MiMeS: Misalignment Mechanism Solver

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Abstract

We introduce a C++ header-only library that solves the Axion equation of motion, MiMeS . MiMeS makes no assumptions regarding the cosmology and the thermal mass of the axion, which allows the user to consider various cosmological scenarios and axion-like models. Moreover, although is written entirely in C++ , MiMeS comes with a convenient python interface, which does not require the user to write any code in C++ .

1 Axion density

The axion field is usually written as

$$\alpha \equiv f_\alpha \; \theta, \tag{1.1}$$

with f_α the scale of the axion that determines the the scale at which the PQ symmetry breaks.

The equation of motion (EOM) for θ is

$$\left(\frac{d^2}{dt} + 3H(t) \; \frac{d}{dt} \right) \theta(t) + m_\alpha^2(t) \; \sin\theta(t) = 0 \; , \tag{1.2}$$

with $H(t)$ the Hubble parameter (determined by the cosmology), and $m_\alpha(t)$ the time (temperature) dependent mass of the axion.

Initial condition

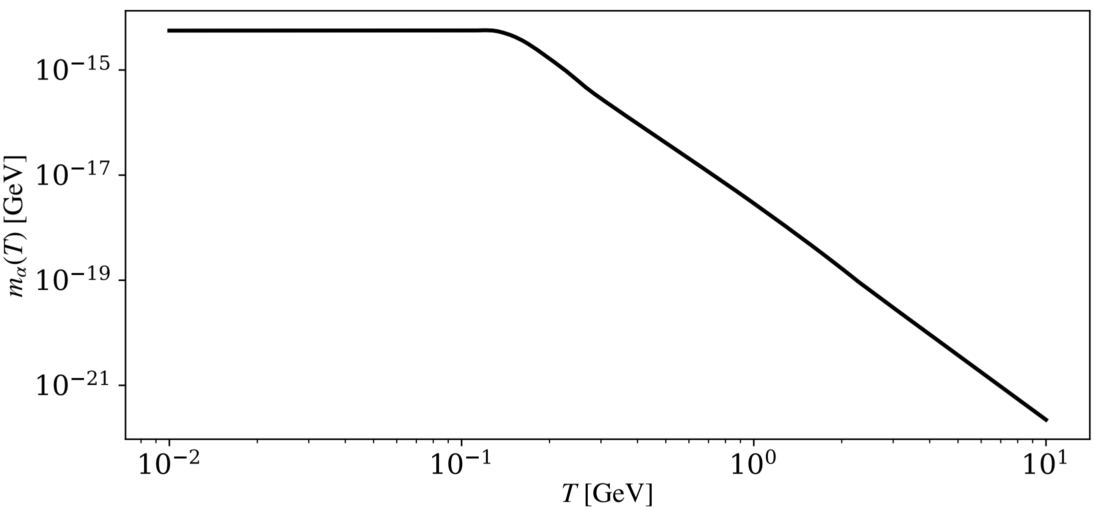

Figure 1: The mass of the axion as a function of the temperature for $f_\alpha = 10^{12}$ GeV, using the data provided in ref. [1].

Assuming that the PQ symmetry breaks before inflation, the initial conditions (*i.e.* at some $t = 0$, after inflation) for the EOM is random. However, we note that $m_\alpha \to 0$ (see Fig. (1)), which means that $m_\alpha \ll H$ at very early times. Therefore, after inflation, the EOM becomes

$$\left(\frac{d^2}{dt} + 3H(t)\ \frac{d}{dt} \right) \theta(t) = 0 \ , \tag{1.3}$$

which is solved by $\theta = \theta_{\rm ini} + C \displaystyle\int_0^t dt\ \left(\frac{R_0}{R(t)} \right)^3$. That is, as the Universe expands, $\theta \approx \theta_{\rm ini}$. Since we would like to calculate the relic abundance of axions, we can integrate eq. (1.2) from a time after inflation (call it $t = t_{\rm ini}$) such that $\dot{\theta}|_{t=t_{\rm ini}} = 0$ and $\theta|_{t=t_{\rm ini}} \approx \theta_{\rm ini}$.

