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Journal of Differential Equations

www.elsevier.com/locate/jde



Non-uniform dependence and well-posedness for the hyperelastic rod equation

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ARTICLE INFO

Article history:

Received 12 November 2009

Available online 18 April 2010

MSC:

35Q53

Keywords:

Hyperelastic rod equation

Periodic

Non-periodic

Initial value problem

Sobolev spaces

Well-posedness

Continuity of solution map

Non-uniform dependence

Approximate solutions

Energy estimates

ABSTRACT

It is shown that the solution map for the hyperelastic rod equation is not uniformly continuous on bounded sets of Sobolev spaces with exponent greater than $3/2$ in the periodic case and greater than 1 in the non-periodic case. The proof is based on the method of approximate solutions and well-posedness estimates for the solution and its lifespan.

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1. Introduction

We consider the initial value problem for the hyperelastic rod (HR) equation

$$\partial_t u - \partial_x \partial_x^2 u + 3u \partial_x u = \gamma (2\partial_x u \partial_x^2 u + u \partial_x^3 u), \quad (1.1)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{T}, \text{ or } \mathbb{R}, \quad t \in \mathbb{R}, \quad (1.2)$$

where γ is a non-zero constant, and prove that the dependence of solutions on initial data is not uniformly continuous in Sobolev spaces $H^s(\mathbb{T})$, $s > 3/2$. Thus, we extend the result proved by Olson

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[28] in the periodic case (for $s \geq 2$ and $\gamma \neq 3$) to $s > 3/2$ (the entire well-posedness range) for HR. Furthermore, motivated by the work of Himonas and Kenig [14], we establish non-uniform dependence in the non-periodic case, where the method of traveling wave solutions used in [28] does not seem to work.

The HR equation was first derived by Dai in [9] as a one-dimensional model for finite-length and small-amplitude axial deformation waves in thin cylindrical rods composed of a compressible Mooney–Rivlin material. The derivation relied upon a reductive perturbation technique, and took into account the nonlinear dispersion of pulses propagating along a rod. It was assumed that each cross-section of the rod is subject to a stretching and rotation in space. The solution $u(x, t)$ to the HR equation represents the radial stretch relative to a pre-stressed state, while γ is a fixed constant depending upon the pre-stress and the material used in the rod, with values ranging from -29.4760 to 3.4174 .

The well-posedness of the HR equation has been studied by several authors. In Yin [31] and Zhou [32], a proof of local well-posedness in Sobolev spaces H^s , $s > 3/2$, is described on the line and the circle, respectively. Their approach is to rewrite the HR equation in its non-local form, and then to verify the conditions needed to apply Kato's semi-group theory [21]. For details on how this is done for CH on the line, see Rodríguez-Blanco [29]. Blow-up criteria is also investigated in [31] and [32], as well as by Constantin and Strauss [8].

Setting $\gamma = 0$ gives the celebrated BBM equation, which was proposed by Benjamin, Bona, and Mahony [3] as a model for the unidirectional evolution of long waves. Solitary wave solutions to this equation are global and orbitally stable (see Benjamin [2,3], and [8]). For more general γ , the existence of global solutions to HR on the line with constant H^1 energy was proved recently by Mustafa [27] using the approach developed by Bressan and Constantin in [5]. Using a vanishing viscosity argument, Coclite, Holden, and Karlsen [7] established existence of a strongly continuous semi-group of global weak solutions of HR on the line for initial data in H^1 . Bendahmane, Coclite, and Karlsen [1] extended this result to traveling wave solutions that are supersonic solitary shock waves. For more information on the existence of global solutions to the HR equation, see Holden and Raynaud [20] and [31].

There is a variety of traveling wave solutions to the HR equation that can be obtained using various combinations of peaks, cusps, compactons, fractal-like waves, and plateaus (see Lenells [25]). Orbital stability of solitary wave solutions was proved in [8]. Solitary shock wave formation was analyzed in Dai and Huo [11] using traveling wave solutions of the HR equation to derive a system of ordinary differential equations, with a vertical singular line in the phase plane corresponding with the formation of shock waves. Head-on collisions between two solitary waves was investigated in the work of Hui-Hui Dai, Shiqiang Dai, and Huo [10] using a reductive perturbation method coupled with the technique of strained coordinates.

In this work we study the continuity of the data-to-solution map for the HR equation. Using the method of traveling wave solutions it was shown in [28] that the data-to-solution map $u_0 \mapsto u$ of the periodic HR equation is not uniformly continuous from any bounded set in $H^s(\mathbb{T})$ into $C([0, T], H^s(\mathbb{T}))$ for $s \geq 2$ and $\gamma \neq 3$. Non-uniform dependence for the non-periodic CH equation in $H^s(\mathbb{R})$ for $s > 1$ was proven in [14] using the method of approximate solutions and well-posedness estimates. The case $s = 1$ for both the line and the circle was proved earlier by Himonas, Misiólek, and Ponce in [19]. Recently in [15] non-uniform continuity of the solution map for the CH equation on the circle has been proved for the whole range of Sobolev exponents for which local well-posedness of CH is known.

We mention that the continuity of the data-to-solution map for CH has been studied in [19,16], and [17], and for the Euler equations in [18]. Continuity of this map for the Benjamin–Ono equation was studied in Koch and Tzvetkov [24]. For related ill-posedness results, we refer the reader to Kenig, Ponce, and Vega [23], Christ, Colliander, and Tao [6], and the references therein.

Here we consider the initial value problem for the HR equation in both the periodic and non-periodic cases and prove non-uniform continuity of the solution map. More precisely, we show the following result:

Theorem 1. *Let γ be a non-zero constant. Then the data-to-solution map $u(0) \mapsto u(t)$ of the Cauchy problem for the HR equation (1.1)–(1.2) is not uniformly continuous from any bounded subset of H^s into $C([-T, T], H^s)$ for $s > 1$ on the line and for $s > 3/2$ on the circle.*

As we mentioned above, when $\gamma = 0$ the HR equation becomes the BBM equation. Bona and Tzvetkov [4] have recently proved that this equation is globally well-posed in Sobolev spaces H^s , if $s \geq 0$, and that its data-to-solution map is smooth.

Our approach for proving Theorem 1 mirrors that in Himonas and Kenig [14] and Himonas, Kenig, and Misiólek [15]. That is, we will choose approximate solutions to the HR equation such that the size of the difference between approximate and actual solutions with identical initial data is negligible. Hence, to understand the degree of dependence, it will suffice to focus on the behavior of the approximate solutions (which will be simple in form), rather than on the behavior of the actual solutions. In order for the method to go through, we will need well-posedness estimates for the size of the actual solutions to the HR equation, as well a lower bound for their lifespan. This will permit us to obtain an upper bound for the size of the difference of approximate and actual solutions. More precisely, we will need the following well-posedness result with estimates, stated in both the periodic and non-periodic case:

Theorem 2. *If $s > 3/2$ then we have:*

- (i) *If $u_0 \in H^s$ then there exists a unique solution to the Cauchy problem (1.1)–(1.2) in $C([-T, T], H^s)$, where the lifespan T depends on the size of the initial data u_0 . Moreover, the lifespan T satisfies the lower bound estimate*

$$T \geq \frac{1}{2c_s \|u_0\|_{H^s}}. \quad (1.3)$$

- (ii) *The flow map $u_0 \mapsto u(t)$ is continuous from bounded sets of H^s into $C([-T, T], H^s)$, and the solution u satisfies the estimate*

$$\|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s}, \quad |t| \leq T. \quad (1.4)$$

A proof of existence, uniqueness, and continuous dependence in this theorem for $\gamma = 1$ (CH) is given by Li and Olver in [26] using a regularization method, and in [29] using Kato's semi-group method [21]. As mentioned above, proofs of existence, uniqueness, and continuous dependence for HR have been outlined in [31] and [32] for the line and circle, respectively. Both outlines rely upon an application of Kato's semi-group method. However, we have not been able to find estimates (1.3) and (1.4) in the literature. Here we shall give a proof of local well-posedness of HR, including estimates (1.3) and (1.4), which are key ingredients in our work, following an alternative approach used for nonlinear hyperbolic equations in Taylor [30].

The paper is structured as follows. In Section 2 we prove Theorem 1 on the line and in Section 3 we prove it on the circle. As mentioned above, we begin with two sequences of appropriate approximate solutions and then we construct actual solutions coinciding at time zero with the approximate solutions. The key step is to show that the H^s -size of the difference between approximate and actual solutions converges to zero (see Propositions 2 and 3). In Section 4 we prove Theorem 2 using a Galerkin-type argument and energy estimates.

2. Proof of Theorem 1 on the line

We begin by outlining the method of the proof, as it has been applied for the case $\gamma = 1$ in [14]. We will show that there exist two sequences of solutions $u_n(t)$ and $v_n(t)$ in $C([-T, T], H^s)$ such that

$$\|u_n(t)\|_{H^s} + \|v_n(t)\|_{H^s} \lesssim 1, \quad (2.1)$$

$$\lim_{n \rightarrow \infty} \|u_n(0) - v_n(0)\|_{H^s} = 0, \quad (2.2)$$

and

$$\liminf_{n \rightarrow \infty} \|u_n(t) - v_n(t)\|_{H^s} \gtrsim |\sin(\gamma t)|, \quad |\gamma t| \leq 1. \quad (2.3)$$

We accomplish this in two steps. First, we will construct two sequences of approximate solutions satisfying the above properties. Then, we will construct two sequences of actual solutions coinciding with the approximate solutions at time zero. The key point of this method is that the difference between solutions and approximate solutions must decay.

For this method, it is more convenient to rewrite the Cauchy problem for the HR equation in the following non-local form

$$\partial_t u = -\gamma u \partial_x u - \Lambda^{-1} \left[\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} (\partial_x u)^2 \right], \quad (2.4)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \quad (2.5)$$

where

$$\Lambda^{-1} = \partial_x (1 - \partial_x^2)^{-1}.$$

2.1. Approximate solutions

Following [14], our approximate solutions $u^{\omega, \lambda} = u^{\omega, \lambda}(x, t)$ to (2.4)–(2.5) will consist of a low frequency and a high frequency part, i.e.

$$u^{\omega, \lambda} = u_\ell + u^h \quad (2.6)$$

where ω is in a bounded set of \mathbb{R} and $\lambda > 0$. The high frequency part is given by

$$u^h = u^{h, \omega, \lambda}(x, t) = \lambda^{-\frac{\delta}{2}-s} \phi\left(\frac{x}{\lambda^\delta}\right) \cos(\lambda x - \gamma \omega t) \quad (2.7)$$

where ϕ is a C^∞ cut-off function such that

$$\phi = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 2, \end{cases}$$

and by Theorem 2 we let the low frequency part $u_\ell = u_{l, \omega, \lambda}(x, t)$ be the unique solution to the Cauchy problem

$$\partial_t u_\ell = -\gamma u_\ell \partial_x u_\ell - \Lambda^{-1} \left[\frac{3-\gamma}{2} (u_\ell)^2 + \frac{\gamma}{2} (\partial_x u_\ell)^2 \right], \quad (2.8)$$

$$u_\ell(x, 0) = \omega \lambda^{-1} \tilde{\phi}\left(\frac{x}{\lambda^\delta}\right), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \quad (2.9)$$

where $\tilde{\phi}$ is a $C_0^\infty(\mathbb{R})$ function such that

$$\tilde{\phi}(x) = 1 \quad \text{if } x \in \text{supp } \phi. \quad (2.10)$$

We remark that for $\lambda \gg 1$ and $\delta < 2$ the approximate solutions $u^{\omega, \lambda}$ share a common lifespan $T \gg 1$. To see why, we first note that the high frequency part $u^{h, \omega, \lambda}$ has infinite lifespan by the following, whose proof can be found in [14]:

Lemma 1. *Let $\psi \in S(\mathbb{R})$, $\alpha \in \mathbb{R}$. Then for $s \geq 0$ we have*

$$\lim_{\lambda \rightarrow \infty} \lambda^{-\frac{\delta}{2}-s} \left\| \psi \left(\frac{x}{\lambda^\delta} \right) \cos(\lambda x - \alpha) \right\|_{H^s(\mathbb{R})} = \frac{1}{\sqrt{2}} \|\psi\|_{L^2(\mathbb{R})}. \quad (2.11)$$

Relation (2.11) remains true if \cos is replaced by \sin .

For the low frequency part $u_{\ell, \omega, \lambda}$, we apply (1.3) and the estimate

$$\left\| \tilde{\phi} \left(\frac{x}{\lambda^\delta} \right) \right\|_{H^k(\mathbb{R})} \leq \lambda^{\frac{\delta}{2}} \|\tilde{\phi}\|_{H^k(\mathbb{R})}, \quad k \geq 0, \quad (2.12)$$

to obtain a lower bound for its lifespan

$$T_{\ell, \omega, \lambda} \geq \frac{1}{2c_s \|u_{\ell, \omega, \lambda}(0)\|_{H^s(\mathbb{R})}} = \frac{1}{2c_s |\omega| \lambda^{\frac{\delta}{2}-1} \|\tilde{\phi}\|_{H^s(\mathbb{R})}} \gg 1.$$

Since ω belongs to a bounded subset of \mathbb{R} , the existence of a common lifespan $T \gg 1$ follows.

