Andrei D. Polyanin
Valentin F. Zaitsev

$$\frac{d^n y}{dx^n}$$

HANDBOOK OF EXACT SOLUTIONS for ORDINARY DIFFERENTIAL EQUATIONS

SECOND EDITION

CHAPMAN & HALL/CRC

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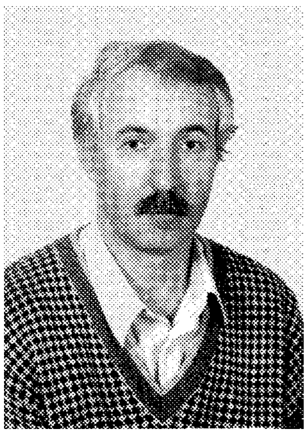
AUTHORS



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A. D. Polyanin graduated from the Department of Mechanics and Mathematics of the Moscow State University in 1974. He earned his Ph.D. degree in 1981 and D.Sc. degree in 1986 at the Institute for Problems in Mechanics of the Russian (former USSR) Academy of Sciences. Since 1975, A. D. Polyanin has been a member of the staff of the Institute for Problems in Mechanics of the Russian Academy of Sciences. He is a member of the Russian National Committee on Theoretical and Applied Mechanics.

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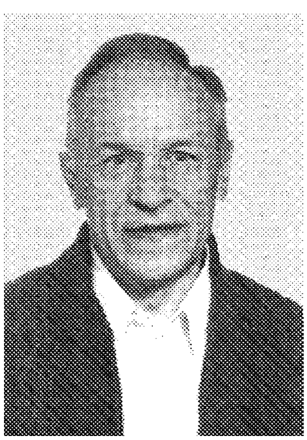

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FOREWORD

Exact solutions of differential equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. These solutions can be used to verify the consistencies and estimate errors of various numerical, asymptotic, and approximate analytical methods.

This book contains nearly 6200 ordinary differential equations and their solutions. A number of new solutions to nonlinear equations are described. In some sections of this book, asymptotic solutions to some classes of equations are also given.

When selecting the material, the authors gave preference to the following two types of equations:

- Equations that have traditionally attracted the attention of many researchers: those of the simplest appearance but involving the most difficulties for integration (Abel equations, Emden–Fowler equations, Painlevé equations, etc.)
- Equations that are encountered in various applications (in the theory of heat and mass transfer, nonlinear mechanics, elasticity, hydrodynamics, theory of nonlinear oscillations, combustion theory, chemical engineering science, etc.)

Special attention is paid to equations that depend on arbitrary functions. All other equations contain one or more arbitrary parameters (in fact, this book deals with whole families of ordinary differential equations), which can be fixed by a reader at will. In total, the handbook contains many more equations than any other book currently available (for example, the number of nonlinear equations of the second and higher order is ten times more than in the well-known E. Kamke's *Handbook on Ordinary Differential Equations*).

For the reader's convenience, the introductory chapter of the book outlines basic definitions, useful formulas, and some transformations. In a concise form, it also presents exact, asymptotic, and approximate analytical methods for solving linear and nonlinear differential equations. Specific examples of utilization of these methods are considered. Formulations of existence and uniqueness theorems are also given. Boundary-value problems and eigenvalue problems are described.

The handbook consists of chapters, sections, subsections, and paragraphs. Equations and formulas are numbered separately in each subsection. The equations within subsections and paragraphs are arranged in increasing order of complexity. The extensive table of contents provides rapid access to the desired equations.

The main material is followed by some supplements, where basic properties of elementary and special functions (Bessel, modified Bessel, hypergeometric, Legendre, etc.) are described.

Here are *three main distinguishing features of the second edition vs. the first edition*:

- 1200 nonlinear equations with solutions have been added.
- An introductory chapter that outlines exact, asymptotic, and approximate analytical methods for solving ordinary differential equations has been included.
- The overwhelming majority of subsections are organized into paragraphs. As a result, the table of contents has been increased threefold to help the readers get faster access to desired equations.

We would like to express our deep gratitude to Alexei Zhurov for fruitful discussions and valuable remarks. We are very grateful to Alain Moussiaux who tested a number of solutions, which allowed us to remove some inaccuracies and misprints.

The authors hope that this book will be helpful for a wide range of scientists, university teachers, engineers, and students engaged in the fields of mathematics, physics, mechanics, control, chemistry, and engineering sciences.

Andrei D. Polyanin
Valentin F. Zaitsev

NOTATIONS AND SOME REMARKS

1. Throughout this book, in the original equations the independent variable is denoted by , and the dependent one is denoted by . In the given solutions, the symbols , $_0$, $_1$, $_2$, stand for arbitrary integration constants. Solutions are often represented in parametric form (e.g., see Subsections 1.3.1 and 2.3.1).

2. Notation for derivatives:

$$' = \frac{}{}, \qquad '' = \frac{2}{\ 2}, \qquad ''' = \frac{3}{\ 3}, \qquad '''' = \frac{4}{\ 4}; \qquad ^{(\ )} = \frac{}{} \quad \text{with} \quad \geq 5.$$

3. Brief notation for partial derivatives:

$$= \frac{}{}, \qquad = \frac{}{}, \qquad = \frac{2}{\ 2}, \qquad = \frac{2}{}, \qquad \text{where} \quad = (\ ,\ ).$$

4. In some cases, we use the operator notation $\frac{}{}\ g$, which is defined by the recurrence relation

$$(\ )\frac{}{}\ g(\ ) = (\ )\frac{}{}\ (\ )\frac{}{}\ ^{-1} g(\ )\ .$$

5. Brief operator notation corresponding to partial derivatives: $= \frac{}{}, \quad = \frac{}{}.$

6. In some sections of the book (e.g., see 1.3, 2.3–2.6, 3.2–3.3), for the sake of brevity, solutions are represented as several formulas containing the terms with the signs " " and "∓." Two formulas are meant—one corresponds to the upper sign and the other to the lower sign. For example, the solution of equation 1.3.1.6 can be written in the parametric form:

$$= a\ ^{-1}\exp(\mp\tau^2), \qquad = a\ ^{-1}\big[\exp(\mp\tau^2)\ \ 2\tau\ \big], \qquad \text{where} \quad = \ \exp(\mp\tau^2)\ \tau - \ , \quad A = \mp 2a^2.$$

This is equivalent so that the solutions of equation 1.3.1.6 are given by the two formulas:

$$= a\ ^{-1}\exp(-\tau^2), \qquad = a\ ^{-1}\big[\exp(-\tau^2) + 2\tau\ \big], \qquad \text{where} \quad = \ \exp(-\tau^2)\ \tau - \ , \quad A = -2a^2$$

(for the upper signs) and

$$= a\ ^{-1}\exp(\tau^2), \qquad = a\ ^{-1}\big[\exp(\tau^2) - 2\tau\ \big], \qquad \text{where} \quad = \ \exp(\tau^2)\ \tau - \ , \quad A = 2a^2$$

(for the lower signs).

7. If a relation contains an expression like $\frac{(\ )}{a-2}$, it is often not stated that the assumption $a \neq 2$ is adopted.

8. In solutions, expressions like $(\ ) = \frac{1}{+1}\ ^{+1}$ can usually be defined so as to cover the case $= -1$ in accordance with the rule $_{-1}(\ ) = \ln|\ |$. This is accounted for by the fact that such expressions arise from the integration of the power-law function $(\ ) =$.

9. The order symbol is used to compare two functions $= (\varepsilon)$ and $g = g(\varepsilon)$, where ε is a small parameter. So $= (g)$ means that $|\ \ g|$ is bounded as $\varepsilon \quad 0$, or and g are of the same order of magnitude as $\varepsilon \quad 0$.

10. When referencing a particular equation, a notation like “4.1.2.5” stands for “equation 5 in Subsection 4.1.2.”

11. The handbooks by Kamke (1977), Murphy (1960), Zaitsev and Polyanin (1993, 2001), Polyanin and Zaitsev (1995) were extensively used in compiling this book; references to these sources are frequently omitted.

12. To highlight portions of the text, the following symbols are used throughout the book:

indicates important information pertaining to a group of equations (Chapters 1–5);

indicates the literature used in the preparation of the text in subsections, paragraphs, and specific equations.

Introduction

Some Definitions, Formulas, Methods, and Transformations

0.1. First-Order Differential Equations

0.1.1. General Concepts. The Cauchy Problem. Uniqueness and Existence Theorems

0.1.1-1. Equations solved for the derivative. General solution.

A *first-order ordinary differential equation** solved for the derivative has the form

$$' = (,). \qquad (1)$$

Sometimes it is represented in terms of differentials as = (,) .

A *solution of a differential equation* is a function ( ) that, when substituted into the equation, turns it into an identity. The *general solution of a differential equation* is the set of all its solutions. In some cases, the general solution can be represented as a function = (,) that depends on one *arbitrary constant* ; specific values of define specific solutions of the equation (*particular solutions*). In practice, the general solution more frequently appears in implicit form, (, ,) = 0, or parametric form, = (,), = (,).

Geometrically, the general solution (also called the general integral) of an equation is a family of curves in the -plane depending on a single parameter ; these curves are called *integral curves* of the equation. To each particular solution (particular integral) there corresponds a single curve that passes through a given point in the plane.

For each point (,), the equation ′ = (,) defines a value of ′, i.e., the slope of the integral curve that passes through this point. In other words, the equation generates a field of directions in the -plane. From the geometrical point of view, the problem of solving a first-order differential equation involves finding the curves, the slopes of which at each point coincide with the direction of the field at this point.

0.1.1-2. The Cauchy problem. The uniqueness and existence theorems.

1 . The *Cauchy problem*: find a solution of equation (1) that satisfies the *initial condition*

$$= {}_0 \quad \text{at} \quad = {}_0, \qquad (2)$$

where $_0$ and $_0$ are some numbers.

Geometrical meaning of the Cauchy problem: find an integral curve of equation (1) that passes through the point ($_0$, $_0$).

Condition (2) is alternatively written ($_0$) = $_0$ or $|_{=0}$ = $_0$.

* In what follows, we often call an ordinary differential equation a "differential equation" or even shorter an "equation."

2 . *Existence theorem* (Peano). Let the function (,) be continuous in an open domain of the -plane. Then there is at least one integral curve of equation (1) that passes through each point ($_0$, $_0$) ; each of these curves can be extended at both ends up to the boundary of any closed domain $_0 \subset$ such that ($_0$, $_0$) belongs to the interior of $_0$.

3 . *Uniqueness theorem.* Let the function (,) be continuous in an open domain and have in a bounded partial derivative with respect to (or the Lipschitz condition holds: | (,) − (, z)| ≤ | − z|, where is some positive number). Then there is a unique solution of equation (1) satisfying condition (2).

0.1.1-3. Equations not solved for the derivative. The existence theorem.

A first-order differential equation not solved for the derivative can generally be written as

$$(\ , \ , \ ') = 0. \tag{3}$$

Existence and uniqueness theorem. There exists a unique solution = ( ) of equation (3) satisfying the conditions $|_{=\ _0} = \ _0$ and $'|_{=\ _0} = \ _0$, where $_0$ is one of the real roots of the equation ($_0$, $_0$, $_0$) = 0, if the following conditions hold in a neighborhood of the point ($_0$, $_0$, $_0$):

1. The function (, ,) is continuous in each of the three arguments.
2. The partial derivative exists and is nonzero.
3. There is a bounded partial derivative with respect to , | | ≤ .

The solution exists for | − $_0$| ≤ a, where a is a (sufficiently small) positive number.

0.1.1-4. Singular solutions.

1 . A point (,) at which the uniqueness of the solution to equation (3) is violated is called a *singular* point. If conditions 1 and 3 of the existence and uniqueness theorem hold, then

$$(\ , \ , \ ) = 0, \qquad (\ , \ , \ ) = 0 \tag{4}$$

simultaneously at each singular point. Relations (4) define a *-discriminant curve* in parametric form. In some cases, the parameter can be eliminated from (4) to give an equation of this curve in implicit form, (,) = 0. If a branch = ( ) of the curve (,) = 0 consists of singular points and, at the same time, is an integral curve, then this branch is called a *singular integral curve* and the function = ( ) is a *singular solution* of equation (3).

2 . The singular solutions can be found by identifying the *envelope of the family of integral curves*, (, ,) = 0, of equation (3). The envelope is part of the *-discriminant curve*, which is defined by the equations

$$(\ , \ , \ ) = 0, \qquad (\ , \ , \ ) = 0.$$

The branch of the -discriminant curve at which

(a) there exist bounded partial derivatives, | | < $_1$ and | | < $_2$, and
(b) | | + | | ≠ 0

is the envelope.

0.1.1-5. Point transformations.

In the general case, a point transformation is defined by

$$= \ (X, \ ), \qquad = \ (X, \ ), \tag{5}$$

where X is the new independent variable, $= (X)$ is the new dependent variable, and and are some (prescribed or unknown) functions.

The derivative $'$ under the point transformation (5) is calculated by

$$' = \frac{+ \qquad '}{+ \qquad '},$$

where the subscripts X and denote the corresponding partial derivatives.

Transformation (5) is invertible if $\qquad - \qquad \neq 0$.

Point transformations are used to simplify equations and reduce them to known equations. Sometimes a point transformation allows the reduction of a nonlinear equation to a linear one. Examples of point transformations can be found in Subsections 0.1.2 and 0.1.4–0.1.6.

References for Subsection 0.1.1: G. M. Murphy (1960), G. A. Korn and T. M. Korn (1968), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1980), D. Zwillinger (1998).

0.1.2. Equations Solved for the Derivative. Simplest Techniques of Integration

0.1.2-1. Equations with separated or separable variables.

1 . An *equation with separated variables* (a *separated equation*) has the form

$$(\;)\ ' = g(\;).$$

Equivalently, the equation can be rewritten as $(\;) \quad = g(\;)$ (the right-hand side depends on alone and the left-hand side on alone). The general solution can be obtained by termwise integration:

$$(\;) \qquad = \qquad g(\;) \qquad + \quad ,$$

where is an arbitrary constant.

2 . An *equation with separable variables* (a *separable equation*) is generally represented by

$$_1(\;)g_1(\;)\ ' = {}_2(\;)g_2(\;).$$

Dividing the equation by $_2(\;)g_1(\;)$, one obtains a separated equation. Integrating yields:

$$\frac{_1(\;)}{_2(\;)} \qquad = \qquad \frac{g_2(\;)}{g_1(\;)} \qquad + \quad .$$

In termwise division of the equation by $_2(\;)g_1(\;)$, solutions corresponding to $_2(\;) = 0$ can be lost.

0.1.2-2. Equation of the form $' = (a \quad + \quad)$.

The substitution $z = a \quad +$ brings the equation to a separable equation, $z' = \quad (z) + a$; see Paragraph 0.1.2-1.

0.1.2-3. Homogeneous equations and equations reducible to them.

1 . A *homogeneous equation* remains the same under simultaneous scaling (dilatation) of the independent and dependent variables in accordance with the rule , , where is an arbitrary constant ($\neq 0$). Such equations can be represented in the form

$$' = \quad - \; .$$

The substitution $=$ brings a homogeneous equation to a separable one, $' = (\;) -$; see Paragraph 0.1.2-1.

2 . The equations of the form

$$' = \frac{a_1 \;+ {}_1 \;+ {}_1}{a_2 \;+ {}_2 \;+ {}_2}$$

can be reduced to a homogeneous equation. To this end, for $a_1 \;+ {}_1 \;\neq k(a_2 \;+ {}_2\;)$, one should use the change of variables $\xi = \;- {}_0$, $\;= \;- {}_0$, where the constants ${}_0$ and ${}_0$ are determined by solving the linear algebraic system

$$a_1 {}_0 + {}_1 {}_0 + {}_1 = 0,$$
$$a_2 {}_0 + {}_2 {}_0 + {}_2 = 0.$$

As a result, one arrives at the following equation for $\;= \;(\xi)$:

$$' = \frac{a_1\xi + {}_1}{a_2\xi + {}_2}.$$

Dividing the argument of by ξ, one obtains a homogeneous equation whose right-hand side depends on the ratio ξ only.

For $a_1 \;+ {}_1 \;= k(a_2 \;+ {}_2\;)$, see the equation of Paragraph 0.1.2-2.

0.1.2-4. Generalized homogeneous equations and equations reducible to them.

1 . A *generalized homogeneous equation* (a homogeneous equation in the generalized sense) remains the same under simultaneous scaling of the independent and dependent variables in accordance with the rule , , where $\neq 0$ is an arbitrary constant and k is some number. Such equations can be represented in the form

$$' = {}^{-1} \;(\;^{-}\;).$$

The substitution $\;= \;^{-}$ brings a generalized homogeneous equation to a separable equation, $' = \;(\;) - k$; see Paragraph 0.1.2-1.

2 . The equations of the form

$$' = \;(e^{\lambda}\;)$$

can be reduced to a generalized homogeneous equation. To this end, one should use the change of variable $z = e$ and set $\lambda = -k$.

0.1.2-5. Linear equation.

A first-order *linear equation* is written as

$$' + \;(\;) \;= g(\;).$$

The solution is sought in the product form $\;= \;v$, where $v = v(\;)$ is any function that satisfies the "truncated" equation $v' + \;(\;)v = 0$ [as $v(\;)$ one takes the particular solution $v = e^{-F}$, where $\;= \;(\;)\;$]. As a result, one obtains the following separable equation for $\;= (\;)$: $v(\;)\;' = g(\;)$. Integrating it yields the general solution:

$$(\;) = e^{-F} \quad e^{F} g(\;) \quad + \quad, \qquad \text{where} \qquad = \quad (\;) \quad .$$

0.1.2-6. Bernoulli equation.

A *Bernoulli equation* has the form

$$' + \;(\;) \;= g(\;) \quad, \qquad a \neq 0,\ 1.$$

The substitution $z = {}^{1-}$ brings it to a linear equation, $z' + (1-a)\;(\;)z = (1-a)g(\;)$, which is discussed in Paragraph 0.1.2-5. With this in view, one can obtain the general integral:

$${}^{1-} = \;e^{-F} + (1-a)e^{-F} \quad e^{F} g(\;) \quad, \qquad \text{where} \qquad = (1-a) \quad (\;) \quad .$$

0.1.2-7. Equation of the form $' = \; + \; (\;)g(\;\;)$.

The substitution $=$ brings the equation to a separable equation, $^{2}\,' = (\;)g(\;)$; see Paragraph 0.1.2-1.

0.1.2-8. Darboux equation.

A *Darboux equation* can be represented as

$$- \; + \; - \; ' = g \; - \; + \; ^{-1} \; - \; .$$

Using the substitution $= z(\;)$ and taking z to be the independent variable, one obtains a Bernoulli equation, which is considered in Paragraph 0.1.2-6:

$$\big[g(z) - z\;(z)\big]\;' = \;(z) + \;^{+1}\;(z).$$

References for Subsection 0.1.2: G. M. Murphy (1960), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1980), A. Moussiaux (1996), D. Zwillinger (1998).

0.1.3. Exact Differential Equations. Integrating Factor

0.1.3-1. Exact differential equations.

An *exact differential equation* has the form

$$(\;,\;)\;\; + g(\;,\;)\;\; = 0, \quad \text{where} \quad \frac{}{} = \frac{g}{}.$$

The left-hand side of the equation is the total differential of a function of two variables $(\;,\;)$.

The general integral, $(\;,\;) =$, where the function is determined from the system:

$$\frac{}{} = \;, \qquad \frac{}{} = g.$$

Integrating the first equation yields $= \int (\;,\;)\;\; + \;(\;)$ (while integrating, the variable is treated as a parameter). On substituting this expression into the second equation, one identifies the function (and hence,). As a result, the general integral of an exact differential equation can be represented in the form:

$$\int_{0} (\xi,\;)\;\xi + \int_{0} g(\;_0,\;)\;\; = \;,$$

where $_0$ and $_0$ are arbitrary numbers.

0.1.3-2. Integrating factor.

An *integrating factor* for the equation

$$(\;,\;)\;\; + g(\;,\;)\;\; = 0$$

is a function $(\;,\;)\;\; 0$ such that the left-hand side of the equation, when multiplied by $(\;,\;)$, becomes a total differential, and the equation itself becomes an exact differential equation.

An integrating factor satisfies the first-order partial differential equation:

$$g\frac{}{} - \;\frac{}{} = \;\Big(\frac{}{} - \frac{g}{}\Big),$$

which is not generally easier to solve than the original equation.

Table 1 lists some special cases where an integrating factor can be found in explicit form.

References for Subsection 0.1.3: G. M. Murphy (1960), N. M. Matveev (1967), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1980), A. Moussiaux (1996), D. Zwillinger (1998).

TABLE 1
An integrating factor $=(\ ,\ )$ for some types of ordinary differential equations $\ +g\ =0$, where $=(\ ,\ )$ and $g=g(\ ,\ )$

No	Conditions for and g	Integrating factor	Remarks
1	$=\ (\ ),\ g=\ (\ )$	$=\frac{1}{-}$	$-\ g\ \ 0$; (z) and (z) are any functions
2	$=g\ ,\quad =-g$	$=\frac{1}{^2+\ ^2}$	$+\ g$ is an analytic function of the complex variable $+$
3	$\frac{y-}{}=\ (\ )$	$=\exp\left[\ (\ )\ \right]$	$(\ )$ is any function
4	$\frac{y-}{}=\ (\ )$	$=\exp\left[-\ (\ )\ \right]$	$(\ )$ is any function
5	$\frac{y-}{-}=\ (\ +\ )$	$=\exp\left[\ (z)\ z\right],\ z=\ +$	(z) is any function
6	$\frac{y-}{-}=\ (\ )$	$=\exp\left[\ (z)\ z\right],\ z=$	(z) is any function
7	$\frac{^2(\ y-\ )}{+}=\ (-$	$=\exp\left[-\ (z)\ z\right],\ z=-$	(z) is any function
8	$\frac{y-}{-}=\ (\ ^2+\ ^2)$	$=\exp\left[\frac{1}{2}\ (z)\ z\right],\ z=\ ^2+\ ^2$	(z) is any function
9	$-g\ =\ (\ )g-\ (\ )$	$=\exp\left[\ (\ )\ +\ (\ )\ \right]$	$(\ )$ and $(\ )$ are any functions
10	$\frac{y-}{-\ \ y}=\ (\omega)$	$=\exp\left[\ (\omega)\ \omega\right]$	$\omega=\omega(\ ,\ )$ is any function of two variables

0.1.4. Riccati Equation

0.1.4-1. General Riccati equation. Simplest integrable cases.

A *Riccati equation* has the general form

$$' = \ _2(\ )\ ^2+\ _1(\ )\ +\ _0(\ ). \tag{1}$$

If $_2\equiv 0$, we have a linear equation (see Paragraph 0.1.2-5), and if $_0\equiv 0$, we have a Bernoulli equation (see Paragraph 0.1.2-6 for $a=2$), whose solutions were given previously. For arbitrary $_2$, $_1$, and $_0$, the Riccati equation is not integrable by quadrature.

Listed below are some special cases where the Riccati equation (1) is integrable by quadrature.

1 . The functions $_2$, $_1$, and $_0$ are proportional, i.e.,

$$' = \ (\ )(a\ ^2+\ \ +\ ),$$

where a, , and are constants. This equation is a separable equation; see Paragraph 0.1.2-1.

2 . The Riccati equation is homogeneous:

$$' = a\frac{^2}{^2}+\ \frac{}{}+\ .$$

See Paragraph 0.1.2-3.

3 . The Riccati equation is generalized homogeneous:

$$' = a \quad ^{2} + - \quad + \quad ^{- -2}.$$

See Paragraph 0.1.2-4 (with $k = - \quad - 1$). The substitution $z = \quad ^{+1}$ brings it to a separable equation: $z' = az^2 + (\quad + \quad + 1)z + \quad$.

4 . The Riccati equation has the form

$$' = a \quad ^{2} \quad ^{2} + \frac{-}{\quad} + \quad ^{2} \quad .$$

By the substitution $\quad = \quad ^{-} z$, the equation is reduced to a separable equation: $^{- -} z' = az^2 + \quad$.

For other Riccati equations integrable by quadrature, see Section 1.2.

0.1.4-2. Polynomial solutions of the Riccati equation.

Let $_2 = 1$, $_1(\ )$, and $_0(\ )$ be polynomials. If the degree of the polynomial $\Delta = {}_1^2 - 2(\ _1)' - 4 \ _0$ is odd, the Riccati equation cannot possess a polynomial solution. If the degree of Δ is even, the equation involved may possess only the following polynomial solutions:

$$= -\tfrac{1}{2}(\ _1 \quad [\ \overline{\Delta}] \ ,$$

where $[\ \overline{\Delta}]$ denotes an integer rational part of the expansion of $\overline{\Delta}$ in decreasing powers of (for example, $[\ \overline{^2 - 2 \quad + 3}] = \quad - 1$).

0.1.4-3. Use of particular solutions to construct the general solution.

1 . Given a particular solution $_0 = \ _0(\ )$ of the Riccati equation (1), the general solution can be written as:

$$= \ _0(\ ) + \ (\ ) \qquad - \qquad (\ ) \ _2(\ ) \qquad ^{-1},$$

where

$$(\ ) = \exp \quad [2 \ _2(\ ) \ _0(\ ) + \ _1(\ )] \qquad .$$

To the particular solution $_0(\ )$ there corresponds $\quad = \quad$.

2 . Let $_1 = \ _1(\ )$ and $_2 = \ _2(\ )$ be two different particular solutions of equation (1). Then the general solution can be calculated by:

$$= \frac{_1 + \ (\ ) \ _2}{+ \ (\ )}, \quad \text{where} \quad (\ ) = \exp \qquad _2(\ _1 - \ _2) \qquad .$$

To the particular solution $_1(\ )$, there corresponds $\quad = \quad$; and to $_2(\ )$, there corresponds $\quad = 0$.

3 . Let $_1 = \ _1(\ )$, $_2 = \ _2(\ )$, and $_3 = \ _3(\ )$ be three distinct particular solutions of equation (1). Then the general solution can be found without quadrature:

$$\frac{- \ _2}{- \ _1} \frac{_3 - \ _1}{_3 - \ _2} = \quad .$$

This means that the Riccati equation has a fundamental system of solutions.

0.1.4-4. Some transformations.

1 . The transformation (, $_1$, $_2$, $_3$, and $_4$ are arbitrary functions)

$$= \;(\xi), \qquad = \frac{_4(\xi) \quad + \quad _3(\xi)}{_2(\xi) \quad + \quad _1(\xi)}$$

reduces the Riccati equation (1) to a Riccati equation for $= \;(\xi)$.

2 . Let $_0 = \;_0(\;)$ be a particular solution of equation (1). Then the substitution $= \;_0 + 1$ leads to a linear equation for $= \;(\;)$:

$$' + \big[2 \;_2(\;)\;_0(\;) + \;_1(\;)\big] \quad + \;_2(\;) = 0.$$

For solution of linear equations, see Paragraph 0.1.2-5.

0.1.4-5. Reduction of the Riccati equation to a second-order linear equation.

The substitution

$$(\;) = \exp \;- \quad _2$$

reduces the general Riccati equation (1) to a second-order linear equation:

$$_2 \;'' \;- \big[(\;_2)' + \;_1\;_2\big] \;' + \;_0\;^2_2 \;= 0,$$

which often may be easier to solve than the original Riccati equation.

0.1.4-6. Reduction of the Riccati equation to the canonical form.

1 . The general Riccati equation (1) can be reduced with the aid of the transformation

$$= \;(\xi), \qquad = \frac{1}{_2} \quad - \frac{1}{2}\frac{_1}{_2} + \frac{1}{2} \quad \frac{1}{_2}\Big)', \quad \text{where} \quad (\xi) = \;(\;)\;', \tag{2}$$

to the canonical form:

$$' = \;^2 + \;(\xi). \tag{3}$$

Here, the function is defined by the formula:

$$(\xi) = \;_0\;_2 - \frac{1}{4}\;^2_1 + \frac{1}{2}\;'_1 - \frac{1}{2}\;_1\frac{'_2}{_2} - \frac{3}{4}\;\frac{'_2}{_2}\Big)^2 + \frac{1}{2}\frac{''_2}{_2};$$

the prime denotes differentiation with respect to ξ.

Transformation (2) depends on a function $= \;(\xi)$ that can be arbitrary. For a specific original Riccati equation, different functions in (2) will generate different functions in equation (3). [In practice, transformation (2) is most frequently used with $(\xi) = \xi$].

2 . In the special case where the original equation has the canonical form

$$' = \;^2 + \;(\;),$$

transformation (2) is written as

$$= \;(\xi), \qquad = \frac{1}{'} \quad - \frac{1}{2}\frac{''}{(\;')^2},$$

and the transformed equation (3) is determined by the function

$$(\xi) = \;(\;)(\;')^2 - \frac{3}{4}\;\frac{''}{'}\Big)^2 + \frac{1}{2}\frac{'''}{'}.$$

If the original Riccati equation is integrable by quadrature, one may obtain, specifying different functions , a variety of different integrable equations of the form (3). In Subsection 1.2.9, some useful transformations are given for specific functions .

References for Subsection 0.1.4: G. M. Murphy (1960), E. L. Ince (1964), W. T. Reid (1972), E. Kamke (1977), W. E. Boyce and R. C. DiPrima (1986), A. D. Polyanin and V. F. Zaitsev (1995).

0.1.5. Abel Equations of the First Kind

0.1.5-1. General form of Abel equations of the first kind. Simplest integrable cases.

An *Abel equation of the first kind* has the general form

$$' = {}_3(\ )\ {}^3 + {}_2(\ )\ {}^2 + {}_1(\ )\ + {}_0(\ ), \qquad {}_3(\ )\quad 0. \tag{1}$$

In the degenerate case ${}_2(\ ) = {}_0(\ ) = 0$, we have a Bernoulli equation (see Paragraph 0.1.2-6 with $a = 3$). The Abel equation (1) is not integrable for arbitrary $(\ )$.

Listed below are some special cases where the Abel equation of the first kind is integrable by quadrature.

1 . If the functions $(\ )\ (\ = 0, 1, 2, 3)$ are proportional, i.e., $(\ ) = a\ g(\ )$, then (1) is a separable equation (see Paragraph 0.1.2-1).

2 . The Abel equation is homogeneous:

$$' = a\frac{3}{3} + \frac{2}{2} + - + \ .$$

See Paragraph 0.1.2-3.

3 . The Abel equation is generalized homogeneous:

$$' = a\ {}^{2\ +1\ 3} + {}^{2} + - + {}^{-\ -2}.$$

See Paragraph 0.1.2-4 for $k = -\ \ - 1$. The substitution $= {}^{+1}$ leads to a separable equation: $' = a\ {}^3 + {}^2 + (\ +\ + 1)\ + \ .$

4 . The Abel equation

$$' = a\ {}^{3\ -}\ {}^3 + {}^2\ {}^2 + \frac{-}{} + {}^2$$

can be reduced with the substitution $= {}^{-} z$ to a separable equation: ${}^{-\ -} z' = az^3 + z^2 + \ .$

5 . Let ${}_0 \equiv 0$, ${}_1 \equiv 0$, and $({}_3\ {}_2)' = a\ {}_2$ for some constant a. Then the substitution $= {}_2\ {}_3^{-1}$ leads to a separable equation: $' = {}_2^2\ {}_3^{-1}(\ {}^3 + {}^2 + a\ )$.

6 . If

$$_0 = \frac{1\ 2}{3\ \ 3} - \frac{2\ \ {}_2^3}{27\ \ {}_3^2} - \frac{1}{3}\frac{}{} - \frac{2}{3}, \qquad = (\ ),$$

then the solution of equation (1) is given by:

$$(\ ) = \quad - 2 \quad {}_3\ {}^2 \quad {}^{-1\ 2} - \frac{2}{3\ \ 3}, \qquad \text{where} \qquad = \exp \quad {}_1 - \frac{{}_2^2}{3\ \ 3}\Big) \quad .$$

For other solvable Abel equations of the first kind, see Subsection 1.4.1.

0.1.5-2. Reduction to the canonical form. Reduction to an Abel equation of the second kind.

1 . The transformation

$$= (\ )\ (\xi) - \frac{2}{3\ \ 3}, \quad \xi = \quad {}_3\ {}^2 \quad , \qquad \text{where} \qquad (\ ) = \exp \quad {}_1 - \frac{{}_2^2}{3\ \ 3}\Big) \quad ,$$

brings equation (1) to the canonical (normal) form:

$$' = {}^3 + (\xi).$$

Here, the function (ξ) is defined parametrically (is the parameter) by the relations:

$$= \frac{1}{\ {}_3\ {}^3} \quad {}_0 - \frac{{}_1\ {}_2}{3\ {}_3} + \frac{2\ {}_2^3}{27\ {}_3^2} + \frac{1}{3}\frac{\ }{}\frac{{}_2}{{}_3}\Big), \qquad \xi = \quad {}_3 \quad {}^2 \quad .$$

2 . Let ${}_0 = {}_0(\ )$ be a particular solution of equation (1). Then the substitution

$$= {}_0 + \frac{(\ )}{z(\ )}, \qquad \text{where} \qquad (\ ) = \exp \quad (3\ {}_3\ {}_0^2 + 2\ {}_2\ {}_0 + {}_1) \qquad ,$$

leads to an Abel equation of the second kind:

$$zz' = -(3\ {}_3\ {}_0 + {}_2)\ z - {}_3 \quad {}^2.$$

For equations of this type, see Subsection 0.1.6.

References for Subsection 0.1.5: E. Kamke (1977), G. M. Murphy (1960).

0.1.6. Abel Equations of the Second Kind

0.1.6-1. General form of Abel equations of the second kind. Simplest integrable cases.

An *Abel equation of the second kind* has the general form

$$[\ + g(\ )]\ ' = {}_2(\ )\ {}^2 + {}_1(\ )\ + {}_0(\ ), \qquad g(\ ) \quad 0. \tag{1}$$

The Abel equation (1) is not integrable for arbitrary $(\ )$. Given below are some special cases where the Abel equation of the second kind is integrable by quadrature.

1 . If $g(\ ) = \text{const}$ and the functions $(\ )\ (\ = 0, 1, 2)$ are proportional, i.e., $(\ ) = a\ \ g(\ )$, then (1) is a separable equation (see Paragraph 0.1.2-1).

2 . The Abel equation is homogeneous:

$$(\ +\ )\ ' = \frac{a}{}\ {}^2 + \quad + \quad .$$

See Paragraph 0.1.2-3.

3 . The Abel equation is generalized homogeneous:

$$(\ +\ )\ ' = \frac{a}{}\ {}^2 + \quad {}^{-1} + \quad {}^{2\ -1}.$$

See Paragraph 0.1.2-4 for $k =$. The substitution $= \quad {}^{-}$ leads to a separable equation: $(\ +\ )\ ' = (a -\ )\ {}^2 + (\ -\ )\ +\ .$

4 . The general solution of the Abel equation

$$(\ + g)\ ' = {}_2\ {}^2 + {}_1\ + {}_1 g - {}_2 g^2, \qquad = (\ ), \ \ g = g(\ ),$$

is given by:

$$= -g + \quad + \quad (\ {}_1 + g' - 2\ {}_2 g)\ {}^{-1} \quad , \qquad \text{where} \qquad = \exp \quad {}_2 \quad .$$

5 . If ${}_1 = 2\ {}_2 g - g'$, the general solution of the Abel equation (1) has the form:

$$= -g \qquad 2\ \ (\ {}_0 + gg' - {}_2 g^2)\ {}^{-2} \quad + \quad {}^{1\ 2}, \qquad \text{where} \qquad = \exp \quad {}_2 \quad .$$

For other solvable Abel equations of the second kind, see Section 1.3.

References for Paragraph 0.1.6-1: G. M. Murphy (1960), E. Kamke (1977).

0.1.6-2. Reduction to the canonical form. Reduction to an Abel equation of the first kind.

1 . The substitution

$$= (\;\; + g) \;\; , \qquad \text{where} \qquad = \exp \; - \;\; {}_2 \;\; , \tag{2}$$

brings equation (1) to the simpler form:

$$' = \;\; {}_1(\;) \;\; + \;\; {}_0(\;), \tag{3}$$

where

$$ {}_1 = (\; {}_1 - 2 \; {}_2 g + g' \,) \;\; , \qquad {}_0 = (\; {}_0 - \; {}_1 g + \; {}_2 g^2) \;\; {}^2 .$$

2 . In turn, equation (3) can be reduced, by the introduction of the new independent variable

$$z = \;\; {}_1(\;) \;\; , \tag{4}$$

to the canonical form:

$$' - \;\; = R(z). \tag{5}$$

Here, the function $R(z)$ is defined parametrically (is the parameter) by the relations

$$R = \frac{{}_0(\;)}{{}_1(\;)}, \qquad z = \;\; {}_1(\;) \;\; .$$

Substitutions (2) and (4), which take the Abel equation to the canonical form, are called canonical.

The transformation $= a \,\hat{}\,$, $z = a\hat{z} +$ brings (5) to a similar equation, $\hat{}\;\hat{}\,'_{\hat{}} - \hat{} = a^{-1} R(a\hat{z} + \;\;)$. Therefore the function $R(z)$ in the right-hand side of the Abel equation (5) can be identified with the two-parameter family of functions $a^{-1} R(az + \;\;)$.

Any Abel equations of the second kind related by linear (in) transformations $= \; {}_1(\;)$, $= \; {}_2(\;) \; + \; {}_3(\;)$ have identical canonical forms (up to the two-parameter family of functions specified in Remark 1).

3 . The substitution $+ g = 1$ leads to an Abel equation of the first kind:

$$' + (\; {}_0 - \; {}_1 g + \; {}_2 g^2) \;\; {}^3 + (\; {}_1 - 2 \; {}_2 g + g' \,) \;\; {}^2 + \; {}_2 \;\; = 0.$$

For equations of this type, see Subsection 0.1.5.

References for Paragraph 0.1.6-2: G. M. Murphy (1960), E. Kamke (1977), V. F. Zaitsev and A. D. Polyanin (1994).

0.1.6-3. Use of particular solutions to construct self-transformations.

1 . Suppose a particular solution ${}_0 = \; {}_0(\;)$ of an Abel equation of the second kind (1) is known. Then the substitution $= 1 \;\; (\; - \; {}_0)$ leads to a similar Abel equation:

$$+ \frac{1}{{}_0 + g} \Big) \;\; ' = \frac{{}'_0 - \; {}_1 - 2 \; {}_2 \; {}_0}{{}_0 + g} \;\; {}^2 - \frac{{}_2}{{}_0 + g} \;\; . \tag{6}$$

If ${}_0 \equiv 0$, equation (1) has the trivial particular solution ${}_0 = 0$. In this case, the change of variable $= 1$ leads to an Abel equation of the form (6) with ${}_0 = 0$.

2 . Given a particular solution ${}_0 = \; {}_0(\;)$ of the Abel equation of the second kind

$$' = \; {}_1(\;) \;\; + \; {}_0(\;), \tag{7}$$

the substitution

$$= \frac{(\;)}{{}^2_0(\; {}_0 - \;)}, \qquad \text{where} \qquad (\;) = \exp \; \frac{1}{{}_0} \;\; \Big), \tag{8}$$

brings (7) to another, similar Abel equation:

$$' = \ _1(\ ) \quad + \ _0(\ ). \tag{9}$$

Here, the functions $_1 = \ _1(\ )$ and $_0 = \ _0(\ )$ are defined by

$$_1 = \frac{(\ _1\ _0 + 3\ _0)}{^4_0}, \qquad _0 = \frac{_0\ ^2}{^6_0}.$$

It is not difficult to verify by direct substitution that equation (9) has a particular solution:

$$_0(\ ) = -\frac{(\ )}{^2_0(\ )}. \tag{10}$$

The transformation based on the particular solution (10) brings the Abel equation (9) to the original equation (7) with $_1$ having the opposite sign.

In general, the canonical forms of equations (1) and (6) and also those of equations (7) and (9) are different. See Paragraph 0.1.6-2.

Given k distinct particular solutions of equation (7), k distinct Abel equations of the second kind related to (7) by known substitutions of the form (8) can be constructed.

References for Paragraph 0.1.6-3: T. A. Alexeeva, V. F. Zaitsev, and T. B. Shvets (1992), V. F. Zaitsev and A. D. Polyanin (1994).

0.1.6-4. Use of particular solutions to construct the general solution.

For some Abel equations of the second kind, the general solution can be found if its distinct particular solutions $= \ (\ )$, $k = 1, \ , \ $, are known.

Below we consider Abel equations of the canonical form

$$' - \ = R(\ ), \tag{11}$$

whose general solutions can be represented in the special form:

$$\prod_{=1} |\ - \ (\ )| \ = \ . \tag{12}$$

Here, the particular solutions $= \ (\ )$ correspond to $= 0$ (if > 0) and $=$ (if < 0).

The logarithmization of (12), followed by the differentiation of the resulting expression and rearrangement, leads to the equation

$$\sum_{=1} (\ ' - \ ')\prod_{\substack{=1\\ \neq}} (\ - \ ) \equiv \ ' \sum_{=1} \ ^{-1} + \sum_{=1} \ ^{-1} = 0, \tag{13}$$

where $' = (\ )'$. We require that equation (13) be equivalent to the Abel equation (11). To this end, we set:

$$= - \ , \qquad _{-1} = -R(\ ) \qquad \text{and equate the other} \quad \text{and} \quad \text{with zero.}$$

Selecting different values $= 1, 2, \ , \ - 1$, we obtain $- 1$ systems of differential-algebraic equations; only one of the systems, corresponding to $\neq 0$ for all $k = 1, \ , \ $ and $\neq$ for $\neq$, leads to a nondegenerate solution of the form (12). Consider the Abel equations (11) corresponding to the simplest solutions of the form (12) in more detail.

1 . *Case* = 2. The system of differential-algebraic equations has the form:

$$\begin{aligned} {}_1 + \ {}_2 &= \ , \\ {}_1\ {}_2 + \ {}_2\ {}_1 &= 0, \\ {}_1\ {}'_1 + \ {}_2\ {}'_2 &= \ , \\ {}_1\ {}'_1\ {}_2 + \ {}_2\ {}_1\ {}'_2 &= - \ R(\ ), \end{aligned} \tag{14}$$

where is an arbitrary constant. It follows from the second and third equations that

$$ {}_1 = \frac{{}_1}{{}^2_1 - {}^2_2}(\quad + N), \qquad {}_2 = -\frac{{}_2}{{}^2_1 - {}^2_2}(\quad + N), $$

where N is an arbitrary constant. Introducing the new constants

$$ A = \frac{{}_1\ {}_2(\ {}_1 + \ {}_2)}{(\ {}_1 - \ {}_2)^2}\quad , \qquad B = \frac{{}_1\ {}_2(\ {}_1 + \ {}_2)}{(\ {}_1 - \ {}_2)^2} N, $$

we find from the last relation in (14) that

$$ R(\ ) = A \quad + B, \tag{15}$$

which means that for = 2 the right-hand side of the Abel equation is a linear function of (see equation 1.3.1.2).

The particular solutions ${}_1$, ${}_2$, and the corresponding exponents ${}_1$, ${}_2$ in the general integral (12), are expressed in terms of the coefficients A, B on the right-hand side (15) of the Abel equation (11) as follows:

$$ {}_1 = \frac{1 + \ \overline{4A+1}}{2A}(A \quad + B), \qquad {}_1 = 2A + 1 + \ \overline{4A+1}, $$
$$ {}_2 = -\frac{1 + \ \overline{4A+1}}{2A + 1 + \ \overline{4A+1}}(A \quad + B), \qquad {}_2 = 2A. $$

2 . *Case* = 3. Equation (13) with = 3 leads to the Abel equation (11) with the right-hand side

$$ R(\ ) = -\tfrac{2}{9} \quad + A + B \ {}^{-1\ 2} \tag{16}$$

(see equation 1.3.1.3).

The particular solutions and the exponents in the general integral (12) are expressed as:

$$ = \frac{2}{3} \quad + \frac{2}{3}\lambda \ {}^{1\ 2} + \frac{3B}{\lambda}, \qquad = \frac{2A}{3(2\lambda^2 - 3A)}, $$

where the λ are roots of the cubic equation

$$ \lambda^3 - \tfrac{9}{2}A\lambda - \tfrac{9}{2}B = 0, \qquad = 1,\ 2,\ 3. $$

3 . *Case* = 4. Equation (13) with = 4 leads to the Abel equation (11) with the right-hand side

$$ R(\ ) = -\tfrac{3}{16} \quad + A \ {}^{-1\ 3} + B \ {}^{-5\ 3} $$

(see equation 1.3.3.61).

The particular solutions and the exponents in (12) are expressed as:

$$ {}_{1,2} = \tfrac{3}{4} \quad \overline{3A + \tfrac{3}{2}\ \overline{-3B}}\ {}^{1\ 3} + \ \overline{-3B}\ {}^{-1\ 3}, \qquad {}_{1,2} = \mp\big(2A - \ \overline{-3B}\ \ , $$
$$ {}_{3,4} = \tfrac{3}{4} \quad \overline{3A - \tfrac{3}{2}\ \overline{-3B}}\ {}^{1\ 3} - \ \overline{-3B}\ {}^{-1\ 3}, \qquad {}_{3,4} = \quad \overline{4A^2 + 3B}. $$

4 . *Case* > 4. The equations for are algebraic equations of degree and, in the general case, are not soluble in radicals. The right-hand of equation (11) is expressed as

$$ R(\ ) = -\frac{\ - 1}{{}^2} \quad + Q(\ ), $$

where the function $Q(\ )$ is bounded as $\qquad$ (Q can be specified in parametric form).

References for Paragraph 0.1.6-4: B. M. Koyalovich (1894), T. A. Alexeeva, V. F. Zaitsev, and T. B. Shvets (1992).

0.1.7. Equations Not Solved for the Derivative

0.1.7-1. The method of “integration by differentiation.”

In the general case, a first-order equation not solved for the derivative,

$$(\;,\; \;,\; \;') = 0, \tag{1}$$

can be rewritten in the equivalent form

$$(\;,\; \;,\;) = 0, \qquad = \;' . \tag{2}$$

We look for a solution in parametric form: $= (\;)$, $= (\;)$. In accordance with the first relation in (2), the differential of is given by:

$$+ \qquad + \qquad = 0. \tag{3}$$

Using the relation $=$, we eliminate successively and from (3). As a result, we obtain the system of two first-order ordinary differential equations:

$$\frac{}{} = -\frac{}{+}, \qquad \frac{}{} = -\frac{}{+}. \tag{4}$$

By finding a solution of this system, one thereby obtains a solution of the original equation (1) in parametric form, $= (\;)$, $= (\;)$.

The application of this method may lead to loss of individual solutions; this issue should be additionally investigated.

0.1.7-2. Equations of the form $= (\;')$.

This equation is a special case of equation (1), with $(\;,\;,\;) = \; - \;(\;)$. The procedure described in Paragraph 0.1.7-1 yields

$$\frac{}{} = \frac{'(\;)}{}, \qquad = (\;). \tag{5}$$

Here, the original equation is used instead of the second equation in system (4); this is valid because the first equation in (4) does not depend on explicitly.

