

Superconducting Quantum Technologies

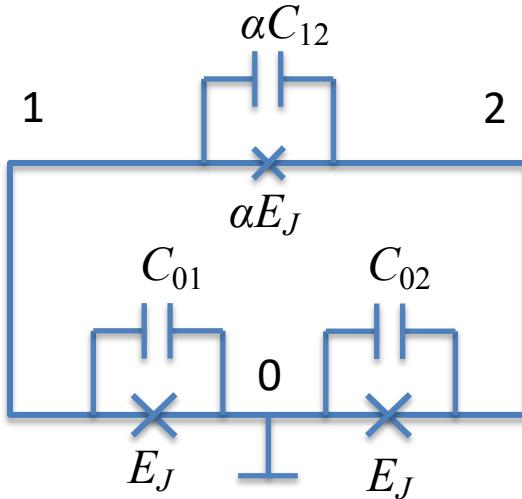
Oleg Astafiev

Lecture 4

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- 3-junction flux qubit (summary)
- Harmonic oscillators
- The number operator and Fock states. Photons in a resonator.
- Coplanar lines and a coplanar resonator
- Filed quantization in coplanar resonators

Flux biased qubit



Josephson potential:

$$U = E_J [2 + \alpha - \cos \varphi_{01} - \cos \varphi_{20} - \alpha \cos(\varphi_{ext} - \varphi_{01} - \varphi_{20})]$$

Charge:

$$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

Potential:

$$\vec{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

Capacitance matrix:

$$C = \begin{pmatrix} C_{01} + C_{12} & -C_{12} \\ -C_{12} & C_{02} + C_{12} \end{pmatrix}$$

$$\vec{n} = \frac{C\vec{V}}{2e} \quad \vec{V} = 2eC^{-1}\vec{n}$$

Electrostatic energy: $T = \frac{(2e)^2}{2} \vec{n} C^{-1} \vec{n}$

The Hamiltonian:

$$H = \frac{(2e)^2}{2} \vec{n} C^{-1} \vec{n} + E_J [2 + \alpha - \cos \varphi_{01} - \cos \varphi_{20} - \alpha \cos(\varphi_{ext} - \varphi_{01} - \varphi_{20})]$$

Phase operators

$$\cos \hat{\varphi} = \frac{e^{i\hat{\varphi}} + e^{-i\hat{\varphi}}}{2}$$

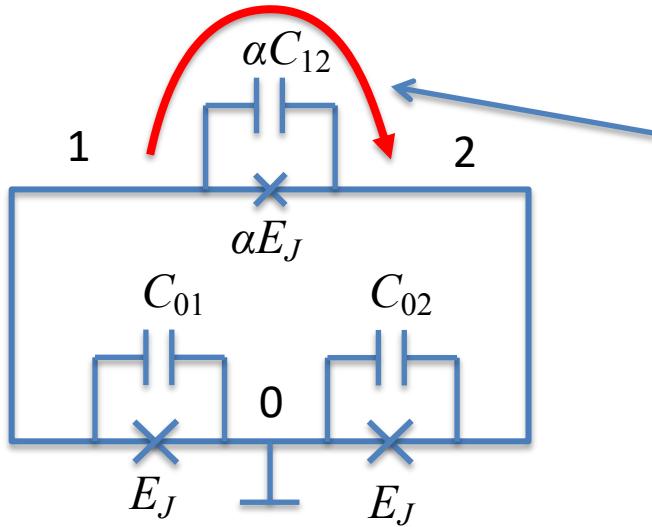
$$e^{i\hat{\varphi}} = |N\rangle\langle N-1| \quad e^{-i\hat{\varphi}} = |N-1\rangle\langle N|$$

$$\cos \hat{\varphi}_{01} = \frac{1}{2}(|N_1\rangle\langle N_1-1| + |N_1-1\rangle\langle N_1|)$$

$$\cos \hat{\varphi}_{20} = \frac{1}{2}(|N_2\rangle\langle N_2-1| + |N_2-1\rangle\langle N_2|)$$

$$\cos(\varphi_{ext} - \hat{\varphi}_{01} - \hat{\varphi}_{20}) = \frac{e^{i(\varphi_{ext} - \hat{\varphi}_{01} - \hat{\varphi}_{20})} + e^{-i(\varphi_{ext} - \hat{\varphi}_{01} - \hat{\varphi}_{20})}}{2}$$

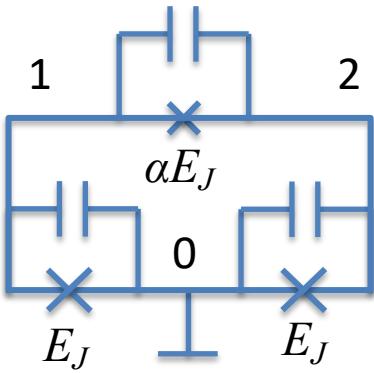
$$e^{i(\varphi_{ext} - \hat{\varphi}_{01} - \hat{\varphi}_{20})} = e^{i\varphi_{ext}} e^{-i\hat{\varphi}_{01}} e^{-i\hat{\varphi}_{20}} = e^{i\varphi_{ext}} |N_1-1\rangle\langle N_1| |N_2\rangle\langle N_2-1| = e^{i\varphi_{ext}} |N_1-1, N_2\rangle\langle N_1, N_2-1|$$



Physical meaning: tunneling of a charge quantum (Cooper pair) from island 1 to island 2

$$\cos(\varphi_{ext} - \hat{\varphi}_{01} - \hat{\varphi}_{20}) = \frac{1}{2}(e^{i\varphi_{ext}} |N_1-1, N_2\rangle\langle N_1, N_2-1| + e^{-i\varphi_{ext}} |N_1, N_2-1\rangle\langle N_1-1, N_2|)$$

