Misalignment **Me**chanism **S**olver

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Recent Progress in Axion Theory and Experiment
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github.com/dkaramit/MiMeS

mimes.hepforge.org

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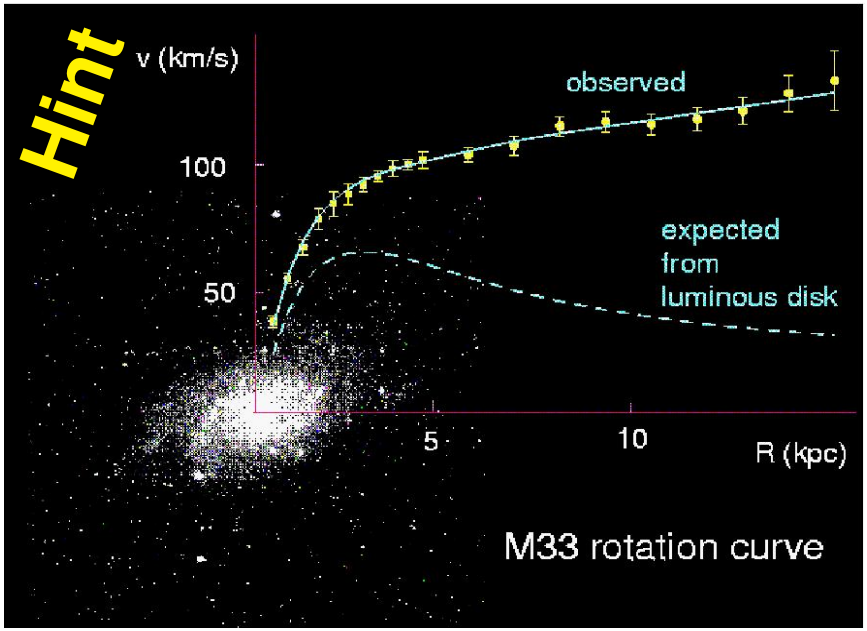
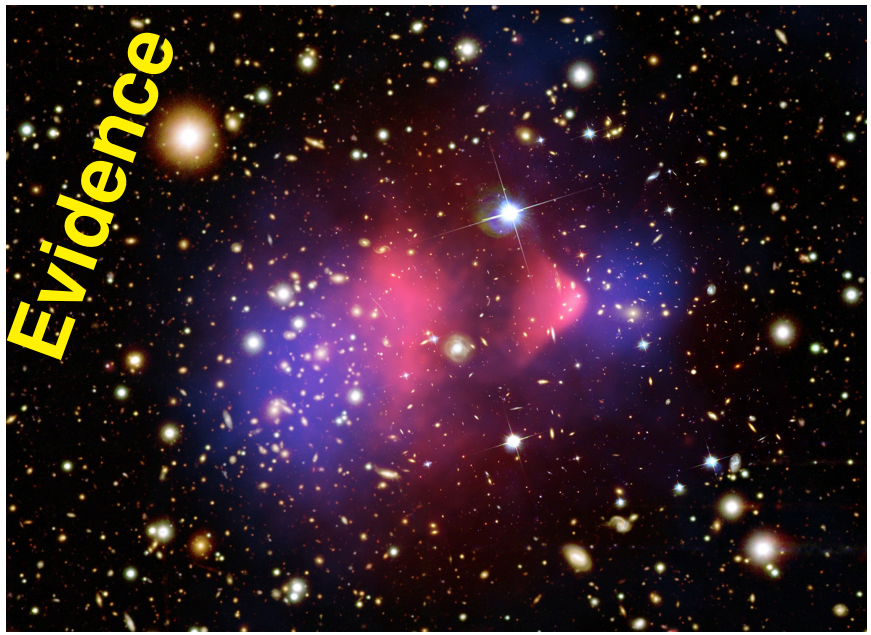
Outline