1.1 The WKB approximation

In order to solve eq. (1.2), we assume $\theta \ll 1$, which results in the linearised EOM

$$\left(\frac{d^2}{dt} + 3H(t)\ \frac{d}{dt} + m_\alpha^2(t) \right) \theta(t) = 0 \ . \tag{1.4}$$

Using a trial solution $\psi = \exp\left[i \displaystyle\int dt\ \Big(u(t) + 3/2\ i\ H(t) \Big) \right]$, and defining $\Omega^2 = m_\alpha^2 - \dfrac{9}{4} H^2 - \dfrac{3}{2}\dot{H}$ we can transform the EOM to

$$u^2 = \Omega^2 + i\ \dot{u} \ , \tag{1.5}$$

which has a formal solution $u = \pm\sqrt{\Omega^2 + i\dot{u}}$. Assuming that $\dot{u} \ll \Omega^2$ and $\dot{\Omega} \ll \Omega^2$, we can make the approximation

$$u \approx \pm\Omega + \frac{i}{2} \frac{d \log \Omega}{dt} \ . \tag{1.6}$$

Substituting everything back, we arrive to the approximate general solution of eq. (1.4)

$$\theta \approx \frac{1}{\sqrt{\Omega}} \exp\left(-\frac{3}{2} \int dt\ H \right) \left[A \cos\left(\int dt\ \Omega \right) + B \sin\left(\int dt\ \Omega \right) \right] \ . \tag{1.7}$$

Applying the initial conditions $\dot{\theta}|_{t=t_{\rm ini}} = 0$ and $\theta|_{t=t_{\rm ini}} \approx \theta_{\rm ini}$, we arrive with the solution

$$\theta(t) \approx \theta_{\rm ini} \sqrt{\frac{\Omega_{\rm ini}}{\Omega(t)}} \exp\left(-\frac{3}{2} \int_{t_{\rm ini}}^{t} dt'\ H(t') \right)\ \cos\left(\int_{t_{\rm ini}}^{t} dt'\ \Omega(t') \right) \ . \tag{1.8}$$

In order to further simplify this approximate result, we note that θ deviates from $\theta_{\rm ini}$ only after the mass becomes comparable to the expansion rate of the Universe, *i.e.* after $m_\alpha|_{t=t_{\rm osc}} = 3H|_{t=t_{\rm osc}}$. This observation allows us to use $t_{\rm ini} = t_{\rm osc}$. Moreover, at $t > t_{\rm osc}$, we can approximate $\Omega \approx m_\alpha$, as H^2 and $\dot{H}$ become much smaller than m_α^2 quickly after $t = t_{\rm osc}$. Finally, the axion angle takes the form [1]

$$\theta(t) \approx \theta_{\rm ini} \sqrt{\frac{m_\alpha|_{t=t_{\rm osc}}}{m_\alpha(t)}} \exp\left(-\frac{3}{2} \int_{t_{\rm osc}}^{t} dt'\ H(t') \right)\ \cos\left(\int_{t_{\rm osc}}^{t} dt'\ m_\alpha(t') \right) \ . \tag{1.9}$$

[1]Note that a more accurate form of the solution is

$$\theta(t) \approx \left(\frac{3}{4} \right)^{1/4} \theta_{\rm osc} \sqrt{\frac{m_\alpha|_{t=t_{\rm osc}}}{m_\alpha(t)}} \exp\left(-\frac{3}{2} \int_{t_{\rm osc}}^{t} dt'\ H(t') \right)\ \cos\left(\int_{t_{\rm osc}}^{t} dt'\ m_\alpha(t') \right) \ ,$$

since $\Omega_{\rm osc}^2 \approx \dfrac{3}{4} \left(m_\alpha|_{t=t_{\rm osc}} \right)^2$ and $\theta_{\rm ini} \neq \theta_{\rm osc}$ (see also section 3).