The flux qubit Hamiltonian in the charge basis

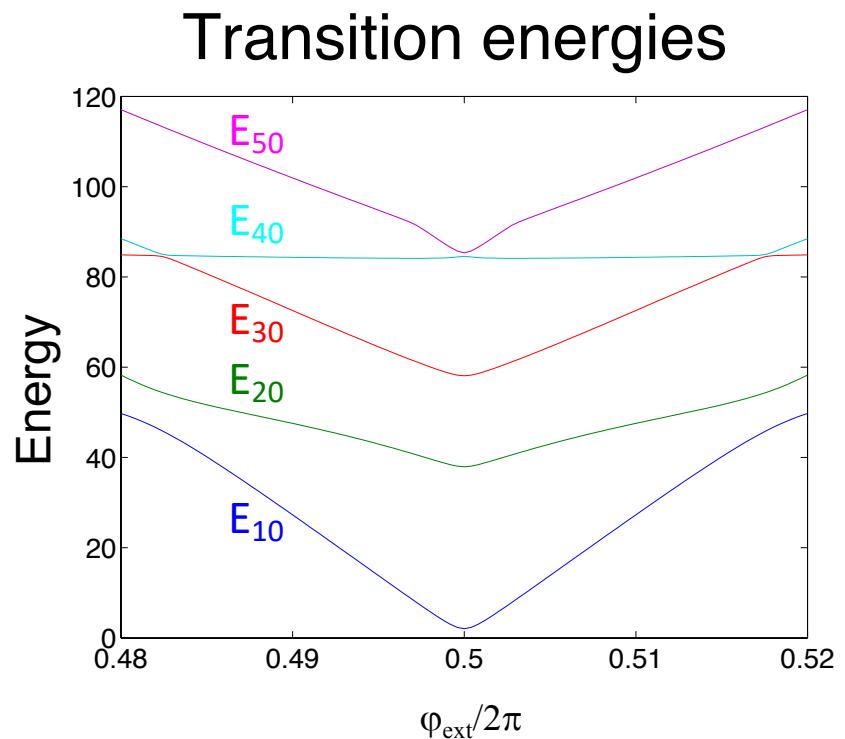
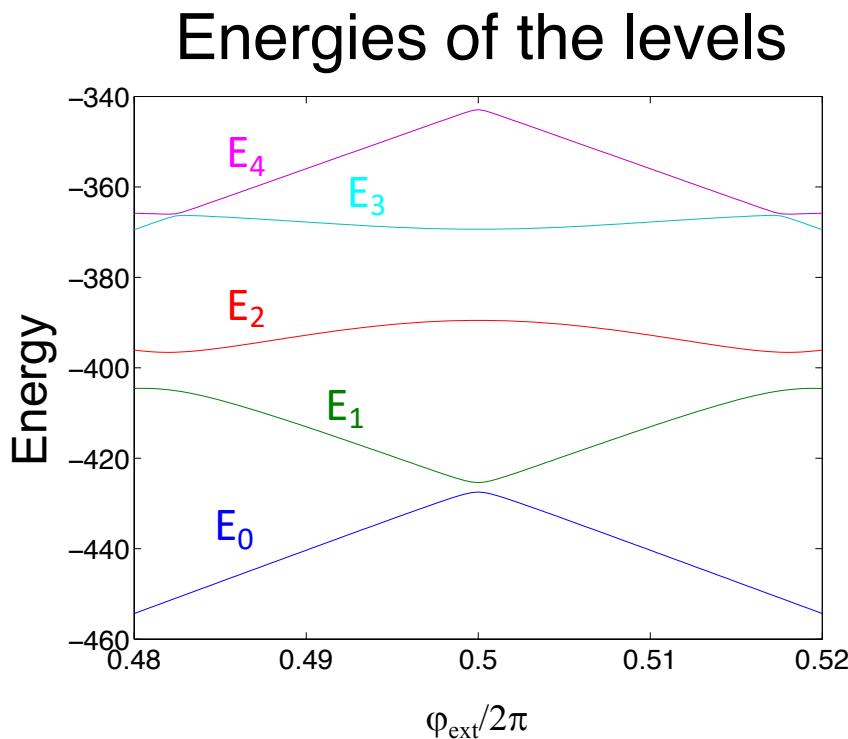


$$H = \frac{(2e)^2}{2} \vec{n} C^{-1} \vec{n} + E_J [2 + \alpha - \cos \varphi_{01} - \cos \varphi_{20} - \alpha \cos(\varphi_{ext} - \varphi_{01} - \varphi_{20})]$$

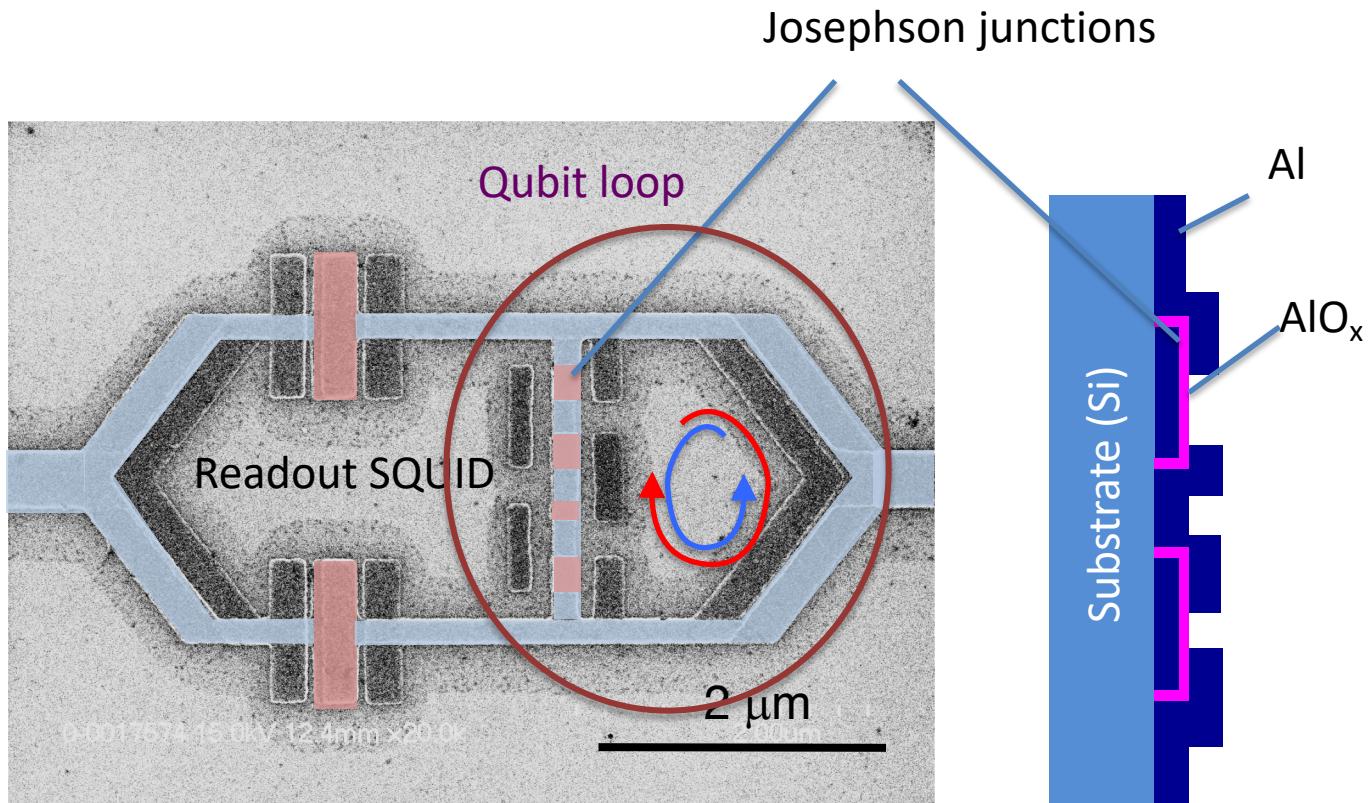
$$U(n_1, n_2) = \frac{(2e)^2}{2} \begin{pmatrix} n_1 & n_2 \end{pmatrix} C^{-1} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

$$H = \begin{pmatrix} \ddots & & & & & \\ U(-1,0) & -\frac{\alpha E_J}{2} e^{i\varphi_{ext}} & -\frac{E_J}{2} & 0 & 0 & \\ -\frac{\alpha E_J}{2} e^{-i\varphi_{ext}} & U(0,-1) & -\frac{E_J}{2} & 0 & 0 & \\ -\frac{E_J}{2} & -\frac{E_J}{2} & U(0,0) & -\frac{E_J}{2} & -\frac{E_J}{2} & \\ 0 & 0 & -\frac{E_J}{2} & U(0,1) & -\frac{\alpha E_J}{2} e^{i\varphi_{ext}} & \\ 0 & 0 & -\frac{E_J}{2} & -\frac{\alpha E_J}{2} e^{-i\varphi_{ext}} & U(1,0) & \\ & & & & & \ddots \end{pmatrix}$$