Integrating the first equation in (5) yields the solution in parametric form:

$$= \int \frac{'(\;)}{} + \;, \qquad = (\;).$$

0.1.7-3. Equations of the form $= (\;')$.

This equation is a special case of equation (1), with $(\;,\;,\;) = \; - \;(\;)$. The procedure described in Paragraph 0.1.7-1 yields

$$= (\;), \qquad \frac{}{} = \;'(\;). \tag{6}$$

Here, the original equation is used instead of the first equation in system (4); this is valid because the second equation in (4) does not depend on explicitly.

Integrating the second equation in (5) yields the solution in parametric form:

$$= (\;), \qquad = \int \;'(\;) \; + \;.$$

0.1.7-4. Clairaut's equation $= \ ' + \ (\ ')$.

Clairaut's equation is a special case of equation (1), with $(\ , \ , \ ) = \ - \ - \ (\ )$. It can be rewritten as

$$= \ + \ (\ ), \qquad = \ ' . \tag{7}$$

Substituting and $'$ in $= \ '$ by their values in accordance with (7), we obtain

$$[\ + \ '(\ )] \ = 0.$$

This equation splits into $= 0$ and $+ \ '(\ ) = 0$. The solution of the first equation is obvious: $=$; it gives the general solution of Clairaut's equation,

$$= \ + \ (\ ), \tag{8}$$

which is a family of straight lines. The second equation generates a solution in parametric form,

$$= - \ '(\ ), \qquad = - \ '(\ ) + \ (\ ),$$

which is a singular solution and is the envelope of the family of lines (8).

0.1.7-5. Lagrange's equation $= \ (\ ') + g(\ ')$.

Lagrange's equation is a special case of equation (1), with $(\ , \ , \ ) = \ - \ (\ ) - g(\ )$. In the special case $(\ ) \equiv$, it coincides with Clairaut's equation; see Paragraph 0.1.7-4.

The procedure described in Paragraph 0.1.7-1 yields

$$\frac{}{} + \frac{'(\ )}{(\ ) - } = \frac{g'(\ )}{ - \ (\ )}, \qquad = \ (\ ) + g(\ ). \tag{9}$$

Here, the original equation is used instead of the second equation in system (4); this is valid because the first equation in (4) does not depend on explicitly.

The first equation of system (9) is linear. Its general solution has the form $= \ (\ ) \ + \ (\ )$; the functions and are defined in Paragraph 0.1.2-5. Substituting this solution into the second equation in (9), we obtain the general solution of Lagrange's equation in parametric form:

$$= \ (\ ) \ + \ (\ ), \qquad = \big[\ (\ ) \ + \ (\ ) \big] \ (\ ) + g(\ ).$$

With the above method, solutions of the form $= \ + g(\ )$, where the are roots of the equation $(\ ) - \ = 0$, may be lost. These solutions can be particular or singular solutions of Lagrange's equation.

References for Subsection 0.1.7: G. M. Murphy (1960), E. Kamke (1977).

0.1.8. Contact Transformations

0.1.8-1. General form of contact transformations.

A contact transformation has the form

$$\begin{aligned} &= \ (X, \ , \ '), \\ &= \ (X, \ , \ '), \end{aligned} \tag{1}$$

where the functions $(X, \ , \ )$ and $(X, \ , \ )$ are chosen so that the derivative $'$ does not depend on $''$:

$$' = \frac{'}{'} = \frac{ + \ ' + \ ''}{ + \ ' + \ ''} = \ (X, \ , \ '). \tag{2}$$

The subscripts X, , and after and denote the respective partial derivatives (it is assumed that 0 and 0).

It follows from (2) that the relation

$$\frac{}{}\ \frac{}{X} + \frac{}{} - \frac{}{}\ \frac{}{X} + \frac{}{} = 0 \tag{3}$$

holds; the derivative is calculated by

$$' = \frac{}{}, \tag{4}$$

where const.

The application of contact transformations preserves the order of differential equations. The inverse of a contact transformation can be obtained by solving system (1) and (4) for X, , $'$.

References for Paragraph 0.1.8-1: D. Zwillinger (1989).

0.1.8-2. A method for the construction of contact transformations.

Suppose the function $= (X, ,)$ in the contact transformation (1) is specified. Then relation (3) can be viewed as a linear partial differential equation for the second function . The corresponding characteristic system of ordinary differential equations (see A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux, 2002),

$$\frac{X}{1} = \frac{}{} = -\frac{}{+}$$

admits the obvious first integral:

$$(X, ,) = {}_1, \tag{5}$$

where ${}_1$ is an arbitrary constant. It follows that, to obtain the general representation of the function $= (X, ,)$, one has to deal with the ordinary differential equation

$$' = , \tag{6}$$

whose right-hand side is defined in implicit form by (5). Let the first integral of equation (6) has the form

$$(X, , {}_1) = {}_2.$$

Then the general representation of $= (X, ,)$ in transformation (1) is given by:

$$= (,),$$

where $(,)$ is an arbitrary function of two variables, $= (X, ,)$, and $= (X, ,)$.

0.1.8-3. Examples of contact transformations linear in the derivative.

Example 1. Legendre transformation (E. Kamke, 1977):

$$= , \quad y = - , \quad y_x = \qquad \text{(direct transformation)};$$
$$= y_x, \quad = y_x - y, \quad = \qquad \text{(inverse transformation)}.$$

Example 2. Contact transformation ($\neq 0$):

$$= + , \quad y = b , \quad y_x = b \qquad \text{(direct transformation)};$$
$$= \frac{1}{} \ln \frac{y_x}{b}, \quad = \frac{1}{} - \frac{y}{y_x}, \quad = \frac{y}{y_x} \qquad \text{(inverse transformation)}.$$

. It is apparent from this example that a contact transformation that is linear in the derivative can have a nonlinear inverse, which is also a contact transformation.

Example 3. Contact transformation ($\neq -1$):

$$= \quad + \frac{}{} \quad , \quad y = \quad ^{+1} \quad - \quad , \qquad y_x = \quad ^{+1} \qquad \text{(direct transformation);}$$

$$= (y_x)^{\frac{1}{\ +1}}, \qquad = \frac{1}{\ +1}(\ y_x - y)(y_x)^{-\frac{}{\ +1}}, \qquad = \frac{y_x + \ y}{(\ +1)y_x} \qquad \text{(inverse transformation).}$$

Example 4. Contact transformation:

$$= \quad + \quad , \quad y = (\quad + \quad) \quad \frac{\varphi}{} d \quad - \varphi \quad , \quad y_x = \quad \frac{\varphi}{} d \quad ,$$

where $= (\ )$ and $= (\ )$ are arbitrary functions, $\varphi = \exp \quad \frac{}{} d \quad$.

Example 5. Contact transformation:

$$= \quad + \quad + \ , \quad y = \quad + (\quad - \quad) \ + \quad - \quad d \quad , \quad y_x = \ ,$$

where $= (\ )$ and $= (\ )$ are arbitrary functions, $= \quad d \ , \ = \quad d \ .$

0.1.8-4. Examples of contact transformations nonlinear in the derivative.

Example 1. Contact transformation:

$$= \quad + \quad , \qquad y = \tfrac{1}{2}(\quad)^2 + \quad , \qquad y_x = \qquad \text{(direct transformation);}$$

$$= \frac{1}{} \quad - y_x \ , \qquad = \frac{1}{2} \ 2y - (y_x)^2 \ , \qquad = y_x \qquad \text{(inverse transformation).}$$

Example 2. Contact transformation:

$$= \quad + \frac{1}{} \ , \qquad y = \ ^2(\quad)^2 - \ ^2, \quad y_x = 2 \ ^2 \qquad \text{(direct transformation);}$$

$$= \frac{1}{} \ \overline{\ y_x - y}, \qquad = \frac{y_x - 2y}{2 \ \overline{\ y_x - y}}, \qquad = \frac{^2 y_x}{2(\ y_x - y)} \qquad \text{(inverse transformation).}$$

Example 3. Contact transformation

$$= \ (\quad - \quad), \qquad y = (\quad)^2 - \ ^2, \qquad y_x = 2 \ ^- \qquad \text{(direct transformation);}$$

$$= \ln \ \overline{\ \overline{y_x - y}} \ , \qquad = \frac{2y - \ y_x}{2 \ \overline{\ y_x - y}}, \qquad = \frac{y_x}{2 \ \overline{\ y_x - y}} \qquad \text{(inverse transformation).}$$

Example 4. Contact transformation:

$$= (\quad)^2 - \ ^2, \quad y = \quad \cosh \quad - \quad \sinh \quad , \quad y_x = \frac{\cosh}{2}.$$

Example 5. Contact transformation:

$$= (\quad)^2 + \ ^2, \quad y = \quad \cos \quad + \quad \sin \quad , \quad y_x = \frac{\cos}{2}.$$

Example 5. Contact transformation ($b \neq 0$):

$$= \ (\quad)^2 - b \quad , \qquad y = 2 \ (\quad)^3 - 3b \quad , \qquad y_x = 3 \qquad \text{(direct transformation);}$$

$$= \frac{}{9b}(y_x)^2 - \frac{1}{b} \ , \qquad = \frac{2}{81b}(y_x)^3 - \frac{1}{3b}y, \qquad = \frac{1}{3}y_x \qquad \text{(inverse transformation).}$$

Example 7. Contact transformation (D. Zwillinger, 1989, p. 169):

$$= \quad - \qquad , \quad y = - \quad \sqrt{(\quad)^2 - 1}, \quad y_x = \frac{}{\sqrt{(\quad)^2 - 1}} \qquad \text{(direct transformation);}$$

$$= \quad - y y_x, \qquad = y \sqrt{(y_x)^2 - 1}, \qquad = - \frac{y_x}{\sqrt{(y_x)^2 - 1}} \qquad \text{(inverse transformation).}$$

Example 8. Contact transformation (D. Zwillinger, 1989, p. 169):

$$= \quad - \frac{}{\sqrt{(\quad)^2 + 1}}, \quad y = \quad + \frac{}{\sqrt{(\quad)^2 + 1}}, \quad y_x = \qquad \text{(direct transformation);}$$

$$= \quad + \frac{y_x}{\sqrt{(y_x)^2 + 1}}, \qquad = y - \frac{}{\sqrt{(y_x)^2 + 1}}, \qquad = y_x \qquad \text{(inverse transformation).}$$

Example 9. Contact transformation ($b \neq 0$, $k \neq -1$):

$$= (\quad) \quad - b \quad , \qquad y = \quad k(\quad)^{\quad +1} - b(k+1) \quad , \qquad y_x = (k+1) \qquad \text{(direct transformation);}$$

$$= \frac{(y_x)}{b(k+1)} - \frac{}{b}, \qquad = \frac{k(y_x)^{\quad +1}}{b(k+1)^{\quad +2}} - \frac{y}{b(k+1)}, \qquad = \frac{y_x}{k+1} \qquad \text{(inverse transformation).}$$

0.1.9. Approximate Analytic Methods for Solution of Equations

0.1.9-1. The method of successive approximations (Picard method).

The method of successive approximations consists of two stages. At the first stage, the Cauchy problem

$$' = (\ ,\ ) \qquad \text{(equation),} \tag{1}$$

$$(\ _0) = \ _0 \qquad \text{(initial condition)} \tag{2}$$

is reduced to the equivalent integral equation:

$$(\ ) = \ _0 + \int_{0} (\ ,\ (\ ))\ . \tag{3}$$

Then a solution of equation (3) is sought using the formula of successive approximations:

$$_{+1}(\ ) = \ _0 + \int_{0} (\ ,\ (\ ))\ ; \qquad = 0,\ 1,\ 2,$$

The initial approximation $_0(\ )$ can be chosen arbitrarily; the simplest way is to take $_0$ to be a number. The iterative process converges as , provided the conditions of the theorems in Paragraph 0.1.1-2 are satisfied.

0.1.9-2. The method of Taylor series expansion in the independent variable.

A solution of the Cauchy problem (1)–(2) can be sought in the form of the Taylor series in powers of $(\ -\ _0)$:

$$(\ ) = (\ _0) + \ '(\ _0)(\ -\ _0) + \frac{''(\ _0)}{2!}(\ -\ _0)^2 + \quad . \tag{4}$$

The first coefficient $(\ _0)$ in solution (4) is prescribed by the initial condition (2). The values of the derivatives of $(\ )$ at $=\ _0$ are determined from equation (1) and its derivative equations (obtained by successive differentiation), taking into account the initial condition (2). In particular, setting $=\ _0$ in (1) and substituting (2), one obtains the value of the first derivative:

$$'(\ _0) = (\ _0,\ _0). \tag{5}$$

Further, differentiating equation (1) yields

$$'' = \ (\ , \ ) + \ (\ , \ ) \ ' . \tag{6}$$

On substituting $\ = \ _0$, as well as the initial condition (2) and the first derivative (5), into the right-hand side of this equation, one calculates the value of the second derivative:

$$'' (\ _0) = \ (\ _0, \ _0) + \ (\ _0, \ _0) \ (\ _0, \ _0).$$

Likewise, one can determine the subsequent derivatives of at $\ = \ _0$.

Solution (4) obtained by this method can normally be used in only some sufficiently small neighborhood of the point $\ = \ _0$.

0.1.9-3. The method of regular expansion in the small parameter.

Consider a general first-order ordinary differential equation with a small parameter ε:

$$' = \ (\ , \ , \varepsilon). \tag{7}$$

Suppose the function is representable as a series in powers of ε:

$$(\ , \ , \varepsilon) = \sum_{=0} \varepsilon \ (\ , \ ). \tag{8}$$

One looks for a solution of the Cauchy problem for equation (7) with the initial condition (2) as $\varepsilon \ \ 0$ in the form of a regular expansion in powers of the small parameter:

$$= \sum_{=0} \varepsilon \ (\ ). \tag{9}$$

Relation (9) is substituted in equation (7) taking into account (8). Then one expands the functions into a power series in ε and matches the coefficients of like powers of ε to obtain a system of equations for $\ (\ )$:

$$'_0 = \ _0(\ , \ _0), \tag{10}$$

$$'_1 = g(\ , \ _0) \ _1 + \ _1(\ , \ _0), \qquad g(\ , \ ) = \frac{\ _0}{\ }. \tag{11}$$

Only the first two equations are written out here. The prime denotes differentiation with respect to . The initial conditions for can be obtained from (2) taking into account (9):

$$_0(\ _0) = \ _0, \qquad _1(\ _0) = 0.$$

Success in the application of this method is primarily determined by the possibility of constructing a solution of equation (10) for the leading term in the expansion of $\ _0$. It is significant that the remaining terms of the expansion, with $\ \geq 1$, are governed by linear equations with homogeneous initial conditions.

Paragraph 0.3.3-2 gives an example of solving a Cauchy problem by the method of regular expansion for a second-order equation and also discusses characteristic features of the method.

The methods of scaled coordinates, two-scale expansions, and matched asymptotic expansions are also used to solve problems defined by first-order differential equations with a small parameter. The basic ideas of these methods are given in Subsection 0.3.4.

References for Subsection 0.1.9: G. M. Murphy (1960), G. A. Korn and T. M. Korn (1968), E. Kamke (1977), D. Zwillinger (1998).

0.1.10. Numerical Integration of Differential Equations

0.1.10-1. The method of Euler polygonal lines.

Consider the Cauchy problem for the first-order differential equation

$$' = (,)$$

with the initial condition $(\ _0) = \ _0$. Our aim is to construct an approximate solution $= (\ )$ of this equation on an interval $[\ _0, \ _*]$.

Let us split the interval $[\ _0, \ _*]$ into equal segments of length $\Delta \ = \frac{\ _* - \ _0}{}$. We seek approximate values $\ _1, \ _2, \ ,$ of the function $(\ )$ at the partitioning points $\ _1, \ _2, \ , \ = \ _*$.

For a given initial value $\ _0 = (\ _0)$ and a sufficiently small Δ , the values of the unknown function $= (\ )$ at the other points $= \ _0 + k\Delta$ are calculated successively by the formula:

$$_{+1} = \ + (\ ,\ )\Delta \qquad (\textit{Euler polygonal line}),$$

where $k = 0, 1, \ , \ -1$. The Euler method is a single-step method of the first-order approximation (with respect to the step Δ).

0.1.10-2. Single-step methods of the second-order approximation.

Two single-step methods for solving the Cauchy problem in the second-order approximation are specified by the recurrence formulas:

$$_{+1} = \ + (\ + \tfrac{1}{2}\Delta\ ,\ + \tfrac{1}{2}\ \Delta\ )\Delta\ ,$$
$$_{+1} = \ + \tfrac{1}{2}\big[\ + (\ _{+1},\ + \ \Delta\ )\big]\Delta\ ,$$

where $= (\ ,\ )$; $k = 0, 1, \ , \ -1$.

0.1.10-3. Runge–Kutta method of the fourth-order approximation.

This is one of the widely used methods. The unknown values are successively found by the formulas:

$$_{+1} = \ + \tfrac{1}{6}(\ _1 + 2\ _2 + 2\ _3 + \ _4)\Delta\ ,$$

where

$$_1 = (\ ,\ ), \qquad _2 = (\ + \tfrac{1}{2}\Delta\ ,\ + \tfrac{1}{2}\ _1\Delta\ ),$$
$$_3 = (\ + \tfrac{1}{2}\Delta\ ,\ + \tfrac{1}{2}\ _2\Delta\ ), \qquad _4 = (\ + \Delta\ ,\ + \ _3\Delta\ ).$$

All methods described in Subsection 0.1.10 are special cases of the Runge—Kutta method (a detailed description of this method can be found in the monographs listed below).

In practice, calculations are performed on the basis of any of the above recurrence formulas with two different steps Δ , $\tfrac{1}{2}\Delta$ and an arbitrarily chosen small Δ . Then one compares the results obtained at common points. If these results coincide within the given order of accuracy, one assumes that the chosen step Δ ensures the desired accuracy of calculations. Otherwise, the step is halved and the calculations are performed with the steps $\tfrac{1}{2}\Delta$ and $\tfrac{1}{4}\Delta$, after which the results are compared again, etc. (Quite often, one compares the results of calculations with steps varying by ten or more times.)

References for Subsection 0.1.10: G. A. Korn and T. M. Korn (1968), N. S. Bakhvalov (1973), E. Kamke (1977), D. Zwillinger (1998).

0.2. Second-Order Linear Differential Equations

0.2.1. Formulas for the General Solution. Some Transformations

0.2.1-1. Homogeneous linear equations. Various representations of the general solution.

1°. Consider a second-order homogeneous linear equation in the general form:

$$f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = 0. \tag{1}$$

The *trivial solution*, $y = 0$, is a particular solution of the homogeneous linear equation.

Let $y_1(x)$, $y_2(x)$ be a fundamental system of solutions (nontrivial linearly independent particular solutions) of equation (1). Then the general solution is given by:

$$y = C_1 y_1(x) + C_2 y_2(x), \tag{2}$$

where C_1 and C_2 are arbitrary constants.

2°. Let $y_1 = y_1(x)$ be any nontrivial particular solution of equation (1). Then its general solution can be represented as:

$$y = C_1 y_1 + C_2 y_1 \int \frac{e^{-F}}{y_1^2}\,dx, \quad \text{where} \quad F = \int \frac{f_1}{f_2}\,dx. \tag{3}$$

3°. Consider the equation

$$y''_{xx} + f(x)y = 0,$$

which is written in the canonical form; see Paragraph 0.2.1-3 for the reduction of equations to this form. Let $y_1(x)$ be any nontrivial partial solution of this equation. The general solution can be constructed by formula (3) with $F = 0$ or formula (2) in which

$$y_2(x) = y_1 \int \frac{[f(x)-1][y_1^2 - (y_1')^2]}{[y_1^2 + (y_1')^2]^2}\,dx + \frac{y_1'}{y_1^2 + (y_1')^2}.$$

Here, $y_1 = y_1(x)$ and the prime denotes differentiation with respect to x. The last formula is suitable where y_1 vanishes at some points.

0.2.1-2. Wronskian determinant and Liouville's formula.

The Wronskian determinant (or Wronskian) is defined by:

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \equiv y_1(y_2)'_x - y_2(y_1)'_x,$$

where $y_1(x)$, $y_2(x)$ is a fundamental system of solutions of equation (1).

Liouville's formula:

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} \frac{f_1(t)}{f_2(t)}\,dt\right).$$

0.2.1-3. Reduction to the canonical form.

1°. The substitution

$$y = u(x)\exp\left(-\frac{1}{2}\int \frac{f_1}{f_2}\,dx\right) \tag{4}$$

brings equation (1) to the canonical (or normal) form:

$$u''_{xx} + f(x)u = 0, \quad \text{where} \quad f = \frac{f_0}{f_2} - \frac{1}{4}\left(\frac{f_1}{f_2}\right)^2 - \frac{1}{2}\left(\frac{f_1}{f_2}\right)'_x. \tag{5}$$

2°. The substitution (4) is a special case of the more general transformation (φ is an arbitrary function)

$$x = \varphi(\xi), \quad y = u(\xi)\sqrt{|\varphi'(\xi)|}\,\exp\left(-\frac{1}{2}\int \frac{f_1(x)}{f_2(x)}\,dx\right),$$

which also brings the original equation to the canonical form.

0.2.1-4. Reduction to the Riccati equation.

The substitution $= \; '$ brings the second-order homogeneous linear equation (1) to the Riccati equation:

$$_2(\;)\;' + \;_2(\;)\;^2 + \;_1(\;)\; + \;_0(\;) = 0,$$

which is discussed in Subsection 0.1.4.

0.2.1-5. Nonhomogeneous linear equations. The existence theorem.

A second-order nonhomogeneous linear equation has the form

$$_2(\;)\;'' + \;_1(\;)\;' + \;_0(\;)\; = g(\;). \tag{6}$$

Existence and uniqueness theorem. On an open interval $a < \; <$, let the functions $_2$, $_1$, $_0$, and g be continuous and $_2 \neq 0$. Also let

$$(\;_0) = A, \qquad '(\;_0) = B$$

be arbitrary initial conditions, where $_0$ is any point such that $a < \;_0 <$, and A and B are arbitrary prescribed numbers. Then a solution of equation (6) exists and is unique. This solutions is defined for all $(a, \;)$.

0.2.1-6. Nonhomogeneous linear equations. Various representations of the general solution.

1 . The general solution of the nonhomogeneous linear equation (6) is the sum of the general solution of the corresponding homogeneous equation (1) and any particular solution of the nonhomogeneous equation (6).

2 . Let $_1 = \;_1(\;)$, $_2 = \;_2(\;)$ be a fundamental system of solutions of the corresponding homogeneous equation, with $g \equiv 0$. Then the general solution of equation (6) can be represented as:

$$= \;_1\;_1 + \;_2\;_2 + \;_2\;\;_1\frac{g}{_2} - \;_1\;\;_2\frac{g}{_2}, \tag{7}$$

where $= \;_1(\;_2)' - \;_2(\;_1)'$ is the Wronskian determinant.

3 . Given a nontrivial particular solution $_1 = \;_1(\;)$ of the homogeneous equation (with $g \equiv 0$), a second particular solution $_2 = \;_2(\;)$ can be calculated from the formula:

$$_2 = \;_1\;\frac{e^{-F}}{_1^2}, \qquad \text{where} \quad = \frac{_1}{_2}, \qquad = e^{-F}. \tag{8}$$

Then the general solution of equation (6) can be constructed by (7).

Subsections 2.1.2–2.1.8 present only homogeneous equations; the solutions of the corresponding nonhomogeneous equations can be obtained using (7) and (8).

4 . Let $\bar{}_1$ and $\bar{}_2$ be respective solutions of the nonhomogeneous differential equations $\mathrm{L}[\bar{}_1] = g_1(\;)$ and $\mathrm{L}[\bar{}_2] = g_2(\;)$, which have the same left-hand side but different right-hand sides; where $\mathrm{L}[\;]$ is the left-hand side of equation (6). Then the function $\bar{} = \bar{}_1 + \bar{}_2$ is a solution of the equation $\mathrm{L}[\bar{}] = g_1(\;) + g_2(\;)$.

0.2.1-7. Reduction to a constant coefficient equation (a special case).

Let $_2 = 1$, $_0 \neq 0$, and the condition

$$\frac{1}{|\;_0|}\;\sqrt{|\;_0|} + \frac{_1}{\sqrt{|\;_0|}} = a = \text{const}$$

be satisfied. Then the substitution $\xi = \sqrt{|\;_0|}$ leads to a constant coefficient linear equation,

$$'' + a\;' + \operatorname{sign}\;_0 = 0.$$

0.2.1-8. Kummer–Liouville transformation.

The transformation

$$= \ (\ ), \qquad = \beta(\ )z + \ (\ ), \tag{9}$$

where ( ), $\beta($ $)$, and ( ) are arbitrary sufficiently smooth functions (β 0), takes any linear differential equation for ( ) to a linear equation for $z = z($ $)$. In the special case $\equiv 0$, a homogeneous equation is transformed to a homogeneous one.

Special cases of transformation (9) are widely used to simplify second- and higher-order linear differential equations.

References for Subsection 0.2.1: G. A. Korn and T. M. Korn (1968), E. Kamke (1977), A. D. Polyanin and V. F. Zaitsev (1995), S. Yu. Dobrokhotov (1998), D. Zwillinger (1998).

0.2.2. Representation of Solutions as a Series in the Independent Variable

0.2.2-1. Equation coefficients are representable in the ordinary power series form.

Let us consider a homogeneous linear differential equation of the general form

$$'' + \ (\ ) \ ' + g(\ ) \ = 0. \tag{1}$$

Assume that the functions ( ) and $g($ $)$ are representable, in the vicinity of a point $= \ _0$, in the power series form,

$$(\ ) = \sum_{=0} A \ (\ - \ _0) \ , \qquad g(\ ) = \sum_{=0} B \ (\ - \ _0) \ , \tag{2}$$

on the interval $| \ - \ _0| < R$, where R stands for the minimum radius of convergence of the two series in (2). In this case, the point $= \ _0$ is referred to as an *ordinary point*, and equation (1) possesses two linearly independent solutions of the form:

$$_1(\ ) = \sum_{=0} a \ (\ - \ _0) \ , \qquad _2(\ ) = \sum_{=0} \ (\ - \ _0) \ . \tag{3}$$

The coefficients a and are determined by substituting the series (2) into equation (1) followed by extracting the coefficients of like powers of $(\ - \ _0)$.*

0.2.2-2. Equation coefficients have poles at some point.

Assume that the functions ( ) and $g($ $)$ are representable, in the vicinity of a point $= \ _0$, in the form

$$(\ ) = \sum_{=-1} A \ (\ - \ _0) \ , \qquad g(\ ) = \sum_{=-2} B \ (\ - \ _0) \ , \tag{4}$$

on the interval $| \ - \ _0| < R$. In this case, the point $= \ _0$ is referred to as a *regular singular point*. Let λ_1 and λ_2 be roots of the quadratic equation

$$\lambda_1^2 + (A_{-1} - 1)\lambda + B_{-2} = 0.$$

There are three cases, depending on the values of the exponents of the singularity.

* Prior to that, the terms containing the same powers $(\ - \ _0)$, $k = 0, 1,$, should be collected.

1. If $\lambda_1 \neq \lambda_2$ and $\lambda_1 - \lambda_2$ is not an integer, equation (1) has two linearly independent solutions of the form:

$$
\begin{aligned}
&{}_1(\ ) = |\ -\ {}_0|^{\lambda_1}\ 1 + \quad\ \underset{=1}{\ }\ a\ (\ -\ {}_0)\quad ,\\
&{}_2(\ ) = |\ -\ {}_0|^{\lambda_2}\ 1 + \quad\ \underset{=1}{\ }\quad (\ -\ {}_0)\quad .
\end{aligned}
\tag{5}
$$

2. If $\lambda_1 = \lambda_2 = \lambda$, equation (1) possesses two linearly independent solutions:

$$
\begin{aligned}
&{}_1(\ ) = |\ -\ {}_0|^{\lambda}\ 1 + \quad\ \underset{=1}{\ }\ a\ (\ -\ {}_0)\quad ,\\
&{}_2(\ ) = \ {}_1(\ )\ln|\ -\ {}_0| + |\ -\ {}_0|^{\lambda}\quad \underset{=0}{\ }\quad (\ -\ {}_0)\ .
\end{aligned}
$$

3. If $\lambda_1 = \lambda_2 + N$, where N is a positive integer, equation (1) has two linearly independent solutions of the form:

$$
\begin{aligned}
&{}_1(\ ) = |\ -\ {}_0|^{\lambda_1}\ 1 + \quad\ \underset{=1}{\ }\ a\ (\ -\ {}_0)\quad ,\\
&{}_2(\ ) = k\ {}_1(\ )\ln|\ -\ {}_0| + |\ -\ {}_0|^{\lambda_2}\quad \underset{=0}{\ }\quad (\ -\ {}_0)\ ,
\end{aligned}
$$

where k may be equal to zero.

To construct the solution in each of the three cases, the following procedure should be performed: substitute the above expressions of $\ _1$ and $\ _2$ into the original equation (1) and equate the coefficients of $(\ -\ _0)$ and $(\ -\ _0)\ \ln|\ -\ _0|$ for different values of to obtain recurrence relations for the unknown coefficients. From these recurrence relations the solution sought can be found.

References for Subsection 0.2.2: G. M. Murphy (1960), G. A. Korn and T. M. Korn (1968), E. Kamke (1977), D. Zwillinger (1989).

0.2.3. Asymptotic Solutions

This subsection presents asymptotic solutions, as $\varepsilon\quad 0$ $(\varepsilon > 0)$, of some second-order linear ordinary differential equations containing arbitrary functions (sufficiently smooth), with the independent variable being real.

0.2.3-1. Equations not containing $\ '$. Leading asymptotic terms.

1 . Consider the equation

$$
\varepsilon^2\ {}''\ -\ (\ )\ \ = 0 \tag{1}
$$

on a closed interval $a \leq\ \ \leq\ \ .$

Case 1. With the condition $\ \neq 0$, the leading terms of the asymptotic expansions of the fundamental system of solutions, as $\varepsilon\quad 0$, are given by the formulas:

$$
\begin{aligned}
&{}_1 = \ {}^{-1\ 4}\exp\left(-\frac{1}{\varepsilon}\quad \overline{\ }\ \right), &&{}_2 = \ {}^{-1\ 4}\exp\left(\frac{1}{\varepsilon}\quad \overline{\ }\ \right) &&\text{if}\quad > 0,\\
&{}_1 = (-\ )^{-1\ 4}\cos\left(\frac{1}{\varepsilon}\quad \overline{-}\ \right), &&{}_2 = (-\ )^{-1\ 4}\sin\left(\frac{1}{\varepsilon}\quad \overline{-}\ \right) &&\text{if}\quad < 0.
\end{aligned}
$$

Case 2. Discuss the asymptotic solution of equation (1) in the vicinity of the point $\;\; = \;_0$, where function $\;(\;)$ vanishes, $\;(\;_0) = 0$ (such a point is referred to as a transition point). We assume that the function can be presented in the form

$$(\;) = (\;_0 - \;)\;(\;), \qquad \text{where} \qquad (\;) > 0.$$

In this case, the fundamental solutions, as $\varepsilon \;\; 0$, are described by three different formulas:

$$_1 = \begin{cases} \dfrac{1}{|\;(\;)|^{1\;4}} \sin \; \dfrac{1}{\varepsilon} \int_0 \; \overline{|\;(\;)|} \;\; + \dfrac{}{4} & \text{if } \; - \;_0 \geq \; , \\ \dfrac{}{[\varepsilon\;\;(\;_0)]^{1\;6}} \operatorname{Ai}(z) & \text{if } |\; - \;_0| \leq \; , \\ \dfrac{1}{2[\;(\;)]^{1\;4}} \exp \; -\dfrac{1}{\varepsilon} \int^{\;0} \; \overline{(\;)} & \text{if } \;_0 - \; \geq \; , \end{cases}$$

$$_2 = \begin{cases} \dfrac{1}{|\;(\;)|^{1\;4}} \cos \; \dfrac{1}{\varepsilon} \int_0 \; \overline{|\;(\;)|} \;\; + \dfrac{}{4} & \text{if } \; - \;_0 \geq \; , \\ \dfrac{}{[\varepsilon\;\;(\;_0)]^{1\;6}} \operatorname{Bi}(z) & \text{if } |\; - \;_0| \leq \; , \\ \dfrac{1}{[\;(\;)]^{1\;4}} \exp \; \dfrac{1}{\varepsilon} \int^{\;0} \; \overline{(\;)} & \text{if } \;_0 - \; \geq \; , \end{cases}$$

where $\operatorname{Ai}(z)$ and $\operatorname{Bi}(z)$ are the Airy functions of the first and second kind, respectively (see equation 2.1.2.2), $z = \varepsilon^{-2\;3}[\;\;(\;_0)]^{1\;3}(\;_0 - \;)$, and $\;\; = \;\;(\varepsilon^{2\;3})$.

0.2.3-2. Equations not containing $\;'$. Two-term asymptotic expansions.

The two-term asymptotic expansions of the solution of equation (1) with $\;\; > 0$, as $\varepsilon \;\; 0$, on a closed interval $a \leq \; \leq \;$, has the form:

$$\begin{aligned} _1 &= \;^{-1\;4} \exp \; -\frac{1}{\varepsilon} \int_0 \; - \;\; 1 - \varepsilon \int_0 \; \frac{1}{8}\frac{\;''}{\;^{3\;2}} - \frac{5}{32}\frac{(\;')^2}{\;^{5\;2}} \;\; + \;(\varepsilon^2) \; , \\ _2 &= \;^{-1\;4} \exp \; \frac{1}{\varepsilon} \int_0 \; - \;\; 1 + \varepsilon \int_0 \; \frac{1}{8}\frac{\;''}{\;^{3\;2}} - \frac{5}{32}\frac{(\;')^2}{\;^{5\;2}} \;\; + \;(\varepsilon^2) \; , \end{aligned} \tag{2}$$

where $\;_0$ is an arbitrary number satisfying the inequality $a \leq \;_0 \leq \;$.

The asymptotic expansions of the fundamental system of solutions of equation (1) with $\;\; < 0$ are derived by separating the real and imaginary parts in either formula (2).

0.2.3-3. Equations of special form not containing $\;'$.

Consider the equation

$$\varepsilon^2\;'' - \;\;^{-2}\;(\;)\; = 0 \tag{3}$$

on a closed interval $a \leq \; \leq \;$, where $a < 0$ and $\;\; > 0$, under the conditions that $\;$ is a positive integer and $\;(\;) \neq 0$. In this case, the leading term of the asymptotic solution, as $\varepsilon \;\; 0$, in the vicinity of the point $\;\; = 0$ is expressed in terms of a simpler model equation, which results from substituting the function $\;(\;)$ in equation (3) by the constant $\;(0)$ (the solution of the model equation is expressed in terms of the Bessel functions of order $1\;\;\;$, see equation 2.1.2.7).

We specify below formulas by which the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (3) with $a < \;\; < 0$ and $0 < \;\; < \;$ are related (excluding a small vicinity of the point $\;\; = 0$). Three different cases can be extracted.

1 . Let be an even integer, and $( ) > 0$. Then,

$$
{}_1 = \begin{cases} [\ (\ )]^{-1\ 4} \exp \frac{1}{\varepsilon}\ {}_0 \ \overline{(\ )} & \text{if} \ < 0, \\ k^{-1}[\ (\ )]^{-1\ 4} \exp \frac{1}{\varepsilon}\ {}_0 \ \overline{(\ )} & \text{if} \ > 0, \end{cases}
$$

$$
{}_2 = \begin{cases} [\ (\ )]^{-1\ 4} \exp -\frac{1}{\varepsilon}\ {}_0 \ \overline{(\ )} & \text{if} \ < 0, \\ k[\ (\ )]^{-1\ 4} \exp -\frac{1}{\varepsilon}\ {}_0 \ \overline{(\ )} & \text{if} \ > 0, \end{cases}
$$

where $= (\ ),\ k = \sin \overline{}$.

2 . Let be an even integer, and $( ) < 0$. Then,

$$
{}_1 = \begin{cases} |\ (\ )|^{-1\ 4} \cos -\frac{1}{\varepsilon}\ {}_0 \ \overline{|\ (\ )|} \ + \frac{}{4} & \text{if} \ < 0, \\ k^{-1}|\ (\ )|^{-1\ 4} \cos \frac{1}{\varepsilon}\ {}_0 \ \overline{|\ (\ )|} \ - \frac{}{4} & \text{if} \ > 0, \end{cases}
$$

$$
{}_2 = \begin{cases} |\ (\ )|^{-1\ 4} \cos -\frac{1}{\varepsilon}\ {}_0 \ \overline{|\ (\ )|} \ - \frac{}{4} & \text{if} \ < 0, \\ k|\ (\ )|^{-1\ 4} \cos \frac{1}{\varepsilon}\ {}_0 \ \overline{|\ (\ )|} \ + \frac{}{4} & \text{if} \ > 0, \end{cases}
$$

where $= (\ ),\ k = \tan \frac{}{2}$.

3 . Let be an odd integer. Then,

$$
{}_1 = \begin{cases} |\ (\ )|^{-1\ 4} \cos -\frac{1}{\varepsilon}\ {}_0 \ \overline{|\ (\ )|} \ + \frac{}{4} & \text{if} \ < 0, \\ \frac{1}{2}k^{-1}[\ (\ )]^{-1\ 4} \exp \frac{1}{\varepsilon}\ {}_0 \ \overline{(\ )} & \text{if} \ > 0, \end{cases}
$$

$$
{}_2 = \begin{cases} |\ (\ )|^{-1\ 4} \cos -\frac{1}{\varepsilon}\ {}_0 \ \overline{|\ (\ )|} \ - \frac{}{4} & \text{if} \ < 0, \\ k[\ (\ )]^{-1\ 4} \exp -\frac{1}{\varepsilon}\ {}_0 \ \overline{(\ )} & \text{if} \ > 0, \end{cases}
$$

where $= (\ ),\ k = \sin \frac{}{2}$.

0.2.3-4. Equations not containing $'$. Equation coefficients are dependent on ε.

Consider an equation of the form

$$
\varepsilon^2 \ '' - (\ ,\varepsilon) \ = 0 \tag{4}
$$

on a closed interval $a \le \ \le$ under the condition that $\ne 0$. Assume that the following asymptotic relation holds:

$$
(\ ,\varepsilon) = \sum_{=0} (\ )\varepsilon \ , \qquad \varepsilon \quad 0.
$$

Then the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (4) are given by the formulas:

$$
{}_1 = {}_0^{-1\ 4}(\ ) \exp -\frac{1}{\varepsilon} \ \overline{{}_0(\ )} \ + \frac{1}{2} \ \frac{{}_1(\ )}{\overline{{}_0(\ )}} \ \big[1 + \ (\varepsilon)\big],
$$

$$
{}_2 = {}_0^{-1\ 4}(\ ) \exp \frac{1}{\varepsilon} \ \overline{{}_0(\ )} \ + \frac{1}{2} \ \frac{{}_1(\ )}{\overline{{}_0(\ )}} \ \big[1 + \ (\varepsilon)\big].
$$

0.2.3-5. Equations containing $'$.

1 . Consider an equation of the form

$$\varepsilon \;'' + g(\;)\;' + (\;)\; = 0$$

on a closed interval $0 \le \;\le 1$. With $g(\;) > 0$, the asymptotic solution of this equation, satisfying the boundary conditions $(0) = {}_1$ and $(1) = {}_2$, can be represented in the form:

$$= ({}_1 - k\;{}_2)\exp\left[-\varepsilon^{-1}g(0)\;\right] + {}_2\exp\;\int^1 \frac{(\;)}{g(\;)} \; + \;(\varepsilon),$$

where $k = \exp \int_0^1 \frac{(\;)}{g(\;)}$.

2 . Now let us take a look at an equation of the form

$$\varepsilon^2\;'' + \varepsilon g(\;)\;' + (\;)\; = 0 \qquad (5)$$

on a closed interval $a \le \;\le$. Assume

$$(\;) \equiv [g(\;)]^2 - 4\;(\;) \ne 0.$$

Then the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (5), as $\varepsilon \;\; 0$, are expressed by

$$_1 = |\;(\;)|^{-1\;4}\exp\; -\frac{1}{2\varepsilon}\;\;\overline{\;(\;)}\; - \frac{1}{2}\;\frac{g'(\;)}{\overline{\;(\;)}}\;\left[1 + \;(\varepsilon)\right],$$

$$_2 = |\;(\;)|^{-1\;4}\exp\; \frac{1}{2\varepsilon}\;\;\overline{\;(\;)}\; - \frac{1}{2}\;\frac{g'(\;)}{\overline{\;(\;)}}\;\left[1 + \;(\varepsilon)\right].$$

0.2.3-6. Equations of the general form.

The more general equation

$$\varepsilon^2\;'' + \varepsilon g(\;,\varepsilon)\;' + (\;,\varepsilon)\; = 0$$

is reducible, with the aid of the substitution $\;=\;\;\exp\; -\frac{1}{2\varepsilon}\;\;g$, to an equation of the form (4),

$$\varepsilon^2\;'' + (\; - \tfrac{1}{4}g^2 - \tfrac{1}{2}\varepsilon g'\;)\; = 0,$$

to which the asymptotic formulas given above in Paragraph 0.2.3-4 are applicable.

References for Subsection 0.2.3: W. Wasov (1965), F. W. J. Olver (1974), A. H. Nayfeh (1973, 1981), M. V. Fedoryuk (1993).

0.2.4. Boundary Value Problems

0.2.4-1. The first, second, third, and mixed boundary value problems ($_1 \le \;\le\; _2$).

We consider the second-order nonhomogeneous linear differential equation

$$'' + \;(\;)\;' + g(\;)\; = \;(\;). \qquad (1)$$

1 . *The first boundary value problem*: Find a solution of equation (1) satisfying the boundary conditions

$$= a_1 \quad\text{at}\quad = {}_1, \qquad = a_2 \quad\text{at}\quad = {}_2. \qquad (2)$$

(The values of the unknown are prescribed at two distinct points $_1$ and $_2$).

2 . *The second boundary value problem*: Find a solution of equation (1) satisfying the boundary conditions

$$' = a_1 \quad \text{at} \quad = {}_1, \qquad ' = a_2 \quad \text{at} \quad = {}_2. \tag{3}$$

(The values of the derivative of the unknown are prescribed at two distinct points $_1$ and $_2$).

3 . *The third boundary value problem*: Find a solution of equation (1) satisfying the boundary conditions

$$\begin{aligned} ' - k_1 \quad &= a_1 \quad \text{at} \quad = {}_1, \\ ' + k_2 \quad &= a_2 \quad \text{at} \quad = {}_2. \end{aligned} \tag{4}$$

4 . *The third boundary value problem*: Find a solution of equation (1) satisfying the boundary conditions

$$= a_1 \quad \text{at} \quad = {}_1, \qquad ' = a_2 \quad \text{at} \quad = {}_2. \tag{5}$$

(The unknown itself is prescribed at one point, and its derivative at another point.)

Conditions (1), (2), (3), and (4) are called *homogeneous* if $a_1 = a_2 = 0$.

0.2.4-2. Simplification of boundary conditions. Reduction of equation to the self-adjoint form.

1 . Nonhomogeneous boundary conditions can be reduced to homogeneous ones by the change of variable $z = A_2 \;^2 + A_1 \; + A_0 +$ (the constants A_2, A_1, and A_0 are selected using the method of undetermined coefficients). In particular, the nonhomogeneous boundary conditions of the first kind (1) can be reduced to homogeneous boundary conditions by the linear change of variable

$$z = \quad - \frac{a_2 - a_1}{{}_2 - {}_1}(\quad - {}_1) - a_1.$$

2 . On multiplying by $(\ ) = \exp \quad (\ )$, one reduces equation (1) to the self-adjoint form:

$$[\ (\ )\ ']' + \ (\ ) \quad = \ (\ ). \tag{6}$$

Without loss of generality, we further consider equation (6) instead of (1). We assume that the functions , $'$, , and are continuous on the interval $_1 \le \quad \le {}_2$, and is positive.

0.2.4-3. The Green's function. Boundary value problems for nonhomogeneous equations.

The *Green's function* of the first boundary value problem for equation (6) with homogeneous boundary conditions (2) is a function of two variables $(\ ,\ )$ that satisfies the following conditions:

1 . $(\ ,\ )$ is continuous in for fixed , with $_1 \le \quad \le {}_2$ and $_1 \le \quad \le {}_2$.

2 . $(\ ,\ )$ is a solution of the homogeneous equation (6), with $= 0$, for all $_1 < \quad < {}_2$ exclusive of the point $=$.

3 . $(\ ,\ )$ satisfies the homogeneous boundary conditions $(\ _1,\ ) = (\ _2,\ ) = 0$.

4 . The derivative $'(\ ,\ )$ has a jump of $1 \quad (\ )$ at the point $=$, that is,

$$'(\ ,\ )\Big|_{\ ,\ >} - \ '(\ ,\ )\Big|_{\ ,\ <} = \frac{1}{(\ )}.$$

For the second, third, and mixed boundary value problems, the Green's function is defined likewise except that in 3 the homogeneous boundary conditions (3), (4), and (5), with $a_1 = a_2 = 0$, are adopted, respectively.

The solution of the nonhomogeneous equation (6) subject to appropriate homogeneous boundary conditions is expressed in terms of the Green's function as follows:*

$$(\ ) = \int_1^2 (\ ,\ )\ (\ ) \quad .$$

* The homogeneous boundary value problem—with $(\ ) = 0$ and $_1 = \ _2 = 0$—is assumed to have only the trivial solution.

0.2.4-4. Representation of the Green's function in terms of particular solutions.

We consider the first boundary value problem. Let $_1 = {}_1(\ )$ and $_2 = {}_2(\ )$ be linearly independent particular solutions of the homogeneous equation (6), with $= 0$, that satisfy the conditions

$$_1(\ _1) = 0, \qquad _2(\ _2) = 0.$$

(Each of the solutions satisfies one of the homogeneous boundary conditions.)

The Green's function is expressed in terms of solutions of the homogeneous equation as follows:

$$(\ ,\ ) = \begin{cases} \dfrac{_1(\ )\ _2(\ )}{(\ )\ \ (\ )} & \text{for } _1 \le\ \le\ , \\ \dfrac{_1(\ )\ _2(\ )}{(\ )\ \ (\ )} & \text{for } \ \le\ \le\ _2, \end{cases} \tag{7}$$

where $(\ ) = {}_1(\ )\ {}'_2(\ ) - {}'_1(\ )\ {}_2(\ )$ is the Wronskian determinant.

Formula (7) can also be used to construct the Green's functions for the second, third, and mixed boundary value problems. To this end, one should find two linearly independent solutions, $_1 = {}_1(\ )$ and $_2 = {}_2(\ )$, of the homogeneous equation; the former satisfies the corresponding homogeneous boundary condition at $= {}_1$ and the latter satisfies the one at $= {}_2$.

References for Subsection 0.2.4: L. E. El'sgol'ts (1961), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1980).

0.2.5. Eigenvalue Problems

0.2.5-1. The Sturm–Liouville problem.