Substituting the approximate solution $u^{\omega, \lambda} = u_\ell + u^h$ into the HR equation, we see that the error E of our approximate solution is given by

$$E = E_1 + E_2 + \cdots + E_8$$

where

$$\begin{aligned} E_1 &= \gamma \lambda^{1-\frac{\delta}{2}-s} [u_\ell(x, 0) - u_\ell(x, t)] \phi \left(\frac{x}{\lambda^\delta} \right) \sin(\lambda x - \gamma \omega t), \\ E_2 &= \gamma \lambda^{-\frac{3\delta}{2}-s} u_\ell(x, t) \cdot \phi' \left(\frac{x}{\lambda^\delta} \right) \cos(\lambda x - \gamma \omega t), \\ E_3 &= \gamma u^h \partial_x u_\ell, \quad E_4 = \gamma u^h \partial_x u^h, \quad E_5 = \frac{3-\gamma}{2} \Lambda^{-1} [(u^h)^2], \\ E_6 &= (3-\gamma) \Lambda^{-1} [u_\ell u^h], \quad E_7 = \frac{\gamma}{2} \Lambda^{-1} [(\partial_x u^h)^2], \quad E_8 = \gamma \Lambda^{-1} [\partial_x u_\ell \partial_x u^h]. \end{aligned} \quad (2.13)$$

Next we prove the decay of the error:

Proposition 1. *Let $1 < \delta < 2$. Then for $s > 1$, bounded ω , and $\lambda \gg 1$ we are assured the decay of the error E of the approximate solutions to the HR equation. Specifically*

$$\|E(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{\frac{\delta}{2}-s}, \quad |t| \leq T. \quad (2.14)$$

It will suffice to estimate the H^1 norms of each E_i . Here we estimate only E_1 . The remaining error terms are estimated like in [14] for the case $\gamma = 1$. We have

$$\begin{aligned} \|E_1\|_{H^1(\mathbb{R})} &= \left\| \gamma \lambda^{1-\frac{\delta}{2}-s} [u_\ell(x, 0) - u_\ell(x, t)] \phi\left(\frac{x}{\lambda^\delta}\right) \sin(\lambda x - \gamma \omega t) \right\|_{H^1(\mathbb{R})} \\ &\lesssim \lambda^{1-\frac{\delta}{2}-s} \left\| [u_\ell(x, 0) - u_\ell(x, t)] \phi\left(\frac{x}{\lambda^\delta}\right) \sin(\lambda x - \gamma \omega t) \right\|_{H^1(\mathbb{R})}. \end{aligned} \quad (2.15)$$

Applying the inequality

$$\|fg\|_{H^1(\mathbb{R})} \leq \sqrt{2} \|f\|_{C^1(\mathbb{R})} \|g\|_{H^1(\mathbb{R})}$$

to estimate (2.15) gives

$$\|E_1\|_{H^1(\mathbb{R})} \lesssim \lambda^{1-\frac{\delta}{2}-s} \left\| \phi\left(\frac{x}{\lambda^\delta}\right) \sin(\lambda x - \gamma \omega t) \right\|_{C^1(\mathbb{R})} \| [u_\ell(x, 0) - u_\ell(x, t)] \|_{H^1(\mathbb{R})}. \quad (2.16)$$

We now estimate the right-hand side of (2.16) in pieces. First, note that routine computations give

$$\left\| \phi\left(\frac{x}{\lambda^\delta}\right) \sin(\lambda x - \gamma \omega t) \right\|_{C^1(\mathbb{R})} \lesssim \lambda. \quad (2.17)$$

Next, we observe that the fundamental theorem of calculus and Minkowski's inequality give

$$\|u_\ell(x, t) - u_\ell(x, 0)\|_{H^1(\mathbb{R})} = \left\| \int_0^t \partial_\tau u_\ell(x, \tau) d\tau \right\|_{H^1(\mathbb{R})} \leq \int_0^t \|\partial_\tau u_\ell(x, \tau)\|_{H^1(\mathbb{R})} d\tau. \quad (2.18)$$

We want to estimate the right-hand side of (2.18). Recalling (2.4), we have

$$\|\partial_\tau u_\ell(x, \tau)\|_{H^1(\mathbb{R})} \leq \|\gamma u_\ell \partial_x u_\ell\|_{H^1(\mathbb{R})} + \left\| \Lambda^{-1} \left[\frac{3-\gamma}{2} (u_\ell)^2 + \frac{\gamma}{2} (\partial_x u_\ell)^2 \right] \right\|_{H^1(\mathbb{R})}. \quad (2.19)$$

Applying the algebra property of Sobolev spaces and the Sobolev Imbedding Theorem, we obtain

$$\|\gamma u_\ell \partial_x u_\ell\|_{H^1(\mathbb{R})} \lesssim \|u_\ell\|_{H^2(\mathbb{R})}^2$$

which yields

$$\|\gamma u_\ell \partial_x u_\ell\|_{H^1(\mathbb{R})} \lesssim \lambda^{-2+\delta} \quad (2.20)$$

by the following:

Lemma 2. Let $0 < \delta < 2$, $\lambda \gg 1$, with ω belonging to a bounded subset of \mathbb{R} . Then the initial value problem (2.8)–(2.9) has a unique solution $u_\ell \in C([-T, T], H^s(\mathbb{R}))$ for all $s \geq 0$ which satisfies

$$\|u_\ell(t)\|_{H^s(\mathbb{R})} \leq c_s \lambda^{-1+\frac{\delta}{2}}, \quad |t| \leq T. \quad (2.21)$$

An analogous result can be found in [14]. Applying the inequality

$$\|\Lambda^{-1}f\|_{H^1(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})},$$

the algebra property of Sobolev spaces, and the Sobolev Imbedding Theorem, we obtain

$$\left\| \Lambda^{-1} \left[\frac{3-\gamma}{2} (u_\ell)^2 + \frac{\gamma}{2} (\partial_x u_\ell)^2 \right] \right\|_{H^1(\mathbb{R})} \lesssim \|u_\ell\|_{H^2(\mathbb{R})}^2$$

which by Lemma 2 gives

$$\left\| \Lambda^{-1} \left[\frac{3-\gamma}{2} (u_\ell)^2 + \frac{\gamma}{2} (\partial_x u_\ell)^2 \right] \right\|_{H^1(\mathbb{R})} \lesssim \lambda^{-2+\delta}, \quad |t| \leq T. \quad (2.22)$$

Substituting (2.20) and (2.22) into the right-hand side of (2.19), and recalling (2.18), we obtain

$$\|u_\ell(x, t) - u_\ell(x, 0)\|_{H^1(\mathbb{R})} \lesssim \lambda^{-2+\delta}, \quad |t| \leq T. \quad (2.23)$$

Grouping estimates (2.16), (2.17), and (2.23), we obtain (2.14). \square

2.2. Construction of solutions

We wish now to estimate the difference between approximate and actual solutions to the HR i.v.p. with common initial data. Let $u_{\omega, \lambda}(x, t)$ be the unique solution to the HR equation with initial data $u^{\omega, \lambda}(x, 0)$. That is, $u_{\omega, \lambda}$ solves the initial value problem

$$\partial_t u_{\omega, \lambda} = -\gamma u_{\omega, \lambda} \partial_x u_{\omega, \lambda} - \Lambda^{-1} \left[\frac{3-\gamma}{2} (u_{\omega, \lambda})^2 + \frac{\gamma}{2} (\partial_x u_{\omega, \lambda})^2 \right], \quad (2.24)$$

$$u_{\omega, \lambda}(x, 0) = u^{\omega, \lambda}(x, 0) = \omega \lambda^{-1} \tilde{\phi}\left(\frac{x}{\lambda^\delta}\right) + \lambda^{-\frac{\delta}{2}-s} \phi\left(\frac{x}{\lambda^\delta}\right) \cos(\lambda x). \quad (2.25)$$

We will now prove that the $H^1(\mathbb{R})$ norm of the difference decays:

Proposition 2. Let $v = u^{\omega, \lambda} - u_{\omega, \lambda}$, with $\lambda \gg 1$. Then, for $s > 1$ and $1 < \delta < 2$ we have

$$\|v(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{\frac{\delta}{2}-s}, \quad |t| \leq T. \quad (2.26)$$

Proof. First we observe that v satisfies

$$\begin{aligned} \partial_t v &= E + \gamma(v \partial_x v - v \partial_x u^{\omega, \lambda} - u^{\omega, \lambda} \partial_x v) \\ &\quad + \Lambda^{-1} \left[\frac{3-\gamma}{2} v^2 + \frac{\gamma}{2} (\partial_x v)^2 - (3-\gamma) u^{\omega, \lambda} v - \gamma \partial_x u^{\omega, \lambda} \partial_x v \right]. \end{aligned}$$

It follows immediately that

$$\begin{aligned} v(1 - \partial_x^2) \partial_t v &= v(1 - \partial_x^2) E + v \gamma (1 - \partial_x^2) (v \partial_x v - v \partial_x u^{\omega, \lambda} - u^{\omega, \lambda} \partial_x v) \\ &\quad + v \partial_x \left[\frac{3-\gamma}{2} v^2 + \frac{\gamma}{2} (\partial_x v)^2 - (3-\gamma) u^{\omega, \lambda} v - \gamma \partial_x u^{\omega, \lambda} \partial_x v \right]. \end{aligned} \quad (2.27)$$

Applying the relation $v \partial_t v = v(1 - \partial_x^2) \partial_t v + v \partial_x^2 \partial_t v$ to (2.27), we obtain

$$\begin{aligned} v \partial_t v &= v(1 - \partial_x^2) E + v \gamma (1 - \partial_x^2) (v \partial_x v - v \partial_x u^{\omega, \lambda} - u^{\omega, \lambda} \partial_x v) \\ &\quad + v \partial_x \left[\frac{3 - \gamma}{2} v^2 + \frac{\gamma}{2} (\partial_x v)^2 - (3 - \gamma) u^{\omega, \lambda} v - \gamma \partial_x u^{\omega, \lambda} \partial_x v \right] + v \partial_x^2 \partial_t v. \end{aligned} \quad (2.28)$$

Adding $\partial_x v \partial_t \partial_x v$ to both sides of (2.28) and integrating gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|_{H^1(\mathbb{R})}^2 &= \int_{\mathbb{R}} [v(1 - \partial_x^2) E] dx \\ &\quad - \gamma \int_{\mathbb{R}} [v(1 - \partial_x^2) (v \partial_x u^{\omega, \lambda} + u^{\omega, \lambda} \partial_x v)] dx \\ &\quad - \int_{\mathbb{R}} [(3 - \gamma) v \partial_x (u^{\omega, \lambda} v) + \gamma v \partial_x (\partial_x u^{\omega, \lambda} \partial_x v)] dx \\ &\quad + \int_{\mathbb{R}} \left[\gamma v(1 - \partial_x^2) (v \partial_x v) + v \partial_x \left(\frac{3 - \gamma}{2} v^2 + \frac{\gamma}{2} (\partial_x v)^2 \right) \right. \\ &\quad \left. + v \partial_x^2 \partial_t v + \partial_x v \partial_t \partial_x v \right] dx. \end{aligned} \quad (2.29)$$

Noting that the last integral can be rewritten as

$$\int_{\mathbb{R}} [\partial_x (v^3) - \gamma \partial_x (v^2 \partial_x^2 v) + \partial_x (v \partial_t \partial_x v)] dx = 0$$

we can simplify (2.29) to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|_{H^1(\mathbb{R})}^2 &= \int_{\mathbb{R}} [v(1 - \partial_x^2) E] dx - \gamma \int_{\mathbb{R}} [v(1 - \partial_x^2) (v \partial_x u^{\omega, \lambda} + u^{\omega, \lambda} \partial_x v)] dx \\ &\quad - \int_{\mathbb{R}} [(3 - \gamma) v \partial_x (u^{\omega, \lambda} v) + \gamma v \partial_x (\partial_x u^{\omega, \lambda} \partial_x v)] dx. \end{aligned} \quad (2.30)$$

