ON THE WELL-POSEDNESS OF THE HYPERELASTIC ROD EQUATION

A Dissertation

Submitted to the Graduate School

of the University of Notre Dame

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

by

David Karapetyan

Alex Himonas, Director

Graduate Program in Mathematics

Notre Dame, Indiana

March 2012

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Abstract

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It is shown that the data-to-solution map for the hyperelastic rod equation is not uniformly continuous on bounded sets of Sobolev spaces with exponent greater than 3/2 in the periodic case and non-periodic cases. The proof is based on the method of approximate solutions and well-posedness estimates for the solution and its lifespan. Building upon this work, we also prove that the data-to-solution map for the hyperelastic rod equation is Hölder continuous from bounded sets of Sobolev spaces with exponent $s > 3/2$ measured in a weaker Sobolev norm with exponent $r < s$ in both the periodic and non-periodic cases. The proof is based on energy estimates coupled with a delicate commutator estimate and multiplier estimate.

To my mama. You have been patient. Be patient a little longer – this is only the beginning.

To zemer Amelda, whose kindnesses give me the strength to never bend, never break.

To Cynthia – loyal and fierce and loving until the end.

CONTENTS

ACKNOWLEDGMENTS

I would like to thank my advisor, Professor Alex Himonas, for his support and guidance. I am also grateful to the math department and the Arthur J. Schmitt Foundation for supporting my graduate studies.

Thanks also to Gerard Misiołek, who was always ready to answer the random analysis question or talk about the arts. To Professors Yongtao Zhang and Bei Hu, for serving on my defense committee, and for suggesting many interesting research projects. To Professor Gerson Petronilho, for the many talks he has given in the analysis seminar, and his patience in collaborating with a junior mathematician. I am also indebted to my friend and collaborator Professor Dan Geba, who is one of the finest human beings I have ever known. He's also not too shabby at tennis.

Many thanks to my officemates Yen Huang and Wenrui Hao, who were always willing to discuss mathematics with me when we weren't horsing around. I must also thank Katie Grayshan for her friendship and many useful discussions, which helped me identify and shore up my weaknesses.

Thanks to Beth VerVelde, who understood me better than most, and is too wise for her own good. To my friends back home, thanks goes to Cynthia Ramirez, who always felt guilty whenever I was sent to voicemail, and called me back as soon as possible. Your patience and loyalty helped me persevere, and will not be forgotten. To Amelda zemer, your kindnesses and enormous heart have given me the strength to never break, no matter the adversity or general nonsense I may encounter.

Lastly, thanks to my mom, Satenik Antonian, who has always had my best inter-

ests at heart, and has never wavered in her support and love. You have suffered more than most, yet take it stoically, with great grace, honor, and strength.

CHAPTER 1

NON-UNIFORM DEPENDENCE AND WELL-POSEDNESS FOR HR

1.1 Introduction

We consider the initial value problem (IVP) for the hyperelastic rod (HR) equation

$$\partial_t u - \partial_t \partial_x^2 u + 3u\partial_x u = \gamma\Big(2\partial_x u \partial_x^2 u + u\partial_x^3 u\Big), \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{T},\ \text{or } \mathbb{R} \quad t \in \mathbb{R}, \tag{2}$$

where γ is a nonzero constant, and prove that the dependence of solutions on initial data is not uniformly continuous in Sobolev spaces $H^s(\mathbb{T})$, $s > 3/2$. Thus, we extend the result proved by Olson [29] in the periodic case (for $s \geq 2$ and $\gamma \neq 3$) to $s > 3/2$ (the entire well-posedness range) for HR. Furthermore, motivated by the work of Himonas and Kenig [15], we establish non-uniform dependence in the non-periodic case, where the method of traveling wave solutions used in [29] does not seem to work.

The HR equation was first derived by Dai in [10] as a one-dimensional model for finite-length and small-amplitude axial deformation waves in thin cylindrical rods composed of a compressible Mooney-Rivlin material. The derivation relied upon a reductive perturbation technique, and took into account the nonlinear dispersion of pulses propagating along a rod. It was assumed that each cross-section of the rod is subject to a stretching and rotation in space. The solution $u(x,t)$ to the HR equation represents the radial stretch relative to a pre-stressed state, while γ is a fixed constant depending upon the pre-stress and the material used in the rod, with values ranging from -29.4760 to 3.4174.

The well-posedness of the HR equation has been studied by several authors. In Yin [34] and Zhou [35], a proof of local well-posedness in Sobolev spaces H^s, $s > 3/2$, is described on the line and the circle, respectively. Their approach is to rewrite the HR equation in its non-local form, and then to verify the conditions needed to apply Kato's semi-group theory [23]. For details on how this is done for CH on the line, see Rodriguez-Blanco [30]. Blow-up criteria is also investigated in [34] and [35], as well as by Constantin and Strauss [9].

Setting $\gamma = 0$ gives the celebrated BBM equation, which was proposed by Benjamin, Bona, and Mahony [1] as a model for the unidirectional evolution of long waves. Solitary-wave solutions to this equation are global and orbitally stable (see Benjamin [4], [1], and [9]). For more general γ, the existence of global solutions to HR on the line with constant H^1 energy was proved recently by Mustafa [28] using the approach developed by Bressan and Constantin in [2]. Using a vanishing viscosity argument, Coclite, Holden, and Karlsen [7] established existence of a strongly continuous semigroup of global weak solutions of HR on the line for initial data in H^1. Bendahmane, Coclite, and Karlsen [3] extended this result to traveling wave solutions that are supersonic solitary shockwaves. For more information on the existence of global solutions to the HR equation, see Holden and Raynaud [21] and [34].

There is a variety of traveling wave solutions to the HR equation that can be obtained using various combinations of peaks, cusps, compactons, fractal-like waves, and plateaus (see Lenells [26]). Orbital stability of solitary wave solutions was proved in [9]. Solitary shock wave formation was analyzed in Dai and Huo [12] using traveling wave solutions of the HR equation to derive a system of ordinary differential equations, with a vertical singular line in the phase plane corresponding with the formation of shock waves. Head-on collisions between two solitary waves was investigated in the work of Hui-Hui Dai, Shiqiang Dai, and Huo [11] using a reductive perturbation method coupled with the technique of strained coordinates.

In this work we study the continuity of the data-to-solution map for the HR equation. Using the method of traveling wave solutions it was shown in [29] that the data-to-solution map $u_0 \mapsto u$ of the periodic HR equation is not uniformly continuous from any bounded set in $H^s(\mathbb{T})$ into $C([0,T], H^s(\mathbb{T}))$ for $s \geq 2$ and $\gamma \neq 3$. Non-uniform dependence for the non-periodic CH equation in $H^s(\mathbb{R})$ for $s > 3/2$ was proven in [15] using the method of approximate solutions and well-posedness estimates. The case $s = 1$ for both the line and the circle was proved earlier by Himonas, Misiołek, and Ponce in [20]. Recently in [16] non-uniform continuity of the solution map for the CH equation on the circle has been proved for the whole range of Sobolev exponents for which local well-posedness of CH is known.

We mention that the continuity of the data-to-solution map for CH has been studied in [20], [17], and [18], and for the Euler equations in [19]. Continuity of this map for the Benjamin-Ono equation was studied in Koch and Tzvetkov [25]. For related ill-posedness results, we refer the reader to Kenig, Ponce, and Vega [24], Christ, Colliander, and Tao [6], and the references therein.

Here we consider the initial value problem for the HR equation in both the periodic and non-periodic cases and prove non-uniform continuity of the solution map. More precisely, we show the following result:

Theorem 1. *Let γ be a nonzero constant. Then the data-to-solution map $u(0) \mapsto u(t)$ of the Cauchy-problem for the HR IVP (1)-(2) is not uniformly continuous from any bounded subset of H^s into $C([-T,T], H^s)$ for $s > 3/2$ on the line and circle.*

As we mentioned above, when $\gamma = 0$ the HR equation becomes the BBM equation. Bona and Tzvetkov [5] have recently proved that this equation is globally well-posed in Sobolev spaces H^s, if $s \geq 0$, and that its data-to-solution map is smooth. Our approach for proving Theorem 1 mirrors that in Himonas and Kenig [15] and Himonas, Kenig, and Misiołek [16]. That is, we will choose approximate solutions to the HR equation such that the size of the difference between approximate and actual solutions with identical initial data is negligible. Hence, to understand the degree of dependence, it will suffice to focus on the behavior of the approximate solutions (which will be simple in form), rather than on the behavior of the actual solutions. In order for the method to go through, we will need well-posedness estimates for the size of the actual solutions to the HR equation, as well a lower bound for their lifespan. This will permit us to obtain an upper bound for the size of the difference of approximate and actual solutions. More precisely, we will need the following well-posedness result with estimates, stated in both the periodic and non-periodic case:

Theorem 2. *If $s > 3/2$ then we have:*

(i) If $u_0 \in H^s$ then there exists a unique solution to the Cauchy problem (1)–(2) in $C([-T,T],H^s)$, where the lifespan T depends on the size of the initial data u_0. Moreover, the lifespan T satisfies the lower bound estimate

$$T \geq \frac{1}{2c_s\|u_0\|_{H^s}}. \tag{3}$$

(ii) The flow map $u_0 \mapsto u(t)$ is continuous from bounded sets of H^s into $C([-T,T],H^s)$, and the solution u satisfies the estimate

$$\|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s}, \quad |t| \leq T. \tag{4}$$

A proof of existence, uniqueness, and continuous dependence in this theorem for $\gamma = 1$ (CH) is given by Li and Olver in [27] using a regularization method, and in [30] using Kato's semi-group method [23]. As mentioned above, proofs of existence, uniqueness, and continuous dependence for HR have been outlined in [34] and [35] for the line and circle, respectively. Both outlines rely upon an application of Kato's semi-group method. However, we have not been able to find estimates (3) and (4) in the literature. Here we shall give a proof of local well-posedness of HR, including estimates (3) and (4), which are key ingredients in our work, following an alternative approach used for nonlinear hyperbolic equations in Taylor [32].

For the Burgers equation, it is also known that for $s > 3/2$, dependence is not better than continuous. Furthermore, Kato [23] showed that for $s > 3/2$ the data-to-solution map $u_0 \mapsto u(t)$ is not Hölder continuous from a closed ball in $H^s(\mathbb{R})$ centered at 0 and measured in the $H^r(\mathbb{R})$ norm, $r < s$, to $C([0,T], H^r(\mathbb{R}))$, where T depends upon the $H^s(\mathbb{R})$ radius of the ball. More precisely, for fixed $0 < \gamma < 1$ and fixed constant $c > 0$, there exist solutions u, v of Burgers with bounded initial data in $H^s(\mathbb{R})$ (and hence, a common lifespan T) and $0 < t_0 < T$ such that

$$\|u(t_0) - v(t_0)\|_{H^r(\mathbb{R})} > c\|u_0 - v_0\|^{\gamma}_{H^r(\mathbb{R})}.$$

However, for certain general quasi-linear hyperbolic systems, Kato also obtained uniform continuity of the data-to-solution map for initial data in Sobolev spaces with integer index, measured in a weaker Sobolev norm. More recently, Tao [31] obtained Lipschitz continuity of the data-to-solution map for the Benjamin-Ono equation for $H^1(\mathbb{R})$ initial data measured in $L^2(\mathbb{R})$. Herr, Ionescu, Kenig, and Koch [14] have also obtained Lipschitz continuity in a weaker topology for the Benjamin-Ono with generalized dispersion. Hence, it is reasonable to ask whether a result similar to these holds for HR. Our main motivation stems from the work of Chen, Liu, and Zhang [8] on the b-family

$$\partial_t u = -u\partial_x u - \partial_x(1-\partial_x^2)^{-1}\left[\frac{b}{2}u^2 + \frac{3-b}{2}(\partial_x u)^2\right], \tag{5}$$

$$u(x,0) = u_0(x),\ x \in \mathbb{T} \text{ or } \mathbb{R},\ t \in \mathbb{R} \tag{6}$$

for which they proved Hölder continuity of the data-to-solution map from a closed ball $B(0,h)$ in $H^s(\mathbb{R})$, $s > 3/2$ (measured in the $H^r(\mathbb{R})$ topology, $r < s$) to $C([0,T], H^r(\mathbb{R}))$, with $T = T(h) > 0$ and Hölder index $\alpha = \alpha(b,s,r)$ given by

$$\alpha = \begin{cases} 1, & (s,r) \in \Omega_1 \\ 1, & b = 3 \ \text{ and } \ (s,r) \in \Omega_2 \\ 2(s-1)/(s-r), & b \neq 3 \ \text{ and } \ (s,r) \in \Omega_2 \\ s-r, & (s,r) \in \Omega_3 \end{cases}$$

where

$$\Omega_1 = \{(s,r) :\ s > 3/2,\ 0 \le r \le s-1,\ r \ge 2-s\}$$

$$\Omega_2 = \{(s,r) :\ 3/2 < s < 2,\ 0 < r < 2-s\}$$

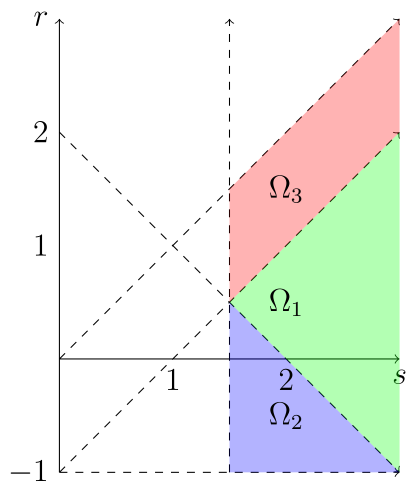
$$\Omega_3 = \{(s,r) :\ s > 3/2,\ s-1 < r < s\}\,.$$

Given this result, and the similarities between the b-family and HR (both can be thought of as weakly dispersive nonlocal perturbations of Burgers), in this work we study the continuity properties of the data-to-solution map for the HR IVP, expanding upon Theorem 1. More precisely, following [8] we show the following result:

Theorem 3. *For $\gamma \neq 0$, the data-to-solution map for HR is Hölder continuous on both the line and circle from $B_{H^s}(R)$ (in the topology of H^r) to $C([0,T], H^r)$, where*

$T = T(R)$, for $s > 3/2$, $-1 \le r < s$. More precisely, consider the following sets

$$\Omega_1 = \Big\{(s,\ r) \in \mathbb{R}^2 :\ s > 3/2,\ -1 \le r \le s-1,\ r \ge 2-s\Big\}$$
$$\Omega_2 = \Big\{(s,\ r) \in \mathbb{R}^2 :\ 3/2 < s < 3,\ -1 \le r < 2-s\Big\}$$
$$\Omega_3 = \Big\{(s,\ r) \in \mathbb{R}^2 :\ s > 3/2,\ s-1 < r < s\Big\}.$$



Then for two initial data $u_0, v_0 \in B_{H^s}(R)$, there exist unique corresponding solutions $u(x,t), v(x,t)$ for $0 \le t \le T = T(R)$ to the HR equation (75) which satisfy

$$\|u(t) - v(t)\|_{H^r} \le C\|u_0 - v_0\|_{H^r}^{\alpha(s,r)}, \quad 0 \le t \le T$$

where

$$\alpha = \begin{cases} 1, & (s,r) \in \Omega_1 \\ 2(s-1)/(s-r), & (s,r) \in \Omega_2 \\ s-r, & (s,r) \in \Omega_3. \end{cases}$$

We remark that this result is sharper then the analogue obtained in [8] for the b-

family. We are confident that the techniques applied here can be applied to sharpen the results obtained in [8].

This document is structured as follows. We first prove Theorem 1 on the line and then on the circle. As mentioned above, we begin with two sequences of appropriate approximate solutions and then we construct actual solutions coinciding at time zero with the approximate solutions. The key step is to show that the H^s-size of the difference between approximate and actual solutions converges to zero (see Proposition 7 and Proposition 9). We then prove Theorem 2 using a Galerkin-type argument and energy estimates. Using the tools (energy estimates, commutator estimate, and multiplier estimate) used to prove Theorem 2, we will then prove Theorem 3.

1.2 Proof of Non-Uniform Dependence on the Line

We begin by outlining the method of the proof, as it has been applied for the case $\gamma = 1$ in [15]. We will show that there there exist two sequences of solutions $u_\lambda(t)$ and $v_\lambda(t)$ in $C([-T,T], H^s(\mathbb{R}))$ such that

$$\|u_\lambda(t)\|_{H^s(\mathbb{R})} + \|v_\lambda(t)\|_{H^s(\mathbb{R})} \lesssim 1, \tag{7}$$

$$\lim_{\lambda\to\infty} \|u_\lambda(0) - v_\lambda(0)\|_{H^s(\mathbb{R})} = 0, \tag{8}$$

and

$$\liminf_{\lambda\to\infty} \|u_\lambda(t) - v_\lambda(t)\|_{H^s(\mathbb{R})} \gtrsim |\sin(\gamma t)|, \quad |\gamma t| \leq 1. \tag{9}$$

We accomplish this in two steps. First, we will construct two sequences of approximate solutions satisfying the above properties. Then, we will construct two sequences of actual solutions coinciding with the approximate solutions at time zero such that for small time the size of the difference of solutions and approximate solutions decays as $\lambda \to \infty$.

For this method, it is more convenient to rewrite the Cauchy problem for the HR equation in the non-local form

$$\partial_t u = -\gamma u \partial_x u - \Lambda^{-1}\left[\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}(\partial_x u)^2\right], \tag{10}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \quad t \in \mathbb{R} \tag{11}$$

where

$$\Lambda^{-1} = \partial_x (1 - \partial_x^2)^{-1}.$$

1.2.1 Approximate Solutions

Following [15], our approximate solutions
$u^{\omega,\lambda} = u^{\omega,\lambda}(x,t)$ to (10)-(11) will consist of a low frequency and a high frequency part, i.e.

$$u^{\omega,\lambda} = u_\ell + u^h \tag{12}$$

where ω is in a bounded set of $\mathbb{R}$ and $\lambda > 0$. The high frequency part is given by

$$u^h = u^{h,\omega,\lambda}(x,t) = \lambda^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{\lambda^\delta}\right)\cos(\lambda x - \gamma\omega t) \tag{13}$$

where ϕ is a C^∞ cut-off function such that

$$\phi = \begin{cases} 1, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 2, \end{cases}$$

and by Theorem 2 we let the low frequency part $u_\ell = u_{l,\omega,\lambda}(x,t)$ be the unique solution to the Cauchy problem

$$\partial_t u_\ell = -\gamma u_\ell \partial_x u_\ell - \Lambda^{-1}\left[\frac{3-\gamma}{2}(u_\ell)^2 + \frac{\gamma}{2}\left(\partial_x u_\ell\right)^2\right], \tag{14}$$

$$u_\ell(x,0) = \omega\lambda^{-1}\tilde{\phi}\left(\frac{x}{\lambda^\delta}\right), \quad x\in\mathbb{R}, \quad t\in\mathbb{R} \tag{15}$$

where $\tilde{\phi}$ is a $C_0^\infty(\mathbb{R})$ function such that

$$\tilde{\phi}(x) = 1 \ \text{ if } \ x \in \text{supp } \phi. \tag{16}$$

We remark that for $\lambda >> 1$ and $\delta < 2$ the approximate solutions $u^{\omega,\lambda}$ share a common lifespan $T >> 1$. To see why, we first note that the high frequency part $u^{h,\omega,\lambda}$ has infinite lifespan by the following, whose proof can be found in [15]:

Lemma 4. *Let $\psi \in S(\mathbb{R})$, $\alpha\in\mathbb{R}$. Then for $s \geq 0$ we have*

$$\lim_{\lambda\to\infty} \lambda^{-\frac{\delta}{2}-s}\|\psi\left(\frac{x}{\lambda^\delta}\right)\cos(\lambda x-\alpha)\|_{H^s(\mathbb{R})} = \frac{1}{\sqrt{2}}\|\psi\|_{L^2(\mathbb{R})}. \tag{17}$$

Relation (17) remains true if cos *is replaced by* sin*.*

For the low frequency part $u_{\ell,\omega,\lambda}$, we apply (3) and the estimate

$$\|\tilde{\phi}\left(\frac{x}{\lambda^\delta}\right)\|_{H^k(\mathbb{R})} \leq \lambda^{\frac{\delta}{2}}\|\tilde{\phi}\|_{H^k(\mathbb{R})}, \quad k \geq 0 \tag{18}$$

to obtain a lower bound for its lifespan

$$T_{\ell,\omega,\lambda} \geq \frac{1}{2c_s\|u_{\ell,\omega,\lambda}(0)\|_{H^s(\mathbb{R})}} = \frac{1}{2c_s|\omega|\lambda^{\frac{\delta}{2}-1}\|\tilde{\phi}\|_{H^s(\mathbb{R})}} >> 1.$$

Since ω belongs to a bounded subset of $\mathbb{R}$, the existence of a common lifespan $T >> 1$

follows.