Consider the second-order homogeneous linear differential equation

$$[\ (\ )\ ']' + [\lambda g(\ ) - \ (\ )]\ = 0 \tag{1}$$

subject to linear boundary conditions of the general form

$$\begin{aligned} a_1\ ' + {}_1\ &= 0 \quad \text{at} \quad = {}_1, \\ a_2\ ' + {}_2\ &= 0 \quad \text{at} \quad = {}_2. \end{aligned} \tag{2}$$

It is assumed that the functions $\ ,\ '$, g, and $\ $ are continuous, and $\ $ and g are positive on an interval $_1 \le\ \le\ _2$. It is also assumed that $|a_1| + |\ _1| > 0$ and $|a_2| + |\ _2| > 0$.

The *Sturm–Liouville problem*: Find the values λ of the parameter λ at which problem (1), (2) has a nontrivial solution. Such λ are called *eigenvalues* and the corresponding solutions $= (\ )$ are called *eigenfunctions* of the Sturm–Liouville problem (1), (2).

0.2.5-2. General properties of the Sturm–Liouville problem (1), (2).

1 . There are infinitely (countably) many eigenvalues. All eigenvalues can be ordered so that $\lambda_1 < \lambda_2 < \lambda_3 <$. Moreover, λ as ; hence, there can only be a finite number of negative eigenvalues. Each eigenvalue has multiplicity 1.

2 . The eigenfunctions are defined up to a constant factor. Each eigenfunction $(\ )$ has precisely -1 zeros on the open interval $(\ _1,\ _2)$.

3 . Any two eigenfunctions $(\ )$ and $(\ )$, $\ne$, are orthogonal with weight $g(\ )$ on the interval $_1 \le\ \le\ _2$:

$$\int_1^2 g(\ )\ \ (\ )\ \ (\ )\ \ = 0 \quad \text{if} \quad \ne\ .$$

4 . An arbitrary function ( ) that has a continuous derivative and satisfies the boundary conditions of the Sturm–Liouville problem can be decomposed into an absolutely and uniformly convergent series in the eigenfunctions:

$$( ) = \sum_{=1} ( ),$$

where the Fourier coefficients of ( ) are calculated by

$$= \frac{1}{\ \ ^2} \int_1^2 g( )\ ( )\ ( ) \ , \qquad ^2 = \int_1^2 g( )\ ^2( ) \ .$$

5 . If the conditions

$$( ) \ge 0, \quad a_1 \ _1 \le 0, \quad a_2 \ _2 \ge 0 \tag{3}$$

hold true, there are no negative eigenvalues. If $\equiv 0$ and $_1 = \ _2 = 0$, the least eigenvalue is $\lambda_1 = 0$, to which there corresponds an eigenfunction $_1 =$ const. In the other cases where conditions (3) are satisfied, all eigenvalues are positive.

6 . The following asymptotic formula is valid for eigenvalues as :

$$\lambda \ = \frac{\ ^2\ ^2}{\Delta^2} + \ (1), \qquad \Delta = \int_1^2 \sqrt{\frac{g( )}{( )}} \ . \tag{4}$$

Paragraphs 0.2.5-3 through 0.2.5-6 will describe special properties of the Sturm–Liouville problem which depend on the specific form of the boundary conditions.

Equation (1) can be reduced to the case where $( ) \equiv 1$ and $g( ) \equiv 1$ by the change of variables

$$= \int \sqrt{\frac{g( )}{( )}} \ , \qquad ( ) = \big[\ ( )g( )\big]^{1\ 4} \ ( ).$$

In this case, the boundary conditions are transformed to boundary conditions of similar form.

The second-order linear equation

$$_2( )\ '' + \ _1( )\ ' + [\lambda + \ _0( )]\ = 0$$

can be represented in the form of equation (1) where $( )$, $g( )$, and $( )$ are given by:

$$( ) = \exp \int \frac{_1( )}{_2( )} \ , \quad g( ) = \frac{1}{_2( )} \exp \int \frac{_1( )}{_2( )} \ , \quad ( ) = -\frac{_0( )}{_2( )} \exp \int \frac{_1( )}{_2( )} \ .$$

0.2.5-3. Problems with boundary conditions of the first kind (case $a_1 = a_2 = 0$ and $_1 = \ _2 = 1$).

Let us note some special properties of the Sturm–Liouville problem that is the first boundary value problem for equation (1) with the boundary conditions

$$= 0 \quad \text{at} \quad = \ _1, \qquad = 0 \quad \text{at} \quad = \ _2. \tag{5}$$

1 . For , the asymptotic relation (4) can be used to estimate the eigenvalues λ . In this case, the asymptotic formula

$$\frac{( )}{} = \left[\frac{4}{\Delta^2\ ( )g( )}\right]^{1\ 4} \sin\left[\frac{}{\Delta} \int_1 \sqrt{\frac{g( )}{( )}} \ \right] + \ \frac{1}{}\ , \quad \Delta = \int_1^2 \sqrt{\frac{g( )}{( )}} \ $$

holds true for the eigenfunctions ( ).

TABLE 2

Example estimates of the first eigenvalue λ_1 in Sturm–Liouville problems with boundary conditions of the first kind $(0) = (1) = 0$ obtained using the Rayleigh–Ritz principle [the right-hand side of relation (6)]

Equation	Test function	λ_1, approximate	λ_1, exact
$'' +\lambda(1+ {}^2)^{-2} =0$	$z=\sin$	15.337	15.0
$'' +\lambda(4- {}^2)^{-2} =0$	$z=\sin$	135.317	134.837
$[(1+)^{-1} \ ']' +\lambda =0$	$z=\sin$	7.003	6.772
$(\overline{1+} \ ')' +\lambda =0$	$z=\sin$	11.9956	11.8985
$'' +\lambda(1+\sin \ ) =0$	$z=\sin$ $z= (1-)$	0.54105 ^{2} 0.55204 ^{2}	0.54032 ^{2} 0.54032 ^{2}

2 . If ≥ 0, the following upper estimate holds for the least eigenvalue (Rayleigh–Ritz principle):

$$\lambda_1 \leq \frac{\int_1^2 \left[( )(z')^2 + ( )z^2 \right]}{\int_1^2 g( )z^2}, \tag{6}$$

where $z = z( )$ is any twice differentiable function that satisfies the conditions $z({}_1) = z({}_2) = 0$. The equality in (6) is attained if $z = {}_1( )$, where ${}_1( )$ is the eigenfunction corresponding to the eigenvalue λ_1. One can take $z = (- {}_1)({}_2 -)$ or $z = \sin \frac{(- {}_1)}{{}_2 - {}_1}$ in (6) to obtain specific estimates.

It is significant to note that the left-hand side of (6) usually gives a fairly precise estimate of the first eigenvalue (see Table 2).

3 . The extension of the interval $[{}_1, {}_2]$ leads to decreasing in eigenvalues.

4 . Let the inequalities

$$0 < {}_{\min} \leq ( ) \leq {}_{\max}, \quad 0 < g_{\min} \leq g( ) \leq g_{\max}, \quad 0 < {}_{\min} \leq ( ) \leq {}_{\max}$$

be satisfied. Then the following bilateral estimates hold:

$$\frac{{}_{\min}}{g_{\max}} \frac{{}^2 \ {}^2}{({}_2 - {}_1)^2} + \frac{{}_{\min}}{g_{\max}} \leq \lambda \ \leq \frac{{}_{\max}}{g_{\min}} \frac{{}^2 \ {}^2}{({}_2 - {}_1)^2} + \frac{{}_{\max}}{g_{\min}}.$$

5 . In engineering calculations for eigenvalues, the approximate formula

$$\lambda = \frac{{}^2 \ {}^2}{\Delta^2} + \frac{1}{{}_2 - {}_1} \int_1^2 \frac{( )}{g( )} \quad , \quad \Delta = \int_1^2 \sqrt{\frac{g( )}{( )}} \tag{7}$$

may be quite useful. This formula provides an exact result if $( )g( ) = \text{const}$ and $( ) \ g( ) = \text{const}$ (in particular, for constant equation coefficients, $= {}_0$, $= {}_0$, and $g = g_0$) and gives a correct asymptotic behavior of (4) for any $( )$, $( )$, and $g( )$. In addition, relation (7) gives two correct leading asymptotic terms as if $( ) = \text{const}$ and $g( ) = \text{const}$ [and also if $( )g( ) = \text{const}$].

6 . Suppose ( ) $= g($ $) = 1$ and the function = ( ) has a continuous derivative. The following asymptotic relations hold for eigenvalues λ and eigenfunctions ( ) as :

$$\overline{\lambda} = \frac{}{{}_2 - {}_1} + \frac{1}{} \quad ({}_1, {}_2) + \quad \frac{1}{{}^2},$$

$$( ) = \sin \frac{(- {}_1)}{{}_2 - {}_1} - \frac{1}{} \quad ({}_1 -) \quad (, {}_2) + ({}_2 -) \quad ({}_1,) \quad \cos \frac{(- {}_1)}{{}_2 - {}_1} + \quad \frac{1}{{}^2},$$

where

$$(, v) = \frac{1}{2} \qquad ( ) \qquad . \tag{8}$$

7 . Let us consider the eigenvalue problem for the equation with a small parameter

$$'' + [\lambda + \varepsilon \ ( )] \ = 0 \qquad (\varepsilon \qquad 0)$$

subject to the boundary conditions (5) with ${}_1 = 0$ and ${}_2 = 1$. We assume that ( ) = (−).

This problem has the following eigenvalues and eigenfunctions:

$$\lambda \ = \ {}^2 \ {}^2 - \varepsilon A \quad + \frac{\varepsilon^2}{{}^2} \sum_{\neq} \frac{A^2}{{}^2 - k^2} + \ (\varepsilon^3), \qquad A \quad = 2 \int_0^1 ( ) \sin(\quad) \sin(\ k \ ) \quad ;$$

$$( ) = \sqrt{2} \sin(\qquad) - \varepsilon \frac{\sqrt{2}}{{}^2} \sum_{\neq} \frac{A}{{}^2 - k^2} \sin(\ k \ ) + \ (\varepsilon^2).$$

Here, the summation is carried out over k from 1 to . The next term in the expansion of can be found in Nayfeh (1973).

0.2.5-4. Problems with boundary conditions of the second kind (case $a_1 = a_2 = 1$ and ${}_1 = {}_2 = 0$).

Let us note some special properties of the Sturm–Liouville problem that is the second boundary value problem for equation (1) with the boundary conditions

$$' = 0 \quad \text{at} \quad = {}_1, \qquad ' = 0 \quad \text{at} \quad = {}_2.$$

1 . If > 0, the upper estimate (6) is valid for the least eigenvalue, with $z = z( )$ being any twice-differentiable function that satisfies the conditions $z'({}_1) = z'({}_2) = 0$. The equality in (6) is attained if $z = {}_1( )$, where ${}_1( )$ is the eigenfunction corresponding to the eigenvalue λ_1.

2 . Suppose ( ) $= g($ $) = 1$ and the function = ( ) has a continuous derivative. The following asymptotic relations hold for eigenvalues λ and eigenfunctions ( ) as :

$$\overline{\lambda} = \frac{(- 1)}{{}_2 - {}_1} + \frac{1}{(- 1)} \quad ({}_1, {}_2) + \quad \frac{1}{{}^2},$$

$$( ) = \cos \frac{(- 1)(- {}_1)}{{}_2 - {}_1} + \frac{1}{(- 1)} \ ({}_1 -) \quad (, {}_2)$$

$$+ ({}_2 -) \quad ({}_1,) \quad \sin \frac{(- 1)(- {}_1)}{{}_2 - {}_1} + \quad \frac{1}{{}^2},$$

where (, v) is given by (8).

0.2.5-5. Problems with boundary conditions of the third kind (case $a_1 = a_2 = 1$ and $\ {}_1\ {}_2 \neq 0$).

We consider the third boundary value problem for equation (1) subject to condition (2) with $a_1 = a_2 = 1$. We assume that $\ (\ ) = g(\ ) = 1$ and the function $\ = \ (\ )$ has a continuous derivative.

The following asymptotic formulas hold for eigenvalues λ eigenfunctions $\ (\ )$ as :

$$\overline{\lambda} = \frac{(\ -1)}{\ {}_2 - \ {}_1} + \frac{1}{(\ -1)}\Big[\ (\ {}_1,\ {}_2) - \ {}_1 + \ {}_2\Big] + \frac{1}{\ {}^2},$$

$$\ (\ ) = \cos\frac{(\ -1)(\ -\ {}_1)}{\ {}_2 - \ {}_1} + \frac{1}{(\ -1)}\ (\ {}_1 - \ )\Big[\ (\ ,\ {}_2) + \ {}_2\Big]$$

$$+ (\ {}_2 - \ )\Big[\ (\ {}_1,\ ) - \ {}_1\Big]\ \sin\frac{(\ -1)(\ -\ {}_1)}{\ {}_2 - \ {}_1} + \frac{1}{\ {}^2},$$

where $\ (\ , v)$ is defined by (8).

0.2.5-6. Problems with mixed boundary conditions (case $a_1 = \ {}_2 = 1$ and $a_2 = \ {}_1 = 0$).

Let us note some special properties of the Sturm–Liouville problem that is the a mixed boundary value problem for equation (1) with the boundary conditions

$$\ ' = 0 \quad \text{at} \quad \ = \ {}_1, \qquad \ = 0 \quad \text{at} \quad \ = \ {}_2.$$

1 . If $\ \geq 0$, the upper estimate (6) is valid for the least eigenvalue, with $z = z(\ )$ being any twice-differentiable function that satisfies the conditions $z'(\ {}_1) = 0$ and $z(\ {}_2) = 0$. The equality in (6) is attained if $z = \ {}_1(\ )$, where $\ {}_1(\ )$ is the eigenfunction corresponding to the eigenvalue λ_1.

2 . Suppose $\ (\ ) = g(\ ) = 1$ and the function $\ = \ (\ )$ has a continuous derivative. The following asymptotic relations hold for eigenvalues λ and eigenfunctions $\ (\ )$ as :

$$\overline{\lambda} = \frac{(2\ -1)}{2(\ {}_2 - \ {}_1)} + \frac{2}{(2\ -1)}\ (\ {}_1,\ {}_2) + \frac{1}{\ {}^2},$$

$$\ (\ ) = \cos\frac{(2\ -1)(\ -\ {}_1)}{2(\ {}_2 - \ {}_1)} + \frac{2}{(2\ -1)}\ (\ {}_1 - \ )\ (\ ,\ {}_2)$$

$$+ (\ {}_2 - \ )\ (\ {}_1,\ )\ \sin\frac{(2\ -1)(\ -\ {}_1)}{2(\ {}_2 - \ {}_1)} + \frac{1}{\ {}^2},$$

where $\ (\ , v)$ is defined by (8).

References for Subsection 0.2.5: L. Collatz (1963), E. Kamke (1977), A. G. Kostyuchenko and I. S. Sargsyan (1979), V. A. Marchenko (1986), B. M. Levitan and I. S. Sargsyan (1988), V. A. Vinokurov and V. A. Sadovnichii (2000), A. D. Polyanin (2002).

0.3. Second-Order Nonlinear Differential Equations

0.3.1. Form of the General Solution. Cauchy Problem

0.3.1-1. Equations solved for the derivative. General solution.

A *second-order ordinary differential equation* solved for the highest derivative has the form

$$\ '' = \ (\ ,\ ,\ '). \tag{1}$$

The general solution of this equation depends on two arbitrary constants, $\ {}_1$ and $\ {}_2$. In some cases, the general solution can be written in explicit form, $\ = \ (\ ,\ {}_1,\ {}_2)$, but more often implicit or parametric forms of the general solution are encountered.

0.3.1-2. Cauchy problem. The existence and uniqueness theorem.

1 . *Cauchy problem*: Find a solution of equation (1) satisfying the *initial conditions*

$$(\ _0) = \ _0, \qquad {}'(\ _0) = \ _1. \tag{2}$$

(At a point $= \ _0$, the value of the unknown function, $_0$, and its derivative, $_1$, are prescribed.)

2 . *Existence and uniqueness theorem*. Let $(\ ,\ ,z)$ be a continuous function in all its arguments in a neighborhood of a point $(\ _0,\ _0,\ _1)$ and let have bounded partial derivatives and in this neighborhood, or the Lipschitz condition is satisfied: $|\ (\ ,\ ,z) - \ (\ ,\bar{\ },\bar{z})| \le A\big(|\ -\bar{\ }| + |z-\bar{z}|\ $, where A is some positive number. Then a solution of equation (1) satisfying the initial conditions (2) exists and is unique.

References for Subsection 0.3.1: G. A. Korn and T. M. Korn (1968), I. G. Petrovskii (1970), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1980).

0.3.2. Equations Admitting Reduction of Order

0.3.2-1. Equations not containing explicitly.

In the general case, an equation that does not contain implicitly has the form

$$(\ ,\ ',\ '') = 0. \tag{1}$$

Such equations remain unchanged under an arbitrary translation of the dependent variable: $+\,\mathrm{const}$. The substitution $' = z(\ )$, $'' = z'(\ )$, brings (1) to a first-order equation: $(\ ,z,z') = 0$.

0.3.2-2. Equations not containing explicitly (autonomous equations).

In the general case, an equation that does not contain implicitly has the form

$$(\ ,\ ',\ '') = 0. \tag{2}$$

Such equations remain unchanged under an arbitrary translation of the independent variable: $+\,\mathrm{const}$. Using the substitution $' = \ (\ )$, where plays the role of the independent variable, and taking into account the relations $'' = \ ' = \ '\ ' = \ '$, one can reduce (2) to a first-order equation: $(\ ,\ ,\ ') = 0$.

The equation $'' = \ (\ + a\ ^2 + \ + \ )$ is reduced by the change of variable $= \ + a\ ^2 + \ + \ $ to an autonomous equation, $'' = \ (\ ) + 2a$.

0.3.2-3. Equations of the form $(a\ +\ ,\ ',\ '') = 0$.

Such equations are invariant under simultaneous translations of the independent and dependent variables in accordance with the rule $+\ $, $-a\ $, where is an arbitrary constant.

For $= 0$, see equation (1). For $\neq 0$, the substitution $= a\ +\ $ leads to equation (2): $(\ ,\ ' - a\ ,\ '') = 0$.

0.3.2-4. Equations of the form $(\ ,\ ' - \ ,\ '') = 0$.

The substitution $(\ ) = \ ' - \ $ leads to a first-order equation: $(\ ,\ ,\ '\ ) = 0$.

0.3.2-5. Homogeneous equations.

1 . The *equations homogeneous in the independent variable* remain unchanged under scaling of the independent variable, , where is an arbitrary nonzero number. In the general case, such equations can be written in the form

$$(\;,\; \;'\;,\; ^{2}\; '') = 0. \qquad (3)$$

The substitution $z(\;) = \;'$ leads to a first-order equation: $(\;, z, zz' - z) = 0$.

2 . The *equations homogeneous in the dependent variable* remain unchanged under scaling of the variable sought, , where is an arbitrary nonzero number. In the general case, such equations can be written in the form

$$(\;,\; ' \;,\; '' \;) = 0. \qquad (4)$$

The substitution $z(\;) = \;'$ leads to a first-order equation: $(\;, z, z' + z^2) = 0$.

3 . The *equations homogeneous in both variables* are invariant under simultaneous scaling (dilatation) of the independent and dependent variables, and , where is an arbitrary nonzero number. In the general case, such equations can be written in the form

$$(\;,\; ',\; '') = 0. \qquad (5)$$

The transformation $= \ln|\;|$, $=$ leads to an autonomous equation (see Paragraph 0.3.2-2): $(\;, \;' + \;, \;'' + \;') = 0$.

0.3.2-6. Generalized homogeneous equations.

1 . The *generalized homogeneous equations* remain unchanged under simultaneous scaling of the independent and dependent variables in accordance with the rule and , where is an arbitrary nonzero number and k is some number. Such equations can be written in the form

$$(^{-}\;,\; ^{1-}\;'\;,\; ^{2-}\;'') = 0. \qquad (6)$$

The transformation $= \ln\;$, $= \;^{-}$ leads to an autonomous equation (see Paragraph 0.3.2-2):

$$\big(\;,\; ' + k\;,\; '' + (2k-1)\;' + k(k-1)\;\;\big) = 0.$$

2 . The most general form of representation of generalized homogeneous equations is as follows:

$$(\;,\; '\;,\; ^{2}\;'' \;) = 0. \qquad (7)$$

The transformation $z = \;$, $= \;'$ brings this equation to the first-order equation

$$\big(z,\;, z(\; + \;)\;' - \; + \;^{2}\;\big) = 0.$$

For $\neq 0$, equation (7) is equivalent to equation (6) in which $k = -$. To the particular values $= 0$ and $= 0$ there correspond equations (3) and (4) homogeneous in the independent and dependent variables, respectively. For $= - \neq 0$, we have an equation homogeneous in both variables (5).

0.3.2-7. Equations invariant under scaling–translation transformations.

1 . The equations of the form

$$(e^{\lambda}\;, e^{\lambda}\;', e^{\lambda}\;'') = 0 \qquad (8)$$

remain unchanged under simultaneous translation and scaling of variables, $+$ and β, where $\beta = e^{-\;\lambda}$ and is an arbitrary number. The substitution $= e^{\lambda}$ brings (8) to an autonomous equation (see Paragraph 0.3.2-2): $(\;, \;' - \lambda\;, \;'' - 2\lambda\;' + \lambda^2\;) = 0$.

2 . The equation

$$(e^{\lambda} \quad , \ ' \quad , \ '' \quad) = 0 \tag{9}$$

is invariant under the simultaneous translation and scaling of variables, $+$ and β , where $\beta = \exp(- \ \lambda \ )$ and is an arbitrary number. The transformation $z = e^{\lambda}$, $=$ $'$ brings (9) to a first-order equation: $\big(z, \ , z(\ + \lambda) \ ' + \ ^2 \ = 0$.

3 . The equation

$$(\ e^{\lambda} , \ \ ', \ ^2 \ '' \ ) = 0 \tag{10}$$

is invariant under the simultaneous scaling and translation of variables, and $+ \beta$, where $= \exp(-\beta\lambda \ )$ and β is an arbitrary number. The transformation $z = \ e^{\lambda}$, $=$ $'$ brings (10) to a first-order equation: $\big(z, \ , z(\lambda \ + \ ) \ ' - \ = 0$.

$\boxed{\text{0.3.2-8. Exact second-order equations.}}$

The second-order equation

$$(\ , \ , \ ', \ '' \ ) = 0 \tag{11}$$

is said to be exact if it is the total differential of some function, $= \ '$, where $= (\ , \ , \ ')$. If equation (11) is exact, then we have a first-order equation for :

$$(\ , \ , \ ') = \ , \tag{12}$$

where is an arbitrary constant.

If equation (11) is exact, then $(\ , \ , \ ', \ '' \ )$ must have the form:

$$(\ , \ , \ ', \ '' \ ) = (\ , \ , \ ') \ '' + g(\ , \ , \ '). \tag{13}$$

Here, and g are expressed in terms of by the formulas:

$$(\ , \ , \ ') = \frac{}{'}, \qquad g(\ , \ , \ ') = \frac{}{} + \frac{}{} \ '. \tag{14}$$

By differentiating (14) with respect to , , and $= \ '$, we eliminate the variable from the two formulas in (14). As a result, we have the following test relations for and g:

$$\begin{gathered} +2 \quad + \ ^2 \quad = g \ + \ g \ - g \ , \\ + \quad +2 \quad = g \ . \end{gathered} \tag{15}$$

Here, the subscripts denote the corresponding partial derivatives.

If conditions (15) hold, then equation (11) with of (13) is exact. In this case, we can integrate the first equation in (14) with respect to $= \ '$ to determine $= (\ , \ , \ ')$:

$$= \quad (\ , \ , \ ) \quad + (\ , \ ), \tag{16}$$

where $(\ , \ )$ is an arbitrary function of integration. This function is determined by substituting (16) into the second equation in (14).

Example. The left-hand side of the equation

$$yy_{xx} + (y_x)^2 + 2 \quad yy_x + \ y^2 = 0 \tag{17}$$

can be represented in the form (13), where $= y$ and $= \ ^2 + 2 \quad y \ + \ y^2$. It is easy to verify that conditions (15) are satisfied. Hence, equation (17) is exact. Using (16), we obtain

$$\varphi = y \ + \ (\ , y). \tag{18}$$

Substituting this expression into the second equation in (14) and taking into account the relation $= \ ^2 + 2 \quad y \ + \ y^2$, we find that $2 \quad y \ + \ y^2 = \ _x + \ $. Since $= (\ , y)$, we have $2 \quad y =$ and $y^2 = \ _x$. Integrating yields $= \ y^2 + \text{const}$. Substituting this expression into (18) and taking into account relation (12), we find a first integral of equation (17):

$$y \ + \ y^2 = \ _1, \qquad \text{where} \quad = y_x. \tag{19}$$

Setting $= y^2$, we arrive at the first-order linear equation $_x + 2 \quad = 2 \ _1$, which is easy to integrate. Thus, we find the solution of the original equation in the form:

$$y^2 = 2 \ _1 \exp - \ ^2 \quad \exp \ ^2 \ d \ + \ _2 \exp - \ ^2 \ .$$

0.3.2-9. Reduction of quasilinear equations to the normal form.

We consider the equation

$$'' + (\ )\ ' + g(\ )\ = (\ ,\ ) \qquad (20)$$

with linear left-hand side and nonlinear right-hand side. Let $_1(\ )$ and $_2(\ )$ form a fundamental system of solutions of the truncated linear equation corresponding to $\ \equiv 0$. The transformation

$$\xi = \frac{_2(\ )}{_1(\ )}, \qquad = \frac{}{_1(\ )} \qquad (21)$$

brings equation (20) to the normal form:

$$'' = (\xi,\ ), \qquad \text{where} \qquad (\xi,\ ) = \frac{^3_1(\ )}{^2(\ )}\ (\ ,\ _1(\ )\ ) \ .$$

Here, $(\ ) = \ _1\ '_2 - \ _2\ '_1$ is the Wronskian of the truncated equation; and the variable must be expressed in terms of ξ using the first relation in (21).

Transformation (21) is convenient for the simplification and classification of equations having the form (20) with $(\ ,\ ) = (\ )\ $, thus reducing the number of functions from three to one: $\{\ , g,\ \}\ \ \{0, 0,\ _1\}$.

References for Subsection 0.3.2: G. M. Murphy (1960), E. Kamke (1977), V. F. Zaitsev and A. D. Polyanin (1993, 2001), A. D. Polyanin and V. F. Zaitsev (1995), D. Zwillinger (1998).

0.3.3. Methods of Regular Series Expansions with Respect to the Independent Variable or Small Parameter

0.3.3-1. Method of expansion in powers of the independent variable.

A solution of the Cauchy problem

$$'' = (\ ,\ ,\ '), \qquad (1)$$

$$(\ _0) = \ _0, \qquad '(\ _0) = \ _1 \qquad (2)$$

can be sought in the form of a Taylor series in powers of the difference $(\ - \ _0)$, specifically:

$$(\ ) = (\ _0) + \ '(\ _0)(\ - \ _0) + \frac{''(\ _0)}{2!}(\ - \ _0)^2 + \frac{'''(\ _0)}{3!}(\ - \ _0)^3 + \ . \qquad (3)$$

The first two coefficients $(\ _0)$ and $'(\ _0)$ in solution (3) are defined by the initial conditions (2). The values of the subsequent derivatives of at the point $\ = \ _0$ are determined from equation (1) and its derivative equations (obtained by successive differentiation of the equation) taking into account the initial conditions (2). In particular, setting $\ = \ _0$ in (1) and substituting (2), we obtain the value of the second derivative:

$$''(\ _0) = (\ _0,\ _0,\ _1). \qquad (4)$$

Further, differentiating (1) yields

$$''' = (\ ,\ ,\ ') + (\ ,\ ,\ ')\ ' + (\ ,\ ,\ ')\ ''\ . \qquad (5)$$

On substituting $\ = \ _0$, the initial conditions (2), and the expression of $''(\ _0)$ of (4) into the right-hand side of equation (5), we calculate the value of the third derivative:

$$'''(\ _0) = (\ _0,\ _0,\ _1) + (\ _0,\ _0,\ _1)\ _1 + (\ _0,\ _0,\ _1)\ \ (\ _0,\ _0,\ _1).$$

The subsequent derivatives of the unknown are determined likewise.

The thus obtained solution (3) can only be used in a small neighborhood of the point $\ = \ _0$.

$\boxed{\text{0.3.3-2. Method of regular (direct) expansion in powers of the small parameter.}}$

We consider an equation of general form with a parameter ε:

$$'' + \ (\ ,\ ,\ ',\varepsilon) = 0. \tag{6}$$

We assume that the function can be represented as a series in powers of ε:

$$(\ ,\ ,\ ',\varepsilon) = \sum_{=0} \varepsilon \ (\ ,\ ,\ '). \tag{7}$$

Solutions of the Cauchy problem and various boundary value problems for equation (6) with $\varepsilon \ \ 0$ are sought in the form of a power series expansion:

$$= \sum_{=0} \varepsilon \ (\ ). \tag{8}$$

One substitutes expression (8) into equation (6) taking into account (7). Then the functions are expanded into power series with respect to the small parameter, and the coefficient of like powers of ε are collected. Equating the resulting expressions (the coefficient of like powers of ε) to zero, one arrives at a system of equations for :

$$''_0 + \ _0(\ ,\ _0,\ '_0) = 0, \tag{9}$$

$$''_1 + \ (\ ,\ _0,\ '_0)\ '_1 + \ (\ ,\ _0,\ '_0)\ _1 + \ _1(\ ,\ _0,\ '_0) = 0, \qquad = \frac{0}{'}, \qquad = \frac{0}{}. \tag{10}$$

Here, only the first two equations are written out. The prime denotes differentiation with respect to . To obtain the initial (or boundary) conditions for , the expansion (8) is taken into account.

The success in the application of this method is primarily determined by the possibility of constructing a solution of equation (9), for the leading term $_0$. It is significant to note that the other terms with ≥ 1 are governed by linear equations with homogeneous initial conditions.

Example. The Duffing equation with initial conditions

$$y_{xx} + y + \ y^3 = 0, \qquad y(0) = \ , \quad y_x(0) = 0,$$

describes the motion of a cubic oscillator, i.e., oscillations of a point mass on a nonlinear spring. Here, y is the deviation of the point mass from the equilibrium and is dimensionless time.

For $\ \ 0$, an approximate solution of the problem is sought in the form of the asymptotic expansion (8). We substitute (8) into the equation and initial conditions and expand in powers of . On equating the coefficients of like powers of the small parameter to zero, we obtain the following problems for y_0 and y_1:

$$y_0 \ + y_0 = 0, \qquad y_0 = \ , \quad y_0 = 0;$$

$$y_1 \ + y_1 = -y_0^3, \qquad y_1 = 0, \quad y_1 = 0.$$

The solution of the problem for y_0 is given by:

$$y_0 = \ \cos \ .$$

Substituting this expression into the equation for y_1 and taking into account the identity $\cos^3 \ = \frac{1}{4}\cos 3 \ + \frac{3}{4}\cos \ $, we obtain

$$y_1 \ + y_1 = -\tfrac{1}{4}\ ^3(\cos 3 \ + 3\cos \ ), \qquad y_1 = 0, \quad y_1 = 0.$$

Integrating yields

$$y_1 = -\tfrac{3}{8}\ ^3 \ \sin \ + \tfrac{1}{32}\ ^3(\cos 3 \ - 3\cos \ ).$$

Thus the two-term solution of the original problem is given by:

$$y = \ \cos \ + \ ^3\Big[-\tfrac{3}{8}\ \sin \ + \tfrac{1}{32}(\cos 3 \ - 3\cos \ )\Big] + \ (\ ^2).$$

1. The term sin causes $y_1/y_0 \ \ \infty$ as $\ \ \infty$. For this reason, the solution obtained is unsuitable at large times. It can only be used for $\ll 1$; this results from the condition of applicability of the expansion, $y_0 \ \ y_1$.

This circumstance is typical of the method of regular series expansions with respect to the small parameter; in other words, the expansion becomes unsuitable at large values of the independent variable. This method is also inapplicable if the expansion (8) begins with negative powers of . Methods that allow avoiding the above difficulties are discussed in Subsection 0.3.4.

. Growing terms as $\ \ \infty$, like sin , that narrow down the domain of applicability of asymptotic expansions are called *secular*.

References for Paragraph 0.3.3-2: A. H. Nayfeh (1973, 1981).

0.3.3-3. Padé approximants.

Suppose the $k+1$ leading coefficients in the Taylor series expansion of a differential equation about the point $\ = 0$ are obtained by the method presented in Paragraph 0.3.3-1, so that

$$_{+1}(\ ) = a_0 + a_1 \ \ + \ \ \ + a \ \ \ . \tag{11}$$

The partial sum (11) pretty well approximates the solution at small but is poor for intermediate and large values of , since the series can be slowly convergent or even divergent. This is also related to the fact that as , while the exact solution can well be bounded.

In many cases, instead of the expansion (11), it is reasonable to consider a Padé approximant $_M(\ )$, which is the ratio of two polynomials of degree N and , specifically,

$$_M(\ ) = \frac{A_0 + A_1 \ \ + \ \ \ + A}{1 + B_1 \ \ + \ \ \ + B_M \ \ ^M}, \qquad \text{where} \quad N + \ \ = k. \tag{12}$$

The coefficients $A_1,\ \ , A$ and $B_1,\ \ , B_M$ are selected so that the $k+1$ leading terms in the Taylor series expansion of (12) coincide with the respective terms of the expansion (11). In other words, the expansions (11) and (12) must be asymptotically equivalent as $\ \ 0$.

In practice, one usually takes $N =$ (the diagonal sequence). It often turns out that formula (12) pretty well approximates the exact solution on the entire range of (for sufficiently large N).

Example 1. Consider the following Cauchy problem for a second-order nonlinear equation:

$$y_{xx} = yy_x + y^3; \qquad y(0) = y_x(0) = 1. \tag{13}$$

The Taylor series expansion of the solution about $\ = 0$ has the form (see Paragraph 0.3.3-1):

$$y = 1 + \ \ + \ ^2 + \ ^3 + \ ^4 + \ \ \ . \tag{14}$$

This geometric series is convergent only for $|\ | < 1$.

The diagonal sequence of Padé approximants corresponding to series (14) is:

$$^1_1(\ ) = \frac{1}{1 - \ }, \qquad ^2_2(\ ) = \frac{1}{1 - \ }, \qquad ^3_3(\ ) = \frac{1}{1 - \ }. \tag{15}$$

It is not difficult to verify that the function $y(\ ) = \dfrac{1}{1 - \ }$ is the exact solution of the Cauchy problem (13). Hence, in this case, the diagonal sequence of Padé approximants recovers the exact solution from only a few terms in the Taylor series.

Example 2. Consider the Cauchy problem for a second-order nonlinear equation:

$$y_{xx} = 2yy_x; \qquad y(0) = 0, \quad y_x(0) = 1. \tag{16}$$

Following the method presented in Paragraph 0.3.3-1, we obtain the Taylor series expansion of the solution to problem (16) in the form:

$$y(\ ) = \ \ + \tfrac{1}{3}\ ^3 + \tfrac{2}{15}\ ^5 + \tfrac{17}{315}\ ^7 + \ \ \ . \tag{17}$$

The exact solution of problem (16) is given by: $y(\ ) = \tan\ $. Hence it has singularities at $\ = \frac{1}{2}(2\ \ + 1)\ $. However, any finite segment of the Taylor series (17) does not have any singularities.

With series (17), we construct the diagonal sequence of Padé approximants:

$$^2_2(\ ) = \frac{3\ }{3 - \ ^2}, \qquad ^3_3(\ ) = \frac{(\ ^2 - 15)}{3(2\ ^2 - 5)}, \qquad ^4_4(\ ) = \frac{5\ (21 - 2\ ^2)}{\ ^4 - 45\ ^2 + 105}. \tag{18}$$

These Padé approximants have singularities (at the points where the denominators vanish):

$$\begin{aligned} &1.732 && \text{for } ^2_2(\ ),\\ &1.581 && \text{for } ^3_3(\ ),\\ &1.571 \ \text{ and } \ \ 6.522 && \text{for } ^4_4(\ ). \end{aligned}$$

It is apparent that the Padé approximants are attempting to recover the singularities of the exact solution at $\ = \ /2$ and $\ = 3\ /2$. The Padé approximant $^4_4(\ )$ gives an accurate numerical approximation of the exact solution on the interval $|\ | \le 2$ (the error is everywhere less than 1%, except for a very small neighborhood of the point $\ = \ /2$).

References for Paragraph 0.3.3-3: G. A. Baker (Jr.) and P. Graves–Morris (1981), D. Zwillinger (1989, pp. 450–453).

0.3.4. Perturbation Methods of Mechanics and Physics

0.3.4-1. Preliminary remarks. A summary table of basic methods.

Perturbation methods are widely used in nonlinear mechanics and theoretical physics for solving problems that are described by differential equations with a small parameter ε. The primary purpose of these methods is to obtain an approximate solution that would be equally suitable at all (small, intermediate, and large) values of the independent variable as $\varepsilon \quad 0$.

Equations with a small parameter can be classified according to the following:

(i) the order of the equation remains the same at $\varepsilon = 0$;
(ii) the order of the equation reduces at $\varepsilon = 0$.

For the first type of equations, solutions of related problems* are sufficiently smooth (little varying as ε decreases). The second type of equation is said to be degenerate at $\varepsilon = 0$, or singularly perturbed. In related problems, thin boundary layers usually arise whose thickness is significantly dependent on ε; such boundary layers are characterized by high gradients of the unknown.

All perturbation methods have a limited domain of applicability; the possibility of using one or another method depends on the type of equations or problems involved. The most commonly used methods are summarized in Table 3 (the method of regular series expansions is set out in Paragraph 0.3.3-2). In subsequent paragraphs, additional remarks and specific examples are given for some of the methods. In practice, one usually confines oneself to few leading terms of the asymptotic expansion.

In many problems of nonlinear mechanics and theoretical physics, the independent variable is dimensionless time . Therefore, in this subsection we use the conventional , $0 \leq \quad <$ instead of .

0.3.4-2. The method of scaled parameters (Lindstedt–Poincaré method).

We illustrate the characteristic features of the method of scaled parameters with a specific example (the transformation of the independent variable we use here as well as the form of the expansion are specified in the first row of Table 3).

Example 1. Consider the Duffing equation:

$$y \quad + y + \quad y^3 = 0. \tag{1}$$

On performing the change of variable $\quad = \quad (1 + \quad {}_1 + \quad)$, we have

$$y \quad + (1 + \quad {}_1 + \quad)^2 (y + \quad y^3) = 0. \tag{2}$$

The solution is sought in the series form $y = y_0(\ ) + \ y_1(\ ) + \quad$. Substituting it into equation (2) and matching the coefficients of like powers of , we arrive at the following system of equation for two leading terms of the series:

$$y_0 \quad + y_0 = 0, \tag{3}$$

$$y_1 \quad + y_1 = -y_0^3 - 2 \quad {}_1 y_0, \tag{4}$$

where the prime denotes differentiation with respect to .

The general solution of equation (3) is given by:

$$y_0 = \quad \cos(\quad + b), \tag{5}$$

where and b are constants of integration. Taking into account (5) and rearranging terms, we reduce equation (4) to

$$y_1 \quad + y_1 = -\tfrac{1}{4} \quad {}^3 \cos\ 3(\quad + b) \ - 2 \quad \tfrac{3}{8} \quad {}^2 + \quad {}_1 \quad \cos(\quad + b). \tag{6}$$

For ${}_1 \neq -\frac{3}{8} \quad {}^2$, the particular solution of equation (6) contains a secular term proportional to $\cos(\quad + b)$. In this case, the condition of applicability of the expansion $y_1/y_0 = \quad (1)$ (see the first row and the last column of Table 3) cannot be satisfied at sufficiently large . For this condition to be met, one should set

$${}_1 = -\tfrac{3}{8} \quad {}^2. \tag{7}$$

* Further on, we assume that the initial and/or boundary conditions are independent of the parameter .

TABLE 3

Perturbation methods of nonlinear mechanics and theoretical physics

(the third column gives leading asymptotic terms with respect to the small parameter ε)

Method name	Examples of problems solved by the method	Form of the solution sought	Additional conditions and remarks
Method of scaled parameters $(0\le t<\ )$	One looks for periodic solutions of the equation $+\ {}^{2}_{0}\ =\ (\ ,\ );$ see also Paragraph 0.3.4-2	$(t)=\ {}^{-1}_{=0}\ (\ ),$ $t=\ \ 1+\ {}^{-1}_{=1}\ \big)$	Unknowns: and ; $_{+1}\ =\ (1);$ secular terms are eliminated through selection of the constants
Method of strained coordinates $(0\le t<\ )$	Cauchy problem: $=\ (t,\ ,\ );\ \ (t_0)=\ _0$ (is of a special form); see also the problem in the method of scaled parameters	$(t)=\ {}^{-1}_{=0}\ (\ ),$ $t=\ +\ {}^{-1}_{=1}\ (\ )$	Unknowns: and ; $_{+1}\ =\ (1),$ $_{+1}\ =\ (1)$
Averaging method $(0\le t<\ )$	Cauchy problem: $+\ {}^{2}_{0}\ =\ (\ ,\ ),$ $(0)=\ _0,\ \ (0)=\ _1;$ for more general problems, see Paragraph 0.3.4-3, Item 2°	$=a(t)\cos\ (t),$ the amplitude a and phase are governed by the equations $—=-—_0\ (a),$ $—=\ _0-—_0\ _c(a)$	Unknowns: a and ; $_s=\frac{1}{2}\ {}^{2}_{0}\ \sin\ ,$ $_c=\frac{1}{2}\ {}^{2}_{0}\ \cos\ ,$ $=\ (a\cos\ ,-a\ _0\sin\ )$
Krylov–Bogolyubov–Mitropolskii method $(0\le t<\ )$	One looks for periodic solutions of the equation $+\ {}^{2}_{0}\ =\ (\ ,\ );$ Cauchy problem for this and other equations	$=a\cos\ +\ {}^{-1}_{=1}\ (a,\ ),$ a and are determined by the equations $—=\ {}_{=1}\ (a),$ $—=\ _0+\ {}_{=1}\ (a)$	Unknowns: , , ; are 2 -periodic functions of ; the are assumed not to contain cos
Method of two-scale expansions $(0\le t<\ )$	Cauchy problem: $+\ {}^{2}_{0}\ =\ (\ ,\ ),$ $(0)=\ _0,\ \ (0)=\ _1;$ for boundary value problems, see Paragraph 0.3.4-4, Item 2°	$=\ {}^{-1}_{=0}\ (\ ,\eta),$ where $=\ t,\ \eta=t\ \ 1+\ {}^{-1}_{=2}\ \big),$ $—=\ \frac{\partial}{\partial}+\big(1+\ ^2\ \ _2+\ \ \big)\frac{\partial}{\partial}$	Unknowns: and ; $_{+1}\ =\ (1);$ secular terms are eliminated through selection of
Multiple scales method $(0\le t<\ )$	One looks for periodic solutions of the equation $+\ {}^{2}_{0}\ =\ (\ ,\ );$ Cauchy problem for this and other equations	$=\ {}^{-1}_{=0}\ $, where $=\ (\ _0,\ _1,\ \ ,\ \ ),\ \ =\ \ t$ $—=\frac{\partial}{\partial\ _0}+\ \frac{\partial}{\partial\ _1}+\ \ +\ \ \frac{\partial}{\partial}$	Unknowns: ; $_{+1}\ =\ (1);$ for $n=1$, this method is equivalent to the averaging method
Method of matched asymptotic expansions $(0\le\ \ \le b)$	Boundary value problem: $_{xx}+\ (\ ,\ )\ _x=\ (\ ,\ ),$ $(0)=\ _0,\ \ (b)=$ (assumed positive); for other problems, see Paragraph 0.3.4-5, Item 2°	Outer expansion: $=\ {}^{-1}_{=0}\ (\ )\ (\ ),\ \ (\ )\le\ \ \le b.$ Inner expansion (=): $=\ {}^{-1}_{=0}\ (\ )\ (\ ),\ 0\le\ \ \le\ (\ )$	Unknowns: , , , ; $_{+1}\ =\ (1),$ $_{+1}\ =\ (1);$ the procedure of matching expansions is used: $(\ \ \ 0)=\ (\ \ \ )$
Method of composite expansions $(0\le\ \ \le b)$	Boundary value problem: $_{xx}+\ (\ ,\ )\ _x=\ (\ ,\ ),$ $(0)=\ _0,\ \ (b)=$ (assumed positive); boundary value problems for other equations	$=\ (\ ,\ )+\ (\ ,\ ),$ $=\ {}^{-1}_{=0}\ (\ )\ (\ ),$ $=\ {}^{-1}_{=0}\ (\ )\ (\ ),\ \ =-.$ Here, 0 as	Unknowns: , , , ; $(b,\ )=\ ,$ $(0,\ )+\ (0,\ )=\ _0;$ two forms of representation of the equation (in terms of and) are used to obtain solutions

In this case, the solution of equation (6) is given by:

$$y_1 = \tfrac{1}{32}\ \ {}^3 \cos\ 3(\ \ + b)\ . \tag{8}$$

Subsequent terms of the expansion can be found likewise.

With (5), (7), and (8), we obtain a solution of the Duffing equation in the form:

$$y = \ \ \cos(\ \ \ + b) + \tfrac{1}{32}\ \ {}^3 \cos\ 3(\ \ \ + b)\ +\ \ (\ {}^2),$$
$$=\ 1 - \tfrac{3}{8}\ \ {}^2 +\ \ (\ {}^2)\ {}^{-1} = 1 + \tfrac{3}{8}\ \ {}^2 +\ \ (\ {}^2).$$

0.3.4-3. Averaging method (Van der Pol–Krylov–Bogolyubov scheme).

1 . The averaging method involved two stages. First, the second-order nonlinear equation

$$'' + \omega_0^2\ \ = \varepsilon\ \ (\ ,\ \ ') \tag{9}$$

is reduced with the transformation

$$= a\cos\ \ ,\qquad ' = -\omega_0 a \sin\ \ ,\qquad \text{where}\quad a = a(\ ),\qquad = \ (\ ),$$

to an equivalent system of two first-order differential equations:

$$a' = -\frac{\varepsilon}{\omega_0}\ \ (a\cos\ \ , -\omega_0 a\sin\ \ )\sin\ \ ,\qquad ' = \omega_0 - \frac{\varepsilon}{\omega_0 a}\ \ (a\cos\ \ , -\omega_0 a\sin\ \ )\cos\ \ . \tag{10}$$

The right-hand sides of equations (10) are periodic in , with the amplitude a being a slow function of time . The amplitude and the oscillation character are changing little during the time the phase changes by 2 .

