\hbar\Omega}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)\begin{pmatrix} 0 & -ie^{-i\omega t} \\ ie^{i\omega t} & 0 \end{pmatrix} = \frac{\hbar\Omega}{2}\begin{pmatrix} 0 & -i + ie^{-2i\omega t} \\ i - ie^{2i\omega t} & 0 \end{pmatrix} \approx \frac{\hbar\Omega}{2}\sigma_y$$

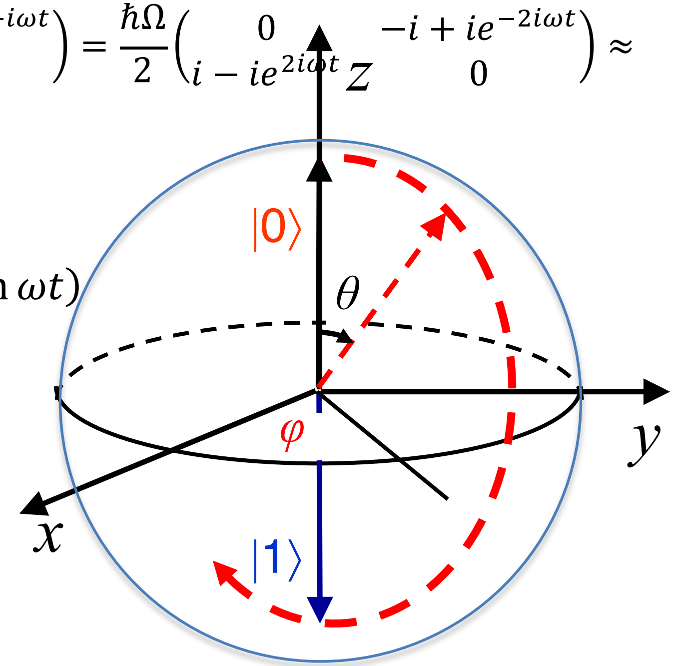
An arbitrary phase shift in the incident wave

$$H_{int} = \hbar\Omega\sigma_x \cos(\omega t - \varphi) = \hbar\Omega\sigma_x(\cos\varphi\cos\omega t + \sin\varphi\sin\omega t)$$

is equivalent to

$$\boxed{H'_{int} = \frac{\hbar\Omega}{2}\left(\sigma_x\cos\varphi + \sigma_y\sin\varphi\right)}$$

It describes rotation around axis in x-y plane at angle φ

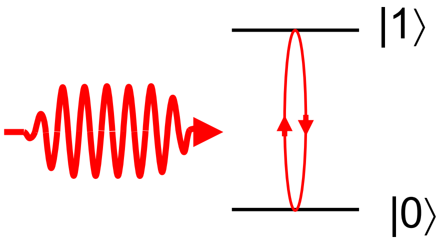


Manipulation with single qubit states

Two-level system in the eigenbasis (diagonal Hamiltonian):

$$H = \begin{pmatrix} -\hbar\omega_0/2 & 0 \\ 0 & \hbar\omega_0/2 \end{pmatrix} = -\frac{\hbar\omega_0}{2}\sigma_z = -\frac{\hbar\omega_0}{2}|0\rangle\langle 0| + \frac{\hbar\omega_0}{2}|1\rangle\langle 1|$$

By applying external field we transform the Hamiltonian:



$$i\hbar\frac{\partial\psi(t)}{\partial t} = \hat{H}\psi(t) \qquad \psi(t) = \exp\left(-i\frac{H}{\hbar}t\right)\psi(0)$$

x-rotation:

$$H = \frac{\hbar\Omega}{2}\sigma_x \qquad \psi(t) = \exp\left(-i\frac{\Omega t}{2}\sigma_x\right)\psi(0) \qquad R_x(t) = \exp\left(-i\frac{\Omega t}{2}\sigma_x\right)$$

y-rotation:

$$H = \frac{\hbar\Omega}{2}\sigma_y \qquad \psi(t) = \exp\left(-i\frac{\Omega t}{2}\sigma_y\right)\psi(0) \qquad R_y(t) = \exp\left(-i\frac{\Omega t}{2}\sigma_y\right)$$

By applying two sequential pulses with $\frac{\pi}{2}$ shifted phases, we can prepare any arbitrary state