Substituting the approximate solution $u^{\omega,\lambda} = u_\ell + u^h$ into the HR equation, we see that the error E of our approximate solution is given by

$$E = E_1 + E_2 + \cdots + E_8$$

where

$$\begin{aligned}
&E_1 = \gamma\lambda^{1-\frac{\delta}{2}-s}\left[u_\ell(x,0) - u_\ell(x,t)\right]\phi\left(\frac{x}{\lambda^\delta}\right)\sin(\lambda x - \gamma\omega t),\\
&E_2 = \gamma\lambda^{-\frac{3\delta}{2}-s}u_\ell(x,t)\cdot\phi'\left(\frac{x}{\lambda^\delta}\right)\cos\left(\lambda x - \gamma\omega t\right),\\
&E_3 = \gamma u^h\partial_x u_\ell,\ \ E_4 = \gamma u^h\partial_x u^h,\ E_5 = \frac{3-\gamma}{2}\Lambda^{-1}\left[\left(u^h\right)^2\right],\\
&E_6 = (3-\gamma)\Lambda^{-1}\left[u_\ell u^h\right],\ E_7 = \frac{\gamma}{2}\Lambda^{-1}\left[\left(\partial_x u^h\right)^2\right],\ E_8 = \gamma\Lambda^{-1}\left[\partial_x u_\ell\partial_x u^h\right].
\end{aligned}\tag{19}$$

Next we prove the decay of the error as $\lambda \to \infty$:

Proposition 5. *Let $1 < \delta < 2$. Then for $s > 1$ and ω in a bounded set*

$$\|E(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{\frac{\delta}{2}-s}, \quad |t| \le T. \tag{20}$$

Proof. It will suffice to estimate the H^1 norms of each E_i. For E_1 we have

$$\begin{aligned}
\|E_1\|_{H^1(\mathbb{R})} &= \|\gamma\lambda^{1-\frac{\delta}{2}-s}\left[u_\ell(x,0) - u_\ell(x,t)\right]\phi\left(\frac{x}{\lambda^\delta}\right)\sin(\lambda x - \gamma\omega t)\|_{H^1(\mathbb{R})}\\
&\lesssim \lambda^{1-\frac{\delta}{2}-s}\|\left[u_\ell(x,0) - u_\ell(x,t)\right]\phi\left(\frac{x}{\lambda^\delta}\right)\sin\left(\lambda x - \gamma\omega t\right)\|_{H^1(\mathbb{R})}.
\end{aligned}\tag{21}$$

Applying the inequality

$$\|fg\|_{H^1(\mathbb{R})} \lesssim \|f\|_{C^1(\mathbb{R})}\|g\|_{H^1(\mathbb{R})} \tag{22}$$

to estimate (21) gives

$$\|E_1\|_{H^1(\mathbb{R})} \lesssim \lambda^{1-\frac{\delta}{2}-s}\|\phi\left(\frac{x}{\lambda^\delta}\right)\sin(\lambda x-\gamma\omega t)\|_{C^1(\mathbb{R})}\|[u_\ell(x,0)-u_\ell(x,t)]\|_{H^1(\mathbb{R})}. \tag{23}$$

We now estimate the right-hand side of (23) in pieces. First, note that routine computations give

$$\|\phi\left(\frac{x}{\lambda^\delta}\right)\sin(\lambda x-\gamma\omega t)\|_{C^1(\mathbb{R})} \lesssim \lambda. \tag{24}$$

Next, we observe that the fundamental theorem of calculus and Minkowski's inequality give

$$\|u_\ell(x,t)-u_\ell(x,0)\|_{H^1(\mathbb{R})} = \|\int_0^t \partial_\tau u_\ell(x,\tau)\,d\tau\|_{H^1(\mathbb{R})} \leq \int_0^t \|\partial_\tau u_\ell(x,\tau)\|_{H^1(\mathbb{R})}\,d\tau. \tag{25}$$

We want to estimate the right-hand side of (25). Recalling (10), we have

$$\|\partial_\tau u_\ell(x,\tau)\|_{H^1(\mathbb{R})} \leq \|\gamma u_\ell \partial_x u_\ell\|_{H^1(\mathbb{R})} + \|\Lambda^{-1}\left[\frac{3-\gamma}{2}(u_\ell)^2+\frac{\gamma}{2}\left(\partial_x u_\ell\right)^2\right]\|_{H^1(\mathbb{R})}. \tag{26}$$

Applying the algebra property of Sobolev spaces, we obtain

$$\|\gamma u_\ell \partial_x u_\ell\|_{H^1(\mathbb{R})} \lesssim \|u_\ell\|^2_{H^2(\mathbb{R})}$$

which yields

$$\|\gamma u_\ell \partial_x u_\ell\|_{H^1(\mathbb{R})} \lesssim \lambda^{-2+\delta} \tag{27}$$

by the following:

Lemma 6. *Let $0 < \delta < 2$, $\lambda >> 1$, with ω belonging to a bounded subset of $\mathbb{R}$. Then the initial value problem (14)-(15) has a unique solution $u_\ell \in C([-T,T], H^s(\mathbb{R}))$ for all $s > 3/2$ which satisfies*

$$\|u_\ell(t)\|_{H^s(\mathbb{R})} \leq c_s \lambda^{-1+\frac{\delta}{2}}, \quad |t| \leq T. \tag{28}$$

An analogous result can be found in [15], and is a simple consequence of Theorem 2 and the profile of our initial data. Applying the inequality

$$\|\Lambda^{-1} f\|_{H^1(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}, \tag{29}$$

and the algebra property of Sobolev spaces, we obtain

$$\|\Lambda^{-1}\left[\frac{3-\gamma}{2}(u_\ell)^2 + \frac{\gamma}{2}\left(\partial_x u_\ell\right)^2\right]\|_{H^1(\mathbb{R})} \lesssim \|u_\ell\|^2_{H^2(\mathbb{R})}$$

which by Lemma 6 gives

$$\|\Lambda^{-1}\left[\frac{3-\gamma}{2}(u_\ell)^2 + \frac{\gamma}{2}\left(\partial_x u_\ell\right)^2\right]\|_{H^1(\mathbb{R})} \lesssim \lambda^{-2+\delta}, \quad |t| \leq T. \tag{30}$$

Substituting (27) and (30) into the right-hand side of (26), and recalling (25), we obtain

$$\|u_\ell(x,t) - u_\ell(x,0)\|_{H^1(\mathbb{R})} \lesssim \lambda^{-2+\delta}, \quad |t| \leq T. \tag{31}$$

Substituting (31) and (24) into (23), we obtain

$$\|E_1\|_{H^1} \lesssim \lambda^{\delta/2-s}. \tag{32}$$

For E_2, we apply (22) to obtain

$$\begin{aligned}\|E_2\|_{H^1(\mathbb{R})} &= \gamma\lambda^{-\frac{3\delta}{2}-s}\|u_\ell(x,t)\cdot\phi'\left(\frac{x}{\lambda^\delta}\right)\cos(\lambda x-\gamma\omega t)\|_{H^1(\mathbb{R})} \\ &\lesssim_s \gamma\lambda^{\frac{-3\delta}{2}-s}\|u_\ell(x,t)\|_{H^1(\mathbb{R})}\|\phi'\left(\frac{x}{\lambda^\delta}\right)\cos(\lambda x-\gamma\omega t\|_{C^1(\mathbb{R})}.\end{aligned} \tag{33}$$

We note that

$$\begin{aligned}&\|\phi'\left(\frac{x}{\lambda^\delta}\right)\cos(\lambda x-\gamma\omega t)\|_{C^1(\mathbb{R})} \\ &\leq \|\phi'\left(\frac{x}{\lambda^\delta}\right)\|_{L^\infty(\mathbb{R})}+\lambda\|\phi'\left(\frac{x}{\lambda^\delta}\right)\|_{L^\infty(\mathbb{R})}+\lambda^{-\delta}\|\phi''\left(\frac{x}{\lambda^\delta}\right)\|_{L^\infty(\mathbb{R})}\end{aligned}$$

which gives

$$\|\phi'\left(\frac{x}{\lambda^\delta}\right)\cos(\lambda x-\gamma\omega t)\|_{C^1(\mathbb{R})}\lesssim\lambda. \tag{34}$$

Applying estimates (34) and (28) to (33), we obtain

$$\|E_2\|_{H^1(\mathbb{R})}\lesssim\lambda^{-\delta-s}.$$

For E_3 we see that (22) implies

$$\|\gamma u^h\partial_x u_\ell\|\lesssim\|u^h\|_{C^1(\mathbb{R})}\|u_\ell\|_{H^1(\mathbb{R})}. \tag{35}$$

Now, note that

$$\begin{aligned}\|u^h\|_{L^\infty(\mathbb{R})} &= \lambda^{-\frac{\delta}{2}-s}\|\phi\left(\frac{x}{\lambda^\delta}\right)\cos\left(\lambda x-\gamma\omega t\right)\|_{L^\infty(\mathbb{R})} \\ &\lesssim \lambda^{-\frac{\delta}{2}-s}\end{aligned} \tag{36}$$

and

$$\begin{aligned}
&\|\partial_x u^h\|_{L^\infty(\mathbb{R})} \\
&= \lambda^{-\frac{\delta}{2}-s}\|\phi\left(\frac{x}{\lambda^\delta}\right)\cdot -\lambda\sin(\lambda x-\gamma\omega t)+\lambda^{-\delta}\phi'\left(\frac{x}{\lambda^\delta}\right)\cos(\lambda x-\gamma\omega t)\|_{L^\infty(\mathbb{R})} \\
&\lesssim \lambda^{1-\frac{\delta}{2}-s}.
\end{aligned} \tag{37}$$

Therefore, from (36) and (37) it follows that

$$\|u^h\|_{C^1(\mathbb{R})} \lesssim \lambda^{-\frac{\delta}{2}-s}+\lambda^{1-\frac{\delta}{2}-s} \sim \lambda^{1-\frac{\delta}{2}-s}. \tag{38}$$

Substituting estimates (38) and (28) into (35) we obtain

$$\|\gamma u^h \partial_x u_\ell\|_{H^1(\mathbb{R})} \lesssim \lambda^{-s}. \tag{39}$$

For E_4 we apply (22) to obtain

$$\|\gamma u^h \partial_x u^h\|_{H^1(\mathbb{R})} \lesssim \|u^h\|_{C^1(\mathbb{R})}\|u^h\|_{H^1(\mathbb{R})}. \tag{40}$$

Substituting in (38), and observing that $\|u^h\|_{H^1(\mathbb{R})} \lesssim 1$ for $\lambda >> 1$ by Lemma 4, we obtain

$$\|u^h \partial_x u^h\|_{H^1(\mathbb{R})} \lesssim \lambda^{1-\frac{\delta}{2}-s}. \tag{41}$$

For E_5 we apply (29) to obtain

$$\begin{aligned}
\|E_5\|_{H^1(\mathbb{R})} &= \|\Lambda^{-1}\left[\frac{3-\gamma}{2}(u^h)^2\right]\|_{H^1(\mathbb{R})} \\
&\lesssim \|u^h\|_{L^\infty(\mathbb{R})}\|u^h\|_{L^2(\mathbb{R})}.
\end{aligned} \tag{42}$$

Substituting (17) and (36) into (42), we conclude that

$$\|E_5\|_{H^1(\mathbb{R})} \lesssim \lambda^{-\frac{\delta}{2}-s}. \tag{43}$$

For E_6 we apply (29) to obtain

$$\begin{aligned}\|E_6\|_{H^1(\mathbb{R})} &= \|\Lambda^{-1}\left[(3-\gamma)u_\ell u^h\right]\|_{H^1(\mathbb{R})} \\ &\lesssim \|u_\ell\|_{L^2(\mathbb{R})}\|u^h\|_{L^\infty(\mathbb{R})}\end{aligned} \tag{44}$$

which by Lemma 6 and (36) reduces to

$$\|E_6\|_{H^1(\mathbb{R})} \lesssim \lambda^{-1-s}. \tag{45}$$

For E_7 we apply (29) to obtain

$$\begin{aligned}\|E_7\|_{H^1(\mathbb{R})} &= \|\Lambda^{-1}\left[\frac{\gamma}{2}\left(\partial_x u\right)^2\right]\|_{H^1(\mathbb{R})} \\ &\lesssim \|\partial_x u^h\|_{L^\infty(\mathbb{R})}\|u^h\|_{H^1(\mathbb{R})}\end{aligned} \tag{46}$$

which by Lemma 4 and (37) reduces to

$$\|E_7\|_{H^1(\mathbb{R})} \lesssim \lambda^{1-\frac{\delta}{2}-s}. \tag{47}$$

For E_8 we apply (29) to obtain

$$\begin{aligned}\|E_8\|_{H^1(\mathbb{R})} &= \|\Lambda^{-1}\left[\gamma\partial_x u_\ell \partial_x u^h\right]\|_{H^1(\mathbb{R})} \\ &\lesssim \|u_\ell\|_{H^2(\mathbb{R})}\|\partial_x u^h\|_{L^\infty(\mathbb{R})}\end{aligned} \tag{48}$$

which by Lemma 6 and (37) reduces to

$$\|E_8\|_{H^1(\mathbb{R})} \lesssim \lambda^{-s}. \tag{49}$$

Collecting all our estimates for the E_i and recalling that we have assumed $1 < \delta < 2$, we see that

$$\|E\|_{H^1(\mathbb{R})} \lesssim \lambda^{\frac{\delta}{2}-s}, \qquad \lambda >> 1$$

which completes the proof. □

1.2.2 Construction of Solutions

We wish now to estimate the difference between approximate and actual solutions to the HR IVP with common initial data. Let $u_{\omega,\lambda}(x,t)$ be the unique solution to the HR equation with initial data $u^{\omega,\lambda}(x,0)$. That is, $u_{\omega,\lambda}$ solves the initial value problem

$$\partial_t u_{\omega,\lambda} = -\gamma u_{\omega,\lambda}\partial_x u_{\omega,\lambda} - \Lambda^{-1}\left[\frac{3-\gamma}{2}\left(u_{\omega,\lambda}\right)^2 + \frac{\gamma}{2}\left(\partial_x u_{\omega,\lambda}\right)^2\right], \tag{50}$$

$$u_{\omega,\lambda}(x,0) = u^{\omega,\lambda}(x,0) = \omega\lambda^{-1}\tilde{\phi}\left(\frac{x}{\lambda^\delta}\right) + \lambda^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{\lambda^\delta}\right)\cos(\lambda x). \tag{51}$$

We will now prove that the $H^1(\mathbb{R})$ norm of the difference decays as $\lambda \to \infty$:

Proposition 7. *Let $v = u^{\omega,\lambda} - u_{\omega,\lambda}$. Then, for $s > 1$ and $1 < \delta < 2$ we have*

$$\|v(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{\frac{\delta}{2}-s}, \quad |t| \leq T. \tag{52}$$

Proof. First we observe that v satisfies

$$\begin{aligned}\partial_t v &= E + \gamma(v\partial_x v - v\partial_x u^{\omega,\lambda} - u^{\omega,\lambda}\partial_x v)\\ &\quad + \Lambda^{-1}\left[\frac{3-\gamma}{2}v^2 + \frac{\gamma}{2}\left(\partial_x v\right)^2 - \left(3-\gamma\right)u^{\omega,\lambda}v - \gamma\partial_x u^{\omega,\lambda}\partial_x v\right].\end{aligned}$$

It follows immediately that

$$\begin{aligned}v(1-\partial_x^2)\partial_t v &= v(1-\partial_x^2)E + v\gamma(1-\partial_x^2)(v\partial_x v - v\partial_x u^{\omega,\lambda} - u^{\omega,\lambda}\partial_x v)\\ &\quad + v\partial_x\left[\frac{3-\gamma}{2}v^2 + \frac{\gamma}{2}(\partial_x v)^2 - (3-\gamma)u^{\omega,\lambda}v - \gamma\partial_x u^{\omega,\lambda}\partial_x v\right].\end{aligned} \tag{53}$$

Applying the relation $v\partial_t v = v(1-\partial_x^2)\partial_t v + v\partial_x^2\partial_t v$ to (53), we obtain

$$\begin{aligned}v\partial_t v &= v(1-\partial_x^2)E + v\gamma(1-\partial_x^2)(v\partial_x v - v\partial_x u^{\omega,\lambda} - u^{\omega,\lambda}\partial_x v)\\ &\quad + v\partial_x\left[\frac{3-\gamma}{2}v^2 + \frac{\gamma}{2}(\partial_x v)^2 - (3-\gamma)u^{\omega,\lambda}v - \gamma\partial_x u^{\omega,\lambda}\partial_x v\right] + v\partial_x^2\partial_t v.\end{aligned} \tag{54}$$

Adding $\partial_x v\partial_t\partial_x v$ to both sides of (54) and integrating gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|^2_{H^1(\mathbb{R})}\\ &= \int_{\mathbb{R}} v(1-\partial_x^2)E dx\\ &- \gamma\int_{\mathbb{R}} v(1-\partial_x^2)(v\partial_x u^{\omega,\lambda} + u^{\omega,\lambda}\partial_x v)dx\\ &- \int_{\mathbb{R}}\left[\left(3-\gamma\right)v\partial_x\left(u^{\omega,\lambda}v\right) + \gamma v\partial_x\left(\partial_x u^{\omega,\lambda}\partial_x v\right)\right]dx\\ &+ \int_{\mathbb{R}}\left[\gamma v\left(1-\partial_x^2\right)\left(v\partial_x v\right) + v\partial_x\left(\frac{3-\gamma}{2}v^2 + \frac{\gamma}{2}\left(\partial_x v\right)^2\right) + v\partial_x^2\partial_t v + \partial_x v\partial_t\partial_x v\right]dx.\end{aligned} \tag{55}$$

Noting that the last integral can be rewritten as

$$\int_{\mathbb{R}}\left[\partial_x(v^3) - \gamma\partial_x(v^2\partial_x^2 v) + \partial_x\left(v\partial_t\partial_x v\right)\right]dx = 0$$

we can simplify (55) to obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v\|_{H^1(\mathbb{R})}^2 = & \int_{\mathbb{R}} v(1-\partial_x^2)E dx \\
& -\gamma \int_{\mathbb{R}} v(1-\partial_x^2)(v\partial_x u^{\omega,\lambda} + u^{\omega,\lambda}\partial_x v) dx \\
& -\int_{\mathbb{R}} \left[(3-\gamma)\, v\partial_x \left(u^{\omega,\lambda} v\right) + \gamma v \partial_x \left(\partial_x u^{\omega,\lambda}\partial_x v\right)\right] dx.
\end{aligned} \tag{56}
$$

We now estimate the three integrals on the right-hand side of (56). Integrating by parts and applying Cauchy-Schwartz, we obtain

$$
\left|\int_{\mathbb{R}} \left[v(1-\partial_x^2)E\right] dx\right| \lesssim \|v\|_{H^1(\mathbb{R})}\|E\|_{H^1(\mathbb{R})} \tag{57}
$$

for the first integral. Applying Parseval and Hölder gives

$$
\begin{aligned}
& \left|-\gamma \int_{\mathbb{R}} \left[v\left(1-\partial_x^2\right)\left(v\partial_x u^{\omega,\lambda} + u^{\omega,\lambda}\partial_x v\right)\right] dx\right| \\
& \lesssim \left(\|u^{\omega,\lambda}\|_{L^\infty(\mathbb{R})}\| + \|\partial_x u^{\omega,\lambda}\|_{L^\infty(\mathbb{R})} + \|\partial_x^2 u^{\omega,\lambda}\|_{L^\infty(\mathbb{R})}\right)\|v\|_{H^1(\mathbb{R})}^2
\end{aligned} \tag{58}
$$

for the second integral, and applying Hölder gives

$$
\begin{aligned}
& \left|-\int_{\mathbb{R}} \left[(3-\gamma)\, v\partial_x \left(u^{\omega,\lambda} v\right) + \gamma v\partial_x \left(\partial_x u^{\omega,\lambda}\partial_x v\right)\right] dx\right| \\
& \lesssim \left(\|u^{\omega,\lambda}\|_{L^\infty(\mathbb{R})} + \|\partial_x u^{\omega,\lambda}\|_{L^\infty(\mathbb{R})}\right)\|v\|_{H^1(\mathbb{R})}^2
\end{aligned} \tag{59}
$$

for the third integral. Combining (57)-(59), we obtain

$$
\begin{aligned}
\frac{d}{dt}\|v(t)\|_{H^1(\mathbb{R})}^2 \lesssim & \left(\|u^{\omega,\lambda}\|_{L^\infty(\mathbb{R})} + \|\partial_x u^{\omega,\lambda}\|_{L^\infty(\mathbb{R})} + \|\partial_x^2 u^{\omega,\lambda}\|_{L^\infty(\mathbb{R})}\right)\|v\|_{H^1(\mathbb{R})}^2 \\
& + \|v\|_{H^1(\mathbb{R})}\|E\|_{H^1(\mathbb{R})}.
\end{aligned} \tag{60}
$$

A straightforward calculation of derivatives yields

$$\|u^h\|_{L^\infty(\mathbb{R})} + \|\partial_x u^h\|_{L^\infty(\mathbb{R})} + \|\partial_x^2 u^h\|_{L^\infty(\mathbb{R})} \lesssim \lambda^{-\frac{\delta}{2}-s+2}.$$

Furthermore, by the Sobolev Imbedding Theorem and Lemma 6, we have

$$\|u_\ell\|_{L^\infty(\mathbb{R})} + \|\partial_x u_\ell\|_{L^\infty(\mathbb{R})} + \|\partial_x^2 u_\ell\|_{L^\infty(\mathbb{R})} \le c_s \|u_\ell\|_{H^3(\mathbb{R})} \lesssim \lambda^{-1+\frac{\delta}{2}}, \quad |t| \le T.$$

Hence

$$\|u^{\omega,\lambda}\|_{L^\infty(\mathbb{R})} + \|\partial_x u^{\omega,\lambda}\|_{L^\infty(\mathbb{R})} + \|\partial_x^2 u^{\omega,\lambda}\|_{L^\infty(\mathbb{R})} \lesssim \lambda^{-\rho_s}, \quad |t| \le T \tag{61}$$

where $\rho_s = \min\left\{\frac{\delta}{2} + s - 2,\ 1 - \frac{\delta}{2}\right\}$. Note that for $s > 1$, we can assure $\rho_s > 0$ by choosing a suitable $1 < \delta < 2$. Substituting (20) and (61) into (60), we get

$$\frac{d}{dt}\|v(t)\|^2_{H(\mathbb{R})} \lesssim \lambda^{-\rho_s}\|v\|^2_{H^1(\mathbb{R})} + \lambda^{-r_s}\|v\|_{H^1(\mathbb{R})}, \quad |t| \le T. \tag{62}$$

Applying Gronwall's inequality completes the proof. □

1.2.3 Non-Uniform Dependence for $s > 3/2$

Let $u_{\pm 1,\lambda}$ be solutions to the HR IVP with initial data $u^{\pm 1,n}(0)$. We wish to show that the H^s norm of the difference of $u_{\pm 1,n}$ and the associated approximate solution $u^{\pm 1,\lambda}$ decays as $\lambda \to \infty$. Note that

$$\begin{aligned}\|u^{\pm 1,\lambda}(t)\|_{H^{2s-1}(\mathbb{R})} &\le \|u_{\ell,\pm 1,\lambda}\|_{H^{2s-1}(\mathbb{R})} + \|\lambda^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{\lambda^\delta}\right)\cos(\lambda x \mp \gamma\omega t)\|_{H^{2s-1}(\mathbb{R})} \\ &\lesssim \lambda^{s-1}, \quad |t| \le T\end{aligned}$$

where the last step follows from Lemma 4 and Lemma 6. Using (4), we have

$$\|u_{\pm 1,\lambda}(t)\|_{H^{2s-1}(\mathbb{R})} \lesssim \|u^{\pm 1,\lambda}(0)\|_{H^{2s-1}(\mathbb{R})}, \quad |t| \leq T.$$

Hence

$$\|u^{\pm 1,\lambda}(t) - u_{\pm 1,\lambda}(t)\|_{H^{2s-1}(\mathbb{R})} \lesssim \lambda^{s-1}, \quad |t| \leq T. \tag{63}$$

Furthermore, by Lemma 7

$$\|u^{\pm 1,\lambda}(t) - u_{\pm 1,\lambda}\|_{H^{1}(\mathbb{R})} \lesssim \lambda^{\frac{\delta}{2}-s}, \quad |t| \leq T. \tag{64}$$

Interpolating between estimates (63) and (64) using the inequality

$$\|\psi\|_{H^s(\mathbb{R})} \leq \left(\|\psi\|_{H^1(\mathbb{R})}\|\psi\|_{H^{2s-1}(\mathbb{R})}\right)^{\frac{1}{2}}$$

gives

$$\|u^{\pm 1,\lambda}(t) - u_{\pm 1,\lambda}(t)\|_{H^{s}(\mathbb{R})} \lesssim \lambda^{\frac{\delta-2}{4}}, \quad |t| \leq T. \tag{65}$$

Next, we will use estimate (65) to prove non-uniform dependence when $s > 3/2$.

