LP Formulation of the Supplier Production Plan

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1 Notation & Problem Data

The total operational horizon of the supply chain and the supplier planning horizon will be denoted as $T \in \mathbb{N}^+$ and $H \in \mathbb{N}^+$, respectively and, by convention, *time will be measured in days*. For notational simplicity, since all suppliers receive and generate the same *type* of information, every day, we may employ generic notation, common to all suppliers, without the use of an explicit "supplier index". Suppose that we are on the t-th day, for some *fixed* $t \in \mathbb{N}_T^+$, and suppose that *Joe* is a supplier. Intuitively, we will call the t-th day *today*. We also make the following explicit assumptions:

- First, we take into account the inevitable existence of non-zero travel times between suppliers; let $\tau_{c_j} \in \mathbb{N}^+$ denote the travel time (in days) between Joe's location and that of *child supplier* $c_j \in \mathcal{N}_{Joe}$, for all $j \in \mathbb{N}^+_{|\mathcal{N}_{Joe}|}$. We also define the *lag vector*

$$\boldsymbol{\tau} \triangleq \left[\tau_{c_1} \, \dots \, \tau_{c_{|\mathcal{N}_{Joe}|}}\right]^{\boldsymbol{T}} \in \left[\mathbb{N}^+\right]^{|\mathcal{N}_{Joe}|}. \tag{1}$$

  Likewise, we let $\tau_p \in \mathbb{N}^+$ be the travel time between Joe and his *parent supplier*.

- Second, we constrain each supplier (and thus, also Joe), so that an incoming part may be used for production *at least one day after its arrival* to the supplier's location.

- Third, implementation of today's production plan (decision) and shipment of produced parts to the parent supplier *both* take place on the *same day (today)*.

Under those considerations, every day, Joe receives the following information:

- From *downstream* (closer to the final product), the *current*, *temporally adjusted* production *forecast* of his parent (the supplier using Joe's produced part). This production plan is denoted by a vector

$$\widetilde{\boldsymbol{x}}^D_{t-1} \triangleq \left[\widetilde{x}^D_{t-1,t+\tau_p+1} \, \widetilde{x}^D_{t-1,t+\tau_p+2} \, \dots \, \widetilde{x}^D_{t-1,t+\tau_p+H}\right]^{\boldsymbol{T}} \in \mathbb{N}^H, \tag{2}$$

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where $\widetilde{x}^{D}_{t-1,t'}$ constitutes the *demand in number of parts* on the $\left(t'-t-\tau_p-1\right)$*-th day from today*, for each $t' \in \mathbb{N}^{t+\tau_p+1}_{t+\tau_p+H}$.

- From *upstream* (closer to basic supplies - leafs of the supply chain tree), the *current*, *temporally adjusted* production *forecasts* from *all* of his children (the suppliers that supply Joe with parts). Those production plans are denoted as

$$\widetilde{\boldsymbol{x}}^{S}_{c,t-1} \triangleq \left[\widetilde{x}^{S}_{c,t-1,t-1}\, \widetilde{x}^{S}_{c,t-1,t}\, \dots\, \widetilde{x}^{S}_{c,t-1,t+H-\tau_c-2}\right]^{\boldsymbol{T}} \in \mathbb{N}^{H-\tau_c}, \quad c \in \mathcal{N}^{-}_{Joe}, \tag{3}$$

where $\mathcal{N}^{-}_{Joe}$ is the set containing the *labels* of Joe's children suppliers (of course, this set can be made bijective to a subset of the naturals), where $\widetilde{x}_{c,t-1,t'}$ constitutes the *projected supply in number of parts* from the child supplier c, on the $\left(t'-t-\tau_c-1\right)$*-th day from today*, for each $t' \in \mathbb{N}^{t+\tau_c+1}_{t+H-2}$. For convenience, we may also define the zero-padded, *extended plans*

$$\widetilde{\boldsymbol{x}}^{S,E}_{c,t-1} \triangleq \left[\boldsymbol{0}_{\tau_c}\, \widetilde{\boldsymbol{x}}^{S}_{c,t-1}\right]^{\boldsymbol{T}} \in \mathbb{N}^{H}, \quad c \in \mathcal{N}^{-}_{Joe}, \tag{4}$$

as well as the vectors of *extended plans across children suppliers*

$$\widetilde{\boldsymbol{x}}^{S,E}_{t-1,t'} \triangleq \left[\widetilde{x}^{S,E}_{c_1,t-1,t'}\, \widetilde{x}^{S,E}_{c_2,t-1,t'}\, \dots\, \widetilde{x}^{S,E}_{c_{\left|\mathcal{N}_{Joe}\right|},t-1,t'}\right]^{\boldsymbol{T}} \in \mathbb{N}^{\left|\mathcal{N}_{Joe}\right|}, \quad t' \in \mathbb{N}^{t}_{t+H-1}, \tag{5}$$

where, for finite set $\mathcal{A}$, $|\mathcal{A}|$ denotes its cardinality, and $c_j \in \mathcal{N}_{Joe}$, for all $j \in \mathbb{N}^{+}_{\left|\mathcal{N}_{Joe}\right|}$.