At the second stage, the right-hand sides of equations (10) are being averaged with respect to . This procedure results in an approximate system of equations:

$$a' = -\frac{\varepsilon}{\omega_0}\ \ {}_{\rm s}(a),\qquad ' = \omega_0 - \frac{\varepsilon}{\omega_0 a}\ \ {}_{\rm c}(a), \tag{11}$$

where

$${}_{\rm s}(a) = \frac{1}{2}\ \int_0^{2\pi} \sin\ \ \ (a\cos\ \ , -\omega_0 a\sin\ \ )\ \ ,\qquad {}_{\rm c}(a) = \frac{1}{2}\ \int_0^{2\pi} \cos\ \ \ (a\cos\ \ , -\omega_0 a\sin\ \ )\ \ .$$

System (11) is substantially simpler than the original system (10)—the first equation in (11), for the oscillation amplitude a, is a separable equation and, hence, can readily be integrated; then the second equation in (11), which is linear in , can also be integrated.

Note that the Krylov–Bogolyubov–Mitropolskii method (see the fourth row in Table 3) generalizes the above approach and allows obtaining subsequent asymptotic terms as ε 0.

2 . Below we outline the general scheme of the averaging method. We consider the second-order nonlinear equation with a small parameter:

$$'' + g(\ ,\ \ ,\ \ ') = \varepsilon\ \ (\ ,\ \ ,\ \ '). \tag{12}$$

Equation (12) should first be transformed to the equivalent system of equations

$$' = \ \ ,\qquad ' + g(\ ,\ \ ,\ \ ) = \varepsilon\ \ (\ ,\ \ ,\ \ ). \tag{13}$$

Suppose the general solution of the "truncated" system (13), with $\varepsilon = 0$, is known:

$${}_0 = \ \ (\ ,\ \ {}_1,\ \ {}_2),\qquad {}_0 = \ \ (\ ,\ \ {}_1,\ \ {}_2), \tag{14}$$

where $_1$ and $_2$ are constants of integration. Taking advantage of the method of variation of constants, we pass from the variables , in (13) to Lagrange's variables $_1$, $_2$ according to the formulas

$$= (, _1, _2), \qquad = (, _1, _2), \tag{15}$$

where and are the same functions that define the general solution of the "truncated" system (14). Transformation (15) allows the reduction of system (13) to the *standard form*:

$$'_1 = \varepsilon \; _1(, _1, _2), \qquad '_2 = \varepsilon \; _2(, _1, _2). \tag{16}$$

Here, the prime denotes differentiation with respect to and

$$_1 = \frac{_2 (, ,)}{_2 \; _1 - \; _1 \; _2}, \qquad _2 = -\frac{_1 (, ,)}{_2 \; _1 - \; _1 \; _2}; \qquad = \frac{}{}, \qquad = \frac{}{},$$
$$= (, _1, _2), \qquad = (, _1, _2).$$

It is significant to note that system (16) is equivalent to the original equation (12). The unknowns $_1$ and $_2$ are slow functions of time.

As a result of averaging, system (16) is replaced by a simpler, approximate autonomous system of equations:

$$'_1 = \varepsilon \; _1(_1, _2), \qquad '_2 = \varepsilon \; _2(_1, _2), \tag{17}$$

where

$$(_1, _2) = \frac{1}{} \int_0^T (, _1, _2) \; , \qquad \text{if} \quad \text{is a} \quad \text{-periodic function of} \; ;$$
$$(_1, _2) = \lim_{T} \frac{1}{} \int_0^T (, _1, _2) \; , \qquad \text{if} \quad \text{is not periodic in} \; .$$

The averaging method is applicable to equations (9) and (12) with nonsmooth right-hand sides.

The averaging method has rigorous mathematical substantiation. There is also a procedure that allows finding subsequent asymptotic terms. For this procedure, e.g., see the books by Bogolyubov and Mitropolskii (1974), Zhuravlev and Klimov (1988), and Arnold, Kozlov, and Neishtadt (1993).

0.3.4-4. Method of two-scale expansions (Cole–Kevorkian scheme).

1 . We illustrate the characteristic features of the method of two-scale expansions with a specific example. Thereafter we outline possible generalizations and modifications of the method.

Example 2. Consider the Van der Pol equation:

$$y \;\; + y = \; (1 - y^2) y \; . \tag{18}$$

The solution is sought in the form (see the fifth row in Table 3):

$$y = y_0(\xi, \;) + \; y_1(\xi, \;) + \; ^2 y_2(\xi, \;) + \;\; , \qquad \xi = \; , \qquad = 1 + \; ^2 \; _2 + \;\; . \tag{19}$$

On substituting (19) into (18) and on matching the coefficients of like powers of , we obtain the following system for two leading terms:

$$\frac{^2 y_0}{^2} + y_0 = 0, \tag{20}$$

$$\frac{^2 y_1}{^2} + y_1 = -2 \frac{^2 y_0}{\xi} + (1 - y_0^2) \frac{y_0}{}. \tag{21}$$

The general solution of equation (20) is given by:

$$y_0 = \quad(\xi)\cos \quad + \quad(\xi)\sin \quad . \tag{22}$$

The dependence of and on the slow variable ξ is not being established at this stage.

We substitute (22) into the right-hand side of equation (21) and perform elementary manipulations to obtain

$$\begin{aligned}\frac{\;^2 y_1}{\;^2} + y_1 = \; & -2 \quad + \tfrac{1}{4} \quad (4 - \;^2 - \;^2) \; \cos \quad + \; 2 \quad - \tfrac{1}{4} \quad (4 - \;^2 - \;^2) \; \sin \\ & + \tfrac{1}{4}(\;^3 - 3 \;^2 \;)\cos 3 \quad + \tfrac{1}{4}(\;^3 - 3 \;^2)\sin 3 \quad .\end{aligned} \tag{23}$$

The solution of this equation must not contain unbounded terms as ∞; otherwise the necessary condition $y_1/y_0 = \;(1)$ is not satisfied. Therefore the coefficients of cos and sin must be set equal to zero:

$$\begin{aligned} -2 \quad + \tfrac{1}{4} \quad (4 - \;^2 - \;^2) &= 0, \\ 2 \quad - \tfrac{1}{4} \quad (4 - \;^2 - \;^2) &= 0. \end{aligned} \tag{24}$$

Equations (24) serve to determine $= \;(\xi)$ and $= \;(\xi)$. We multiply the first equation in (24) by $-$ and the second by and add them together to obtain

$$- \tfrac{1}{8} \quad (4 - \;^2) = 0, \qquad \text{where} \quad ^2 = \;^2 + \;^2. \tag{25}$$

The integration by separation of variables yields:

$$^2 = \frac{4 \;_0^2}{\;_0^2 + (4 - \;_0^2) \; ^{-}}, \tag{26}$$

where $_0$ is the initial oscillation amplitude.

On expressing and in terms of the amplitude and phase φ, we have $= \cos\varphi$ and $= - \sin\varphi$. Substituting these expressions into either of the two equations in (24) and using (25), we find that $\varphi \; = 0$ or $\varphi = \varphi_0 = \text{const}$. Therefore the leading asymptotic term can be represented as:

$$y_0 = \quad(\xi)\cos(\quad + \varphi_0),$$

where $\xi =$ and $=$, and the function (ξ) is determined by (26).

2 . The method of two-scale expansions can also be used for solving boundary value problems where the small parameter appears together with the highest derivative as a factor (such problems for $0 \le \quad \le a$ are indicated in the seventh row of Table 3 and in Paragraph 0.3.4-5). In the case where a boundary layer arises near the point $= 0$ (and its thickness has an order of magnitude of ε), the solution is sought in the form:

$$\begin{aligned} &= \;_0(\xi, \;) + \varepsilon \;_1(\xi, \;) + \varepsilon^2 \;_2(\xi, \;) + \quad , \\ \xi = \; , \qquad &= \varepsilon^{-1}\big[g_0(\;) + \varepsilon g_1(\;) + \varepsilon^2 g_2(\;) + \quad \big], \end{aligned}$$

where the functions $= \;(\xi, \;)$ and $g \; = g \;(\;)$ are to be determined. The derivative with respect to is calculated in accordance with the rule:

$$\frac{}{} = \frac{}{\xi} + \;' \frac{}{} = \frac{}{\xi} + \frac{1}{\varepsilon}\big(g_0' + \varepsilon g_1' + \varepsilon^2 g_2' + \quad \frac{}{}.$$

Additional conditions are imposed on the asymptotic terms in the domain under consideration; namely, $_{+1} \quad = \;(1)$ and $g \;_{+1} \; g \; = \;(1)$ for $k = 0, 1,$, and $g_0(\;) \quad$ as $\quad 0$.

The two-scale method is also used to solve problems that arise in mechanics and physics and are described by partial differential equations.

0.3.4-5. Method of matched asymptotic expansions.

1 . We illustrate the characteristic features of the method of matched asymptotic expansions with a specific example (the form of the expansions is specified in the seventh row of Table 3). Thereafter we outline possible generalizations and modifications of the method.

Example 3. Consider the linear boundary value problem

$$y_{xx} + y_x + \ (\ )y = 0, \tag{27}$$

$$y(0) = \ , \quad y(1) = b, \tag{28}$$

where $0 < \ (0) < \infty$.

At $\ = 0$ equation (27) degenerates; the solution of the resulting first-order equation

$$y_x + \ (\ )y = 0 \tag{29}$$

cannot meet the two boundary conditions (28) simultaneously. It can be shown that the condition at $\ = 0$ has to be omitted in this case (a boundary layer arises near this point).

The leading asymptotic term of the outer expansion, $y = y_0(\ ) + \ (\ )$, is determined by equation (29). The solution of (29) that satisfies the second boundary condition in (28) is given by:

$$y_0(\ ) = b \exp \int_x^1 \ (\xi)\, d\xi \Big]. \tag{30}$$

We seek the leading term of the inner expansion, in the boundary layer adjacent to the left boundary, in the following form (see the seventh row and third column in Table 3):

$$y = y_0(\ ) + \ (\ ), \qquad = \ / \ , \tag{31}$$

where $\ $ is the extended variable. Substituting (31) into (27) and extracting the coefficient of $\ ^{-1}$, we obtain

$$y_0 + y_0 = 0, \tag{32}$$

where the prime denotes differentiation with respect to $\ $. The solution of equation (32) that satisfies the first boundary condition in (28) is given by:

$$y_0 = \ - \ + \ ^{-}\ . \tag{33}$$

The constant of integration $\ $ is determined from the condition of matching the leading terms of the outer and inner expansions:

$$y_0(\ \ 0) = y_0(\ \ \infty). \tag{34}$$

Substituting (30) and (33) into condition (34) yields

$$= \ - b \ ^{\langle\ \rangle}, \qquad \text{where} \ \ \langle\ \rangle = \int_0^1 \ (\ )\, d\ . \tag{35}$$

Taking into account relations (30), (31), (33), and (35), we represent the approximate solution in the form:

$$y = \begin{cases} b \ ^{\langle\ \rangle} + \ - b \ ^{\langle\ \rangle} \ ^{-x/} & \text{for } 0 \le \ \le \ (\ ), \\ b \exp \int_x^1 \ (\xi)\, d\xi \Big] & \text{for } \ (\ ) \le \ \le 1. \end{cases} \tag{36}$$

It is apparent that inside the thin boundary layer, whose thickness is proportional to $\ $, the solution rapidly changes by a finite value, $\ = b \ ^{\langle\ \rangle} - \ $.

To determine the function y on the entire interval $\ [0, 1]$ using formula (36), one has to "switch" at some intermediate point $\ = \ _0$ from one part of the solution to the other. Such switching is not convenient and, in practice, one often resorts to a *composite solution* instead of using the double formula (36). In similar cases, a composite solution is defined as:

$$y = y_0(\ ) + y_0(\ ) - \ , \qquad = \lim_{x \ 0} y_0(\ ) = \lim \ y_0(\ ).$$

In the problem under consideration, we have $\ = b \ ^{\langle\ \rangle}$ and hence the composite solution becomes:

$$y = \ - b \ ^{\langle\ \rangle} \ ^{-x/} + b \exp \int_x^1 \ (\xi)\, d\xi \Big].$$

For $\ \ll \ \le 1$, this solution transforms to the outer solution $y_0(\ )$ and for $0 \le \ \ll \ $, to the inner solution, thus providing an approximate representation of the unknown over the entire domain.

2 . We now consider an equation of the general form

$$\varepsilon \ '' = \ (\ ,\ ,\ '\ ) \tag{37}$$

subject to boundary conditions (28).

For the leading term of the outer expansion $\ = \ _0(\ ) + \ $, we have the equation:

$$(\ ,\ _0,\ '_0) = 0.$$

In the general case, when using the method of matched asymptotic expansions, the position of the boundary layer and the form of the inner (extended) variable have to be determined in the course of the solution of the problem.

First we assume that the boundary layer is located near the left boundary. In (37), we make a change of variable $z = \quad (\varepsilon)$ and rewrite the equation as

$$'' = \frac{^2}{\varepsilon} \quad z, \ , \frac{1}{\ } \ ' \ . \tag{38}$$

The function $= \ (\varepsilon)$ is selected so that the right-hand side of equation (38) has a nonzero limit value as $\varepsilon \quad 0$, provided that z, , and $'$ are of the order of 1.

Example 4. For $(\ , y, y_x) = -k \ ^\lambda y_x + y$, where $0 \leq \ < 1$, the substitution $= \ / \ (\ )$ brings equation (37) to

$$y \quad = -\frac{^{1+\lambda}}{\ } k \ ^\lambda y \ + \frac{^2}{\ } y.$$

In order that the right-hand side of this equation has a nonzero limit value as $\quad 0$, one has to set $^{1+\lambda} / \ = 1$ or $^{1+\lambda} / \ = \text{const}$, where const is any positive number. It follows that $= \ ^{\frac{1}{1+\lambda}}$.

The leading asymptotic term of the inner expansion in the boundary layer, $y = y_0(\ ) +$, is determined by the equation $y_0 \ + k \ ^\lambda y_0 = 0$, where the prime denotes differentiation with respect to .

If the position of the boundary layer is selected incorrectly, the outer amd inner expansions cannot be matched. In this situation, one should consider the case where an arbitrary boundary layer is located on the right (this case is reduced to the previous one with the change of variable $= 1 - z$). In example 4 above, the boundary layer is on the left if $k > 0$ and on the right if $k < 0$.

There is a procedure for matching subsequent asymptotic terms of the expansion (see the seventh row and last column in Table 3). In its general form, this procedure can be represented as:

inner expansion of the outer expansion (*-expansion for* 0)

= *outer expansion of the inner expansion* (*-expansion for z*).

The method of matched asymptotic expansions can also be applied to construct periodic solutions of singularly perturbed equations (e.g., in the problem of relaxation oscillations of the Van der Pol oscillator).

Two boundary layers can arise in some problems (e.g., in cases where the right-hand side of equation (37) does not explicitly depend on $'$).

The method of matched asymptotic expansions is also used for solving equations (in semi-infinite domains) that do not degenerate at $\varepsilon = 0$. In such cases, there are no boundary layers; the original variable is used in the inner domain, and an extended coordinate is introduced in the outer domain.

The method of matched asymptotic expansions is successfully applied for the solution of various problems in mathematical physics that are described by partial differential equations; in particular, it plays an important role in the theory of heat and mass transfer and in hydrodynamics.

References for Subsection 0.3.4: M. Van Dyke (1964), A. Blaquiere (1966), G. D. Cole (1968), G. E. O. Giacaglia (1972), A. H. Nayfeh (1973, 1981), N. N. Bogolyubov and Yu. A. Mitropolskii (1974) J. Kevorkian and J. D. Cole (1981, 1996), P. A. Lagerstrom (1988), V. Ph. Zhuravlev and D. M. Klimov (1988), V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt (1993).

0.3.5. Galerkin Method and Its Modifications (Projection Methods)

0.3.5-1. General form of an approximate solution.

Consider a boundary value problem for the equation

$$\mathfrak{F}[\ ] - \ (\ ) = 0 \tag{1}$$

with linear homogeneous boundary conditions at the points $=\ _1$ and $=\ _2$ ($_1 \le\ \ \le\ _2$). Here, $\mathfrak{F}$ is a linear or nonlinear differential operator of the second order (or a higher order operator); $=\ (\ )$ is the unknown function and $=\ (\ )$ is a given function. It is assumed that $\mathfrak{F}[0] = 0$.

Let us choose a sequence of linearly independent functions

$$= \quad (\ ) \qquad (\ = 1,\ 2,\ \quad ,\ N) \tag{2}$$

satisfying the same boundary conditions as $=\ (\ )$. According to all methods that will be considered below, an approximate solution of equation (1) is sought as a linear combination

$$= \quad A \quad (\ ), \qquad {}_{=1} \tag{3}$$

with the unknown coefficients A to be found in the process of solving the problem.

The finite sum (3) is called an approximation function. The remainder term R obtained after the finite sum has been substituted into the equation (1) has the form

$$R \quad = \mathfrak{F}[\quad] - \ . \tag{4}$$

If the remainder R is identically equal to zero, then the function is the exact solution of equation (1). In general, $R \quad 0$.

0.3.5-2. Galerkin method.

In order to find the coefficients A in (3), consider another sequence of linearly independent functions

$$= \quad (\ ) \qquad (k = 1,\ 2,\ \quad ,\ N). \tag{5}$$

Let us multiply both sides of (4) by and integrate the resulting relation over the region $=\{\ _1 \le\ \ \le\ _2\}$, in which we seek the solution of equation (1). Next, we equate the corresponding integrals to zero (for the exact solutions, these integrals are equal to zero). Thus, we obtain the following system of linear algebraic equations for the unknown coefficients A :

$$^{2}_{1} \quad R \qquad = 0 \qquad (k = 1,\ 2,\ \quad ,\ N). \tag{6}$$

Relations (6) mean that the approximation function (3) satisfies equation (1) "on the average" (i.e., in the integral sense) with weights . Introducing the scalar product $\langle g,\ \rangle = {}^{2}_{1}\ g$ of arbitrary functions g and , we can consider equations (6) as the condition of orthogonality of the remainder R to all weight functions .

The Galerkin method can be applied not only to boundary value problems but also to eigenvalue problems (in the latter case, one takes $= \lambda$ and seeks eigenfunctions , together with eigenvalues λ).

Mathematical justification of the Galerkin method for specific boundary value problems can be found in the literature listed at the end of Subsection 0.3.5. Below we describe some other methods, which are in fact special cases of the Galerkin method.

As a rule, one takes suitable sequences of polynomials or trigonometric functions as $(\ )$ in the approximation function (3).

0.3.5-3. The Bubnov–Galerkin method, the moment method, and the least squares method.

1 . The sequences of functions (2) and (5) in the Galerkin method can be chosen arbitrarily. In the case of equal functions,

$$(\ ) = \quad (\ ) \qquad (k = 1,\ 2,\ \quad ,\ N), \tag{7}$$

the method is often called the Bubnov–Galerkin method.

2 . The moment method is the Galerkin method with the weight functions (5) being powers of ,

$$= \quad . \tag{8}$$

3 . Sometimes, the functions are expressed in terms of by the relations

$$= \mathfrak{F}[\quad] \qquad (k = 1,\ 2,\quad),$$

where $\mathfrak{F}$ is the differential operator of equation (1). This version of the Galerkin method is called the least squares method.

0.3.5-4. Collocation method.

In the collocation method, one chooses a sequence of points , $k = 1,\quad , N$, and imposes the condition that the remainder (4) be zero at these points,

$$R \quad = 0 \quad \text{at} \qquad = \qquad\qquad (k = 1,\quad , N). \tag{9}$$

When solving a specific problem, the points , at which the remainder R is set equal to zero, are regarded as most significant. The number of collocation points N is taken equal to the number of the terms of the series (3). This allows one to obtain a complete system of algebraic equations for the unknown coefficients A (for linear boundary value problems, this algebraic system is linear).

Note that the collocation method is a special case of the Galerkin method with the sequence (5) consisting of the Dirac delta functions:

$$= \ (\ - \quad).$$

In the collocation method, there is no need to calculate integrals, and this essentially simplifies the procedure of solving nonlinear problems (although usually this method yields less accurate results than other modifications of the Galerkin method).

0.3.5-5. The method of partitioning the domain.

The domain $= \{ \ _1 \le \quad \le \ _2\}$ is split into N subdomains: $= \{ \ _1 \le \quad \le \ _2\}$, $k = 1, \quad , N$. In this method, the weight functions are chosen as follows:

$$(\ ) = \begin{cases} 1 & \text{for} \qquad , \\ 0 & \text{for} \quad \notin \quad . \end{cases}$$

The subdomains are chosen according to the specific properties of the problem under consideration and can generally be arbitrary (the union of all subdomains may differ from the domain , and some and may overlap).

0.3.5-6. The least squared error method.

Sometimes, in order to find the coefficients A of the approximation function (3), one uses the least squared error method based on the minimization of the functional:

$$= \int_{1}^{2} R^2 \qquad \min . \tag{10}$$

For given functions in (3), the integral is a quadratic polynomial with respect to the coefficients A . In this case, the necessary conditions of minimum in (10) have the form:

$$\frac{\quad}{A} = 0 \qquad (\ = 1, \quad , N).$$

This is a system of linear algebraic equations for the coefficients A .

References for Subsection 0.3.5: L. V. Kantorovich and V. I. Krylov (1962), M. A. Krasnoselskii, G. M. Vainikko, P. P. Zabreiko et al. (1969), S. G. Mikhlin (1970), B. A. Finlayson (1972), D. Zwillinger (1989).

0.3.6. Iteration and Numerical Methods

0.3.6-1. The method of successive approximations (Cauchy problem).

The method of successive approximations is implemented in two steps. First, the Cauchy problem

$$'' = (\;,\; ,\; ') \qquad \text{(equation)}, \tag{1}$$

$$({}_0) = {}_0, \quad '({}_0) = {}'_0 \qquad \text{(boundary conditions)} \tag{2}$$

is reduced to an equivalent system of integral equations by the introduction of the new variable $(\;) = \;'$. These integral equations have the form

$$(\;) = {}'_0 + \int_0 \big(\;, (\;), (\;) \big) \;, \qquad (\;) = {}_0 + \int_0 (\;) \;. \tag{3}$$

Then the solution of system (3) is sought by means of successive approximations defined by the following recurrence formulas:

$$_{+1}(\;) = {}'_0 + \int_0 \big(\;, \; (\;), \; (\;) \big) \;, \qquad _{+1}(\;) = {}_0 + \int_0 (\;) \;; \qquad = 0,\ 1,\ 2,$$

As the initial approximation, one can take $_0(\;) = {}_0$, $_0(\;) = {}'_0$.

References for Paragraph 0.3.6-1: G. A. Korn and T. M. Korn (1968), N. S. Bakhvalov (1973), E. Kamke (1977).

0.3.6-2. The Runge–Kutta method (Cauchy problem).

For the numerical integration of the Cauchy problem (1)–(2), one often uses the Runge–Kutta method.

Let Δ be sufficiently small. We introduce the following notation:

$$= {}_0 + k\Delta \;, \qquad = (\;), \qquad ' = '(\;), \qquad = (\;, \;, \;'); \qquad k = 0,\ 1,\ 2,$$

The desired values and $'$ are successively found by the formulas:

$$_{+1} = \; + \;'\Delta \; + \tfrac{1}{6}({}_1 + {}_2 + {}_3)(\Delta \;)^2,$$
$$'_{+1} = \;' + \tfrac{1}{6}({}_1 + 2 {}_2 + 2 {}_3 + {}_4)\Delta \;,$$

where

$$_1 = \big(\;, \;, \;' \;,$$
$$_2 = \big(\; + \tfrac{1}{2}\Delta \;, \; + \tfrac{1}{2} \;'\Delta \;, \;' + \tfrac{1}{2} {}_1\Delta \;,$$
$$_3 = \big(\; + \tfrac{1}{2}\Delta \;, \; + \tfrac{1}{2} \;'\Delta \; + \tfrac{1}{4} {}_1(\Delta \;)^2, \;' + \tfrac{1}{2} {}_2\Delta \;,$$
$$_4 = \big(\; + \Delta \;, \; + \;'\Delta \; + \tfrac{1}{2} {}_2(\Delta \;)^2, \;' + {}_3\Delta \;.$$

In practice, the step Δ is determined in the same way as for first-order equations (see Remark 2 in Paragraph 0.1.10-3).

References for Paragraph 0.3.6-2: G. A. Korn and T. M. Korn (1968), N. S. Bakhvalov (1973), E. Kamke (1977), D. Zwillinger (1989).

0.3.6-3. Shooting method (boundary value problems).

In order to solve the boundary value problem for equation (1) with the boundary conditions

$$({}_1) = {}_1, \qquad ({}_2) = {}_2, \tag{4}$$

one considers an auxiliary Cauchy problem for equation (1) with the initial conditions

$$({}_1) = {}_1, \qquad '({}_1) = a. \tag{5}$$

(The solution of this Cauchy problem can be obtained by the Runge–Kutta method or some other numerical method.) The parameter a is chosen so that the value of the solution $= (\ ,a)$ at the point $=\ _2$ coincides with the value required by the second boundary condition in (4):

$$(\ _2, a) =\ _2.$$

In a similar way one constructs the solution of the boundary value problem with mixed boundary conditions

$$(\ _1) =\ _1, \qquad {}'(\ _2) + k\ (\ _2) =\ _2. \tag{6}$$

In this case, one also considers the auxiliary Cauchy problem (1), (5). The parameter a is chosen so that the solution $= (\ , a)$ satisfies the second boundary condition in (6) at the point $=\ _2$.

References for Paragraph 0.3.6-3: S. K. Godunov and V. S. Ryaben'kii (1973), N. N. Kalitkin (1978).

0.3.6-4. Method of accelerated convergence in eigenvalue problems.

Consider the Sturm–Liouville problem for the second-order nonhomogeneous linear equation

$$[\ (\ )\ ']' + [\lambda g(\ ) -\ (\ )]\ = 0 \tag{7}$$

with linear homogeneous boundary conditions of the first kind

$$(0) =\ (1) = 0. \tag{8}$$

It is assumed that the functions $\ ,\ ', g,$ are continuous and $> 0,\ g > 0$.

First, using the Rayleigh–Ritz principle, one finds an upper estimate for the first eigenvalue λ_1^0 [this value is determined by the right-hand side of relation (6) from Paragraph 0.2.5-3]. Then, one solves numerically the Cauchy problem for the auxiliary equation

$$[\ (\ )\ ']' + [\lambda_1^0 g(\ ) -\ (\ )]\ = 0 \tag{9}$$

with the boundary conditions

$$(0) = 0, \qquad {}'(0) = 1. \tag{10}$$

The function $(\ , \lambda_1^0)$ satisfies the condition $(\ _0, \lambda_1^0) = 0$, where $_0 < 1$. The criterion of closeness of the exact and approximate solutions, λ_1 and λ_1^0, has the form of the inequality $|1 -\ _0| \le\ $, where is a sufficiently small given constant. If this inequality does not hold, one constructs a refinement for the approximate eigenvalue on the basis of the formula:

$$\lambda_1^1 = \lambda_1^0 - \varepsilon_0\ (1)\frac{[\ '(1)]^2}{\ ^2}, \qquad \varepsilon_0 = 1 -\ _0, \tag{11}$$

where $^2 = \int_0^1 g(\ )\ ^2(\ )\ $. Then the value λ_1^1 is substituted for λ_1^0 in the Cauchy problem (9)–(10). As a result, a new solution and a new point $_1$ are found; and one has to check whether the criterion $|1 -\ _1| \le$ holds. If this inequality is violated, one refines the approximate eigenvalue by means of the formula:

$$\lambda_1^2 = \lambda_1^1 - \varepsilon_1\ (1)\frac{[\ '(1)]^2}{\ ^2}, \qquad \varepsilon_1 = 1 -\ _1, \tag{12}$$

and repeat the above procedure.

Formulas of the type (11) are obtained by a perturbation method based on a transformation of the independent variable (see Paragraph 0.3.4-1). If > 1, the functions $\ , g,$ and are smoothly extended to the interval $(1, \xi]$, where $\xi \ge$.

The algorithm described above has the property of accelerated convergence $\varepsilon\ _{+1}\ \ \varepsilon^2$, which ensures that the relative error of the approximate solution becomes 10^{-4} to 10^{-8} after two or three iterations for $\varepsilon_0\ \ 0.1$. This method is quite effective for high-precision calculations, is fail-safe, and guarantees against accumulation of roundoff errors.

In a similar way, one can compute subsequent eigenvalues λ , $= 2, 3,$ (to that end, a suitable initial approximation λ^0 should be chosen).

A similar computation scheme can also be used in the case of boundary conditions of the second and the third kinds, periodic boundary conditions, etc. (see the references below).

Example 1. The eigenvalue problem for the equation

$$y_{xx} + \ (1 + \ ^2)^{-2} y = 0$$

with the boundary conditions (8) admits an exact analytic solution and has eigenvalues $_1 = 15$, $_2 = 63$, , $= 16\ ^2 - 1$.

According the Rayleigh–Ritz principle, formula (6) of Paragraph 0.2.5-3 for $= \sin(\ )$ yields the approximate value $^0_1 = 15.33728$. The solution of the Cauchy problem (9)–(10) with $(\ ) = 1$, $(\ ) = \ (1 + \ ^2)^{-2}$, $h(\ ) = 0$ yields $_0 = 0.983848$, $1 - \ _0 = 0.016152$, $y\ ^2 = 0.024585$, $y_x(\ _0) = -0.70622822$.

The first iteration for the first eigenvalue is determined by (11) and results in the value $^1_1 = 14.99245$ with the relative error $/\ ^1_1 = 5 \times 10^{-4}$.

The second iteration results in $^2_1 = 14.999986$ with the relative error $/\ ^2_1 < 10^{-6}$.

Example 2. Consider the eigenvalue problem for the equation

$$(\sqrt{1 + \ } \ y_x)_x + \ y = 0$$

with the boundary conditions (8).

The Rayleigh–Ritz principle yields $^0_1 = 11.995576$. The next two iterations result in the values $^1_1 = 11.898578$ and $^2_1 = 11.898458$. For the relative error we have $/\ ^2_1 < 10^{-5}$.

References for Paragraph 0.3.6-4: L. D. Akulenko and S. V. Nesterov (1996, 1997).

For more details about finite-difference methods and other numerical methods, see, for instance, the books by Lambert (1973), Keller (1976), and Zwillinger (1998).

0.4. Linear Equations of Arbitrary Order

0.4.1. Linear Equations with Constant Coefficients

0.4.1-1. Homogeneous linear equations.

An th-order homogeneous linear equation with constant coefficients has the general form

$$^{(\ )} + a_{\ -1}\ ^{(\ -1)} + \quad + a_1\ ' + a_0\ = 0. \tag{1}$$

The general solution of this equation is determined by the roots of the characteristic equation:

$$(\lambda) = 0, \qquad \text{where} \qquad (\lambda) = \lambda\ + a_{\ -1}\lambda^{\ -1} + \quad + a_1\lambda + a_0. \tag{2}$$

The following cases are possible:

1 . All roots $\lambda_1, \lambda_2, \quad , \lambda$ of the characteristic equation (2) are real and distinct. Then the general solution of the homogeneous linear differential equation (1) has the form:

$$= \ _1 \exp(\lambda_1\ ) + \ _2 \exp(\lambda_2\ ) + \quad + \quad \exp(\lambda\ ).$$

2 . There are equal real roots $\lambda_1 = \lambda_2 = \quad = \lambda$ $(\ \le\ )$, and the other roots are real and distinct. In this case, the general solution is given by:

$$\begin{aligned} &= \exp(\lambda_1\ )(\ _1 + \ _2\ + \quad + \qquad ^{-1}) \\ &\quad + \quad _{+1} \exp(\lambda_{\ +1}\ ) + \quad _{+2} \exp(\lambda_{\ +2}\ ) + \quad + \quad \exp(\lambda\ ). \end{aligned}$$

3 . There are equal complex conjugate roots $\lambda = \quad \beta$ $(2\ \le\ )$, and the other roots are real and distinct. In this case, the general solution is:

$$\begin{aligned} &= \exp(\ ) \cos(\beta\ )(A_1 + A_2\ + \quad + A\qquad ^{-1}) \\ &\quad + \exp(\ ) \sin(\beta\ )(B_1 + B_2\ + \quad + B\qquad ^{-1}) \\ &\quad + \ _{2\ +1} \exp(\lambda_{2\ +1}\ ) + \ _{2\ +2} \exp(\lambda_{2\ +2}\ ) + \quad + \quad \exp(\lambda\ ), \end{aligned}$$

where $A_1, \quad , A\ , B_1, \quad , B\ , \ _{2\ +1}, \quad ,$ are arbitrary constants.

4 . In the general case, where there are different roots $\lambda_1, \lambda_2,\ \ , \lambda$ of multiplicities $\ _1,\ \ _2,\ \ ,\ \ $, respectively, the right-hand side of the characteristic equation (2) can be represented as the product

$$(\lambda) = (\lambda - \lambda_1)^{\ _1} (\lambda - \lambda_2)^{\ _2} \quad (\lambda - \lambda\ )\ \ ,$$

where $\ _1 + \ _2 + \quad + \ \ = \ $. The general solution of the original equation is given by the formula:

$$= \sum_{=1} \exp(\lambda\ \ )(\ _{,0} + \ _{,1}\ \ + \quad + \ _{,\ -1}\ \ ^{-1}),$$

where $\ _{,}$ are arbitrary constants.

If the characteristic equation (2) has complex conjugate roots, then in the above solution, one should extract the real part on the basis of the relation $\exp(\quad \beta) = e\ (\cos\beta \quad \sin\beta)$.

0.4.1-2. Nonhomogeneous linear equations.

An th-order nonhomogeneous linear equation with constant coefficients has the general form

$$^{(\ )} + a_{\ -1}\ ^{(\ -1)} + \quad + a_1\ ' + a_0\ \ = \ (\ ). \tag{3}$$

The general solution of this equation is the sum of the general solution of the corresponding homogeneous equation (see Paragraph 0.4.1-1) and any particular solution of the nonhomogeneous equation.

If all the roots $\lambda_1, \lambda_2,\ \ , \lambda$ of the characteristic equation (2) are different, equation (3) has the general solution:

$$= \sum_{=1} \ e^{\lambda\ } + \sum_{=1} \frac{e^{\lambda}}{\ '_{\lambda}(\lambda\ )} \int \ (\ ) e^{-\lambda}$$

(for complex roots, the real part should be taken).

Table 4 lists the forms of particular solutions corresponding to some special forms of functions on the right-hand side of the linear nonhomogeneous equation.

In the general case, a particular solution can be constructed on the basis of the formulas from Paragraph 0.4.2-3.

References for Subsection 0.4.1: G. A. Korn and T. M. Korn (1968), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1980), D. Zwillinger (1989).

0.4.2. Linear Equations with Variable Coefficients

0.4.2-1. Homogeneous linear equations. Structure of the general solution.

The general solution of the th-order homogeneous linear differential equation

$$(\ )\ ^{(\ )} + \ _{-1}(\ )\ ^{(\ -1)} + \quad + \ _1(\ )\ ' + \ _0(\ )\ \ = 0 \tag{1}$$

has the form:

$$= \ _1\ _1(\ ) + \ _2\ _2(\ ) + \quad + \quad (\ ). \tag{2}$$

Here, $\ _1(\ ),\ \ _2(\ ),\ \ ,\ \ (\ )$ is a fundamental system of solutions (the are linearly independent particular solutions, 0); $\ _1,\ \ _2,\ \ ,\ $ are arbitrary constants.

TABLE 4

Forms of particular solutions of the constant coefficient nonhomogeneous linear equation $^{( )} + a_{-1}\,^{(-1)} + \quad + a_1\,' + a_0 \quad = \ (\ )$ that correspond to some special forms of the function $(\ )$

Form of the function $(\ )$	Roots of the characteristic equation $\lambda \ + a_{-1}\lambda^{-1} + \quad + a_1\lambda + a_0 = 0$	Form of a particular solution $= \ (\ )$
$(\ )$	Zero is not a root of the characteristic equation (i.e., $a_0 \neq 0$)	$(\ )$
	Zero is a root of the characteristic equation (multiplicity $\ $)	$(\ )$
$(\ )e$ (is a real constant)	is not a root of the characteristic equation	$(\ )e$
	is a root of the characteristic equation (multiplicity $\ $)	$(\ )e$
$(\ )\cos\beta$ $+Q \ (\ )\sin\beta$	β is not a root of the characteristic equation	$(\ )\cos\beta$ $+Q \ (\ )\sin\beta$
	β is a root of the characteristic equation (multiplicity $\ $)	$[\ \ (\ )\cos\beta$ $+Q \ (\ )\sin\beta \ ]$
$[\ \ (\ )\cos\beta$ $+Q \ (\ )\sin\beta \ ]e$	$+ \ \beta$ is not a root of the characteristic equation	$[\ \ (\ )\cos\beta$ $+Q \ (\ )\sin\beta \ ]e$
	$+ \ \beta$ is a root of the characteristic equation (multiplicity $\ $)	$[\ \ (\ )\cos\beta$ $+Q \ (\ )\sin\beta \ ]e$

Notation: and Q are polynomials of degrees and with given coefficients; , , and Q are polynomials of degrees and whose coefficients are determined by substituting the particular solution into the basic equation; $= \max(\ , \ )$; and and β are real numbers, $^2 = -1$.

0.4.2-2. Utilization of particular solutions for reducing the order of the original equation.

1 . Let $_1 = \ _1(\ )$ be a nontrivial particular solution of equation (1). The substitution

$$= \ _1(\ ) \quad z(\ )$$

results in a linear equation of order $\ - 1$ for the function $z(\ )$.

2 . Let $_1 = \ _1(\ )$ and $_2 = \ _2(\ )$ be two nontrivial linearly independent solutions of equation (1). The substitution

$$= \ _1 \quad _2 \quad - \ _2 \quad _1$$

results in a linear equation of order $\ - 2$ for $(\ )$.

3 . Suppose that linearly independent solutions $_1(\ )$, $_2(\ )$, , $(\ )$ of equation (1) are known. Then one can reduce the order of the equation to $\ - \ $ by successive application of the following procedure. The substitution $= \ (\ ) \quad z(\ )$ leads to an equation of order $\ - 1$ for the function $z(\ )$ with known linearly independent solutions:

$$z_1 = \frac{\ _1\ }{\ }', \quad z_2 = \frac{\ _2\ }{\ }', \quad \ , \quad z_{-1} = \frac{\ _{-1}\ }{\ }'.$$

The substitution $z = z_{-1}(\ )\quad(\ )$ yields an equation of order -2. Repeating this procedure times, we arrive at a homogeneous linear equation of order $-$.

0.4.2-3. Wronskian determinant and Liouville formula.

The Wronskian determinant (or simply, Wronskian) is the function defined as:

$$(\ ) = \begin{matrix} {}_1(\ ) & (\ ) \\ {}'_1(\ ) & {}'(\ ) \\ \\ {}^{(\ -1)}_1(\ ) & {}^{(\ -1)}(\ ) \end{matrix}, \tag{3}$$

where ${}_1(\ ),\quad,\quad(\ )$ is a fundamental system of solutions of the homogeneous equation (1); ${}^{(\ )}(\ ) = \text{———},\quad = 1,\quad,\ -1;\ k = 1,\quad,\ .$

The following Liouville formula holds:

$$(\ ) = \quad(\ {}_0)\exp\ -\ {}_0\ \frac{{}_{-1}(\ )}{(\ )}\quad .$$

0.4.2-4. Nonhomogeneous linear equations. Construction of the general solution.

1 . The general nonhomogeneous th-order linear differential equation has the form

$$(\ )\ {}^{(\ )} + \ {}_{-1}\ {}^{(\ -1)} + \quad + \ {}_1(\ )\ ' + \ {}_0(\ )\ = g(\ ). \tag{4}$$

The general solution of the nonhomogeneous equation (4) can be represented as the sum of its particular solution and the general solution of the corresponding homogeneous equation (1).

2 . Let ${}_1(\ ),\quad,\quad(\ )$ be a fundamental system of solutions of the homogeneous equation (1), and let $(\ )$ be the Wronskian determinant (3). Then the general solution of the nonhomogeneous linear equation (4) can be represented as:

$$= \sum_{=1}\quad(\ ) + \sum_{=1}\quad(\ )\quad \frac{(\ )}{(\ )\ (\ )},$$

where $(\ )$ is the determinant obtained by replacing the th column of the matrix (3) by the column vector with the elements $0, 0,\quad, 0, g$.

References for Subsection 0.4.2: G. A. Korn and T. M. Korn (1968), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1980), D. Zwillinger (1989).

0.4.3. Asymptotic Solutions of Linear Equations

This subsection presents asymptotic solutions, as $\varepsilon \quad 0$ ($\varepsilon > 0$), of some higher-order linear ordinary differential equations containing arbitrary functions (sufficiently smooth), with the independent variable being real.

0.4.3-1. Fourth-order linear equations.

1 . Consider the equation

$$\varepsilon^4\ {}'''' \ -\ (\ )\ = 0$$

on a closed interval $a \le \quad \le \ $. With the condition $\quad > 0$, the leading terms of the asymptotic expansions of the fundamental system of solutions, as $\varepsilon \quad 0$, are given by the formulas:

$$ _1 = [\ (\ )]^{-3\ 8} \exp \ -\frac{1}{\varepsilon} \int [\ (\ )]^{1\ 4} \qquad , \qquad _2 = [\ (\ )]^{-3\ 8} \exp \ \frac{1}{\varepsilon} \int [\ (\ )]^{1\ 4} \qquad ,$$

$$ _3 = [\ (\ )]^{-3\ 8} \cos \ \frac{1}{\varepsilon} \int [\ (\ )]^{1\ 4} \qquad , \qquad _4 = [\ (\ )]^{-3\ 8} \sin \ \frac{1}{\varepsilon} \int [\ (\ )]^{1\ 4} \qquad .$$

2 . Now consider the "biquadratic" equation:

$$\varepsilon^4 \ '''' \ - 2\varepsilon^2 g(\ )\ '' \ - \ (\ ) \ = 0. \tag{1}$$

Introduce the notation:

$$(\ ) = [g(\ )]^2 + \ (\ ).$$

In the range where the conditions $(\ ) \neq 0$ and $(\ ) \neq 0$ are satisfied, the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (1) are described by the formulas:

$$ = [\lambda\ (\ )]^{-1\ 2}[\ (\ )]^{-1\ 4} \exp \ \frac{1}{\varepsilon} \int \lambda\ (\ ) \quad - \frac{1}{2} \int \frac{[\lambda\ (\ )]'}{\overline{(\ )}} \qquad ; \quad k = 1,\ 2,\ 3,\ 4,$$

where

$$\lambda_1(\ ) = \overline{g(\ ) + \overline{(\ )}}, \qquad \lambda_2(\ ) = -\overline{g(\ ) + \overline{(\ )}},$$
$$\lambda_3(\ ) = \overline{g(\ ) - \overline{(\ )}}, \qquad \lambda_4(\ ) = -\overline{g(\ ) - \overline{(\ )}}.$$

0.4.3-2. Higher-order linear equations.

1 . Consider an equation of the form

$$\varepsilon \quad ^{(\ )} - \ (\ ) \ = 0$$

on a closed interval $a \le \quad \le \ $. Assume that $\quad \neq 0$. Then the leading terms of the asymptotic expansions of the fundamental system of solutions, as $\varepsilon \quad 0$, are given by:

$$ = [\ (\ )]^{-\frac{1}{2} + \frac{1}{2}} \exp \ \frac{\omega}{\varepsilon} \int [\ (\ )]^{\frac{1}{}} \qquad [1 + \ (\varepsilon)],$$

where $\omega_1, \omega_2, \quad , \omega$ are roots of the equation $\omega \quad = 1$:

$$\omega \quad = \cos \frac{2}{\quad} \ + \ \sin \frac{2}{\quad} \ , \qquad = 1,\ 2, \quad , \ .$$

2 . Now consider an equation of the form

$$\varepsilon \quad ^{(\ )} + \varepsilon \ ^{-1} \ _{-1}(\ )\ ^{(\ -1)} + \quad + \varepsilon \ _1(\ )\ ' + \ _0(\ ) \ = 0 \tag{2}$$

on a closed interval $a \le \quad \le \ $. Let $\lambda \ = \lambda\ (\ )\ (\ = 1, 2, \quad , \ )$ be the roots of the characteristic equation:

$$(\ , \lambda) \equiv \lambda \ + \ _{-1}(\ )\lambda^{\ -1} + \quad + \ _1(\ )\lambda + \ _0(\ ) = 0.$$

Let all the roots of the characteristic equation be different on the interval $a \le \quad \le \ $, i.e., the conditions $\lambda\ (\ ) \neq \lambda\ (\ ), \quad \neq k$, are satisfied, which is equivalent to the fulfillment of the conditions $_\lambda(\ , \lambda\ ) \neq 0$. Then the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (2), as $\varepsilon \quad 0$, are given by:

$$ = \exp \ \frac{1}{\varepsilon} \int \lambda\ (\ ) \quad - \frac{1}{2} \int [\lambda\ (\ )]' \frac{_{\lambda\lambda}(\ , \lambda\ (\ )}{_\lambda(\ , \lambda\ (\ )} \qquad ,$$

where

$$_\lambda(\ , \lambda) \equiv \frac{\quad}{\lambda} = \ \lambda^{\ -1} + (\ - 1)\ _{-1}(\ )\lambda^{\ -2} + \quad + 2\lambda\ _2(\ ) + \ _1(\ ),$$

$$_{\lambda\lambda}(\ , \lambda) \equiv \frac{^2}{\lambda^2} = \ (\ - 1)\lambda^{\ -2} + (\ - 1)(\ - 2)\ _{-1}(\ )\lambda^{\ -3} + \quad + 6\lambda\ _3(\ ) + 2\ _2(\ ).$$

References for Subsection 0.4.3: W. Wasov (1965), M. V. Fedoryuk (1993).

0.5. Nonlinear Equations of Arbitrary Order

0.5.1. Structure of the General Solution. Cauchy Problem

0.5.1-1. Equations solved for the highest derivative. General solution.

An th-order differential equation solved for the highest derivative has the form

$$^{(\ )} = (\ ,\ ,\ ',\quad ,\ ^{(\ -1)}). \tag{1}$$

The general solution of this equation depends on arbitrary constants $_1,\quad ,\quad$. In some cases, the general solution can be written in explicit form as $= (\ ,\ _1,\quad ,\quad)$.

0.5.1-2. The Cauchy problem. The existence and uniqueness theorem.

1 . The *Cauchy problem*: find a solution of equation (1) with the *initial conditions*

$$(\ _0) = \ _0,\qquad '(\ _0) = {}_0^{(1)},\qquad ,\qquad ^{(\ -1)}(\ _0) = {}_0^{(\ -1)}. \tag{2}$$

(At a point $_0$, the values of the unknown function $(\ )$ and all its derivatives of orders $\le\ -1$ are prescribed.)

2 . The *existence and uniqueness theorem*. Suppose the function $(\ ,\ , z_1,\quad , z_{\ -1})$ is continuous in all its arguments in a neighborhood of the point $(\ _0,\ _0,\ {}_0^{(1)},\quad ,\ {}_0^{(\ -1)})$ and has bounded derivatives with respect to $,\ z_1,\quad ,\ z_{\ -1}$ in this neighborhood. Then a solution of equation (1) satisfying the initial conditions (2) exists and is unique.