1.2.3.1 Behavior at time $t = 0$

We have

$$\|u_{1,\lambda}(0) - u_{-1,\lambda}(0)\|_{H^s(\mathbb{R})} = \|u^{1,\lambda}(0) - u^{-1,\lambda}(0)\|_{H^s(\mathbb{R})} = 2\lambda^{-1}\|\tilde{\phi}\left(\frac{x}{\lambda^{\delta}}\right)\|_{H^s(\mathbb{R})}.$$

Applying (18) and recalling that $1 < \delta < 2$, we conclude that

$$\|u_{1,\lambda}(0) - u_{-1,\lambda}(0)\|_{H^s(\mathbb{R})} \leq 2\lambda^{\frac{\delta}{2}-1}\|\tilde{\phi}\|_{H^s(\mathbb{R})} \to 0 \ \text{ as } \ \lambda \to \infty. \tag{66}$$

1.2.3.2 Behavior at time $t > 0$

Using the reverse triangle inequality, we have

$$\begin{aligned}\|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s(\mathbb{R})} \geq\ & \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s(\mathbb{R})} \\ & - \|u^{1,\lambda}(t) - u_{1,\lambda}(t)\|_{H^s(\mathbb{R})} \\ & - \| - u^{-1,\lambda}(t) + u_{-1,\lambda}(t)\|_{H^s(\mathbb{R})}.\end{aligned} \tag{67}$$

Using estimate (65) for the last two terms of the right-hand side of (67) and letting $\lambda \to \infty$ yields

$$\liminf_{n\to\infty} \|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s(\mathbb{R})} \geq \liminf_{n\to\infty} \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s(\mathbb{R})}. \tag{68}$$

Using the identity

$$\cos\alpha - \cos\beta = -2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$$

gives

$$u^{1,\lambda}(t) - u^{-1,\lambda}(t) = u_{\ell,1,\lambda}(t) - u_{\ell,-1,\lambda}(t) + 2\lambda^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{\lambda^\delta}\right)\sin(\lambda x)\sin(\gamma t). \tag{69}$$

Hence, applying the reverse triangle inequality to (69), we obtain

$$\begin{aligned}&\|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s(\mathbb{R})} \\ &\geq 2\lambda^{-\frac{\delta}{2}-s}\|\phi\left(\frac{x}{\lambda^\delta}\right)\sin(\lambda x)\|_{H^s(\mathbb{R})}|\sin\gamma t| - \|u_{\ell,-1,\lambda}(t) - u_{\ell,1,\lambda}(t)\|_{H^s(\mathbb{R})}.\end{aligned} \tag{70}$$

Letting $\lambda \to \infty$, we see that Lemma 4, Lemma 6, and (70) give

$$\liminf_{\lambda\to\infty} \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s(\mathbb{R})} \gtrsim |\sin \gamma t|, \quad |t| \leq T. \tag{71}$$

This completes the proof of Theorem 1 for the non-periodic case. $\square$

1.3 Proof of Non-Uniform Dependence on the Circle

Here we follow the proof in [16]. We will show that there there exist two sequences of solutions $u_n(t)$ and $v_n(t)$ in $C([-T,T], H^s(\mathbb{T}))$ such that

$$\|u_n(t)\|_{H^s(\mathbb{T})} + \|v_n(t)\|_{H^s(\mathbb{T})} \lesssim 1, \tag{72}$$

$$\lim_{n\to\infty} \|u_n(0) - v_n(0)\|_{H^s(\mathbb{T})} = 0, \tag{73}$$

and

$$\liminf_{n\to\infty} \|u_n(t) - v_n(t)\|_{H^s(\mathbb{T})} \gtrsim |\sin(\gamma t)|, \quad |\gamma t| \leq 1. \tag{74}$$

As in the non-periodic case, we will first construct two sequences of approximate solutions satisfying the above properties. Then, we will construct two sequences of actual solutions coinciding with the approximate solutions at time zero such that for small time the size of the difference of solutions and approximate solutions decays as $n \to \infty$.

Again, we rewrite the Cauchy problem for the HR equation in the following non-local form

$$\partial_t u = -\gamma u \partial_x u - \Lambda^{-1}\left[\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}(\partial_x u)^2\right], \tag{75}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{T}, \quad t \in \mathbb{R}. \tag{76}$$

1.3.1 Approximate Solutions

Following [16], we shall now take approximate solutions of the form

$$u^{\omega,n}(x,t) = \omega n^{-1} + n^{-s}\cos\left(nx - \gamma\omega t\right), \tag{77}$$

where n is a positive integer and ω is in a bounded subset of $\mathbb{R}$. We remark that the approximate solutions are in $C^\infty(\mathbb{T})$ for all $t \in \mathbb{R}$, and hence have infinite lifespan in $H^s(\mathbb{T})$ for $s \geq 0$. Furthermore, we have

$$\|u^{\omega,n}\|_{H^s(\mathbb{T})} \sim 1 \tag{78}$$

from the inequality

$$\|\cos(k(nx-c))\|_{H^s(\mathbb{T})} \simeq n^s, \quad k \in \mathbb{R}\setminus\{0\}. \tag{79}$$

Note that for $\gamma = 1$ one gets the approximate solutions used for the CH equation in [16]. Substituting the approximate solutions into (75), we obtain the error

$$E = E_1 + E_2 + E_3 \tag{80}$$

where

$$E_1 = -\frac{\gamma}{2} n^{-2s+1} \sin\left[2\left(nx - \gamma\omega t\right)\right], \tag{81}$$

$$E_2 = -\Lambda^{-1}\left[\frac{3-\gamma}{2}\left(n^{-2s+1}\sin\left(2(nx-\gamma\omega t) + 2\omega n^{-s}\sin(2(nx-\gamma\omega t)\right)\right)\right], \tag{82}$$

$$E_3 = \frac{\gamma}{4} n^{-2s+2}\left[1 - \cos\left(\frac{nx - \gamma\omega t}{2}\right)\right]. \tag{83}$$

Next we will prove that the error decays as $n \to \infty$:

Lemma 8. *Let $u^{\omega,n}$ be an approximate solution to the HR IVP, with $\sigma \leq 1$ and ω in a bounded set. Then*

$$\|E(t)\|_{H^\sigma(\mathbb{T})} \lesssim n^{-r_s} \quad \textit{where} \quad r_s = \begin{cases} 2(s-1) & \textit{if} \quad s \leq 3, \\ s+1 & \textit{if} \quad s > 3. \end{cases} \tag{84}$$

Proof. It follows from the inequality

$$\|\Lambda^{-1} f\|_{H^k(\mathbb{T})} \leq \|f\|_{H^{k-1}(\mathbb{T})}$$

and (79). □

1.3.2 Construction of Solutions

We wish now to estimate the difference between approximate and actual solutions to the HR IVP with common initial data. Let $u_{\omega,n}(x,t)$ be the unique solution to the HR equation with initial data $u^{\omega,n}(x,0)$. That is, $u_{\omega,n}$ solves the initial value problem

$$\partial_t u_{\omega,n} = -\gamma u_{\omega,n}\partial_x u_{\omega,n} - n^{-1}\left[\frac{3-\gamma}{2}\left(u_{\omega,n}\right)^2 + \frac{\gamma}{2}\left(\partial_x u_{\omega,n}\right)^2\right],$$

$$u_{\omega,n}(x,0) = u^{\omega,n}(x,0) = \omega n^{-1}.$$

We will now prove that the size of the difference decays as $n \to \infty$:

Proposition 9. *Let $v = u^{\omega,n} - u_{\omega,n}$. If $s > 3/2$ and $\sigma = 1/2 + \varepsilon$ for a sufficiently small $\varepsilon = \varepsilon(s) > 0$, then*

$$\|v(t)\|_{H^\sigma(\mathbb{T})} \lesssim n^{-r_s}, \quad |t| \le T. \tag{85}$$

Proof. The difference $v = u^{\omega,n} - u_{\omega,n}$ satisfies the IVP

$$\begin{aligned}
\partial_t v = E - \frac{\gamma}{2}\partial_x \left[\left(u^{\omega,n} + u_{\omega,n} \right) v \right] \\
- \Lambda^{-1} \left[\frac{3-\gamma}{2} \left(u^{\omega,n} + u_{\omega,n} \right) v + \frac{\gamma}{2} \left(\partial_x u^{\omega,n} + \partial_x u_{\omega,n} \right) \partial_x v \right],
\end{aligned} \tag{86}$$

$$v(x,0) = 0. \tag{87}$$

For any $\sigma \in \mathbb{R}$ let $D^\sigma = (1 - \partial_x^2)^{\sigma/2}$ be the operator defined by

$$\widehat{D^\sigma f}(\xi) \doteq (1+\xi^2)^{\sigma/2} \widehat{f}(\xi),$$

where $\widehat{f}$ is the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{T}} e^{-i\xi x} f(x)\ dx.$$

Applying D^σ to both sides of (86), multiplying by $D^\sigma v$, and integrating, we obtain the relation

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{H^\sigma(\mathbb{T})} = & \int_{\mathbb{T}} D^\sigma E \cdot D^\sigma v\ dx \\
& - \frac{\gamma}{2} \int_{\mathbb{T}} D^\sigma \partial_x \left[\left(u^{\omega,n} + u_{\omega,n} \right) v \right] \cdot D^\sigma v\ dx \\
& - \frac{3-\gamma}{2} \int_{\mathbb{T}} D^{\sigma-2} \partial_x \left[\left(u^{\omega,n} + u_{\omega,n} \right) v \right] \cdot D^\sigma v\ dx \\
& - \frac{\gamma}{2} \int_{\mathbb{T}} D^{\sigma-2} \partial_x \left[\left(\partial_x u^{\omega,n} + \partial_x u_{\omega,n} \right) \cdot \partial_x v \right] \cdot D^\sigma v\ dx.
\end{aligned} \tag{88}$$

We now estimate each integral of the right-hand side of (88). For the first integral, we applying Cauchy-Schwartz and obtain

$$\left|\int_{\mathbb{T}} D^\sigma E \cdot D^\sigma v \; dx\right| \leq \|E\|_{H^\sigma(\mathbb{T})}\|v\|_{H^\sigma(\mathbb{T})}. \tag{89}$$

For the second integral, we rewrite

$$\begin{aligned} -\frac{\gamma}{2}\int_{\mathbb{T}} D^\sigma \partial_x \left[\left(u^{\omega,n} + u_{\omega,n}\right) v\right] \cdot D^\sigma v \; dx = & -\frac{\gamma}{2}\int_{\mathbb{T}} \left[D^\sigma \partial_x, u^{\omega,n} + u_{\omega,n}\right] v \cdot D^\sigma v \; dx \\ & -\frac{\gamma}{2}\int_{\mathbb{T}} (u^{\omega,n} + u_{\omega,n}) D^\sigma \partial_x v \cdot D^\sigma v \; dx. \end{aligned} \tag{90}$$

We now estimate (90). Integration by parts and Cauchy-Schwartz gives

$$\left|\frac{\gamma}{2}\int_{\mathbb{T}} (u^{\omega,n} + u_{\omega,n}) D^\sigma \partial_x v \cdot D^\sigma v \; dx\right| \lesssim \|\partial_x(u^{\omega,n} + u_{\omega,n})\|_{L^\infty(\mathbb{T})}\|v\|^2_{H^\sigma(\mathbb{T})}. \tag{91}$$

We now need the following result taken from [16]:

Lemma 10. *If $\rho > 3/2$ and $0 \leq \sigma + 1 \leq \rho$, then*

$$\|[D^\sigma \partial_x, f]v\|_{L^2} \leq C\|f\|_{H^\rho}\|v\|_{H^\sigma}. \tag{92}$$

Let $\sigma = 1/2 + \varepsilon$ and $\rho = 3/2 + \varepsilon$, where $\varepsilon > 0$ is arbitrarily small. Applying Cauchy-Schwartz and Lemma 10, we obtain

$$\left|-\frac{\gamma}{2}\int_{\mathbb{T}} [D^\sigma \partial_x, u^{\omega,n} + u_{\omega,n}] v \cdot D^\sigma v \; dx\right| \lesssim \|u^{\omega,n} + u_{\omega,n}\|_{H^\rho(\mathbb{T})}\|v\|^2_{H^\sigma(\mathbb{T})}. \tag{93}$$

Combining estimates (91) and (93) we conclude that

$$\begin{aligned}&\left|-\frac{\gamma}{2}\int_{\mathbb{T}} D^{\sigma}\partial_x\left[\left(u^{\omega,n}+u_{\omega,n}\right)v\right]\cdot D^{\sigma}v\ dx\right|\\&\lesssim \left(\|u^{\omega,n}+u_{\omega,n}\|_{H^{\rho}(\mathbb{T})}+\|\partial_x u^{\omega,n}+\partial_x u_{\omega,n}\|_{L^{\infty}(\mathbb{T})}\right)\cdot\|v\|^2_{H^{\sigma}(\mathbb{T})}.\end{aligned} \tag{94}$$

For the third integral, we use Cauchy-Schwartz and recall that $\sigma = 1/2+\varepsilon$ to obtain

$$\left|-\frac{3-\gamma}{2}\int_{\mathbb{T}} D^{\sigma-2}\partial_x\left[(u^{\omega,n}+u_{\omega,n})v\right]\cdot D^{\sigma}v\ dx\right|\lesssim \|u^{\omega,n}+u_{\omega,n}\|_{L^{\infty}(\mathbb{T})}\|v\|^2_{H^{\sigma}(\mathbb{T})}. \tag{95}$$

For the fourth integral, we will need the following result.

Lemma 11. *For $s>3/2$, $r\le s$, $s+r\ge 2$, we have*

$$\|fg\|_{H^{r-1}}\lesssim \|f\|_{H^{r-1}}\|g\|_{H^{s-1}}. \tag{96}$$

Proof. For the periodic case we have

$$\|fg\|^2_{H^{r-1}}\le \sum_{n\in\mathbb{Z}}(1+n^2)^{r-1}\left[\sum_{k\in\mathbb{Z}}|\widehat{f}(k)||\widehat{g}(n-k)|(1+k^2)^{\frac{1-s}{2}}(1+k^2)^{\frac{s-1}{2}}\right]^2.$$

Applying Cauchy Schwartz in k, we bound this by

$$\|f\|^2_{H^{s-1}}\sum_{n\in\mathbb{Z}}(1+n^2)^{r-1}\sum_{k\in\mathbb{Z}}\frac{|\widehat{g}(n-k)|^2}{(1+k^2)^{s-1}}. \tag{97}$$

But by change of variables and Fubini

$$\sum_{n\in\mathbb{Z}}(1+n^2)^{r-1}\sum_{k\in\mathbb{Z}}\frac{|\widehat{g}(n-k)|^2}{(1+k^2)^{s-1}}=\sum_{k\in\mathbb{Z}}|\widehat{g}(k)|^2\sum_{n\in\mathbb{Z}}\frac{1}{(1+n^2)^{s-1}[1+(n-k)^2]^{1-r}}. \tag{98}$$

Without loss of generality, we assume $k \geq 0$ and write

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \frac{1}{(1+n^2)^{s-1}[1+(n-k)^2]^{1-r}} \\
&= \sum_{0\leq n\leq 2k} \frac{1}{(1+n^2)^{s-1}[1+(n-k)^2]^{1-r}} + \sum_{n>2k} \frac{1}{(1+n^2)^{s-1}[1+(n-k)^2]^{1-r}} \\
&+ \sum_{n\geq 0} \frac{1}{(1+n^2)^{s-1}[1+(n+k)^2]^{1-r}} \\
&\doteq I + II + III.
\end{aligned}$$

We have the estimate

$$\begin{aligned}
II &\leq \sup_{n>2k} \frac{1}{[1+(n-k)^2]^{1-r}} \sum_{n>2k} \frac{1}{(1+n^2)^{s-1}} \\
&\lesssim (1+k^2)^{r-1}, \quad s > 3/2.
\end{aligned} \tag{99}$$

Similarly

$$III \lesssim (1+k^2)^{r-1}, \quad s > 3/2.$$

To estimate I, we assume without loss of generality that k is even and write

$$\begin{aligned}
I &= \sum_{0\leq n\leq k/2} \frac{1}{(1+n^2)^{s-1}[1+(n-k)^2]^{1-r}} + \sum_{k/2<n\leq 3k/2} \frac{1}{(1+n^2)^{s-1}[1+(n-k)^2]^{1-r}} \\
&+ \sum_{3k/2<n\leq 2k} \frac{1}{(1+n^2)^{s-1}[1+(n-k)^2]^{1-r}} \\
&\doteq i + ii + iii.
\end{aligned}$$

Hence, estimating as in (99), we have

$$i, iii \lesssim (1+k^2)^{r-1}, \quad s > 3/2$$

and

$$ii \le \sup_{k/2\le n\le 3k/2} \frac{1}{(1+n^2)^{s-1}} \sum_{k/2\le n\le 3k/2} \frac{1}{[1+(n-k)^2]^{1-r}}$$
$$\lesssim \frac{1}{(1+k^2)^{s-1}}, \quad r\le 1/2.$$

Therefore,

$$I + II + III \lesssim (1+k^2)^{1-s} + (1+k^2)^{r-1}, \quad r\le 1/2,\ s>3/2$$
$$\lesssim (1+k^2)^{r-1}, \quad r-1\ge 1-s.$$

Applying this estimate to (98) and recalling (97), we obtain

$$\|fg\|_{H^{r-1}} \lesssim \|f\|_{H^{s-1}}\|g\|_{H^{r-1}}, \quad s>3/2,\ r\le 1/2,\ s+r\ge 2. \tag{100}$$

We now need the following result taken from Taylor [33].

Lemma 12 (Sobolev Interpolation)**.** *For fixed $j\le k, m\le n$ suppose that $T: H^j\to H^m$ continuously and $T: H^k\to H^n$. Then $T: H^{\theta j+(1-\theta)k}\to H^{\theta m+(1-\theta)n}$ continuously for all $\theta\in(0,1]$.*

To apply Lemma 12, we note that (100) and the algebra property of the Sobolev space H^t, $t>1/2$ imply that for $s>3/2$

$$\|fg\|_{H^{r-1}} \lesssim \|g\|_{H^{r-1}},\ \text{where } r=1/2 \text{ or } r=s,\ \|f\|_{H^{s-1}}=1.$$

That is, for fixed $f\in H^{s-1}$ with $\|f\|_{H^{s-1}}=1$, the map $g\mapsto Tg=fg$ is linear continuous from $H^{-1/2}$ to $H^{-1/2}$ and from H^{s-1} to H^{s-1}. Therefore, by Lemma 12, it is continuous from $H^{\theta(s-1)+(1-\theta)(-1/2)}$ to $H^{\theta(s-1)+(1-\theta)(-1/2)}$ for all $\theta\in[0,1)$. Setting $\theta=(r-1/2)/(s-1/2)$, $1/2\le r<s$, we obtain that T is continuous from H^{r-1} to

H^{r-1}. Since T is also linear from H^{r-1} to H^{r-1}, we see that

$$\|fg\|_{H^{r-1}} \lesssim \|g\|_{H^{r-1}}, \quad 1/2 \leq r \leq s,\ s > 3/2,\ \|f\|_{H^{s-1}} = 1$$

and so for general $f \in H^{s-1}$ we have

$$\|fg\|_{H^{r-1}} \lesssim \|f\|_{H^{s-1}} \|g\|_{H^{r-1}}, \quad 1/2 \leq r \leq s,\ s > 3/2. \tag{101}$$

Combining (100) and (101) completes the proof in the periodic case. For the non-periodic case we have

$$\|fg\|_{H^{r-1}}^2 \leq \int_{\mathbb{R}} (1+\xi^2)^{r-1} \left[\int_{\mathbb{R}} |\widehat{f}(\eta)||\widehat{g}(\xi-\eta)|(1+\eta^2)^{\frac{1-s}{2}} (1+\eta^2)^{\frac{s-1}{2}} d\eta \right]^2 d\xi.$$

Applying Cauchy Schwartz in η, we bound this by

$$\|f\|_{H^{s-1}}^2 \int_{\mathbb{R}} (1+\xi^2)^{r-1} \int_{\mathbb{R}} \frac{|\widehat{g}(\xi-\eta)|^2}{(1+\eta^2)^{s-1}} d\eta d\xi.$$

We now wish to bound the integral term. Applying a change of variable, we see it is equal to

$$\int_{\mathbb{R}} (1+\xi^2)^{r-1} \int_{\mathbb{R}} \frac{|\widehat{g}(\eta)|^2}{[1+(\xi-\eta)^2]^{s-1}} d\eta d\xi$$

which by Fubini is equal to

$$\begin{aligned} &\int_{\mathbb{R}} |\widehat{g}(\eta)|^2 \int_{\mathbb{R}} \frac{1}{[1+(\xi-\eta)^2]^{s-1} (1+\xi^2)^{1-r}} d\xi d\eta \\ &\lesssim \int_{\mathbb{R}} |\widehat{g}(\eta)|^2 \int_{\mathbb{R}} \frac{1}{[1+|\xi-\eta|]^{2(s-1)} (1+|\xi|)^{2(1-r)}} d\xi d\eta. \end{aligned} \tag{102}$$

We now need the following lemma:

Lemma 13. *Fix $p, q > 0$ such that $p + q > 1$, and let $r = \min\{p - \varepsilon_q, q - \varepsilon_p, p + q - 1\}$, where $\varepsilon_j > 0$ is arbitrarily small for $j = 1$ and $\varepsilon_j = 0$ for $j \neq 1$. Adopt the notation $\langle x - \alpha \rangle \doteq 1 + |x - \alpha|$. Then*

$$\int_{\mathbb{R}} \frac{1}{\langle x-\alpha\rangle^p \langle x-\beta\rangle^q} dx \le \frac{c_r}{\langle \alpha - \beta \rangle^r}.$$

To be able to apply Lemma 13 to the integral term in (102), we must first check that its conditions are met. Let $s = 3/2 + \varepsilon$, $r = 1 - \delta$, $\varepsilon > 0$, $\delta \geq 0$ and observe that

$$\begin{aligned} 2(s-1) + 2(1-r) &= 2(s-r) \\ &= 2[3/2 + \varepsilon - (1-\delta)] \\ &= 2(1/2 + \varepsilon + \delta) \\ &= 1 + 2\varepsilon + 2\delta > 1. \end{aligned}$$

Furthermore, $2(s-1), 2(1-r) > 0$. Hence, Lemma 13 is applicable. Note that since $s > 3/2$, we see that $2(s-1) \neq 1$. However, it is possible that $2(1-r) = 1$; hence we must now separate the cases $r \neq 1/2$ and $r = 1/2$. Suppose $r \neq 1/2$. Then

$$\begin{aligned} \min\{2(s-1), 2(1-r), 2(s-1) + 2(1-r) - 1\} &= \min\{1 + 2\varepsilon, 2\delta, 2\varepsilon + 2\delta\} \\ &= \min\{1 + 2\varepsilon, 2\delta\} \\ &= 2\delta, \quad \delta \le 1/2 + \varepsilon. \end{aligned}$$

If $r = 1/2$, then since $s > 3/2$, we can choose $\eta > 0$ sufficiently small such that

$$\begin{aligned}\min\{2(s-1)-\eta, 2(1-r), 2(1-r)+2(s-1)-1\} &= 1 \\ &= 2(1-r) \\ &= 2\delta.\end{aligned}$$

Hence, for $0 \leq \delta \leq 1/2 + \varepsilon$, $\varepsilon > 0$, (102) is bounded by

$$\begin{aligned}C_{s,r}\int_{\mathbb{R}} |\widehat{g}(\eta)|^2 \int_{\mathbb{R}} \frac{1}{(1+|\eta|)^{2\delta}} d\xi d\eta &\lesssim \|g\|_{H^{-\delta}}^2 \\ &= \|g\|_{H^{r-1}}^2.\end{aligned}$$

Our restriction on δ is equivalent to the restriction

$$1 - r \leq 1/2 + s - 3/2, \quad r \leq 1, \ s > 3/2,$$

or

$$s + r \geq 2, \quad r \leq 1, \ s > 3/2.$$

Therefore,

$$\|fg\|_{H^{r-1}} \lesssim \|f\|_{H^{s-1}} \|g\|_{H^{r-1}}, \quad s + r \geq 2, \ s > 3/2, \ r \leq 1.$$

The remainder of the proof is analogous to that in the periodic case. $\square$

Proof of Lemma 13. By the change of variable $x \mapsto x/2 + (\alpha+\beta)/2$, we have

$$\begin{aligned}
\int_{\mathbb{R}} \frac{1}{\langle x-\alpha\rangle^p \langle x-\beta\rangle^q} dx &\simeq \int_{\mathbb{R}} \frac{1}{\langle x/2-(\alpha-\beta)/2\rangle^p \langle x/2+(\alpha-\beta)/2\rangle^q} dx \\
&\lesssim \int_{\mathbb{R}} \frac{1}{\langle x-(\alpha-\beta)\rangle^p \langle x+(\alpha-\beta)\rangle^q} dx \\
&= \int_{\mathbb{R}} \frac{1}{\langle a-x\rangle^p \langle a+x\rangle^q} dx, \quad a = \alpha-\beta
\end{aligned} \tag{103}$$

which for $a = 0$ reduces to

$$\begin{aligned}
\int_{\mathbb{R}} \frac{1}{\langle x\rangle^{p+q}} dx &= 2\int_0^\infty \frac{1}{(1+x)^{p+q}} dx \\
&= \frac{2}{p+q-1}.
\end{aligned}$$