Energy bands in the flux quantum system

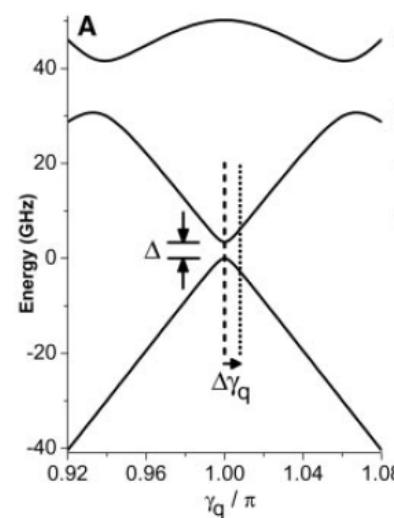
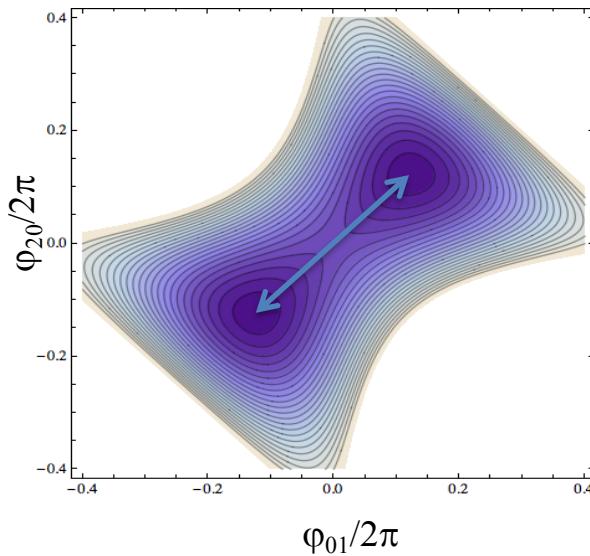
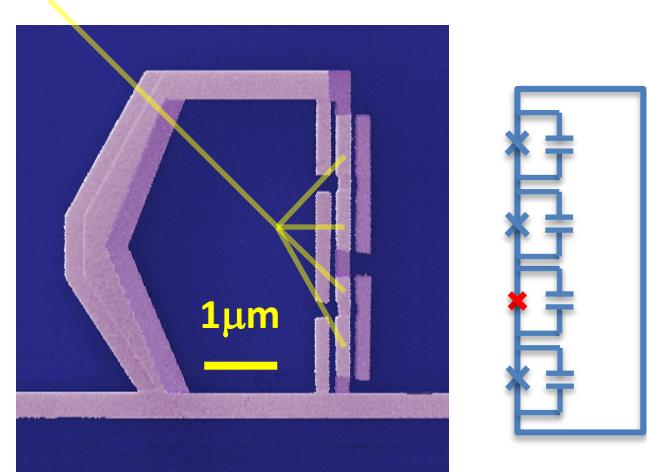
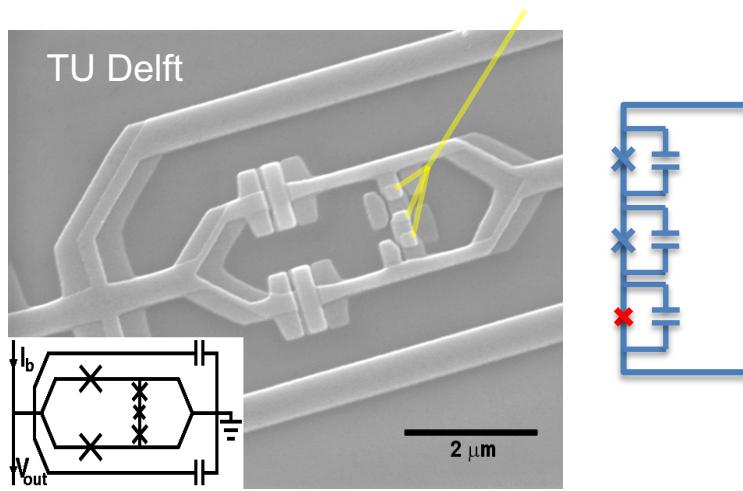


The four-junction flux qubit



Josephson junction flux qubit

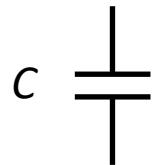
Josephson junctions



Quantum mechanics of electrical circuits.
Capacitance, inductance, resonators

Electric circuit

Capacitor



Inductor



Voltage, current:

V, I

Charge, magnetic flux:

$Q = CV \quad \Phi = LI$

$I = \dot{Q} \quad V = \dot{\Phi}$

Potential energy

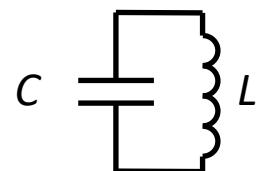
$$U = \frac{CV^2}{2}$$

$$U = \frac{Q^2}{2C}$$

Kinetic energy

$$T = \frac{LI^2}{2}$$

$$T = \frac{\Phi^2}{2L}$$

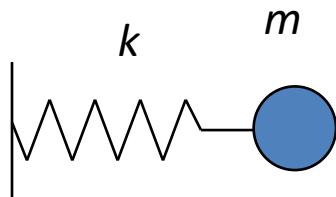


Total energy of the system

$$E = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

$$E = \frac{CV^2}{2} + \frac{LI^2}{2}$$

Classical harmonic oscillators



$$E = \frac{kx^2}{2} + \frac{p^2}{2m}$$

$$E = \frac{kx^2}{2} + \frac{m\dot{x}^2}{2}$$

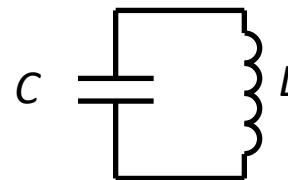
Coordinate: x

Momentum: $p = m\dot{x}$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$E = \frac{m}{2}(\omega_0^2 x^2 + \dot{x}^2)$$

$$x = Ae^{i\omega t} + Be^{-i\omega t}$$



$$E = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

$$E = \frac{Q^2}{2C} + \frac{L\dot{Q}^2}{2}$$

$$Q$$

$$p' = L\dot{Q} \rightarrow \Phi$$

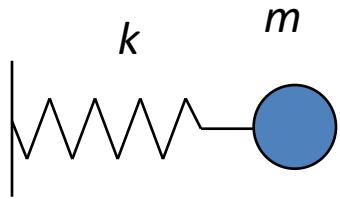
$$E = \frac{L}{2}(\omega_0^2 Q^2 + \dot{Q}^2) \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = Ae^{i\omega t} + Be^{-i\omega t}$$

The equations are transformed from one to another with the substitutions

$x \rightarrow Q$	$p \rightarrow \Phi$	$k \rightarrow \frac{1}{C}$	$m \rightarrow L$
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Classical harmonic oscillators (alternative approach)

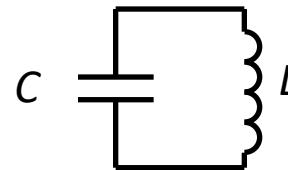


$$E = \frac{kx^2}{2} + \frac{p^2}{2m}$$

$$E = \frac{kx^2}{2} + \frac{m\dot{x}^2}{2}$$

Coordinate: x

Momentum: $p = m\dot{x}$



$$E = \frac{\Phi^2}{2L} + \frac{Q^2}{2C}$$

$$E = \frac{\Phi^2}{2L} + \frac{C\dot{\Phi}^2}{2}$$

Φ

$p' = C\dot{\Phi} \rightarrow Q$

$$\omega_0 = \sqrt{\frac{k}{m}} \quad E = \frac{m}{2} (\omega_0^2 x^2 + \dot{x}^2)$$

$$E = \frac{L}{2} (\omega_0^2 \Phi^2 + \dot{\Phi}^2) \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$x = Ae^{i\omega t} + Be^{-i\omega t}$$

$$Q = Ae^{i\omega t} + Be^{-i\omega t}$$

The equations are identical with the following substitutions

$$x \rightarrow \Phi \quad p \rightarrow Q \quad k \rightarrow \frac{1}{L} \quad m \rightarrow C$$