Axion Dark Matter

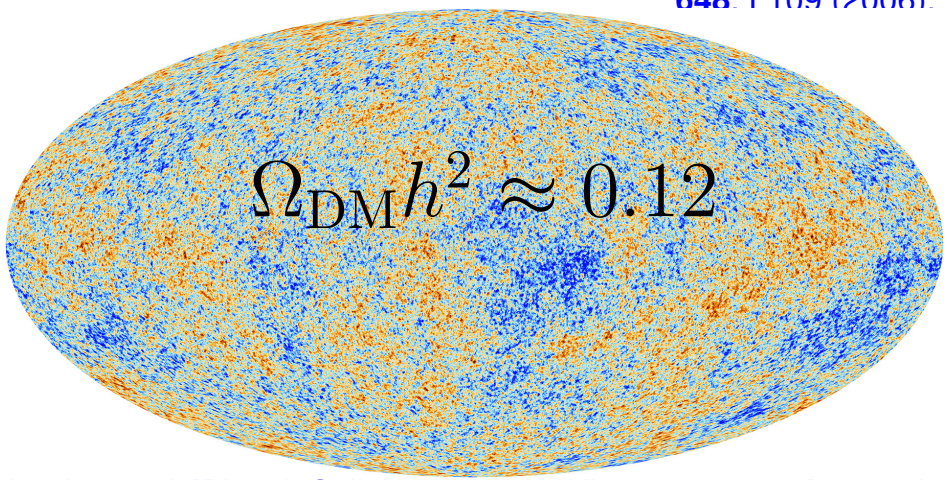
Why particle dark matter



E. Corbelli and P. Salucci, *Mon. Not. Roy. Astron. Soc.* **311** *441* (2000), arXiv:astro-ph/9909252.



M. Markevitch, ESA Spec. Publ. **604** (2006) 723, astro-ph/0511345.Clowe, Bradac, *et. al.* Astrophys. J. **648** L109 (2006), astro-ph/0608407



N. Aghanim *et al.* [Planck Collaboration], arXiv:1807.06209 [astro-ph.CO].

The dark matter particle

"I know one thing, that I know nothing."

–Socrates

- Gravitational interactions.
- Mostly electrically neutral.
- Stable or very slow decay rate.
- Non-Baryonic.
- Cold/Warm and non-relativistic today.

The axion (like) particle

Notably, the original axion was originally introduced in order to solve the *strong-CP problem* of the SM. Axion-like-particles (ALPs) arise in a number of new physics models, beyond the SM.

Axions and ALPs generally:

- Have suppressed interactions with photons.
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Maybe DM has axionic nature!

Calculating the Relic Abundance

The axion EOM

Axions and ALPs follow a similar equation of motion (EOM):

$$\left(\frac{d^2}{dt^2} + 3H(t)\,\frac{d}{dt}\right)\theta(t) + \tilde{m}_a{}^2(t)\,\sin\theta(t) = 0\ ,$$

where $\theta = A\,f_a$, with A the axion filed, and f_a some energy scale that characterises the potential (Peccei-Quinn breaking scale).

How hard can it be?

Hard (in general).

The classical analogue is the dumped pendulum with both frequency (length) and friction being time-dependent:

- There is no closed form solution.
- There are no constants of motion (wait a minute...).
- No package/library/program available!

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`MiMeS` *simulates the evolution of the axion/ALP, for (virtually) any cosmological scenario and axion/ALP (thermal) mass.*

Initial conditions

Some time at the very early Universe, $\tilde{m}_a \ll H(T)$, [1] with

$$\ddot{\theta} + 3H\,\dot{\theta} \approx 0\ .$$

The solution is

$$\theta = \theta_{\rm ini} + C\int_0^t dt'\ \left(\frac{a(t'=0)}{a(t')}\right)^3\ .$$

So, $\dot{\theta} \sim a^{-3}$. Since we are interested in θ at much later times (once the potential becomes relevant), $\dot{\theta} \approx 0$.Therefore, we can begin integration at some point with $3H \gg \tilde{m}_a$, and set $\theta(t = t_{\rm ini}) = \theta_{\rm ini}$ and $\dot{\theta}(t = t_{\rm ini}) = 0$.

[1] This is an assumption that `MiMeS` has to make for the sake of generality.

(Bad) Approximations

"If a man knows not which port he sails, no wind is favourable."

–Seneca

- Assume $\theta \ll 1$, and linearise the EOM. *Not general.*
- Assume that at $\tilde{m}_a(T_{\rm osc}) \approx 3H(T_{\rm osc})$ we have $\dot{\theta}(T_{\rm osc}) = 0$. *Not very precise.*
- Assume that $\theta_{\rm osc} \approx \theta_{\rm ini}$. *Generally quite bad.*

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These result in "WKB"-approximate solution

$$\theta(t) \approx \theta_{\rm ini} \left(\frac{3}{4}\right)^{1/4} \sqrt{\frac{\tilde{m}_a(T_{\rm osc})}{\tilde{m}_a(T)}} \left(\frac{a}{a_{\rm osc}}\right)^{-3/2} \cos\left(\int_{t_{\rm osc}}^{t} dt'\, \tilde{m}_a(t')\right) ,$$

which gives us:

$$\rho_{a,0} = \gamma^{-1} \frac{s_0}{s_{\rm osc}} \frac{1}{2} {f_a}^2 m_a \tilde{m}_{a,\rm osc} {\theta_{\rm ini}}^2 ,$$

where γ the amount of entropy injection between $T_{\rm osc}$ and today.

Need for speed, accuracy, and automation

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Life before `MiMeS`:

- No available tool that can help us reproduce published results obtained by numerical integration. *Reproducing results means reproducing effort*.
- The approximations can be tested against numerical results in a case-by-case basis. *No measure of accuracy*.
- Simply checking if an ALP model is compatible with a cosmological scenario is *slow* or *inaccurate*.

MiMeS

MiMeS: Why?