We now handle the case $a \neq 0$. Note that by the change of variable $x \mapsto -x$ we may restrict our attention to the case $a > 0$ without loss of generality. Split

$$\begin{aligned}
\int_{\mathbb{R}} \frac{1}{\langle a+x\rangle^p \langle a-x\rangle^q} dx &= \int_{-2a}^{2a} \frac{1}{\langle a+x\rangle^p \langle a-x\rangle^q} dx \\
&\quad + \int_{|x|\geq 2a} \frac{1}{\langle a+x\rangle^p \langle a-x\rangle^q} dx \\
&= I + II.
\end{aligned}$$

Then

$$I = \int_0^{2a} \frac{1}{\langle a+x\rangle^p \langle a-x\rangle^q} dx + \int_{-2a}^{0} \frac{1}{\langle a+x\rangle^p \langle a-x\rangle^q} dx.$$

We bound the first term by

$$\begin{aligned}
\sup_{0\le x\le 2a}\frac{1}{\langle a+x\rangle^p}\int_0^{2a}\frac{1}{\langle a-x\rangle^q}dx &= \frac{1}{\langle a\rangle^p}\int_0^{2a}\frac{1}{(1+|a-x|)^q}dx\\
&= \frac{2}{\langle a\rangle^p}\int_0^{a}\frac{1}{(1+a-x)^q}dx\\
&\lesssim \begin{cases} 1/\langle a\rangle^p\left|1-1/(1+a)^{q-1}\right|, & q\neq 1\\ \log(1+a)/\langle a\rangle^p, & q=1.\end{cases}
\end{aligned}$$

But

$$\frac{1}{\langle a\rangle^p}\left|1-\frac{1}{(1+a)^{q-1}}\right|\lesssim\begin{cases}1/\langle a\rangle^p, & q>1\\ 1/\langle a\rangle^{p+q-1}, & q<1\end{cases}$$

and

$$\frac{\log(1+a)}{\langle a\rangle^p}\le\frac{c_\varepsilon}{\langle a\rangle^{p-\varepsilon}}\text{ for any }\varepsilon>0.$$

For the second term, we bound by

$$\begin{aligned}
&\sup_{-2a\le x\le 0}\frac{1}{\langle a-x\rangle^q}\int_{-2a}^{0}\frac{1}{\langle a+x\rangle^p}dx\\
&= \frac{1}{\langle a\rangle^q}\int_{-2a}^{0}\frac{1}{(1+|a+x|)^p}dx\\
&= \frac{2}{\langle a\rangle^q}\int_{-a}^{0}\frac{1}{(1+a+x)^p}dx\\
&\lesssim \begin{cases} 1/\langle a\rangle^q\left|1-1/(1+a)^{p-1}\right|, & p\neq 1\\ \log(1+a)/\langle a\rangle^q, & p=1.\end{cases}
\end{aligned}$$

But

$$\frac{1}{\langle a\rangle^q}\left|1-\frac{1}{(1+a)^{p-1}}\right| \lesssim \begin{cases} 1/\langle a\rangle^q, & p>1 \\ 1/\langle a\rangle^{p+q-1}, & p<1 \end{cases}$$

and

$$\frac{\log(1+a)}{\langle a\rangle^q} \le \frac{c_\varepsilon}{\langle a\rangle^{q-\varepsilon}} \text{ for any } \varepsilon>0.$$

Therefore,

$$I \le \frac{c_{p,q,\varepsilon}}{\langle a\rangle^{\min\{p-\varepsilon_q, q-\varepsilon_p, p+q-1\}}}.$$

Also

$$\begin{aligned} II &= \int_{x\ge 2a} \frac{1}{(1+x-a)^p(1+x+a)^q}dx \\ &\le \int_{x\ge 2a} \frac{1}{(1+x-a)^{p+q}}dx \\ &\simeq \frac{1}{\langle a\rangle^{p+q-1}}, \qquad p+q>1. \end{aligned}$$

Collecting our estimates for I and II we see that for $p,q>0$ such that $p+q>1$, and $r=\min\{p-\varepsilon_q, q-\varepsilon_p, p+q-1\}$, we have

$$\int_{\mathbb{R}} \frac{1}{\langle a-x\rangle^p\langle a+x\rangle^q}dx \le \frac{c_r}{\langle a\rangle^r}.$$

Recalling (103), the proof is complete. □

Noting that (96) implies

$$\|fg\|_{H^{\sigma-1}} \le C\|f\|_{H^\sigma}\|g\|_{H^{\sigma-1}}, \quad \sigma>1/2 \tag{104}$$

and applying Cauchy-Schwartz and (104), we obtain

$$\begin{aligned}
&\left| -\frac{\gamma}{2} \int_{\mathbb{T}} D^{\sigma-2} \partial_x \left[(\partial_x u^{\omega,n} + \partial_x u_{\omega,n}) \cdot \partial_x v \right] \cdot D^{\sigma} v \; dx \right| \\
&\lesssim \| \partial_x u^{\omega,n} + \partial_x u_{\omega,n} \|_{H^{\sigma}(\mathbb{T})} \| v \|^2_{H^{\sigma}(\mathbb{T})}.
\end{aligned} \tag{105}$$

Collecting estimates (89), (94), (95), and (105), and applying the Sobolev Imbedding Theorem, we deduce

$$\frac{1}{2} \frac{d}{dt} \| v \|^2_{H^{\sigma}(\mathbb{T})} \lesssim \| u^{\omega,n} + u_{\omega,n} \|_{H^{\rho}(\mathbb{T})} \| v \|^2_{H^{\sigma}(\mathbb{T})} + \| E \|_{H^{\sigma}(\mathbb{T})} \| v \|_{H^{\sigma}(\mathbb{T})}. \tag{106}$$

It follows from (3) and (78) that the solutions $u_{\omega,n}$ have a common lifespan T. Hence, applying the triangle inequality, (4), and (79) we obtain

$$\| u^{\omega,n} + u_{\omega,n} \|_{H^{\rho}(\mathbb{T})} \lesssim n^{\rho - s}, \quad |t| \leq T. \tag{107}$$

Using Lemma 8 and substituting (84) and (107) into (106), we obtain

$$\frac{1}{2} \frac{d}{dt} \| v \|^2_{H^{\sigma}(\mathbb{T})} \lesssim n^{\rho - s} \| v \|^2_{H^{\sigma}(\mathbb{T})} + n^{-r_s} \| v \|_{H^{\sigma}(\mathbb{T})}, \quad |t| \leq T. \tag{108}$$

Applying Gronwall's inequality gives (85), concluding the proof. □

1.3.3 Non-Uniform Dependence for $s > 3/2$

Let $u_{\pm 1,n}$ be solutions to the HR IVP with common initial data $u^{\pm 1,n}(0)$, respectively. We wish to show that the H^s norm of the difference of $u_{\pm 1,n}$ and the associated approximate solution $u^{\pm 1,n}$ decays as $n \to \infty$. We assume $s > 3/2$ and $\sigma = 1/2 + \varepsilon$

for a sufficiently small $\varepsilon = \varepsilon(s) > 0$. Then by Proposition 9 we have

$$\|u^{\pm 1,n}(t) - u_{\pm 1,n}(t)\|_{H^\sigma(\mathbb{T})} \lesssim n^{-r_s}, \quad |t| \le T. \tag{109}$$

Furthermore, by (79) we obtain

$$\|u^{\pm 1,n}(t)\|_{H^{2s-\sigma}(\mathbb{T})} \lesssim n^{s-\sigma} \tag{110}$$

while (4) and (110) give

$$\|u_{\pm 1,n}(t)\|_{H^{2s-\sigma}(\mathbb{T})} \lesssim n^{s-\sigma}, \quad |t| \le T. \tag{111}$$

Therefore, (110), (111), and the triangle inequality yield

$$\|u^{\pm 1,n}(t) - u_{\pm 1,n}(t)\|_{H^{2s-\sigma}(\mathbb{T})} \lesssim n^{s-\sigma}, \quad |t| \le T. \tag{112}$$

Interpolating between estimates (109) and (112) using the inequality

$$\|\psi\|_{H^s(\mathbb{T})} \le \left(\|\psi\|_{H^\sigma(\mathbb{T})} \|\psi\|_{H^{2s-\sigma}(\mathbb{T})}\right)^{\frac{1}{2}}$$

we obtain

$$\|u^{\pm 1,n}(t) - u_{\pm 1,n}(t)\|_{H^s(\mathbb{T})} \lesssim n^{-\varepsilon(s)/2}, \quad |t| \le T. \tag{113}$$

The remainder of the proof of non-uniform dependence on the circle is analogous to that on the real line. $\square$

1.4 Well-Posedness for HR in the Periodic Case

We will now prove well-posedness for the periodic case, after which we will provide the necessary details to extend the argument to the non-periodic case.

1.4.1 Existence

Here we will prove the existence of a solution to the HR IVP and inequalities (3) and (4). We begin by mollifying the HR equation, so that we may apply the following ODE theorem, taken from Dieudonnè [13]:

Theorem 14. *Let Y be a Banach space, $X \subset Y$ be an open subset, $J \subset \mathbb{R}$, and $f : J \times X \to Y$ a continuously differentiable map. Then for any $t_0 \in J$ and $x_0 \in X$ there exists an open ball $I \subset J$ and a unique differentiable mapping $u : I \to Y$ such that for all $t \in I$, $u'(t) = f(t, u)$ and $u(t_0) = x_0$.*

To see why we cannot apply the Banach Space ODE Theorem to the HR equation as is, let $u = x^{-1/2}\chi_{[0,1]}$. Then $u \in L^2$ but $u\partial_x u \notin L^2$. Hence, returning to the general case, we see that the HR equation as is can not be thought as an ODE on the space H^s. To deal with this problem we will replace the HR IVP by

$$\partial_t u_\varepsilon = -\gamma J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon - \partial_x(1-\partial_x^2)^{-1}\left[\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}(\partial_x u)^2\right], \tag{114}$$

$$u_\varepsilon(x,0) = u_0(x), \quad x \in \mathbb{T}, \quad t \in \mathbb{R} \tag{115}$$

where J_ε is the "Friedrichs mollifier" defined by

$$J_\varepsilon f(x) = j_\varepsilon * f(x), \quad \varepsilon > 0. \tag{116}$$

Here

$$j_\varepsilon(x) = \frac{1}{2\pi}\sum_{\xi\in\mathbb{Z}} \widehat{j}(\varepsilon\xi)e^{i\xi x}, \quad \varepsilon > 0 \tag{117}$$

where $\widehat{j}(\xi) \in S(\mathbb{R})$. Notice that the map $f \mapsto J_\varepsilon f$ is bounded from $H^s(\mathbb{T})$ to $H^s(\mathbb{T})$. In order to apply the ODE Theorem, we will also need to show that it is a continuously differentiable map:

Lemma 15. *Let $f_\varepsilon : H^s(\mathbb{T}) \to H^s(\mathbb{T})$ be given by*

$$f_\varepsilon(u) = -\gamma J_\varepsilon u\partial_x J_\varepsilon u - \partial_x(1-\partial_x^2)^{-1}\left[\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}(\partial_x u)^2\right]. \tag{118}$$

Then f_ε is a continuously differentiable map.

Proof. We explicitly calculate the derivative of f_ε at an arbitrary $w \in H^s(\mathbb{T})$:

$$\begin{aligned}[][Df_\varepsilon(u)](w) = &-\gamma(J_\varepsilon w \cdot \partial_x J_\varepsilon u + J_\varepsilon u \cdot \partial_x J_\varepsilon w)\\ &-(1-\partial_x^2)^{-1}\partial_x\left[(3-\gamma)wu + \gamma\partial_x w\partial_x u\right].\end{aligned}$$

Let $w_n \xrightarrow{H^s(\mathbb{T})} w$. Then it is easy to check that

$$\begin{aligned}&-\gamma(J_\varepsilon w_n \cdot \partial_x J_\varepsilon u + J_\varepsilon u \cdot \partial_x J_\varepsilon w_n) + (1-\partial_x^2)^{-1}\partial_x\left[(3-\gamma)w_n u + \gamma\partial_x w_n\partial_x u\right]\\ &\xrightarrow{H^s(\mathbb{T})} -\gamma(J_\varepsilon w \cdot \partial_x J_\varepsilon u + J_\varepsilon u \cdot \partial_x J_\varepsilon w) + (1-\partial_x^2)^{-1}\partial_x\left[(3-\gamma)wu + \gamma\partial_x w\partial_x u\right].\end{aligned} \tag{119}$$

This concludes the proof. ∎

Hence, by Theorem 14, for each $\varepsilon > 0$ there exists a unique solution $u_\varepsilon \in C(I, H^s(\mathbb{T}))$ satisfying the Cauchy-problem (114)-(115). Next, we analyze the size and lifespan of the family $\{u_\varepsilon\}$ of solutions.

1.4.1.1 Energy Estimate for u_ε

We will show that there is a lower bound T for T_ε, which is independent of $\varepsilon \in (0,1]$. This is based on the following differential inequality for the solution u_ε:

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon(t)\|^2_{H^s(\mathbb{T})} \leq c_s\|u_\varepsilon(t)\|^3_{H^s(\mathbb{T})}, \quad |t| \leq T_\varepsilon. \tag{120}$$

We will prove this inequality by following the approach used for quasilinear symmetric hyperbolic systems in Taylor [32]. In what follows we will suppress the t parameter for the sake of clarity. For any $s \in \mathbb{T}$ let $D^s = (1-\partial_x^2)^{s/2}$ be the operator defined by

$$\widehat{D^s f}(\xi) \doteq (1+\xi^2)^{s/2}\widehat{f}(\xi),$$

where $\widehat{f}$ is the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{T}} e^{-i\xi x} f(x)\ dx.$$

Applying the operator D^s to both sides of (114), then multiplying the resulting equation by $D^s J_\varepsilon u_\varepsilon$ and integrating it for $x \in \mathbb{T}$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_\varepsilon\|^2_{H^s} =& -\gamma\int_{\mathbb{T}} D^s(J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon)\cdot D^s J_\varepsilon u_\varepsilon\ dx\\ &-\frac{3-\gamma}{2}\int_{\mathbb{T}} D^{s-2}\partial_x(u_\varepsilon^2)\cdot D^s J_\varepsilon u_\varepsilon\ dx\\ &-\frac{\gamma}{2}\int_{\mathbb{T}} D^{s-2}\partial_x(\partial_x u_\varepsilon)^2\cdot D^s J_\varepsilon u_\varepsilon\ dx.\end{aligned} \tag{121}$$

We will estimate the right hand side of (121) in parts. In what follows next we use the fact that D^s and J_ε commute and that J_ε satisfies the properties

$$(J_\varepsilon f, g)_{L^2(\mathbb{T})} = (f, J_\varepsilon g)_{L^2(\mathbb{T})} \tag{122}$$

and

$$\|J_\varepsilon u\|_{H^s(\mathbb{T})} \leq \|u\|_{H^s(\mathbb{T})}. \tag{123}$$

Letting

$$v = J_\varepsilon u_\varepsilon \tag{124}$$

we have

$$\begin{aligned}
&-\gamma \int_{\mathbb{T}} D^s(J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon) \cdot D^s J_\varepsilon u_\varepsilon \ dx \\
&= -\gamma \int_{\mathbb{T}} D^s(v\partial_x v) \cdot D^s v \ dx \\
&= -\gamma \int_{\mathbb{T}} \left[D^s(v\partial_x v) - vD^s(\partial_x v)\right] D^s v \ dx - \gamma \int_{\mathbb{T}} vD^s(\partial_x v) D^s v \ dx.
\end{aligned} \tag{125}$$

We now estimate (125) in parts. Applying the Cauchy-Schwarz inequality gives

$$\begin{aligned}
&\left| -\gamma \int_{\mathbb{T}} \Big[D^s(v\partial_x v) - vD^s(\partial_x v)\Big] D^s v \, dx \right| \\
&\lesssim \cdot \|D^s(v\partial_x v) - vD^s(\partial_x v)\|_{L^2(\mathbb{T})} \|D^s v\|_{L^2(\mathbb{T})} \\
&= \|D^s(v\partial_x v) - vD^s(\partial_x v)\|_{L^2(\mathbb{T})} \|v\|_{H^s(\mathbb{T})} \\
&\leq c_s \|\partial_x v\|_{L^\infty(\mathbb{T})} \|v\|^2_{H^s(\mathbb{T})},
\end{aligned} \tag{126}$$

where the last step follows from

$$\|D^s(v\partial_x v) - vD^s(\partial_x v)\|_{L^2(\mathbb{T})} \leq c_s \|\partial_x v\|_{L^\infty(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}, \tag{127}$$

which we prove below by using the following Kato-Ponce commutator estimate:

Lemma 16. *[Kato-Ponce] If $s > 0$ then there is $c_s > 0$ such that*

$$\|D^s\big(fg) - fD^s g\|_{L^2(\mathbb{T})} \leq c_s\Big(\|D^s f\|_{L^2(\mathbb{T})}\|g\|_{L^\infty(\mathbb{T})} + \|\partial_x f\|_{L^\infty(\mathbb{T})}\|D^{s-1} g\|_{L^2(\mathbb{T})}\Big). \tag{128}$$

In fact, applying this estimate with $f = v$ and $g = \partial_x v$ gives

$$\begin{aligned}
&\|D^s(v\partial_x v) - vD^s(\partial_x v)\|_{L^2(\mathbb{T})} \\
&\le c_s\Big(\|D^s v\|_{L^2(\mathbb{T})}\|\partial_x v\|_{L^\infty(\mathbb{T})} + \|\partial_x v\|_{L^\infty(\mathbb{T})}\|D^{s-1}\partial_x v\|_{L^2(\mathbb{T})}\Big) \\
&\lesssim c_s\|\partial_x v\|_{L^\infty(\mathbb{T})}\|v\|_{H^s(\mathbb{T})},
\end{aligned} \tag{129}$$

which is the desired estimate (127). Next, we have

$$\begin{aligned}
\left| -\gamma\int_{\mathbb{T}} vD^s(\partial_x v)\cdot D^s v\ dx\right| &\simeq \left|\int_{\mathbb{T}} v\partial_x\left(D^s v\right)^2 dx\right| \\
&\simeq \left|\int_{\mathbb{T}} \partial_x v\left(D^s v\right)^2\ dx\right| \\
&\le \int_{\mathbb{T}} \left|\partial_x v\left(D^s v\right)^2\right|\ dx \\
&\le \|\partial_x v\|_{L^\infty(\mathbb{T})}\|v\|^2_{H^s(\mathbb{T})}.
\end{aligned} \tag{130}$$

Combining inequalities (126) and (130) and applying the Sobolev Imbedding Theorem, we have

$$\begin{aligned}
\left| -\gamma\int_{\mathbb{T}} D^s(v\partial_x v)\cdot D^s v\, dx\right| &\lesssim_s \|\partial_x v\|_{L^\infty(\mathbb{T})}\|v\|^2_{H^s(\mathbb{T})} \\
&\le \|v\|_{C^1(\mathbb{T})}\|v\|^2_{H^s(\mathbb{T})} \\
&\lesssim_s \|v\|^3_{H^s(\mathbb{T})} \\
&\le \|u_\varepsilon\|^3_{H^s(\mathbb{T})}.
\end{aligned} \tag{131}$$

Next we estimate

$$\begin{aligned}
\left|-\frac{3-\gamma}{2}\int_{\mathbb{T}} D^{s-2}\partial_x u_\varepsilon^2\cdot D^s J_\varepsilon u_\varepsilon\ \right| dx &\lesssim \int_{\mathbb{T}}\left|D^{s-2}\partial_x u_\varepsilon^2\cdot D^s J_\varepsilon u_\varepsilon\ dx\right| \\
&\le \|D^{s-2}\partial_x u_\varepsilon^2\|_{L^2(\mathbb{T})}\|D^s J_\varepsilon u_\varepsilon\|_{L^2(\mathbb{T})} \\
&\le \|D^{s-1}u_\varepsilon^2\|_{L^2(\mathbb{T})}\|D^s u_\varepsilon\|_{L^2(\mathbb{T})} \\
&\lesssim \|u_\varepsilon^2\|_{H^s(\mathbb{T})}\|u_\varepsilon\|_{H^s(\mathbb{T})}.
\end{aligned} \tag{132}$$

Applying the algebra property, we obtain

$$\left|-\frac{3-\gamma}{2}\int_{\mathbb{T}} D^{s-2}\partial_x u_\varepsilon^2 \cdot D^s J_\varepsilon u_\varepsilon \; dx\right| \lesssim_s \|u_\varepsilon\|_{H^s(\mathbb{T})}^3. \tag{133}$$

We also have

$$\begin{aligned}\left|-\frac{\gamma}{2}\int_{\mathbb{T}} D^{s-2}\partial_x(\partial_x u_\varepsilon)^2 \cdot D^s J_\varepsilon u_\varepsilon \; dx\right| &\lesssim \int_{\mathbb{T}} \left|D^{s-2}\partial_x(\partial_x u_\varepsilon)^2\right| \cdot |D^s J_\varepsilon u_\varepsilon| \; dx \\ &\leq \|D^{s-1}(\partial_x u_\varepsilon)^2\|_{L^2(\mathbb{T})}\|D^s J_\varepsilon u_\varepsilon\|_{L^2(\mathbb{T})} \\ &\lesssim \|(\partial_x u_\varepsilon)^2\|_{H^{s-1}(\mathbb{T})}\|J_\varepsilon u_\varepsilon\|_{H^{s-1}(\mathbb{T})} \\ &\lesssim \|(\partial_x u_\varepsilon)^2\|_{H^{s-1}(\mathbb{T})}\|u_\varepsilon\|_{H^{s-1}(\mathbb{T})}\end{aligned} \tag{134}$$

and applying the algebra property yields

$$\begin{aligned}\left|-\frac{\gamma}{2}\int_{\mathbb{T}} D^{s-2}(\partial_x u_\varepsilon)^2 \cdot D^s J_\varepsilon u_\varepsilon \; dx\right| &\lesssim_s \|\partial_x u_\varepsilon\|_{H^{s-1}(\mathbb{T})}^2\|u_\varepsilon\|_{H^s(\mathbb{T})} \\ &\leq \|u_\varepsilon\|_{H^s(\mathbb{T})}^3.\end{aligned} \tag{135}$$

Combining (131), (133), and (135), we obtain (120).