Quantum mechanics of harmonic oscillators

$$H = \frac{\hbar\omega_0}{2} \left(\frac{\hat{x}^2}{x_0^2} + \frac{\hat{p}^2}{p_0^2} \right)$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega_0}} \quad p_0 = \frac{\hbar}{x_0}$$

$$H = \frac{\hbar\omega_0}{2} \left(\frac{\hat{Q}^2}{q_0^2} + \frac{\hat{\Phi}^2}{\phi_0^2} \right)$$

$$q_0 = \sqrt{\frac{\hbar}{L\omega_0}} = \sqrt{\hbar C \omega_0} \quad \phi_0 = \frac{\hbar}{q_0} = \sqrt{\hbar L \omega_0}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Because $CV = Q$ $LI = \Phi$ we can rewrite the Hamiltonian of LC-resonator in the form

$$H = \frac{\hbar\omega_0}{2} \left(\frac{\hat{V}^2}{V_0^2} + \frac{\hat{I}^2}{I_0^2} \right)$$

$$V_0 = \frac{q_0}{C} = \sqrt{\frac{\hbar C \omega_0}{C^2}} = \boxed{\sqrt{\frac{\hbar \omega_0}{C}}} \quad I_0 = \frac{\phi_0}{L} = \sqrt{\frac{\hbar L \omega_0}{L^2}} = \boxed{\sqrt{\frac{\hbar \omega_0}{L}}}$$

Hint to remember: $\frac{CV_0^2}{2} = \frac{LI_0^2}{2} = \frac{\hbar\omega_0}{2}$

Position operator: \hat{x}

Momentum operator: $\hat{p} = -i\hbar \frac{\partial}{\partial x}$

Commutation relations: $[\hat{p}, \hat{x}] = -i\hbar$

$$y = \frac{x}{x_0}$$

$$y = \frac{Q}{q_0} \quad \text{or} \quad y = \frac{V}{V_0}$$

Voltage operator: $\hat{V} = \hat{Q}/C$

Current operator: $\hat{I} = \frac{\hat{\Phi}}{L} = -i \frac{\hbar}{L} \frac{\partial}{\partial Q} = -i \frac{\hbar}{LC} \frac{\partial}{\partial V}$

$$[L\hat{I}, C\hat{V}] = -i\hbar$$

$$\hat{H} = \frac{\hbar\omega_0}{2} \left(y^2 - \frac{\partial^2}{\partial y^2} \right)$$

Quantum mechanics of harmonic oscillators

$$H = \frac{\hbar\omega_0}{2} \left(\frac{\hat{x}^2}{x_0^2} + \frac{\hat{p}^2}{p_0^2} \right)$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega_0}} \quad p_0 = \frac{\hbar}{x_0}$$

Position operator: \hat{x}

Momentum operator: $\hat{p} = -i\hbar \frac{\partial}{\partial x}$

Commutation relations: $[\hat{p}, \hat{x}] = -i\hbar$

$$H = \frac{\hbar\omega_0}{2} \left(\frac{\hat{V}^2}{V_0^2} + \frac{\hat{I}^2}{I_0^2} \right)$$

$$V_0 = \sqrt{\frac{\hbar\omega_0}{C}} \quad I_0 = \sqrt{\frac{\hbar\omega_0}{L}}$$

Voltage operator: \hat{V}

Current operator: $\hat{I} = -i \frac{\hbar}{LC} \frac{\partial}{\partial V}$

$$[L\hat{I}, C\hat{V}] = -i\hbar$$

$$y = \frac{x}{x_0}$$

$$y = \frac{V}{V_0}$$

$$\hat{H} = \frac{\hbar\omega_0}{2} \left(y^2 - \frac{\partial^2}{\partial y^2} \right)$$

Number operator

$$H = \frac{\hbar\omega_0}{2} \left(y^2 - \frac{\partial^2}{\partial y^2} \right)$$

$$H\psi_n(y) = E_n\psi_n(y)$$

Hermit polynomials

$$\psi_n(y) = \frac{1}{\pi^{\frac{1}{4}} \sqrt{2^n n!}} e^{-\frac{y^2}{2}} H_n(y)$$

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} (e^{-y^2})$$

$$H_0(y) = 1$$

$$H_1(y) = 2y$$

$$H_2(y) = 4y^2 - 2$$

$$H_3(y) = 8y^3 - 12y$$

$$E_n = \hbar\omega_0 \left(n + \frac{1}{2} \right) \quad H = \frac{\hbar\omega_0}{2} \left(y^2 - \frac{\partial^2}{\partial y^2} \right)$$

Number operator

$$\hat{N} = \frac{1}{2} \left(y^2 - \frac{\partial^2}{\partial y^2} - 1 \right)$$

$$\hat{N}\psi_n = \frac{1}{2} \left(y^2 - \frac{\partial^2}{\partial y^2} - 1 \right) \psi_n = n\psi_n$$

$$\hat{N}|n\rangle = n$$

$$H = \hbar\omega_0 \left(\hat{N} + \frac{1}{2} \right) \quad \hat{N} = n|n\rangle\langle n|$$

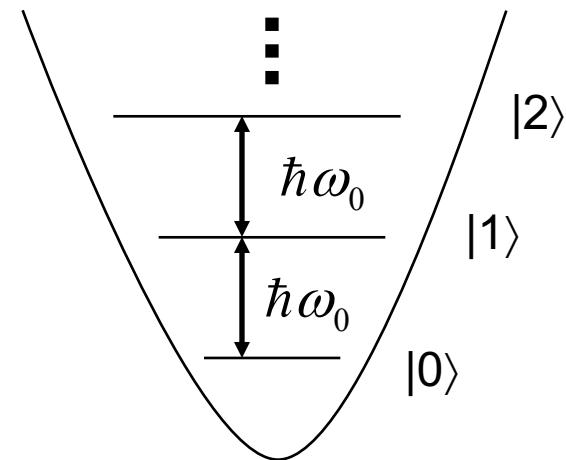
Number operator and Fock states

Harmonic oscillator Hamiltonian

$$\hat{H} = \hbar\omega_0 \left(\hat{N} + \frac{1}{2} \right)$$

$$\hat{H}\Psi_n = E_n \Psi_n \quad E_n = \hbar\omega_0 \left(n + \frac{1}{2} \right)$$

Photons



Photon number operator

$$\hat{N} = \sum_{n=0}^{\infty} n |n\rangle \langle n| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ & & & \ddots \end{pmatrix}$$

Fock states are photon number states

$$\Psi_n = |n\rangle$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix} \quad \dots$$

a and a^\dagger operators

$$H = \frac{\hbar\omega_0}{2} \left(y^2 - \frac{\partial^2}{\partial y^2} \right)$$

$$[\xi, \eta] = \frac{1}{\hbar} [x, \hat{p}] = i$$

$$\xi = y \quad \eta = -i \frac{\partial}{\partial \xi}$$

$$a = \frac{1}{\sqrt{2}} \left(y + \frac{\partial}{\partial y} \right)$$

$$a^\dagger = \frac{1}{\sqrt{2}} \left(y - \frac{\partial}{\partial y} \right)$$

$$y = \frac{Q}{Q_0} = \frac{V}{V_0} \quad \text{or} \quad y = \frac{\Phi}{\Phi_0} = \frac{I}{I_0}$$