At the t-th day, implementing some policy, Joe generates his own production plan, denoted as

$$\widetilde{\boldsymbol{x}}_{t} \triangleq \left[\widetilde{x}_{tt}\, \widetilde{x}_{t,t+1}\, \dots\, \widetilde{x}_{t,t+H-1}\right]^{\boldsymbol{T}} \in \mathbb{N}^{H}, \tag{6}$$

where $x_{tt'}$ is the *number of parts to be produced* on the $\left(t'-t\right)$*-th day from today*, for all $t' \in \mathbb{N}^{t}_{t+H-1}$.

In order to produce, Joe needs certain quantities of parts from his children suppliers. Those quantities are assumed to be fixed in time, known and stored in the *quantity vector*, defined as

$$\boldsymbol{Q} \triangleq \left[Q_{c_1}\, Q_{c_2}\, \dots\, Q_{c_{\left|\mathcal{N}_{Joe}\right|}}\right]^{\boldsymbol{T}} \in \left[\mathbb{N}^{+}\right]^{\left|\mathcal{N}_{Joe}\right|}, \tag{7}$$

where Q_{c_j} denotes the quantity of parts required from child supplier $c_j \in \mathcal{N}_{Joe}$, $j \in \mathbb{N}^{+}_{\left|\mathcal{N}_{Joe}\right|}$, required so that Joe can produce *precisely one* of his parts.

Additionally, Joe maintains an *input (or part) inventory* for the parts that have arrived *and* are available for production. In particular, under the setting established above, if the vector $\boldsymbol{R}^{I}_{t} \triangleq \widetilde{\boldsymbol{R}}^{I}_{tt} \in \mathbb{N}^{\left|\mathcal{N}_{Joe}\right|}$ stacks the resources (per part) available on the current day (or the 0*-th day from today*), *both* from the previous day *and* including any additions of incoming parts shipped to Joe earlier, Joe's input inventory evolves *deterministically*, according to the differential equation

$$\widetilde{\boldsymbol{R}}^{I}_{t,t'+1} \equiv \widetilde{\boldsymbol{R}}^{I}_{tt'} - \boldsymbol{Q}\widetilde{x}_{tt'} + \widetilde{\boldsymbol{x}}^{S,E}_{t-1,t'} \in \mathbb{Z}^{\left|\mathcal{N}_{Joe}\right|}, \quad \forall t' \in \mathbb{N}^{t}_{t+H-1}. \tag{8}$$

Note that $\widetilde{\boldsymbol{R}}^{I}_{tt'}$ denotes the inventory levels at the *start* of day t', *before* all production and part additions due to (projected) arrivals have taken place. Also, we do not require $\widetilde{\boldsymbol{R}}^{I}_{tt'}$ to be nonnegative by definition; instead, this requirement will be imposed as a constraint while designing the plan $\widetilde{\boldsymbol{x}}_t$ (see below).

Except for the input inventory, Joe maintains an *output (or production, or stock) inventory*, for the parts that have been produced by Joe, but have not yet shipped downstream. Similarly to the description of the input inventory, if $R^O_t \triangleq \widetilde{R}^O_{tt} \in \mathbb{N}$ denotes the stock available on the current day, Joe's stock inventory is assumed to evolve according to the differential equation

$$\widetilde{R}^O_{t,t'+1} \equiv \widetilde{R}^O_{tt'} + \widetilde{x}_{tt'} - \left(\widetilde{x}^D_{t-1,t'} - \widetilde{x}^{UD}_{tt'}\right) \in \mathbb{Z}, \quad \forall t' \in \mathbb{N}^t_{t+H-1}, \tag{9}$$