We need accurate code that solves the EOM, but most importantly we need *reproducible* results!

MiMeS is:

- MiMeS is a C++ header-only library that contains various templated classes.
- MiMeS comes with a python interface.
- Is *easy* to use; anyone can run it and see if their model can work or check against the literature.
- Is reasonably fast; less than $0.05\ s$ for the scenarios tested.
- Provides full access to results and their errors, which can help determine if the results are accurate.
- Asks the user to decide when to start, stop, and when adiabaticity is reached.

MiMeS: Under the hood

"It is the empty space which makes a bowl useful."

–Laozi

MiMeS is built as minimally as possible:

- MiMeS relies on NaBBODES [2] SimpleSplines [3].
- You only need to have the standard C++ library.
- The two libraries are developed by myself, so their integration with MiMeS is seamless.
- There is always going to be a compatible version of these libraries that works with MiMeS.

[2] https://github.com/dkaramit/NaBBODES.
[3] https://github.com/dkaramit/SimpleSplines.

Sidenote: adiabatic invariant – I

Given a system with Hamiltonian $\mathcal{H}(\theta, p; t)$, the equations of motion are

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial \theta} \ , \quad \dot{\theta} = \frac{\partial \mathcal{H}}{\partial p} \ .$$

Also,

$$d\mathcal{H} = \dot{\theta} \ dp - \dot{p} \ d\theta + \frac{\partial \mathcal{H}}{\partial t} \ dt \ .$$

If this system exhibits closed orbits (*e.g.* if it oscillates), we define

$$J \equiv C \oint p \ d\theta \ ,$$

where the integral is over a closed path (*e.g.* a period, T), and C indicates that J can always be rescaled with a constant. If the Hamiltonian varies slowly during a cycle,

$$\frac{dJ}{dt} = C \oint \left(\dot{p} \ d\theta + p \ d\dot{\theta} \right) = C \int_t^{t+T} \frac{\partial \mathcal{H}}{\partial t'} \ dt' \approx T \left. \frac{\partial \mathcal{H}(t')}{\partial t'} \right|_{t'=t} \approx 0 \ .$$

So, J is an adiabatic invariant!

Sidenote: adiabatic invariant – II

The Hamiltonian that results in the EOM

$$\mathcal{H} = \frac{1}{2}\frac{p^2}{{f_a}^2\ a^3} + V(\theta)\ a^3\ ,$$

with

$$p = {f_a}^2\ a^3\ \dot{\theta}\ , \quad V(\theta) = \tilde{m}_a{}^2 {f_a}^2 (1 - \cos\theta)\ .$$

If $\mathcal{H}$ varies slowly ($\dot{\tilde{m}}_a(T)/\tilde{m}_a \ll \tilde{m}_a$ and $H \ll \tilde{m}_a$):

$$\begin{aligned}
J &= \frac{\oint p\ d\theta}{\pi {f_a}^2} = \frac{1}{\pi {f_a}^2} \oint \sqrt{2\left(\mathcal{H}(\theta) - V(\theta)\ a^3\right)\ {f_a}^2 a^3}\ d\theta \\
&= \frac{2}{\pi {f_a}^2} \int_{-\theta_{\rm peak}}^{\theta_{\rm peak}} \sqrt{2\left(\mathcal{H}(\theta_{\rm peak}) - V(\theta)\ a^3\right)\ {f_a}^2 a^3}\ d\theta \\
&= \frac{2\sqrt{2}}{\pi f_a} \int_{-\theta_{\rm peak}}^{\theta_{\rm peak}} \sqrt{V(\theta_{\rm peak}) - V(\theta)} a^3 d\theta \\
&= \frac{2\sqrt{2}}{\pi}\ \tilde{m}_a\ a^3 \int_{-\theta_{\rm peak}}^{\theta_{\rm peak}} \sqrt{\cos\theta - \cos\theta_{\rm peak}}\ d\theta\ ,
\end{aligned}$$

is the adiabatic invariant – up to a multiplication with a constant.

MiMeS solves the transformed EOM: [4]

$$\frac{d\zeta}{du} + \left[\frac{1}{2}\frac{d\log H^2}{du} + 3\right]\zeta + \left(\frac{\tilde{m}_a}{H}\right)^2 \sin\theta = 0 \,.$$

$$\frac{d\theta}{du} - \zeta = 0 \,.$$

The parameter u is "time" [5] $u \equiv \log(a/a_{\rm ini})$. Integration starts at $T_{\rm ini}$ such that $3H(T_{\rm ini})/m_a(T_{\rm ini}) = x \gg 1$, [6] with initial conditions are $\zeta(T_{\rm ini}) = 0$ and $\theta(T_{\rm ini}) = \theta_{\rm ini}$. MiMeS stops integrating when J becomes acceptably constant.

[4] Suitable for Runge-Kutta.