+1

$$a = \frac{1}{\sqrt{2}} (\xi + i\eta)$$

$$a^\dagger = \frac{1}{\sqrt{2}} (\xi - i\eta)$$

$$aa^\dagger = \frac{1}{2} (\xi^2 + \eta^2 - i[\xi, \eta])$$

$$a^\dagger a = \frac{1}{2} (\xi^2 + \eta^2 + i[\xi, \eta])$$

$$H = \frac{\hbar\omega_0}{2} (a^\dagger a + aa^\dagger)$$

$$[a, a^\dagger] = 1$$

$$\hat{H} = \hbar\omega_0 \left(a^\dagger a + \frac{1}{2} \right)$$

-1

The solution of the equation $\hat{H}\psi_n = E_n\psi_n$ is $E_n = \hbar\omega_0 \left(n + \frac{1}{2} \right)$

$$\hat{N} = \frac{1}{2} \left(y^2 - \frac{\partial^2}{\partial y^2} - 1 \right)$$

$$\hat{H} = \hbar\omega_0 \left(\hat{N} + \frac{1}{2} \right)$$

$$\hat{N} = a^\dagger a$$

Derivation of explicit forms of operators a and a^\dagger

$$\hat{N}a|n\rangle = a^\dagger aa|n\rangle = (aa^\dagger - 1)a|n\rangle = (aa^\dagger a - a)|n\rangle = a(a^\dagger a - 1)|n\rangle = a(\hat{N} - 1)|n\rangle = a(n - 1)|n\rangle$$

Therefore $a = c_n |n-1\rangle \langle n|$

$$\hat{N}a^\dagger|n\rangle = a^\dagger aa^\dagger|n\rangle = a^\dagger(a^\dagger a + 1)|n\rangle = a^\dagger(n + 1)|n\rangle = a(a^\dagger a - 1)|n\rangle = (n + 1)a^\dagger|n\rangle$$

Therefore $a^\dagger = d_n |n+1\rangle \langle n|$

We verify:

$$\hat{N}a|n\rangle = c_n(n - 1)|n - 1\rangle \langle n - 1|n - 1\rangle \langle n| = c_n(n - 1)|n - 1\rangle \langle n| = (n - 1)a|n\rangle$$

$$\hat{N}a^\dagger|n\rangle = d_n(n + 1)|n + 1\rangle \langle n + 1|n + 1\rangle \langle n| = d_n(n + 1)|n + 1\rangle \langle n| = (n + 1)a^\dagger|n\rangle$$

From $a^\dagger a = \hat{N}$ we find

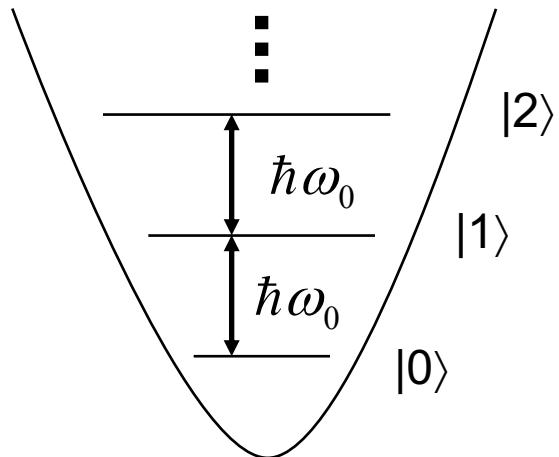
$$a^\dagger a|n\rangle = d_{n-1}|n\rangle \langle n-1|c_n|n-1\rangle \langle n|n\rangle = d_{n-1}c_n|n\rangle = n|n\rangle \text{ therefore } d_{n-1}c_n = n$$

From $a^\dagger = (a)^\ast = c_n|n\rangle \langle n-1| \equiv d_{n-1}|n\rangle \langle n-1|$ we find $d_{n-1} = c_n = \sqrt{n}$

Finally
$$a = \sqrt{n}|n-1\rangle \langle n| \quad a^\dagger = \sqrt{n+1}|n+1\rangle \langle n|$$

Explicit form of a -operators

Harmonic oscillator



\hat{a} - creation operator

\hat{a}^+ - annihilation operator

$$a^+ = \sum_{n=0}^{\infty} \sqrt{n+1} |n+1\rangle\langle n| \quad a = \sum_{n=0}^{\infty} \sqrt{n} |n-1\rangle\langle n|$$

$$a^+ |n\rangle = \sqrt{n+1} |n+1\rangle \quad a |n\rangle = \sqrt{n} |n-1\rangle$$

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 & \\ 0 & 0 & \sqrt{2} & 0 & \\ 0 & 0 & 0 & \sqrt{3} & \\ 0 & 0 & 0 & 0 & \ddots \end{pmatrix} \quad a^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & \\ 0 & \sqrt{2} & 0 & 0 & \\ 0 & 0 & \sqrt{3} & 0 & \ddots \end{pmatrix}$$

$$H = \hbar\omega_0 \left(\sum_{n=0}^{\infty} n |n\rangle\langle n| + \frac{1}{2} \right) = \hbar\omega_0 \left(\hat{N} + \frac{1}{2} \right)$$

Photon number operator:

$$\hat{N} = a^+ a = \sum_{n=0}^{\infty} n |n\rangle\langle n|$$

Quantized fields in LC-resonator

$$H = \frac{C\hat{V}^2}{2} + \frac{L\hat{I}^2}{2}$$

$$H = \frac{\hbar\omega_0}{2} \left(\frac{\hat{V}^2}{V_0^2} + \frac{\hat{I}^2}{I_0^2} \right) \quad V_0 = \sqrt{\frac{\hbar\omega_0}{C}} \quad I_0 = \sqrt{\frac{\hbar\omega_0}{L}}$$

Voltage operator: \hat{V}

Current operator: $\hat{I} = -i \frac{\hbar}{LC} \frac{\partial}{\partial V}$

$$a = \frac{1}{\sqrt{2}} \left(\frac{\hat{V}}{V_0} + i \frac{\hat{I}}{I_0} \right) \quad a^\dagger = \frac{1}{\sqrt{2}} \left(\frac{\hat{V}}{V_0} - i \frac{\hat{I}}{I_0} \right)$$