References for Subsection 0.5.1: G. A. Korn and T. M. Korn (1968), I. G. Petrovskii (1970), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1980).

0.5.2. Equations Admitting Reduction of Order

0.5.2-1. Equations not containing $,\ ',\quad ,\ ^{(\ )}$ explicitly.

An equation that does not explicitly contain the unknown function and its derivatives up to order k inclusive can generally be written as

$$\big(\ ,\ ^{(\ +1)},\quad ,\ ^{(\ )}\big) = 0 \qquad (1 \le k+1 < \ ). \tag{1}$$

Such equations are invariant under arbitrary translations of the unknown function, $+\mathrm{const}$ (the form of such equations is also preserved under the transformation $(\ ) = \ + a\quad +\quad + a_1\ + a_0$, where the a are arbitrary constants). The substitution $z(\ ) = \ ^{(\ +1)}$ reduces (1) to an equation whose order is by $k+1$ smaller than that of the original equation, $\big(\ , z, z',\quad , z^{(\ -\ -1)}\big) = 0$.

0.5.2-2. Equations not containing explicitly (autonomous equations).

An equation that does not explicitly contain has in the general form

$$\big(\ ,\ ',\quad ,\ ^{(\ )}\big) = 0. \tag{2}$$

Such equations are invariant under arbitrary translations of the independent variable, $+\mathrm{const}$. The substitution $' = (\ )$ (where plays the role of the independent variable) reduces by one the order of an autonomous equation. Higher derivatives can be expressed in terms of and its derivatives with respect to the new independent variable, $'' = \ '$, $''' = \ ^2\ '' + (\ ')^2$, 0.5.2-3. Equations of the form $(a \; + \; , \; ', \; , \; ^{(\;)} = 0.$

Such equations are invariant under simultaneous translations of the independent variable and the unknown function, $+$ and $- a$, where is an arbitrary constant.

For $=0$, see equation (1). For $\neq 0$, the substitution $(\;) = \; + (a \;)$ leads to an autonomous equation of the form (2).

0.5.2-4. Equations of the form $(\, , \; ' - \; , \; '' , \; , \; ^{(\;)} = 0$ and its generalizations.

The substitution $(\;) = \; ' -$ reduces the order of this equation by one.

This equation is a special case of the equation

$$(\, , \; ' - \; , \; ^{(\; +1)}, \; , \; ^{(\;)} = 0, \qquad \text{where} \qquad = 1,\, 2,\, \; ,\; - 1. \tag{3}$$

The substitution $(\;) = \; ' -$ reduces by one the order of equation (3).

0.5.2-5. Homogeneous equations.

1 . *Equations homogeneous in the independent variable* are invariant under scaling of the independent variable, , where is an arbitrary constant ($\neq 0$). In general, such equations can be written in the form

$$(\, , \; ', \; ^2 \; '' , \; , \; ^{(\;)} = 0.$$

The substitution $z(\;) = \; '$ reduces by one the order of this equation.

2 . *Equations homogeneous in the unknown function* are invariant under scaling of the unknown function, , where is an arbitrary constant ($\neq 0$). Such equations can be written in the general form

$$(\, , \; ' \; , \; '' \; , \; , \; ^{(\;)} = 0.$$

The substitution $z(\;) = \; '$ reduces by one the order of this equation.

3 . *Equations homogeneous in both variables* are invariant under simultaneous scaling (dilatation) of the independent and dependent variables, and , where is an arbitrary constant ($\neq 0$). Such equations can be written in the general form

$$(\; , \; ', \; '' , \; , \; ^{-1} \; ^{(\;)} = 0.$$

The transformation $= \ln|\;|$, $=$ leads to an autonomous equation considered in Paragraph 0.5.2-2.

0.5.2-6. Generalized homogeneous equations.

1 . *Generalized homogeneous equations* (*equations homogeneous in the generalized sense*) are invariant under simultaneous scaling of the independent variable and the unknown function, and , where $\neq 0$ is an arbitrary constant and k is a given number. Such equations can be written in the general form

$$(\; ^{-} \; , \; ^{1-} \; ', \; , \; ^{-} \; ^{(\;)} = 0.$$

The transformation $= \ln \;$, $= \; ^{-}$ leads to an autonomous equation considered in Paragraph 0.5.2-2.

2 . The most general form of generalized homogeneous equations is

$$(\; , \; ' \; , \; , \; ^{(\;)} = 0.$$

The transformation $z = \;$, $= \; '$ reduces the order of this equation by one.

0.5.2-7. Equations of the form $\left(e^{\lambda}\quad,\ '\quad,\ ''\quad,\quad,\ ^{(\ )}\right)=0.$

Such equations are invariant under simultaneous translation and scaling of variables, $+$ and β, where $\beta=\exp(-\ \lambda\ )$, and is an arbitrary constant. The transformation $z=e^{\lambda}$, $=\ '$ leads to an equation of order -1.

0.5.2-8. Equations of the form $\left(\ e^{\lambda},\ ',\ ^{2}\ '',\quad,\ ^{(\ )}\right)=0.$

Such equations are invariant under simultaneous scaling and translation of variables, and $+\beta$, where $=\exp(-\beta\lambda\ )$, and β is an arbitrary constant. The transformation $z=\ e^{\lambda}$, $=\ '$ leads to an equation of order -1.

0.5.2-9. Other equations.

Consider the nonlinear differential equation

$$\left(\ ,\mathrm{L}_1[\ ],\quad,\mathrm{L}\ [\ ]\right)=0, \tag{4}$$

where the $\mathrm{L}\ [\ ]$ are linear homogeneous differential forms,

$$\mathrm{L}\ [\ ]=\sum_{=0}^{s}\ ^{(\ )}(\ )\ ^{(\ )},\qquad =1,\quad,k.$$

Let $_0=\ _0(\ )$ be a common particular solution of the linear equations:

$$\mathrm{L}\ [\ _0]=0\qquad(\ =1,\quad,k).$$

Then the substitution

$$=\ (\ )\left[\ _0(\ )\ '-\ '_0(\ )\ \right] \tag{5}$$

with an arbitrary function $(\ )$ reduces by one the order of equation (4).

Example 1. Consider the third-order equation

$$y_{xxx}=\ (\ y_x-2y).$$

It can be represented in the form (4) with

$$k=2,\qquad(\ ,\ ,\ )=\quad-\ (\ ),\quad \mathrm{L}_1[y]=y_{xxx},\quad \mathrm{L}_2[y]=\ y_x-2y.$$

The linear equations $\mathrm{L}\ [y]=0$ are

$$y_{xxx}=0,\qquad y_x-2y=0.$$

These equations have a common particular solution $y_0=\ ^2$. Therefore, the substitution $=\ y_x-2y$ (see formula (5) with $\varphi(\ )=1/\ $) leads to an autonomous second-order equation of the form 2.9.1.1: $_{xx}=\ (\ )$.

Example 2. The th-order equation

$$y_x^{(\ )}=\ (\ ,y_{xx}-y)+y$$

can be represented in the form (4) with

$$k=2,\qquad(\ ,\ ,\ )=\ -\ (\ ,\ ),\quad \mathrm{L}_1[y]=y_x^{(\ )}-y,\quad \mathrm{L}_2[y]=y_{xx}-y.$$

The linear equations

$$\mathrm{L}_1[y]\equiv y_x^{(\ )}-y=0,\quad \mathrm{L}_2[y]\equiv y_{xx}-y=0$$

have a common particular solution, $y_0=\ ^x$. Therefore, the substitution $=y_x-y$ (see formula (5) with $\varphi(\ )=\ ^{-x}$) leads to an $(\ -1)$st-order equation.

References for Subsection 0.5.2: G. M. Murphy (1960), G. A. Korn and T. M. Korn (1968), E. Kamke (1977), V. F. Zaitsev and A. D. Polyanin (1993, 2001), A. D. Polyanin and V. F. Zaitsev (1995), D. Zwillinger (1998).

0.5.3. A Method for Construction of Solvable Equations of General Form

0.5.3-1. Description of the method.

Consider a function

$$= (\ , \ _1, \ _2, \ \ , \ _{+1}) \tag{1}$$

depending on $+1$ free parameters . Differentiating relation (1) times, we obtain the following sequence of equations:

$$^{(\ )} = \ ^{(\ )}(\ , \ _1, \ _2, \ \ , \ _{+1}), \qquad k = 1, 2, \ \ , \ . \tag{2}$$

Treating relations (1), (2) as an algebraic (transcendental) system of equations for the parameters $_1, \ _2, \ \ , \ _{+1}$ and solving this system, we obtain

$$= \ \big(\ , \ , \ ', \ \ , \ ^{(\ )} \ , \qquad k = 1, 2, \ \ , \ +1. \tag{3}$$

Consider a general th-order equation of the form

$$(\ _1, \ _2, \ \ , \ _{+1}) = 0, \tag{4}$$

where is an arbitrary function of $(\ +1)$ variables and $= \ \big(\ , \ , \ ', \ \ , \ ^{(\ )}$ are the functions from (3). Equation (4) is satisfied by the function (1), where the $(\ +1)$ arbitrary parameters $_1, \ _2, \ \ , \ _{+1}$ are related by a single constraint:

$$(\ _1, \ _2, \ \ , \ _{+1}) = 0.$$

Equation (4) may also have singular solutions depending on a smaller number of arbitrary constants. In order to examine these solutions, one should differentiate equation (4); see Example 1.

Instead of (4), one can consider a more general equation

$$(\ _1, \ _2, \ \ , \ _{+1}) = 0, \qquad \text{where} \qquad = \ (\ _1, \ _2, \ \ , \ _{+1}).$$

The original expression (1) can be specified in an implicit form.

The original expression (1) can be written as an th-order differential equation $(\ < \ )$ with $- \ +1$ free parameters . The solution of the th-order differential equation obtained in this way can be expressed in terms of the solution of an th-order differential equation (see Example 4).

0.5.3-2. Examples.

Example 1. Consider the function

$$y = - \ _1 \ ^{-x} + \ _2. \tag{5}$$

By differentiation we obtain

$$y_x = \ _1 \ ^{-x}. \tag{6}$$

Let us solve equations (5)–(6) for the parameters $_1$ and $_2$. We have

$$_1 = \ ^{x} y_x, \qquad _2 = y_x + y.$$

Using the above method, we construct an equation in accordance with (4):

$$^{x} y_x, \ y_x + y \ = 0. \tag{7}$$

This equation admits a solution of the form (5) with constants $_1$ and $_2$ related by the constraint $(\ _1, \ _2) = 0$.

Singular solution. Differentiating equation (7) with respect to , we get

$$(y_{xx}+y_x)(\ {}^{x}\quad + \quad)=0, \tag{8}$$

where the subscripts and indicate the respective partial derivatives of the function $=\ (\ ,\ )$. Equating the first factor in (8) to zero, we obtain solution (5). Equating the second factor to zero, we obtain an expression which, combined with equation (7), yields a singular solution in parametric form:

$$(\ ,\ )=0,\qquad {}^{x}\quad + \quad =0,\qquad \text{where}\quad =\ {}^{x}\ ,\qquad =\ +y.$$

One should eliminate $=y_x$ from these expressions.

Example 2. Consider the function

$$y=\ {}_{1}\ {}^{2}+\ {}_{2}\ +\ {}_{3}. \tag{9}$$

Differentiating this function twice, we get

$$\begin{aligned} y_x &= 2\ {}_{1}\ +\ {}_{2},\\ y_{xx} &= 2\ {}_{1}. \end{aligned} \tag{10}$$

Solving (9)–(10) for the parameters , we find that

$${}_{1}=\tfrac{1}{2}y_{xx},\qquad {}_{2}=y_x-\ y_{xx},\qquad {}_{3}=y-\ y_x+\tfrac{1}{2}\ {}^{2}y_{xx}.$$

These relations lead to a second-order equation of general form:

$$\tfrac{1}{2}y_{xx},\ y_x-\ y_{xx},\ y-\ y_x+\tfrac{1}{2}\ {}^{2}y_{xx}\quad =0,$$

which has a solution of the type (9) with the three constants ${}_{1}$, ${}_{2}$, and ${}_{3}$ related by the constraint $(\ {}_{1},\ {}_{2},\ {}_{3})=0$.

Example 3. In Example 2, one can choose the functions of the form (see Remark 2)

$${}_{1}=2\varphi_1,\qquad {}_{2}=-\varphi_2,\qquad {}_{3}=4\varphi_1\varphi_3-\varphi_2^2,$$

where $\varphi_1=\frac{1}{2}y_{xx}$, $\varphi_2=y_x-\ y_{xx}$, $\varphi_3=y-\ y_x+\frac{1}{2}\ {}^{2}y_{xx}$. As a result, we obtain the differential equation:

$$y_{xx},\quad y_{xx}-y_x,\ 2yy_{xx}-(y_x)^2\quad =0.$$

Its solution is given by (9) with three constants ${}_{1}$, ${}_{2}$, and ${}_{3}$ related by a single constraint $(2\ {}_{1},-\ {}_{2},4\ {}_{1}\ {}_{3}-\ {}_{3}^{2})=0$.

Example 4. Consider the autonomous equation

$$y_{xx}=\ {}_{1}y^{-}\ +\ {}_{2}. \tag{11}$$

Its solution can be represented in implicit form (see 2.4.2.1 and 2.9.1.1). Differentiating (11), we obtain

$$y_{xxx}=-\ {}_{1}y^{-\ -1}y_x. \tag{12}$$

Let us solve equations (11)–(12) for the parameters ${}_{1}$ and ${}_{2}$:

$${}_{1}=-y^{\ +1}\frac{y_{xxx}}{y_x},\qquad {}_{2}=y_{xx}+y\frac{y_{xxx}}{y_x}.$$

Taking ${}_{1}=-\ \varphi_1$ and ${}_{2}=\ \varphi_2$ (see Remark 2), we obtain the equation:

$$y^{\ +1}\frac{y_{xxx}}{y_x},\ y\frac{y_{xxx}}{y_x}+\ y_{xx}\quad =0.$$

This equation is satisfied by the solutions of a second-order autonomous equation of the form (11), where the constants ${}_{1}$ and ${}_{2}$ are related by the constraint $(-\ {}_{1},\ {}_{2})=0$.

0.6. Lie Group and Discrete-Group Methods

0.6.1. Lie Group Method. Point Transformations

0.6.1-1. Local one-parameter Lie group of transformations. Invariance condition.

Here, we examine transformations of the ordinary differential equation

$${}^{(\ )}=\ \big(\ ,\ ,\ \ ,\ {}^{(\ -1)}\big). \tag{1}$$

Consider the set of transformations

$$\mathrm{T} \;=\; \begin{cases} \bar{\ } = \ (\ ,\ ,\varepsilon), & \bar{\ }|_{=0} = \ , \\ \bar{\ } = \ (\ ,\ ,\varepsilon), & \bar{\ }|_{=0} = \ , \end{cases} \tag{2}$$

where , are smooth functions of their arguments and ε is a real parameter. The set T is called a continuous one-parameter Lie group of point transformations if, for any ε_1 and ε_2, the following relation holds: $\mathrm{T}_{1}\ \mathrm{T}_{1} = \mathrm{T}_{1+2}$, i.e., consecutive application of two transformations of the form (1) with parameters ε_1 and ε_2 is equivalent to a single transformation of the same form with parameter $\varepsilon_1 + \varepsilon_2$.

In what follows, we consider local continuous one-parameter Lie groups of point transformations (briefly called point groups) corresponding to an infinitesimal transformation (2) for ε 0. Taylor's expansion of $\bar{\ }$ and $\bar{\ }$ in (2) with respect to the parameter ε about $\varepsilon = 0$ yields:

$$\bar{\ } \quad + \xi(\ ,\ )\varepsilon, \qquad \bar{\ } \quad + \ (\ ,\ )\varepsilon, \tag{3}$$

where

$$\xi(\ ,\ ) = \frac{(\ ,\ ,\varepsilon)}{\varepsilon}\bigg|_{=0}, \qquad (\ ,\ ) = \frac{(\ ,\ ,\varepsilon)}{\varepsilon}\bigg|_{=0}.$$

At each point $(\ ,\ )$, the vector $(\xi,\ )$ is tangent to the curve described by the transformed points $(\bar{\ },\bar{\ })$.

The first-order linear differential operator

$$\mathrm{X} = \xi(\ ,\ )\frac{}{} + \ (\ ,\ )\frac{}{} \tag{4}$$

corresponding to the infinitesimal transformation (3), is called the infinitesimal operator (or infinitesimal generator) of the group.

By definition, the universal invariant (briefly, invariant) of the group (2) and the operator (4) is a function $_0(\ ,\ )$, satisfying the condition $_0(\bar{\ },\bar{\ }) = \ _0(\ ,\ )$. Taylor's expansion with respect to the small parameter ε yields the following linear partial differential equation for $_0$:

$$\mathrm{X}\ _0 = \xi(\ ,\ )\frac{\ _0}{} + \ (\ ,\ )\frac{\ _0}{} = 0. \tag{5}$$

Equation (1) will be treated as a relation for $+2$ variables $\ ,\ ,\ ',\ \ ,\ ^{(\ )}$ with the differential constraints

$$^{(\ +1)} = \frac{^{(\ )}}{}. \tag{6}$$

The space of these $+2$ variables is called the space of th prolongation; and in order to work with differential equations, one has to define the action of operator (4) on the "new" variables $\ ',\ \ ,\ ^{(\ )}$, taking into account the differential constraints (6). For example, let us calculate the infinitesimal transformation of the first derivative. We have

$$\frac{\bar{\ }}{\bar{\ }} = \frac{(\ +\ \varepsilon)}{(\ +\xi\varepsilon)} \quad \frac{\ ' + (\ \ +\ \ \ ')\varepsilon}{1 + (\xi\ \ +\xi\ \ ')\varepsilon}$$

$(\ \ =\ \ +\ '\ \ +\ \ $ is the operator of total derivative). Expanding the right-hand side into a power series with respect to the parameter ε and preserving the first-order terms, we obtain

$$\bar{\ }' \quad \ ' + \ _1(\ ,\ ,\ ')\varepsilon,$$

where

$$_1 = \ \ + (\ \ -\xi\ \ )\ ' - \xi\ (\ ')^2 = \ \ (\ ) - \ '\ \ (\xi).$$

The action of the group on higher-order derivatives is determined by the recurrence formula:

$$_{+1} = \ \ (\ ) - \ ^{(\ +1)}\ \ (\xi).$$

To a prolonged group there corresponds a prolonged operator:

$$\mathrm{X} = \xi(\ ,\ )\frac{}{} + (\ ,\ )\frac{}{} + \sum_{=1} (\ ,\ ,\ ',\ \ ,\ ^{(\ )}) \frac{}{(\ )}. \tag{7}$$

The ordinary differential equation (1) admits the group (2) if

$$\mathrm{X}\big[\ ^{(\ )} - (\ ,\ ,\ ',\ \ ,\ ^{(\ -1)})\big]_{(\ )=F} = 0. \tag{8}$$

Relation (8) is called the invariance condition.

The invariant $_0$, which is a solution of equation (5), also satisfies the equation $\mathrm{X}\ _0 = 0$.

0.6.1-2. Group analysis of second-order equations. Structure of an admissible operator.

For second-order nonlinear equations

$$'' = (\ ,\ ,\ '), \tag{9}$$

the invariance condition (8) is written in the form

$$\begin{aligned} &+ (2\ \ - \xi\ \ )\ ' + (\ \ - 2\xi\ \ )(\ ')^2 - \xi\ \ (\ ')^3 \\ &= (2\xi\ \ -\ \ + 3\xi\ \ ')\ \ + \xi\ \ +\ \ + [\ \ + (\ \ - \xi\ )\ ' - \xi\ (\ ')^2]\ \ , \end{aligned}$$

where $= (\ ,\ ,\ ')$. This condition is in fact a second-order partial differential equation for two unknown functions $\xi(\ ,\ )$ and $(\ ,\ )$. Since the unknown functions do not depend on the derivative $'$, this equation can be represented (after has been expanded in a power series with respect to $'$, unless it is already a polynomial) in the form

$$\sum_{=0} (\ ')\ \ = 0, \tag{10}$$

with the independent of $'$. In order to ensure that condition (10) holds identically, one should set $= 0$, $k = 0, 1,$ Thus, the invariance condition for a second-order equation can be "split" and represented as a system of equations (whose number can generally be infinite).

Example 1. If $= (\ , y)$, i.e., the right-hand side of equation (9) does not depend on y_x, then the determining equation can be "split" and represented as the system:

$$\begin{aligned} &\xi\ \ = 0, \\ &\ \ - 2\xi_x\ \ = 0, \\ &2\ _x - \xi_{xx} - 3\ (\ , y)\xi\ \ = 0, \\ &_{xx} + (\ \ - 2\xi_x)\ (\ , y) - \ _x(\ , y)\xi - \ (\ , y)\ \ = 0. \end{aligned}$$

From the first two equations we find that

$$\xi = (\ )y + b(\ ), \qquad = (\ )y^2 + (\ )y + d(\ ),$$

where $(\ )$, $b(\ )$, $(\ )$, and $d(\ )$ are arbitrary functions. Substituting these expressions into the third and the fourth equations, we get

$$\begin{aligned} &3\ \ y + 2\ \ - b\ \ - 3\ (\ , y)\ \ = 0, \\ &y^2 + \ \ y + d\ \ + (\ - 2b\ )\ \ - (\ y + b)\ _x - (\ y^2 + \ y + d)\ \ \ = 0. \end{aligned} \tag{11}$$

In what follows, it is assumed that the function $(\ , y)$ is nonlinear with respect to the second argument. Then from the first equation in (11), we find that $= 0$ and $= \frac{1}{2}b\ +\ $, where is an arbitrary constant. The second equation in (11) becomes

$$\tfrac{1}{2}b\ \ y + d\ \ + \ \ - \tfrac{3}{2}b\ \ \ - b\ \ _x - \ \tfrac{1}{2}b\ + \ \ y + d\ \ \ = 0. \tag{12}$$

Equation (12) allows us to solve two different problems.

1°. If the function $(\ , y)$ is given, then, splitting equation (12) with respect to powers of y (the unknown functions b and d are independent of y), we obtain a new system, from which b, d, and can be found; i.e., we ultimately obtain an admissible operator.

2°. Assuming that the functions b, d and the constant are known but arbitrary, one can regard relation (12) as an equation for the unknown function $(\ , y)$. Solving this equation, we obtain a class of equations admitting a point operator.

Example 2. Let $(\ , y) = \quad y$, i.e., we are dealing with the Emden–Fowler equation. Then equation (12) becomes

$$\tfrac{1}{2}b \quad y + d \quad + \quad - \tfrac{3}{2}b \qquad y \quad - b \qquad ^{-1}y \quad - \quad \tfrac{1}{2}b \ + \quad y + d \qquad y \quad ^{-1} = 0.$$

This relation must be satisfied identically by any function $y = y(\ )$, and therefore, the coefficients of different powers of y must be equal to zero. As a result, we obtain a new system whose structure essentially depends on the value of .

1°. It was assumed above that $(\ , y)$ is nonlinear in its second argument, and therefore, $\neq 0$ and $\neq 1$. Let $\neq 2$. Then the system has the form:

$$\begin{aligned}
&d \quad = 0,\\
&b \quad = 0,\\
&d = 0,\\
&\quad (1 - \quad) - \tfrac{1}{2}(3 - \quad)b \qquad - \quad b = 0.
\end{aligned}$$

It follows that $d = 0$ and $b(\ ) = b_2 \ ^2 + b_1 \quad + b_0$, and the last equation of the system can be written in the form

$$(\quad + \quad + 3)b_2 \ ^2 + \tfrac{1}{2}(\quad + 2 \quad + 3)b_1 + \quad (\quad - 1) \quad + \quad b_0 = 0. \tag{13}$$

To ensure relation (13), we equate all coefficients of this quadratic trinomial to zero to obtain

$$(\quad + \quad + 3)b_2 = 0, \quad \tfrac{1}{2}(\quad + 2 \quad + 3)b_1 + \quad (\quad - 1) = 0, \qquad b_0 = 0. \tag{14}$$

Analysis of system (14) yields solutions of the determining system corresponding to three different operators:

$$\begin{aligned}
&\mathrm{X}_1 = (\quad - 1) \quad _x - (\quad + 2)y && \text{if} \quad \text{and} \quad \text{are arbitrary},\\
&\mathrm{X}_2 = \quad _x && \text{if} \quad = 0,\\
&\mathrm{X}_3 = \quad ^2 \quad _x + \quad y && \text{if} \quad + \quad + 3 = 0.
\end{aligned}$$

2°. Let $= 2$. Then equation (12) becomes

$$d \quad + \tfrac{1}{2}b \quad - 2 \quad d \qquad y - \quad \tfrac{5}{2}b \ + \qquad + \quad b \qquad ^{-1}y^2 = 0.$$

Equating the term d and the coefficient of y in parentheses to zero, we get

$$\begin{aligned}
&d(\ ) = d_1 \quad + d_0,\\
&b(\ ) = \frac{4 \ d_1 \qquad ^{+4}}{(\quad + 2)(\quad + 3)(\quad + 4)} + \frac{4 \ d_0 \qquad ^{+3}}{(\quad + 1)(\quad + 2)(\quad + 3)} + b_2 \ ^2 + b_1 \quad + b_0, \qquad \neq -1, -2, -3, -4.
\end{aligned}$$

The expression in square brackets (the coefficient of y^2) can be split with respect to powers of and we obtain an algebraic system which, to within nonzero coefficients, has the form:

$$\begin{aligned}
&(7 \quad + 20)d_1 = 0,\\
&(7 \quad + 15)d_0 = 0,\\
&(\quad + 5)b_2 = 0,\\
&(2 \quad + 5)b_1 + 2 \quad = 0,\\
&\quad b_0 = 0.
\end{aligned}$$

The last three equations coincide with the corresponding equations of system (14), whose solutions are already known. The first two equations yield two cases of prolongation of the admissible group:

$$\begin{aligned}
&\mathrm{X}_1 = 343 \qquad ^{8/7} \quad _x + 4 \ \ 49 \qquad ^{1/7}y - 3 && \text{if} \quad = -\tfrac{20}{7},\\
&\mathrm{X}_2 = 343 \qquad ^{6/7} \quad _x + 3 \ \ 49 \qquad ^{-1/7}y + 4 && \text{if} \quad = -\tfrac{15}{7}.
\end{aligned}$$

0.6.1-3. Utilization of local groups for reducing the order of equations and their integration.

Suppose that an ordinary differential equation (1) admits an infinitesimal operator X of the form (4). Then the order of the equation can be reduced by one. Below we describe two methods for reducing the order of an equation.

1 . *The first method.* The transformation

$$= \;(\;,\;), \qquad = g(\;,\;), \tag{15}$$

with and g (g 0) being arbitrary particular solutions of the first-order linear partial differential equations

$$\begin{aligned} &\xi(\;,\;)\text{——} + \;(\;,\;)\text{——} = k, \\ &\xi(\;,\;)\frac{g}{\;} + \;(\;,\;)\frac{g}{\;} = 0, \end{aligned} \tag{16}$$

reduces equation (1) to an autonomous equation (the constant $k \neq 0$ can be chosen arbitrarily). The function $g = g(\;,\;)$ is a universal invariant of the operator X.

Suppose that the general solution of the characteristic equation

$$\frac{\;}{\xi(\;,\;)} = \frac{\;}{\;(\;,\;)}$$

has the form

$$(\;,\;) = \;,$$

where is an arbitrary constant. Then the general solutions of equations (16) are given by (see A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux, 2002):

$$\begin{aligned} &= k \quad \frac{\;}{\bar{\xi}(\;,\;)} + \;_1(\;), \\ &g = \;_2(\;), \qquad = \;(\;,\;), \end{aligned}$$

where $_1(\;)$ and $_2(\;)$ are arbitrary functions, $\bar{\xi}(\;,\;(\;,\;)) \equiv \xi(\;,\;)$, and in the integral is regarded as a parameter.

Example 3. The Emden–Fowler equation $y_{xx} = \;^{-15/7}y^2$ admits the operator (cf. the operator X_2 in Item 2° of Example 2):

$$X = \xi(\;,y)\text{——} + \;(\;,y)\frac{\;}{y}, \quad \text{where} \quad \xi(\;,y) = 343\;^{6/7}, \quad (\;,y) = 147\;^{-1/7}y + 12.$$

Equations (16) for $k = 49$ admit the particular solutions

$$= \;^{1/7}, \qquad = \;^{-3/7}y + \frac{6}{49}\;^{-2/7}.$$

Solving (15) for and y, we obtain the transformation

$$= \;^7, \quad y = \;^3\; - \frac{6}{49}\; ,$$

which reduces the original equation to the autonomous equation

$$= 49\;^2,$$

which can easily be integrated by quadrature.

2 . *The second method.* Suppose that we know two invariants of the admissible operator X:

$$\begin{aligned} &_0 = \;_0(\;,\;) && \text{(universal invariant)}, && (17) \\ &_1 = \;_1(\;,\;,\;') && \text{(first differential invariant*)}. && (18) \end{aligned}$$

---

* By definition, an th-order differential invariant of the operator X is a function $=\;(\;,y,y_x,\;,y_x^{(\;)})$, satisfying the linear partial differential equation $X\;=0$ with the operator X defined by (7).

Then the second differential invariant can be found by differentiation,

$$_2(\ , \ , \ ', \ '' \ ) = \frac{1}{0}, \tag{19}$$

where $\quad = (\qquad)$. Using (18)–(19), let us eliminate the derivatives $'$ and $''$ from the original equation and take into account relation (17). Thus we obtain the first-order equation:

$$\frac{1}{0} = \ (\ _0, \ _1).$$

Example 4. The Emden–Fowler equation $y_{xx} = \ ^{-6}y^3$ (see 2.3.1.3, the special case $\ = 3$) admits an operator whose first prolongation has the form:

$$\underset{1}{\mathrm{X}} = \ ^2 \ _x + \ y \ + (y - \ y) \ .$$

This operator admits the invariants:

$$_0 = y/ \ , \qquad _1 = \ y_x - y, \tag{20}$$

which form an integral basis of the first-order linear partial differential equation

$$^2 \frac{}{} + \ y \frac{}{y} + (y - \ y) \frac{}{y} = 0.$$

Using (19) and (20), we find the second invariant:

$$_2 = \frac{d \ _1}{d \ _0} = \frac{^3 y_{xx}}{y_x - y}. \tag{21}$$

Let us express the unknown function and its derivatives from (20)–(21) to obtain

$$y = \ , \quad y_x = \frac{+}{}, \quad y_{xx} = \frac{}{3}, \qquad \text{where} \quad = \ _0, \quad = \ _1.$$

Substituting these expressions into the original equation, we see that the variable is canceled and the equation takes the form

$$= \ ^3,$$

i.e., it becomes a first-order separable equation.

References for Subsection 0.6.1: G. W. Bluman and J. D. Cole (1974), L. V. Ovsiannikov (1982), J. M. Hill (1982), P. J. Olver (1986), G. W. Bluman and S. Kumei (1989), H. Stephani (1989), N. H. Ibragimov (1994).

0.6.2. Contact Transformations. Bäcklund Transformations. Formal Operators. Factorization Principle

0.6.2-1. Contact transformations.

The set of transformations

$$\mathrm{T} = \begin{cases} \bar{\ } = \ (\ , \ , \ ', \varepsilon), & \bar{\ }|_{=0} = \ , \\ \bar{\ } = \ (\ , \ , \ ', \varepsilon), & \bar{\ }|_{=0} = \ , \\ \bar{\ }' = \chi(\ , \ , \ ', \varepsilon), & \bar{\ }'|_{=0} = \ ' \end{cases} \tag{1}$$

(here, $\ , \ , \chi$ are smooth functions of their arguments and ε is a real parameter) is called a continuous one-parameter Lie group of tangential transformations (or simply, a tangential or contact group) if $\mathrm{T}_{\ 1} \ \mathrm{T}_{\ 2} = \mathrm{T}_{\ 1+\ 2}$, i.e., if successive application of transformations (1) with parameters ε_1 and ε_2 is equivalent to the same transformation with parameter $\varepsilon_1 + \varepsilon_2$. The transformed derivative $\bar{\ }'$ depends only on the first derivative $'$ and does not depend on the second derivative. Thus, the functions and in (1) cannot be arbitrary but are related by (see Paragraph 0.1.8-1):

$$\frac{}{'} \left(\frac{}{} + \ ' \frac{}{} \right) - \frac{}{'} \left(\frac{}{} + \ ' \frac{}{} \right) = 0,$$

where the function χ is defined by

$$\chi = \frac{}{\ \ '}\quad \frac{}{\ \ '}.$$

Proceeding as in Paragraph 0.6.1-1, we consider the Taylor expansions of $\bar{}$, $\bar{}$, and $\bar{}'_{\bar{}}$ in (1) with respect to the parameter ε about $\varepsilon = 0$, preserving only the first-order terms. We have

$$\bar{}\quad + \xi(\ ,\ ,\ ')\varepsilon,\quad \bar{}\quad + \ (\ ,\ ,\ ')\varepsilon,\quad \bar{}'_{\bar{}}\quad ' + \ (\ ,\ ,\ ')\varepsilon,$$

where

$$\xi(\ ,\ ,\ ') = \frac{(\ ,\ ,\ ',\varepsilon)}{\varepsilon}\bigg|_{=0},\quad (\ ,\ ,\ ') = \frac{(\ ,\ ,\ ',\varepsilon)}{\varepsilon}\bigg|_{=0},\quad (\ ,\ ,\ ') = \frac{\chi(\ ,\ ,\ ',\varepsilon)}{\varepsilon}\bigg|_{=0}.$$

On the other hand,

$$\bar{}'_{\bar{}} \equiv \frac{\bar{}}{\bar{}} = \frac{(\ +\ \varepsilon)}{(\ +\xi\varepsilon)}\quad \frac{' + (\ \ +\ \ \ ' +\ \ \ '')\varepsilon}{1 + (\xi\ \ +\xi\ \ \ ' +\xi\ \ \ '')\varepsilon}. \tag{2}$$

Expanding (2) with respect to ε and requiring that be independent of $''$, we find that three the functions ξ, , and are expressed in terms of a single function $(\ ,\ ,\ ')$ as follows:

$$\xi = -\frac{}{\ \ '},\qquad = \quad - \ '\frac{}{\ \ '},\qquad = \frac{}{} + \ '\frac{}{}. \tag{3}$$

To an infinitesimal tangential transformation (1) there corresponds the infinitesimal operator:

$$\mathrm{X} = \xi(\ ,\ ,\ ')\frac{}{} + \ (\ ,\ ,\ ')\frac{}{} + \ (\ ,\ ,\ ')\frac{}{\ \ '} \tag{4}$$

whose coordinates satisfy relations (3).

The action of the group on higher derivatives is determined by the recurrence formula:

$$_{+1} = \quad (\ ) - \ ^{(\ +1)}\quad (\xi),$$

where $_1 =$. The invariance condition and the algorithm of finding tangential operators (4) admitted by ordinary differential equations are similar as those for point operators. The only difference is that the coordinates of the tangential operator depend on the first derivative; therefore, the determining equation can be split and reduced to a system only in the case of equations whose order is greater or equal to three.

There are no tangential transformations of finite order $k > 1$ other than prolonged point transformations and contact transformations [these transformations are described by formulas similar to (1) and, in addition to $'$, $\bar{}'_{\bar{}}$, contain higher derivatives of up to order k inclusive].

0.6.2-2. Bäcklund transformations. Formal operators and nonlocal variables.

1 . If the coordinates of the infinitesimal operator are allowed to depend on the derivatives of arbitrary (up to infinity) orders, we obtain Lie–Bäcklund groups (of tangential transformations of infinite order). However, on the manifold determined by an ordinary differential equation, all higher derivatives are expressed through finitely many lower derivatives, as dictated by the structure of the equation itself and the differential relations obtained from the equation. The substitution of the right-hand side of equation (1) into an infinite series with derivatives usually results in very cumbersome formulas hardly suitable for practical calculations. For this reason, the Lie–Bäcklund groups are widely used only for the investigation of partial differential equations, whereas in the case of ordinary differential equations, a more effective approach is that based on the canonical form of an operator and the notion of a formal operator.

2 . The canonical form X is defined by the relation

$$\mathrm{X} = \mathrm{X} - \xi(\,,\,) \qquad = [\ (\,,\,) - \xi(\,,\,)\ '\,] \text{—},$$

where $\mathrm{X} = \xi(\,,\,)\text{—} + \ (\,,\,)\text{—}$ is the infinitesimal operator of the group [see formula (4) in Paragraph 0.6.1-1], and is the operator of total derivative. The operators X and X are equivalent in the sense that if one of them is admissible for the equation, then the other is also admissible (the operator of total derivative is admissible for any ordinary differential equation). The function $_0(\,,\,) \equiv$ is an invariant of any operator in canonical form.

The action of the group on higher order derivatives for an operator in canonical form is determined by the simple recurrence formula $_{+1} = \ (\ )$. The order of an equation that admits an operator in canonical form can be reduced on the basis of the algorithm described in Paragraph 0.6.1-3 (see Item 2 , *the second method*).

3 . By definition, a formal operator is an infinitesimal operator of the form

$$X = \qquad , \tag{5}$$

where the function depends on $\,,\,,\,'\,,\ \ ,\ ^{(\ )}$ (with k smaller than the order of the equation under investigation) and auxiliary variables whose definition involves the symbol of indefinite integral, for instance,

$$(\,,\,,\, ')$$

(the integration is with respect to the variable which is involved both explicitly and implicitly, through the dependence of on). Such auxiliary variables are called nonlocal, in contrast to the coordinates of the prolonged space defined pointwise. The nonlocal variables depend on derivatives of arbitrarily high order, for instance,

$$= \sum_{=0} (-1) \frac{\ ^{+1}}{(\ +1)!}\ ^{(\ )}.$$

This formula is obtained by successive integration by parts of its left-hand side. Thus, a nonlocal variable can be represented as an infinite formal series; and this allows us to express the coordinates of the Lie–Bäcklund operator in concise form.

A formal operator is a far-reaching generalization of an operator in canonical form. The function $_0(\,,\,) \equiv$ is an invariant of the formal operator (5) for any .

When solving the direct problem, one usually prescribes the nonlocal operator in the general form

$$\mathrm{X} = \ _1 \exp \qquad + \ _2 \qquad \text{or} \qquad \mathrm{X} = \ _1 \qquad + \ _2 \quad , \tag{6}$$

and then, in order to find an admissible operator, one uses a search algorithm similar to that described in Paragraph 0.6.1-2. The coordinates of the prolonged operator are found by the formulas $= \ (\ _{-1})$, where $_0 =$. In contrast to the method of finding a point operator, in the present case, there are three unknown functions $(\ _1,\ _2,\ )$; and the splitting procedure to obtain a system can be realized with respect to all "independent" variables, in particular, the nonlocal variables.

Suppose that the differential equation

$$^{(\ )} = \ (\,,\,,\, '\,,\ \ ,\ ^{(\ -1)}) \tag{7}$$

can be written in new variables $= \ _0$, $z = \ _1(\,,\,,\, ')$, z', z'', $\ldots$, $z^{(\ -1)}$, where $_0$ and $_1$ are invariants of an admissible operator of the form (5). Then the coordinate of this operator satisfies the equation

$$\frac{\ _1}{\ } + \frac{\ _1}{\ '}\ [\ ] = 0,$$

which is an analogue of a linear ordinary differential equation for a function of several variables, since it involves the total derivative of the unknown function (exact differential equation). Its solution has the form:

$$= \exp \; - \quad \frac{1}{1 \quad '} \quad , \tag{8}$$

where the integral is taken with respect to involved explicitly and implicitly (through the dependence of , $'$, on), which means that this representation of an operator through a nonlocal variable is most universal. The function (8) generates a nonlocal exponential operator of the form (5) [the class of nonlocal exponential operators is specified by the first expression in (6) with $_2 \equiv 0$].

Example 1. The equation

$$y_{xx} = 0$$

admits two Lie–Bäcklund operators:

$$\mathrm{X}_1 = \xi(\;, y, y_x) D_x, \quad \mathrm{X}_2 = \sum_{=0} D_x \quad y \;\; (y - \;\; y_x, y_x) + h(y - \;\; y_x, y_x)\Big] \frac{}{y_x^{(\;)}},$$

where ξ, , h are arbitrary functions of their variables. The first operator is trivial (the operator of total derivative is admissible for any differential equation), while the second operator determines the maximal group of contact transformations admitted by the equation under consideration.

A Lie–Bäcklund operator admitted by an ordinary differential equation can by found by three methods:

(i) in the form of an infinite formal series;

(ii) by passing to an equivalent system of ordinary first-order differential equations:

$$'_1 = \;_2, \quad '_2 = \;_3, \quad \;, \quad ' = \;(\;, \;_1, \;_2, \quad, \quad),$$

and finding an admissible point group;

(iii) by its representation as a formal operator whose coordinates depend on nonlocal variables (the general form of the operator is chosen by the investigator).

In all cases, the search algorithm amounts to solving the determining system which is constructed by a procedure similar to that of Subsection 0.6.1. From the standpoint of simplicity and the possibility of integrating equations, the third method seems to be the most effective if one takes into account that an equation admitting an operator can be written in terms of new variables—invariants of the admissible operator—as a new ordinary differential equation whose order is by one less than that of the original equation.

0.6.2-3. Factorization principle.

The use of formal operators allows us to formulate universal principles for reducing the order of an equation, independently of the specific structure of the operator (it can be a point operator, a tangential or nonlocal operator, or a Lie–Bäcklund operator).

Theorem 1. An arbitrary th-order differential equation (7) can be factorized to a system of special form

$$\begin{aligned} z^{(\;-1)} &= \;\big(\;, z, z', \quad, z^{(\;-2)} \big), \\ z &= \;(\;, \;, \;'), \end{aligned} \tag{9}$$

if and only if equation (7) admits the nonlocal exponential operator:

$$\mathrm{X} = \exp \; - \quad \frac{}{} \quad \frac{}{}. \tag{10}$$

The function $(\; , \; , \; ')$ is the first differential invariant of the operator (10). Therefore, having found an admissible operator (10) of the form

$$\mathrm{X} = \quad \frac{\quad}{\quad}, \qquad \equiv \exp \quad Q(\; , \; , \; ') \quad , \tag{11}$$

we can calculate by solving the first-order linear partial differential equation $\quad + Q \quad = 0$. The function $Q(\; , \; , \; ')$ is found as a solution of the determining system obtained by "splitting" the invariance condition for operator (11):

$$\mathrm{X}\Big[\;^{(\;)} - \quad \big(\; , \; , \; ', \quad , \;^{(\;-1)} \big) \Big]_{\;(\;)=F} = 0,$$

where

$$\mathrm{X} = \sum_{=0} \quad \frac{\quad}{(\;)}, \qquad = \qquad _{-1}, \qquad _0 = \quad , \qquad = \frac{\quad}{\quad} + \; ' \frac{\quad}{\quad} + \; '' \frac{\quad}{\;'} + \quad .$$

Theorem 1 generalizes the classical Lie algorithm, which is restricted to the case of unconditional solvability of the second equation of system (9). On the other hand, the introduction of the factor system (9) allows for two more cases, since the first equation is independent of . These cases are the following:

1. The first equation of system (9) allows for the reduction of the order or is solvable.

2. The first equation of system (9) has some special properties, for instance, admits a fundamental system of solutions.

Example 2. The equation

$$y_{xx} = \; (\;)y + \; _x(\;)y^{-1} - [\; (\;)]^2 y^{-3} \tag{12}$$

for arbitrary functions $(\;)$ and $(\;)$ is the only equation of the form (its uniqueness is to within a Kummer–Liouville equivalence transformation; see Paragraph 0.2.1-8)

$$y_{xx} = \; (\; , y)$$

admitting the nonlocal exponential operator:

$$\mathrm{X} = \exp \quad \zeta \, d \quad \frac{\quad}{y} = \exp \quad \zeta + \frac{\;_x + \quad y_x}{\quad} \quad d \; \Big] \frac{\quad}{y}, \qquad = \; (\; , y), \quad \zeta = \zeta(\; , y).$$

The second prolongation of the operator X has the form:

$$\underset{2}{\mathrm{X}} = \exp \quad \zeta \, d \qquad + (\;_x + \quad y_x + \quad \zeta)$$
$$+ \quad _{xx} + 2\zeta \; _x + \quad \zeta_x + \zeta^2 \quad + (2 \; _x \quad + 2\zeta \quad + \; \zeta \;) y_x + \quad (y_x)^2 + \quad y_{xx} \qquad .$$

Applying this operator to the equation $y_{xx} = \; (\; , y)$ and replacing all instances of y_{xx} by $\quad = \; (\; , y)$, we obtain the invariance condition in the form:

$$_{xx} + 2\zeta \; _x + \quad \zeta_x + \zeta^2 \quad + \qquad - \qquad + (2 \; _x \quad + 2\zeta \quad + \; \zeta \;) y_x + \quad (y_x)^2 = 0.$$

Splitting this relation with respect to powers of the "independent" variable y_x, we obtain the following system of three equations for the functions $\;$, ζ, and $\;$:

$$\begin{aligned} &\quad = 0,\\ 2 \; _x \quad &+ 2\zeta \quad + \; \zeta \quad = 0,\\ _{xx} + 2\zeta \; _x + \quad \zeta_x &+ \zeta^2 \quad + \qquad - \qquad = 0. \end{aligned}$$

From the first two equations it follows that

$$= \; (\;)y + b(\;),$$
$$\zeta = -\frac{\quad y^2 + 2 \quad by + \; (\;)}{(\; y + b)^2},$$

where $= (\ )$, $b = b(\ )$, and $= (\ )$ are arbitrary functions. The third equation can be treated as a first-order linear differential equation for the unknown function $= (\ , y)$:

$$\frac{d}{dy} - \frac{}{} = \frac{1}{}(\ _{xx} + 2\zeta\ _x + \ \zeta_x + \ \zeta^2).$$

Substituting the above expressions of and ζ into this relation and integrating the result, we obtain

$$(\ , y) = (\ y + b)\ (\ ) + \frac{[\quad -2(\quad)^2]b - (\ b \ -2 \ b\ )}{^3} - \frac{[\quad -3(\quad)^2]b^2 + 2 \quad bb \ -(\quad -2 \quad)}{2\ ^3(\ y+b)} - \frac{(\ b^2 - \quad)^2}{4\ ^3(\ y+b)^3},$$

where $(\ )$ is an arbitrary function.