[5] With $a_{\rm ini}$ the initial value of the scale factor. It corresponds to $T_{\rm ini}$.

[6] User defined x. $T_{\rm ini}$ is automatically determined.

Using MiMeS

How to get MiMeS

There are several ways you can get a stable version of MiMeS:

1. `git clone -b stable https://github.com/dkaramit/MiMeS.git`. This is the preferred way, as it is guaranteed to be the latest stable version.
2. Go to mimes.hepforge.org/downloads, and download it.
3. Go to github.com/dkaramit/MiMeS/releases, and download a released version.

You can get the most up-to-date code – not always the most stable one – including the latest version of NaBBODES and SimpleSplines, by running

```
1  git clone https://github.com/dkaramit/MiMeS.git
2  cd MiMeS
3  git submodule init
4  git submodule update --remote
```

Configure (and make)

There is no need to install anything if you are going to use `MiMeS` in a `C++` program. The only thing you *must* do is run

```
1     bash configure.sh
```

Alter that, you can include the header file `MiMeS/MiMeS.hpp`, and you are good to go.
However, you can also run

- `make lib`, in order to produce the (shared) libraries. This is needed in order to run the `python` interface.
- `make examples`, in order to compile the examples in `MiMeS/UserSpace/Cpp`.
- `make exec`, in order to produce some test executables (in `MiMeS/exec`). You just need to run then in order to see if you get any segfaults.

Classes

There are three classes useful to the user. [7]

[7] There are various arguments that need to be passed to the constructors, and the are all listed and explained in the Appendix of the documentation.

[8] K. Saikawa and S. Shirai, JCAP **08** (2020), 011 [arXiv:2005.03544 [hep-ph]].

[9] S. Borsanyi, Z. Fodor, J. Guenther, K. H. Kampert, S. D. Katz, T. Kawanai, T. G. Kovacs, S. W. Mages, A. Pasztor and F. Pittler, *et al.* Nature **539** (2016) no.7627, 69-71 [arXiv:1606.07494 [hep-lat]].

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- `mimes::Axion<LD,Solver,Method>`: This is responsible for actually solving the EOM.

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Template arguments

You need to choose what numeric type to use. This is done by the template argument `LD` which should be `double` (fast) or `long double` (accurate). [10]

You also need to tell `MiMeS` which integration strategy to use. This is done by choosing template arguments:

- `Solver` can be set to `1` for Rosenbrock (semi-implicit Runge-Kutta). The `Method` argument in this case can be:
  - `RODASPR2<LD>` (4th order).
  - `ROS34PW2<LD>` (3rd order).
  - `ROS3W<LD>` (2rd order, *very* bad).
- `Solver` can be set to `2` for explicit RK. The `Method` argument can be:
  - `DormandPrince<LD>` (7th order)
  - `CashKarpRK45<LD>` (5th order, *very* bad).
  - `RK45<LD>` (5th order, *very* bad).

[10] You could choose `float`, but we live in 2021.

MiMeS from python

In order to call the python interface of MiMeS, we need to first call make lib in the root directory of MiMeS.

Before that, we can take some time to decide what the template arguments and compilation options should be. In the file MiMeS/Definitions.mk, you can change the variables:

- LONGpy=long will compile the library with long double numeric types. LONGpy= will compile the library with double numeric types.
- SOLVER and METHOD, as in the template arguments.

Also, in the same file, you can change compilation options:

- Compiler:
  - CC=g++ in order to use the GNU C++ compiler.
  - CC=clang -lstdc++ in order to use the clang C++ compiler.
- Optimization level:
  - OPT=O0: No optimization.
  - O=O1, O2, or O3: all these perform mostly the same (read the compiler documentation for more information on the optimization).
  - OPT=Ofast: full optimization (fast, but dangerous).

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2. $\zeta(0) = 0$. This will be changed in the future.
3. The energy density of the axion/ALP is always subdominant.
4. Only the EOM determines the energy density (no annihilations, no strings, etc.).

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4. Relative difference of J between a given number of peaks at which we consider adiabaticity to have been reached.
5. Other input, such as the temperature at which integration exits.