We now estimate the three integrals in the right-hand side of (2.30). Integrating by parts and applying Cauchy–Schwartz, we obtain

$$\left| \int_{\mathbb{R}} [v(1 - \partial_x^2) E] dx \right| \lesssim \|v\|_{H^1(\mathbb{R})} \|E\|_{H^1(\mathbb{R})} \quad (2.31)$$

for the first integral,

$$\begin{aligned} &\left| -\gamma \int_{\mathbb{R}} [v(1 - \partial_x^2) (v \partial_x u^{\omega, \lambda} + u^{\omega, \lambda} \partial_x v)] dx \right| \\ &\lesssim (\|u^{\omega, \lambda}\|_{L^\infty(\mathbb{R})} + \|\partial_x u^{\omega, \lambda}\|_{L^\infty(\mathbb{R})} + \|\partial_x^2 u^{\omega, \lambda}\|_{L^\infty(\mathbb{R})}) \|v\|_{H^1(\mathbb{R})}^2 \end{aligned} \quad (2.32)$$

for the second integral, and

$$\begin{aligned} & \left| - \int_{\mathbb{R}} [(3 - \gamma)v \partial_x(u^{\omega, \lambda} v) + \gamma v \partial_x(\partial_x u^{\omega, \lambda} \partial_x v)] dx \right| \\ & \lesssim (\|u^{\omega, \lambda}\|_{L^\infty(\mathbb{R})} + \|\partial_x u^{\omega, \lambda}\|_{L^\infty(\mathbb{R})}) \|v\|_{H^1(\mathbb{R})}^2 \end{aligned} \quad (2.33)$$

for the third integral. Combining (2.31)–(2.33), we obtain

$$\begin{aligned} \frac{d}{dt} \|v(t)\|_{H^1(\mathbb{R})}^2 & \lesssim (\|u^{\omega, \lambda}\|_{L^\infty(\mathbb{R})} + \|\partial_x u^{\omega, \lambda}\|_{L^\infty(\mathbb{R})} + \|\partial_x^2 u^{\omega, \lambda}\|_{L^\infty(\mathbb{R})}) \|v\|_{H^1(\mathbb{R})}^2 \\ & + \|v\|_{H^1(\mathbb{R})} \|E\|_{H^1(\mathbb{R})}. \end{aligned} \quad (2.34)$$

Assume $\lambda \gg 1$. A straightforward calculation of derivatives yields

$$\|u^h\|_{L^\infty(\mathbb{R})} + \|\partial_x u^h\|_{L^\infty(\mathbb{R})} + \|\partial_x^2 u^h\|_{L^\infty(\mathbb{R})} \lesssim \lambda^{-\frac{\delta}{2}-s+2}.$$

Furthermore, by the Sobolev Imbedding Theorem and Lemma 2, we have

$$\|u_\ell\|_{L^\infty(\mathbb{R})} + \|\partial_x u_\ell\|_{L^\infty(\mathbb{R})} + \|\partial_x^2 u_\ell\|_{L^\infty(\mathbb{R})} \leq c_s \|u_\ell\|_{H^3(\mathbb{R})} \lesssim \lambda^{-1+\frac{\delta}{2}}, \quad |t| \leq T.$$

Hence

$$\|u^{\omega, \lambda}\|_{L^\infty(\mathbb{R})} + \|\partial_x u^{\omega, \lambda}\|_{L^\infty(\mathbb{R})} + \|\partial_x^2 u^{\omega, \lambda}\|_{L^\infty(\mathbb{R})} \lesssim \lambda^{-\rho_s}, \quad |t| \leq T, \quad (2.35)$$

where $\rho_s = \min\{\frac{\delta}{2} + s - 2, 1 - \frac{\delta}{2}\}$. Note that for $s > 1$, we can assure $\rho_s > 0$ by choosing a suitable $1 < \delta < 2$. Substituting (2.14) and (2.35) into (2.34), we get

$$\frac{d}{dt} \|v(t)\|_{H(\mathbb{R})}^2 \lesssim \lambda^{-\rho_s} \|v\|_{H^1(\mathbb{R})}^2 + \lambda^{-r_s} \|v\|_{H^1(\mathbb{R})}, \quad |t| \leq T. \quad (2.36)$$

Applying Gronwall's inequality completes the proof. \square

2.3. Non-uniform dependence for $s > 1$

Let $u_{\pm 1, \lambda}$ be solutions to the HR i.v.p. with initial data $u^{\pm 1, n}(0)$. We wish to show that the H^s norm of the difference of $u_{\pm 1, n}$ and the associated approximate solution $u^{\pm 1, \lambda}$ decays as $\lambda \rightarrow \infty$. Note that

$$\begin{aligned} \|u^{\pm 1, \lambda}(t)\|_{H^{2s-1}(\mathbb{R})} & \leq \|u_{\ell, \pm 1, \lambda}\|_{H^{2s-1}(\mathbb{R})} + \left\| \lambda^{-\frac{\delta}{2}-s} \phi\left(\frac{x}{\lambda^\delta}\right) \cos(\lambda x \mp \gamma \omega t) \right\|_{H^{2s-1}(\mathbb{R})} \\ & \lesssim \lambda^{s-1}, \quad |t| \leq T, \end{aligned}$$

where the last step follows from Lemmas 1 and 2. Using (1.4), we have

$$\|u_{\pm 1, \lambda}(t)\|_{H^{2s-1}(\mathbb{R})} \leq 2 \|u^{\pm 1, \lambda}(0)\|_{H^{2s-1}(\mathbb{R})}, \quad |t| \leq T.$$

Hence

$$\|u^{\pm 1, \lambda}(t) - u_{\pm 1, \lambda}(t)\|_{H^{2s-1}(\mathbb{R})} \lesssim \lambda^{s-1}, \quad |t| \leq T. \quad (2.37)$$

Furthermore, by Proposition 2

$$\|u^{\pm 1, \lambda}(t) - u_{\pm 1, \lambda}(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{\frac{\delta}{2}-s}, \quad |t| \leq T. \quad (2.38)$$

Interpolating between estimates (2.37) and (2.38) using the inequality

$$\|\psi\|_{H^s(\mathbb{R})} \leq (\|\psi\|_{H^1(\mathbb{R})} \|\psi\|_{H^{2s-1}(\mathbb{R})})^{\frac{1}{2}}$$

gives

$$\|u^{\pm 1, \lambda}(t) - u_{\pm 1, \lambda}(t)\|_{H^s(\mathbb{R})} \lesssim \lambda^{\frac{\delta-2}{4}}, \quad |t| \leq T. \quad (2.39)$$

Next, we will use estimate (2.39) to prove non-uniform dependence when $s > 1$.

2.4. Behavior at time $t = 0$

We have

$$\|u_{1, \lambda}(0) - u_{-1, \lambda}(0)\|_{H^s(\mathbb{R})} = \|u^{1, \lambda}(0) - u^{-1, \lambda}(0)\|_{H^s(\mathbb{R})} = 2\lambda^{-1} \left\| \tilde{\phi}\left(\frac{x}{\lambda^\delta}\right) \right\|_{H^s(\mathbb{R})}.$$

Applying (2.12) and recalling that $1 < \delta < 2$, we conclude that

$$\|u_{1, \lambda}(0) - u_{-1, \lambda}(0)\|_{H^s(\mathbb{R})} \leq 2\lambda^{\frac{\delta}{2}-1} \|\tilde{\phi}\|_{H^s(\mathbb{R})} \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty. \quad (2.40)$$

2.5. Behavior at time $t > 0$

Using the reverse triangle inequality, we have

$$\begin{aligned} \|u_{1, \lambda}(t) - u_{-1, \lambda}(t)\|_{H^s(\mathbb{R})} &\geq \|u^{1, \lambda}(t) - u^{-1, \lambda}(t)\|_{H^s(\mathbb{R})} - \|u^{1, \lambda}(t) - u_{1, \lambda}(t)\|_{H^s(\mathbb{R})} \\ &\quad - \|-u^{-1, \lambda}(t) + u_{-1, \lambda}(t)\|_{H^s(\mathbb{R})}. \end{aligned} \quad (2.41)$$

Using estimate (2.39) for the last two terms of the right-hand side of (2.41) we obtain

$$\|u_{1, \lambda}(t) - u_{-1, \lambda}(t)\|_{H^s(\mathbb{R})} \geq \|u^{1, \lambda}(t) - u^{-1, \lambda}(t)\|_{H^s(\mathbb{R})} - c\lambda^{\frac{\delta-2}{4}} \quad (2.42)$$

where c is a positive, non-zero constant. Letting λ go to ∞ in (2.42) yields

$$\liminf_{n \rightarrow \infty} \|u_{1, \lambda}(t) - u_{-1, \lambda}(t)\|_{H^s(\mathbb{R})} \geq \liminf_{n \rightarrow \infty} \|u^{1, \lambda}(t) - u^{-1, \lambda}(t)\|_{H^s(\mathbb{R})}. \quad (2.43)$$

Using the identity

$$\cos \alpha - \cos \beta = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$$

gives

$$u^{1,\lambda}(t) - u^{-1,\lambda}(t) = u_{\ell,1,\lambda}(t) - u_{\ell,-1,\lambda}(t) + 2\lambda^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{\lambda^\delta}\right)\sin(\lambda x)\sin(\gamma t). \quad (2.44)$$

Now, by Lemma 2 we have

$$\|u_{\ell,1,\lambda}(t) - u_{\ell,-1,\lambda}(t)\|_{H^s(\mathbb{R})} \lesssim \lambda^{-1+\frac{\delta}{2}}.$$

Hence, applying the reverse triangle inequality to (2.44), we obtain

$$\begin{aligned} & \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s(\mathbb{R})} \\ & \geq 2\lambda^{-\frac{\delta}{2}-s} \left\| \phi\left(\frac{x}{\lambda^\delta}\right)\sin(\lambda x) \right\|_{H^s(\mathbb{R})} |\sin \gamma t| - \|u_{\ell,-1,\lambda}(t) - u_{\ell,1,\lambda}(t)\|_{H^s(\mathbb{R})} \\ & \gtrsim \lambda^{-\frac{\delta}{2}-s} \left\| \phi\left(\frac{x}{\lambda^\delta}\right)\sin(\lambda x) \right\|_{H^s(\mathbb{R})} |\sin \gamma t| - \lambda^{-1+\frac{\delta}{2}}. \end{aligned} \quad (2.45)$$

Letting λ go to ∞ , Lemma 1 with (2.45) gives

$$\liminf_{\lambda \rightarrow \infty} \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s(\mathbb{R})} \gtrsim |\sin \gamma t|, \quad |t| \leq T. \quad (2.46)$$

Combining (2.43) with (2.46), and recalling that $T \gg 1$, we obtain (2.3). This completes the proof of Theorem 1 for the non-periodic case.

3. Proof of Theorem 1 on the circle

Here we follow the proof in [15]. Consider the periodic Cauchy problem for the HR equation

$$\partial_t u = -\gamma u \partial_x u - \Lambda^{-1} \left[\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} (\partial_x u)^2 \right], \quad (3.1)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{T}, \quad t \in \mathbb{R}. \quad (3.2)$$

In this case the approximate solutions are of the form

$$u^{\omega,n}(x, t) = \omega n^{-1} + n^{-s} \cos(nx - \gamma \omega t), \quad (3.3)$$

where n is a positive integer and ω is in a bounded subset of \mathbb{R} . We remark that the approximate solutions are in $C^\infty(\mathbb{T})$ for all $t \in \mathbb{R}$, and hence have infinite lifespan in $H^s(\mathbb{T})$ for $s \geq 0$. Furthermore, for $n \gg 1$ we have

$$\|u^{\omega,n}\|_{H^s(\mathbb{T})} \approx 1 \quad (3.4)$$

from the inequality

$$\|\cos(k(nx - c))\|_{H^s(\mathbb{T})} \simeq n^s, \quad k \in \mathbb{R} \setminus \{0\}. \quad (3.5)$$

Note that for $\gamma = 1$ one gets the approximate solutions used for the CH equation in [15]. Substituting the approximate solutions into (3.1), we obtain the error

$$E = E_1 + E_2 + E_3 \quad (3.6)$$

where

$$E_1 = -\frac{\gamma}{2} n^{-2s+1} \sin[2(nx - \gamma \omega t)], \quad (3.7)$$

$$E_2 = -\Lambda^{-1} \left[\frac{3-\gamma}{2} (n^{-2s+1} \sin(2(nx - \gamma \omega t)) + 2\omega n^{-s} \sin(2(nx - \gamma \omega t))) \right], \quad (3.8)$$

$$E_3 = \frac{\gamma}{4} n^{-2s+2} \left[1 - \cos\left(\frac{nx - \gamma \omega t}{2}\right) \right]. \quad (3.9)$$

Next we will prove a decaying estimate for the error:

Lemma 3. Let $u^{\omega,n}$ be an approximate solution to the HR i.v.p., with $\sigma \leq 1$, ω bounded, and $n \gg 1$. Then for the error E we have

$$\|E(t)\|_{H^\sigma(\mathbb{T})} \lesssim n^{-r_s} \quad \text{where } r_s = \begin{cases} 2(s-1) & \text{if } s \leq 3, \\ s+1 & \text{if } s > 3. \end{cases} \quad (3.10)$$

Proof. It follows from (3.5) and the inequality

$$\|\Lambda^{-1} f\|_{H^k(\mathbb{T})} \leq \|f\|_{H^{k-1}(\mathbb{T})}. \quad \square$$

We are now prepared to prove a decaying estimate for the difference of approximate and actual solutions:

Proposition 3. Let $v = u^{\omega,n} - u_{\omega,n}$, $n \gg 1$, where $u_{\omega,n}$ denotes a solution to the Cauchy problem (3.1)–(3.2) with initial data $u_0(x) = u^{\omega,n}(x, 0)$. If $s > 3/2$ and $\sigma = 1/2 + \varepsilon$ for a sufficiently small $\varepsilon = \varepsilon(s) > 0$, then

$$\|v(t)\|_{H^\sigma(\mathbb{T})} \lesssim n^{-r_s}, \quad |t| \leq T. \quad (3.11)$$

Proof. The difference $v = u^{\omega,n} - u_{\omega,n}$ satisfies the i.v.p.

$$\begin{aligned} \partial_t v &= E - \frac{\gamma}{2} \partial_x [(u^{\omega,n} + u_{\omega,n})v] \\ &\quad - \Lambda^{-1} \left[\frac{3-\gamma}{2} (u^{\omega,n} + u_{\omega,n})v + \frac{\gamma}{2} (\partial_x u^{\omega,n} + \partial_x u_{\omega,n}) \partial_x v \right], \end{aligned} \quad (3.12)$$

$$v(x, 0) = 0. \quad (3.13)$$

For any $\sigma \in \mathbb{R}$ let $D^\sigma = (1 - \partial_x^2)^{\sigma/2}$ be the operator defined by

$$\widehat{D^\sigma f}(\xi) \doteq (1 + \xi^2)^{\sigma/2} \hat{f}(\xi)$$

where \hat{f} is the Fourier transform

$$\hat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-i\xi x} f(x) dx.$$

Applying D^σ to both sides of (3.12), multiplying by $D^\sigma v$, and integrating, we obtain the relation

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^\sigma(\mathbb{T})}^2 &= \int_{\mathbb{T}} D^\sigma E \cdot D^\sigma v \, dx \\ &\quad - \frac{\gamma}{2} \int_{\mathbb{T}} D^\sigma \partial_x [(u^{\omega,n} + u_{\omega,n})v] \cdot D^\sigma v \, dx \\ &\quad - \frac{3-\gamma}{2} \int_{\mathbb{T}} D^{\sigma-2} \partial_x [(u^{\omega,n} + u_{\omega,n})v] \cdot D^\sigma v \, dx \\ &\quad - \frac{\gamma}{2} \int_{\mathbb{T}} D^{\sigma-2} \partial_x [(\partial_x u^{\omega,n} + \partial_x u_{\omega,n}) \cdot \partial_x v] \cdot D^\sigma v \, dx. \end{aligned} \quad (3.14)$$

We now estimate each integral of the right-hand side of (3.14).

Estimate of integral 1. Applying Cauchy–Schwartz, we obtain

$$\left| \int_{\mathbb{T}} D^\sigma E \cdot D^\sigma v \, dx \right| \leq \|E\|_{H^\sigma(\mathbb{T})} \|v\|_{H^\sigma(\mathbb{T})}. \quad (3.15)$$

Estimate of integral 2. We can rewrite

$$\begin{aligned} &-\frac{\gamma}{2} \int_{\mathbb{T}} D^\sigma \partial_x [(u^{\omega,n} + u_{\omega,n})v] \cdot D^\sigma v \, dx \\ &= -\frac{\gamma}{2} \int_{\mathbb{T}} [D^\sigma \partial_x, u^{\omega,n} + u_{\omega,n}]v \cdot D^\sigma v \, dx - \frac{\gamma}{2} \int_{\mathbb{T}} (u^{\omega,n} + u_{\omega,n}) D^\sigma \partial_x v \cdot D^\sigma v \, dx. \end{aligned} \quad (3.16)$$

We now estimate (3.16). Integration by parts and Cauchy–Schwartz gives

$$\left| \frac{\gamma}{2} \int_{\mathbb{T}} (u^{\omega,n} + u_{\omega,n}) D^\sigma \partial_x v \cdot D^\sigma v \, dx \right| \lesssim \|\partial_x (u^{\omega,n} + u_{\omega,n})\|_{L^\infty(\mathbb{T})} \|v\|_{H^\sigma(\mathbb{T})}^2. \quad (3.17)$$

We now need the following result taken from [15]:

Lemma 4. *If $\rho > 3/2$ and $0 \leq \sigma + 1 \leq \rho$, then*

$$\|[D^\sigma \partial_x, f]v\|_{L^2} \leq C \|f\|_{H^\rho} \|v\|_{H^\sigma}. \quad (3.18)$$

Let $\sigma = 1/2 + \varepsilon$ and $\rho = 3/2 + \varepsilon$, where $\varepsilon > 0$ is arbitrarily small. Applying Cauchy–Schwartz and Lemma 4, we obtain

$$\left| -\frac{\gamma}{2} \int_{\mathbb{T}} [D^\sigma \partial_x, u^{\omega,n} + u_{\omega,n}]v \cdot D^\sigma v \, dx \right| \lesssim \|u^{\omega,n} + u_{\omega,n}\|_{H^\rho(\mathbb{T})} \|v\|_{H^\sigma(\mathbb{T})}^2. \quad (3.19)$$

Combining estimates (3.17) and (3.19) we conclude that

$$\begin{aligned} & \left| -\frac{\gamma}{2} \int_{\mathbb{T}} D^{\sigma} \partial_x [(u^{\omega,n} + u_{\omega,n})v] \cdot D^{\sigma} v \, dx \right| \\ & \lesssim (\|u^{\omega,n} + u_{\omega,n}\|_{H^{\rho}(\mathbb{T})} + \|\partial_x u^{\omega,n} + \partial_x u_{\omega,n}\|_{L^{\infty}(\mathbb{T})}) \cdot \|v\|_{H^{\sigma}(\mathbb{T})}^2. \end{aligned} \quad (3.20)$$

Estimate of integral 3. Using Cauchy–Schwartz, and recalling that $\sigma = 1/2 + \varepsilon$, we obtain

$$\left| -\frac{3-\gamma}{2} \int_{\mathbb{T}} D^{\sigma-2} \partial_x [(u^{\omega,n} + u_{\omega,n})v] \cdot D^{\sigma} v \, dx \right| \lesssim \|u^{\omega,n} + u_{\omega,n}\|_{L^{\infty}(\mathbb{T})} \|v\|_{H^{\sigma}(\mathbb{T})}^2. \quad (3.21)$$

Estimate of integral 4. We will need the following result whose proof can be found in [15]:

Lemma 5. If $1/2 < \sigma < 1$ then

$$\|fg\|_{H^{\sigma-1}} \leq C \|f\|_{H^{\sigma}} \|g\|_{H^{\sigma-1}}. \quad (3.22)$$

Applying Cauchy–Schwartz and Lemma 5, we obtain

$$\begin{aligned} & \left| -\frac{\gamma}{2} \int_{\mathbb{T}} D^{\sigma-2} \partial_x [(\partial_x u^{\omega,n} + \partial_x u_{\omega,n}) \cdot \partial_x v] \cdot D^{\sigma} v \, dx \right| \\ & \lesssim \|\partial_x u^{\omega,n} + \partial_x u_{\omega,n}\|_{H^{\sigma}(\mathbb{T})} \|v\|_{H^{\sigma}(\mathbb{T})}^2. \end{aligned} \quad (3.23)$$

Collecting estimates (3.15), (3.20), (3.21), and (3.23), and applying the Sobolev Imbedding Theorem, we deduce

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^{\sigma}(\mathbb{T})}^2 \lesssim \|u^{\omega,n} + u_{\omega,n}\|_{H^{\rho}(\mathbb{T})} \|v\|_{H^{\sigma}(\mathbb{T})}^2 + \|E\|_{H^{\sigma}(\mathbb{T})} \|v\|_{H^{\sigma}(\mathbb{T})}. \quad (3.24)$$

It follows from (1.3) and (3.4) that the solutions $u_{\omega,n}$ have a common lifespan T . Hence, applying the triangle inequality, (1.4), and (3.5) we obtain

$$\|u^{\omega,n} + u_{\omega,n}\|_{H^{\rho}(\mathbb{T})} \lesssim n^{\rho-s}, \quad |t| \leq T. \quad (3.25)$$

Using Lemma 3 and substituting (3.10) and (3.25) into (3.24), we obtain

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^{\sigma}(\mathbb{T})}^2 \lesssim n^{\rho-s} \|v\|_{H^{\sigma}(\mathbb{T})}^2 + n^{-rs} \|v\|_{H^{\sigma}(\mathbb{T})}, \quad |t| \leq T. \quad (3.26)$$

Applying Gronwall's inequality gives (3.11), concluding the proof. \square

3.1. Non-uniform dependence for $s > 3/2$

Let $u_{\pm 1,n}$ be solutions to the HR i.v.p. with common initial data $u^{\pm 1,n}(0)$, respectively. We wish to show that the H^s norm of the difference of $u_{\pm 1,n}$ and the associated approximate solution $u^{\pm 1,n}$ decays. We assume $s > 3/2$ and $\sigma = 1/2 + \varepsilon$ for a sufficiently small $\varepsilon = \varepsilon(s) > 0$. Then by Proposition 3 we have

$$\|u^{\pm 1,n}(t) - u_{\pm 1,n}(t)\|_{H^{\sigma}(\mathbb{T})} \lesssim n^{-rs}, \quad |t| \leq T. \quad (3.27)$$

Furthermore, by (3.5) we obtain

$$\|u^{\pm 1,n}(t)\|_{H^{2s-\sigma}(\mathbb{T})} \lesssim n^{s-\sigma} \quad (3.28)$$

while (1.4) and (3.28) give

$$\|u_{\pm 1,n}(t)\|_{H^{2s-\sigma}(\mathbb{T})} \lesssim n^{s-\sigma}, \quad |t| \leq T. \quad (3.29)$$

Therefore, (3.28), (3.29), and the triangle inequality yield

$$\|u^{\pm 1,n}(t) - u_{\pm 1,n}(t)\|_{H^{2s-\sigma}(\mathbb{T})} \lesssim n^{s-\sigma}, \quad |t| \leq T. \quad (3.30)$$

Interpolating between estimates (3.27) and (3.30) using the inequality

$$\|\psi\|_{H^s(\mathbb{T})} \leq \left(\|\psi\|_{H^\sigma(\mathbb{T})} \|\psi\|_{H^{2s-\sigma}(\mathbb{T})} \right)^{\frac{1}{2}}$$

we obtain

$$\|u^{\pm 1,n}(t) - u_{\pm 1,n}(t)\|_{H^s(\mathbb{T})} \lesssim n^{-\varepsilon(s)/2}, \quad |t| \leq T. \quad (3.31)$$

The remainder of the proof of non-uniform dependence on the circle is analogous to that on the real line.

4. Well-posedness for HR

We will now prove well-posedness for HR. Since the proofs for the circle and the line are similar, we will provide it only for circle. For the line, we present only the needed modifications.

We will prove existence by using an abstract ODE theorem (see Dieudonné [12]) in H^s . Unfortunately, the right-hand side of the HR i.v.p. (3.1)–(3.2) is not a map from H^s to H^s . For this we will consider the following mollification of (3.1)–(3.2)

$$\partial_t u_\varepsilon = -\gamma J_\varepsilon (J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon) - \Lambda^{-1} \left[\frac{3-\gamma}{2} (u_\varepsilon)^2 + \frac{\gamma}{2} (\partial_x u_\varepsilon)^2 \right], \quad (4.1)$$

$$u_\varepsilon(x, 0) = u_0(x), \quad (4.2)$$

where J_ε is defined by

$$J_\varepsilon f(x) = j_\varepsilon * f(x), \quad \varepsilon > 0,$$

with

$$j_\varepsilon(x) = \frac{1}{\varepsilon} j\left(\frac{x}{\varepsilon}\right)$$

for non-negative $j(x) \in \mathcal{S}(\mathbb{R})$. Notice that f_ε given by

$$f_\varepsilon(u) = -\gamma J_\varepsilon (J_\varepsilon u \partial_x J_\varepsilon u) - \Lambda^{-1} \left[\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} (\partial_x u)^2 \right]$$

is a map from $H^s(\mathbb{T})$ into $H^s(\mathbb{T})$. Therefore, (4.1) is an ODE in H^s . Furthermore, f_ε has a continuous total derivative $Df_\varepsilon(u) : H^s(\mathbb{T}) \rightarrow H^s(\mathbb{T})$ given by

$$[Df_\varepsilon(u)](w) = -\gamma J_\varepsilon(J_\varepsilon w \partial_x J_\varepsilon u + J_\varepsilon u \partial_x J_\varepsilon w) - \Lambda^{-1}[(3 - \gamma)wu + \gamma \partial_x w \partial_x u].$$

Hence, by the Cauchy Existence Theorem (see [12]), for each $\varepsilon > 0$ there exists a unique solution $u_\varepsilon \in C(I, H^s(\mathbb{T}))$ satisfying the Cauchy problem (4.1)–(4.2). Next, we analyze the size and lifespan of the family $\{u_\varepsilon\}$ of solutions.