The differential invariant of the operator X satisfies the linear partial differential equation

$$\frac{}{y} + (\ _x + \ \ y_x + \ \zeta)\frac{}{y_x} = 0$$

(obtained after the division by $\exp(\ \zeta\, d\ )$). Substituting the above and ζ into this equation, we pass to the characteristic equation

$$\frac{d}{dy} = \frac{}{y+b} + \frac{2\quad y^2 + (3\quad b + \ b\ )y + bb \ +}{(\ y+b)},$$

where $= y_x$. Integrating this equation, we find the differential invariant:

$$= \frac{y_x}{y+b} - \frac{b - \ b}{^2(\ y+b)} + \frac{b^2 -}{2\ ^2(\ y+b)^2}.$$

Having calculated the derivative $_x$, one can find y_{xx} and, taking into account the known structure of the function $(\ , y)$, one obtains the factorization of the original equation:

$$\begin{gathered} _x + \ ^2 + (\ /\ )\ = \ , \\ (\ y+b)y_x = (\ y+b)^2 \ + \ ^{-2}(\ b - \ b\ )(\ y+b) - \tfrac{1}{2}\ ^{-2}(\ b^2 - \quad). \end{gathered}$$

An equivalence transformation of the form $y + b \quad y$, combined with the corresponding transformation of the independent variable and changed notation, yields:

$$\begin{gathered} _x + \ ^2 = \ (\ ), \\ y_x = \ y + \ (\ )y^{-1}. \end{gathered} \tag{13}$$

The first equation of system (13) is the Riccati equation. Its general solution can be represented in terms of a fundamental system of solutions of the "truncated" linear equation:

$$y_{xx} = \ (\ )y, \tag{14}$$

which coincides with (12) for $\equiv 0$. The second equation of system (13) is a Bernoulli equation. It can be integrated by quadrature for an arbitrary function $= (\ ,\ )$. Therefore, the general solution of equation (12) can be expressed in terms of a fundamental system of solutions of the linear equation (14). Note that in the general case, equation (12) admits no point groups.

If an operator admitted by equation (1) has no differential invariants of the first-order, then it is possible to apply the general factorization principle.

Theorem 2. An arbitrary th-order differential equation (1) can be factorized to the system of special structure

$$\begin{gathered} z^{(\ -\ )} = \ \big(\ , z, z', \quad , z^{(\ -\ -1)}\big), \\ z = \ \big(\ ,\ ,\ ',\ \quad ,\ ^{(\ )}\big), \quad \frac{}{^{(\ )}} \neq 0, \end{gathered} \tag{15}$$

provided that equation (7) admits a formal operator (5) for which $\big(\ ,\ ,\ ',\ \quad ,\ ^{(\ )}\big)$ is a lower-order differential invariant on the manifold given by (7). The coordinate of this operator satisfies the linear equation with total derivatives:

$$\frac{z}{} + \quad [\ ]\frac{z}{'} + \quad + \ ^{(\ )}[\ ]\frac{z}{^{(\ )}} = 0. \tag{16}$$

Equation (16) plays a crucial role in both the direct and inverse problems. It can be regarded as an equation for the determination of the coordinate of the canonical operator (if one knows the invariant z). It can also be regarded as an equation for the determination of an invariant (if one knows the coordinate). In the latter case, this is a first-order partial differential equation.

Example 3. The third-order nonlinear equation

$$yy_{xxx} + \;\; y_{xx} \;\; {}^2 - y_x y_{xx} - \;\; (\;\;) y^2 = 0 \tag{17}$$

admits two operators

$$\mathrm{X}_1 = y \quad , \quad \mathrm{X}_2 = \quad y \quad y^{-2}\, d \quad , \tag{18}$$

which can be found with the help of the direct algorithm, if the structure of the operator is specified by the second expression in (6). The first operator, X_1, is the usual point operator of scaling (the original equation is homogeneous) and provides the usual reduction of order of equation (17) by one. The second operator, X_2, is nonlocal.

Let us construct differential invariants of the operator X_2. To this end, we should solve the equations:

$$\begin{aligned} &\frac{\;1}{y} + D_x[\;\;]\frac{\;1}{y_x} = 0, \qquad = y \quad y^{-2}\, d \;\;, \\ &\frac{\;2}{y} + D_x[\;\;]\frac{\;2}{y_x} + D_x^2[\;\;]\frac{\;2}{y_{xx}} = 0. \end{aligned} \tag{19}$$

After differentiation with respect to , the first equation in (19) becomes

$$y \quad y^{-2}\, d \quad \frac{\;1}{y} + \;\; y^{-1} + \;\; y_x \quad y^{-2}\, d \quad \Big] \frac{\;1}{y_x} = 0.$$

Let us show that this equation admits no solutions depending only on , y, y_x, and $\;_1/\;\; y_x \neq 0$, i.e., there are no first-order differential invariants. The nonlocal expression $\;\; y^{-2}\, d\;\;$ depends on derivatives of arbitrarily high orders and can be regarded as an independent quantity. Therefore, the first equation (19) can be split and we obtain the system:

$$y \frac{\;1}{y} + y_x \frac{\;1}{y_x} = 0, \qquad y^{-1} \frac{\;1}{y_x} = 0.$$

It follows that $\;_1/\;\; y_x = 0$.

Let us find a second-order differential invariant. After differentiation with respect to , the second equation in (19) becomes

$$y \quad y^{-2}\, d \quad \frac{\;2}{y} + \;\; y^{-1} + \;\; y_x \quad y^{-2}\, d \quad \Big] \frac{\;2}{y_x} + \;\; y_{xx} \quad y^{-2}\, d \quad \frac{\;2}{y_{xx}} = 0.$$

Splitting this equation with respect to the nonlocal variable $\;\; y^{-2}\, d\;$, we establish that $\;_2/\;\; y_x = 0$. In the remaining equation, the nonlocal variable is canceled,

$$y \frac{\;2}{y} + y_{xx} \frac{\;2}{y_{xx}} = 0.$$

Hence, we find that $\;_2 = \;\; = y_{xx}/y$, and equation (17) is factorized to the system:

$$\begin{aligned} &{}_x + \;\; {}^2 = \;\; (\;\;), \\ &y_{xx} - y \;\; = 0. \end{aligned}$$

References for Subsection 0.6.2: R. L. Anderson and N. H. Ibragimov (1979), O. N. Pavlovskii and G. N. Yakovenko (1982), N. H. Ibragimov (1985), P. J. Olver (1986), V. F. Zaitsev (2001).

0.6.3. First Integrals (Conservation Laws)

A function $\;\; = \;\; (\;,\;,\;',\;\;,\;{}^{(\;-1)}$ is called a first integral (conservation law) of the ordinary differential equation

$${}^{(\;)} = \;\; (\;,\;,\;\;,\;{}^{(\;-1)} \tag{1}$$

if the total derivative of the function along the trajectories of equation (1) is zero or, equivalently, if

$$[\;\;] \equiv \;\; (\;,\;,\;',\;\;,\;{}^{(\;-1)} \big[\;{}^{(\;)} - \;\; (\;,\;,\;\;,\;{}^{(\;-1)} \big] = 0, \tag{2}$$

where is an integrating factor. From this definition it is clear that $\;\; = \frac{\quad}{{}^{(\;-1)}}$.

The algorithm of finding a first integral is similar to that of finding an admissible operator. It is necessary to prescribe the desired structure of the first integral (or the integrating factor) and substitute it into the determining equation (2). Subsequent splitting with respect to lower derivatives (assumed to be independent variables) leads to the determining system.

An arbitrary function of first integrals is also a first integral of the same equation. Therefore, having found a first integral depending on $(\; ')$, one has to make sure that it is nontrivial; i.e., it cannot be represented as the product of first integrals depending on lower powers of the derivative.

If the equation has k functionally independent first integrals, then its order can be reduced by k by successively excluding higher derivatives (see Example 3).

For second-order equations

$$'' = (\; , \; , \; '), \tag{3}$$

the determining equation (2) can be written in the form

$$\frac{}{} + \; ' \frac{}{} + (\; , \; , \; ') \frac{}{'} = 0. \tag{4}$$

In this case, one can solve the direct problem (find for the given equation), as well as the inverse problem (find possible for the given structure of the first integral).

Example 1. Let us find all equations of the form

$$y_{xx} = (\; , y) \tag{5}$$

admitting a first integral that is quadratic with respect to the first derivative:

$$= R(\; , y)(y_x)^2 + (\; , y)y_x + (\; , y).$$

Then the left-hand side of the determining equation (4) is a cubic polynomial with respect to y_x. The procedure of splitting with respect to powers of y_x yields the system of four equations:

$$\begin{aligned}
&R \quad = 0,\\
&R_x + \quad = 0,\\
&{}_x + \quad + 2R \quad = 0,\\
&{}_x + \quad = 0.
\end{aligned}$$

The solution of this system for is given by:

$$\begin{aligned}
(\; , y) &= R^{-3/2} \; (\;) + \frac{1}{2}R^{-2} \; RR_{xx} - \frac{1}{2}R_x^2 \; y - R\varphi_x + \frac{1}{2}R_x\varphi\Big],\\
&= R^{-1/2}y + \frac{1}{2} \; \varphi R^{-3/2} \, d \; ,
\end{aligned}$$

where $= (\;)$, $R = R(\;)$, and $\varphi = \varphi(\;)$ are arbitrary functions. The first integral has the form:

$$= R(y_x)^2 - (R_x y - \varphi)y_x + \frac{1}{4}R^{-1}(R_x)^2y^2 - \frac{1}{2}R^{-1}R_x\varphi y + \frac{1}{4}R^{-1}\varphi^2 - 2 \; (\;)d \; .$$

Example 2. Consider the equation

$$y_{xx} = \; y^{-1/2}.$$

Let us find its first integral, which is a cubic polynomial with respect to the first derivative:

$$= R(\; , y)(y_x)^3 + (\; , y)(y_x)^2 + (\; , y)y_x + (\; , y).$$

In this case, the left-hand side of the determining equation (4) is a fourth-order polynomial in y_x, and hence the determining system consists of five equations:

$$\begin{aligned}
&R \quad = 0,\\
&R_x + \quad = 0,\\
&{}_x + \quad + 3 \; y^{-1/2}R = 0,\\
&{}_x + \quad + 2 \; y^{-1/2} \quad = 0,\\
&{}_x + \; y^{-1/2} \quad = 0.
\end{aligned}$$

Solving this system, we obtain the first integral in the form:

$$= (y_x)^3 - 6 \quad y^{1/2} y_x + 4 \quad y^{3/2} + 2 \quad {}^2 \quad {}^3.$$

Example 3. The equation

$$y_{xxxx} = \quad y^{-5/3}$$

admits three first integrals:

$$\begin{aligned} {}_1 &= y_x y_{xxx} - \tfrac{1}{2}(y_{xx})^2 + \tfrac{3}{2} \quad y^{-2/3}, \\ {}_2 &= \quad {}_1 - \tfrac{3}{2} y y_{xxx} + \tfrac{1}{2} y_x y_{xx}, \\ {}_3 &= \quad {}_2 - \tfrac{1}{2} \quad {}^2 \quad {}_1 + \tfrac{3}{2} y y_{xx} - (y_x)^2. \end{aligned}$$

Equating these expressions to independent constants $\ _1$, $\ _2$, $\ _3$ and eliminating y_{xxx} and y_{xx}, we obtain a first-order equation (see 4.2.1.1).

References for Subsection 0.6.3: P. J. Olver (1986), A. D. Polyanin and V. F. Zaitsev (1995).

0.6.4. Discrete-Group Method. Point Transformations

Consider transformations of the class of ordinary differential equations

$$^{(\ )} = \quad \big(\ , \ , \quad , \ ^{(\ -1)}, \mathbf{a} \ \big), \tag{1}$$

whose elements are uniquely defined by a vector of essential parameters $\mathbf{a}$.

Any set of invertible transformations

$$= (\ ,\ ), \qquad = g(\ ,\ ) \qquad (\ g - \ g \neq 0), \tag{2}$$

mapping each equation of class (1) into some (other) equation of the same class

$$^{(\ )} = \quad \big(\ , \ , \quad , \ ^{(\ -1)}, \mathbf{b} \ \big), \tag{3}$$

and containing the identical transformation is called a discrete point group of transformations admitted by the class (1). Transformation (2) maps any solution of equation (1) to a solution of equation (3). Therefore, knowing the discrete group of transformations for some class of equations and having a set of solvable equations of this class, one can construct new solvable cases.

Point transformations (2) can be found by a direct method — namely, if one substitutes an arbitrary transformation of the form (2) into equation (1) and imposes condition (3), one arrives at a determining equation containing partial derivatives up to order of the unknown functions and g and having variable coefficients depending on , , $'$, , $^{(\ -1)}$. Since the functions and g do not depend on the derivatives, the determining equation can be "split" with respect to the "independent" variables $'$, , $^{(\ -1)}$, and we obtain an overdetermined system which is nonlinear, in contrast to that obtained by the Lie method (see Subsection 0.6.1).

Example 1. For second-order equations

$$y_{xx} = \quad (\ , y, y_x, \mathbf{a}), \tag{4}$$

the substitution of (2) into (4) yields

$$\begin{aligned} &(\quad - \quad) \ + (\quad - \quad)(\ )^3 + (\quad - \quad + 2 \quad - 2 \quad)(\ )^2 \\ &\quad + (\quad - \quad + 2 \quad - 2 \quad) \ + \quad - \quad = (\ + \quad)^3 \quad \Big(\ , \ , \frac{+}{+}, \mathbf{a} \Big). \end{aligned} \tag{5}$$

Let us require that the transformed equation (5) belong to the class (4), i.e.,

$$= \quad (\ , \ , \ , \mathbf{b}). \tag{6}$$

Condition (6) imposed on the determining equation (5), i.e., the replacement of by the right-hand side of equation (6), leads us to the relation

$$\begin{aligned} &(\quad - \quad) \ (\ , \ , \ , \mathbf{b}) + (\quad - \quad)(\ )^3 + (\quad - \quad + 2 \quad - 2 \quad)(\ )^2 \\ &\quad + (\quad - \quad + 2 \quad - 2 \quad) \ + \quad - \quad = (\ + \quad)^3 \quad \Big(\ , \ , \frac{+}{+}, \mathbf{a} \Big), \end{aligned} \tag{7}$$

which contains the “independent” variable . Expanding the function into a series in powers of , we can represent (7) in the form

$$, y, [\ ], [\ ]\ (\ )\ = 0, \qquad (8)$$
$$=0$$

where the symbols [ ] and [ ] indicate dependence on the functions , and their partial derivatives involved in (7). The sum in (8) is finite if is a polynomial with respect to the third variable [for a polynomial of degree ≥ 4, both sides of the equation must be first multiplied by $(\ + \ )^{-3}$]. Condition (8) is satisfied if the following equations hold:

$$= 0, \quad k = 0,\ 1,\ 2,$$

Example 2. Consider a special case of equation (4) with the right-hand side independent of the derivative y_x:

$$y_{xx} = \ (\ , y, \mathbf{a}). \qquad (9)$$

Relation (7) has the form:

$$(\quad - \quad)\ (\ , \ , \mathbf{b}) + (\quad - \quad)(\ )^3 + (\quad - \quad + 2 \quad - 2 \quad)(\ )^2$$
$$+ (\quad - \quad + 2 \quad - 2 \quad) \ + \quad - \quad = (\ + \ )^3 \ (\ , \ , \mathbf{a}).$$

In this case, the sum (8) is finite and the determining system has the form:

$$
\begin{aligned}
- \quad &= {}^3 \ (\ , \ , \mathbf{a}),\\
- \quad + 2 \quad - 2 \quad &= 3 \ {}^2 \ (\ , \ , \mathbf{a}),\\
- \quad + 2 \quad - 2 \quad &= 3 \ {}^2 \ (\ , \ , \mathbf{a}),\\
- \quad + (\quad - \quad) \ (\ , \ , \mathbf{b}) &= {}^3 \ (\ , \ , \mathbf{a}).
\end{aligned} \qquad (10)
$$

It can be shown that for $\neq 0$, solving system (10) is equivalent to solving the original equation (9).

Consider the case $= 0$. In this case, the first equation of the system holds identically and the system becomes

$$
\begin{aligned}
&= 0,\\
- 2 \quad &= 0,\\
- \quad + \quad (\ , \ , \mathbf{b}) &= {}^3 \ (\ , \ , \mathbf{a}).
\end{aligned} \qquad (11)
$$

Since $\neq 0$, the first two equations yield

$$(\ , \ ) = \ (\ ) \ + \Theta(\ ), \qquad = \ [\ (\ )]^2. \qquad (12)$$

Substituting (12) into the last equation of system (11) and “splitting” the resulting relation with respect to powers of the “independent” variable , we obtain a new system of (ordinary) differential equations. Solving this system, we find the unknown functions and Θ, and finally, the desired discrete group of transformations. In order to give calculation details, one has to know the specific structure of the function $(\ , y)$, for in the general case it was only shown that any discrete point group of transformations of equation (9) for $= 0$ consists of Kummer–Liouville transformations (12).

Example 3. Consider the generalized Emden–Fowler equation:

$$y_{xx} = \quad y \quad (y_x) \ . \qquad (13)$$

Here, $\mathbf{a} = \{ \ , \ , l\}$ is the vector of essential parameters, and is an unessential parameter (it can be made equal to unity by scaling the independent variable and the unknown function).

1°. First, we note that equation (13) admits a discrete group of transformations determined by the hodograph transformation, i.e., by passing to the inverse function:

$$= \ , \quad y = \ , \qquad \text{where} \quad = \ (\ ). \qquad (14)$$

This transformation is a consequence of the invariance of equation (13) with respect to the transformation y, , $l \quad 3-l$, $-$ (note that the hodograph transformation changes the sign of the unessential parameter). Denoting the transformation (14) by , let us schematically represent its action on the parameters of the equation as follows:

$$\{ \ , \ , l\} \quad -- \quad \{ \ , \ , 3-l\} \qquad \textit{transformation} \ . \qquad (15)$$

Double application of the transformation yields the original equation.

2°. For $l = 0$, equation (13) is of the class (8), and the last equation of system (11) becomes

$$[\quad -2(\quad)^2] + \Theta \quad -2 \quad \Theta + \quad {}^2 \quad = \quad {}^2(\quad +\Theta) \quad , \tag{16}$$

where , , and are the parameters of the transformed equation = , and

$$(\ ) = \quad [\ (\ )]^2 d\ .$$

Let , $\neq 0, 1, 2$. Then relation (16) is possible only if $\Theta(\ ) \equiv 0$. Splitting with respect to powers of leads us to the system:

$$\begin{aligned} -2(\quad)^2 &= 0, \\ = \quad {}^2 \quad {}^{+3} \quad &. \end{aligned} \tag{17}$$

By integration we find that $= {}^{-1}$, $= {}^{-1}$ (to within unessential coefficients). Thus, we arrive at the transformation

$$= {}^{-1}, \quad y = {}^{-1} \quad , \qquad \text{where} \quad = (\ ). \tag{18}$$

Denoting the transformation (18) by , let us schematically represent its action on the parameters of the equation:

$$\{\ ,\ \ , 0\} \quad \longrightarrow \quad \{-\ -\quad -3,\quad , 0\} \qquad \textit{transformation} \quad . \tag{19}$$

Double application of the transformation yields the original equation.

3°. Let $l = 0$ and $= 2$. Then, $= 2$ and the splitting procedure for equation (16) yields the system of three equations:

$$\begin{aligned} -2(\quad)^2 &= 2 \quad {}^2\ {}^6\Theta \quad , \\ \Theta \quad -2 \quad \Theta &= \quad {}^2\ {}^5\Theta^2 \quad , \\ = \quad {}^2\ {}^5 \quad &. \end{aligned}$$

Its solution gives us the transformation

$$\begin{aligned} = {}^r, \quad y = \quad + \quad & \qquad \text{transformation of the variables,} \quad = (\ ); \\ \{\ , 2, 0\} \quad \longrightarrow \quad \{\ , 2, 0\} & \qquad \text{transformation of the vector of essential parameters;} \end{aligned}$$

where we use the notation:

$$= (8 \ {}^2 + 40 \quad + 49)^{-1/2}, \quad k = \frac{\ -1}{2}, \quad = \frac{1}{2}[\ (2 \quad +5) - 5], \quad s = -\ (\quad +2), \quad = \frac{(\quad +2)(\quad +3)}{\quad}.$$

Example 4. Likewise, for the class of equations

$$y_{xx} = (\ )\ (y) h(y_x)$$

we find two transformations:

$$\begin{aligned} &: \{\ ,\ \ , h\} \quad -- \quad \{\ ,\ \ , -(y_x)^3 h(1/y_x)\} && \text{transformation of the variables; see (14);} \\ &: \{\ , y \ \ , 1\} \quad \longrightarrow \quad \{\ {}^{-}\ {}^{-3}\ (\ {}^{-1}), y \ \ , 1\} && \text{transformation of the variables; see (18).} \end{aligned}$$

References for Subsection 0.6.4: V. F. Zaitsev and A. D. Polyanin (1994), A. D. Polyanin and V. F. Zaitsev (1995).

0.6.5. Discrete-Group Method. The Method of RF-Pairs

The direct method (see Subsection 0.6.4) is unsuitable for finding nonpoint transformations of second-order equations (i.e., transformations containing derivatives), since the determining equation cannot be split into equations forming an overdetermined system. Therefore, instead of searching for Bäcklund transformations in the form of arbitrary functions $= (\ ,\ ,\ ')$, $= g(\ ,\ ,\ ')$, one uses the superposition of some "standard" transformation containing the derivative and a point transformation which can be found by the direct method. The "standard" dependence on the derivative can be introduced by means of an RF-pair, which amounts to a transformation of successively increasing and decreasing the order of the equation (this transformation is not equivalent to the identity transformation). An additional point-transformation is necessary, since the equation obtained by an RF-pair is usually outside the original class.

1 . Suppose that any equation of the original class can be solved for the independent variable :

$$(\ , \ ', \ '' \ ) = \ .$$

Termwise differentiation of this equation with respect to yields the following autonomous equation:

$$\frac{}{} \ ' + \frac{}{\ '} \ '' + \frac{}{\ ''} \ ''' = 1,$$

whose order can be reduced with the substitution $' = z(\ )$. This pair of transformations is called a first RF-pair.

2 . Suppose that any equation of the original class can be solved for the dependent variable :

$$(\ , \ ', \ '' \ ) = \ .$$

Then, termwise differentiation of this equation with respect to brings us to the following equation which does not explicitly contain :

$$\frac{}{} + \frac{}{\ '} \ '' + \frac{}{\ ''} \ ''' = \ '.$$

The order of this equation can be reduced by means of the substitution $' = z(\ )$. This pair of transformations is called a second RF-pair.

Example 1. Consider transformations of the class of generalized Emden–Fowler equations:

$$y_{xx} = \ y \ (y_x) \ . \tag{1}$$

This class will be briefly denoted by the vector of essential parameters $\{ \ , \ , l\}$. Application of the first RF-pair transforms this equation to

$$= (l-1) \ ^{-1}(\ )^2 + \ y^{-1} \ + \ \frac{1}{y} \frac{}{} \ ^{\frac{- \ -1}{}}(\ )^{\frac{-1}{}}. \tag{2}$$

Now we have to find a point transformation that maps class (2) into class (1) (with another vector of parameters):

$$= \ (\ )^{\lambda}. \tag{3}$$

Note that in this case, the desired transformation does not map the given class into itself as in Subsection 0.6.4, but is a mapping of the equations classes (2) — (1). Nevertheless, the method for finding transformations

$$y = \ (\ , \ ), \quad = (\ , \ ) \quad (\ - \ \neq 0)$$

is completely the same and involves solving the determining equation:

$$\begin{aligned} (\ - \ ) \ (\ )^{\lambda} + (\ - \ )(\ )^3 + (\ - \ +2 \ -2 \ )(\ )^2 \\ + (\ - \ +2 \ -2 \ ) \ + \ - \ = \frac{l-1}{}(\ + \ )(\ + \ )^2 \\ + \frac{}{}(\ + \ )^2(\ + \ ) + \ \frac{1}{} \frac{}{} \ ^{\frac{- \ -1}{}}(\ + \ )^{\frac{2 \ +1}{}}(\ + \ )^{\frac{-1}{}}. \end{aligned} \tag{4}$$

Following the procedure set out in Subsection 0.6.4, we omit the general case $\neq 0$ and consider transformations for which at least one of the above partial derivatives is zero.

1°. *Case* $= 0, \ = 0$. Equation (4) has the form

$$\begin{aligned} (\ )^{\lambda} + \ (\ )^2 - \ = \frac{l-1}{} \ (\ )^2(\ )^2 \\ + \frac{}{}(\ )^2 \ + \ \frac{1}{} \frac{}{} \ ^{\frac{- \ -1}{}}(\ )^{\frac{2 \ +1}{}}(\ )^{\frac{-1}{}}(\ )^{\frac{-1}{}}. \end{aligned}$$

and for $\neq 0, -1, \ \neq 1, 2$ can easily be solved by splitting,

$$= \frac{1}{m+1}, \quad = \frac{1}{-2}.$$

As a result, using an RF-pair, we obtain:

$$= (\ \ )^{\frac{1}{}},\quad y = \frac{1}{+1},\quad y_x = \frac{1}{2-}\qquad \text{transformation of the variables},\quad = (\ );$$
$$\{\ ,\ \ ,l\} \longrightarrow \Big\{ -\frac{}{+1}, \frac{1}{l-2}, \frac{-1}{} \Big\}\qquad \text{transformation of the vector of essential parameters}. \tag{5}$$

2°. *Case* $= 0$, $= 0$. Similar calculations bring us to the formulas:

$$= (\ \ )^{-\frac{1}{}},\quad y = \frac{1}{+1},\quad y_x = \frac{1}{2-}\qquad \text{transformation of the variables},\quad = (\ );$$
$$\{\ ,\ \ ,l\} \longrightarrow \Big\{ \frac{1}{1-l}, -\frac{}{+1}, \frac{2\ \ +1}{} \Big\}\qquad \text{transformation of the vector of essential parameters}. \tag{6}$$

Transformation (6) can be obtained by successive application of transformation (5) and the hodograph transformation (see Subsection 0.6.4).

The inverse transformations have a similar structure. For instance, the inverse of transformation (5) can be written (after changing notation) as follows:

$$= \frac{1}{+1},\quad y = (\ \ )^{-\frac{1}{}},\quad y_x = \frac{1}{1-},\qquad \text{where}\quad = (\ ). \tag{7}$$

Denoting the transformation (7) by , let us schematically represent its action on the parameters of the equation:

$$\{\ ,\ \ ,l\} \longrightarrow \Big\{ \frac{1}{1-l}, -\frac{}{+1}, \frac{2\ \ +1}{} \Big\}\qquad \textit{transformation}\ . \tag{8}$$

Applying the transformation three times, we obtain the original equation.

It can be shown that all transformations which can be found from equation (4), without additional restrictions on the parameters of the original and the transformed equations, are obtained by superposition of the transformations and (see Subsection 0.6.4, Example 3), which form a group of order 6. The parameters of these equations are given in Figure 1.

Example 2. Suppose that $l = 0$ in equation (1). Then, on the class of Emden–Fowler equations

$$y_{xx} = \quad y \qquad \text{(briefly denoted by } \{\ ,\ \ ,0\}\text{)}, \tag{9}$$

one can define the transformation (see Subsection 0.6.4, Example 3). Therefore, in this case, the group considered in the previous example is prolonged to a group of order 12 (see Figure 2).

This prolongation takes place each time the third component of the parameter vector becomes equal to zero. This happens, for instance, if $= 1$ in equation (9). In this case, the order of the group is equal to 24 (see Figure 3).

Example 3. The class of second-order equations

$$y_{xx} = (\ )\ (y)h(y_x) \tag{10}$$

admits a discrete group of transformations similar to that for the generalized Emden–Fowler equation. Most simply, this group can be obtained by inverting the transformation (6). Thus, we seek the parameters of the transformation as functions of a single variable,

$$= \varphi(\ \ ),\quad y = (\ ),\quad y_x = \chi(\ ).$$

Introducing a point generator (see Subsection 0.6.4), we find a discrete group of transformations relating the equations shown in Figure 4. The functions $_1(\ _1)$, $_1(y_1)$, $h_1(y_{x_1})$ determine the original equation, while the corresponding functions for the transformed equations, $(\ \ )$, $(y\ \ )$, $h\ (y_x\ \ )$ with $k = 2, 3$, are determined by the parametric formulas:

$$\begin{aligned}
&_2(\ _2) = \ _1, && _2 = \frac{d\ _1}{h_1(\ _1)},\\
&_2(y_2) = \frac{1}{_1(\ _1)}, && y_2 = \int\ _1(\ _1)\,d\ ,\\
&h_2(\ _2) = -\frac{1}{[\ _1(y_1)]^3}\frac{d\ _1}{dy_1}, && _2 = \frac{1}{_1(y_1)}
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&_3(\ _3) = \frac{1}{_1(y_1)}, && _3 = \int\ _1(y_1)\,dy_1,\\
&_3(y_3) = \frac{1}{_1}, && y_3 = \int\frac{_1\,d\ _1}{h_1(\ _1)},\\
&h_3(\ _3) = \frac{d\ _1}{d\ _1}, && _3 = \ _1(\ _1),
\end{aligned} \tag{12}$$

where $= y_x$, $k = 1, 2, 3$.

The above example allows us to eliminate "singular points" of the group of transformations defined by (7) for $= -1$, $= -1$, $l = 1, 2$. For these values of the parameters, the form (10) and the transformations (11), (12) should be used.

References for Subsection 0.6.5: V. F. Zaitsev and A. D. Polyanin (1994), A. D. Polyanin and V. F. Zaitsev (1995).

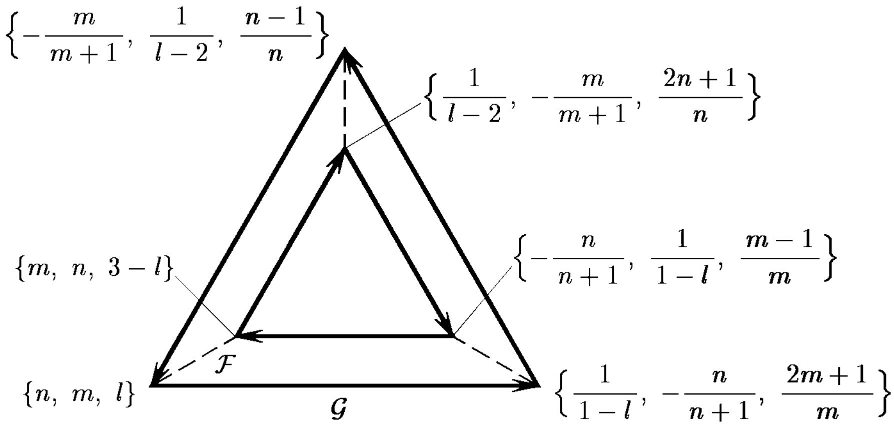

Figure 1

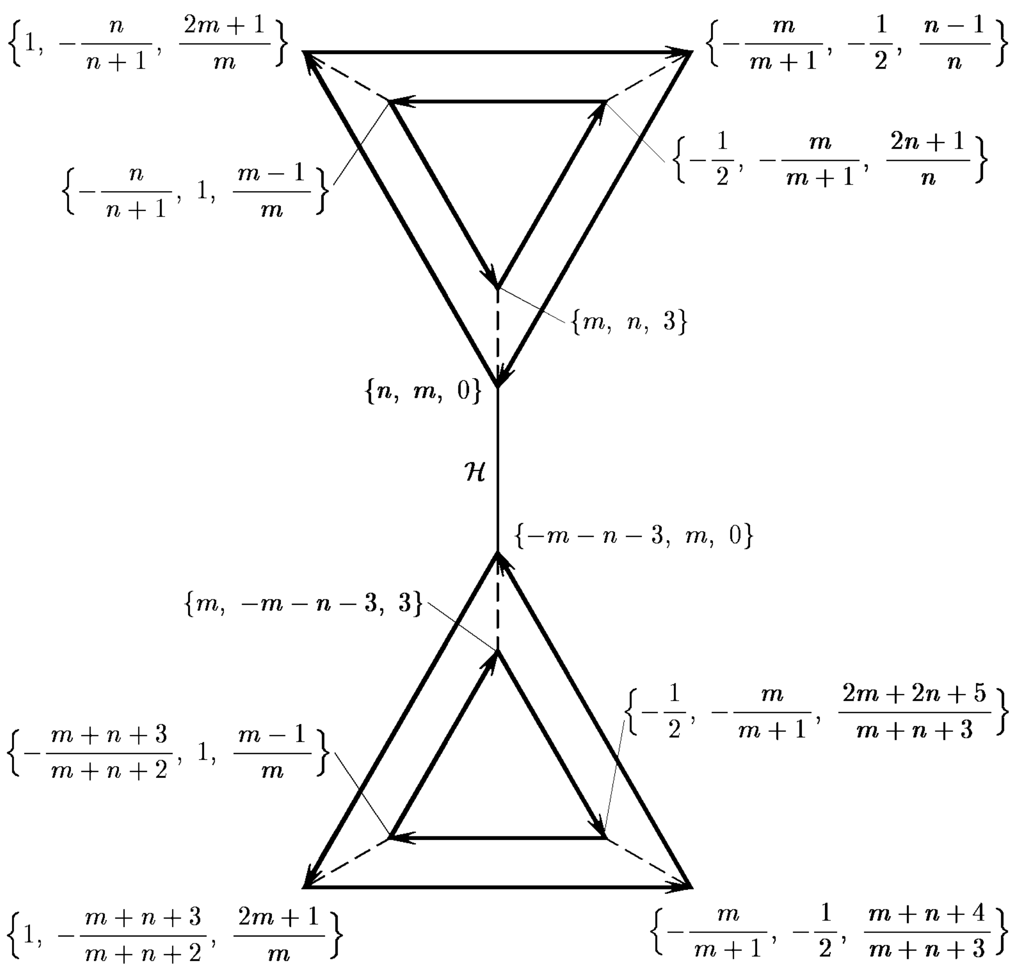

Figure 2

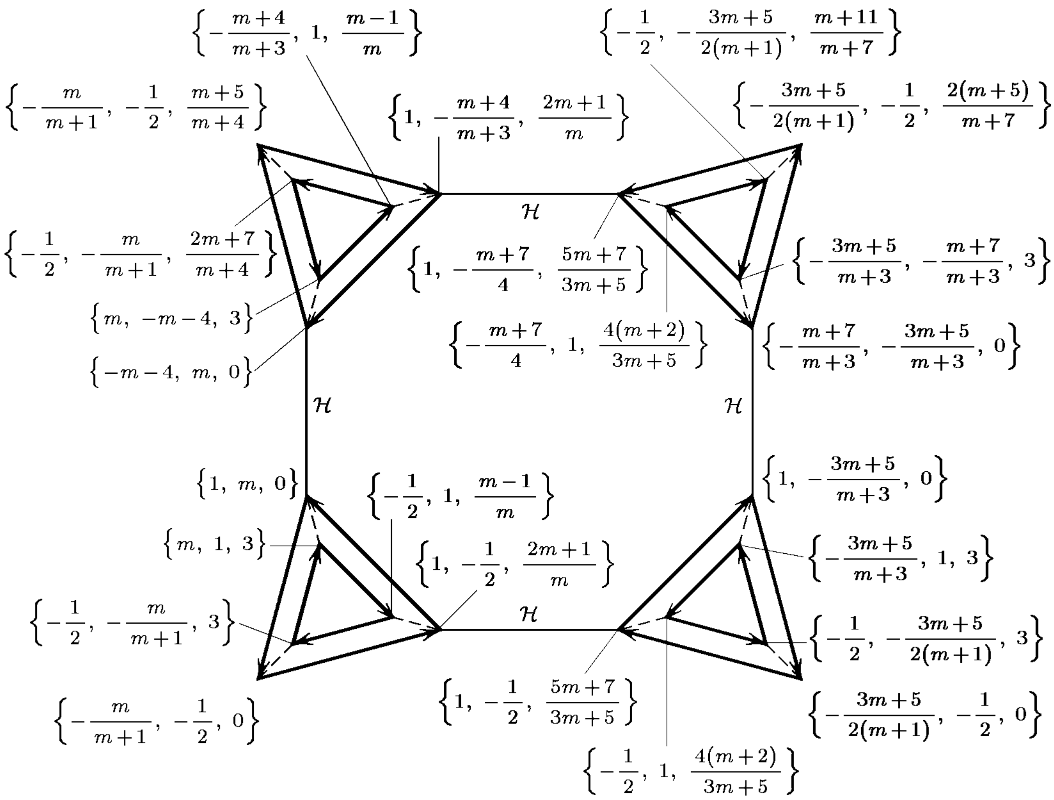

Figure 3

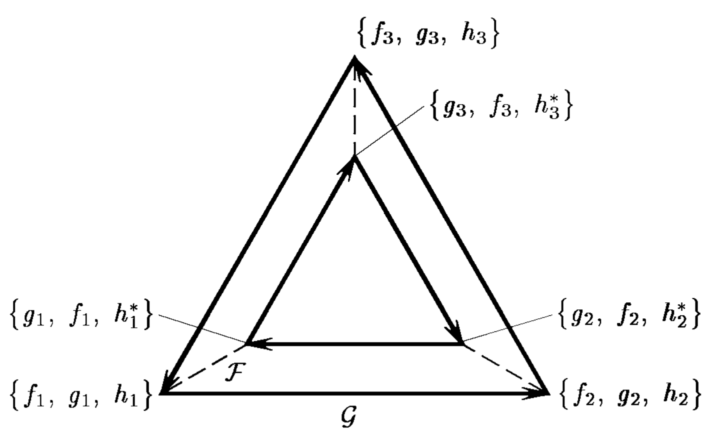

Figure 4

Chapter 1

First-Order Differential Equations

1.1. Simplest Equations with Arbitrary Functions Integrable in Closed Form*

1.1.1. Equations of the Form $= \ (\ )$

Solution:** $= \ (\ ) \ + \ .$

1.1.2. Equations of the Form $= \ (\ )$

Solution: $= \dfrac{}{(\ )} + \ .$

Particular solutions: $= A$, where A are roots of the algebraic (transcendental) equation $(A \ ) = 0.$

1.1.3. Separable Equations $= \ (\ )g(\ )$

Solution: $\dfrac{}{g(\ )} = \ (\ ) \ + \ .$

Particular solutions: $= A$, where A are roots of the algebraic (transcendental) equation $g(A \ ) = 0.$

The equation of the form $_1(\ )g_1(\ ) \ ' = \ _2(\ )g_2(\ )$ is reduced to the form 1.1.3 by dividing both sides by $_1g_1$.

1.1.4. Linear Equation $g(\ ) \ = \ _1(\ ) \ + \ _0(\ )$

Solution:

$$= e^F + e^F \quad e^{-F} \frac{_0(\ )}{g(\ )} \quad , \quad \text{where} \quad (\ ) = \frac{_1(\ )}{g(\ )} \quad .$$

1.1.5. Bernoulli Equation $g(\ ) \ = \ _1(\ ) \ + \ (\ )$

Here, is an arbitrary number. The substitution $(\ ) = \ ^{1-}$ leads to a linear equation: $' = (1- \ ) \ _1(\ ) \ + (1- \ ) \ (\ ).$

Solution:

$$^{1-} = e^F + (1- \ )e^F \quad e^{-F} \frac{(\ )}{g(\ )} \quad , \quad \text{where} \quad (\ ) = (1- \ ) \quad \frac{_1(\ )}{g(\ )} \quad .$$

---

* Special cases of equations 1.1.1–1.1.5 for specific functions , $_0$, $_1$, , and g are not discussed in this book; such cases can readily be recognized by the appearance of equations investigated, and the solution can be obtained using the general formulas given in Section 1.1.

** Hereinafter we shall often use the term “solution” to mean “general solution.”

1.1.6. Homogeneous Equation $= (\quad)$

The substitution $(\ ) =$ leads to a separable equation: $' = (\ ) - $.

Solution: $\dfrac{}{(\ ) -} = \ln|\ | +$.

Particular solutions: $= A$, where A are roots of the algebraic (transcendental) equation $A - (A\ ) = 0$.

1.2. Riccati Equation $(\ )y = f_2(\ )y^2 + f_1(\ )y + f_0(\ )$

1.2.1. Preliminary Remarks

For $_2 \equiv 0$, we obtain a linear equation (see Subsection 1.1.4); and for $_0 \equiv 0$, we have a Bernoulli equation (see Subsection 1.1.5 with $= 2$), whose solutions were given previously. Below we discuss equations with $_0\ _2\ \ 0$.

1 . Given a particular solution $_0 = {}_0(\ )$ of the Riccati equation, the general solution can be written as:

$$= {}_0(\ ) + (\ ) \quad - \quad (\ )\frac{_2(\ )}{g(\ )} \quad ^{-1}, \quad \text{where} \quad (\ ) = \exp \quad \left[2\ _2(\ )\ _0(\ ) + {}_1(\ )\right]\frac{}{g(\ )} \quad .$$

To the particular solution $_0(\ )$ there corresponds $=$.

Often only particular solutions will be given for the specific equations presented below in Subsections 1.2.2–1.2.8. The general solutions of these equations can be obtained by the above formulas.

2 . The substitution

$$(\ ) = \exp \ - \ \frac{_2}{g}$$

reduces the Riccati equation to a second-order linear equation:

$$_2 g^2\ '' + g\left[\ _2 g' - g(\ _2)' - \ _1\ _2\right]\ ' + \ _0\ _2^2 \ = 0.$$

The latter often may be easier to solve than the original Riccati equation. Specific second-order linear equations are outlined in Section 2.1.

1.2.2. Equations Containing Power Functions

1.2.2-1. Equations of the form $g(\ )\ ' = {}_2(\ )\ ^2 + {}_0(\ )$.

1. $= \ ^2 + \ \ +$.

For $= 0$, we have a separable equation of the form 1.1.2. For $\neq 0$, the substitution $= \ +$ leads to an equation of the form 1.2.2.4: $' = a\ ^2 +$.

2. $= \ ^2 - \ ^2\ ^2 + 3$.

Particular solution: $_0 = a\ - \ ^{-1}$.

3. $= \ ^2 + \ ^2 + \ +$.

This is a special case of equation 1.2.2.27 with $= 0$ and $\beta = 0$.

4. $= \ {}^2 + \ .$

Special Riccati equation, is an arbitrary number.

Solution: $= -\frac{1}{a}\frac{\ '}{\ }$, where $(\ ) = \ - \ _{1}\ _{\frac{1}{2}}\ \frac{1}{k}\ \overline{a}\ + \ _{2}\ _{\frac{1}{2}}\ \frac{1}{k}\ \overline{a}\ $, $k = \frac{1}{2}(\ +2)$; (z) and (z) are the Bessel functions, $\neq -2$. For the case $= -2$, see equation 1.2.2.13.

5. $= \ {}^2 + \ {}^{-1} - \ {}^2\ {}^2\ .$

Particular solution: $_0 = a\ .$

6. $= \ {}^2 + \ {}^2 + \ {}^{-1}.$

For the case $= -1$, see equation 1.2.2.13. For $\neq -1$, the transformation $\xi = \frac{1}{\ +1}\ {}^{+1}$, $= \ {}^{-}$ leads to an equation of the form 1.2.2.38: $\xi\ ' + a\xi\ {}^2 + \frac{\ }{\ +1} = \ \xi + \frac{\ }{\ +1}.$

7. $= \ {}^2 + \ {}^{-}\ {}^{-2}.$

Solution: $\overline{a}\ \ln\ = \ \frac{\ }{\ {}^2 + \beta\ +1} + \ $, where $= \sqrt{\frac{a}{\ }}\ {}^{+1}$, $\beta = \frac{\ +1}{\overline{a}}.$

8. $= \ {}^2 + \ .$

1 . For $\neq -1$, the substitution $\xi = \ {}^{+1}$ leads to a Riccati equation of the form 1.2.2.4: $' = \frac{a}{\ +1}\ {}^2 + \frac{\ }{\ +1}\xi^{\frac{-}{+1}}.$

2 . For $= -1$ and $\neq -1$, the transformation $= \ {}^{+1}$, $= -1$ leads to a Riccati equation of the form 1.2.2.4: $' = \frac{\ }{\ +1}\ {}^2 + \frac{a}{\ +1}\ {}^{-1}.$

3 . For $= \ = -1$, the original equation is a separable equation. In this case we have the solution: $\ln|\ | = \frac{\ }{a\ {}^2 + \ } + \ .$

9. $= \ {}^2 + (\ + \ )(\ + \ )^{-\ -4}.$

The transformation $\xi = \frac{a\ +\ }{\ +\ }$, $= \frac{1}{\Delta}[(\ + \ )^2\ + (\ + \ )]$, where $\Delta = a\ -\ $, leads to an equation of the form 1.2.2.4: $' = \ {}^2 + k\Delta^{-2}\xi\ .$

10. $= \ {}^2 + \ {}^{-1} - \ {}^2\ {}^{+2}\ .$

Particular solution: $_0 = \ .$

11. $= (\ {}^2 + \ {}^{-1})\ {}^2 + \ .$

The substitution $= -1$ leads to an equation of the form 1.2.2.6: $' = \ {}^2 + a\ {}^2 + \ {}^{-1}.$

12. $(\ _2\ + \ _2)(\ + \ {}^2) + \ _0\ + \ _0 = 0.$

The substitution $\lambda\ = \ '$ leads to a second-order linear equation of the form 2.1.2.108: $(a_2\ + \ _2)\ '' + \lambda(a_0\ + \ _0)\ = 0.$

13. $^2\ = \ {}^2\ {}^2 + \ .$

Solution: $= \frac{\lambda}{\ } - \ {}^{2\lambda}\ \frac{a}{2a\lambda+1}\ {}^{2\lambda} + \ {}^{-1}$, where λ is a root of the quadratic equation $a\lambda^2 + \lambda + \ = 0.$

14. $\quad {}^{2} \quad = \quad {}^{2}\,{}^{2} - \quad {}^{2}\,{}^{4} + \quad (1-2\quad)\;{}^{2} - \quad(\quad+1).$

Particular solution: $\;{}_{0} = a \quad + \quad {}^{-1}.$

15. $\quad {}^{2} \quad = \quad {}^{2}\,{}^{2} + \qquad + \quad .$

The substitution $\quad = \quad + A$, where A is a root of the quadratic equation $aA^{2} - A + \quad = 0$, leads to an equation of the form 1.2.2.35: $\quad {}' = a \quad {}^{2} + (1-2aA) \quad + \qquad .$

16. $\quad {}^{2} \quad = \quad {}^{2}\,{}^{2} + \quad {}^{2}\;(\qquad + \;)\; + \tfrac{1}{4}(1 - \quad {}^{2}).$

The transformation $\xi = \qquad + \;,\quad = \dfrac{1}{\quad}\;{}^{1-} \qquad + \dfrac{1-}{2}\;{}^{-}$ leads to an equation of the form 1.2.2.4: $\quad {}' = \quad {}^{2} + a(\quad)^{-2}\xi \quad .$

17. $(\;{}_{2}\;{}^{2} + \;{}_{2}\; + \;{}_{2})(\quad + \quad {}^{2}) + \quad {}_{0} = 0.$

The substitution $\lambda \quad = \quad {}'$ leads to a second-order linear equation of the form 2.1.2.179: $(\;{}_{2}\;{}^{2} + \;{}_{2}\; + a_{2})\;{}'' \; + \lambda a_{0} \quad = 0.$

18. $\quad {}^{4} \quad = - \quad {}^{4}\,{}^{2} - \quad {}^{2}.$

Solution: $\quad = \dfrac{1}{\quad} + \dfrac{a}{\quad {}^{2}} \tan \dfrac{a}{\quad} + \qquad .$

19. $\quad {}^{2}(\quad -1)^{2}(\quad + \quad {}^{2}) + \quad {}^{2} + \quad + s = 0.$

The substitution $\lambda \quad = \quad {}'$ leads to a second-order linear equation of the form 2.1.2.218: $a \quad {}^{2}(\quad -1)^{2}\;{}'' \; + \lambda(\quad {}^{2} + \quad + \;)\quad = 0.$

20. $(\quad {}^{2} + \quad + \;)^{2}(\quad + \quad {}^{2}) + \quad = 0.$

The substitution $\quad = \quad {}'$ leads to a second-order linear equation of the form 2.1.2.234: $(a \quad {}^{2} + \quad + \;)^{2}\;{}'' \; + A \quad = 0.$

21. $\quad {}^{+1} \quad = \quad {}^{2} \quad {}^{2} + \qquad + \quad .$

The substitution $\quad = \quad + A$, where A is a root of the quadratic equation $aA^{2} - \quad A + \quad = 0$, leads to an equation of the form 1.2.2.35: $\quad {}' = a \quad {}^{2} + (\quad - 2aA) \quad + \qquad .$

22. $(\quad + \;) \quad = \quad {}^{2} + \quad {}^{-2}.$

Particular solution: $\;{}_{0} = -1 \quad .$

23. $(\qquad + \qquad + \;)(\quad - \quad {}^{2}) + \quad (\quad -1) \quad {}^{-2} + \quad (\quad -1) \quad {}^{-2} = 0.$

Particular solution: $\;{}_{0} = -\dfrac{a \quad {}^{-1} + \quad {}^{-1}}{a \quad + \quad + \quad}.$

1.2.2-2. Other equations.