Examples

Define everything and solve in just a few lines of code!

```
1  from time import time; from sys import stderr #you need these in order to print the time in stderr
2
3  #add the relative path for MiMeS/src
4  from sys import path as sysPath; sysPath.append('../src')
5
6  from interfacePy.AxionMass import AxionMass #import the AxionMass class
7  from interfacePy.Axion import Axion #import the Axion class
8  from interfacePy.Cosmo import mP #import the Planck mass
9
10 def main():
11
12     # AxionMass instance
13     axionMass = AxionMass(r'../src/data/chi.dat',0,mP)
14
15     # define ma2 for T <= Tmin
16     TMin, chiMin=axionMass.getTMin(), axionMass.getChiMin()
17
18     axionMass.set_ma2_MIN( lambda T,fa: chiMin/fa/fa )
19
20     # define ma2 for T >= Tmax
21     TMax, chiMax=axionMass.getTMax(), axionMass.getChiMax()
22
23     axionMass.set_ma2_MAX( lambda T,fa: chiMax/fa/fa*pow(TMax/T,8.16))
24
25     #in python it is more convenient to use relative paths
26     inputFile="../UserSpace/InputExamples/MatterInput.dat"
27
28     ax = Axion(0.1, 1e16, 500, 1e-4, 1e3, 10, 1e-2, inputFile, axionMass,
29         1e-2, 1e-8, 1e-2, 1e-10, 1e-10, 0.85, 1.5, 0.85, int(1e7))
30
31     ax.solveAxion()
32
33     print("theta_i=",ax.theta_i,"\t\t\t\t","f_a=",ax.fa,"GeV\n","theta_osc~=",
34             ax.theta_osc,"\t","T_osc~=",ax.T_osc,"GeV␣\n","Omega␣h^2=",ax.relic)
35
36     #once we are done we should run the destructor
37     del ax,axionMass
38
39 if __name__ == '__main__':
40     _=time()
41     main()
42     print(round(time()-_,3),file=stderr)
```

Notice: `C++` and `python` are quite similar!

```
1  #include<iomanip>
2  #include"MiMeS.hpp"
3
4  using numeric = long double;//make life easier if you want to change to double
5
6  int main(){
7      mimes::util::Timer _timer_;//use this to time it!
8
9      // use chi_PATH to interpolate the axion mass.
10     mimes::AxionMass<numeric> axionMass(chi_PATH,0,mimes::Cosmo<numeric>::mP);
11
12     /*set m̃_a^2 for T ≥ T_max*/
13     numeric TMax=axionMass.getTMax(), chiMax=axionMass.getChiMax();
14
15     axionMass.set_ma2_MAX(
16         [&chiMax,&TMax](numeric T, numeric fa){ return chiMax/fa/fa*std::pow(T/TMax,-8.16);}
17     );
18
19     /*set m̃_a^2 for T ≤ T_min*/
20     numeric TMin=axionMass.getTMin(), chiMin=axionMass.getChiMin();
21
22     axionMass.set_ma2_MIN(
23         [&chiMin,&TMin](numeric T, numeric fa){ return chiMin/fa/fa;}
24     );
25
26     /*this path contains the cosmology*/
27     std::string inputFile = std::string(rootDir)+
28         std::string("/UserSpace/InputExamples/MatterInput.dat");
29
30     /*declare an instance of Axion*/
31     mimes::Axion<numeric, 1, RODASPR2<numeric> > ax(0.1, 1e16, 500, 1e-4, 1e3, 10, 1e-2,
32                         inputFile, &axionMass, 1e-2, 1e-8, 1e-2, 1e-10, 1e-10, 0.85, 1.5, 0.85,
33                         int(1e7) );
34     /*solve the EOM!*/
35     ax.solveAxion();
36
37     std::cout<<std::setprecision(5)
38     <<"theta_i="<<ax.theta_i<<std::setw(25)<<"f_a="<<ax.fa<<" GeV\n"<<"theta_osc~="<<ax.theta_osc
39     <<std::setw(20)<<"T_osc~="<<ax.T_osc<<"GeV \n"<<"Omega h^2="<<ax.relic<<"\n";
40
41     return 0;
42 }
```

Outlook

Outlook

What we saw:

- `MiMeS` solves the axion/ALP EOM.
- `MiMeS` treats both the mass and the underlying cosmology as user inputs.
- `MiMeS` allows the user to change a number of other things, from the plasma RDOFs to the convergence conditions.

`MiMeS` may be amended in the future because:

- `MiMeS` should allow the user to consider different initial value of ζ; the "kinematic" `MiMeS` might come soon.
- `MiMeS` should be able to handle non-vanishing RHS; *i.e.* solve the "driven" dumped time-dependent pendulum.
- `MiMeS` should be able to compare against searches on the fly.

Thank you!

Breakdown of MiMeS:

Language	files	comment	code
C/C++ Header	35	444	1595
C++	20	198	1106
Python	22	367	1000
SUM:	77	1009	3701

Backup

(equations, derivations, tables)

WKB – I

$$\left(\frac{d^2}{dt^2} + 3H(t)\,\frac{d}{dt} + \tilde{m}_a{}^2(t)\right)\theta(t) = 0\ .$$

Reparametrize by introducing

$$\theta_{\rm trial} = \exp\left[i\int dt\,\left(\psi(t) + 3/2\ i\ H(t)\right)\right]\ .$$