4.1. Estimates for the Lifespan and Sobolev norm of u_ε

We will show that there is a lower bound T for T_ε which is independent of $\varepsilon \in (0, 1]$. This is based on the differential inequality

$$\frac{1}{2} \frac{d}{dt} \|u_\varepsilon(t)\|_{H^s(\mathbb{T})}^2 \leq c_s \|u_\varepsilon(t)\|_{H^s(\mathbb{T})}^3, \quad |t| \leq T_\varepsilon, \quad (4.3)$$

which we now prove by following the approach used for quasilinear symmetric hyperbolic systems in Taylor [30]. In what follows we will suppress the t parameter for the sake of clarity. Applying D^s to both sides of (4.1), multiplying the resulting equation by $D^s u_\varepsilon$, integrating it for $x \in \mathbb{T}$, and noting that D^s and J_ε commute and that J_ε satisfies

$$(J_\varepsilon f, g)_{L^2} = (f, J_\varepsilon g)_{L^2} \quad (4.4)$$

we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_\varepsilon\|_{H^s(\mathbb{T})}^2 &= -\gamma \int_{\mathbb{T}} D^s(J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon) \cdot D^s J_\varepsilon u_\varepsilon dx - \frac{3-\gamma}{2} \int_{\mathbb{T}} D^{s-2} \partial_x (u_\varepsilon)^2 \cdot D^s J_\varepsilon u_\varepsilon dx \\ &\quad - \frac{\gamma}{2} \int_{\mathbb{T}} D^{s-2} \partial_x (\partial_x u_\varepsilon)^2 \cdot D^s J_\varepsilon u_\varepsilon dx. \end{aligned} \quad (4.5)$$

We will estimate the right-hand side of (4.5) in parts. Letting $v = J_\varepsilon u_\varepsilon$ we can rewrite the first integral on the right-hand side of (4.5) as

$$\begin{aligned} &-\gamma \int_{\mathbb{T}} D^s(J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon) \cdot D^s J_\varepsilon u_\varepsilon dx \\ &= -\gamma \int_{\mathbb{T}} [D^s(v \partial_x v) - v D^s(\partial_x v)] \cdot D^s v dx - \gamma \int_{\mathbb{T}} v D^s(\partial_x v) \cdot D^s v dx. \end{aligned} \quad (4.6)$$

We now estimate (4.6) in parts. Applying the Cauchy–Schwartz inequality gives

$$\begin{aligned} &\left| -\gamma \int_{\mathbb{T}} [D^s(v \partial_x v) - v D^s(\partial_x v)] \cdot D^s v dx \right| \\ &\lesssim \|D^s(v \partial_x v) - v D^s(\partial_x v)\|_{L^2(\mathbb{T})} \|v\|_{H^s(\mathbb{T})} \lesssim \|\partial_x v\|_{L^\infty(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}^2, \end{aligned} \quad (4.7)$$

where the last step follows from

$$\|D^s(v\partial_x v) - vD^s(\partial_x v)\|_{L^2(\mathbb{T})} \leq 2c_s \|\partial_x v\|_{L^\infty(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}, \quad (4.8)$$

which is a simple corollary of the following Kato–Ponce commutator estimate, whose proof can be found in [22]:

Lemma 6 (Kato–Ponce). *If $s > 0$ then there is $c_s > 0$ such that*

$$\|D^s(fg) - fD^s g\|_{L^2(\mathbb{T})} \leq c_s (\|D^s f\|_{L^2(\mathbb{T})} \|g\|_{L^\infty(\mathbb{T})} + \|\partial_x f\|_{L^\infty(\mathbb{T})} \|D^{s-1} g\|_{L^2(\mathbb{T})}). \quad (4.9)$$

A proof by Kato and Ponce can be found in [22]. Recalling (4.6), we apply Cauchy–Schwartz to estimate the remaining integral

$$\left| -\gamma \int_{\mathbb{T}} v D^s(\partial_x v) \cdot D^s v \, dx \right| \lesssim \|\partial_x v\|_{L^\infty(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}^2. \quad (4.10)$$

Recalling that $v = J_\varepsilon u_\varepsilon$, combining inequalities (4.7) and (4.10) and applying the Sobolev Imbedding Theorem and the estimate

$$\|J_\varepsilon u_\varepsilon\|_{H^s(\mathbb{T})} \leq \|u_\varepsilon\|_{H^s(\mathbb{T})}$$

we obtain

$$\left| -\gamma \int_{\mathbb{T}} D^s(J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon) \cdot D^s J_\varepsilon u_\varepsilon \, dx \right| \lesssim \|u_\varepsilon\|_{H^s(\mathbb{T})}^3. \quad (4.11)$$

For the remaining integrals of the right-hand side of (4.5), Cauchy–Schwartz and the algebra property of Sobolev spaces give

$$\left| -\frac{3-\gamma}{2} \int_{\mathbb{T}} D^{s-2} \partial_x u_\varepsilon^2 \cdot D^s J_\varepsilon u_\varepsilon \, dx \right| \lesssim \|u_\varepsilon\|_{H^s(\mathbb{T})}^3 \quad (4.12)$$

and

$$\left| -\frac{\gamma}{2} \int_{\mathbb{T}} D^{s-2} (\partial_x u_\varepsilon)^2 \cdot D^s J_\varepsilon u_\varepsilon \, dx \right| \lesssim \|u_\varepsilon\|_{H^s(\mathbb{T})}^3. \quad (4.13)$$

Recalling (4.5) and combining (4.11), (4.12), and (4.13), we obtain (4.3), which is an ordinary differential inequality. Solving it, we obtain the following:

Lemma 7. *Let $u_0(x) \in H^s(\mathbb{T})$, $s > 3/2$. Then for any $\varepsilon \in (0, 1]$ the i.v.p. for the mollified HR equation*

$$\partial_t u_\varepsilon = -\gamma J_\varepsilon (J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon) - \Lambda^{-1} \left[\frac{3-\gamma}{2} (u_\varepsilon)^2 + \frac{\gamma}{2} (\partial_x u_\varepsilon)^2 \right], \quad (4.14)$$

$$u_\varepsilon(x, 0) = u_0(x) \quad (4.15)$$

has a unique solution $u_\varepsilon(t) \in C([-T, T], H^s(\mathbb{T}))$. In particular,

$$T = \frac{1}{2c_s \|u_0\|_{H^s(\mathbb{T})}} \quad (4.16)$$

is a lower bound for the lifespan of $u_\varepsilon(t)$ and

$$\|u_\varepsilon(t)\|_{H^s(\mathbb{T})} \leq 2\|u_0\|_{H^s(\mathbb{T})}, \quad |t| \leq T. \quad (4.17)$$

Furthermore, $u_\varepsilon(t) \in C^1([T, T], H^{s-1}(\mathbb{T}))$ and satisfies

$$\|\partial_t u_\varepsilon(t)\|_{H^{s-1}(\mathbb{T})} \lesssim \|u_0\|_{H^s(\mathbb{T})}^2, \quad |t| \leq T. \quad (4.18)$$

Here c_s is a constant depending only on s .

4.2. Choosing a convergent subsequence

Next we shall show that the family $\{u_\varepsilon\}$ has a convergent subsequence whose limit u solves the HR i.v.p. Let $I = [-T, T]$. By Lemma 7 and the compactness of I we have a uniformly bounded family

$$\{u_\varepsilon\} \subset C(I, H^s(\mathbb{T})) \cap C^1(I, H^{s-1}(\mathbb{T})).$$

By the Riesz Lemma, we can identify $H^s(\mathbb{R})$ with $(H^s(\mathbb{R}))^*$, where for $w, \psi \in H^s(\mathbb{R})$ the duality is defined by

$$T_w(\psi) = \langle w, \psi \rangle_{H^s(\mathbb{R})} = \int_{\mathbb{R}} \hat{w}(\xi, t) \bar{\hat{\psi}}(\xi, t) \cdot (1 + \xi^2)^s d\xi.$$

Applying the Riesz Representation Theorem, it follows that we can identify $L^\infty(I, H^s(\mathbb{T}))$ with the dual space of $L^1(I, H^s(\mathbb{T}))$, where for $v \in L^\infty(I, H^s(\mathbb{T}))$ and $\phi \in L^1(I, H^s(\mathbb{T}))$ the duality is defined by

$$T_v(\phi) = \int_I \langle v(t), \phi(t) \rangle_{H^s(\mathbb{R})} dt = \int_I \int_{\mathbb{R}} \hat{v}(\xi, t) \bar{\hat{\phi}}(\xi, t) \cdot (1 + \xi^2)^s d\xi dt. \quad (4.19)$$

By Alaoglu's Theorem (see Folland [13]), the bounded family $\{u_\varepsilon\}$ is compact in the weak* topology of $L^\infty(I, H^s(\mathbb{T}))$. More precisely, there is a subsequence $\{u_{\varepsilon_k}\}$ converging weakly to a $u \in L^\infty(I, H^s(\mathbb{T}))$. That is

$$\lim_{n \rightarrow \infty} T_{u_{\varepsilon_k}}(\phi) = T_u(\phi) \quad \text{for all } \phi \in L^1(I, H^s(\mathbb{T})).$$

In order to show that u solves the HR i.v.p. we need to obtain a subsequence of $\{u_\varepsilon\}$ with a stronger convergence, so that we can take the limit in the mollified HR equation. First we will need the following interpolation result:

Lemma 8 (Interpolation). *Let $s > \frac{3}{2}$. If $v \in C(I, H^s(\mathbb{T})) \cap C^1(I, H^{s-1}(\mathbb{T}))$ then $v \in C^\sigma(I, H^{s-\sigma}(\mathbb{T}))$ for $0 < \sigma < 1$.*

Applying Lemma 8 gives

$$\sup_{t \neq t'} \frac{\|u_\varepsilon(t) - u_\varepsilon(t')\|_{H^{s-\sigma}(\mathbb{T})}}{|t - t'|^\sigma} < c$$

or

$$\|u_\varepsilon(t) - u_\varepsilon(t')\|_{H^{s-\sigma}(\mathbb{T})} < c|t - t'|^\sigma \quad \text{for all } t, t' \in I,$$

which shows that the family $\{u_\varepsilon\}$ is equicontinuous in $C(I, H^{s-\sigma}(\mathbb{T}))$. Furthermore, since the inclusion $H^s(\mathbb{T}) \subset H^{s-\sigma}(\mathbb{T})$ is compact and $\{u_\varepsilon(t)\}$ is a uniformly bounded family by (4.17), it follows that $\{u_\varepsilon(t)\}$ is precompact in $H^{s-\sigma}(\mathbb{T})$. Hence, we can apply Ascoli's Theorem [12] to conclude that there exists a subsequence $\{u_{\varepsilon_n}\}$ such that

$$u_{\varepsilon_n} \rightarrow u \quad \text{in } C(I, H^{s-\sigma}(\mathbb{T})). \quad (4.20)$$

4.3. Verifying that u solves the HR equation

Using (4.20) and the Sobolev Imbedding Theorem, we see that

$$\begin{aligned} & -\gamma J_\varepsilon(J_{\varepsilon_n} u_{\varepsilon_n} J_{\varepsilon_n} \partial_x u_{\varepsilon_n}) - \Lambda^{-1} \left(\frac{3-\gamma}{2} (u_{\varepsilon_n})^2 + \frac{\gamma}{2} (\partial_x u_{\varepsilon_n})^2 \right) \\ & \rightarrow -\gamma u \partial_x u - \Lambda^{-1} \left(\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} (\partial_x u)^2 \right) \quad \text{in } C(I, C(\mathbb{T})). \end{aligned} \quad (4.21)$$

Furthermore, since (4.20) holds, we have

$$T_{u_{\varepsilon_n}}(\phi) \rightarrow T_u(\phi) \quad \text{for all } \phi \in L^1(I, H^s(\mathbb{T})) \quad (4.22)$$

which implies

$$T_{\partial_t u_{\varepsilon_n}}(\phi) \rightarrow T_{\partial_t u}(\phi) \quad \text{for all } \phi \in L^1(I, H^s(\mathbb{T})). \quad (4.23)$$

It follows from the uniqueness of the limit in (4.21) that

$$\partial_t u = -\gamma u \partial_x u - \Lambda^{-1} \left[\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} (\partial_x u)^2 \right]. \quad (4.24)$$

Thus we have constructed a solution $u \in L^\infty(I, H^s(\mathbb{T}))$ to the HR i.v.p. It remains to prove that $u \in C(I, H^s(\mathbb{T}))$.