1.4.1.2 Lifespan Estimate for u_ε

To derive an explicit formula for T_ε we proceed as follows. Letting $y(t) = \|u_\varepsilon(t)\|_{H^s(\mathbb{T})}^2$ inequality (120) takes the form

$$\frac{1}{2}y^{-3/2}\frac{dy}{dt} \leq c_s, \qquad y(0) = y_0 = \|u_0\|_{H^s(\mathbb{T})}^2. \tag{136}$$

Suppose t is non-negative. Integrating (136) from 0 to t gives

$$\frac{1}{\sqrt{y_0}} - \frac{1}{\sqrt{y(t)}} \leq c_s t. \tag{137}$$

Replacing $y(t)$ with $\|u_\varepsilon(t)\|^2_{H^s(\mathbb{T})}$ and solving for $\|u_\varepsilon(t)\|_{H^s(\mathbb{T})}$ we obtain the formula

$$\|u_\varepsilon(t)\|_{H^s(\mathbb{T})} \leq \frac{\|u_0\|_{H^s(\mathbb{T})}}{1 - c_s\|u_0\|_{H^s(\mathbb{T})}t}, \quad t \geq 0. \tag{138}$$

Now, from (138) we see that $\|u_\varepsilon(t)\|_{H^s(\mathbb{T})}$ is finite if

$$c_s\|u_0\|_{H^s(\mathbb{T})}t < 1,$$

or

$$t < \frac{1}{c_s\|u_0\|_{H^s(\mathbb{T})}}. \tag{139}$$

Similarly, by integrating (136) from $-t$ to 0 gives

$$\|u_\varepsilon(t)\|_{H^s(\mathbb{T})} \leq \frac{\|u_0\|_{H^s(\mathbb{T})}}{1 + c_s\|u_0\|_{H^s(\mathbb{T})}t}, \quad t \leq 0 \tag{140}$$

from which it follows that $\|u_\varepsilon(t)\|_{H^s(\mathbb{T})}$ is finite if

$$t > \frac{-1}{c_s\|u_0\|_{H^s(\mathbb{T})}}. \tag{141}$$

Therefore, the solution $u_\varepsilon(t)$ to the mollified CH Cauchy problem exists for $|t| < T_0$, where

$$T_0 = \frac{1}{c_s\|u_0\|_{H^s(\mathbb{T})}}. \tag{142}$$

1.4.1.3 Uniform Regularity of $\{u_\varepsilon\}$

If we choose $T = \frac{1}{2}T_0$, that is

$$T = \frac{1}{2c_s\|u_0\|_{H^s(\mathbb{T})}}, \tag{143}$$

then for $|t| \leq T$, estimates (138) and (140) imply

$$\|u_\varepsilon(t)\|_{H^s(\mathbb{T})} \leq \frac{\|u_0\|_{H^s(\mathbb{T})}}{1 - (c_s\|u_0\|_{H^s(\mathbb{T})})/(2c_s\|u_0\|_{H^s(\mathbb{T})})},$$

or

$$\|u_\varepsilon(t)\|_{H^s(\mathbb{T})} \leq 2\|u_0\|_{H^s(\mathbb{T})}, \quad |t| \leq T. \tag{144}$$

Thus we have obtained a lower bound for T_ε and an upper bound for $\|u_\varepsilon(t)\|_{H^s(\mathbb{T})}$ independent of $\varepsilon \in (0,1]$. The following lemma summarizes these results and provides an estimate for the $H^{s-1}(\mathbb{T})$ norm of $\partial_t u_\varepsilon(t)$:

Lemma 17. *Let $u_0(x) \in H^s(\mathbb{T})$, $s > 3/2$. Then for any $\varepsilon \in (0,1]$ the IVP for the mollified HR equation*

$$\partial_t u_\varepsilon = -\gamma(J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon) - \partial_x(1-\partial_x^2)^{-1}\left[\frac{3-\gamma}{2}(u_\varepsilon)^2 + \frac{\gamma}{2}(\partial_x u_\varepsilon)^2\right], \tag{145}$$

$$u_\varepsilon(x,0) = u_0(x) \tag{146}$$

has a unique solution $u_\varepsilon(t) \in C([-T,T]; H^s(\mathbb{T}))$. In particular,

$$T = \frac{1}{2c_s\|u_0\|_{H^s(\mathbb{T})}} \tag{147}$$

is independent of ε and is a lower bound for the lifespan of $u_\varepsilon(t)$ and

$$\|u_\varepsilon(t)\|_{H^s(\mathbb{T})} \leq 2\|u_0\|_{H^s(\mathbb{T})}, \quad |t| \leq T. \tag{148}$$

Furthermore, $u_\varepsilon(t) \in C^1([T,T]; H^{s-1}(\mathbb{T}))$ and satisfies

$$\|\partial_t u_\varepsilon(t)\|_{H^{s-1}(\mathbb{T})} \lesssim \|u_0\|^2_{H^s(\mathbb{T})}, \quad |t| \leq T. \tag{149}$$

Here c_s is a constant depending only on s.

Proof. It suffices to prove (149). Using (145), for any $t \in [-T, T]$ we have

$$\begin{aligned}
&\|\partial_t u_\varepsilon(t)\|_{H^{s-1}(\mathbb{T})} \\
&= \| -\gamma(J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon) - \partial_x(1-\partial_x^2)^{-1}\left[\frac{3-\gamma}{2}(u_\varepsilon)^2 + \frac{\gamma}{2}(\partial_x u_\varepsilon)^2\right] \|_{H^{s-1}(\mathbb{T})} \\
&\lesssim \|J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon\|_{H^{s-1}(\mathbb{T})} + \|\partial_x(1-\partial_x^2)^{-1}(u_\varepsilon)^2\|_{H^{s-1}(\mathbb{T})} \\
&+ \|\partial_x(1-\partial_x^2)^{-1}(\partial_x u_\varepsilon)^2\|_{H^{s-1}(\mathbb{T})}.
\end{aligned}$$

We break this into three parts:

$$\begin{aligned}
\|J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon\|_{H^{s-1}(\mathbb{T})} &= \frac{1}{2}\|\partial_x[(J_\varepsilon u_\varepsilon)^2]\|_{H^{s-1}(\mathbb{T})} \\
&\lesssim \|(J_\varepsilon u_\varepsilon)^2\|_{H^s(\mathbb{T})}.
\end{aligned} \tag{150}$$

Applying the algebra property of Sobolev spaces and estimate (148) to (150) gives

$$\begin{aligned}
\|J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon\|_{H^{s-1}(\mathbb{T})} &\lesssim \|J_\varepsilon u_\varepsilon\|_{H^s(\mathbb{T})}^2 \\
&\lesssim \|u_\varepsilon\|_{H^s(\mathbb{T})}^2 \\
&\lesssim \|u_0\|_{H^s(\mathbb{T})}^2.
\end{aligned} \tag{151}$$

We also have

$$\|\partial_x(1-\partial_x^2)^{-1}(u_\varepsilon)^2\|_{H^{s-1}(\mathbb{T})} \le \|(u_\varepsilon)^2\|_{H^{s-1}(\mathbb{T})}$$

which by the algebra property and estimate (148) gives

$$\begin{aligned}
\|\partial_x(1-\partial_x^2)^{-1}(u_\varepsilon)^2\|_{H^{s-1}(\mathbb{T})} &\lesssim \|u_\varepsilon\|_{H^s(\mathbb{T})}^2 \\
&\lesssim \|u_0\|_{H^s(\mathbb{T})}^2.
\end{aligned} \tag{152}$$

Similarly,

$$\begin{aligned}\|\partial_x(1-\partial_x^2)^{-1}(\partial_x u_\varepsilon)^2\|_{H^{s-1}(\mathbb{T})} &\lesssim \|\partial_x u_\varepsilon\|^2_{H^{s-1}(\mathbb{T})}\\ &\lesssim \|u_\varepsilon\|^2_{H^s(\mathbb{T})}\\ &\lesssim \|u_0\|^2_{H^s(\mathbb{T})}.\end{aligned} \tag{153}$$

Combining (151), (152), and (153), we obtain (149). □

1.4.1.4 Choosing a Convergent Subsequence

Next we shall show that the family $\{u_\varepsilon\}$ has a convergent subsequence whose limit u solves the hyperelastic rod IVP. Let $I = [-T, T]$. By Lemma 17 we have

$$\{u_\varepsilon\} \subset C(I, H^s(\mathbb{T})) \cap C^1(I, H^{s-1}(\mathbb{T})) \tag{154}$$

and bounded. Since I is compact, we have

$$\{u_\varepsilon\} \subset L^\infty(I, H^s(\mathbb{T})) \cap C^1(I, H^{s-1}(\mathbb{T})). \tag{155}$$

Now, by the Riesz Lemma, we can identify $H^s(\mathbb{T})$ with $(H^s(\mathbb{T}))^\star$, where for $w, \psi \in H^s(\mathbb{T})$ the duality is defined by

$$T_w(\psi) = < w, \psi >_{H^s(\mathbb{T})} .$$

Hence, by the Riesz Representation Theorem it follows that we can identify $L^\infty(I, H^s(\mathbb{T}))$ with the dual space of $L^1(I, H^s(\mathbb{T})$, where for $v \in L^\infty(I, H^s(\mathbb{T}))$ and

$\phi \in L^1(I, H^s(\mathbb{T}))$ the duality is defined by

$$T_v(\phi) = \int_I < v(t), \phi(t) >_{H^s(\mathbb{R})} dt = \int_I \sum_{\xi \in \mathbb{Z}} \widehat{v}(\xi, t) \overline{\widehat{\phi}}(\xi, t) \cdot (1 + \xi^2)^s dt. \tag{156}$$

Next, we recall Aloaglu's Theorem:

Theorem 18. *If X is a normed vector space, the closed unit ball $B^\star = \{f \in X^\star : \|f\| \leq 1\}$ in $X^\star$ is compact in the weak* topology.*

Therefore the bounded family $\{u_\varepsilon\}$ is compact in the weak* topology of $L^\infty(I, H^s(\mathbb{T}))$. More precisely, there is a sequence $\{u_{\varepsilon_n}\}$ converging weakly to a $u \in L^\infty(I, H^s(\mathbb{T}))$. That is,

$$\lim_{n \to \infty} T_{u_{\varepsilon_n}}(\phi) = T_u(\phi) \quad \text{for all} \quad \phi \in L^1(I, H^s(\mathbb{T})). \tag{157}$$

In order to show that u solves the HR IVP we need to obtain a stronger convergence for u_{ε_n} so that we can take the limit in the mollified HR equation. In fact we will prove that

$$u_{\varepsilon_n} \longrightarrow u \quad \text{in} \;\; C(I, H^{s-\sigma}(\mathbb{T})), \text{ for any } \; 0 < \sigma < 1. \tag{158}$$

For this we will need the following interpolation result:

Lemma 19. *(Interpolation) Let $s > \frac{3}{2}$. If $v \in C(I, H^s(\mathbb{T})) \cap C^1(I, H^{s-1}(\mathbb{T}))$ then $v \in C^\sigma(I, H^{s-\sigma}(\mathbb{T}))$ for $0 < \sigma < 1$.*

Proof. We have

$$\begin{aligned}
&\frac{\|v(t)-v(t')\|^2_{H^{s-\sigma}}}{|t-t'|^{2\sigma}} \\
&= \sum_{\xi\in\mathbb{Z}}(1+\xi^2)^{s-\sigma}\frac{|\hat{v}(\xi,t)-\hat{v}(\xi,t')|^2}{|t-t'|^{2\sigma}}d\xi \\
&= \sum_{\xi\in\mathbb{Z}}(1+\xi^2)^{s}\left(\frac{1}{(1+\xi^2)|t-t'|^2}\right)^{\sigma}|\hat{v}(\xi,t)-\hat{v}(\xi,t')|^2 d\xi \\
&\le \sum_{\xi\in\mathbb{Z}}(1+\xi^2)^{s}\left(1+\frac{1}{(1+\xi^2)|t-t'|^2}\right)|\hat{v}(\xi,t)-\hat{v}(\xi,t')|^2 d\xi \\
&\le \sum_{\xi\in\mathbb{Z}}(1+\xi^2)^{s}|\hat{v}(\xi,t)-\hat{v}(\xi,t')|^2 d\xi + \sum_{\xi\in\mathbb{Z}}(1+\xi^2)^{s-1}\frac{|\hat{v}(\xi,t)-\hat{v}(\xi,t')|^2}{|t-t'|^2} \\
&\le \sup_{t\in I}\|v(t)\|^2_{H^s(\mathbb{T})} + \sup_{t\in I}\|\partial_t v(t)\|^2_{H^{s-1}(\mathbb{T})} \\
&< \infty
\end{aligned}$$

which completes the proof. □

Next, using this lemma we will show that the family $\{u_\varepsilon\}$ is equicontinuous in $C(I, H^{s-\sigma}(\mathbb{T}))$, $0<\sigma<1$. We will follow this by proving that there exists a subfamily $\{u_{\varepsilon_n}\}$ that is precompact in $C(I, H^{s-\sigma}(\mathbb{T}))$. These two facts, in conjunction with Ascoli's Theorem, will yield

$$u_{\varepsilon_n} \to u \text{ in } C(I, H^{s-\sigma}(\mathbb{T})), \quad 0<\sigma<1. \tag{159}$$

Proceeding, we observe that Lemma 19 gives

$$\sup_{t\neq t'}\frac{\|u_\varepsilon(t)-u_\varepsilon(t')\|_{H^{s-\sigma}(\mathbb{T})}}{|t-t'|^{\sigma}} < \infty \tag{160}$$

or

$$\|u_\varepsilon(t)-u_\varepsilon(t')\|_{H^{s-\sigma}(\mathbb{T})} < c|t-t'|^{\sigma}, \text{ for all } t,t'\in I, \tag{161}$$

which shows that the family $\{u_\varepsilon\}$ is equicontinuous in $C(I, H^{s-\sigma}(\mathbb{T}))$. Now recall that

$$\|u_\varepsilon(t)\|_{H^s(\mathbb{T})} \leq 2\|u_0\|_{H^s(\mathbb{T})}, \quad t \in I. \tag{162}$$

By Kondrachov's Theorem, the inclusion $H^s(\mathbb{T}) \subset H^{s-\sigma}(\mathbb{T})$ is compact. By (162), it follows that $\{u_\varepsilon(t)\}$ is precompact in $H^{s-\sigma}(\mathbb{T})$. We are now in a position to apply a result of Ascoli:

Lemma 20. *(Ascoli) Let X be a Banach space, I be a compact metric space, and $C(I, X)$ be the set of continuous functions $f : I \longrightarrow X$. Suppose $S \subset C(I, X)$ has the following properties:*

(i) S is equicontinuous.

(ii) For each $x \in M$ that the set $S(x) = \{f(x)\}$ is precompact in X.

Then S is precompact in $C(I, X)$.

Compiling our previous results on equicontinuity and precompactness and applying Lemma 20, we conclude that there exists a subfamily $\{u_{\varepsilon_n}\}$ such that

$$u_{\varepsilon_n} \to u \text{ in } C(I, H^{s-\sigma}(\mathbb{T})). \tag{163}$$

1.4.1.5 Verifying that the Limit u Solves the HR Equation

We shall need the following.

Proposition 21.

$$J_{\varepsilon_n} u_{\varepsilon_n} \to u \quad in \quad C(I, H^{s-\sigma}(\mathbb{T})), \tag{164}$$

$$J_{\varepsilon_n} \partial_x u_{\varepsilon_n} \to \partial_x u \quad in \quad C(I, H^{s-\sigma-1}(\mathbb{T})). \tag{165}$$

Proof. Note that

$$\begin{aligned} &\|u - J_{\varepsilon_n} u_{\varepsilon_n}\|_{C(I,H^{s-\sigma}(\mathbb{T}))} \\ &= \|u - J_{\varepsilon_n} u_{\varepsilon_n} \pm u_{\varepsilon_n}\|_{C(I,H^{s-\sigma}(\mathbb{T}))} \\ &= \|u - u_{\varepsilon_n}\|_{C(I,H^{s-\sigma}(\mathbb{T}))} + \|(I - J_{\varepsilon_n}) u_{\varepsilon_n}\|_{C(I,H^{s-\sigma}(\mathbb{T}))}. \end{aligned} \tag{166}$$

Applying the estimates

$$\begin{aligned} &\|I - J_{\varepsilon_n}\|_{L(H^{s-\sigma}(\mathbb{T}),H^{s-\sigma}(\mathbb{T}))} = o(1), \\ &\|u_{\varepsilon_n}\|_{H^{s-\sigma}(\mathbb{T})} \leq 2\|u_0\|_{H^{s-\sigma}(\mathbb{T})} \end{aligned}$$

to (166) gives

$$\|u - J_{\varepsilon_n} u_{\varepsilon_n}\|_{H^{s-\sigma}(\mathbb{T})} \leq \left(\|u - u_{\varepsilon_n}\|_{C(I,H^{s-\sigma}(\mathbb{T}))} + o(1) \cdot \|u_0\|_{H^{s-\sigma}(\mathbb{T})}\right). \tag{167}$$

Letting $\varepsilon_n \to 0$ in (167) and applying (163) gives (164). Furthermore, note that

$$\begin{aligned} \|\partial_x u - J_\varepsilon \partial_x u_{\varepsilon_n}\|_{C(I,H^{s-\sigma-1}))} &= \|\partial_x u - \partial_x J_\varepsilon u_{\varepsilon_n}\|_{C(I,H^{s-\sigma-1}(\mathbb{T}))} \\ &\leq \|u - J_\varepsilon u_{\varepsilon_n}\|_{C(I,H^{s-\sigma}(\mathbb{T}))}. \end{aligned}$$

Applying (164) completes the proof of Proposition 21. □

Observe that Proposition 21 implies

$$J_{\varepsilon_n} u_{\varepsilon_n} \cdot J_{\varepsilon_n} \partial_x u_{\varepsilon_n} \to u \partial_x u \ \text{ in } \ C(I, H^{s-\sigma-1}(\mathbb{T})). \tag{168}$$

Furthermore, since $\|\partial_x(1-\partial_x^2)^{-1}\|_{L(H^s(\mathbb{T}),H^s(\mathbb{T}))} \leq 1$ for all $s \in \mathbb{R}$, it follows immediately from (163) that

$$\begin{aligned}&\partial_x(1-\partial_x^2)^{-1}\left(\frac{3-\gamma}{2}(u_{\varepsilon_n})^2+\frac{\gamma}{2}(\partial_x u_{\varepsilon_n})^2\right)\\&\to\partial_x(1-\partial_x^2)^{-1}\left(\frac{3-\gamma}{2}u^2+\frac{\gamma}{2}(\partial_x u)^2\right)\quad\text{in}\quad C(I,H^{s-\sigma-1}(\mathbb{T})).\end{aligned}\tag{169}$$

Combining (168) and (169), and applying the Sobolev Imbedding Theorem, we deduce

$$\begin{aligned}&-\gamma(J_{\varepsilon_n}u_{\varepsilon_n}\cdot J_{\varepsilon_n}\partial_x u_{\varepsilon_n})-\partial_x(1-\partial_x^2)^{-1}\left(\frac{3-\gamma}{2}(u_{\varepsilon_n})^2+\frac{\gamma}{2}(\partial_x u_{\varepsilon_n})^2\right)\\&\to-\gamma u\partial_x u-\partial_x(1-\partial_x^2)^{-1}\left(\frac{3-\gamma}{2}u^2+\frac{\gamma}{2}(\partial_x u)^2\right)\quad\text{in}\quad C(I,C(\mathbb{T})).\end{aligned}\tag{170}$$

Furthermore, we note that the convergence

$$T_{u_{\varepsilon_n}}(\phi)\longrightarrow T_u(\phi)\ \text{ for all }\ \phi\in L^1(I,H^s(\mathbb{T}))\tag{171}$$

implies

$$u_{\varepsilon_n}\longrightarrow u\quad\text{ in }\ \mathcal{D}'(I\times\mathbb{T})\tag{172}$$

and so

$$\partial_t u_{\varepsilon_n}\longrightarrow\partial_t u\quad\text{ in }\ \mathcal{D}'(I\times\mathbb{T}).\tag{173}$$

Since for all n we have

$$\partial_t u_{\varepsilon_n}=-\gamma(J_{\varepsilon_n}u_{\varepsilon_n}\cdot J_{\varepsilon_n}\partial_x u_{\varepsilon_n})-\partial_x(1-\partial_x^2)^{-1}\left[\frac{3-\gamma}{2}(u_\varepsilon)^2+\frac{\gamma}{2}(\partial_x u_\varepsilon)^2\right]\tag{174}$$

by the uniqueness of the limit in (170) we must have

$$\partial_t u=-\gamma u\partial_x u-\partial_x(1-\partial_x^2)^{-1}\left[\frac{3-\gamma}{2}u^2+\frac{\gamma}{2}(\partial_x u)^2\right].\tag{175}$$

Thus we have constructed a solution $u \in L^\infty(I, H^s(\mathbb{T}))$ to the HR IVP. In fact, $u \in L^\infty(I, H^s(\mathbb{T})) \cap \mathrm{Lip}(I, H^{s-1})$. To see this, we observe that $u_{\varepsilon_n}(t) \to u(t)$ strongly in H^s, and so

$$u_{\varepsilon_n}(t_1) \to u(t_1) \text{ in } H^{s-1}$$
$$u_{\varepsilon_n}(t_2) \to u(t_2) \text{ in } H^{s-1}.$$

Applying this in conjunction with the triangle inequality we obtain

$$\|u(t_1) - u(t_2)\|_{H^{s-1}} \leq \lim_{n\to\infty} \|u_{\varepsilon_n}(t_1) - u_{\varepsilon_n}(t_2)\|_{H^{s-1}} \leq C|t_1 - t_2|.$$

where the last step follows from (149) and the mean value theorem.

1.4.1.6 Proof that $u \in C(I, H^s(\mathbb{T}))$

We first outline our strategy. Since $u \in L^\infty(I, H^s(\mathbb{T})) \bigcap \mathrm{Lip}(I, H^{s-1})$, it is a continuous function from I to $H^s(\mathbb{T})$ with respect to the weak topology on H^s; that is, for $\{t_n\} \subset I$ such that $t_n \to t$, we have

$$< u(t_n),\ v >_{H^s(\mathbb{T})} \longrightarrow < u(t),\ v >_{H^s(\mathbb{T})}, \quad \forall v \in H^s(\mathbb{T}). \tag{176}$$

Indeed, let $\{t_n\}$ be a bounded sequence in I converging to t. Then since $u \in L^\infty(I, H^s(\mathbb{T}))$ and H^s is a separable Hilbert space, there exists a subsequence t_{n_j} such that

$$u(t_{n_j}) \rightharpoonup w(t) \text{ in } H^s.$$

But

$$u(t_{n_j}) \to u(t) \text{ in } H^{s-1}$$

so by the uniqueness of the limit we must have $w(t) = u(t)$. Next, note that

$$\begin{aligned}
\|u(t) - u(t_n)\|^2_{H^s(\mathbb{T})} &=< u(t) - u(t_n),\ u(t) - u(t_n) >_{H^s(\mathbb{T})} \\
&= \|u(t)\|^2_{H^s(\mathbb{T})} + \|u(t_n)\|^2_{H^s(\mathbb{T})} \\
&- < u(t_n),\ u(t) >_{H^s(\mathbb{T})} - < u(t), u(t_n) >_{H^s(\mathbb{T})} .
\end{aligned} \tag{177}$$

Applying (176) and (177), we see that

$$\lim_{n\to\infty} \|u(t) - u(t_n)\|^2_{H^s(\mathbb{T})} = \left[\lim_{n\to\infty} \|u(t_n)\|^2_{H^s(\mathbb{T})}\right] - \|u(t)\|^2_{H^s(\mathbb{T})}. \tag{178}$$

Hence, by (178), to prove that $u \in C(I, H^s(\mathbb{T}))$, it will be enough to show that the map $t \mapsto \|u(t)\|_{H^s(\mathbb{T})}$ is a continuous function of t. However, this will follow from the energy estimate

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{H^s(\mathbb{T})} \le c_s \|u(t)\|^3_{H^s(\mathbb{T})}, \quad |t| \le T \tag{179}$$

which we now derive. Applying D^s to both sides of (175), multiplying the resulting equation by $D^s u$, and integrating for $x \in \mathbb{T}$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|^2_{H^s} =& -\gamma \int_{\mathbb{T}} D^s(u\partial_x u) \cdot D^s u \ dx \\
&- \frac{3-\gamma}{2} \int_{\mathbb{T}} D^{s-2}\partial_x(u^2) \cdot D^s u \ dx \\
&- \frac{\gamma}{2} \int_{\mathbb{T}} D^{s-2}\partial_x(\partial_x u)^2 \cdot D^s u \ dx.
\end{aligned} \tag{180}$$

First we estimate

$$
\begin{aligned}
\left|-\frac{3-\gamma}{2}\int_{\mathbb{T}} D^{s-2}\partial_x(u^2)\cdot D^s u\,dx\right| &\lesssim \int_{\mathbb{T}} \left|D^{s-2}\partial_x(u^2)\cdot D^s u\right| dx \\
&\leq \|D^{s-2}\partial_x(u^2)\|_{L^2(\mathbb{T})}\|D^s u\|_{L^2(\mathbb{T})} \\
&\leq \|D^{s-1}(u^2)\|_{L^2(\mathbb{T})}\|D^s u\|_{L^2(\mathbb{T})} \\
&= \|u^2\|_{H^{s-1}(\mathbb{T})}\|u\|_{H^s(\mathbb{T})} \\
&\leq \|u^2\|_{H^s(\mathbb{T})}\|u\|_{H^s(\mathbb{T})}.
\end{aligned} \tag{181}
$$