The Eome, then becomes just

$$\psi^2 = \Omega^2 + i\,\dot{\psi}\ ,$$

with $\Omega^2 = \tilde{m}_a{}^2 - \frac{9}{4}H^2 - \frac{3}{2}\dot{H}$. The solution takes the form $\psi = \pm\sqrt{\Omega^2 + i\dot{\psi}}$. However, for $\dot{\psi} \ll \Omega^2$ and $\dot{\Omega} \ll \Omega^2$, it can be approximated as

$$\psi \approx \pm\Omega + \frac{i}{2}\frac{d\log\Omega}{dt}\ .$$

WKB – II

So, after applying the initial conditions, the EOM is solved by

$$\theta(t) \approx \theta_{\rm ini} \sqrt{\frac{\Omega_{\rm ini}}{\Omega(t)}} \left(\frac{a}{a_{\rm ini}}\right)^{-3/2} \cos\left(\int_{t_{\rm ini}}^{t} dt'\, \Omega(t')\right) \ .$$

Taking $t_{\rm ini} = t_{\rm osc}$ (*i.e.* $\dot{\theta}(t_{\rm osc}) = 0$, which is not generally good), have

$$\theta(t) \approx \theta_{\rm osc} \left(\frac{3}{4}\right)^{1/4} \sqrt{\frac{\tilde{m}_a|_{t=t_{\rm osc}}}{\tilde{m}_a(t)}} \left(\frac{a}{a_{\rm osc}}\right)^{-3/2} \cos\left(\int_{t_{\rm osc}}^{t} dt'\, \tilde{m}_a(t')\right) \ ,$$

where $\theta_{\rm osc} = \theta|_{t=t_{\rm osc}}$. This equation is further simplified if we assume that $\theta_{\rm osc} \approx \theta_{\rm ini}$ (again not really good), *i.e.*

$$\theta(t) \approx \theta_{\rm ini} \left(\frac{3}{4}\right)^{1/4} \sqrt{\frac{\tilde{m}_a|_{t=t_{\rm osc}}}{\tilde{m}_a(t)}} \left(\frac{a}{a_{\rm osc}}\right)^{-3/2} \cos\left(\int_{t_{\rm osc}}^{t} dt'\, \tilde{m}_a(t')\right) \ .$$

Adiabatic invariant – I

Given a system with Hamiltonian $\mathcal{H}(\theta, p; t)$, the equations of motion are

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial \theta} \ , \quad \dot{\theta} = \frac{\partial \mathcal{H}}{\partial p} \ .$$

Also,

$$d\mathcal{H} = \dot{\theta} \ dp - \dot{p} \ d\theta + \frac{\partial \mathcal{H}}{\partial t} \ dt \ .$$

If this system exhibits closed orbits (*e.g.* if it oscillates), we define

$$J \equiv C \oint p \ d\theta \ ,$$

where the integral is over a closed path (*e.g.* a period, T), and C indicates that J can always be rescaled with a constant. If the Hamiltonian varies slowly during a cycle,

$$\frac{dJ}{dt} = C \oint \left(\dot{p} \ d\theta + p \ d\dot{\theta} \right) = C \int_t^{t+T} \frac{\partial \mathcal{H}}{\partial t'} \ dt' \approx T \left. \frac{\partial \mathcal{H}(t')}{\partial t'} \right|_{t'=t} \approx 0 \ .$$

So, J is an adiabatic invariant!

Adiabatic invariant – II

The Hamiltonian that results in the EOM

$$\mathcal{H} = \frac{1}{2}\frac{p^2}{{f_a}^2\ a^3} + V(\theta)\ a^3\ ,$$

with

$$p = {f_a}^2\ a^3\ \dot{\theta}$$
$$V(\theta) = \tilde{m}_a{}^2 {f_a}^2 (1 - \cos\theta)\ .$$

If $\mathcal{H}$ varies slowly – $\dot{\tilde{m}}_a(T)/\tilde{m}_a \ll \tilde{m}_a$ and $H \ll \tilde{m}_a$, then

$$\begin{aligned}
J &= \frac{\oint p\ d\theta}{\pi {f_a}^2} = \frac{1}{\pi {f_a}^2}\oint \sqrt{2\left(\mathcal{H}(\theta) - V(\theta)\ a^3\right)\ {f_a}^2 a^3}\ d\theta \\
&= \frac{2}{\pi {f_a}^2}\int_{-\theta_{\rm peak}}^{\theta_{\rm peak}} \sqrt{2\left(\mathcal{H}(\theta_{\rm peak}) - V(\theta)\ a^3\right)\ {f_a}^2 a^3}\ d\theta \\
&= \frac{2\sqrt{2}}{\pi f_a}\int_{-\theta_{\rm peak}}^{\theta_{\rm peak}} \sqrt{V(\theta_{\rm peak}) - V(\theta)}\, a^3 d\theta \\
&= \frac{2\sqrt{2}}{\pi}\ \tilde{m}_a\ a^3 \int_{-\theta_{\rm peak}}^{\theta_{\rm peak}} \sqrt{\cos\theta - \cos\theta_{\rm peak}}\ d\theta\ ,
\end{aligned}$$

is the adiabatic invariant – up to a multiplication with a constant.