24. $\quad = \quad {}^{2} + \quad + \quad + \quad .$

The substitution $\quad = -\dfrac{\quad {}'}{a \quad}$ leads to a second-order linear equation of the form 2.1.2.12: $\quad {}'' \quad = \quad {}' - a(\quad + k) \quad .$

25. $\quad = \quad {}^{2} + \qquad + \quad {}^{-1}.$

Particular solution: $\;{}_{0} = -1 \quad .$

26. $= \;^{2} + \quad + \quad ^{-1}.$

The substitution $= - \;'$ leads to a second-order linear equation of the form 2.1.2.45: $'' - a \quad ' + \quad ^{-1} \quad = 0.$

27. $= \;^{2} + (\quad + \;) \; + \quad ^{2} + \quad + \; .$

The substitution $= - \;'$ leads to a second-order linear equation of the form 2.1.2.31: $'' - (\quad + \beta) \; ' + (a \;^{2} + \quad + \;) \; = 0.$

28. $= \;^{2} + \quad - \quad - \;^{2}.$

Particular solution: $_{0} = \; .$

29. $= -(\; + 1) \quad ^{2} + \quad ^{+ \quad +1} - \quad .$

Particular solution: $_{0} = \;^{- \; -1}.$

30. $= \quad ^{2} + \quad + \quad - \;^{2} \; .$

Particular solution: $_{0} = - \; .$

31. $= \quad ^{2} - \quad (\quad + \;) \; + \quad ^{-1}.$

Particular solution: $_{0} = \quad + \; .$

32. $= - \quad ^{-1} \;^{2} + \quad (\quad + \;) \; - \quad .$

Particular solution: $_{0} = (a \quad + \;)^{-1}.$

33. $= \quad ^{2} + \quad + \quad ^{k-1} - \quad ^{+k} - \;^{2} \quad ^{+2k}.$

Particular solution: $_{0} = \quad .$

34. $= \;^{2} + \quad + \;^{2} \; .$

The transformation $= \;^{b}, \quad = \;^{-b}$ leads to a separable equation: $' = a \;^{2} + \; .$

35. $= \;^{2} + \quad + \quad .$

The transformation $\xi = \;^{b}, \quad = \quad ^{-b}$ leads to the special Riccati equation of the form 1.2.2.4: $' = \frac{a}{} \;^{2} + - \xi \;$, where $= - - 2.$

36. $= \;^{2} + (\; + \quad) \; + \;^{2} \; .$

The substitution $=$ leads to a separable equation: $' = \quad ^{-1}(a \;^{2} + \quad + \;).$

37. $= \;^{2} + \quad + \quad .$

The substitution $= - \;'$ leads to a second-order linear equation of the form 2.1.2.67: $'' - a \; ' + \quad = 0.$

38. $+ \;_{3} \;^{2} + \;_{2} + \;_{1} + \;_{0} = 0.$

The substitution $a_3 \quad = \;'$ leads to a second-order linear equation of the form 2.1.2.64: $'' + a_2 \; ' + a_3(a_1 \; + a_0) \; = 0.$

39. $= \quad ^{2} + \quad + \;^{-} \; .$

The substitution $=$ leads to a separable equation: $' = a \;^{2} + (\; + \;) \; + \; .$

40. $= \quad ^{2} + \quad - \;^{2} \quad ^{+2} \; .$

Particular solution: $_{0} = \quad .$

41. $= {}^2 \quad {}^2 + (\quad - \quad) \quad + \quad {}^2 \quad .$

Solution: $= \quad {}^{-} \tan \dfrac{+}{+} + \quad .$

42. $= \quad {}^2 + \quad + \quad .$

The transformation $\xi = \quad {}^{-b}, \quad = \quad {}^{b}$ leads to a special Riccati equation of the form 1.2.2.4: $(\quad + \quad) \quad ' = a \quad {}^2 + \quad \xi \quad$, where $k = \dfrac{- \quad -2}{+}$.

43. $= \quad {}^2 \quad {}^2 + (\quad - \quad) \quad + \quad .$

For $\quad = 0$, this is a separable equation. For $\quad \neq 0$, the solution is:

$$\frac{\quad}{a \quad {}^2 + \quad + \quad} = \quad + \quad , \qquad \text{where} \quad = \quad .$$

44. $= \quad {}^{2} \quad {}^{+} \quad {}^2 + (\quad {}^{+} \quad - \quad) \quad + \quad .$

The substitution $\quad = \quad$ leads to a separable equation: $\quad ' = \quad {}^{+} \quad {}^{-1}(a \quad {}^2 + \quad + \quad).$

45. $(\quad _2 \quad + \quad _2)(\quad + \quad {}^2) + (\quad _1 \quad + \quad _1) \quad + \quad _0 \quad + \quad _0 = 0.$

The substitution $\lambda \quad = \quad '$ leads to a second-order linear equation of the form 2.1.2.108: $(a_2 \quad + \quad _2) \quad '' + (a_1 \quad + \quad _1) \quad ' + \lambda(a_0 \quad + \quad _0) \quad = 0.$

46. $(\quad + \quad) \quad = (\quad + \quad)^2 + \quad (\quad + \quad) - \quad + \gamma.$

The substitution $\quad = a \quad +$ leads to a first-order linear equation with respect to $\quad = \quad (\quad)$: $(\quad a \quad {}^2 + \beta a \quad + \quad a + \quad) \quad ' = a \quad + \quad .$

47. $2 \quad {}^2 \quad = 2 \quad {}^2 + \quad - 2 \quad {}^2 \quad .$

Particular solution: $\quad _0 = a \quad \overline{\quad}.$

48. $2 \quad {}^2 \quad = 2 \quad {}^2 + 3 \quad - 2 \quad {}^2 \quad .$

Particular solution: $\quad _0 = a \quad \overline{\quad} - \tfrac{1}{2} \quad .$

49. $\quad {}^2 \quad = \quad {}^2 \quad {}^2 + \quad + \quad .$

The substitution $\quad = \quad$ leads to a separable equation: $\quad ' = a \quad {}^2 + (\quad + 1) \quad + \quad .$

50. $\quad {}^2 \quad = \quad {}^2 \quad {}^2 + (\quad {}^2 + \quad) \quad + \quad {}^2 + \quad + \gamma.$

The substitution $\quad = - \quad '$ leads to a second-order linear equation of the form 2.1.2.139: $\quad {}^2 \quad '' - (a \quad + \quad) \quad ' + (\quad {}^2 + \beta \quad + \quad) \quad = 0.$

51. $\quad {}^2 \quad = \quad {}^2 \quad {}^2 + \quad + \quad + s.$

The substitution $a \quad = - \quad '$ leads to a second-order linear equation of the form 2.1.2.132: $\quad {}^2 \quad '' - \quad ' + a(\quad + \quad) \quad = 0.$

52. $\quad {}^2 \quad = \quad {}^2 \quad {}^2 + \quad + \quad {}^2 \quad + s \quad .$

The substitution $a \quad = - \quad '$ leads to a second-order linear equation of the form 2.1.2.133: $\quad {}^2 \quad '' - \quad ' + a \quad (\quad + \quad) \quad = 0.$

53. $\quad {}^2 \quad = \quad {}^2 \quad {}^2 + (\quad + \quad) \quad + \quad {}^2 + \quad + \gamma.$

The substitution $\quad = - \quad '$ leads to a second-order linear equation of the form 2.1.2.146: $\quad {}^2 \quad '' - (a \quad + \quad) \quad ' + (\quad {}^2 + \beta \quad + \quad) \quad = 0.$

54. $\quad ^2 \quad = (\quad ^2 + \quad + \gamma) \quad ^2 + (\quad + \quad) \quad + \quad ^2.$

The substitution $\quad = -1 \quad$ leads to an equation of the form 1.2.2.53: $\quad ^2 \quad ' = \quad ^2 \quad ^2 - (a \quad + \quad) \quad + \quad ^2 + \beta \quad + \quad .$

55. $(\quad ^2 - 1) \quad + \quad (\quad ^2 - 2 \quad + 1) = 0.$

The substitution $\quad = \dfrac{2\lambda - 1}{\lambda} \quad + \dfrac{1 - \lambda}{\lambda} \dfrac{1}{(\ )}$ leads to an equation of the same form:

$$(\quad ^2 - 1) \quad ' + (\lambda - 1)(\quad ^2 - 2 \quad + 1) = 0.$$

If $\lambda = \quad$ is a positive integer, then by using the above substitution, the original equation can be reduced to an equation of the same form in which $\lambda = 1$, i.e., to an equation of the form 1.2.2.58 with $a = 1$, $\quad = -1$.

56. $(\quad ^2 + \quad) \quad + \quad ^2 + \quad + \text{—}(\quad + \quad) = 0.$

Particular solution: $\quad _0 = -\dfrac{a + \beta}{\quad} \quad .$

57. $(\quad ^2 + \quad) \quad + \quad ^2 + \quad + \gamma = 0.$

The substitution $\quad = -\dfrac{a + \beta}{\quad} \quad - \dfrac{1}{(\ )}$ leads to an equation of the same form: $(a \quad ^2 + \quad) \quad ' + \quad - \dfrac{a + \beta}{\quad} \quad ^2 + (2a + \beta) \quad + \quad = 0.$

58. $(\quad ^2 + \quad) \quad + \quad ^2 - 2 \quad + (1 - \quad) \quad ^2 - \quad = 0.$

Solution: $\quad = \quad + \quad \dfrac{\quad}{a \quad ^2 + \quad} + \quad ^{-1} \quad .$

59. $(\quad ^2 + \quad + \quad) \quad = \quad ^2 + (2 \quad + \quad) \quad + \quad (\quad - \quad) \quad ^2 + \quad .$

Particular solutions: $\quad _0 = -\lambda \quad + A$, where $A = \tfrac{1}{2}\bigl(- \quad \sqrt{\quad ^2 - 4 \quad - 4\lambda \quad}\bigr.$.

60. $(\quad ^2 + \quad + \quad) \quad = \quad ^2 + (\quad + \quad) \quad - \quad ^2 \quad ^2 + \quad (\quad - \quad) \quad + \quad .$

Particular solution: $\quad _0 = \lambda \quad .$

61. $(\quad _2 \quad ^2 + \quad _2 \quad + \quad _2) \quad = \quad ^2 + (\quad _1 \quad + \quad _1) \quad - \quad (\quad + \quad _1 - \quad _2) \quad ^2 + \quad (\quad _2 - \quad _1) \quad + \quad _2.$

Particular solution: $\quad _0 = \lambda \quad .$

62. $(\quad _2 \quad ^2 + \quad _2 \quad + \quad _2) \quad = \quad ^2 + (\quad _1 \quad + \quad _1) \quad + \quad _0 \quad ^2 + \quad _0 \quad + \quad _0.$

Let λ and β be roots of the system of the quadratic equations

$$\lambda^2 + \lambda(a_1 - a_2) + a_0 = 0, \qquad \beta^2 + \beta \quad _1 + \quad _0 - \lambda \quad _2 = 0,$$

where the first equation is solved independently (in the general case there are four roots). If some roots satisfy the condition $2\lambda\beta + \lambda \quad _1 + \beta a_1 + \quad _0 - \lambda \quad _2 = 0$, the original equation possesses a particular solution: $\quad _0 = \lambda \quad + \beta.$

63. $\mathbf{(-)(-) \quad + {}^2 + \ (+ -)(+ -) = 0.}$

To the case $k=0$ there corresponds a separable equation. To $k=-1$ there corresponds a linear equation. For $k \neq -1$ and $k \neq 0$, with the aid of the substitution $k \ (\ ) = \ + k(\ + \ )$, we obtain the general solution:

$$\frac{\ + k(\ + \ - a)}{\ + k(\ + \ - \ )} \quad \frac{\ - a}{\ -} \quad = \qquad \text{if } a \neq \ ,$$

$$\frac{1}{\ + k(\ + \ - a)} + \frac{1}{\ - a} = \qquad \text{if } a = \ .$$

64. $\mathbf{({}_2 \ {}^2 + {}_2 \ + {}_2)(\ + \ {}^2) + ({}_1 \ + {}_1) \ + {}_0 = 0.}$

The substitution $\lambda \ = \ '$ leads to a second-order linear equation of the form 2.1.2.179: $({}_2 \ {}^2 + {}_2 \ + a_2) \ '' \ + ({}_1 \ + a_1) \ ' + \lambda a_0 \ = 0.$

65. $\mathbf{{}^3 \ = \ {}^3 \ {}^2 + (\ {}^2 + \ ) \ + s \ .}$

The substitution $a \ = - \ '$ leads to a second-order linear equation of the form 2.1.2.183: ${}^3 \ '' \ - (\ {}^2 + \ ) \ ' + a \ = 0.$

66. $\mathbf{{}^3 \ = \ {}^3 \ {}^2 + \ (\ + \ ) \ + \ + \ .}$

The substitution $a \ = - \ '$ leads to a second-order linear equation of the form 2.1.2.186: ${}^3 \ '' \ - \ (\ + \ ) \ ' + a(\ + \beta) \ = 0.$

67. $\mathbf{({}^2 + \ )(\ + \ {}^2) + (\ {}^2 + \ ) \ + s \ = 0.}$

The substitution $\lambda \ = \ '$ leads to a second-order linear equation of the form 2.1.2.190: $(\ {}^2 + a) \ '' \ + (\ {}^2 + \ ) \ ' + \lambda \ = 0.$

68. $\mathbf{{}^2(\ + \ )(\ + \ {}^2) + \ (\ + \ ) \ + \ + \ = 0.}$

The substitution $\lambda \ = \ '$ leads to a second-order linear equation of the form 2.1.2.194: ${}^2(\ + a) \ '' \ + \ (\ + \ ) \ ' + \lambda(\ + \beta) \ = 0.$

69. $\mathbf{(\ {}^2 + \ + \ )(\ - \ ) - \ {}^2 + \ {}^2 = 0.}$

Solution: $\ln \dfrac{\ -}{\ +} \ = \ + 2 \ \dfrac{}{a \ {}^2 + \ + \ }.$

70. $\mathbf{{}^2(\ {}^2 + \ )(\ + \ {}^2) + \ (\ {}^2 + \ ) \ + s = 0.}$

The substitution $\lambda \ = \ '$ leads to a second-order linear equation of the form 2.1.2.219: ${}^2(\ {}^2 + a) \ '' \ + \ (\ {}^2 + \ ) \ ' + \lambda \ = 0.$

71. $\mathbf{(\ {}^2 - 1)^2(\ + \ {}^2) + \ (\ {}^2 - 1) \ + \ {}^2 + \ + s = 0.}$

The substitution $\lambda \ = \ '$ leads to a second-order linear equation of the form 2.1.2.227: $a(\ {}^2 - 1)^2 \ '' \ + \ (\ {}^2 - 1) \ ' + \lambda(\ {}^2 + \ + \ ) \ = 0.$

72. $\mathbf{{}^{+1} \ = \ {}^2 \ {}^2 + \ + \ + \ .}$

The substitution $\ = \ + A$, where A is a root of the quadratic equation $aA^2 - (\ + \ )A + \ = 0$, leads to an equation of the form 1.2.2.35: $\ ' = a \ {}^2 + (\ + \ - 2aA) \ + \ .$

73. $\mathbf{(\ {}^k + \ ) \ = \ {}^2 + (\ - \ {}^k) \ + \gamma \ {}^- \ .}$

The transformation $\ = \ , z = \ {}^-$ leads to a separable equation: $[\ {}^2 + (\beta + \ ) \ + \ ]z' = -k(\ z + a).$

74. $^2(\quad -1)(\quad + \quad ^2) + (p \quad + q) \quad + \quad + s = 0.$

The substitution $\lambda \quad = \quad '$ leads to a second-order linear equation of the form 2.1.2.254: $^2(a \quad -1) \quad '' \; + (\quad + \;) \quad ' + \lambda(\quad + \;) \quad = 0.$

75. $(\quad + \quad + \;) \quad = \quad ^2 - \quad ^{-1} \; + \quad ^{-2}.$

Particular solution: $_0 = -1 \quad .$

76. $(\quad + \quad + \;) \quad = \quad ^{-2} \; ^2 + \quad ^{-1} \; + \; .$

Particular solution: $_0 = \quad .$

77. $(\quad + \quad + \;) \quad = \quad ^k \; ^2 + \quad ^s \; - \quad ^2 \; ^k + \quad ^s.$

Particular solution: $_0 = -\lambda.$

78. $(\quad + \quad + \;)(\quad - \;) + s \quad ^k(\; ^2 - \quad ^2) = 0.$

Particular solutions: $_0 = \quad \overline{\lambda}.$

1.2.3. Equations Containing Exponential Functions

1.2.3-1. Equations with exponential functions.

1. $\quad = \quad ^2 + \quad ^\lambda \; .$

The substitution $\quad = e^\lambda$ leads to an equation of the form 1.2.2.35: $\lambda \quad ' = a \quad ^2 + \quad .$

2. $\quad = \quad ^2 + \quad ^\lambda \; - \; ^2 \; ^{2\lambda} \; .$

Particular solution: $_0 = ae^\lambda \; .$

3. $\quad = \quad ^2 + \quad + \quad ^\lambda + \quad ^{2\lambda} \; .$

The substitution $\sigma \quad = - \quad '$ leads to a second-order linear equation of the form 2.1.3.5: $'' \; + \sigma(a + \; e^\lambda \; + \; e^{2\lambda} \;) \quad = 0.$

4. $\quad = \quad ^2 + \quad + \quad + \; .$

The substitution $\sigma \quad = - \quad '$ leads to a second-order linear equation of the form 2.1.3.10: $'' \; - a \quad ' + \sigma(\; e \; + \;) \quad = 0.$

5. $\quad = \quad ^2 + \quad + \; (\; - \;) \; ^\lambda \; - \; ^2 \; ^{2\lambda} \; .$

Particular solution: $_0 = ae^\lambda \; .$

6. $\quad = \quad ^2 + \quad ^\lambda \; - \quad ^\lambda \; - \; ^2.$

Particular solution: $_0 = \quad .$

7. $\quad = \quad ^2 + \quad ^{2\lambda} \; (\; ^\lambda \; + \;) \; - \frac{1}{4} \; ^2.$

The transformation $\xi = e^\lambda \; + \; , \quad = \dfrac{1}{\lambda} \; e^{-\lambda} \quad - \dfrac{\lambda}{2} e^{-\lambda}$ leads to an equation of the form 1.2.2.4: $' = \quad ^2 + a\lambda^{-2}\xi \; .$

8. $\quad = \quad ^2 + \quad ^{8\lambda} \; + \quad ^{6\lambda} \; + \quad ^{4\lambda} \; - \; ^2.$

The transformation $\xi = e^{2\lambda} \; , \quad = e^{-2\lambda} \quad \dfrac{}{2\lambda} - \dfrac{1}{2}$ leads to an equation of the form 1.2.2.3: $' = \quad ^2 + (2\lambda)^{-2}(a\xi^2 + \; \xi + \;).$

9. $= {}^{k}\ {}^{2} + {}^{s},\quad \neq 0.$

The substitution $= e$ leads to an equation of the form 1.2.2.4: $k\ ' = a\ {}^{2} + \ {}^{-}$.

10. $= {}^{2} + {}^{\lambda} - {}^{2}\ {}^{(\ +2\lambda)}$.

Particular solution: ${}_{0} = ae^{\lambda}$.

11. $= {}^{\lambda}\ {}^{2} + \ + {}^{-\lambda}$.

The substitution $z = e^{\lambda}$ leads to a separable equation: $z' = az^{2} + (\ + \lambda)z +$.

12. $= {}^{2} + \ - {}^{2}\ {}^{(\ +2\lambda)}$.

Particular solution: ${}_{0} = \ e^{\lambda}$.

13. $= {}^{\lambda}\ {}^{2} + \ + {}^{(\ -\lambda)}$.

Particular solution: ${}_{0} = -\lambda e^{-\lambda}$.

14. $= - {}^{\lambda}\ {}^{2} + \ - {}^{(\ -\lambda)}$.

Particular solution: ${}_{0} = e^{-\lambda}$.

15. $= {}^{2} + {}^{(\lambda+\ )} - {}^{\lambda}$.

Particular solution: ${}_{0} = -\ e^{\lambda}$.

16. $= {}^{k}\ {}^{2} + \ + {}^{s} + {}^{-k}$.

The substitution $= e$ leads to an equation of the form 1.2.2.51: $k\ {}^{2}\ ' = a\ {}^{2}\ {}^{2} + \ + {}^{(\ +\ )} +$.

17. $= {}^{(2\lambda+\ )}\ {}^{2} + [\ {}^{(\lambda+\ )} - \ ] +$.

The substitution $= e^{\lambda}$ leads to a separable equation: $' = e^{(\lambda+\ )}(a\ {}^{2} + \ + \ )$.

18. $= {}^{k}\ {}^{2} + \ + {}^{k} + {}^{k(2\ +1)}$.

The substitution $= e$ leads to an equation of the form 1.2.2.52: $k\ {}^{2}\ ' = a\ {}^{2}\ {}^{2} + \ + {}^{+1} + {}^{2(\ +1)}$.

19. $= (\ - {}^{\lambda})^{2} + {}^{\lambda}$.

Particular solution: ${}_{0} = \ e^{\lambda}$.

20. $(\ {}^{\lambda} + \ + \ ) = {}^{2} + {}^{\nu} - {}^{2} + {}^{\nu}.$

Particular solution: ${}_{0} = -$.

21. $(\ {}^{\lambda} + \ + \ )(\ - {}^{2}) + {}^{2}\ {}^{\lambda} + {}^{2} = 0.$

Particular solution: ${}_{0} = -\dfrac{a\lambda e^{\lambda} + \ e}{ae^{\lambda} + e + }$.

$\boxed{\text{1.2.3-2. Equations with power and exponential functions.}}$

22. $= {}^{2} + {}^{\lambda} + {}^{\lambda}$.

Particular solution: ${}_{0} = -1$.

23. $= \lambda \; ^2 + \; ^{-\lambda}$.

Solution: $\dfrac{z}{az^2 + \lambda z +} = \; + \;$, where $z = e^{\lambda}$.

24. $= \lambda \; ^2 + \; ^{-1} - \; ^2 \; \lambda \; ^2$.

Particular solution: $_0 =$.

25. $= \lambda \; ^2 + \; + \; ^{-\lambda}$.

Particular solution: $_0 = -\lambda e^{-\lambda}$.

26. $= - \; \lambda \; ^2 + \; \lambda \; -$.

Particular solution: $_0 = e^{-\lambda}$.

27. $= \lambda \; ^2 - \; \lambda \; + \; ^{-1}$.

Particular solution: $_0 =$.

28. $= \; ^2 + \; \lambda \; - \; ^2 \; 2\lambda$.

Particular solution: $_0 = \; e^{\lambda}$.

29. $= \; ^2 + \; - \; ^2 \; 2\lambda$.

Particular solution: $_0 = \; e^{\lambda}$.

30. $= \; ^2 - \; \lambda \; + \; \lambda$.

Particular solution: $_0 = \; e^{\lambda}$.

31. $= -(\; + 1) \; ^k \; ^2 + \; ^{k+1} \; \lambda \; - \; \lambda$.

Particular solution: $_0 = \; ^{- \; -1}$.

32. $= \; ^2 - \; (\; \lambda \; + \;) \; + \; \lambda$.

Particular solution: $_0 = \; e^{\lambda} \; + \;$.

33. $= \; ^{2\lambda} \; ^2 + (\; \lambda \; - \;) \; +$.

The substitution $= e^{\lambda}$ leads to a separable equation: $' = \; e^{\lambda} \; (a \; ^2 + \; + \;)$.

34. $= \lambda \; (\; - \; - \;)^2 + \; ^{-1}$.

Particular solution: $_0 = \; + \;$.

35. $= \lambda \; ^2 + \; + \; ^2 \; ^{2k} \; \lambda$ **.**

Solution: $= \; \tan \; a \; ^{-1} e^{\lambda} \; + \;$.

36. $= \; ^2 \; \lambda \; ^2 + (\; \lambda \; - \;) \; + \; \lambda$ **.**

Solution: $\dfrac{}{a \; ^2 + \; +} = \; ^{-1} e^{\lambda} \; + \;$, where $=$.

37. $= \; ^2 + 2 \; \lambda \; ^2 - \; ^2 \; 2\lambda \; ^2$.

Particular solution: $_0 = ae^{\lambda \; ^2}$.

38. $= \;\; {}^{-\lambda\;\;2\;\;2} + \quad + \quad {}^{2}.$

Solution: $= e^{\lambda\;\;2\;2} \tan\; a \quad e^{-\lambda\;\;2\;2} \quad + \quad .$

39. $= \quad {}^{2} + \quad + \quad {}^{2} \quad \lambda\;{}^{2}.$

Solution: $= e^{\lambda\;\;2\;2} \tan\; a \quad e^{\lambda\;\;2\;2} \quad + \quad .$

40. ${}^{4}(\quad - \quad {}^{2}) = \quad + \quad \exp(\quad) + \quad \exp(2 \quad).$

The transformations $\xi = 1 \quad , \quad = - \quad {}^{2} \quad -$ leads to a Riccati equation of the form 1.2.3.3: ${}' = \quad {}^{2} + a + \quad e \quad + \quad e^{2} \quad .$

1.2.4. Equations Containing Hyperbolic Functions

1.2.4-1. Equations with hyperbolic sine and cosine.

1. $= \quad {}^{2} - \quad {}^{2} + \quad \sinh(\quad) - \quad {}^{2} \sinh^{2}(\quad).$

Particular solution: ${}_{0} = a \cosh(\lambda \quad).$

2. $= \quad {}^{2} + \quad \sinh(\quad) \quad + \quad \sinh(\quad) - \quad {}^{2}.$

Particular solution: ${}_{0} = - \quad .$

3. $= \quad {}^{2} + \quad \sinh \quad (\quad) \quad + \quad \sinh \quad (\quad).$

Particular solution: ${}_{0} = -1 \quad .$

4. $= \quad \sinh(\quad) \quad {}^{2} - \quad \sinh^{3}(\quad).$

Particular solution: ${}_{0} = \cosh(\lambda \quad).$

5. $= [\quad \sinh^{2}(\quad) - \quad] \quad {}^{2} - \quad \sinh^{2}(\quad) + \quad - \quad .$

Particular solution: ${}_{0} = \coth(\lambda \quad).$

6. $[\quad \sinh(\quad) + \quad] \quad = \quad {}^{2} + \quad \sinh(\quad) \quad - \quad {}^{2} + \quad \sinh(\quad).$

Particular solution: ${}_{0} = - \quad .$

7. $[\quad \sinh(\quad) + \quad](\quad - \quad {}^{2}) + \quad {}^{2} \sinh(\quad) = 0.$

Particular solution: ${}_{0} = -\dfrac{a\lambda \cosh(\lambda \quad)}{a \sinh(\lambda \quad) + \quad }.$

8. $= \quad {}^{2} + \quad + \gamma \cosh \quad .$

The transformation $\quad = 2 \; , \quad = - \quad {}'$ leads to the modified Mathieu equation 2.1.4.9: ${}'' - (a - 2 \quad \cosh 2 \;) \quad = 0$, where $a = -4 \quad \beta, \quad = 2 \quad .$

9. $= \quad {}^{2} + \quad \cosh(\quad) \quad + \quad \cosh(\quad) - \quad {}^{2}.$

Particular solution: ${}_{0} = - \quad .$

10. $= \quad {}^{2} + \quad \cosh \quad (\quad) \quad + \quad \cosh \quad (\quad).$

Particular solution: ${}_{0} = -1 \quad .$

11. $= [\quad \cosh^{2}(\quad) - \quad] \quad {}^{2} + \quad + \quad - \quad \cosh^{2}(\quad).$

Particular solution: ${}_{0} = \tanh(\lambda \quad).$

12. $\mathbf{2 \quad = [\ - \ + \ \cosh(\ )]\ ^2 + \ + \ - \ \cosh(\ ).}$

Particular solution: $_0 = \tanh\big(\tfrac{1}{2}\lambda \quad .$

13. $\mathbf{= \ ^2 - \ ^2 + \ \cosh\ (\ )\sinh^{- \ -4}(\ ).}$

The transformation $\xi = \coth(\lambda \ )$, $\ = -\dfrac{1}{\lambda} \sinh^2(\lambda \ ) \ - \sinh(\lambda \ )\cosh(\lambda \ )$ leads to an equation of the form 1.2.2.4: $\ ' = \ ^2 + \lambda^{-2}\xi \ .$

14. $\mathbf{= \ \sinh(\ )\ ^2 + \ \sinh(\ )\cosh\ (\ ).}$

The transformation $\xi = \cosh(\lambda \ )$, $\ = \dfrac{a}{\lambda}$ leads to an equation of the form 1.2.2.4: $\ ' = \ ^2 + a \ \lambda^{-2}\xi \ .$

15. $\mathbf{= \ \cosh(\ )\ ^2 + \ \cosh(\ )\sinh\ (\ ).}$

The transformation $\xi = \sinh(\lambda \ )$, $\ = \dfrac{a}{\lambda}$ leads to an equation of the form 1.2.2.4: $\ ' = \ ^2 + a \ \lambda^{-2}\xi \ .$

16. $\mathbf{[\ \cosh(\ ) + \ ] \ = \ ^2 + \ \cosh(\ ) \ - \ ^2 + \ \cosh(\ ).}$

Particular solution: $_0 = - \ .$

17. $\mathbf{[\ \cosh(\ ) + \ ](\ - \ ^2) + \ ^2\cosh(\ ) = 0.}$

Particular solution: $_0 = -\dfrac{a\lambda \sinh(\lambda \ )}{a\cosh(\lambda \ ) + \ }.$

1.2.4-2. Equations with hyperbolic tangent and cotangent.

18. $\mathbf{= \ ^2 + \ - \ (\ + \ )\tanh^2(\ ).}$

Particular solution: $_0 = a\tanh(\lambda \ ).$

19. $\mathbf{= \ ^2 + 3 \ - \ ^2 - \ (\ + \ )\tanh^2(\ ).}$

Particular solution: $_0 = a\tanh(\lambda \ ) - \lambda\coth(\lambda \ ).$

20. $\mathbf{= \ ^2 + \ \tanh\ (\ ) \ + \ \tanh\ (\ ).}$

Particular solution: $_0 = -1 \ .$

21. $\mathbf{[\ \tanh(\ ) + \ ] \ = \ ^2 + \ \tanh(\ ) \ - \ ^2 + \ \tanh(\ ).}$

Particular solution: $_0 = - \ .$

22. $\mathbf{= \ ^2 + \ - \ (\ + \ )\coth^2(\ ).}$

Particular solution: $_0 = a\coth(\lambda \ ).$

23. $\mathbf{= \ ^2 - \ ^2 + 3 \ - \ (\ + \ )\coth^2(\ ).}$

Particular solution: $_0 = a\coth(\lambda \ ) - \lambda\tanh(\lambda \ ).$

24. $\mathbf{= \ ^2 + \ \coth\ (\ ) \ + \ \coth\ (\ ).}$

Particular solution: $_0 = -1 \ .$

25. **[coth( ) +] = ^{2} + coth( ) − ^{2} + coth( ).**

Particular solution: $_0$ = − .

26. **= ^{2} − 2 ^{2} tanh^2( ) − 2 ^{2} coth^2( ).**

Particular solution: $_0 = \lambda \tanh(\lambda\ ) + \lambda \coth(\lambda\ )$.

27. **= ^{2} + + − 2 − (+) tanh^2( ) − (+) coth^2( ).**

Particular solution: $_0 = a \tanh(\lambda\ ) +\ \coth(\lambda\ )$.

1.2.5. Equations Containing Logarithmic Functions

1.2.5-1. Equations of the form $g(\ )\ ' =\ _2(\ )\ ^2 +\ _0(\ )$.

1. **= (ln) ^{2} + $^{-1}$ − ^{2} ^{2} (ln) .**

Particular solution: $_0$ = .

2. **= ^{2} + ln + .**

The substitution = e leads to an equation of the form 1.2.2.1: $' = a\ ^2 + \ + \ $.

3. **= ^{2} + ln^k + ln^{2k+2} .**

The substitution = ln leads to an equation of the form 1.2.2.6 with k = − 1: $' = a\ ^2 + \ + \ ^{2\ +2}$.

4. **= ^{2} − ^{2} ln^2( ) + .**

Particular solution: $_0 = a \ln(\beta\ )$.

5. **= ^{2} − ^{2} ln^{2k}( ) + ln^{k-1}( ).**

Particular solution: $_0 = a \ln\ (\beta\ )$.

6. **= ^{2} + − ^{2} ln^2 .**

Particular solution: $_0$ = ln .

7. **^{2} = ^{2} ^{2} + ln^2 + ln + .**

The transformation ξ = ln , = + $\frac{1}{2}$ leads to an equation of the form 1.2.2.3: $' =\ ^2 + a\xi^2 +\ \xi +\ - \frac{1}{4}$.

8. **^{2} = ^{2} ^{2} + (ln +) + $\frac{1}{4}$.**

The transformation ξ = ln + , = − + $\dfrac{1}{2}$ leads to an equation of the form 1.2.2.4: $' =\ ^2 + a\ ^{-2}\xi\ $.

9. **^{2} ln( )(− ^{2}) = 1.**

Particular solution: $_0 = -[\ \ln(a\ )]^{-1}$.

1.2.5-2. Equations of the form $g(\ )\ ' =\ _2(\ )\ ^2 +\ _1(\ )\ ' +\ _0(\ )$.

10. **= ^{2} + ln( ) − ln( ) − ^{2}.**

Particular solution: $_0$ = .

11. $= {}^2 + \ \ln(\ ) + \ \ln(\ ).$

Particular solution: $_0 = -1\ .$

12. $= \ {}^2 - \ {}^{+1}\ln \ + \ \ln \ + \ .$

Particular solution: $_0 = \ \ln\ .$

13. $= -(\ + 1)\ {}^2 + \ {}^{+1}(\ln\ ) \ - (\ln\ )\ .$

Particular solution: $_0 = {}^{-\ -1}.$

14. $= (\ln\ )\ {}^2 - \ (\ln\ )\ {}^{+1} + \ \ln \ + \ .$

Particular solution: $_0 = \ \ln\ .$

15. $= (\ln\ )^k(\ - \ - \ )^2 + \ {}^{-1}.$

Particular solution: $_0 = \ + \ .$

16. $= (\ln\ )\ {}^2 + (\ln\ )\ + (\ln\ )\ - {}^2(\ln\ )\ .$

Particular solution: $_0 = -\ .$

17. $= (\ + \ \ln\ )^2.$

Solution: $\ln\ = \dfrac{z}{az^2 + } + \ ,$ where $z = a\ + \ \ln\ .$

18. $= \ln(\ )\ {}^2 + \ + \ {}^2\ {}^{2k}\ln(\ ).$

Solution: $= \ \tan a\ \ {}^{-1}\ln(\lambda\ ) + \ .$

19. $= (\ + \ \ln\ )^2 - \ .$

Solution: $\dfrac{1}{\ + \ \ln\ } + \dfrac{a}{\ } = \ .$

20. $= {}^2(\ln\ )\ {}^2 + (\ \ln\ - \ )\ + \ \ln\ .$

Solution: $\dfrac{}{a\ {}^2 + \ + \ } = \ {}^{-1}\ln\ + \ ,$ where $= \ .$

21. ${}^2\ = {}^2\ {}^2\ {}^2 - \ + {}^2\ln\ .$

The substitution $a^2\ = -\ '$ leads to a second-order linear equation of the form 2.1.5.27: ${}^2\ '' + \ ' + (a\ )^2\ln\ = 0.$

22. $(\ \ln\ + \ )\ = {}^2 + (\ln\ )\ - {}^2 + \ (\ln\ )\ .$

Particular solution: $_0 = -\lambda.$

23. $(\ \ln\ + \ )\ = (\ln\ )\ {}^2 + \ - {}^2(\ln\ ) + \ .$

Particular solution: $_0 = -\lambda.$

1.2.6. Equations Containing Trigonometric Functions

1.2.6-1. Equations with sine.

1. $= {}^2 + \ + \gamma\sin(\ ).$

The substitution $2\ = 2\lambda\ + \ $ leads to an equation of the form 1.2.6.14: $\lambda\ ' = {}^2 + \beta + \ \cos\ .$

2. $\quad = {}^{2} - {}^{2} + \quad \sin(\quad) + {}^{2}\sin^{2}(\quad).$

Particular solution: $\;_{0} = -a\cos(\lambda\;).$

3. $\quad = {}^{2} + {}^{2} + \;\sin\;(\quad + \;)\sin^{-\;-4}(\quad + \;).$

The transformation $\xi = \dfrac{\sin(\lambda\;\; + a)}{\sin(\lambda\;\; + \;)}, \quad = \dfrac{\sin^{2}(\lambda\;\; + \;)}{\sin(\;\; - a)}\;\dfrac{}{\lambda} + \cot(\lambda\;\; + \;)$ leads to an equation of the form 1.2.2.4: $\;' = \;{}^{2} + A\xi\;$, where $A = \;[\lambda\sin(\;\; - a)]^{-2}$.

4. $\quad = {}^{2} + \;\sin(\quad)\; + \quad\sin(\quad) - {}^{2}.$

Particular solution: $\;_{0} = -\;.$

5. $\quad = {}^{2} + \quad\sin\;(\quad)\; + \;\sin\;(\quad).$

Particular solution: $\;_{0} = -1\quad.$

6. $\quad = \;\sin(\quad)\;{}^{2} + \;\sin^{3}(\quad).$

Particular solution: $\;_{0} = -\cos(\lambda\;).$

7. $2\quad = [\;+\;-\;\sin(\quad)]\;{}^{2} + \;-\;-\;\sin(\quad).$

Particular solution: $\;_{0} = \tan\left(\tfrac{1}{2}\lambda\;\; + \tfrac{1}{4}\;\right.\;.$

8. $\quad = [\;+\;\sin^{2}(\quad)]\;{}^{2} + \;-\;+\;\sin^{2}(\quad).$

Particular solution: $\;_{0} = -\cot(\lambda\;).$

9. $\quad = -(\;+1)\;{}^{k}\;{}^{2} + \quad{}^{k+1}(\sin\;)\quad - \;(\sin\;)\quad.$

Particular solution: $\;_{0} = \quad{}^{-\;-1}.$

10. $\quad = \;\sin^{k}(\quad + \;)(\;-\quad-\;)^{2} + \qquad{}^{-1}.$

Particular solution: $\;_{0} = \quad + \;.$

11. $\quad = \;\sin\;(\quad)\;{}^{2} + \quad + \quad{}^{2}\;{}^{2k}\sin\;(\quad).$

Solution: $\quad = \quad\tan\;a\qquad{}^{-1}\sin\;\;(\lambda\;)\quad + \quad.$

12. $[\;\sin(\quad) + \;]\quad = {}^{2} + \;\sin(\quad)\; - \;{}^{2} + \quad\sin(\quad).$

Particular solution: $\;_{0} = -\;.$

13. $[\;\sin(\quad) + \;](\quad - \;{}^{2}) - \quad{}^{2}\sin(\quad) = 0.$

Particular solution: $\;_{0} = -\dfrac{a\lambda\cos(\lambda\;)}{a\sin(\lambda\;) + \;}.$

1.2.6-2. Equations with cosine.

14. $\quad = \quad{}^{2} + \quad + \gamma\cos\;\;.$

The transformation $\;= 2\;,\quad = -\;'\;$ leads to a Mathieu equation of the form 2.1.6.29: $\;'' + (a - 2\;\cos 2\;)\; = 0$, where $a = 4\;\beta,\quad = -2\quad.$

15. $\quad = {}^{2} - {}^{2} + \quad\cos(\quad) + {}^{2}\cos^{2}(\quad).$

Particular solution: $\;_{0} = a\sin(\lambda\;).$

16. $= {}^2 + {}^2 + \cos(\;+\;)\cos^{-\;-4}(\;+\;).$

The substitution $\lambda \;= \lambda z - \frac{\pi}{2}$ leads to an equation of the form 1.2.6.3: $' = \;{}^2 + \lambda^2 +$ $\sin\;(\lambda z + a)\sin^{-\;-4}(\lambda z + \;).$

17. $= {}^2 + \cos(\;) + \cos(\;) - {}^2.$

Particular solution: $_0 = -\;.$

18. $= {}^2 + \cos(\;) + \cos(\;).$

Particular solution: $_0 = -1\;.$

19. $= \cos(\;)\,{}^2 + \cos^3(\;).$

Particular solution: $_0 = \sin(\lambda\;).$

20. $2\; = [\;+\;+\cos(\;)]\,{}^2 + \;-\;+\cos(\;).$

Particular solution: $_0 = \tan\left(\frac{1}{2}\lambda\right.\;.$

21. $= [\;+\cos^2(\;)]\,{}^2 + \;-\;+\cos^2(\;).$

Particular solution: $_0 = \tan(\lambda\;).$

22. $= -(\;+1)\;{}^{k}\;{}^2 + \;{}^{k+1}(\cos\;)\; - (\cos\;)\;.$

Particular solution: $_0 = \;{}^{-\;-1}.$

23. $= \cos^k(\;+\;)(\;-\;-\;)^2 + \;{}^{-1}.$

Particular solution: $_0 = \;+\;.$

24. $= \cos(\;)\,{}^2 + \;+\;{}^2\;{}^{2k}\cos(\;).$

Solution: $= \tan\;a\;{}^{-1}\cos\;(\lambda\;)\;+\;.$

25. $[\;\cos(\;) + \;]\; = {}^2 + \cos(\;) - {}^2 + \cos(\;).$

Particular solution: $_0 = -\;.$

26. $[\;\cos(\;) + \;](\;-\;{}^2) - \;{}^2\cos(\;) = 0.$

Particular solution: $_0 = \dfrac{a\lambda\sin(\lambda\;)}{a\cos(\lambda\;) + \;}.$

1.2.6-3. Equations with tangent.

27. $= {}^2 + \;+(\;-\;)\tan^2(\;).$

Particular solution: $_0 = a\tan(\lambda\;).$

28. $= {}^2 + {}^2 + 3\; + (\;-\;)\tan^2(\;).$

Particular solution: $_0 = a\tan(\lambda\;) - \lambda\cot(\lambda\;).$

29. $= {}^2 + \tan\; + \;.$

The substitution $a\; = -\;'$ leads to a second-order linear equation of the form 2.1.6.53: $'' - \tan\;' + a\; = 0.$

30. $= {}^2 + 2\;\tan\; + (\;-1)\tan^2\;.$

The substitution $= \;+\tan$ leads to a separable equation of the form 1.1.2: $' = a\;{}^2 + \;.$

31. $= {}^{2} + \ \tan(\ ) + \ \tan(\ ) - {}^{2}.$

Particular solution: $_{0} = - \ .$

32. $= {}^{2} + \ \tan \ (\ ) + \tan \ (\ ).$

Particular solution: $_{0} = -1 \ .$

33. $= -(\ + 1) \ {}^{k} \ {}^{2} + \ {}^{k+1}(\tan \ ) \ - (\tan \ ) \ .$

Particular solution: $_{0} = \ {}^{- \ -1}.$

34. $= \tan \ (\ ) \ {}^{2} - \ {}^{2} \tan \ {}^{+2}(\ ) + \ \tan^{2}(\ ) + \ .$

Particular solution: $_{0} = \ \tan(\lambda \ ).$

35. $= \tan^{k}(\ + \ )(\ - \ - \ )^{2} + \ {}^{-1}.$

Particular solution: $_{0} = \ + \ .$

36. $= \tan \ (\ ) \ {}^{2} + \ + \ {}^{2} \ {}^{2k} \tan \ (\ ).$

Solution: $= \ \tan \ a \ {}^{-1} \tan \ (\lambda \ ) \ + \ .$

37. $[\ \tan(\ ) + \ ] \ = {}^{2} + \ \tan(\ ) \ - {}^{2} + \ \tan(\ ).$

Particular solution: $_{0} = - \ .$

1.2.6-4. Equations with cotangent.

38. $= {}^{2} + \ + (\ - \ ) \cot^{2}(\ ).$

Particular solution: $_{0} = -a \cot(\lambda \ ).$

39. $= {}^{2} + {}^{2} + 3 \ + (\ - \ ) \cot^{2}(\ ).$

Particular solution: $_{0} = \lambda \tan(\lambda \ ) - a \cot(\lambda \ ).$

40. $= {}^{2} - 2 \ \cot(\ ) \ + {}^{2} - {}^{2}.$

Particular solution: $_{0} = a \cot(a \ ) - \ \cot(\ ).$

41. $= {}^{2} + \ \cot(\ ) + \ \cot(\ ) - {}^{2}.$

Particular solution: $_{0} = - \ .$

42. $= {}^{2} + \ \cot \ (\ ) + \cot \ (\ ).$

Particular solution: $_{0} = -1 \ .$

43. $= -(\ + 1) \ {}^{k} \ {}^{2} + \ {}^{k+1}(\cot \ ) \ - (\cot \ ) \ .$

Particular solution: $_{0} = \ {}^{- \ -1}.$

44. $= \cot^{k}(\ + \ )(\ - \ - \ )^{2} + \ {}^{-1}.$

Particular solution: $_{0} = \ + \ .$

45. $= \cot \ (\ ) \ {}^{2} + \ + \ {}^{2} \ {}^{2k} \cot \ (\ ).$

Solution: $= \ \tan \ a \ {}^{-1} \cot \ (\lambda \ ) \ + \ .$

46. $[\ \cot(\ ) + \ ] \ = {}^{2} + \ \cot(\ ) \ - {}^{2} + \ \cot(\ ).$

Particular solution: $_{0} = - \ .$

1.2.6-5. Equations containing combinations of trigonometric functions.