Applying the algebra property, we obtain

$$
\left|-\frac{3-\gamma}{2}\int_{\mathbb{T}} D^{s-2}\partial_x u^2\cdot D^s u\,dx\right| \lesssim_s \|u\|^3_{H^s(\mathbb{T})}. \tag{182}
$$

We also have

$$
\begin{aligned}
\left|-\frac{\gamma}{2}\int_{\mathbb{T}} D^{s-2}\partial_x(\partial_x u)^2\cdot D^s u\,dx\right| &\lesssim \int_{\mathbb{T}} \left|D^{s-2}\partial_x(\partial_x u)^2\cdot D^s u\right|\,dx \\
&\leq \|D^{s-2}\partial_x(\partial_x u)^2\|_{L^2(\mathbb{T})}\|D^s u\|_{L^2(\mathbb{T})} \\
&\leq \|(\partial_x u)^2\|_{H^{s-1}(\mathbb{T})}\|u\|_{H^s(\mathbb{T})}
\end{aligned} \tag{183}
$$

and applying the algebra property yields

$$
\left|-\frac{\gamma}{2}\int_{\mathbb{T}} D^{s-2}(\partial_x u)^2\cdot D^s u\,dx\right| \lesssim_s \|u\|^3_{H^s(\mathbb{T})}. \tag{184}
$$

It remains to estimate

$$
-\gamma\int_{\mathbb{T}} \left[D^s(u\partial_x u)\cdot D^s u\right] dx.
$$

We have

$$
\begin{aligned}
-\gamma \int_{\mathbb{T}} \left[D^s(u\partial_x u) \cdot D^s u \right] \, dx =& -\gamma \int_{\mathbb{T}} \left[D^s(u\partial_x u) \cdot D^s u \right] \, dx \\
=& -\gamma \int_{\mathbb{T}} \Big[D^s(u\partial_x u) - uD^s(\partial_x u) \Big] \cdot D^s u \, dx \\
& -\gamma \int_{\mathbb{T}} uD^s(\partial_x u) \cdot D^s u \, dx.
\end{aligned}
\tag{185}
$$

We now estimate (185) in parts. Applying the Cauchy-Schwarz inequality gives

$$
\begin{aligned}
&\left| -\gamma \int_{\mathbb{T}} \Big[D^s(u\partial_x u) - uD^s(\partial_x u) \Big] \cdot D^s u \, dx \right| \\
&\lesssim \| D^s(u\partial_x u) - uD^s(\partial_x u) \|_{L^2(\mathbb{T})} \| D^s u \|_{L^2(\mathbb{T})} \\
&= \| D^s(u\partial_x u) - uD^s(\partial_x u) \|_{L^2(\mathbb{T})} \| u \|_{H^s(\mathbb{T})}
\end{aligned}
$$

Applying (127), we obtain

$$
\left| -\gamma \int_{\mathbb{T}} \Big[D^s(u\partial_x u) - uD^s(\partial_x u) \Big] D^s u \, dx \right| \lesssim_s \| \partial_x u \|_{L^\infty(\mathbb{T})} \| u \|_{H^s(\mathbb{T})}^2.
$$

Next, we apply Cauchy-Schwartz and the Sobolev Imbedding Theorem to deduce

$$
\begin{aligned}
\left| \int_{\mathbb{T}} uD^s(\partial_x u) \cdot D^s u dx \right| &\simeq \left| \int_{\mathbb{T}} u\partial_x \left(D^s u \right)^2 \, dx \right| \\
&\leq \int_{\mathbb{T}} \left| \partial_x u \left(D^s u \right)^2 \right| \, dx \\
&\leq \| \partial_x u \|_{L^\infty(\mathbb{T})} \| u \|_{H^s(\mathbb{T})}^2. \\
&\lesssim_s \| u \|_{H^s(\mathbb{T})}^3.
\end{aligned}
\tag{186}
$$

Combining (182), (184), and (186), we obtain (179), as desired. Letting $y(t) = \|u(t)\|_{H^s(\mathbb{T})}^2$ inequality (179) takes the form

$$
\frac{1}{2} y^{-3/2} \frac{dy}{dt} \leq c_s, \qquad y(0) = y_0 = \|u_0\|_{H^s(\mathbb{T})}^2.
\tag{187}
$$

Suppose t is non-negative. Then integrating (187) from 0 to t gives

$$\frac{1}{\sqrt{y_0}} - \frac{1}{\sqrt{y(t)}} \leq c_s t.$$

Replacing $y(t)$ with $\|u(t)\|^2_{H^s(\mathbb{T})}$ and solving for $\|u(t)\|_{H^s(\mathbb{T})}$ we obtain the formula

$$\|u(t)\|_{H^s(\mathbb{T})} \leq \frac{\|u_0\|_{H^s(\mathbb{T})}}{1 - c_s\|u_0\|_{H^s(\mathbb{T})} t}. \tag{188}$$

Now, note that our solution u inherits the common lifespan T of the family $\{u_\varepsilon\}$; that is, u has lifespan

$$T = \frac{1}{2c_s\|u_0\|_{H^s(\mathbb{T})}}.$$

Substituting into (188) we obtain

$$\|u(t)\|_{H^s(\mathbb{T})} \leq \frac{\|u_0\|_{H^s(\mathbb{T})}}{1 - (c_s\|u_0\|_{H^s(\mathbb{T})})/(2c_s\|u_0\|_{H^s(\mathbb{T})})},$$

which simplifies to

$$\|u(t)\|_{H^s(\mathbb{T})} \leq 2\|u_0\|_{H^s(\mathbb{T})}, \quad 0 \leq t \leq T.$$

Similarly, for negative t, we have

$$\|u(t)\|_{H^s(\mathbb{T})} \leq 2\|u_0\|_{H^s(\mathbb{T})}, \quad -T \leq t < 0.$$

Hence,

$$\|u(t)\|_{H^s(\mathbb{T})} \leq 2\|u_0\|_{H^s(\mathbb{T})}, \quad |t| \leq T. \tag{189}$$

Derivating the left hand side of (179) and simplifying, we obtain

$$\frac{d}{dt}\|u(t)\|_{H^s(\mathbb{T})} \leq c_s\|u(t)\|^2_{H^s(\mathbb{T})}, \quad |t| \leq T. \tag{190}$$

Since $\|u(t)\|_{H^s(\mathbb{T})}$ is uniformly bounded for $|t| \le T$ by (189), we conclude from (190) that the map $t \mapsto \|u(t)\|_{H^s(\mathbb{T})}$ is Lipschitz continuous in t, for $|t| \le T$. Therefore, by (178), $u \in C(I, H^s(\mathbb{T}))$. $\square$

1.4.2 Uniqueness

Let $u, \omega \in C(I, H^s(\mathbb{T})),\ s > 3/2$ be two solutions to the Cauchy-problem (75)-(76) with common initial data. Let $v = u - w$; since

$$\begin{aligned}
\partial_t u &= -\gamma u \partial_x u - D^{-2}\partial_x \left[\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}\left(\partial_x u\right)^2\right] \\
\partial_t w &= -\gamma w \partial_x w - D^{-2}\partial_x \left[\frac{3-\gamma}{2}w^2 + \frac{\gamma}{2}(\partial_x w)^2\right]
\end{aligned}$$

we subtract the two equations to obtain

$$\partial_t v = -\frac{\gamma}{2}\partial_x[v(u+w)] - D^{-2}\partial_x\left\{\frac{3-\gamma}{2}[v(u+w)] + \frac{\gamma}{2}[\partial_x v \cdot \partial_x(u+w)]\right\}$$

and hence

$$D^\sigma \partial_t v = -\frac{\gamma}{2}D^\sigma \partial_x[v(u+w)] - D^{\sigma-2}\partial_x\left\{\frac{3-\gamma}{2}[v(u+w)] + \frac{\gamma}{2}[\partial_x v \cdot \partial_x(u+w)]\right\}. \tag{191}$$

Multiplying both sides of (191) by $D^\sigma v$ and integrating, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v\|^2_{H^\sigma(\mathbb{T})} = &-\frac{\gamma}{2}\int_{\mathbb{T}} D^\sigma \partial_x[v(u+w)] \cdot D^\sigma v\ dx \\
&- \frac{3-\gamma}{2}\int_{\mathbb{T}} D^{\sigma-2}\partial_x[v(u+w)] \cdot D^\sigma v\ dx \\
&- \frac{\gamma}{2}\int_{\mathbb{T}} D^{\sigma-2}\partial_x[\partial_x v \cdot \partial_x(u+w)] \cdot D^\sigma v\ dx.
\end{aligned} \tag{192}$$

To estimate the first integral, we rewrite

$$\begin{aligned}
&-\frac{\gamma}{2}\int_{\mathbb{T}} D^{\sigma}\partial_x[v(u+w)]\cdot D^{\sigma}v\ dx \\
&= -\frac{\gamma}{2}\int_{\mathbb{T}} [D^{\sigma}\partial_x,\ u+w]\, v\cdot D^{\sigma}v\ dx - \frac{\gamma}{2}\int_{\mathbb{T}} (u+w)D^{\sigma}\partial_x v\cdot D^{\sigma}v\ dx.
\end{aligned} \tag{193}$$

We now estimate (193) in pieces. Observe that by integrating by parts and applying Cauchy-Schwartz we have

$$\begin{aligned}
\left|\frac{\gamma}{2}\int_{\mathbb{T}}(u+w)D^{\sigma}\partial_x v\cdot D^{\sigma}v\ dx\right| &\simeq \left|\int_{\mathbb{T}}\partial_x(u+w)D^{\sigma}v\cdot D^{\sigma}v\ dx\right| \\
&\lesssim \|\partial_x(u+w)D^{\sigma}v\|_{L^2(\mathbb{T})}\|D^{\sigma}v\|_{L^2(\mathbb{T})} \\
&\lesssim \|\partial_x(u+w)\|_{L^{\infty}(\mathbb{T})}\|v\|^2_{H^{\sigma}(\mathbb{T})}.
\end{aligned} \tag{194}$$

To estimate the remaining piece of (193), we choose $3/2 < \rho < s,\ 1/2 < \sigma < \rho - 1$ and obtain

$$\begin{aligned}
\left|-\frac{\gamma}{2}\int_{\mathbb{T}}[D^{\sigma}\partial_x,\ u+w]v\cdot D^{\sigma}v\ dx\right| &\lesssim \|[D^{\sigma}\partial_x,\ u+w]v\|_{L^2(\mathbb{T})}\|v\|_{H^{\sigma}(\mathbb{T})} \\
&\lesssim \|u+w\|_{H^{\rho}(\mathbb{T})}\|v\|^2_{H^{\sigma}(\mathbb{T})}.
\end{aligned} \tag{195}$$

Combining (194) and (195) and applying the Sobolev Imbedding Theorem, we obtain the estimate

$$|-\frac{\gamma}{2}\int_{\mathbb{T}} D^{\sigma}\partial_x[v(u+w)]\cdot D^{\sigma}v\ dx| \lesssim \|u+w\|_{H^{\rho}(\mathbb{T})}\|v\|^2_{H^{\sigma}(\mathbb{T})}. \tag{196}$$

For the second integral, we apply Cauchy-Schwartz to obtain

$$\begin{aligned}
|\frac{3-\gamma}{2}\int_{\mathbb{T}} D^{\sigma-2}\partial_x[v(u+w)]\cdot D^{\sigma}v\ dx| &\lesssim \|D^{\sigma-2}\partial_x[v(u+w)]\cdot D^{\sigma}v\|_{L^1(\mathbb{T})} \\
&\leq \|D^{\sigma-2}\partial_x[v(u+w)]\|_{L^2(\mathbb{T})}\|v\|_{H^{\sigma}(\mathbb{T})} \\
&\leq \|v(u+w)\|_{H^{\sigma-1}(\mathbb{T})}\|v\|_{H^{\sigma}(\mathbb{T})}
\end{aligned}$$

which by the algebra property gives

$$|\frac{3-\gamma}{2}\int_{\mathbb{T}} D^{\sigma-2}\partial_x[v(u+w)]\cdot D^\sigma v\ dx| \lesssim_s \|u+w\|_{H^{\sigma-1}(\mathbb{T})}\|v\|^2_{H^\sigma(\mathbb{T})}. \tag{197}$$

For the third integral, we first apply Cauchy-Schwartz:

$$\begin{aligned}|\frac{\gamma}{2}\int_{\mathbb{T}} D^{\sigma-2}\partial_x[\partial_x v\cdot\partial_x(u+w)]\cdot D^\sigma v\ dx| &\lesssim \|D^{\sigma-2}\partial_x[\partial_x v\cdot\partial_x(u+w)]\cdot D^\sigma v\|_{L^1(\mathbb{T})}\\ &\leq \|D^{\sigma-2}\partial_x[\partial_x v\cdot\partial_x(u+w)]\|_{L^2(\mathbb{T})}\|v\|_{H^\sigma(\mathbb{T})}\\ &\leq \|[\partial_x v\cdot\partial_x(u+w)]\|_{H^{\sigma-1}(\mathbb{T})}\|v\|_{H^\sigma(\mathbb{T})}.\end{aligned}$$

Restrict $\sigma > 1/2$. Then applying (104), we conclude that

$$\begin{aligned}|\frac{\gamma}{2}\int_{\mathbb{T}} D^{\sigma-2}\partial_x[\partial_x v\cdot\partial_x(u+w)]\cdot D^\sigma v\ dx| &\lesssim \|\partial_x(u+w)\|_{H^\sigma(\mathbb{T})}\|\partial_x v\|_{H^{\sigma-1}(\mathbb{T})}\|v\|_{H^\sigma(\mathbb{T})}\\ &\lesssim_s \|u+w\|_{H^{\sigma+1}(\mathbb{T})}\|v\|^2_{H^\sigma(\mathbb{T})}.\end{aligned} \tag{198}$$

Recall (192). Grouping (196)-(198), and applying the Sobolev Imbedding Theorem, we see that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{H^\sigma(\mathbb{T})} \lesssim \|u+w\|_{H^\rho(\mathbb{T})}\|v\|^2_{H^\sigma(\mathbb{T})}. \tag{199}$$

By Gronwall's inequality, (199) gives

$$\|v\|_{H^\sigma(\mathbb{T})} \lesssim e^{\int_0^t \|u+w\|_{H^\rho}}\|v_0\|_{H^\sigma(\mathbb{T})}, \qquad |t|\leq T. \tag{200}$$

First, note that $v_0 = u_0 - w_0 = 0$; secondly, $\|u+w\|_{H^\rho} \le \|u+w\|_{H^s(\mathbb{T})} < \infty$ for $|t| \le T$ by the triangle inequality and (4). Hence, from (200) we obtain

$$\begin{aligned}\|v\|_{H^\sigma(\mathbb{T})} &\lesssim \|v_0\|_{H^\sigma(\mathbb{T})}, \quad |t| \le T \\ &= 0.\end{aligned}$$

We conclude that solutions to the HR IVP with initial data in $H^s(\mathbb{T})$ are unique for $s > 3/2$. □

1.4.3 Continuous Dependence

Let $\{u_{0,n}\} \subset H^s(\mathbb{T})$ be a uniformly bounded sequence converging to u_0 in $H^s(\mathbb{T})$. Consider solutions u, u^ε, u_n^ε, and u_n to the Cauchy-problem (75)-(76) with associated initial data u_0, $J_\varepsilon u_0$, $J_\varepsilon u_{0,n}$, and $u_{0,n}$, respectively, where J_ε is the operator defined by

$$J_\varepsilon f(x) = j_\varepsilon * f(x), \quad \varepsilon > 0. \tag{201}$$

Here

$$j_\varepsilon(x) = \frac{1}{2\pi}\sum_{\xi\in\mathbb{Z}} \widehat{j}(\varepsilon\xi)e^{i\xi x}, \quad \varepsilon > 0 \tag{202}$$

where $\widehat{j}(\xi) \in \mathcal{S}(\mathbb{R})$ is chosen such that

$$0 \le \widehat{j}(\xi) \le 1 \ \text{ and } \ \widehat{j}(\xi) = 1 \ \text{ if } \ |\xi| \le 1. \tag{203}$$

We remark that it follows immediately from (202) that

$$\widehat{j}_{\varepsilon}(\xi) = \widehat{j}(\varepsilon\xi), \quad \varepsilon > 0. \tag{204}$$

This will prove crucial later on. Next, applying the triangle inequality, we obtain

$$\|u - u_n\|_{H^s(\mathbb{T})} \leq \|u - u^{\varepsilon}\|_{H^s(\mathbb{T})} + \|u^{\varepsilon} - u_n^{\varepsilon}\|_{H^s(\mathbb{T})} + \|u_n^{\varepsilon} - u_n\|_{H^s(\mathbb{T})}.$$

Let $\eta > 0$. To prove continuous dependence, it will be enough to show that we can find $\varepsilon > 0$ and $N \in \mathbb{N}$ such that for all $n > N$

$$\|u(t) - u^{\varepsilon}(t)\|_{H^s(\mathbb{T})} < \eta/3, \quad |t| \leq T, \tag{205}$$

$$\|u^{\varepsilon}(t) - u_n^{\varepsilon}(t)\|_{H^s(\mathbb{T})} < \eta/3, \quad |t| \leq T, \tag{206}$$

$$\|u_n^{\varepsilon}(t) - u_n(t)\|_{H^s(\mathbb{T})} < \eta/3, \quad |t| \leq T. \tag{207}$$

The proof of (207) will be analogous to that of (205), so we will omit the details.

Proof of (205). Consider two solutions u and u^{ε} to the Cauchy-problem (75)-(76) with associated initial data u_0 and $J_{\varepsilon}u_0$, respectively. Set $v = u - u^{\varepsilon}$. Then v solves the Cauchy-problem

$$\begin{aligned} \partial_t v = &-\gamma(v\partial_x v + v\partial_x u^{\varepsilon} + u^{\varepsilon}\partial_x v) \qquad (208)\\ &- D^{-2}\partial_x \left\{ \left(\frac{3-\gamma}{2}\right)(v^2 + 2u^{\varepsilon}v) + \frac{\gamma}{2}\left[(\partial_x v)^2 + 2\partial_x u^{\varepsilon}\partial_x v\right] \right\}, \\ v(0) = &\ (I - J_{\varepsilon})u_0. \qquad (209) \end{aligned}$$

Applying the operator D^s to both sides of (208), then multiplying by $D^s v$ and integrating gives

$$\frac{1}{2}\frac{d}{dt}\|v\|_{H^s(\mathbb{T})} = A + B \tag{210}$$

where

$$\begin{aligned} A = &- \gamma \int_{\mathbb{T}} D^s(v\partial_x v) \cdot D^s v \ dx \\ &- \frac{3-\gamma}{2} \int_{\mathbb{T}} D^{s-2}\partial_x(v^2) \cdot D^s v \ dx \\ &- \frac{\gamma}{2} \int_{\mathbb{T}} D^{s-2}\partial_x(\partial_x v)^2 \cdot D^s v \ dx \end{aligned} \tag{211}$$

and

$$\begin{aligned} B = &- \gamma \int_{\mathbb{T}} D^s(v\partial_x u^\varepsilon) \cdot D^s v \ dx \\ &- \gamma \int_{\mathbb{T}} D^s(u^\varepsilon \partial_x v) \cdot D^s v \ dx \\ &- \ (3-\gamma) \int_{\mathbb{T}} D^{s-2}\partial_x(u^\varepsilon v) \cdot D^s v \ dx \\ &- \gamma \int_{\mathbb{T}} D^{s-2}\partial_x(\partial_x u^\varepsilon \cdot \partial_x v) \cdot D^s v \ dx. \end{aligned} \tag{212}$$

Using estimates analogous to those in (125)-(135), we obtain

$$|A| \lesssim \|v\|^3_{H^s(\mathbb{T})}, \quad |t| \leq T. \tag{213}$$

Next we estimate B in parts: For the first integral, we can rewrite

$$\begin{aligned} -\gamma \int_{\mathbb{T}} D^s(v\partial_x u^\varepsilon) \cdot D^s v \ dx = &- \gamma \int_{\mathbb{T}} [D^s(v\partial_x u^\varepsilon) - vD^s\partial_x u^\varepsilon] \cdot D^s v \ dx \\ &- \gamma \int_{\mathbb{T}} vD^s\partial_x u^\varepsilon \cdot D^s v \ dx. \end{aligned} \tag{214}$$

Applying Cauchy-Schwartz, the Kato-Ponce estimate (128), and the Sobolev Imbedding Theorem, we obtain

$$|-\gamma\int_{\mathbb{T}}[D^s(v\partial_x u^\varepsilon)-vD^s\partial_x u^\varepsilon]\cdot D^s v\ dx|\lesssim\|u^\varepsilon\|_{H^s(\mathbb{T})}\|v\|^2_{H^s(\mathbb{T})}. \tag{215}$$

For the remaining integral of (214), Cauchy-Schwartz, Hölder, and the Sobolev Imbedding Theorem give

$$|-\gamma\int_{\mathbb{T}}vD^s\partial_x u^\varepsilon\cdot D^s v\ dx|\lesssim\|u^\varepsilon\|_{H^{s+1}(\mathbb{T})}\|v\|_{H^{s-1}(\mathbb{T})}\|v\|_{H^s(\mathbb{T})}. \tag{216}$$

Combining estimates (215) and (216) we conclude that

$$\left|-\gamma\int_{\mathbb{T}}D^s(v\partial_x u^\varepsilon)\cdot D^s v\ dx\right|\lesssim\|u^\varepsilon\|_{H^s(\mathbb{T})}\|v\|^2_{H^s(\mathbb{T})}+\|u^\varepsilon\|_{H^{s+1}(\mathbb{T})}\|v\|_{H^{s-1}(\mathbb{T})}\|v\|_{H^s(\mathbb{T})}. \tag{217}$$

For the second integral, we rewrite

$$\begin{aligned}-\gamma\int_{\mathbb{T}}D^s(u^\varepsilon\partial_x v)\cdot D^s v\ dx=&-\gamma\int_{\mathbb{T}}[D^s(u^\varepsilon\partial_x v)-u^\varepsilon D^s\partial_x v]\cdot D^s v\ dx\\&-\gamma\int_{\mathbb{T}}u^\varepsilon D^s\partial_x v\cdot D^s v\ dx.\end{aligned} \tag{218}$$

Applying Cauchy-Schwartz, the Kato-Ponce estimate (128), and the Sobolev Imbedding Theorem to the first integral, we obtain

$$|-\gamma\int_{\mathbb{T}}[D^s(u^\varepsilon\partial_x v)-u^\varepsilon D^s\partial_x v]\cdot D^s v\ dx|\lesssim\|u^\varepsilon\|_{H^s(\mathbb{T})}\|v\|^2_{H^s(\mathbb{T})}. \tag{219}$$

For the remaining integral of (218), integration by parts, Cauchy-Schwartz, and the Sobolev Imbedding Theorem give

$$|-\gamma\int_{\mathbb{T}}u^\varepsilon D^s\partial_x v\cdot D^s v\ dx|\lesssim\|u^\varepsilon\|_{H^s(\mathbb{T})}\|v\|^2_{H^s(\mathbb{T})}. \tag{220}$$

Combining estimates (219) and (220) we conclude that

$$\left|-\gamma \int_{\mathbb{T}} D^s(u^\varepsilon \partial_x v) \cdot D^s v \; dx\right| \lesssim \|u^\varepsilon\|_{H^s(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}^2. \tag{221}$$

For the third integral, we apply Cauchy-Schwartz, the algebra property of Sobolev spaces, and the Sobolev Imbedding Theorem to obtain

$$\left|-\;(3-\gamma) \int_{\mathbb{T}} D^{s-2}\partial_x(u^\varepsilon v) \cdot D^s v \; dx\right| \lesssim \|u^\varepsilon\|_{H^s(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}^2. \tag{222}$$

For the fourth integral, we apply Cauchy-Schwartz, the algebra property of Sobolev spaces, and the Sobolev Imbedding Theorem to obtain

$$\left|-\gamma \int_{\mathbb{T}} D^{s-2}\partial_x(\partial_x u^\varepsilon \cdot \partial_x v) \cdot D^s v \; dx\right| \lesssim \|u^\varepsilon\|_{H^s(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}^2.$$

Hence, collecting our estimates for integrals 1-4, we obtain

$$|B| \lesssim \|u^\varepsilon\|_{H^s(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}^2 + \|u^\varepsilon\|_{H^{s+1}(\mathbb{T})} \|v\|_{H^{s-1}(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}. \tag{223}$$

Combining estimates (213) and (223) and recalling (210), we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|_{H^s(\mathbb{T})}^2 \lesssim \|v\|_{H^s(\mathbb{T})}^3 + \|u^\varepsilon\|_{H^s(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}^2 + \|u^\varepsilon\|_{H^{s+1}(\mathbb{T})} \|v\|_{H^{s-1}(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}$$

which simplifies to

$$\frac{d}{dt}\|v\|_{H^s(\mathbb{T})} \lesssim \|v\|_{H^s(\mathbb{T})}^2 + \|v\|_{H^s(\mathbb{T})} + \varepsilon^{-1}\|v\|_{H^{s-1}(\mathbb{T})} \tag{224}$$

by differentiating the left-hand side and applying the following lemma:

Lemma 22. *For $r \geq s > 3/2$ and $0 < \varepsilon < 1$,*

$$\|u^\varepsilon(t)\|_{H^r(\mathbb{T})} \lesssim \varepsilon^{s-r}. \tag{225}$$

Proof. Recalling the construction of J_ε in (201)-(204), we have

$$|\widehat{J_\varepsilon u_0}(\xi)| = |\widehat{j_\varepsilon}(\xi)\widehat{u_0}(\xi)| = |\widehat{j}(\varepsilon\xi)\widehat{u_0}(\xi)| \leq \begin{cases} |\widehat{u_0}(\xi)|, & |\xi| \leq 1/\varepsilon \\ |\varepsilon\xi|^{s-r}|\widehat{u_0}(\xi)|, & |\xi| \geq 1/\varepsilon. \end{cases} \tag{226}$$

Applying (4) and (226), the result follows. □

We now aim to prove decay as $\varepsilon \to 0$ for the $\varepsilon^{-1}\|v\|_{H^{s-1}(\mathbb{T})}$ term in (224). To do so, we will first obtain an estimate for $\|v\|_{H^\sigma(\mathbb{T})}$ for suitably chosen $\sigma < s-1$. Then, interpolating between $\|v\|_{H^\sigma(\mathbb{T})}$ and $\|v\|_{H^s(\mathbb{T})}$, we will show that $\|v\|_{H^{s-1}(\mathbb{T})}$ experiences $o(\varepsilon)$ decay. This will imply $o(1)$ decay of $\varepsilon^{-1}\|v\|_{H^{s-1}(\mathbb{T})}$.