C++ Input

AxionMass class – Definition via file

In order to define an instance of the `AxionMass` class that interpolates the $\tilde{m}_a$, use the constructor:

```
1 template<class LD>
2 mimes::AxionMass<LD>(std::string chi_PATH, LD minT=0, LD maxT=mimes::Cosmo::mP)
```

The arguments are:

1. `chi_Path`: Relative or absolute path to data file with T (in GeV), $\chi(T)$ (in GeV^4).
2. `minT`, `maxT`: Interpolation limits. These are used in order to stop the interpolation at the closest temperatures that exist in the data file. This means that the actual interpolation limits are $T_{\min} \geq$`minT` and $T_{\max} \leq$`maxT`. Beyond these limits hat axion mass is assumed to be constant.

The definition of ${\tilde{m}_a}^2$ beyond $T_{\min}$ and $T_{\max}$ can be changed to realistic function, using mimes::AxionMass<LD>::set_ma2_MIN(std::function<LD(LD,LD)> ma2_MIN) and mimes::AxionMass<LD>::set_ma2_MAX(std::function<LD(LD,LD)> ma2_MAX). These definitions may need the actual values of $T_{\min,\max}$ and $\chi(T_{\min,\max})$. These are obtained from

- template<class LD> LD mimes::AxionMass<LD>::getTMin(): This function returns the minimum interpolation temperature, $T_{\min}$.
- template<class LD> LD mimes::AxionMass<LD>::getTMax(): This function returns the maximum interpolation temperature, $T_{\max}$.
- template<class LD> LD mimes::AxionMass<LD>::getChiMin(): This function returns $\chi(T_{\min})$.
- template<class LD> LD mimes::AxionMass<LD>::getChiMax(): This function returns $\chi(T_{\max})$.

Note that all std::function<LD(LD,LD)> can be any callable object that takes T and f_a and returns ${\tilde{m}_a}^2$.

AxionMass class – Definition via function

In order to define an instance of the `AxionMass` class via a function, use the constructor:

```
1 template<class LD>
2 mimes::AxionMass<LD>(std::function<LD(LD,LD)> ma2)
```

Here, ma2 can be any callable object that takes T and f_a and returns ${\tilde{m}_a}^2$.

Axion class – Expected input

The constructor of the `Axion` class is

```
template<class LD, const int Solver, class Method>
mimes::Axion<LD, Solver, Method>(LD theta_i, LD fa, LD umax, LD TSTOP,
                LD ratio_ini, unsigned int N_convergence_max, LD convergence_lim,
                std::string inputFile, AxionMass<LD> *axionMass, LD initial_step_size=1e-2,
                LD minimum_step_size=1e-8, LD maximum_step_size=1e-2,
                LD absolute_tolerance=1e-8, LD relative_tolerance=1e-8, LD beta=0.9,
                LD fac_max=1.2, LD fac_min=0.8, unsigned int maximum_No_steps=10000000)
```

The input that `MiMeS` expects is:

1. `theta_i`: Initial angle.
2. `fa` The PQ scale.
3. `umax`: Once $u = \log a/a_i >$`umax`, the integration stops. Typical value: ~ 500.
4. `TSTOP`: Once $T <$`TSTOP`, integration stops. Typical value: 10^{-4} GeV.
5. `ratio_ini`: Integration starts at u with $3H/\tilde{m}_a \approx$`ratio_ini`. Typical value: $\sim 10^3$.
6. `N_convergence_max`, `convergence_lim`: Integration stops when the relative difference between two consecutive peaks is less than `convergence_lim` for `N_convergence_max` consecutive peaks.
7. `inputFile`: Relative (or absolute) path to a file that describes the cosmology. The columns should be: u T [GeV] $\log H$, with acceding u. Entropy injection should have stopped before the lowest temperature of given in `inputFile`.
8. `axionMass`: Instance of `mimes::AxionMass<LD>` class. In `C++` this instance is passed as a pointer to the constructor of the `mimes::Axion<LD,Solver,Method>` class, while in `python` it is simply passed as a variable.

Axion class – Optional input

The optional input, relative to the RK algorithm, is:

1. `initial_stepsize`: Initial step-size of the solver. Default value: 10^{-2}.
2. `minimum_stepsize`: Lower limit of the step-size. Default value: 10^{-8}.
3. `maximum_stepsize`: Upper limit of the step-size. Default value: 10^{-2}.
4. `absolute_tolerance`: Absolute tolerance of the RK solver. Default value: 10^{-8}.
5. `relative_tolerance`: Relative tolerance of the RK solver. Default value: 10^{-8}.
6. `beta`: Aggressiveness of the adaptation strategy. Default value: 0.9.
7. `fac_max`, `fac_min`: The step-size does not change more than `fac_max` and less than `fac_min` within a trial step. Default values: 1.2 and 0.8, respectively.
8. `maximum_No_steps`: If integration needs more than `maximum_No_steps` integration stops. Default value: 10^7.

python Input

AxionMass class – Definition via file

The actual constructor of the `AxionMass` in the `python` interface is AxionMass(*args). However, it is intended to be used in *only* two ways.
In order to define an instance of the `AxionMass` class that interpolates the $\tilde{m}_a$, use the constructor as:

```
1 AxionMass(chi_PATH, minT=0, maxT=Cosmo.mP)
```

The arguments are the same as in the `C++` case.