Axion relic abundance

The energy density of the axion is

$$\rho_\alpha = \frac{1}{2} {f_\alpha}^2 \left[\dot{\theta}^2 + {m_\alpha}^2 \theta^2\right] \ . \tag{1.10}$$

For the relic abundance of axions, we need to calculate their energy density at very late times. That is, $\dot{m_\alpha} = 0$, $m_\alpha \gg H$ and $\dot{H} \ll H^2$. After some algebra, we obtain the approximate form of the energy density (as a function of the scale factor R)

$$\rho_\alpha \approx \frac{m_{\alpha,0}}{2} \ {f_\alpha}^2 \ {\theta_{\rm ini}}^2 \ m_\alpha(R_{\rm osc}) \ \left(\frac{R_{\rm osc}}{R}\right)^3 \ , \tag{1.11}$$

which shows that the energy density of axions at late times scales as the energy density of matter. Thus, today, the energy density of axions can be found by calculating the entropy injection (γ) between $t_{\rm osc}$ and today, *i.e.*

$$R^3 \ s = \gamma \ R_{\rm osc}^3 \ s_{\rm osc} \Rightarrow \left(\frac{R_{\rm osc}}{R}\right)^3 = \gamma^{-1} \frac{s}{s_{\rm osc}} \ , \tag{1.12}$$

which results in the energy density today

$$\rho_{\alpha,0} = \gamma^{-1} \frac{s_0}{s_{\rm osc}} \ \frac{1}{2} \ {f_\alpha}^2 \ m_{\alpha,0} \ m_{\alpha,{\rm osc}} \ {\theta_{\rm ini}}^2 \ . \tag{1.13}$$

2 Adiabatic invariant and the anharmonic factor

In oscillatory systems with varying period, the energy is not conserved, and it is usually useful to define an "adiabatic invariant", which is an approximate constant of motion.

Definition of the adiabatic invariant

Given a system with Hamiltonian $\mathcal{H}(\theta, p; t)$, the equations of motion are

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial \theta} \ , \quad \dot{\theta} = \frac{\partial \mathcal{H}}{\partial p} \ . \tag{2.1}$$

Moreover, we note that

$$d\mathcal{H} = \dot{\theta} \ dp - \dot{p} \ d\theta + \frac{\partial \mathcal{H}}{\partial t} \ dt \ . \tag{2.2}$$

If this system exhibits closed orbits (*e.g.* if it oscillates), we define

$$J \equiv \oint p \ d\theta \ , \tag{2.3}$$

where the integral is over a closed path (*e.g.* a period, T). This quantity is the adiabatic invariant of the system, if the Hamiltonian varies slowly during a cycle. That is,

$$\frac{dJ}{dt} = \oint \left(\dot{p} \ d\theta + p \ d\dot{\theta}\right) = \int_t^{t+T} \frac{\partial \mathcal{H}}{\partial t'} \ dt' \approx T \ \frac{\partial \mathcal{H}(t')}{\partial t'}\Big|_{t'=t} \approx 0 \ .$$

It is also noteworthy that this definition can help us identify the time it takes for an orbit to complete, since

$$\frac{dJ}{dt} \approx T \ \frac{\partial \mathcal{H}}{\partial t} \Rightarrow$$
$$J(t) \approx \mathcal{H}(t) \ T + \text{const.} \Rightarrow$$
$$T \approx \frac{dJ}{d\mathcal{H}} \ .$$