Proof that $u \in C(I, H^s(\mathbb{T}))$. Since $u \in L^\infty(I, H^s(\mathbb{T}))$, it is a continuous function from I to $H^s(\mathbb{T})$ with respect to the weak topology on $H^s(\mathbb{T})$. That is, for $\{t_n\} \subset I$ such that $t_n \rightarrow t$, we have

$$\langle u(t_n), v \rangle_{H^s(\mathbb{T})} \rightarrow \langle u(t), v \rangle_{H^s(\mathbb{T})}, \quad \forall v \in H^s(\mathbb{T}). \quad (4.25)$$

Next, note that

$$\begin{aligned} \|u(t) - u(t_n)\|_{H^s(\mathbb{T})}^2 &= \langle u(t) - u(t_n), u(t) - u(t_n) \rangle_{H^s(\mathbb{T})} \\ &= \|u(t)\|_{H^s(\mathbb{T})}^2 + \|u(t_n)\|_{H^s(\mathbb{T})}^2 - \langle u(t_n), u(t) \rangle_{H^s(\mathbb{T})} \\ &\quad - \langle u(t), u(t_n) \rangle_{H^s(\mathbb{T})}. \end{aligned} \quad (4.26)$$

Applying (4.25) to (4.26), we see that

$$\lim_{n \rightarrow \infty} \|u(t) - u(t_n)\|_{H^s(\mathbb{T})}^2 = \left[\lim_{n \rightarrow \infty} \|u(t_n)\|_{H^s(\mathbb{T})}^2 \right] - \|u(t)\|_{H^s(\mathbb{T})}^2. \quad (4.27)$$

Hence, to prove that $u \in C(I, H^s(\mathbb{T}))$, it will be enough to show that the map $t \mapsto \|u(t)\|_{H^s(\mathbb{T})}$ is a continuous function of t . This will follow from the energy estimate

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^s(\mathbb{T})}^2 \leq c_s \|u(t)\|_{H^s(\mathbb{T})}^3, \quad |t| \leq T, \quad (4.28)$$

which we now derive. Applying D^s to both sides of (4.24), multiplying the resulting equation by $D^s u$, and integrating for $x \in \mathbb{T}$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{H^s(\mathbb{T})}^2 &= -\gamma \int_{\mathbb{T}} D^s(u \partial_x u) \cdot D^s u \, dx - \frac{3-\gamma}{2} \int_{\mathbb{T}} D^{s-2} \partial_x(u^2) \cdot D^s u \, dx \\ &\quad - \frac{\gamma}{2} \int_{\mathbb{T}} D^{s-2} \partial_x(\partial_x u)^2 \cdot D^s u \, dx. \end{aligned} \quad (4.29)$$

Using estimates analogous to those in (4.6)–(4.13), we obtain (4.28). Derivating the left-hand side of (4.28) and simplifying gives

$$\frac{d}{dt} \|u(t)\|_{H^s(\mathbb{T})} \leq c_s \|u(t)\|_{H^s(\mathbb{T})}^2, \quad |t| \leq T. \quad (4.30)$$

Solving this ordinary differential inequality yields an upper bound

$$\|u(t)\|_{H^s(\mathbb{T})} \leq 2\|u_0\|_{H^s(\mathbb{T})}, \quad |t| \leq T, \quad (4.31)$$

for the size of the solution. Since $\|u(t)\|_{H^s(\mathbb{T})}$ is uniformly bounded for $|t| \leq T$ by (4.31), we conclude from (4.30) that the map $t \mapsto \|u(t)\|_{H^s(\mathbb{T})}$ is Lipschitz continuous in t , for $|t| \leq T$. \square

4.4. Uniqueness

Let $u, \omega \in C(I, H^s(\mathbb{T}))$, $s > 3/2$, be two solutions to the Cauchy problem (3.1)–(3.2) with common initial data. Set $v = u - \omega$. Then v solves the Cauchy problem

$$\partial_t v = -\frac{\gamma}{2} \partial_x[v(u + \omega)] - D^{-2} \partial_x \left\{ \frac{3-\gamma}{2} [v(u + \omega)] + \frac{\gamma}{2} [\partial_x v \cdot \partial_x(u + \omega)] \right\}, \quad (4.32)$$

$$v(x, 0) = 0. \quad (4.33)$$

Applying D^σ to both sides of (4.32), then multiplying both sides by $D^\sigma v$ and integrating, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|_{H^\sigma(\mathbb{T})}^2 &= -\frac{\gamma}{2} \int_{\mathbb{T}} D^\sigma \partial_x[v(u + \omega)] \cdot D^\sigma v \, dx - \frac{3-\gamma}{2} \int_{\mathbb{T}} D^{\sigma-2} \partial_x[v(u + \omega)] \cdot D^\sigma v \, dx \\ &\quad - \frac{\gamma}{2} \int_{\mathbb{T}} D^{\sigma-2} \partial_x[\partial_x v \cdot \partial_x(u + \omega)] \cdot D^\sigma v \, dx. \end{aligned} \quad (4.34)$$

We now estimate (4.34) in parts.

Estimate of integral 1. Note that

$$\begin{aligned} & \left| -\frac{\gamma}{2} \int_{\mathbb{T}} D^{\sigma} \partial_x [v(u+w)] \cdot D^{\sigma} v \, dx \right| \\ &= \left| -\frac{\gamma}{2} \int_{\mathbb{T}} [D^{\sigma} \partial_x, u+w] v \cdot D^{\sigma} v \, dx - \frac{\gamma}{2} \int_{\mathbb{T}} (u+w) D^{\sigma} \partial_x v \cdot D^{\sigma} v \, dx \right| \\ &\leq \left| -\frac{\gamma}{2} \int_{\mathbb{T}} [D^{\sigma} \partial_x, u+w] v \cdot D^{\sigma} v \, dx \right| + \left| \frac{\gamma}{2} \int_{\mathbb{T}} (u+w) D^{\sigma} \partial_x v \cdot D^{\sigma} v \, dx \right|. \end{aligned} \quad (4.35)$$

Observe that by integrating by parts gives

$$\left| \frac{\gamma}{2} \int_{\mathbb{T}} (u+w) D^{\sigma} \partial_x v \cdot D^{\sigma} v \, dx \right| \lesssim \|\partial_x(u+w)\|_{L^{\infty}(\mathbb{T})} \|v\|_{H^{\sigma}(\mathbb{T})}^2. \quad (4.36)$$

To estimate the remaining integral of (4.35), we first choose $3/2 < \rho < s$ and $1/2 < \sigma \leq \rho - 1$. An application of Cauchy–Schwartz and Lemma 4 then yields

$$\left| -\frac{\gamma}{2} \int_{\mathbb{T}} [D^{\sigma} \partial_x, u+w] v \cdot D^{\sigma} v \, dx \right| \lesssim \|u+w\|_{H^{\rho}(\mathbb{T})} \|v\|_{H^{\sigma}(\mathbb{T})}^2. \quad (4.37)$$

Combining (4.36) and (4.37) and applying the Sobolev Imbedding Theorem, we obtain the estimate

$$\left| -\frac{\gamma}{2} \int_{\mathbb{T}} D^{\sigma} \partial_x [v(u+w)] \cdot D^{\sigma} v \, dx \right| \lesssim \|u+w\|_{H^{\rho}(\mathbb{T})} \|v\|_{H^{\sigma}(\mathbb{T})}^2. \quad (4.38)$$

Estimate of integral 2. Applying Cauchy–Schwartz, the algebra property of Sobolev spaces, and the Sobolev Imbedding Theorem, we obtain

$$\left| -\frac{3-\gamma}{2} \int_{\mathbb{T}} D^{\sigma-2} \partial_x [v(u+w)] \cdot D^{\sigma} v \, dx \right| \lesssim \|u+w\|_{H^{\sigma-1}(\mathbb{T})} \|v\|_{H^{\sigma}(\mathbb{T})}^2. \quad (4.39)$$

Estimate of integral 3. We first apply Cauchy–Schwartz and the Sobolev Imbedding Theorem to obtain

$$\left| -\frac{\gamma}{2} \int_{\mathbb{T}} D^{\sigma-2} \partial_x [\partial_x v \cdot \partial_x (u+w)] \cdot D^{\sigma} v \, dx \right| \lesssim \|[\partial_x v \cdot \partial_x (u+w)]\|_{H^{\sigma-1}(\mathbb{T})} \|v\|_{H^{\sigma}(\mathbb{T})}.$$

Restrict $1/2 < \sigma < 1$. Then applying Lemma 5 and the Sobolev Imbedding Theorem, we conclude that

$$\left| -\frac{3-\gamma}{2} \int_{\mathbb{T}} D^{\sigma-2} \partial_x [v(u+w)] \cdot D^{\sigma} v \, dx \right| \lesssim \|u+w\|_{H^{\sigma-1}(\mathbb{T})} \|v\|_{H^{\sigma}(\mathbb{T})}^2. \quad (4.40)$$

Grouping (4.38), (4.39), and (4.40), and applying the Sobolev Imbedding Theorem, we obtain

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^\sigma(\mathbb{T})}^2 \lesssim \|u + w\|_{H^\rho(\mathbb{T})} \|v\|_{H^\sigma(\mathbb{T})}^2. \quad (4.41)$$

Since $v_0 = 0$ and $\|u + w\|_{H^\rho} \leq \|u + w\|_{H^s(\mathbb{T})} < \infty$ for $|t| \leq T$, we deduce from (4.41) and an application of Gronwall's inequality that $v = 0$. \square

4.5. Continuous dependence

Let $\{u_{0,n}\}_n \subset H^s(\mathbb{T})$ be a uniformly bounded sequence converging to u_0 in $H^s(\mathbb{T})$. Consider solutions u , u^ε , u_n^ε , and u_n to the Cauchy problem (3.1)–(3.2) with associated initial data u_0 , $J_\varepsilon u_0$, $J_\varepsilon u_{0,n}$, and $u_{0,n}$, respectively, where J_ε is the operator defined by

$$J_\varepsilon f(x) = j_\varepsilon * f(x), \quad \varepsilon > 0. \quad (4.42)$$

Here

$$j_\varepsilon(x) = \sum_{\xi \in \mathbb{Z}} \hat{j}(\varepsilon\xi) e^{i\xi x}, \quad \varepsilon > 0, \quad (4.43)$$

where $\hat{j}(\xi) \in \mathcal{S}(\mathbb{R})$ is chosen such that

$$0 \leq \hat{j}(\xi) \leq 1 \quad \text{and} \quad \hat{j}(\xi) = 1 \quad \text{if} \quad |\xi| \leq 1. \quad (4.44)$$

We remark that it follows immediately from (4.43) that

$$\hat{j}_\varepsilon(\xi) = \hat{j}(\varepsilon\xi), \quad \varepsilon > 0. \quad (4.45)$$

This will prove crucial later on. Next, applying the triangle inequality, we obtain

$$\|u - u_n\|_{H^s(\mathbb{T})} \leq \|u - u^\varepsilon\|_{H^s(\mathbb{T})} + \|u^\varepsilon - u_n^\varepsilon\|_{H^s(\mathbb{T})} + \|u_n^\varepsilon - u_n\|_{H^s(\mathbb{T})}.$$

Let $\eta > 0$. To prove continuous dependence, it will be enough to show that we can find $\varepsilon > 0$ and $N \in \mathbb{N}$ such that for all $n > N$

$$\|u(t) - u^\varepsilon(t)\|_{H^s(\mathbb{T})} < \eta/3, \quad |t| \leq T, \quad (4.46)$$

$$\|u^\varepsilon(t) - u_n^\varepsilon(t)\|_{H^s(\mathbb{T})} < \eta/3, \quad |t| \leq T, \quad (4.47)$$

$$\|u_n^\varepsilon(t) - u_n(t)\|_{H^s(\mathbb{T})} < \eta/3, \quad |t| \leq T. \quad (4.48)$$

The proof of (4.48) will be analogous to that of (4.46), so we will omit the details.