The definition of ${\tilde{m}_a}^2$ beyond $T_{\rm min}$ and $T_{\rm max}$ can be changed using AxionMass.set_ma2_MIN(ma2_MIN) and AxionMass.set_ma2_MAX(ma2_MAX). These definitions may need the actual values of $T_{\rm min,max}$ and $\chi(T_{\rm min,max})$. These are obtained from

- AxionMass.getTMin(): This function returns the minimum interpolation temperature, $T_{\rm min}$.
- AxionMass.getTMax(): This function returns the maximum interpolation temperature, $T_{\rm max}$.
- AxionMass.getChiMin(): This function returns $\chi(T_{\rm min})$.
- AxionMass.getChiMax(): This function returns $\chi(T_{\rm max})$.

The difference between the `C++` case is that ma2 cannot be any callable object; it has to be a regular function that takes T and f_a and returns ${\tilde{m}_a}^2$.

AxionMass class – Definition via function

In order to define an instance of the `AxionMass` class via a function, use the constructor as:

```
1 AxionMass(ma2)
```

The difference between the `C++` case is that ma2 cannot be any callable object; it has to be a regular function that takes T and f_a and returns ${\tilde{m}_a}^2$.

Axion class

The constructor of the `Axion` class is

```
Axion(theta_i, fa, umax, TSTOP, ratio_ini, N_convergence_max, convergence_lim, inputFile,
    axionMass, initial_step_size=1e-2, minimum_step_size=1e-8, maximum_step_size=1e-2,
    absolute_tolerance=1e-8, relative_tolerance=1e-8, beta=0.9, fac_max=1.2, fac_min=0.8,
    maximum_No_steps=10000000)
```

All the arguments are the same as in the `C++` case. The only difference is that the AxionMass instance (axionMass) is not passed as a pointer, as there is no direct way to do it in `python`. However, the underlying object is the same, as it is converted internally using ctypes.

Files and compilation variables

Paths.mk

There are some paths to file that the user can provide in order to use different data for the RDOF, anharmonic factor, and χ (optional).
These paths are stored as strings in `MiMeS/src/misc_dir/path.hpp` when `bash configure.sh` is run.
These paths can be changed by changing the following variables in `MiMeS/Paths.mk`:

- `cosmoDat`: Relative path to data file with T (in GeV), h_{eff}, g_{eff}.
- `axMDat`: Relative path to data file with T (in GeV), h_{eff}, g_{eff}. This variable can be ommitted if the user indents to define all masses via functions.
- `anFDat`: Relative path to data file with θ_{peak}, $f(\theta_{\mathrm{peak}})$.

It is advisable that if the paths change `bash configure.sh` and `make` should be run.

Template arguments

You need to choose what numeric type to use. This is done by the template argument `LD` which should be `double` (fast) or `long double` (accurate). [11]

You also need to tell `MiMeS` which integration strategy to use. This is done by choosing template arguments:

- `Solver` can be set to `1` for Rosenbrock (semi-implicit Runge-Kutta). The `Method` argument in this case can be:
  - `RODASPR2<LD>` (4th order).
  - `ROS34PW2<LD>` (3rd order).
  - `ROS3W<LD>` (2rd order, *very* bad).
- `Solver` can be set to `2` for explicit RK. The `Method` argument can be:
  - `DormandPrinceRK45<LD>` (7th order)
  - `CashKarpRK45<LD>` (5th order, *very* bad).
  - `RK45<LD>` (5th order, *very* bad).

[11] You could choose `float`, but we live in 2021.

Definitions.mk

In order to call the `python` interface of `MiMeS`, we need to first call `make lib` in the root directory of `MiMeS`.

Before that, we can take some time to decide what the template arguments and compilation options should be. In the file `MiMeS/Definitions.mk`, you can change the variables:

- `LONGpy=long` will compile the library with `long double` numeric types. `LONGpy=` will compile the library with `double` numeric types.
- `SOLVER` and `METHOD`, as in the template arguments.

Also, in the same file, you can change compilation options:

- Compiler:
  - `CC=g++` in order to use the `GNU C++` compiler.
  - `CC=clang -lstdc++` in order to use the `clang C++` compiler.
- Optimization level:
  - `OPT=O0`: No optimization.
  - `O=O1`, `O2`, or `O3`: all these perform mostly the same (read the compiler documentation for more information on the optimization).
  - `OPT=Ofast`: full optimization (fast, but dangerous).