$$\hat{V} = \sqrt{\frac{\hbar\omega_0}{2C}} (a + a^\dagger)$$

$$\hat{I} = i \sqrt{\frac{\hbar\omega_0}{2L}} (a^\dagger - a)$$

$$H = \hbar\omega_0 \left(a^\dagger a + \frac{1}{2} \right)$$

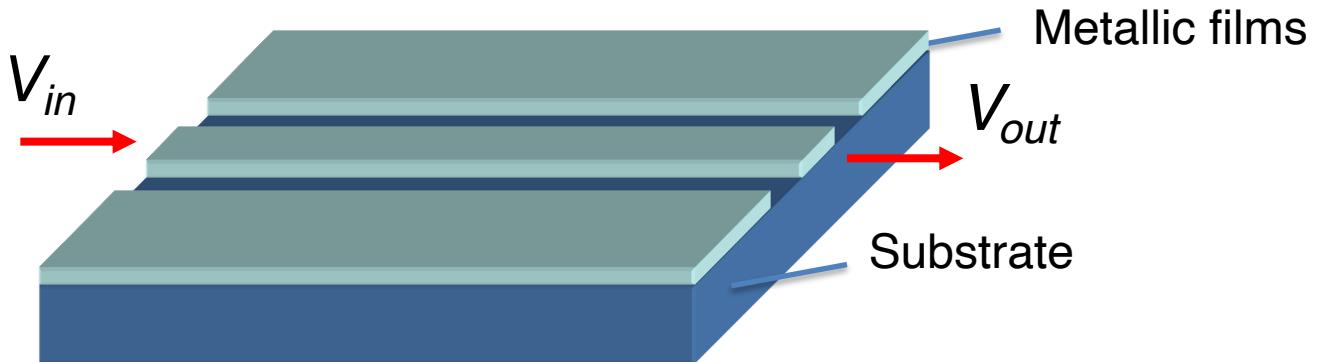
$$\hat{N} = a^\dagger a$$

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

Coplanar waveguide transmission lines

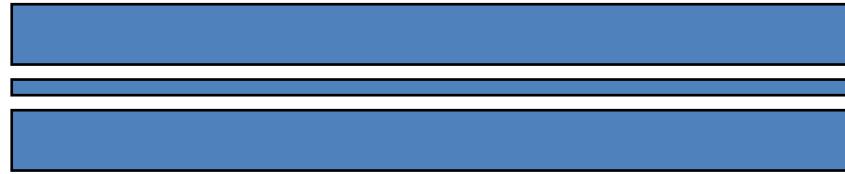
Coplanar transmission line



Right propagating wave

$$I_R = \frac{V_R}{Z} \quad V_+ e^{ikx-i\omega t}$$

$$I_+ e^{ikx-i\omega t}$$

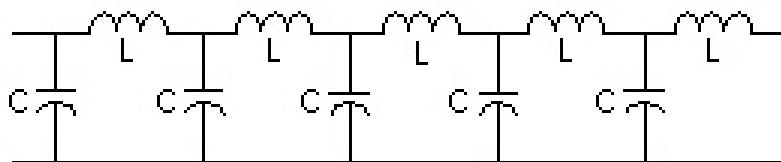


Left propagating wave (e.g. relected wave)

$$V_- e^{-ikx-i\omega t}$$

$$I_- e^{-ikx-i\omega t}$$

$$I_L = -\frac{V_L}{Z}$$



l – inductance per length

c – capacitance per length

Telegrapher's equations:

$$\frac{\partial V}{\partial x} = -l \frac{\partial I}{\partial t}$$

$$ikV_0 e^{ikx-i\omega t} = li\omega I_0 e^{ikx-i\omega t}$$

$$\frac{\partial I}{\partial x} = -c \frac{\partial V}{\partial t}$$

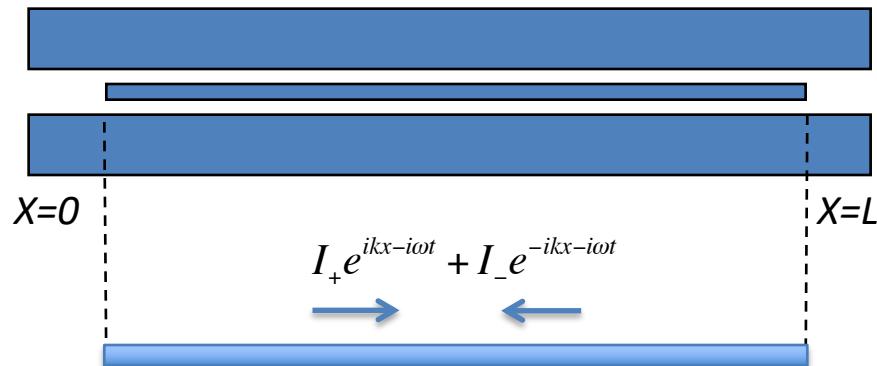
$$ikI_0 e^{ikx-i\omega t} = \omega icV_0 e^{ikx-i\omega t}$$

$$Z = \frac{V_0}{I_0} = \frac{\omega l}{k}$$

$$Z = \frac{V_0}{I_0} = \frac{k}{\omega c}$$

$$Z = \sqrt{\frac{l}{c}}$$

Quantum mechanics of coplanar resonators



$$x = 0 : I_+ + I_- = 0$$

$$I_- = -I_+$$

$$x = L : I_+ e^{ikL} + I_- e^{-ikL} = 0$$

$$I_- = -I_+ e^{i2kL}$$

Boundary conditions:
No current at the ends

$$e^{i2kL} = 1 \rightarrow kL = \pi n \rightarrow k = \frac{\pi n}{L}$$

$$I_+ e^{i\frac{\pi n}{L}x-i\omega t} - I_+ e^{-i\frac{\pi n}{L}x-i\omega t} = I_0 \sin\left(\frac{\pi n}{L}x\right) e^{-i\omega t}$$

$n = 1, 2, 3, \dots$

$$\frac{\partial I}{\partial x} = -c \frac{\partial V}{\partial t} \rightarrow \frac{\partial I}{\partial x} = i\omega c V$$

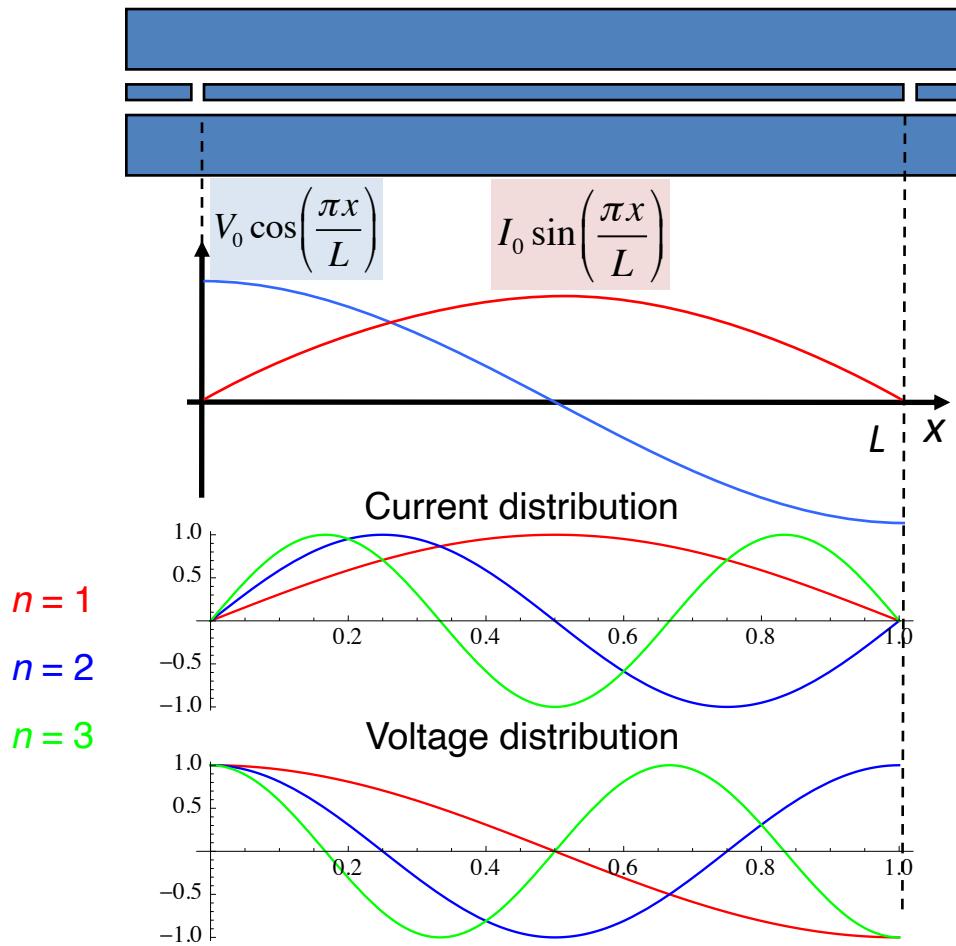
$$V = V_0 \cos\left(\frac{\pi n}{L}x\right) e^{-i\omega t - i\frac{\pi}{2}}$$