Proof of (4.46). Consider two solutions u and u^ε to the Cauchy problem (3.1)–(3.2) with associated initial data u_0 and $J_\varepsilon u_0$, respectively. Set $v = u - u^\varepsilon$. Then v solves the Cauchy problem

$$\begin{aligned} \partial_t v = & -\gamma(v \partial_x v + v \partial_x u^\varepsilon + u^\varepsilon \partial_x v) \\ & - D^{-2} \partial_x \left\{ \left(\frac{3-\gamma}{2} \right) (v^2 + 2u^\varepsilon v) + \frac{\gamma}{2} [(\partial_x v)^2 + 2\partial_x u^\varepsilon \partial_x v] \right\}, \end{aligned} \quad (4.49)$$

$$v(0) = (I - J_\varepsilon)u_0. \quad (4.50)$$

Applying the operator D^s to both sides of (4.49), then multiplying by $D^s v$ and integrating gives

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^s(\mathbb{T})}^2 = A + B \quad (4.51)$$

where

$$\begin{aligned} A = & -\gamma \int_{\mathbb{T}} D^s(v \partial_x v) \cdot D^s v \, dx - \frac{3-\gamma}{2} \int_{\mathbb{T}} D^{s-2} \partial_x(v^2) \cdot D^s v \, dx \\ & - \frac{\gamma}{2} \int_{\mathbb{T}} D^{s-2} \partial_x(\partial_x v)^2 \cdot D^s v \, dx \end{aligned} \quad (4.52)$$

and

$$\begin{aligned} B = & -\gamma \int_{\mathbb{T}} D^s(v \partial_x u^\varepsilon) \cdot D^s v \, dx - \gamma \int_{\mathbb{T}} D^s(u^\varepsilon \partial_x v) \cdot D^s v \, dx \\ & - (3-\gamma) \int_{\mathbb{T}} D^{s-2} \partial_x(u^\varepsilon v) \cdot D^s v \, dx - \gamma \int_{\mathbb{T}} D^{s-2} \partial_x(\partial_x u^\varepsilon \cdot \partial_x v) \cdot D^s v \, dx. \end{aligned} \quad (4.53)$$

Using estimates analogous to those in (4.6)–(4.13), we obtain

$$|A| \lesssim \|v\|_{H^s(\mathbb{T})}^3, \quad |t| \leq T. \quad (4.54)$$

Next we estimate B in parts:

Estimate of integral 1. We can rewrite

$$\begin{aligned} & -\gamma \int_{\mathbb{T}} D^s(v \partial_x u^\varepsilon) \cdot D^s v \, dx \\ = & -\gamma \int_{\mathbb{T}} [D^s(v \partial_x u^\varepsilon) - v D^s \partial_x u^\varepsilon] \cdot D^s v \, dx - \gamma \int_{\mathbb{T}} v D^s \partial_x u^\varepsilon \cdot D^s v \, dx. \end{aligned} \quad (4.55)$$

Applying Cauchy–Schwartz, the Kato–Ponce estimate (4.9), and the Sobolev Imbedding Theorem, we obtain

$$\left| -\gamma \int_{\mathbb{T}} [D^s(v \partial_x u^\varepsilon) - v D^s \partial_x u^\varepsilon] \cdot D^s v \, dx \right| \lesssim \|u^\varepsilon\|_{H^s(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}^2. \quad (4.56)$$

For the remaining integral of (4.55), Cauchy–Schwartz and the Sobolev Imbedding Theorem give

$$\left| -\gamma \int_{\mathbb{T}} u^\varepsilon D^s \partial_x v \cdot D^s v \, dx \right| \lesssim \|u^\varepsilon\|_{H^{s+1}(\mathbb{T})} \|v\|_{H^{s-1}(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}. \quad (4.57)$$

Combining estimates (4.56) and (4.57) we conclude that

$$\left| -\gamma \int_{\mathbb{T}} D^s(v \partial_x u^\varepsilon) \cdot D^s v \, dx \right| \lesssim \|u^\varepsilon\|_{H^s(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}^2 + \|u^\varepsilon\|_{H^{s+1}(\mathbb{T})} \|v\|_{H^{s-1}(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}. \quad (4.58)$$

Estimate of integral 2. We can rewrite

$$\begin{aligned} & -\gamma \int_{\mathbb{T}} D^s(u^\varepsilon \partial_x v) \cdot D^s v \, dx \\ &= -\gamma \int_{\mathbb{T}} [D^s(u^\varepsilon \partial_x v) - u^\varepsilon D^s \partial_x v] \cdot D^s v \, dx - \gamma \int_{\mathbb{T}} u^\varepsilon D^s \partial_x v \cdot D^s v \, dx. \end{aligned} \quad (4.59)$$

Applying Cauchy–Schwartz, the Kato–Ponce estimate (4.9), and the Sobolev Imbedding Theorem to the first integral, we obtain

$$\left| -\gamma \int_{\mathbb{T}} [D^s(u^\varepsilon \partial_x v) - u^\varepsilon D^s \partial_x v] \cdot D^s v \, dx \right| \lesssim \|u^\varepsilon\|_{H^s(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}^2. \quad (4.60)$$

For the remaining integral of (4.59), integration by parts, Cauchy–Schwartz, and the Sobolev Imbedding Theorem give

$$\left| -\gamma \int_{\mathbb{T}} u^\varepsilon D^s \partial_x v \cdot D^s v \, dx \right| \lesssim \|u^\varepsilon\|_{H^s(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}^2. \quad (4.61)$$

Combining estimates (4.60) and (4.61) we conclude that

$$\left| -\gamma \int_{\mathbb{T}} D^s(u^\varepsilon \partial_x v) \cdot D^s v \, dx \right| \lesssim \|u^\varepsilon\|_{H^s(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}^2. \quad (4.62)$$

Estimate of integral 3. Applying Cauchy–Schwartz, the algebra property of Sobolev spaces, and the Sobolev Imbedding Theorem gives

$$\left| -(3 - \gamma) \int_{\mathbb{T}} D^{s-2} \partial_x(u^\varepsilon v) \cdot D^s v \, dx \right| \lesssim \|u^\varepsilon\|_{H^s(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}^2. \quad (4.63)$$

Estimate of integral 4. Applying Cauchy–Schwartz, the algebra property of Sobolev spaces, and the Sobolev Imbedding Theorem, we obtain

$$\left| -\gamma \int_{\mathbb{T}} D^{s-2} \partial_x(\partial_x u^\varepsilon \cdot \partial_x v) \cdot D^s v \, dx \right| \lesssim \|u^\varepsilon\|_{H^s(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}^2.$$

Hence, collecting our estimates for integrals 1–4, we obtain

$$|B| \lesssim \|u^\varepsilon\|_{H^s(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}^2 + \|u^\varepsilon\|_{H^{s+1}(\mathbb{T})} \|v\|_{H^{s-1}(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}. \quad (4.64)$$

Combining estimates (4.54) and (4.64) and recalling (4.51), we obtain

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^s(\mathbb{T})}^2 \lesssim \|v\|_{H^s(\mathbb{T})}^3 + \|u^\varepsilon\|_{H^s(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}^2 + \|u^\varepsilon\|_{H^{s+1}(\mathbb{T})} \|v\|_{H^{s-1}(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}$$

which simplifies to

$$\frac{d}{dt} \|v\|_{H^s(\mathbb{T})} \lesssim \|v\|_{H^s(\mathbb{T})}^2 + \|v\|_{H^s(\mathbb{T})} + \varepsilon^{-1} \|v\|_{H^{s-1}(\mathbb{T})} \quad (4.65)$$

by differentiating the left-hand side and applying the following lemma:

Lemma 9. For $r \geq s > 3/2$ and $0 < \varepsilon < 1$,

$$\|u^\varepsilon(t)\|_{H^r(\mathbb{T})} \lesssim \varepsilon^{s-r}. \quad (4.66)$$

Proof. Recalling the construction of J_ε in (4.42)–(4.45), we have

$$|\widehat{J_\varepsilon u_0}(\xi)| = |\hat{j}_\varepsilon(\xi) \hat{u}_0(\xi)| = |\hat{j}(\varepsilon \xi) \hat{u}_0(\xi)| \leq c_r |\varepsilon \xi|^{s-r} \hat{u}_0(\xi), \quad r \geq s, \xi \neq 0. \quad (4.67)$$

Applying (1.4) and (4.67), the result follows. \square

We now aim to prove decay for the $\varepsilon^{-1} \|v\|_{H^{s-1}(\mathbb{T})}$ term in (4.65). To do so, we will first obtain an estimate for $\|v\|_{H^\sigma(\mathbb{T})}$ for suitably chosen $\sigma < s - 1$. Then, interpolating between $\|v\|_{H^\sigma(\mathbb{T})}$ and $\|v\|_{H^s(\mathbb{T})}$, we will show that $\|v\|_{H^{s-1}(\mathbb{T})}$ experiences $o(\varepsilon)$ decay. This will imply $o(1)$ decay of $\varepsilon^{-1} \|v\|_{H^{s-1}(\mathbb{T})}$.

Proposition 4. For σ such that $1/2 < \sigma < 1$ and $\sigma + 1 \leq s$, we have

$$\|v\|_{H^\sigma(\mathbb{T})} = o(\varepsilon^{s-\sigma}), \quad |t| \leq T. \quad (4.68)$$

Proof. Recall that v solves the Cauchy problem (4.49)–(4.50). Applying D^σ to both sides of (4.49), then multiplying by $D^\sigma v$ and integrating, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^\sigma(\mathbb{T})}^2 &= -\frac{\gamma}{2} \int_{\mathbb{T}} D^\sigma \partial_x [(u + u^\varepsilon)v] \cdot D^\sigma v \, dx \\ &\quad - \frac{3-\gamma}{2} \int_{\mathbb{T}} D^{\sigma-2} \partial_x [(u + u^\varepsilon)v] \cdot D^\sigma v \, dx \\ &\quad - \frac{\gamma}{2} \int_{\mathbb{T}} D^{\sigma-2} \partial_x [(\partial_x u + \partial_x u^\varepsilon) \cdot \partial_x v] \cdot D^\sigma v \, dx. \end{aligned}$$

Repeating calculations (3.14)–(3.23), with E set to zero, $u^{\omega,n}$ replaced by u , $u_{\omega,n}$ replaced by u^ε , and σ and ρ chosen such that

$$1/2 < \sigma < 1 \quad \text{and} \quad \sigma + 1 \leq \rho \leq s$$

yields

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^\sigma(\mathbb{T})}^2 \lesssim (\|u^\varepsilon + u\|_{H^\rho(\mathbb{T})} + \|\partial_x(u^\varepsilon + u)\|_{H^\sigma(\mathbb{T})}) \cdot \|v\|_{H^\sigma(\mathbb{T})}^2.$$

By the Sobolev Imbedding Theorem, it follows that

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^\sigma(\mathbb{T})}^2 \lesssim \|u^\varepsilon + u\|_{H^s(\mathbb{T})} \cdot \|v\|_{H^\sigma(\mathbb{T})}^2. \quad (4.69)$$

Hence, applying the triangle inequality, (1.4), and the estimate

$$\|J_\varepsilon f\|_{H^s(\mathbb{T})} \leq \|f\|_{H^s(\mathbb{T})} \quad (4.70)$$

to the right-hand side of (4.69) yields

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^\sigma(\mathbb{T})}^2 \leq C \|v\|_{H^\sigma(\mathbb{T})}^2$$

where $C = C(\|u_0\|_{H^s(\mathbb{T})})$. Gronwall's inequality then gives

$$\|v\|_{H^\sigma(\mathbb{T})} \leq e^{Ct} \|v(0)\|_{H^\sigma(\mathbb{T})} = e^{Ct} \|u_0 - J_\varepsilon u_0\|_{H^\sigma(\mathbb{T})} = o(\varepsilon^{s-r})$$

where the last step follows from the operator norm estimate provided below. \square

Lemma 10. For $r \leq s$ and $\varepsilon > 0$

$$\|I - J_\varepsilon\|_{L(H^s(\mathbb{T}), H^r(\mathbb{T}))} = o(\varepsilon^{s-r}). \quad (4.71)$$

Proof. Let $u \in H^s(\mathbb{T})$ and $r, s \in \mathbb{R}$ such that $r \leq s$. Recalling the construction of J_ε in (4.42)–(4.45), we have

$$\|u - J_\varepsilon u\|_{H^r(\mathbb{T})}^2 = \sum_{\xi \in \mathbb{Z}} |[1 - \hat{j}(\varepsilon\xi)] \cdot \hat{u}(\xi)|^2 (1 + \xi^2)^r \quad \text{and} \quad (4.72)$$

$$|1 - \hat{j}(\varepsilon\xi)| \leq |\varepsilon\xi|^{s-r}, \quad \xi \in \mathbb{R}, \quad \varepsilon > 0. \quad (4.73)$$

Applying (4.73) to (4.72) we obtain

$$\|u - J_\varepsilon u\|_{H^r(\mathbb{T})} \lesssim \varepsilon^{(s-r)}$$

while a dominated convergence argument gives

$$\|u - J_\varepsilon u\|_{H^s(\mathbb{T})} = o(1).$$

Applying the interpolation estimate

$$\|f\|_{H^{k_2}(\mathbb{T})} \leq \|f\|_{H^{k_1}(\mathbb{T})}^{(s-k_2)/(s-k_1)} \|f\|_{H^s(\mathbb{T})}^{1-(s-k_2)/(s-k_1)}, \quad k_1 < k_2 \leq s, \quad (4.74)$$

completes the proof. \square

We now return to analyzing the $\varepsilon^{-1} \|v\|_{H^{s-1}(\mathbb{T})}$ term of (4.65). Applying (4.74) and Proposition 4, we obtain

$$\|v\|_{H^{s-1}(\mathbb{T})} \lesssim o(\varepsilon) \|v\|_{H^s(\mathbb{T})}^{1-1/(s-\sigma)}.$$

Note that $\|v(t)\|_{H^s(\mathbb{T})}$ is uniformly bounded for all $\varepsilon > 0$. More precisely, by the triangle inequality, (4.70), and (1.4), we have

$$\|v(t)\|_{H^s(\mathbb{T})} \leq 4\|u_0\|_{H^s(\mathbb{T})}, \quad |t| \leq T. \quad (4.75)$$

Hence

$$\|v\|_{H^{s-1}(\mathbb{T})} = o(\varepsilon)$$

which implies

$$\varepsilon^{-1}\|v\|_{H^{s-1}(\mathbb{T})} = o(1). \quad (4.76)$$

Substituting (4.76) into (4.65), we obtain

$$\frac{d}{dt}\|v\|_{H^s(\mathbb{T})} \lesssim \|v\|_{H^s(\mathbb{T})}^2 + \|v\|_{H^s(\mathbb{T})} + o(1). \quad (4.77)$$