Proposition 23. *For σ such that $\sigma > 1/2$ and $\sigma + 1 \leq s$, we have*

$$\|v\|_{H^\sigma(\mathbb{T})} = o(\varepsilon^{s-\sigma}), \quad |t| \leq T. \tag{227}$$

Proof. Recall that v solves the Cauchy-problem (208)-(209). Applying D^σ to both sides of (208), then multiplying by $D^\sigma v$ and integrating, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{H^\sigma(\mathbb{T})} = &-\frac{\gamma}{2}\int_{\mathbb{T}} D^\sigma\partial_x\left[(u+u^\varepsilon)\,v\right]\cdot D^\sigma v\ dx \\ &-\frac{3-\gamma}{2}\int_{\mathbb{T}} D^{\sigma-2}\partial_x\left[(u+u^\varepsilon)\,v\right]\cdot D^\sigma v\ dx \\ &-\frac{\gamma}{2}\int_{\mathbb{T}} D^{\sigma-2}\partial_x\left[(\partial_x u+\partial_x u^\varepsilon)\cdot\partial_x v\right]\cdot D^\sigma v\ dx. \end{aligned}$$

Repeating calculations (88)-(105), with E set to zero, $u^{\omega,n}$ replaced by u, $u_{\omega,n}$ replaced by u^ε, and σ and ρ chosen such that

$$\sigma > 1/2, \quad \text{and} \quad \sigma + 1 \leq \rho \leq s$$

yields

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{H^\sigma(\mathbb{T})} \lesssim (\|u^\varepsilon + u\|_{H^\rho(\mathbb{T})} + \|\partial_x(u^\varepsilon + u)\|_{H^\sigma(\mathbb{T})}) \cdot \|v\|^2_{H^\sigma(\mathbb{T})}.$$

By the Sobolev Imbedding Theorem, it follows that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{H^\sigma(\mathbb{T})} \lesssim \|u^\varepsilon + u\|_{H^s(\mathbb{T})} \cdot \|v\|^2_{H^\sigma(\mathbb{T})}. \tag{228}$$

Hence, applying the triangle inequality, (4), and the estimate

$$\|J_\varepsilon f\|_{H^s(\mathbb{T})} \leq \|f\|_{H^s(\mathbb{T})} \tag{229}$$

to the right-hand side of (228) yields

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{H^\sigma(\mathbb{T})} \leq C\|v\|^2_{H^\sigma(\mathbb{T})}$$

where $C = C(\|u_0\|_{H^s(\mathbb{T})})$. Gronwall's inequality then gives

$$\|v\|_{H^\sigma(\mathbb{T})} \leq e^{Ct}\|v(0)\|_{H^\sigma(\mathbb{T})} = e^{Ct}\|u_0 - J_\varepsilon u_0\|_{H^\sigma(\mathbb{T})} = o(\varepsilon^{s-r})$$

where the last step follows from the operator norm estimate provided below. $\square$

Lemma 24. *For $r \leq s$ and $\varepsilon > 0$*

$$\|I - J_\varepsilon\|_{L(H^s(\mathbb{T}), H^r(\mathbb{T}))} = o(\varepsilon^{s-r}). \tag{230}$$

Proof. Let $u \in H^s(\mathbb{T})$ and $r, s \in \mathbb{R}$ such that $r \leq s$. Recalling the construction of J_ε in (201)-(204), we have

$$\|u - J_\varepsilon u\|^2_{H^r(\mathbb{T})} = \sum_{\xi \in \mathbb{Z}} |[1 - \widehat{j}(\varepsilon\xi)] \cdot \widehat{u}(\xi)|^2 (1 + \xi^2)^r, \quad \text{and} \tag{231}$$

$$|1 - \widehat{j}(\varepsilon\xi)| \leq |\varepsilon\xi|^{s-r}, \quad \xi \in \mathbb{R},\ \varepsilon > 0. \tag{232}$$

Applying (232) to (231) we obtain

$$\|u - J_\varepsilon u\|_{H^r(\mathbb{T})} \lesssim \varepsilon^{s-r}$$

while a dominated convergence argument gives

$$\|u - J_\varepsilon u\|_{H^s(\mathbb{T})} = o(1).$$

Applying the interpolation estimate

$$\|f\|_{H^{k_2}(\mathbb{T})} \leq \|f\|^{(s-k_2)/(s-k_1)}_{H^{k_1}(\mathbb{T})} \|f\|^{1-(s-k_2)/(s-k_1)}_{H^s(\mathbb{T})}, \quad k_1 < k_2 \leq s \tag{233}$$

completes the proof. □

We now return to analyzing the $\varepsilon^{-1}\|v\|_{H^{s-1}(\mathbb{T})}$ term of (224). Applying (233) and Proposition 23, we obtain

$$\|v\|_{H^{s-1}(\mathbb{T})} \lesssim o(\varepsilon)\|v\|^{1-1/(s-\sigma)}_{H^s(\mathbb{T})}.$$

Note that $\|v(t)\|_{H^s(\mathbb{T})}$ is uniformly bounded for all $\varepsilon > 0$. More precisely, by the triangle inequality, (229), and (4), we have

$$\|v(t)\|_{H^s(\mathbb{T})} \leq 4\|u_0\|_{H^s(\mathbb{T})}, \quad |t| \leq T. \tag{234}$$

Hence

$$\|v\|_{H^{s-1}(\mathbb{T})} = o(\varepsilon)$$

which implies

$$\varepsilon^{-1}\|v\|_{H^{s-1}(\mathbb{T})} = o(1). \tag{235}$$

Substituting (235) into (224), we obtain

$$\frac{d}{dt}\|v\|_{H^s(\mathbb{T})} \lesssim \|v\|^2_{H^s(\mathbb{T})} + \|v\|_{H^s(\mathbb{T})} + o(1). \tag{236}$$

Letting $y(t) = \|v(t)\|_{H^s(\mathbb{T})}$, we can factor the right-hand side to obtain

$$\frac{dy}{dt} \lesssim (y-\alpha)(y-\beta) \tag{237}$$

where

$$\alpha = \frac{-1+\sqrt{1-o(1)}}{2} \qquad \text{and} \qquad \beta = \frac{-1-\sqrt{1-o(1)}}{2}. \tag{238}$$

Rewriting (237) yields

$$\left(\frac{1}{y-\alpha} - \frac{1}{y-\beta}\right)\frac{dy}{dt} \lesssim \sqrt{1-o(1)} \sim 1.$$

Noting that $1/(y-\alpha) - 1/(y-\beta)$ is positive, and integrating from 0 to t, we obtain

$$\ln\left(\frac{y(t)-\alpha}{y(t)-\beta}\cdot\frac{y(0)-\beta}{y(0)-\alpha}\right) \le ct.$$

Exponentiating both sides and rearranging gives

$$\frac{y(t)-\alpha}{y(t)-\beta} \le e^{ct}\cdot\frac{y(0)-\alpha}{y(0)-\beta}$$

which implies

$$y(t) \le e^{ct}\cdot\frac{[y(0)-\alpha]\,[y(t)-\beta]}{y(0)-\beta} + \alpha \lesssim [y(0)-\alpha]\,[y(t)-\beta] + \alpha, \quad |t| \le T$$

where the last step follows from the fact that $1/2 \le -\beta \le 1$. Substituting back in $\|v\|_{H^s(\mathbb{T})}$, we obtain

$$\|v\|_{H^s(\mathbb{T})} \lesssim \Big[\|v(0)\|_{H^s(\mathbb{T})} - \alpha\Big]\Big[\|v\|_{H^s(\mathbb{T})} - \beta\Big] + \alpha. \tag{239}$$

Noting that $\|v\|_{H^s(\mathbb{T})}$ is uniformly bounded in ε by (234), $\alpha \to 0$, and

$$\|v(0)\|_{H^s(\mathbb{T})} = \|u_0 - J_\varepsilon u_0\|_{H^s(\mathbb{T})} \to 0$$

by Lemma 24, we conclude from (239) that

$$\|v(t)\|_{H^s(\mathbb{T})} = \|u(t) - u^\varepsilon(t)\|_{H^s(\mathbb{T})} = o(1), \quad |t| \le T. \tag{240}$$

Choosing ε sufficiently small gives $\|v(t)\|_{H^s(\mathbb{T})} < \eta/3$, completing the proof of (205).

□

Proof of (206). Let $v = u_n^\varepsilon - u^\varepsilon$. Then v solves the Cauchy problem

$$\partial_t v = -\gamma(v\partial_x v + v\partial_x u^\varepsilon + u^\varepsilon \partial_x v) - D^{-2}\partial_x \left\{ \left(\frac{3-\gamma}{2}\right)(v^2 + 2u^\varepsilon v) + \frac{\gamma}{2}\left[(\partial_x v)^2 + 2\partial_x u^\varepsilon \partial_x v\right] \right\}, \tag{241}$$

$$v(0) = J_\varepsilon(u_{0,n} - u_0). \tag{242}$$

Applying the operator D^s to both sides of (241), multiplying by D^s and integrating, and estimating as in (213)-(223), we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{H^s(\mathbb{T})} \lesssim \|v\|^3_{H^s(\mathbb{T})} + \|u^\varepsilon\|_{H^s(\mathbb{T})}\|v\|^2_{H^s(\mathbb{T})} + \|u^\varepsilon\|_{H^{s+1}(\mathbb{T})}\|v\|_{H^{s-1}(\mathbb{T})}\|v\|_{H^s(\mathbb{T})}$$

which by differentiating the left-hand side and applying Lemma 22 to the right-hand side simplifies to

$$\frac{d}{dt}\|v\|_{H^s(\mathbb{T})} \lesssim \|v\|^2_{H^s(\mathbb{T})} + \|v\|_{H^s(\mathbb{T})} + \varepsilon^{-1}\|v\|_{H^{s-1}(\mathbb{T})}. \tag{243}$$

We now aim to control of the growth the $\varepsilon^{-1}\|v\|_{H^{s-1}(\mathbb{T})}$ term of (243). To do so, we will need an estimate for $\|v\|_{H^{s-1}(\mathbb{T})}$, which we will obtain using interpolation. First, we will need the following:

Proposition 25. *For σ such that $1/2 < \sigma < 1$ and $\sigma + 1 \le s$,*

$$\|v\|_{H^\sigma(\mathbb{T})} = \|u_n^\varepsilon - u^\varepsilon\|_{H^\sigma(\mathbb{T})} \lesssim \|u_0 - u_{0,n}\|_{H^s(\mathbb{T})}, \quad |t| \le T. \tag{244}$$

Proof. Repeating calculations (88)-(105), with E set to zero, $u^{\omega,n}$ replaced by u_n^ε, $u_{\omega,n}$

replaced by u^ε, and σ and ρ chosen such that

$$1/2 < \sigma < 1 \quad \text{and} \quad \sigma + 1 \leq \rho \leq s \tag{245}$$

yields

$$\frac{1}{2}\frac{d}{dt}\|v\|_{H^\sigma(\mathbb{T})}^2 \lesssim (\|u_n^\varepsilon + u^\varepsilon\|_{H^\rho(\mathbb{T})} + \|\partial_x(u_n^\varepsilon + u^\varepsilon)\|_{H^\sigma(\mathbb{T})}) \cdot \|v\|_{H^\sigma(\mathbb{T})}^2.$$

Since $u_{0,n} \to u_0$ in $H^s(\mathbb{T})$, it follows that

$$\frac{1}{2}\frac{d}{dt}\|v\|_{H^\sigma(\mathbb{T})}^2 \leq C\|v\|_{H^\sigma(\mathbb{T})}^2 \tag{246}$$

where $C = C(\|u_0\|_{H^s(\mathbb{T})})$. Applying Gronwall's inequality to (246), we obtain

$$\|v\|_{H^\sigma(\mathbb{T})} \leq e^{Ct}\|v(0)\|_{H^\sigma(\mathbb{T})} = e^{Ct}\|u^\varepsilon(0) - u_n^\varepsilon(0)\|_{H^\sigma(\mathbb{T})} \leq e^{Ct}\|u_0 - u_{0,n}\|_{H^\sigma(\mathbb{T})}$$

concluding the proof. □

We now return to analyzing the $\varepsilon^{-1}\|v\|_{H^{s-1}(\mathbb{T})}$ term of (243). Applying the interpolation estimate (233) and Proposition 25 gives

$$\|v\|_{H^{s-1}(\mathbb{T})} \lesssim \|u_0 - u_{0,n}\|_{H^s(\mathbb{T})}^{1/(s-\sigma)}\|v\|_{H^s(\mathbb{T})}^{1-1/(s-\sigma)}. \tag{247}$$

Note that the triangle inequality, (4), and (229) imply that $\|v\|_{H^s(\mathbb{T})}$ is uniformly bounded in n *and* ε. That is

$$\|v\|_{H^s(\mathbb{T})} \lesssim \|u_0\|_{H^s(\mathbb{T})}, \quad |t| \leq T.$$

Hence, (247) gives

$$\|v\|_{H^{s-1}(\mathbb{T})} \lesssim \|u_0 - u_{0,n}\|_{H^s(\mathbb{T})}^{1/(s-\sigma)}. \tag{248}$$

Fix $\varepsilon, \rho > 0$. Since $\|u_0 - u_{0,n}\|_{H^s(\mathbb{T})} \to 0$, we can find $N \in \mathbb{N}$ such that for all $n > N$

$$\varepsilon^{-1}\|u_0 - u_{0,n}\|_{H^s(\mathbb{T})}^{1/(s-\sigma)} < \rho$$

which by (248) implies

$$\varepsilon^{-1}\|v\|_{H^{s-1}(\mathbb{T})} \lesssim \rho. \tag{249}$$

Since ρ can be chosen to be arbitrarily small, the remainder of the proof is analogous to that of (205). □

1.5 Well-Posedness for HR in the Non-Periodic Case

The method will be analogous to that of the periodic case, with two major modifications. First, we must choose a different mollifier J_ε for the initial data in the proof of continuous dependence. Secondly, in the proof of existence, we will have difficulties in arranging that the solutions $\{u_\varepsilon\}$ to the mollified HR IVP converge in $C(I, H^{s-\sigma}(\mathbb{R}))$, $0 < \sigma < 1$ to a candidate solution u of the HR IVP. We will get around this by considering the family $\{\varphi u_\varepsilon\}$ instead, where $\varphi \in S(\mathbb{R})$.

1.5.1 Existence

Again, as in the non-periodic we see that the HR equation as is can not be thought as an ODE on the space H^s. To deal with this problem we will replace the HR IVP by

$$\partial_t u_\varepsilon = -\gamma J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon - \partial_x (1-\partial_x^2)^{-1} \left[\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} (\partial_x u)^2 \right], \tag{250}$$

$$u_\varepsilon(x,0) = u_0(x) \quad x \in \mathbb{T}, \quad t \in \mathbb{R}. \tag{251}$$

where J_ε is defined as follows: Pick a function $j(x) \in \mathcal{S}(\mathbb{R})$ and let

$$j_\varepsilon(x) = \frac{1}{\varepsilon} j\left(\frac{x}{\varepsilon}\right).$$

We then define J_ε to be the "Friedrichs mollifier"

$$J_\varepsilon f(x) = j_\varepsilon * f(x), \quad \varepsilon > 0.$$

Using an analogous argument to that in the periodic case, one can check that the map $f \mapsto J_\varepsilon f$ is continuously differentiable from $H^s(\mathbb{T})$ to $H^s(\mathbb{T})$. Hence, by Theorem 14, for each $\varepsilon > 0$ there exists a unique solution $u_\varepsilon \in C(I, H^s(\mathbb{R}))$ satisfying the Cauchy-problem (250)-(251). In fact, Lemma 17 (uniform regularity) holds for the non-periodic case. The proof is analogous to that in the periodic case, and so we omit it.

1.5.1.1 Choosing a Convergent Subsequence

Mirroring the argument in the periodic case, we see that the bounded family $\{u_\varepsilon\}$ is compact in the weak* topology of $L^\infty(I, H^s(\mathbb{R}))$, where $I \subset \mathbb{R}$ is compact. More precisely, there is a sequence $\{u_{\varepsilon_n}\}$ converging weak* to a $u \in L^\infty(I, H^s(\mathbb{R}))$; that is

$$\lim_{n\to\infty} T_{u_{\varepsilon_n}}(\varphi) = T_u(\varphi) \quad \text{for all} \quad \varphi \in L^1(I, H^s(\mathbb{R}))$$

where

$$T_v(\varphi) = \int_I < v(t), \varphi(t) >_{H^s(\mathbb{R})} dt = \int_I \int_{\mathbb{R}} \widehat{v}(\xi,t)\bar{\widehat{\varphi}}(\xi,t) \cdot (1+\xi^2)^s \, d\xi \, dt. \tag{252}$$

Similarly, $\{\partial_x u_{\varepsilon_n}\}$ is compact in the weak* topology of $L^\infty(I, H^{s-1}(\mathbb{R}))$ and converges weak* to $\partial_x u$. Hence, for any $k \in \mathbb{N}$, we have

$$(u_{\varepsilon_n})^k \xrightarrow{\text{weak}^\star} u^k \quad \text{on} \quad L^\infty(I, H^s(\mathbb{R})), \tag{253}$$

$$(\partial_x u_{\varepsilon_n})^k \xrightarrow{\text{weak}^\star} (\partial_x u)^k \quad \text{on} \quad L^\infty(I, H^{s-1}(\mathbb{R})). \tag{254}$$

In order to show that u solves the HR IVP, it would suffice to obtain a stronger convergence for u_{ε_n} so that we could take the limit in the mollified HR equation. However, this is difficult, and unnecessary. Rather, our approach will be to show that for any pseudo-differential operator $P \in \Psi^0$ and arbitrary $\varphi \in S(\mathbb{R})$, $k \in \mathbb{N}$, $0 < \sigma < 1$, we have

$$\varphi P[(u_{\varepsilon_n})^k] \longrightarrow \varphi P[u^k] \quad \text{in} \ \ C(I, H^{s-\sigma}(\mathbb{R})), \tag{255}$$

$$\varphi P[(\partial_x u_{\varepsilon_n})^k] \longrightarrow \varphi P[(\partial_x u)^k] \quad \text{in} \ \ C(I, H^{s-\sigma-1}(\mathbb{R})), \tag{256}$$

which will then be applied to a rewritten version of the HR IVP. Our focus will be on proving (255); since the proof of (256) is similar, we will omit the details. First, we will need the following interpolation result:

Lemma 26. *(Interpolation) Let $s > \frac{3}{2}$. If $v \in C(I, H^s(\mathbb{R})) \cap C^1(I, H^{s-1}(\mathbb{R}))$ then*

$v \in C^\sigma(I, H^{s-\sigma}(\mathbb{R}))$ for $0 < \sigma < 1$.