$$V_0 = ZI_0$$

Field distribution

$$I_0 \sin\left(\frac{\pi n}{L} x\right)$$

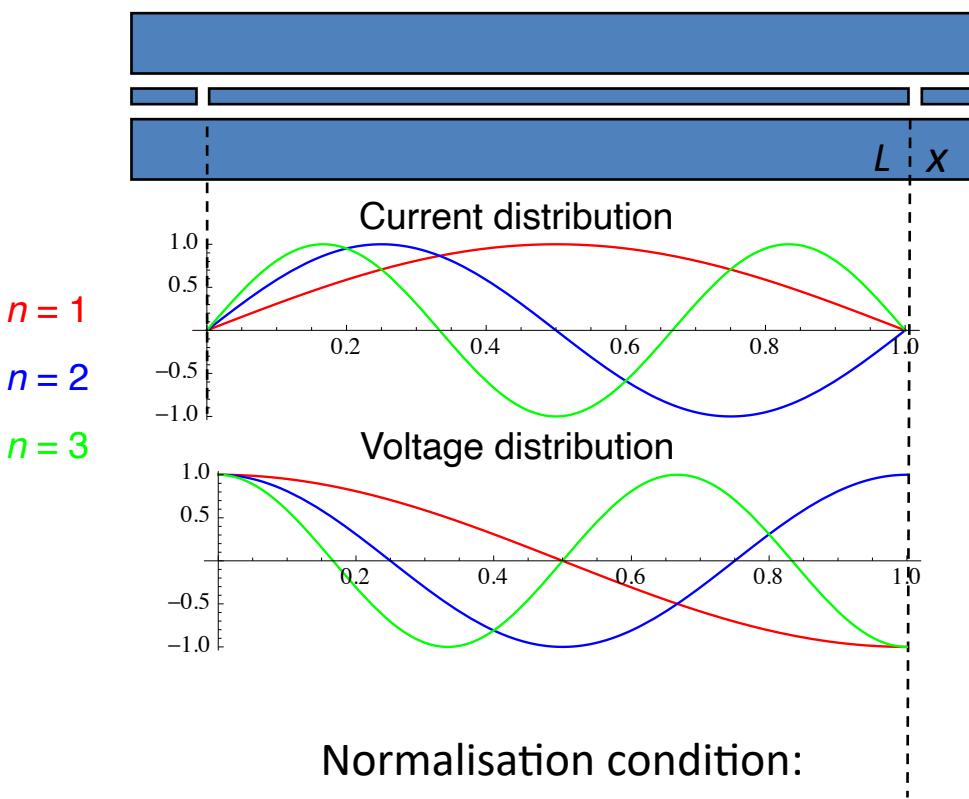
$$V = V_0 \cos\left(\frac{\pi n}{L} x\right)$$



Field quantisation in coplanar resonators

$$I_0 \sin\left(\frac{\pi n}{L}x\right)$$

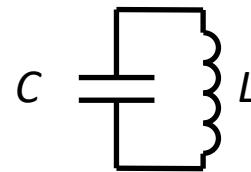
$$V = V_0 \cos\left(\frac{\pi n}{L}x\right)$$



$$\int_0^L \cos^2\left(\frac{\pi n x}{L}\right) dx = \frac{1}{2}$$

$$\int_0^L \sin^2\left(\frac{\pi n x}{L}\right) dx = \frac{1}{2}$$

Therefore $\sqrt{2} \cos\left(\frac{\pi x}{L}\right)$ and $\sqrt{2} \sin\left(\frac{\pi x}{L}\right)$ account space distribution of voltage and current



$$\hat{V} = \sqrt{\frac{\hbar\omega_0}{2C}} (a^\dagger + a)$$

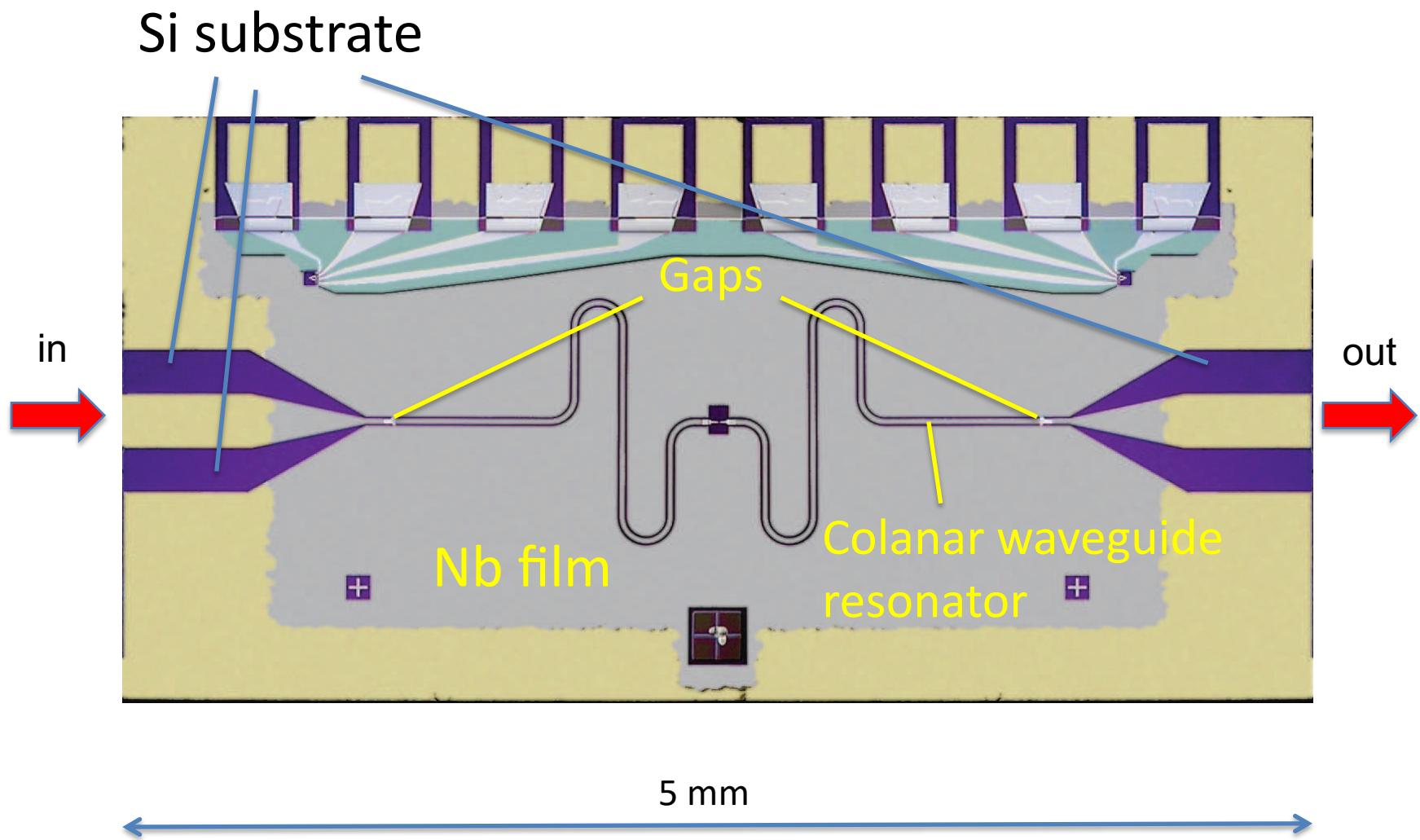
$$\hat{I} = i\sqrt{\frac{\hbar\omega_0}{2L}} (a^\dagger - a)$$

Voltage and current operators in a coplanar resonator:

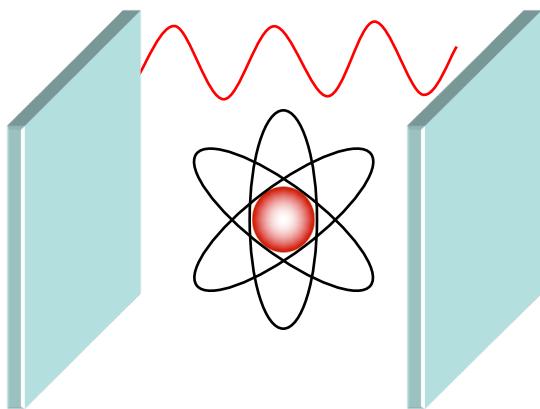
$$\hat{V} = \sqrt{\frac{\hbar\omega_0}{C}} (a + a^\dagger) \cos\left(\frac{\pi n}{L}x\right)$$

$$\hat{I} = i\sqrt{\frac{\hbar\omega_0}{C}} (a^\dagger - a) \sin\left(\frac{\pi n}{L}x\right)$$

Coplanar waveguide resonator on silicon substrate



Atom in cavity versus atom in on-chip resonator Cavity QED versus Circuit QED

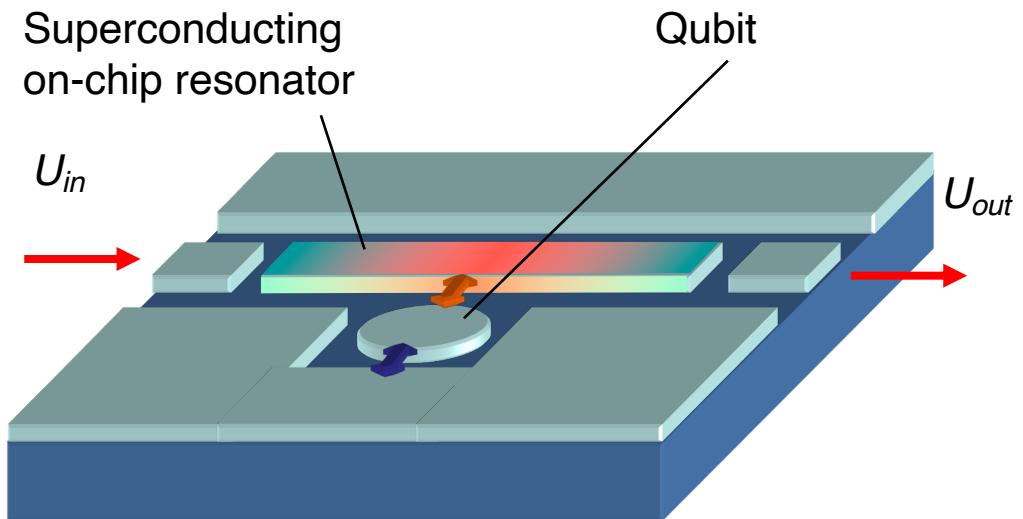
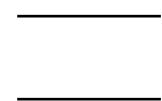


Wavelength $\lambda \sim 1 \mu\text{m}$

The atom Hamiltonian

$$H_a = -\frac{\Delta E}{2} \sigma_z$$

$$|n_a\rangle$$

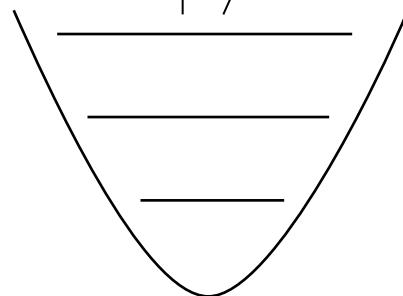


Wavelength $\lambda \sim 1 \text{ cm}$

The resonator Hamiltonian

$$H_r = \hbar\omega \left(\hat{N} + \frac{1}{2} \right)$$

$$|N\rangle$$



Two-level atom creation/annihilation operators

To describe interaction of our atom with external fields we introduce operators of creation/annihilation for the atomic excited state

$$\sigma^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 1|$$

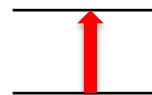
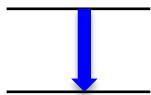
$$\sigma^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0|$$

$$\sigma^- |1\rangle = |0\rangle \quad \sigma^- |0\rangle = 0$$

$$\sigma^+ |0\rangle = |1\rangle \quad \sigma^+ |1\rangle = 0$$

$$|1\rangle \rightarrow |0\rangle$$

$$|0\rangle \rightarrow |1\rangle$$



$$\sigma^- = \frac{\sigma_x + i\sigma_y}{2}$$

$$\sigma^+ = \frac{\sigma_x - i\sigma_y}{2}$$

$$\sigma_x = \sigma^+ + \sigma^-$$

Atom-resonator interaction

The system is described by two quantum numbers:

- qubit state n_a : (0, 1)
- photon number state N : (0, 1, 2, 3, ...)

The system state:

$$|n_a N\rangle$$

Two-level system
(atom)

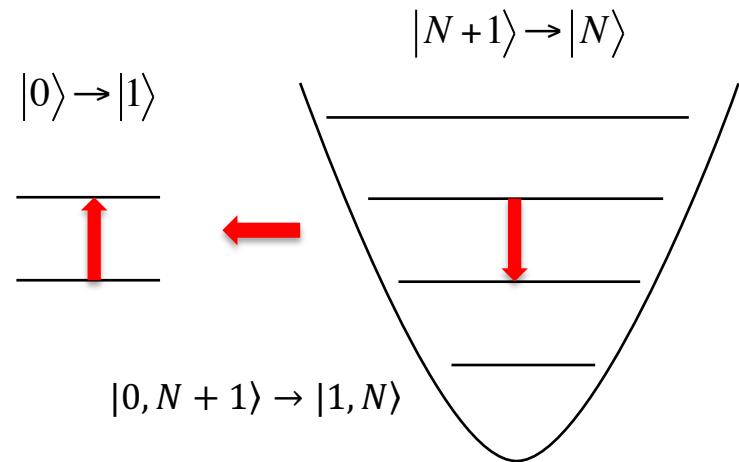
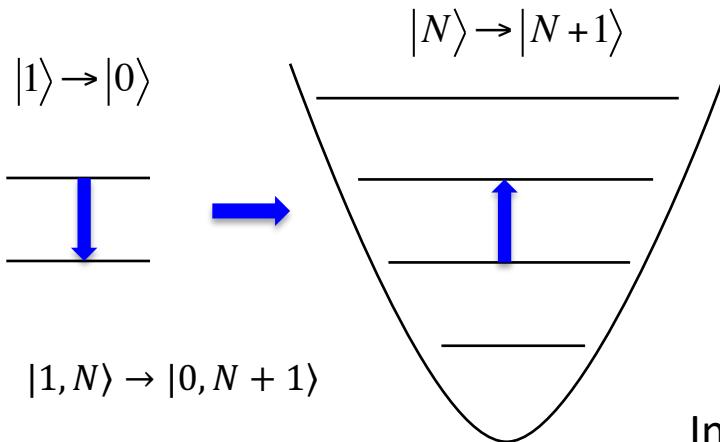
$$\begin{array}{c} |n_a\rangle \\ \hline \\ \hline \end{array}$$

Harmonic oscillator
(resonator)

$$|N\rangle$$

The Hamiltonian of the total system:

$$H_{tot} = H_a + H_r + H_{int}$$



Interaction:

$$|0, N+1\rangle\langle 1, N| + |1, N\rangle\langle 0, N+1|$$

$$a^+\sigma^- + a^-\sigma^+$$