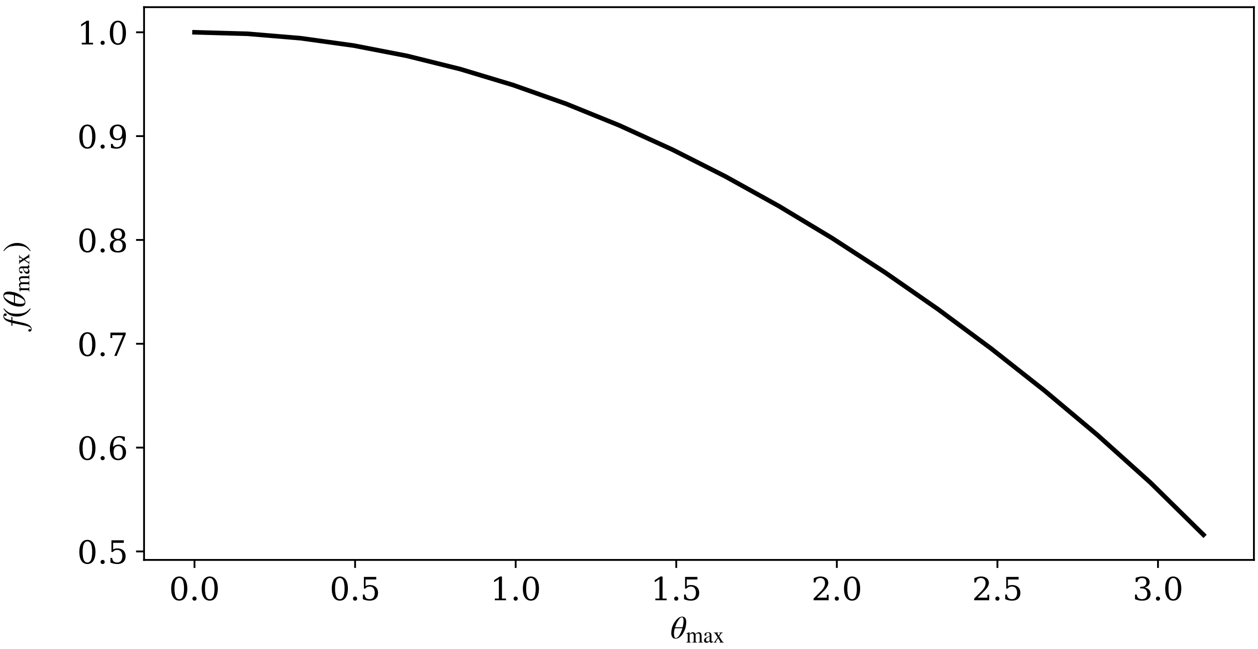

Figure 2: The anharmonic factor for $0 \leq \theta_{\max} < \pi$.

Application to the axion

The Hamiltonian that results in the EOM of eq. (1.2) is

$$\mathcal{H} = \frac{1}{2} \frac{p^2}{f_\alpha{}^2 \; R^3} + V(\theta) \; R^3 \; , \tag{2.4}$$

with

$$p = f_\alpha{}^2 \; R^3 \; \dot{\theta} \tag{2.5}$$

$$V(\theta) = m_\alpha{}^2 f_\alpha{}^2 (1 - \cos\theta) \; . \tag{2.6}$$

Notice that the Hamiltonian varies slowly if $\dot{m_\alpha}/m_\alpha \ll m_\alpha$ and $H \ll m_\alpha$, which are the adiabatic conditions. When these conditions are met, the adiabatic invariant for this system becomes

$$\begin{aligned} J = \quad & \oint p \; d\theta = \oint \sqrt{2 \left(\mathcal{H}(\theta) - V(\theta) \; R^3\right) \; f_\alpha{}^2 R^3} \; d\theta = \oint \sqrt{2 \left(\mathcal{H}(\theta_{\max}) - V(\theta) \; R^3\right) \; f_\alpha{}^2 R^3} \, d\theta = \\ & 2 \textstyle\int_{-\theta_{\max}}^{\theta_{\max}} \sqrt{2} \sqrt{\left(\mathcal{H}(\theta_{\max}) - V(\theta) \; R^3\right) \; f_\alpha{}^2 R^3} \, d\theta = f_\alpha 2\sqrt{2} \textstyle\int_{-\theta_{\max}}^{\theta_{\max}} \sqrt{V(\theta_{\max}) - V(\theta)} R^3 d\theta = \\ & 2\sqrt{2} \; f_\alpha{}^2 \; m_\alpha \, R^3 \; \int_{-\theta_{\max}}^{\theta_{\max}} \sqrt{\cos\theta - \cos\theta_{\max}} \; d\theta \; , \end{aligned} \tag{2.7}$$

where we have defined $\theta_{\max}$ the maximum θ during its oscillation, which corresponds to $p = 0$ (by definition). Also, we have used $\mathcal{H}(\theta, p) = \mathcal{H}(\theta_{\max}, p = 0) = V(\theta_{\max}) \; R^3$ (remember that $\mathcal{H}$ is constant during one cycle). Moreover, for the final equality we have used the adiabatic conditions, *i.e.* negligible change of m_α and R during one period.