Proof. It is analogous to the proof in the periodic case. $\square$

Fix $k \in \mathbb{N}$. Using Lemma 26, we will show that the family

$$\{\varphi P[(u_\varepsilon)^k]\}$$

is equicontinuous in $C(I, H^{s-\sigma}(\mathbb{R}))$ for $0 < \sigma < 1$ and $\varphi \in \mathcal{S}(\mathbb{R})$. We will follow this by proving that there exists a sub-family $\{\varphi P[(u_{\varepsilon_n}(t))^k]\}$ that is precompact in $H^{s-\sigma}(\mathbb{R})$ for $\sigma > 0$. These two facts, in conjunction with Ascoli's Theorem, will yield

$$\varphi P[(u_\varepsilon)^k] \to \tilde{u} \ \text{ in } \ C(I, H^{s-\sigma}(\mathbb{R}))$$

for $0 < \sigma < 1$. We will then show that $\tilde{u} = \varphi P[u^k]$, from which it will follow that

$$\varphi P[(u_\varepsilon)^k] \to \varphi P[u^k] \ \text{ in } \ C(I, H^{s-\sigma}(\mathbb{R})).$$

Proceeding, we observe that since $\varphi \in \mathcal{S}(\mathbb{R})$, the map $u \mapsto \varphi u$ is a bounded linear function on $H^s(\mathbb{R})$, for arbitrary $s \in \mathbb{R}$, where

$$\|\varphi u\|_{H^s(\mathbb{R})} \leq C(s, \varphi)\|u\|_{H^s(\mathbb{R})}, \quad \forall s \in \mathbb{R}. \tag{257}$$

Furthermore,

$$P : H^s(\mathbb{R}) \to H^s(\mathbb{R})$$

is bounded and linear, with

$$\|P\|_{L(H^s(\mathbb{R}), H^s(\mathbb{R}))} \leq 1. \tag{258}$$

Hence, the map

$$
\begin{aligned}
&T : H^s(\mathbb{R}) \to H^s(\mathbb{R}), \\
&T(u) = \varphi P u
\end{aligned}
\tag{259}
$$

is bounded and linear, with

$$
\|T\|_{L(H^s(\mathbb{R}),H^s(\mathbb{R}))} \leq C(s,\varphi). \tag{260}
$$

Therefore, applying Lemma 26 gives

$$
\begin{aligned}
&\sup_{t\neq t'} \frac{\|\varphi P[(u_\varepsilon(t))^k] - \varphi P[(u_\varepsilon(t'))^k]\|_{H^{s-\sigma}(\mathbb{R})}}{|t-t'|} \\
&\leq \sup_{t\neq t'} \frac{\|\varphi P\|_{L(H^{s-\sigma}(\mathbb{R}),H^{s-\sigma}(\mathbb{R}))} \cdot \|[u_\varepsilon(t)]^k - [u_\varepsilon(t')]^k\|_{H^{s-\sigma}(\mathbb{R})}}{|t-t'|} \\
&\leq C(s,\varphi) \cdot \sup_{t\neq t'} \frac{\|[u_\varepsilon(t)]^k - [u_\varepsilon(t')]^k\|_{H^{s-\sigma}(\mathbb{R})}}{|t-t'|} \\
&< c
\end{aligned}
$$

or

$$
\|\varphi P[(u_\varepsilon(t))^k] - \varphi P[(u_\varepsilon(t'))^k\|_{H^{s-\sigma}(\mathbb{R})} < c|t-t'|, \text{ for all } \ t,t' \in I,
$$

which shows that the family $\{\varphi P[(u_\varepsilon)^k]\}$ is equicontinuous in $C(I, H^{s-\sigma}(\mathbb{R}))$. Furthermore, observe that by the algebra property of Sobolev Spaces, and (259)-(260), we have

$$
\begin{aligned}
\|\varphi P[(u_\varepsilon(t))^k]\|_{H^s(\mathbb{R})} &\leq C(s,\varphi) \cdot \|[u_\varepsilon(t)]^k\|_{H^s(\mathbb{R})} \\
&\leq C(s,\varphi) \cdot \|u_\varepsilon(t)\|^k_{H^s(\mathbb{R})}.
\end{aligned}
\tag{261}
$$

Letting $|t| \leq T$, we now apply Lemma 17 to (261) to obtain

$$\|\varphi P[(u_\varepsilon(t))^k]\|_{H^s(\mathbb{R})} \leq 2^k C(s,\varphi) \cdot \|u_0\|^k_{H^s(\mathbb{R})} < \infty.$$

Therefore, by Rellich's Theorem, the family $\left\{\varphi P[(u_\varepsilon(t))^k]\right\}$ is precompact in $H^{s-\sigma}(\mathbb{R})$ for all $\sigma > 0$ and $|t| \leq T$. Hence, compiling our previous results on equicontinuity and precompactness and applying Ascoli's Theorem, we conclude that we can find $\tilde{u}$ and a subfamily $\left\{\varphi P[(u_{\varepsilon_n})^k]\right\}$ such that

$$\varphi P[(u_{\varepsilon_n})^k] \to \tilde{u} \text{ in } C(I, H^{s-\sigma}(\mathbb{R})). \tag{262}$$

We would now like to find out what $\tilde{u}$ is:

Lemma 27. *For arbitrary $k \in \mathbb{N}$,*

$$\varphi P[(u_{\varepsilon_n})^k] \xrightarrow{weak^\star} \varphi P[u^k] \quad on \quad L^\infty(I, H^{s-\sigma}(\mathbb{R})). \tag{263}$$

Proof. Fix $k \in \mathbb{N}$ and recall that the operators

$$T_\varphi : H^s(\mathbb{R}) \to H^s(\mathbb{R})$$
$$T_\varphi u = \varphi u$$

and

$$P : H^s(\mathbb{R}) \to H^s(\mathbb{R})$$

are continuous; therefore

$$T_\varphi P : H^s(\mathbb{R}) \to H^s(\mathbb{R})$$

continuously. Since H^s, it follows that the adjoint operator $(T_\varphi P)^\star$ exists and

$$(T_\varphi P)^\star : H^s(\mathbb{R}) \to H^s(\mathbb{R})$$

continuously. Therefore, applying (253), we conclude that

$$\begin{aligned}&\int_I < \varphi P[u^k] - \varphi P[(u_{\varepsilon_n})^k],\ f >_{H^{s-\sigma}(\mathbb{R})} dt \\ &= \int_I < u^k - (u_{\varepsilon_n})^k,\ (T_\varphi P)^\star f >_{H^{s-\sigma}(\mathbb{R})} \to 0\end{aligned} \tag{264}$$

completing the proof. □

Now, recalling (262) and applying Lemma 27, we obtain

$$\varphi P[(u_{\varepsilon_n})^k] \to \varphi P[u^k] \ \text{ in } \ C(I, H^{s-\sigma}(\mathbb{R})) \tag{265}$$

for arbitrary $k \in \mathbb{N}$. Using precisely the same strategy we used to prove (265) (applied now to the family $\{\varphi P[(\partial_x u_\varepsilon)^k]\}$), one can also show

$$\varphi P[(\partial_x u_{\varepsilon_n})^k] \to \varphi P[(\partial_x u)^k] \ \text{ in } \ C(I, H^{s-\sigma-1}(\mathbb{R})). \tag{266}$$

We summarize our result below:

Proposition 28. *Let $P \in \Psi^0$ be a pseudo-differential operator. Then for arbitrary $k \in \mathbb{N}$,*

$$\begin{aligned}&\varphi P[(u_{\varepsilon_n})^k] \to \varphi P[u^k] \ \ \textit{in} \ \ C(I, H^{s-\sigma}(\mathbb{R})), \\ &\varphi P[(\partial_x u_{\varepsilon_n})^k] \to \varphi P[(\partial_x u)^k] \ \ \textit{in} \ \ C(I, H^{s-\sigma-1}(\mathbb{R})).\end{aligned} \tag{267}$$

1.5.1.2 Verifying that the Limit u Solves the HR Equation

We recall the mollified HR IVP

$$\partial_t u_{\varepsilon_n} = -\gamma(J_{\varepsilon_n} u_{\varepsilon_n} \cdot \partial_x J_{\varepsilon_n} u_{\varepsilon_n}) - \partial_x(1-\partial_x^2)^{-1}\left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}(\partial_x u)^2\right) \tag{268}$$

$$u(x,0) = u_0(x). \tag{269}$$

Multiplying both sides of (268) by φ and rewriting, we obtain

$$\partial_t(u_{\varepsilon_n}\varphi) = -\gamma\varphi(J_{\varepsilon_n} u_{\varepsilon_n} \cdot J_{\varepsilon_n}\partial_x u_{\varepsilon_n}) - \partial_x(1-\partial_x^2)^{-1}\left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}(\partial_x u)^2\right). \tag{270}$$

The following lemma will play a crucial role in our proof of the existence of a solution to the HR IVP

Lemma 29. *For $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\varphi^{\frac{1}{2}} \in \mathcal{S}(\mathbb{R})$, we have*

$$\varphi(J_{\varepsilon_n} u_{\varepsilon_n} \cdot J_{\varepsilon_n}\partial_x u_{\varepsilon_n}) \to \varphi u \partial_x u \ \ in \ \ C(I, H^{s-\sigma-1}(\mathbb{R})). \tag{271}$$

Proof. We will need a couple of lemmas:

Lemma 30. *For arbitrary $\varphi \in \mathcal{S}(\mathbb{R})$*

$$\varphi J_{\varepsilon_n} u_{\varepsilon_n} \to \varphi u \ \ \ in \ \ C(I, H^{s-\sigma}(\mathbb{R})). \tag{272}$$

Proof. Note that

$$\begin{aligned} &\|\varphi u - \varphi J_{\varepsilon_n} u_{\varepsilon_n}\|_{C(I,H^{s-\sigma}(\mathbb{R}))} \\ &= \|\varphi u - \varphi J_{\varepsilon_n} u_{\varepsilon_n} \pm \varphi u_{\varepsilon_n}\|_{C(I,H^{s-\sigma}(\mathbb{R}))} \\ &= \|\varphi u - \varphi u_{\varepsilon_n}\|_{C(I,H^{s-\sigma}(\mathbb{R}))} + \|\varphi(I - J_{\varepsilon_n}) u_{\varepsilon_n}\|_{C(I,H^{s-\sigma}(\mathbb{R}))}. \end{aligned} \tag{273}$$

Applying (257) and the estimates

$$\|I - J_{\varepsilon_n}\|_{L(H^{s-\sigma}(\mathbb{R}),H^{s-\sigma}(\mathbb{R}))} = o(1),$$
$$\|u_{\varepsilon_n}\|_{H^{s-\sigma}(\mathbb{R})} \leq 2\|u_0\|_{H^{s-\sigma}(\mathbb{R})}$$

to (273) gives

$$\|\varphi u - \varphi J_{\varepsilon_n} u_{\varepsilon_n}\|_{H^{s-\sigma}(\mathbb{R})} \leq \left(\|\varphi u - \varphi u_{\varepsilon_n}\|_{C(I,H^{s-\sigma}(\mathbb{R}))} + C(s,\varphi)\cdot o(1)\cdot \|u_0\|_{H^{s-\sigma}(\mathbb{R})}\right). \tag{274}$$

Letting $\varepsilon \to 0$ in (274) and applying Proposition 28 completes the proof. □

Lemma 31. *For arbitrary $\varphi \in \mathcal{S}(\mathbb{R})$,*

$$\varphi J_{\varepsilon_n}\partial_x u_{\varepsilon_n} \to \varphi u \quad in \quad C(I, H^{s-\sigma-1}(\mathbb{R})). \tag{275}$$

Proof. The result follows from Proposition 28. The proof is nearly identical to that of Lemma 30, with $s-1$ substituted for s and $\partial_x u_{\varepsilon_n}$ substituted for u_{ε_n}. □

We now have enough tools to prove Lemma 29. Restrict the choice of φ such that $\varphi^{\frac{1}{2}} \in S(\mathbb{R})$ (such Schwartz functions exist; as an example, take the square of the Gaussian). Using this fact, and applying Lemma 30 and Lemma 31, we conclude that

$$\begin{aligned}\varphi J_{\varepsilon_n} u_{\varepsilon_n}\partial_x J_{\varepsilon_n} u_{\varepsilon_n} &= \varphi^{\frac{1}{2}} J_{\varepsilon_n} u_{\varepsilon_n}\cdot \varphi^{\frac{1}{2}}\partial_x J_{\varepsilon_n} u_{\varepsilon_n}\\ &\to \varphi^{\frac{1}{2}} u\cdot\varphi^{\frac{1}{2}}\partial_x u = \varphi u\partial_x u\end{aligned}$$

completing the proof of Lemma 29. □

By Proposition 28 it follows immediately that

$$\begin{aligned}&\varphi\partial_x(1-\partial_x^2)^{-1}\left(\frac{3-\gamma}{2}(u_{\varepsilon_n})^2+\frac{\gamma}{2}(\partial_x u_{\varepsilon_n})^2\right)\\&\to\varphi\partial_x(1-\partial_x^2)^{-1}\left(\frac{3-\gamma}{2}u^2+\frac{\gamma}{2}(\partial_x u)^2\right)\quad\text{in}\quad C(I,H^{s-\sigma-1}(\mathbb{R})).\end{aligned}\tag{276}$$

Combining (271) and (276), and applying the Sobolev Imbedding Theorem, we deduce

$$\begin{aligned}&-\gamma\varphi(J_{\varepsilon_n}u_{\varepsilon_n}\cdot J_{\varepsilon_n}\partial_x u_{\varepsilon_n})-\varphi\partial_x(1-\partial_x^2)^{-1}\left(\frac{3-\gamma}{2}(u_{\varepsilon_n})^2+\frac{\gamma}{2}(\partial_x u_{\varepsilon_n})^2\right)\\&\to-\gamma\varphi u\partial_x u-\varphi\partial_x(1-\partial_x^2)^{-1}\left(\frac{3-\gamma}{2}u^2+\frac{\gamma}{2}(\partial_x u)^2\right)\quad\text{in}\quad C(I,C(\mathbb{R})).\end{aligned}\tag{277}$$

Next, we note that the convergence

$$T_{\varphi u_{\varepsilon_n}}(f)\longrightarrow T_{\varphi u}(f)\quad\text{for all}\quad f\in L^1(I,H^{-s}(\mathbb{R}))\tag{278}$$

can be restated as

$$\varphi u_{\varepsilon_n}\longrightarrow\varphi u\quad\text{in}\quad\mathcal{D}'(I\times\mathbb{R}).\tag{279}$$

This implies

$$\partial_t(\varphi u_{\varepsilon_n})\longrightarrow\partial_t(\varphi u)\quad\text{in}\quad\mathcal{D}'(I\times\mathbb{R}).\tag{280}$$

Since for all n we have

$$\begin{aligned}\partial_t(\varphi u_{\varepsilon_n})=&-\gamma\varphi(J_{\varepsilon_n}u_{\varepsilon_n}\cdot J_{\varepsilon_n}\partial_x u_{\varepsilon_n})\\&-\varphi\partial_x(1-\partial_x^2)^{-1}\left(\frac{3-\gamma}{2}(u_{\varepsilon_n})^2+\frac{\gamma}{2}(\partial_x u_{\varepsilon_n})^2\right)\end{aligned}\tag{281}$$

it follows from (280) and the uniqueness of the limit in (277) that

$$\partial_t(\varphi u)=-\gamma\varphi u\partial_x u-\varphi\partial_x(1-\partial_x^2)^{-1}\left(\frac{3-\gamma}{2}u^2+\frac{\gamma}{2}(\partial_x u)^2\right).\tag{282}$$

Further restricting $\varphi \in \mathcal{S}(\mathbb{R})$ to be nonzero in $\mathbb{R}$, we can divide both sides of (282) by φ to obtain

$$\partial_t u = -\gamma u \partial_x u - \partial_x (1-\partial_x^2)^{-1} \left(\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} (\partial_x u)^2 \right). \tag{283}$$

Thus we have constructed a solution $u \in L^\infty(I, H^s(\mathbb{R}))$ to the HR IVP.

1.5.1.3 Proof that $u \in C(I, H^s(\mathbb{R}))$

The proof is analogous to that in the periodic case.

1.5.2 Uniqueness

The proof is analogous to that in the periodic case.

1.5.3 Continuous Dependence

The proof is analogous to that in the periodic case, with one important caveat: we must choose an appropriate mollifier J_ε for the initial data. Pick a function $j(x) \in \mathcal{S}(\mathbb{R})$ such that

$$0 \le \widehat{j}(\xi) \le 1,$$
$$\widehat{j}(\xi) = 1 \quad \text{if} \quad |\xi| \le 1.$$

Letting

$$j_\varepsilon(x) = \frac{1}{\varepsilon} j\left(\frac{x}{\varepsilon}\right)$$

we then define J_ε to be the "Friedrichs mollifier"

$$J_\varepsilon f(x) = j_\varepsilon * f(x), \quad \varepsilon > 0.$$

Given this construction, the proofs of Lemma 22 and Lemma 24 for the non-periodic case are analogous to those in the periodic case.

Hence, how we construct the mollifier J_ε plays a critical role in the proofs of well-posedness for the HR IVP in both the periodic and non-periodic cases.

CHAPTER 2

HÖLDER CONTINUITY FOR HR IN THE WEAK TOPOLOGY

2.1 Proof of Hölder Continuity

We note that the only significant difference between the proof of Hölder continuity in the periodic and non-periodic cases is in the proof of Lemma 11, which we have already addressed. Hence, we focus our attention on the proof of Hölder continuity in the periodic case.

2.1.1 Region Ω_1

Let $u_0(x), v_0(x) \in B_{H^s}(R),\ s > 3/2$ be two initial data. Then from the well-posedness theory for HR [22], we know that there exists unique corresponding solutions $u, v \in C(I, B_{H^s}(2R))$ to HR. Set $v = u - w$. Then v solves the Cauchy-problem

$$\begin{aligned}
\partial_t v = &-\frac{\gamma}{2}\partial_x[v(u+w)] \qquad (1)\\
&- \partial_x(1-\partial_x^2)^{-1}\left\{\frac{3-\gamma}{2}[v(u+w)] + \frac{\gamma}{2}[\partial_x v \cdot \partial_x(u+w)]\right\},\\
v(x,0) = &\ u_0(x) - v_0(x). \qquad (2)
\end{aligned}$$

Let

$$D^m \doteq (1-\partial_x^2)^{m/2}, \quad m \in \mathbb{R}.$$

Applying D^r to both sides of (1), then multiplying both sides by $D^r v$ and integrating, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|_{H^r}^2 = & -\frac{\gamma}{2}\int_{\mathbb{T}} D^r\partial_x[v(u+w)]\cdot D^r v\ dx \\ & -\frac{3-\gamma}{2}\int_{\mathbb{T}} D^{r-2}\partial_x[v(u+w)]\cdot D^r v\ dx \\ & -\frac{\gamma}{2}\int_{\mathbb{T}} D^{r-2}\partial_x[\partial_x v\cdot\partial_x(u+w)]\cdot D^r v\ dx.\end{aligned} \tag{3}$$

We now estimate (3) in parts. For the first integral, we note that

$$\begin{aligned}&\left|-\frac{\gamma}{2}\int_{\mathbb{T}} D^r\partial_x[v(u+w)]\cdot D^r v\ dx\right| \\ &= \left|-\frac{\gamma}{2}\int_{\mathbb{T}}[D^r\partial_x,\ u+w]\,v\cdot D^r v\ dx - \frac{\gamma}{2}\int_{\mathbb{T}}(u+w)D^r\partial_x v\cdot D^r v\ dx\right| \\ &\lesssim \left|\int_{\mathbb{T}}[D^r\partial_x,\ u+w]\,v\cdot D^r v\ dx\right| + \left|\int_{\mathbb{T}}(u+w)D^r\partial_x v\cdot D^r v\ dx\right|.\end{aligned} \tag{4}$$

Observe that integrating by parts gives

$$\left|\int_{\mathbb{T}}(u+w)D^r\partial_x v\cdot D^r v\ dx\right| \leq \|\partial_x(u+w)\|_{L^\infty}\|v\|_{H^r}^2. \tag{5}$$

An application of Cauchy-Schwartz and Lemma 10 then yields

$$\left|\int_{\mathbb{T}}[D^r\partial_x,\ u+w]v\cdot D^r v\ dx\right| \lesssim \|u+w\|_{H^s}\|v\|_{H^r}^2. \tag{6}$$

Combining (5) and (6) and applying the Sobolev embedding theorem, we obtain the estimate

$$\left|-\frac{\gamma}{2}\int_{\mathbb{T}} D^r\partial_x[v(u+w)]\cdot D^r v\ dx\right| \lesssim \|u+w\|_{H^s}\|v\|_{H^r}^2, \quad s>3/2,\ -1\leq r\leq s-1. \tag{7}$$

For the second integral, we apply Cauchy-Schwartz and Lemma 11 to obtain

$$\left|-\frac{3-\gamma}{2}\int_{\mathbb{T}} D^{r-2}\partial_x[v(u+w)]\cdot D^r v\ dx\right| \lesssim \|u+w\|_{H^{r-1}}\|v\|_{H^r}^2$$

which implies

$$\left|-\frac{3-\gamma}{2}\int_{\mathbb{T}} D^{r-2}\partial_x[v(u+w)]\cdot D^r v\ dx\right| \lesssim \|u+w\|_{H^s}\|v\|_{H^r}^2 \tag{8}$$

for $s > 3/2,\ r \leq s,$ and $s + r \geq 2$. For the third integral, we first apply Cauchy-Schwartz to obtain

$$\left|-\frac{\gamma}{2}\int_{\mathbb{T}} D^{r-2}\partial_x[\partial_x v\cdot\partial_x(u+w)]\cdot D^r v\ dx\right| \lesssim \|[\partial_x v\cdot\partial_x(u+w)]\|_{H^{r-1}}\|v\|_{H^r}.$$

Applying Lemma 11 and the inequality $\|f_x\|_{H^{m-1}} \leq \|f\|_{H^m}$, we conclude that

$$\left|-\frac{\gamma}{2}\int_{\mathbb{T}} D^{r-2}\partial_x[\partial_x v\cdot\partial_x(u+w)]\cdot D^r v\ dx\right| \lesssim \|u+w\|_{H^s}\|v\|_{H^r}^2 \tag{9}$$

for $s > 3/2,\ r \leq s,$ and $s + r \geq 2$. Grouping (7)-(9), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|_{H^r}^2 &\lesssim \|u+w\|_{H^s}\|v\|_{H^r}^2, \quad |t| < T\\ &\leq 4R\|v\|_{H^r}^2.\end{aligned}$$

Letting $y(t) = \|v\|_{H^r}^2$, we obtain

$$\frac{dy}{dt} \leq cy$$

where $c = c(s, r, R) > 0$. Hence

$$y(t) \leq y(0)e^{ct}, \quad |t| < T$$

which implies

$$y(t) \leq y(0)e^{cT}.$$

Substituting back in for y, we see that

$$\|v\|_{H^r}^2 \leq \|v(0)\|_{H^r}^2 e^{cT}$$

or

$$\begin{aligned} &\|u(t) - w(t)\|_{H^r} \leq C\|u_0 - w_0\|_{H^r}, \\ &\text{for } |t| < T, \ s > 3/2, \ -1 \leq r \leq s-1, \ s+r \geq 2. \end{aligned} \tag{10}$$

Hence, in region Ω_1, the data-to-solution map is locally Lipschitz from $B_{H^s}(R)$ (measured with the H^r norm) to $C([-T, T], H^r)$, with Lipschitz constant $C = C(s, r, R)$.

2.1.2 Region Ω_2

We have the estimate

$$\|u(t) - w(t)\|_{H^r} < \|u(t) - w(t)\|_{H^{2-s}}. \tag{11}$$

We see that (10) is valid for $r = 2 - s$, $3/2 < s \le 3$. Hence, applying (10) to (11), we obtain

$$\|u(t) - w(t)\|_{H^r} \lesssim \|u_0 - w_0\|_{H^{2-s}}.$$

Applying the interpolation estimate

$$\|f\|_{H^m} \le \|f\|_{H^{m_1}}^{(m_2-m)/(m_2-m_1)} \|f\|_{H^{m_2}}^{(m-m_1)/(m_2-m_1)}, \quad m_1 < m < m_2 \tag{12}$$

with $m_1 = r$, $m = 2 - s$, and $m_2 = s$ (notice $m_2 > m$ for $s > 1$), we bound

$$\begin{aligned}\|u_0 - w_0\|_{H^{2-s}} &\le \|u_0 - w_0\|_{H^r}^{\frac{2(s-1)}{s-r}} \|u_0 - w_0\|_{H^s}^{\frac{2-s-r}{s-r}} \\ &\lesssim \|u_0 - w_0\|_{H^r}^{\frac{2(s-1)}{s-r}}.\end{aligned}$$

We conclude that

$$\|u(t) - w(t)\|_{H^r} \lesssim \|u_0 - w_0\|_{H^r}^{\frac{2(s-1)}{s-r}}.$$

2.1.3 Region Ω_3

Applying (12) with $m_1 = s - 1$, $m = r$ and $m_2 = s$, and using the estimate

$$\|u - w\|_{H^s} \le 4R$$

we obtain

$$\begin{aligned}\|u(t) - w(t)\|_{H^r} &\lesssim \|u(t) - w(t)\|_{H^{s-1}}^{s-r} \|u(t) - w(t)\|_{H^s}^{1-s+r} \\ &\simeq \|u(t) - w(t)\|_{H^{s-1}}^{s-r}.\end{aligned} \tag{13}$$

We see that (10) is valid for $r = s - 1$, $s \geq 3/2$. Hence, applying (10) to (13) gives

$$\begin{aligned}\|u(t) - w(t)\|_{H^r} &\lesssim \|u_0 - w_0\|_{H^{s-1}}^{s-r} \\ &\leq \|u_0 - w_0\|_{H^r}^{s-r}.\end{aligned}$$

This completes the proof of Theorem 3. $\square$

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