where $\widetilde{x}^{UD}_{tt'} \in \mathbb{N}$ constitutes the *projected unmet demand* on the $(t'-t)$*-th day from today*, for each $t' \in \mathbb{N}^t_{t+H-1}$. Therefore, the difference $\widetilde{x}^D_{t-1,t'} - \widetilde{x}^{UD}_{tt'}$ expresses the *projected met demand*, for each $t' \in \mathbb{N}^t_{t+H-1}$. Except for the requirement that $\widetilde{x}^{UD}_{tt'}$ is a nonnegative integer, we must also have that $\widetilde{x}^D_{t-1,t'} - \widetilde{x}^{UD}_{tt'} \in \mathbb{N}$, for all $t' \in \mathbb{N}^t_{t+H-1}$, and that $\widetilde{R}^O_{t,t'+1} \in \mathbb{N}$, for all $t' \in \mathbb{N}^t_{t+H-1}$, which is trivially equivalent to the constraint

$$\mathbb{Z} \ni \widetilde{x}^{UD}_{tt'} \geq \widetilde{x}^D_{tt'} - \widetilde{R}^O_{tt'} - \widetilde{x}_{tt'}, \quad \forall t' \in \mathbb{N}^t_{t+H-1}. \tag{10}$$

Finally, it would not be reasonable to assume that Joe has an infinite production capacity. Instead, we assume that Joe's production capacity is fixed on a *per day basis*; for simplicity, this capacity is taken as constant relative to time, and will be denoted by $C \in \mathbb{N}$.

2 Production Policy

Every day, Joe's objective will be to come up will a *feasible* production plan, not only for today, but also for all remaining days in a horizon of length H, so that, in general, some combination of downstream demand and inventory maintenance costs are somehow meaningfully optimized.

For simplicity, let us first *relax integer constraints*, for all problem variables. Then, based on the our definitions above, and imposing the aforementioned suitable requirements on the values of the inventories over time, one possible formulation could be

$$\begin{array}{ll}
\underset{\substack{\widetilde{\boldsymbol{R}}^I_{tt'}, \widetilde{R}^O_{tt'}, \widetilde{x}_{tt'}, \widetilde{x}^{UD}_{tt'} \\ \forall t' \in \mathbb{N}^{t+1}_{t+H}}}{\text{minimize}} & \displaystyle\sum_{t' \in \mathbb{N}^t_{t+H-1}} \theta_{tt'} \widetilde{x}^{UD}_{tt'} + \sum_{t' \in \mathbb{N}^t_{t+H-1}} \kappa^O \widetilde{R}^O_{t,t'+1} + \sum_{t' \in \mathbb{N}^t_{t+H-1}} \left(\boldsymbol{\kappa}^I\right)^{\boldsymbol{T}} \widetilde{\boldsymbol{R}}^I_{t,t'+1} \\
\text{subject to} & \widetilde{\boldsymbol{R}}^I_{t,t'+1} \equiv \widetilde{\boldsymbol{R}}^I_{tt'} - \boldsymbol{Q}\widetilde{x}_{tt'} + \widetilde{\boldsymbol{x}}^{S,E}_{t-1,t'} \\
& \widetilde{\boldsymbol{R}}^I_{t,t'+1} \geq \boldsymbol{0} \\
& 0 \leq \widetilde{x}_{tt'} \leq C \\
& \widetilde{R}^O_{t,t'+1} \equiv \widetilde{R}^O_{tt'} + \widetilde{x}_{tt'} - \left(\widetilde{x}^D_{t-1,t'} - \widetilde{x}^{UD}_{tt'}\right) \\
& \widetilde{R}^O_{t,t'+1} \geq 0 \\
& 0 \leq \widetilde{x}^{UD}_{tt'} \leq \widetilde{x}^D_{tt'}, \quad \forall t' \in \mathbb{N}^t_{t+H-1}
\end{array}, \tag{11}$$

where $\theta_{tt'} \geq 0$ is a tunable parameter, penalizing unmet demand $\widetilde{x}^{UD}_{tt'}$, for all $t' \in \mathbb{N}^t_{t+H-1}$, $\kappa^S \geq 0$ denotes the *stock inventory storage cost per unit* and, likewise,

$$\boldsymbol{\kappa}^I \triangleq \left[\kappa^I_1 \ \ldots \ \kappa^I_{\left|\mathcal{N}_{Joe}\right|}\right]^{\boldsymbol{T}} \in \mathbb{R}^{\left|\mathcal{N}_{Joe}\right|}_+, \tag{12}$$

with κ_j^I being the *input inventory storage cost per unit of the j-th part*, $j \in \mathbb{N}^+_{|\mathcal{N}_{Joe}|}$.

References

[1] R. B. Ash and C. Doléans-Dade, *Probability and Measure Theory*. Academic Press, 2000.

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