For the sake of consistency with the literature, we define a rescaled adiabatic invariant

$$I \equiv R^3 \; m_\alpha \; \theta_{\max}{}^2 \, f(\theta_{\max}) \; , \tag{2.8}$$

where

$$f(\theta_{\max}) = \frac{2\sqrt{2}}{\pi \theta_{\max}{}^2} \int_{-\theta_{\max}}^{\theta_{\max}} d\theta \sqrt{\cos\theta - \cos\theta_{\max}} \; , \tag{2.9}$$

is called the anharmonic factor, with $0.5 \lesssim f(\theta_{\max}) \leq 1$ (see Fig. (2)).

Improving the accuracy of the axion energy density

The adiabatic invariant allows us to calculate the maximum value of the angle θ at late times from its corresponding value at some point after the adiabatic conditions where met. It allows us to calculate the energy density of the axions field taking for $\theta \gtrsim 1$ since we have taken into account the exact potential.

How to use the adiabatic invariant for numerical calculations In order to do this, we need to integrate numerically eq. (1.2), and identify the maxima of θ. Once the adiabatic conditions are fulfilled, we can stop the integration at $\theta_{\max,*}$ (which corresponds to T_* and R_*). Then, the value of the maximum angle today ($\theta_{\max,0} \ll 1$) is related to $\theta_{\max,*}$ via

$$\theta_{\max,0}^{\;2} = \left(\frac{R_*}{R_0}\right)^3 \frac{m_{\alpha,*}}{m_{\alpha,0}} f(\theta_{\max,*})\, \theta_{\max,*}^{\;2} = \gamma^{-1} \frac{s_0}{s_*} \frac{m_{\alpha,*}}{m_{\alpha,0}} f(\theta_{\max,*})\, \theta_{\max,*}^{\;2} \,. \tag{2.10}$$

Using this, we can determine the energy density of axions today from eq. (1.10). That is,

$$\rho_{\alpha,0} = \gamma^{-1} \frac{s_0}{s_*} m_{\alpha,0}\, m_{\alpha,*} \frac{1}{2} f_\alpha^{\;2}\, \theta_{\max,*}^{\;2}\; f(\theta_{\max,*}) \,. \tag{2.11}$$

Notice that this is similar to the result using the WKB approximation 3.7 multiplied by the anharmonic factor, with $t_{\rm osc} \to t_*$.

3 Investigation beyond the WKB approximation

The WKB approximation is very useful in order to understand the evolution of the axion field. However, it fails to explain how the oscillation begins before $\dot{\Omega} \ll \Omega^2$ is reached. In this section we will try to understand the evolution of the axion as generally as possible.

3.1 Useful variables and notation

The EOM 1.2 depends on time, which is not very useful in cosmology. Therefore, we introduce

$$u = \log \frac{R_{\rm ini}}{R} \,, \tag{3.1}$$

which gives us

$$\begin{aligned} \frac{dF}{dt} &= -H \frac{dF}{du} \\ \frac{d^2F}{dt^2} &= H^2 \left(\frac{d^2F}{du^2} + \frac{1}{2}\frac{d\log H^2}{du}\frac{dF}{du} \right) . \end{aligned} \tag{3.2}$$

The EOM in terms of u, then, become

$$\frac{d^2\theta}{du^2} + \left[\frac{1}{2}\frac{d\log H^2}{du} - 3\right] \frac{d\theta}{du} + \left(\frac{m_\alpha}{H}\right)^2 \sin\theta = 0 \,. \tag{3.3}$$

Notice that in a radiation dominated Universe

$$\frac{d\log H^2}{du} = \left(\frac{d\log g_*}{d\log T} + 4\right) \delta_h^{-1} \,,$$

with $\delta_h = 1 + \frac{1}{3}\frac{d\log h}{d\log T}$, such that $\frac{ds}{dT} = \delta_h \frac{3\; s}{T}$.