Letting $y(t) = \|v(t)\|_{H^s(\mathbb{T})}$, we can factor the right-hand side to obtain

$$\frac{dy}{dt} \lesssim (y - \alpha)(y - \beta) \quad (4.78)$$

where

$$\alpha = \frac{-1 + \sqrt{1 - o(1)}}{2} \quad \text{and} \quad \beta = \frac{-1 - \sqrt{1 - o(1)}}{2}. \quad (4.79)$$

Rewriting (4.78) yields

$$\left(\frac{1}{y - \alpha} - \frac{1}{y - \beta} \right) \frac{dy}{dt} \lesssim \sqrt{1 - o(1)} \approx 1.$$

Noting that $1/(y - \alpha) - 1/(y - \beta)$ is positive, and integrating from 0 to t , we obtain

$$\ln \left(\frac{y(t) - \alpha}{y(t) - \beta} \cdot \frac{y(0) - \beta}{y(0) - \alpha} \right) \leq ct.$$

Exponentiating both sides and rearranging gives

$$\frac{y(t) - \alpha}{y(t) - \beta} \leq e^{ct} \cdot \frac{y(0) - \alpha}{y(0) - \beta}$$

which implies

$$y(t) \leq e^{ct} \cdot \frac{[y(0) - \alpha][y(t) - \beta]}{y(0) - \beta} + \alpha \lesssim [y(0) - \alpha][y(t) - \beta] + \alpha, \quad |t| \leq T,$$

where the last step follows from the fact that $1/2 \leq -\beta \leq 1$. Substituting back in $\|v\|_{H^s(\mathbb{T})}$, we obtain

$$\|v\|_{H^s(\mathbb{T})} \lesssim [\|v(0)\|_{H^s(\mathbb{T})} - \alpha][\|v\|_{H^s(\mathbb{T})} - \beta] + \alpha. \quad (4.80)$$

Noting that $\|v\|_{H^s(\mathbb{T})}$ is uniformly bounded in ε by (4.75), $\alpha \rightarrow 0$, and

$$\|v(0)\|_{H^s(\mathbb{T})} = \|u_0 - J_\varepsilon u_0\|_{H^s(\mathbb{T})} \rightarrow 0$$

by Lemma 10, we conclude from (4.80) that

$$\|v(t)\|_{H^s(\mathbb{T})} = \|u(t) - u^\varepsilon(t)\|_{H^s(\mathbb{T})} = o(1), \quad |t| \leq T. \quad (4.81)$$

Choosing ε sufficiently small gives $\|v(t)\|_{H^s(\mathbb{T})} < \eta/3$, completing the proof of (4.46). \square

Proof of (4.47). Let $v = u_n^\varepsilon - u^\varepsilon$. Then v solves the Cauchy problem

$$\begin{aligned} \partial_t v = & -\gamma(v \partial_x v + v \partial_x u^\varepsilon + u^\varepsilon \partial_x v) \\ & - D^{-2} \partial_x \left\{ \left(\frac{3-\gamma}{2} \right) (v^2 + 2u^\varepsilon v) + \frac{\gamma}{2} [(\partial_x v)^2 + 2\partial_x u^\varepsilon \partial_x v] \right\}, \end{aligned} \quad (4.82)$$

$$v(0) = J_\varepsilon(u_{0,n} - u_0). \quad (4.83)$$

Applying the operator D^s to both sides of (4.82), multiplying by D^s and integrating, and estimating as in (4.54)–(4.64), we obtain

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^s(\mathbb{T})}^2 \lesssim \|v\|_{H^s(\mathbb{T})}^3 + \|u^\varepsilon\|_{H^s(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}^2 + \|u^\varepsilon\|_{H^{s+1}(\mathbb{T})} \|v\|_{H^{s-1}(\mathbb{T})} \|v\|_{H^s(\mathbb{T})}$$

which by differentiating the left-hand side and applying Lemma 9 to the right-hand side simplifies to

$$\frac{d}{dt} \|v\|_{H^s(\mathbb{T})} \lesssim \|v\|_{H^s(\mathbb{T})}^2 + \|v\|_{H^s(\mathbb{T})} + \varepsilon^{-1} \|v\|_{H^{s-1}(\mathbb{T})}. \quad (4.84)$$

We now aim to control of the growth the $\varepsilon^{-1} \|v\|_{H^{s-1}(\mathbb{T})}$ term of (4.84). To do so, we will need an estimate for $\|v\|_{H^{s-1}(\mathbb{T})}$, which we will obtain using interpolation. First, we will need the following:

Proposition 5. For σ such that $1/2 < \sigma < 1$ and $\sigma + 1 \leq s$,

$$\|v\|_{H^\sigma(\mathbb{T})} = \|u_n^\varepsilon - u^\varepsilon\|_{H^\sigma(\mathbb{T})} \lesssim \|u_0 - u_{0,n}\|_{H^s(\mathbb{T})}, \quad |t| \leq T. \quad (4.85)$$

Proof. Repeating calculations (3.14)–(3.23), with E set to zero, $u^{\omega,n}$ replaced by u_n^ε , $u_{\omega,n}$ replaced by u^ε , and σ and ρ chosen such that

$$1/2 < \sigma < 1 \quad \text{and} \quad \sigma + 1 \leq \rho \leq s \quad (4.86)$$

yields

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^\sigma(\mathbb{T})}^2 \lesssim (\|u_n^\varepsilon + u^\varepsilon\|_{H^\rho(\mathbb{T})} + \|\partial_x(u_n^\varepsilon + u^\varepsilon)\|_{H^\sigma(\mathbb{T})}) \cdot \|v\|_{H^\sigma(\mathbb{T})}^2.$$

Since $\{u_{0,n}\}$ belongs to a bounded subset of $H^s(\mathbb{T})$, it follows that

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^\sigma(\mathbb{T})}^2 \leq C \|v\|_{H^\sigma(\mathbb{T})}^2 \quad (4.87)$$

where $C = C(\|u_0\|_{H^s(\mathbb{T})}, \limsup_{n \rightarrow \infty} \|u_{0,n}\|_{H^s(\mathbb{T})})$. Applying Gronwall's inequality to (4.87), we obtain

$$\|v\|_{H^\sigma(\mathbb{T})} \leq e^{Ct} \|v(0)\|_{H^\sigma(\mathbb{T})} = e^{Ct} \|u^\varepsilon(0) - u_n^\varepsilon(0)\|_{H^\sigma(\mathbb{T})} \leq e^{Ct} \|u_0 - u_{0,n}\|_{H^\sigma(\mathbb{T})}$$

concluding the proof. \square

We now return to analyzing the $\varepsilon^{-1} \|v\|_{H^{s-1}(\mathbb{T})}$ term of (4.84). Applying the interpolation estimate (4.74) and Proposition 5 gives

$$\|v\|_{H^{s-1}(\mathbb{T})} \lesssim \|u_0 - u_{0,n}\|_{H^s(\mathbb{T})}^{1/(s-\sigma)} \|v\|_{H^s(\mathbb{T})}^{1-1/(s-\sigma)}. \quad (4.88)$$

Note that the triangle inequality, (1.4), and (4.70) imply that $\|v\|_{H^s(\mathbb{T})}$ is uniformly bounded in n and ε . That is

$$\|v\|_{H^s(\mathbb{T})} \leq 2 \left[\|u_0\|_{H^s(\mathbb{T})} + \limsup_{n \rightarrow \infty} \|u_{0,n}\|_{H^s(\mathbb{T})} \right], \quad |t| \leq T.$$

Hence, (4.88) gives

$$\|v\|_{H^{s-1}(\mathbb{T})} \lesssim \|u_0 - u_{0,n}\|_{H^s(\mathbb{T})}^{1/(s-\sigma)}. \quad (4.89)$$

Fix $\varepsilon, \rho > 0$. Since $\|u_0 - u_{0,n}\|_{H^s(\mathbb{T})} \rightarrow 0$, we can find $N \in \mathbb{N}$ such that for all $n > N$

$$\varepsilon^{-1} \|u_0 - u_{0,n}\|_{H^s(\mathbb{T})}^{1/(s-\sigma)} < \rho$$

which by (4.89) implies

$$\varepsilon^{-1} \|v\|_{H^{s-1}(\mathbb{T})} \lesssim \rho. \quad (4.90)$$

Since ρ can be chosen to be arbitrarily small, the remainder of the proof is analogous to that of (4.46). \square

4.6. Extending well-posedness to the non-periodic case

In the proof of existence on the line, we will have difficulties in arranging that the solutions $\{u_\varepsilon\}$ to the mollified HR i.v.p. converge in $C(I, H^{s-\sigma}(\mathbb{R}))$, $0 < \sigma < 1$, to a candidate solution u of the HR i.v.p., since the inclusion $H^s(\mathbb{R}) \subset H^{s-\sigma}(\mathbb{R})$ is not compact for $\sigma > 0$ (contrast this with the situation on the circle). However, by Rellich's Theorem, the map $f \mapsto \varphi f$ is a compact operator from $H^s(\mathbb{R})$ to $H^{s-\sigma}(\mathbb{R})$ for any $\varphi \in \mathcal{S}(\mathbb{R})$. Hence, considering the family $\{\varphi u_\varepsilon\}$ instead, it can be shown that for arbitrary $k \in \mathbb{N}$

$$\begin{aligned} \varphi \Lambda^{-1}[(u_{\varepsilon_n})^k] &\rightarrow \varphi \Lambda^{-1}[u^k] \quad \text{in } C(I, H^{s-\sigma}(\mathbb{R})), \\ \varphi \Lambda^{-1}[(\partial_x u_{\varepsilon_n})^k] &\rightarrow \varphi \Lambda^{-1}[(\partial_x u)^k] \quad \text{in } C(I, H^{s-\sigma-1}(\mathbb{R})). \end{aligned} \quad (4.91)$$

To utilize this result, we multiply both sides of (4.14) by φ and rewrite to obtain the Cauchy problem

$$\partial_t(\varphi u_{\varepsilon_n}) = -\gamma \varphi J_\varepsilon(J_{\varepsilon_n} u_{\varepsilon_n} J_{\varepsilon_n} \partial_x u_{\varepsilon_n}) - \varphi \Lambda^{-1} \left(\frac{3-\gamma}{2} (u_{\varepsilon_n})^2 + \frac{\gamma}{2} (\partial_x u_{\varepsilon_n})^2 \right), \quad (4.92)$$

$$u_{\varepsilon_n}(x, 0) = u_0(x). \quad (4.93)$$

Choosing $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\varphi^{\frac{1}{2}} \in \mathcal{S}(\mathbb{R})$, (4.91) and the Sobolev Imbedding Theorem gives

$$\begin{aligned} & -\gamma \varphi J_{\varepsilon}(J_{\varepsilon_n} u_{\varepsilon_n} J_{\varepsilon_n} \partial_x u_{\varepsilon_n}) - \varphi \Lambda^{-1} \left(\frac{3-\gamma}{2} (u_{\varepsilon_n})^2 + \frac{\gamma}{2} (\partial_x u_{\varepsilon_n})^2 \right) \\ & \rightarrow -\gamma \varphi u \partial_x u - \varphi \Lambda^{-1} \left(\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} (\partial_x u)^2 \right) \quad \text{in } C(I, C(\mathbb{R})). \end{aligned} \quad (4.94)$$

Restricting φ to be non-zero, and using an argument analogous to that in the periodic case, it follows from (4.94) that u is a solution to the HR i.v.p. (2.4)–(2.5). Proofs of $u \in C(I, H^s(\mathbb{R}))$ and uniqueness are analogous to the proofs in the periodic case.

For the proof of continuous dependence, the method mirrors that of the periodic case. However, we must choose a different mollifier J_{ε} . Define

$$J_{\varepsilon} f(x) = j_{\varepsilon} * f(x), \quad \varepsilon > 0, \quad (4.95)$$

where

$$j_{\varepsilon}(x) = \frac{1}{\varepsilon} j\left(\frac{x}{\varepsilon}\right). \quad (4.96)$$

Here $j(x) \in \mathcal{S}(\mathbb{R})$ such that

$$0 \leq \hat{j}(\xi) \leq 1 \quad \text{and} \quad \hat{j}(\xi) = 1 \quad \text{if } |\xi| \leq 1. \quad (4.97)$$

From (4.96) it follows that

$$\hat{j}_{\varepsilon}(\xi) = \hat{j}(\varepsilon \xi), \quad \varepsilon > 0. \quad (4.98)$$

Given this construction, the proofs of Lemmas 9 and 10 for the non-periodic case will be analogous to those in the periodic case. Hence, how we construct the mollifier J_{ε} plays a critical role in the proofs of well-posedness for the HR i.v.p. in both the periodic and non-periodic cases.

Acknowledgments

The author thanks his advisor Alex Himonas for his guidance and many helpful remarks. The author also thanks the Department of Mathematics at the University of Notre Dame and the Arthur J. Schmitt Foundation for supporting his doctoral studies.

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