In general, away from particle annihilations, the expansion rate is dominated by an energy density that scales as $\rho \sim R^{-c}$, *i.e.* $\frac{d\log H^2}{du} = c$ (for example, for radiation domination $c = 4$).

3.2 Behaviour close to the initial condition

Now that we have found the initial conditions, it would be useful to find if θ tends to increase or decrease once the mass is turned on. For this, we look at time $t = \delta t$, with

$$\ddot{\theta} \approx \frac{\dot{\theta}(\delta t) - \dot{\theta}(t_{\rm ini})}{\delta t} = \frac{\dot{\theta}(\delta t)}{\delta t} \; .$$

$$\dot{\theta} \approx -\frac{\delta t}{1 + 3\ H\ \delta t}\ {m_\alpha}^2 \sin\theta \approx -\frac{\delta t}{1 + 3\ H\ \delta t}\ {m_\alpha}^2 \sin\theta_{\rm ini} \; ,$$

which means that $\dot{\theta} < 0$, *i.e.* the angle θ decreases (if $\theta_{\rm ini} > 0$) once the mass is activated. Expressing $\dot{\theta}(\delta t) = \dfrac{\theta - \theta_{\rm ini}}{\delta t}$, we can also write

$$\theta \ \approx\ \theta_{\rm ini} - \frac{\delta t^2}{1 + 3\ H\ \delta t}\ {m_\alpha}^2 \sin\theta \approx \theta_{\rm ini} - \delta t^2\ {m_\alpha}^2\ \sin\theta_{\rm ini} \; . \tag{3.4}$$

Using the eom 3.3, eq. (3.4) takes the form

$$\theta \approx \theta_{\rm ini} - \delta u^2\ \left(\frac{m_\alpha}{H}\right)^2_{t=t_{\rm ini}} \sin\theta_{\rm ini} \; , \tag{3.5}$$

which can be used to estimate the angle at the oscillation temperature

$$\theta_{\rm osc} \approx \theta_{\rm ini} - \left(\frac{m_\alpha}{H}\right)^2_{t=t_{\rm ini}} \left[\left(\frac{h_{\rm osc}}{\gamma_{\rm osc}\ h_{\rm ini}}\right)^{1/3} \frac{T_{\rm osc}}{T_{\rm ini}} - 1\right]^2 \sin\theta_{\rm ini} \; , \tag{3.6}$$

where $\gamma_{\rm osc}$ the entropy injection between $t_{\rm ini}$ and $t_{\rm osc}$.

Notice that in the derivation of eq. (1.9) we used $\theta_{\rm osc} = \theta_{\rm ini}$ as our first approximation. Thus, eq. (3.6) provides a correction that takes into account the deviation between $\theta_{\rm osc}$ and $\theta_{\rm ini}$, and eq. (3.7) becomes

$$\rho_{\alpha,0} = \gamma^{-1} \frac{s_0}{s_{\rm osc}}\ \frac{1}{2}\ {f_\alpha}^2\ m_{\alpha,0}\ m_{\alpha,{\rm osc}}\ {\theta_{\rm ini}}^2 \left[1 - 2\ \left(\frac{m_\alpha}{H}\right)^2_{t=t_{\rm ini}} \left(\left(\frac{h_{\rm osc}}{\gamma_{\rm osc}\ h_{\rm ini}}\right)^{1/3} \frac{T_{\rm osc}}{T_{\rm ini}} - 1\right)^2\right]^2 , \tag{3.7}$$

which shows that the WKB approximation overestimates the energy density of the axion, especially if entropy is injected close to the oscillation temperature.

References

[1] S. Borsanyi et al., *Calculation of the axion mass based on high-temperature lattice quantum chromodynamics*, *Nature* **539** (2016), no. 7627 69–71, [arXiv:1606.07494].