47. $= \;^2 + \;^2 + \sin(\;)\cos^{-\;-4}(\;)$.

This is a special case of equation 1.2.6.3 with $a = 0$ and $= \;2$.

48. $= \sin(\;)\;^2 + \sin(\;)\cos(\;)$.

The transformation $\xi = \cos(\lambda\;)$, $= -\dfrac{a}{\lambda}$ leads to an equation of the form 1.2.2.4: $' = \;^2 + a\;\lambda^{-2}\xi\;$.

49. $= \sin(\;)\;^2 + \cos(\;) - \cos^{-1}(\;)$.

Particular solution: $_0 = 1\;\cos(\lambda\;)$.

50. $= \cos(\;)\;^2 + \cos(\;)\sin(\;)$.

The transformation $\xi = \sin(\lambda\;)$, $= \dfrac{a}{\lambda}$ leads to an equation of the form 1.2.2.4: $' = \;^2 + a\;\lambda^{-2}\xi\;$.

51. $= \sin(\;)\;^2 + \;\cos(\;) - \;$.

Particular solution: $_0 = 1\;\cos(\lambda\;)$.

52. $\sin^{\;+1}(2\;) = \;^2\sin^2\; + \cos^2\;$.

The substitution $z = \;\tan\;$ leads to a separable equation: $2\;\sin(2\;)z' = az^2 + \;2^{\;+1}z + \;$.

53. $= \;^2 - \tan\; + (1 - \;)\cot^2\;$.

Particular solution: $_0 = -a\cot\;$.

54. $= \;^2 - \;\tan\; + \;^2\cos^2\;$.

Solution: $= -\;\cos\;\cot\;\cos\; + \;$.

55. $= \;^2 + \;\cot\; + \;^2(\sin\;)^2\;$.

Solution: $= -\;\sin\;\cot\;\sin\; + \;$.

56. $= \;^2 - 2\;^2\tan^2(\;) - 2\;^2\cot^2(\;)$.

Particular solution: $_0 = \lambda\cot(\lambda\;) - \lambda\tan(\lambda\;)$.

57. $= \;^2 + \; + \; + 2\; + (\; - \;)\tan^2(\;) + (\; - \;)\cot^2(\;)$.

Particular solution: $_0 = a\tan(\lambda\;) - \;\cot(\lambda\;)$.

58. $= \;^2 - \tfrac{1}{2}\;^2 - \tfrac{3}{4}\;^2\tan^2(\;) + \;\cos^2(\;)\sin(\;)$.

The transformation $\xi = \sin(\lambda\;)$, $= \dfrac{}{\lambda\cos(\lambda\;)} + \dfrac{\sin(\lambda\;)}{2\cos^2(\lambda\;)}$ leads to an equation of the form 1.2.2.4: $' = \;^2 + a\lambda^{-2}\xi\;$.

59. $= \sin(\;)\;^2 + \sin(\;) - \tan(\;)$.

Particular solution: $_0 = 1\;\cos(\lambda\;)$.

1.2.7. Equations Containing Inverse Trigonometric Functions

1.2.7-1. Equations containing arcsine.

1. $= \ {}^2 + \ (\arcsin\ ) \ - \ {}^2 + \ (\arcsin\ )\ .$

Particular solution: $_0 = -a$.

2. $= \ {}^2 + \ (\arcsin\ ) \ + \ (\arcsin\ )\ .$

Particular solution: $_0 = -1 \ .$

3. $= -(\ + 1) \ {}^{k} \ {}^{2} + \ (\arcsin\ ) \ (\ {}^{k+1} \ - 1).$

Particular solution: $_0 = \ {}^{-\ -1}.$

4. $= \ (\arcsin\ ) \ {}^2 + \ + \ - \ {}^2 \ (\arcsin\ )\ .$

Particular solution: $_0 = - \ .$

5. $= \ (\arcsin\ ) \ {}^2 - \ (\arcsin\ ) \ + \ {}^{-1}.$

Particular solution: $_0 = \ .$

6. $= \ (\arcsin\ ) \ {}^2 + \ {}^{-1} - \ {}^2 \ {}^2 \ (\arcsin\ )\ .$

Particular solution: $_0 = \beta \ .$

7. $= \ (\arcsin\ ) \ (\ - \ - \ )^2 + \ {}^{-1}.$

Particular solution: $_0 = a \ + \ .$

8. $= \ (\arcsin\ ) \ {}^2 + \ + \ {}^2 \ {}^{2k}(\arcsin\ )\ .$

Solution: $= \ \tan \lambda \ {}^{-1}(\arcsin\ ) \ + \ .$

9. $= (\ {}^2 \ {}^2 + \ + \ )(\arcsin\ ) \ - \ .$

The substitution $z =$ leads to a separable equation: $z' = \ {}^{-1}(\arcsin\ ) \ (az^2 + \ z + \ ).$

1.2.7-2. Equations containing arccosine.

10. $= \ {}^2 + \ (\arccos\ ) \ - \ {}^2 + \ (\arccos\ )\ .$

Particular solution: $_0 = -a$.

11. $= \ {}^2 + \ (\arccos\ ) \ + \ (\arccos\ )\ .$

Particular solution: $_0 = -1 \ .$

12. $= -(\ + 1) \ {}^{k} \ {}^{2} + \ (\arccos\ ) \ (\ {}^{k+1} \ - 1).$

Particular solution: $_0 = \ {}^{-\ -1}.$

13. $= \ (\arccos\ ) \ {}^2 + \ + \ - \ {}^2 \ (\arccos\ )\ .$

Particular solution: $_0 = - \ .$

14. $= \ (\arccos\ ) \ {}^2 - \ (\arccos\ ) \ + \ {}^{-1}.$

Particular solution: $_0 = \ .$

15. = (arccos) ^{2} + $^{-1}$ − ^{2} ^{2} (arccos) .

Particular solution: $_0 = \beta$.

16. = (arccos) (− −)^2 + $^{-1}$.

Particular solution: $_0 = a$ + .

17. = (arccos) ^{2} + + ^{2} ^{2k}(arccos) .

Solution: = tan λ $^{-1}$(arccos) + .

18. = (^{2} ^{2} + +)(arccos) − .

The substitution $z =$ leads to a separable equation: $z' =$ $^{-1}$(arccos) $(az^2 +\ z +\ )$.

1.2.7-3. Equations containing arctangent.

19. = ^{2} + (arctan) − ^{2} + (arctan) .

Particular solution: $_0 = -a$.

20. = ^{2} + (arctan) + (arctan) .

Particular solution: $_0 = -1$.

21. = −(+ 1) ^{k} ^{2} + (arctan) ($^{k+1}$ − 1).

Particular solution: $_0 =$ $^{-\ -1}$.

22. = (arctan) ^{2} + + − ^{2} (arctan) .

Particular solution: $_0 = -$.

23. = (arctan) ^{2} − (arctan) + $^{-1}$.

Particular solution: $_0 =$.

24. = (arctan) ^{2} + $^{-1}$ − ^{2} ^{2} (arctan) .

Particular solution: $_0 =$.

25. = (arctan) (− −)^2 + $^{-1}$.

Particular solution: $_0 = a$ + .

26. = (arctan) ^{2} + + ^{2} ^{2k}(arctan) .

Solution: = tan λ $^{-1}$(arctan) + .

27. = (^{2} ^{2} + +)(arctan) − .

The substitution $z =$ leads to a separable equation: $z' =$ $^{-1}$(arctan) $(az^2 +\ z +\ )$.

1.2.7-4. Equations containing arccotangent.

28. = ^{2} + (arccot) − ^{2} + (arccot) .

Particular solution: $_0 = -a$.

29. $= \ {}^{2} + \quad (\mathrm{arccot}\ ) \quad + \ (\mathrm{arccot}\ )\ .$

Particular solution: $_0 = -1 \quad .$

30. $= -(\ + 1) \ {}^{k} \ {}^{2} + \ (\mathrm{arccot}\ ) \ (\ {}^{k+1} \ - 1).$

Particular solution: $_0 = \ {}^{- \ -1}.$

31. $= \ (\mathrm{arccot}\ ) \ {}^{2} + \quad + \quad - \ {}^{2} \ (\mathrm{arccot}\ )\ .$

Particular solution: $_0 = - \ .$

32. $= \ (\mathrm{arccot}\ ) \ {}^{2} - \quad (\mathrm{arccot}\ ) \quad + \quad {}^{-1}.$

Particular solution: $_0 = \quad .$

33. $= \ (\mathrm{arccot}\ ) \ {}^{2} + \quad {}^{-1} - \ {}^{2} \ {}^{2} \ (\mathrm{arccot}\ )\ .$

Particular solution: $_0 = \quad .$

34. $= \ (\mathrm{arccot}\ ) \ (\ - \quad - \ )^{2} + \quad {}^{-1}.$

Particular solution: $_0 = a \quad + \ .$

35. $= \ (\mathrm{arccot}\ ) \ {}^{2} + \quad + \ {}^{2} \ {}^{2k}(\mathrm{arccot}\ )\ .$

Solution: $= \quad \tan \lambda \quad {}^{-1}(\mathrm{arccot}\ ) \quad + \quad .$

36. $= (\ {}^{2} \ {}^{2} + \quad + \ )(\mathrm{arccot}\ ) \ - \quad .$

The substitution $z = \quad$ leads to a separable equation: $z' = \quad {}^{-1}(\mathrm{arccot}\ ) \ (az^{2} + \ z + \ ).$

1.2.8. Equations with Arbitrary Functions

Notation: $= \ (\ )$ *and* $g = g(\ )$ *are arbitrary functions;* a, , , *and* λ *are arbitrary parameters.*

1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).

1. $= \ {}^{2} + f \ - \ {}^{2} - \ f.$

Particular solution: $_0 = a.$

2. $= f \ {}^{2} + \quad - \quad - \ {}^{2} f.$

Particular solution: $_0 = \ .$

3. $= \ {}^{2} + \ f \ + f.$

Particular solution: $_0 = -1 \quad .$

4. $= f \ {}^{2} - \quad f \ + \quad {}^{-1}.$

Particular solution: $_0 = a \quad .$

5. $= f \ {}^{2} + \quad {}^{-1} - \ {}^{2} \ {}^{2} \ f.$

Particular solution: $_0 = a \quad .$

6. $= -(\ + 1) \quad {}^{2} + \quad {}^{+1} f \ - f.$

Particular solution: $_0 = \ {}^{- \ -1}.$

7. $= f\ ^{2} + \quad + \quad ^{2}\ f.$

Solution: $= \begin{cases} \overline{a}\quad \tan\quad \overline{a}\qquad ^{-1} \qquad + & \text{if } a > 0, \\ \overline{|a|}\quad \tanh\ -\ \overline{|a|}\qquad ^{-1}\qquad + & \text{if } a < 0. \end{cases}$

8. $= \ ^{2}\ f\ ^{2} + (\quad f - \ )\ +\ f.$

The substitution $z =$ leads to a separable equation: $z' = \quad ^{-1}\ (\ )(z^2 + az + \ ).$

9. $= f\ ^{2} + \quad - \ ^{2} f - \quad .$

Particular solution: $_{0} = a.$

10. $= f\ ^{2} + \quad + \qquad ^{-1} - \qquad - \ ^{2} f\ ^{2}\ .$

Particular solution: $_{0} = a \quad .$

11. $= f\ ^{2} - \qquad + \qquad ^{-1} + \ ^{2}\ ^{2}\ (\ - f).$

Particular solution: $_{0} = a \quad .$

12. $= \quad ^{\lambda}\quad ^{2} + \quad ^{\lambda}\ f \ +\ f.$

Particular solution: $_{0} = -\dfrac{\lambda}{a} e^{-\lambda} \ .$

13. $= f\ ^{2} - \quad ^{\lambda}\ f \ + \qquad ^{\lambda}\ .$

Particular solution: $_{0} = ae^{\lambda}\ .$

14. $= f\ ^{2} + \qquad ^{\lambda} - \ ^{2}\ ^{2\lambda}\ f.$

Particular solution: $_{0} = ae^{\lambda}\ .$

15. $= f\ ^{2} + \quad + \quad ^{2\lambda}\ f.$

Solution: $= \begin{cases} \overline{a}\, e^{\lambda}\ \tan\quad \overline{a}\quad e^{\lambda}\qquad + & \text{if } a > 0, \\ \overline{|a|}\, e^{\lambda}\ \tanh\ -\ \overline{|a|}\quad e^{\lambda}\qquad + & \text{if } a < 0. \end{cases}$

16. $= f\ ^{2} - f(\quad ^{\lambda} + \ )\ + \qquad ^{\lambda}\ .$

Particular solution: $_{0} = ae^{\lambda} + \ .$

17. $= \ ^{\lambda}\ f\ ^{2} + (\ f - \ )\ + \quad ^{-\lambda}\ f.$

The substitution $z = e^{\lambda}$ leads to a separable equation: $z' = \ (\ )(z^2 + az + \ ).$

18. $= f\ ^{2} + \quad + \qquad ^{\lambda} - \quad ^{\lambda} \quad - \ ^{2}\ ^{2\lambda}\ f.$

Particular solution: $_{0} = ae^{\lambda}\ .$

19. $= f\ ^{2} - \quad ^{\lambda} \qquad + \qquad ^{\lambda} + \ ^{2}\ ^{2\lambda}\ (\ - f).$

Particular solution: $_{0} = ae^{\lambda}\ .$

20. $= f\ ^{2} + 2 \qquad ^{\lambda}\ ^{2} - \ ^{2} f\ ^{2\lambda}\ ^{2}.$

Particular solution: $_{0} = ae^{\lambda\ ^{2}}.$

21. $= f\ {}^{2} + \quad + \ f\ {}^{\lambda}\ {}^{2}.$

Solution: $= \begin{cases} \overline{a}\, e^{\lambda\ {}^2\ 2} \tan \quad \overline{a} \quad e^{\lambda\ {}^2\ 2} \quad + & \text{if } a > 0, \\ \overline{|a|}\, e^{\lambda\ {}^2\ 2} \tanh - \overline{|a|} \quad e^{\lambda\ {}^2\ 2} \quad + & \text{if } a < 0. \end{cases}$

22. $= f\ {}^{2} - \ \tanh^2(\quad)(\ f + \ ) + \quad .$

Particular solution: ${}_0 = a \tanh(\lambda\ ).$

23. $= f\ {}^{2} - \ \coth^2(\quad)(\ f + \ ) + \quad .$

Particular solution: ${}_0 = a \coth(\lambda\ ).$

24. $= f\ {}^{2} - \ {}^{2} f + \quad \sinh(\quad) - \ {}^{2} f \sinh^2(\quad).$

Particular solution: ${}_0 = a \cosh(\lambda\ ).$

25. $= f\ {}^{2} + \quad - \ {}^{2} f (\ln\ )^2.$

Particular solution: ${}_0 = a \ln\ .$

26. $= f(\ + \ \ln\ )^2 - \ .$

Solution: $\dfrac{1}{+ a \ln} + \dfrac{(\ )}{} \quad = \ .$

27. $= f\ {}^{2} - \quad \ln\ f\ + \ \ln\ + \ .$

Particular solution: ${}_0 = a \quad \ln\ .$

28. $= - \ \ln \quad {}^{2} + \ f(\ \ln\ - \ ) - f.$

Particular solution: ${}_0 = \dfrac{1}{a(\ \ln\ - \ )}.$

29. $= \ \sin(\quad)\ {}^{2} + f \cos(\quad) - f.$

Particular solution: ${}_0 = 1 \quad \cos(\lambda\ ).$

30. $= f\ {}^{2} - \ {}^{2} f + \quad \sin(\quad) + \ {}^{2} f \sin^2(\quad).$

Particular solution: ${}_0 = -a \cos(\lambda\ ).$

31. $= f\ {}^{2} - \ {}^{2} f + \quad \cos(\quad) + \ {}^{2} f \cos^2(\quad).$

Particular solution: ${}_0 = a \sin(\lambda\ ).$

32. $= f\ {}^{2} - \ \tan^2(\quad)(\ f - \ ) + \quad .$

Particular solution: ${}_0 = a \tan(\lambda\ ).$

33. $= f\ {}^{2} - \ \cot^2(\quad)(\ f - \ ) + \quad .$

Particular solution: ${}_0 = -a \cot(\lambda\ ).$

1.2.8-2. Equations containing arbitrary functions and their derivatives.

34. $= \ {}^{2} - f^2 + f\ .$

Particular solution: ${}_0 = \ .$

35. $= f\ {}^{2} - f \quad + \quad .$

Particular solution: ${}_0 = g.$

36. $= -f \quad {}^2 + f \quad - \; .$

Particular solution: $\quad_0 = 1 \quad .$

37. $= \;(\; - f)^2 + f \; .$

Particular solution: $\quad_0 = \quad .$

38. $= \frac{f}{\quad} \; {}^2 - \frac{\quad}{f}.$

Particular solution: $\quad_0 = -g \quad .$

39. $f^2 \quad - f \quad {}^2 + \; (\; - f) = 0.$

Particular solution: $\quad_0 = \quad .$

40. $= f \quad {}^2 + \quad {}^\lambda \; f \quad + \quad {}^\lambda \; .$

Particular solution: $\quad_0 = -1 \quad .$

41. $= f \; {}^2 + \quad + \; f \; {}^2 \; .$

Solution: $= \begin{cases} \overline{a}\, e \;\; \tan \;\; \overline{a} \quad e \quad + & \text{if } a > 0, \\ \overline{|a|}\, e \;\; \tanh \; - \; \overline{|a|} \quad e \quad + & \text{if } a < 0. \end{cases}$

42. $= \; {}^2 - \frac{f}{f}.$

Particular solution: $\quad_0 = - \; {}' \quad .$

1.2.9. Some Transformations

Notation: , g, and are arbitrary composite functions of their argument, which is written in parentheses following the function name (the argument is a function of).

1. $= \; {}^2 + \; {}^2 f(\quad + \;).$

The transformation $\xi = a \quad + \; , \quad = \quad a$ leads to the equation $\; {}' = \; {}^2 + \; (\xi).$

2. $= \; {}^2 + \; {}^{-4} f(1 \quad).$

The transformation $\xi = 1 \quad , \quad = - \; {}^2 \; - \;$ leads to the equation $\; {}' = \; {}^2 + \; (\xi).$

3. $= \; {}^2 + \frac{1}{(\quad + \;)^4} f \; \frac{\quad + \quad}{\quad + \quad} \; .$

The transformation

$$\xi = \frac{a \quad +}{\quad +}, \qquad = \frac{1}{\Delta}[(\quad + \;)^2 \; + \; (\quad + \;)], \quad \text{where} \quad \Delta = a \; - \quad ,$$

leads to a simpler equation: $\; {}' = \; {}^2 + \Delta^{-2} \; (\xi).$

4. $\; {}^2 \quad = \; {}^4 f(\;) \; {}^2 + 1.$

The substitution $\quad = -\frac{1}{\; {}^2} - \frac{1}{\quad}$ leads to the equation $\; {}' = \; {}^2 + \; (\;).$

5. $\quad {}^2 \quad = \; {}^2\,{}^2 + {}^2 f(\quad + \;) + \frac{1}{4}(1 - \; {}^2).$

The transformation $\xi = a \quad + \;,\quad = \frac{1}{a} \; {}^{1-} \quad + \frac{1-}{2a} \; {}^{-}$ leads to a simpler equation: $\; {}' = \; {}^2 + (a \;)^{-2} \; (\xi).$

6. $\quad = f(\;)\; {}^2 + \;(\;)\; + \;(\;).$

The substitution $\; = -1\;$ leads to an equation of the same form: $\; {}' = \;(\;)\; {}^2 - g(\;)\; + \;(\;).$

7. $\quad = \; {}^2 + \; {}^{2\lambda} f(\; {}^{\lambda}\;) - \frac{1}{4}\; {}^2.$

The transformation $\xi = e^{\lambda}\;,\quad = \frac{1}{\lambda} e^{-\lambda}\; - \frac{1}{2} e^{-\lambda}$ leads to a simpler equation: $\; {}' = \; {}^2 + \lambda^{-2}\; (\xi).$

8. $\quad = \; {}^2 - \frac{{}^2}{4} + \frac{{}^{2\lambda}}{(\; {}^{\lambda} + \;)^4} f\left(\frac{\; {}^{\lambda} + \;}{\; {}^{\lambda} + \;}\right).$

The transformation

$$\xi = \frac{ae^{\lambda} + }{e^{\lambda} + },\qquad = \frac{(\; e^{\lambda} + \;)^2}{\Delta\lambda e^{\lambda}} + \frac{{}^2 e^{2\lambda} - \; {}^2}{2\Delta e^{\lambda}},\qquad \text{where}\quad \Delta = a \; - \;,$$

leads to a simpler equation: $\; {}' = \; {}^2 + (\Delta\lambda)^{-2}\; (\xi).$

9. $\quad = \; {}^2 - \; {}^2 + \sinh^{-4}(\;) f\big(\coth(\;)\big).$

The transformation $\xi = \coth(\lambda\;),\quad = -\lambda^{-1}\sinh^2(\lambda\;)\; - \frac{1}{2}\sinh(2\lambda\;)$ leads to a simpler equation: $\; {}' = \; {}^2 + \lambda^{-2}\; (\xi).$

10. $\quad = \; {}^2 - \; {}^2 + \cosh^{-4}(\;) f\big(\tanh(\;)\big).$

The transformation $\xi = \tanh(\lambda\;),\quad = \lambda^{-1}\cosh^2(\lambda\;)\; + \frac{1}{2}\sinh(2\lambda\;)$ leads to a simpler equation: $\; {}' = \; {}^2 + \lambda^{-2}\; (\xi).$

11. $\quad {}^2 \quad = \; {}^2\,{}^2 + f(\;\ln\; + \;) + \frac{1}{4}.$

The transformation $\xi = a\ln\; + \;,\quad = \frac{1}{a}\; + \frac{1}{2a}$ leads to a simpler equation: $\; {}' = \; {}^2 + a^{-2}\; (\xi).$

12. $\quad = \; {}^2 + \; {}^2 + \sin^{-4}(\;) f(\cot(\;)).$

The transformation $\xi = \cot(\lambda\;),\quad = -\sin^2(\lambda\;)\frac{}{\lambda} + \cot(\lambda\;)$ leads to a simpler equation: $\; {}' = \; {}^2 + \lambda^{-2}\; (\xi).$

13. $\quad = \; {}^2 + \; {}^2 + \cos^{-4}(\;) f\big(\tan(\;)\big).$

The transformation $\xi = \tan(\lambda\;),\quad = \cos^2(\lambda\;)\frac{}{\lambda} - \tan(\lambda\;)$ leads to a simpler equation: $\; {}' = \; {}^2 + \lambda^{-2}\; (\xi).$

14. $\quad = \; {}^2 + \; {}^2 + \sin^{-4}(\; + \;) f\left(\frac{\sin(\; + \;)}{\sin(\; + \;)}\right).$

The transformation $\xi = \dfrac{\sin(\lambda\; + a)}{\sin(\lambda\; + \;)},\quad = \dfrac{\sin^2(\lambda\; + \;)}{\sin(\; - a)}\dfrac{}{\lambda} + \cot(\lambda\; + \;)$ leads to a simpler equation: $\; {}' = \; {}^2 + [\lambda\sin(\; - a)]^{-2}\; (\xi).$

1.3. Abel Equations of the Second Kind

1.3.1. Equations of the Form – = ( )

1.3.1-1. Preliminary remarks. Classification tables.

For the sake of convenience, listed in Tables 5–8 are all the Abel equations discussed in Section 1.3. Tables 5–7 classify Abel equations in which the functions are of the same form; Table 8 gives other Abel equations. In Table 5, equations are arranged in accordance with the growth of the parameter . In Table 6, equations are arranged in accordance with the growth of the parameter . In Table 7, equations are arranged in accordance with the growth of the parameter . The rightmost column of the tables indicates the equation numbers where the corresponding solutions are written out.

TABLE 5
Solvable Abel equations of the form $' - = + A$, A is an arbitrary parameter

		Equation
arbitrary	$-\dfrac{2(\ \ +1)}{(\ \ +3)^2}$	1.3.1.10
−7	15 4	1.3.1.56
−4	6	1.3.1.54
−5 2	12	1.3.1.47
−2	0	1.3.1.33
−2	2	1.3.1.19
−5 3	−3 16	1.3.1.30
−5 3	−9 100	1.3.1.23
−5 3	63 4	1.3.1.48
−7 5	−5 36	1.3.1.27
−1	0	1.3.1.16
−1 2	−2 9	1.3.1.26
−1 2	−4 25	1.3.1.22
−1 2	0	1.3.1.32
−1 2	20	1.3.1.55
0	arbitrary	1.3.1.2
0	0	1.3.1.1
1 2	−12 49	1.3.1.53
2	−6 25	1.3.1.45
2	6 25	1.3.1.46

TABLE 6
Solvable Abel equations of the form $' - = + \ A \ + \beta A^2$, A is an arbitrary parameter

				β	Equation
−1	−3	arbitrary	1	−1	1.3.1.5
−1	−3	$\dfrac{2\ \ +1}{4\ \ ^2}$	1	−1	1.3.1.13
−1	−3	0	1	−1	1.3.1.7
−3 5	−7 5	−5 36	arbitrary	arbitrary	1.3.1.62
−5 11	−13 11	−33 196	$286A$ 3	$-770A$ 9	1.3.1.69
−1 3	−5 3	−3 16	arbitrary	arbitrary	1.3.1.61
−1 3	−5 3	−3 16	3	−12	1.3.1.40
−1 3	−5 3	−3 16	5	−12	1.3.1.15
−1 3	−5 3	15 4	6	−3	1.3.1.60
−1 5	−4 5	−10 49	$13A$ 5	$-7A$ 20	1.3.1.68
0	−1 2	−2 9	arbitrary	arbitrary	1.3.1.3
2	3	4 9	2	2	1.3.1.14

TABLE 7
Solvable Abel equations of the form $\ ' - \ = \ + \sigma A(\ ^{1\ 2} + \beta A + \ A^{2\ -1\ 2})$,
A is an arbitrary parameter

	σ		β		Equation
arbitrary ≠ 0	arbitrary	0	arbitrary	0	1.3.1.2
$\frac{2(\ -1)}{(\ -3)^2}$	$\frac{2}{(\ -3)^2}$	$(\ +3)$	$4\ ^2+3\ +9$	$3\ (\ +3)$	1.3.1.12
−1 4	1 4	1	5	3	1.3.1.17
−30 121	3 242	21	35	6	1.3.1.29
−12 49	arbitrary	arbitrary	0	0	1.3.1.53
−12 49	1 98	25	41	10	1.3.1.25
−12 49	6 49	1	8	5	1.3.1.38
−12 49	2 49	5	34	15	1.3.1.24
−12 49	4 49	−10	27	10	1.3.1.31
−12 49	1 49	5	262	65	1.3.1.52
−12 49	6 49	−3	23	12	1.3.1.28
−12 49	2 49	1	166	55	1.3.1.58
−12 49	1	$3\ 49+3B$	$12\ 49-15B\ 2$	$15\ 196+75B\ 16$	1.3.1.64
−6 25	2 25	2	19	6	1.3.1.20
−6 25	6 25	2	7	4	1.3.1.39
−28 121	2 121	5	106	15	1.3.1.51
−2 9	arbitrary	0	arbitrary	arbitrary	1.3.1.3
−2 9	arbitrary	0	0	arbitrary	1.3.1.26
−2 9	6	0	1	2	1.3.1.11
−10 49	2 49	4	61	12	1.3.1.57
−4 25	arbitrary	0	0	arbitrary	1.3.1.22
−4 25	1 50	7	49	6	1.3.1.59
0	arbitrary	0	0	arbitrary	1.3.1.32
0	1	1	2	arbitrary	1.3.1.36
0	$+2$	1	$2(\ +2)$	$(\ +1)(\ +3)$	1.3.1.34
0	$+2$	1	$2(\ +2)$	$2\ +3$	1.3.1.35
0	1	−1	2	0	1.3.1.37
0	2	1	4	3	1.3.1.4
0	arbitrary	0	arbitrary	0	1.3.1.1
2	2	−10	19	30	1.3.1.50
2	2	10	31	30	1.3.1.49
20	arbitrary	0	0	arbitrary	1.3.1.55

Given below in this section are all solvable Abel equations known so far. The equations are arranged into groups, in which all solutions are expressed in terms of the same functions. Notation is given before each group.

In most cases the solutions are presented in parametric form:

$$\ = \ _1(\tau,\ ), \qquad \ = \ _2(\tau,\ ),$$

where τ is the parameter and is an arbitrary constant.

TABLE 8

Other solvable Abel equations of the form $\;' - \; = \;(\;)$

Function $\;(\;)$	Equation
$A\quad^{-1} - kB\quad + kB^2\quad^{2\;-1}$	1.3.1.6 (particular solution)
$A\quad^{2} - \frac{9}{625}A^{-1}$	1.3.1.44
$\frac{3}{4}\quad - \frac{3}{2}A\quad^{1\;3} + \frac{3}{4}A^2\quad^{-1\;3} - \frac{27}{625}A^4\quad^{-5\;3}$	1.3.1.66
$-\frac{6}{25}\quad + \frac{7}{5}A\quad^{1\;3} + \frac{31}{3}A^2\quad^{-1\;3} - \frac{100}{3}A^4\quad^{-5\;3}$	1.3.1.67
$-\frac{6}{25}\quad + a\quad^{1\;3} + \quad + \quad^{-1\;3} + \quad^{-2\;3}$ (coefficients a, , , and are related by an equality)	1.3.1.65
$-\frac{21}{100}\quad + \frac{7}{9}A^2\bigl(123\quad^{-1\;7} + 280\,A\quad^{-5\;7} - 400\,A^2\quad^{-9\;7}$	1.3.1.70
$\dfrac{k}{\overline{A\quad^{2} + B\quad + }}$	1.3.1.63
$\dfrac{A}{\overline{\quad^{2} + 4A}}$	1.3.1.18
$-\dfrac{3}{32}\quad + \dfrac{9a^2 - 6\quad^{2}}{64\quad\overline{\quad^{2} + a^2}}$	1.3.1.43
$\dfrac{3}{8}\quad + \dfrac{6\quad^{2} + 5a^2}{16\quad\overline{\quad^{2} + a^2}}$	1.3.1.21
$\dfrac{3}{8}\quad + \dfrac{6\quad^{2} + 9A}{16\quad\overline{\quad^{2} + A}}$	1.3.1.41
$\dfrac{9}{32}\quad + \dfrac{30\quad^{2} + 33A}{64\quad\overline{\quad^{2} + A}}$	1.3.1.42
$A + B\exp(-2\quad A)$	1.3.1.8
$A[\exp(2\quad A) - 1]$	1.3.1.9
$a^2\lambda e^{2\lambda}\quad - a(\;\lambda + 1)e^{\lambda}\quad +$	1.3.1.73 (particular solution)
$a^2\lambda e^{2\lambda}\quad + a\lambda\quad e^{\lambda}\quad + \quad e^{\lambda}$	1.3.1.74 (particular solution)
$2a^2\lambda\sin(2\lambda\quad) + 2a\sin(\lambda\quad)$	1.3.1.75 (particular solution)

1.3.1-2. Solvable equations and their solutions.

1. $\quad - \quad = \quad .$

Solution: $\quad = \quad - A\ln|\quad + A| + \quad .$

2. $\quad - \quad = \quad + \boldsymbol{B}, \qquad \neq \mathbf{0}.$

Solution in parametric form:

$$= \quad \exp\ -\quad \frac{\tau\quad\tau}{\tau^2 - \tau - A} - \frac{B}{A}, \qquad = \tau\exp\ -\quad \frac{\tau\quad\tau}{\tau^2 - \tau - A} \quad .$$

3. $\quad - \quad = -\frac{2}{9} \quad + \quad + B \quad ^{-1} \; ^{2}.$

1 . Solution in parametric form with $A > 0$:

$$= a \; \frac{(2k-1) \quad \tau \quad -(k-2)\tau - k - 1}{\tau \quad + \tau + 1}^{\;2}, \qquad = -6a \frac{(k-1)^2 \quad \tau \quad ^{+1} + k^2 \quad \tau \quad + \tau}{\tau \quad + \tau + 1},$$

where $A = \frac{2}{3}a(k^2 - k + 1)$, $B = \frac{2}{3}a^3 \; ^2(2k-1)(k-2)(k+1)$.

2 . Solution in parametric form with $A < 0$:

$$= \xi[2\lambda e^{-\lambda\tau} - (\quad _1\lambda - 3 \quad _2\omega)\sin\omega\tau - (3 \quad _1\omega + \quad _2\lambda)\cos\omega\tau]^2,$$
$$= 6\xi \;\; (\quad ^2_1 + \quad ^2_2)\omega^2 - [\quad _1(\lambda^2 - \omega^2) - 2 \quad _2\omega\lambda]e^{-\lambda\tau}\sin\omega\tau$$
$$- [2 \quad _1\omega\lambda + \quad _2(\lambda^2 - \omega^2)]e^{-\lambda\tau}\cos\omega\tau \quad ,$$

where $\xi = a(e^{-\lambda\tau} + \quad _1\sin\omega\tau + \quad _2\cos\omega\tau)^{-2}$, $A = \frac{1}{9}a(3\omega^2 - \lambda^2)$, $B = \frac{2}{27}a\lambda(9\omega^2 + 5\lambda^2)$.

3 . For the case $A = 0$, see equation 1.3.1.26.

4. $\quad - \quad = 2 \quad (\quad ^{1} \; ^{2} + 4 \quad + 3 \quad ^{2} \quad ^{-1} \; ^{2}).$

Solution in parametric form:

$$= \tfrac{1}{4}a(3 \quad 2\tau L \quad)^2, \qquad = \quad aL \;\; (R^2 L \quad + \tau), \qquad A = -\tfrac{1}{2}a^{1} \; ^{2},$$

where

$$L_+ = \begin{cases} \dfrac{\tau}{1+\tau^2} = \arctan\tau - \quad , & R_+ = \overline{\quad 1+\tau^2}, \\ \dfrac{\tau}{\tau^2 - 1} = \dfrac{1}{2}\ln\dfrac{\tau-1}{\tau+1} - \quad , & R_+ = \overline{\quad \tau^2 - 1}, \end{cases}$$

$$L_- = \frac{\tau}{1-\tau^2} = \frac{1}{2}\ln\frac{1+\tau}{1-\tau} - \quad , \qquad R_- = \overline{\quad 1-\tau^2}.$$

5. $\quad - \quad = \quad + B \quad ^{-1} - B^2 \quad ^{-3}.$

Solution in parametric form:

$$= \quad \text{—} \quad ^{-1} \; ^{2}, \qquad = (\tau+1) \quad \text{—} \quad ^{-1} \; ^{2} - B \quad \text{—} \quad ^{1} \; ^{2}.$$

Here,

$$= \begin{cases} (\tau^2 + \tau - A)\exp \quad \dfrac{2}{\overline{-\Delta}}\arctan\dfrac{2\tau+1}{\overline{-\Delta}}\Big) & \text{if } \Delta < 0, \\ (\tau^2 + \tau - A)\exp \quad -\dfrac{2}{2\tau+1}\Big) & \text{if } \Delta = 0, \\ (\tau^2 + \tau - A) \quad \dfrac{2\tau + 1 - \quad \overline{\Delta}}{2\tau + 1 + \quad \overline{\Delta}}\Big)^{\frac{1}{\overline{\quad}}} & \text{if } \Delta > 0, \end{cases}$$

$$= \begin{cases} \quad + 2B \quad \exp \quad \dfrac{2}{\overline{-\Delta}}\arctan\dfrac{2\tau+1}{\overline{-\Delta}}\Big) \quad \tau & \text{if } \Delta < 0, \\ \quad + 2B \quad \exp \quad -\dfrac{2}{2\tau+1}\Big) \quad \tau & \text{if } \Delta = 0, \\ \quad + 2B \quad \dfrac{2\tau + 1 - \quad \overline{\Delta}}{2\tau + 1 + \quad \overline{\Delta}}\Big)^{\frac{1}{\overline{\quad}}} \quad \tau & \text{if } \Delta > 0, \end{cases}$$

where $\Delta = 4A + 1$.

6. $\quad - \; = \quad {}^{k-1} - \; B \; {}^{k} + \; B^2 \; {}^{2k-1}.$

Particular solution: $\;_0 = \; -B \quad - \dfrac{A}{kB}.$

7. $\quad - \; = \quad {}^{-1} - \quad {}^{2} \; {}^{-3}.$

Solution in parametric form:

$$= a\tau^{-1}(\tau - \ln|1+\tau| - \quad)^{1\;\;2}, \qquad = a\;\frac{1+\tau}{\tau}(\tau - \ln|1+\tau| - \quad)^{1\;\;2} - \frac{1}{2}\tau(\tau - \ln|1+\tau| - \quad)^{-1\;\;2}\;,$$

where $A = a^2\;\;2.$

8. $\quad - \; = \quad + B \; {}^{-2} \quad .$

Solution in parametric form:

$$= A\ln\frac{\sqrt{\tau^2 + AB}}{A\ln\tau + \sqrt{\tau^2+AB} \; +} \;, \qquad = \tau\frac{A\ln\tau + \sqrt{\tau^2+AB} \; +}{\sqrt{\tau^2+AB}} - A.$$

9. $\quad - \; = \quad (\; {}^{2} \quad - 1).$

Solution in parametric form:

$$= A\ln\frac{\tau^2+1}{\tau}(\arctan\tau - \quad)\;, \qquad = \frac{A}{\tau}\big[\tau + (\tau^2 - 1)(\arctan\tau - \quad)\big].$$

In the solutions of equations 10–15, the following notation is used:

$$_{,} \; = \quad (1 \quad \tau \;\; {}^{+1})^{\frac{1}{-2}} \;\; \tau - \quad , \qquad = \quad _{,0} = \quad (1 \quad \tau \;\; {}^{+1})^{-1\;\;2} \;\; \tau - \quad ,$$

$$R \quad = \quad \sqrt{1 \quad \tau \;\; {}^{+1}}, \qquad = R \qquad - \tau.$$

10. $\quad - \quad = -\dfrac{2(\quad + 1)}{(\quad + 3)^2} \quad + \qquad .$

Solution in parametric form:

$$= \frac{+3}{-1} a\tau \;\; {}^{\frac{2}{-1}}, \qquad = a \;\; {}^{\frac{2}{-1}} \;\; R \qquad + \frac{2}{-1}\tau \;\; ,$$

where $A = \quad \dfrac{+1}{2} \quad \dfrac{-1}{+3} \quad {}^{+1} \; a^{1-} \quad .$

11. $\quad - \quad = -\frac{2}{9} \quad + 6 \quad {}^{2}(1 + 2 \quad {}^{-1\;\;2}), \qquad > 0.$

Solution in parametric form:

$$= A^2R^{-4} \quad {}^{-2}(R^2 \qquad 6\tau^{1\;\;2})^2, \qquad = -12A^2R^{-4} \quad {}^{-2}(R^2 \quad - 2\tau),$$

where $\quad = \quad _{-1\;\;2,\,3\;\;2}, \;\; R = R_{-1\;\;2}.$

12. $\quad - \quad = \dfrac{2(\quad -1)}{(\quad -3)^2} \quad + \dfrac{2}{(\quad -3)^2}\big[\quad (\quad +3) \;\; {}^{1\;\;2} + (4 \quad {}^{2} + 3 \quad +9) \quad +3 \quad (\quad +3) \quad {}^{2\;\;-1\;\;2}\big].$

Solution in parametric form:

$$= \frac{a}{(\quad -3)^2}\tau^{-2}[(\quad -3)R \qquad +3\tau]^2, \qquad = \frac{a}{-3}\tau^{-2} \qquad [\;(\quad -1)\tau \;\; {}^{+1} \qquad -2 \qquad +2\tau R \quad],$$

where $A = -\dfrac{a^{1\;\;2}}{-3}.$

13. $- \;= \dfrac{2 \quad +1}{4 \quad ^{2}} + \quad ^{-1} - \quad ^{2} \quad ^{-3}.$

Solution in parametric form:

$$= \frac{1 \quad 2}{a\tau^{1 \quad 2} R^2}, \qquad = \frac{\tau - \left[1 \mp (2 \quad +1)\tau \quad ^{+1}\right] R^2}{2a \quad \tau^{1 \quad 2} R^2 \quad 1 \quad 2}, \qquad \text{where} \quad a^2 = -2 \quad A, \qquad = \quad _{,3 \quad 2}.$$

14. $- \;= \frac{4}{9} \quad + 2 \quad ^{2} + 2 \quad ^{2} \quad ^{3}.$

Solution in parametric form:

$$= \frac{1}{3A}\tau^{-1} \quad _3, \qquad = \frac{1}{9A}\tau^{-2} \quad _3(\tau R_3 - \quad _3 \quad \tau^4 \quad _3).$$

15. $- \;= -\frac{3}{16} \quad + 5 \quad ^{-1 \quad 3} - 12 \quad ^{2} \quad ^{-5 \quad 3}.$

Solution in parametric form:

$$= a\tau^{1 \quad 2} \quad ^{-3 \quad 2} \quad ^{3 \quad 2}, \qquad = \tfrac{1}{4} a\tau^{1 \quad 2} \quad ^{-3 \quad 2} \quad ^{-1 \quad 2}(\quad ^{2} - 2\tau \quad - \tau^{-2 \quad 3} \quad ^{2}),$$

where $A = \frac{1}{24} a^{4 \quad 3}, \quad = \quad _{-5 \quad 3}, \quad = \quad _{-5 \quad 3}.$

In the solutions of equations 16–18, the following notation is used:

$$= \quad \exp(\mp\tau^2) \quad \tau - \quad , \qquad g = 2\tau \quad \exp(\mp\tau^2) \quad \tau - \qquad \exp(\mp\tau^2).$$

16. $- \;= \quad ^{-1}.$

Solution in parametric form:

$$= a \quad ^{-1} \exp(\mp\tau^2), \qquad = a \quad ^{-1}\left[\exp(\mp\tau^2) \quad 2\tau \quad \right], \qquad \text{where} \quad A = \mp 2a^2.$$

17. $- \;= -\frac{1}{4} \quad + \frac{1}{4} \quad (\quad ^{1 \quad 2} + 5 \quad + 3 \quad ^{2} \quad ^{-1 \quad 2}).$

Solution in parametric form:

$$= \tfrac{1}{16} a\left[3 \quad 8\tau \quad \exp(\quad \tau^2)\right]^2, \qquad = a \quad \exp(\quad \tau^2)\left[(2\tau^2 \quad 1) \quad \exp(\quad \tau^2) \quad \tau\right], \qquad \text{where } A = \tfrac{1}{4} \quad \overline{\quad a}.$$

18. $- \;= \dfrac{2 \quad ^{2}}{\overline{\quad ^{2} \quad 8 \quad ^{2}}}.$

Solution in parametric form:

$$= \quad a(\quad g)^{-1}(g^2 \mp 2 \quad ^{2}), \qquad = a(\quad g)^{-1}[\exp(\mp\tau^2)g - 2 \quad ^{2}].$$

In the solutions of equations 19–21, the following notation is used:

$$= \quad \overline{\tau(\tau+1)} - \ln \quad \left(\quad \overline{\tau} + \quad \overline{\tau+1} \quad \right), \qquad R = \sqrt{\frac{\tau+1}{\tau}}, \qquad = 1 - \sqrt{\frac{\tau+1}{\tau}} \ln \quad \left(\quad \overline{\tau} + \quad \overline{\tau+1} \quad \right).$$

19. $- \;= 2 \quad + \quad ^{-2}.$

Solution in parametric form:

$$= \tfrac{1}{3} a \quad ^{-2 \quad 3}\tau, \qquad = a \quad ^{-2 \quad 3}\left(\tfrac{2}{3}\tau - R \quad \right), \qquad \text{where} \quad A = -\tfrac{243}{2} a^3.$$

20. $- \;= -\frac{6}{25} \quad + \frac{2}{25} \quad (2 \quad ^{1 \quad 2} + 19 \quad + 6 \quad ^{2} \quad ^{-1 \quad 2}).$

Solution in parametric form:

$$= a\tau^{-2}(5R \quad - 3\tau)^2, \qquad = 5a\tau^{-3} \quad [(2\tau+3) \quad - 2\tau^2 R], \qquad \text{where} \quad A = - \quad \overline{\quad a}.$$

21. $- \ = \frac{3}{8} \ + \frac{3}{8} \ \overline{\ ^2 + \ ^2} - \frac{\ ^2}{16 \ \overline{\ ^2 + \ ^2}}.$

Solution in parametric form:

$$= \frac{a}{2\ \overline{2}} \frac{\ ^2 - 2\tau^2}{\tau \ \overline{\ \ }}, \qquad = \frac{a}{4\ \overline{2}} \frac{4\tau \ ^2 - \ ^2}{\tau \ \overline{\ \ }}.$$

In the solutions of equations 22–25, the following notation is used:

$$_2 = \ (\tau^2 - 1), \qquad _3 = \tau^3 - 3\tau + \ , \qquad _4 = \ (\tau^4 - 6\tau^2 + 4 \ \tau - 3).$$

22. $- \ = -\frac{4}{25} \ + \ ^{-1\ 2}.$

Solution in parametric form:

$$= 5a \ _2^2 \ _3^{-4\ 3}, \qquad = 4a \ _3^{-4\ 3}(\ _2^2 - \tau \ _3), \qquad \text{where} \quad A = \ \tfrac{4}{5}a \ \overline{5a}.$$

23. $- \ = -\frac{9}{100} \ + \ ^{-5\ 3}.$

Solution in parametric form:

$$= 10a \ _3^{3\ 2} \ _4^{-9\ 8}, \qquad = 9a \ _3^{-1\ 2} \ _4^{-9\ 8}(\ _3^2 - \ _2 \ _4), \qquad \text{where} \quad A = \ 9a^2(10a)^{2\ 3}.$$

24. $- \ = -\frac{12}{49} \ + \frac{2}{49} \ (5 \ ^{1\ 2} + 34 \ + 15 \ ^2 \ ^{-1\ 2}).$

Solution in parametric form:

$$= a \ _2^{-4}(14\tau \ _3 - 9 \ _2^2)^2, \qquad = 28a \ _2^{-4} \ _3(4\tau^2 \ _3 - 3\tau \ _2^2 \mp \ _2 \ _3), \qquad \text{where} \quad A = -3 \ \overline{a}.$$

25. $- \ = -\frac{12}{49} \ + \frac{1}{98} \ (25 \ ^{1\ 2} + 41 \ + 10 \ ^2 \ ^{-1\ 2}).$

Solution in parametric form:

$$= a \ _3^{-4}(21 \ _2 \ _4 - 16 \ _3^2)^2, \qquad = 21a \ _3^{-4} \ _4(9 \ _2^2 \ _4 \mp \ _4^2 - 8 \ _2 \ _3^2), \qquad \text{where} \quad A = -8 \ \overline{a}.$$

In the solutions of equations 26–29, the following notation is used:

$$_1 = \exp(\ \overline{3}\tau) + \ \sin\tau, \qquad _2 = 2\exp(\ \overline{3}\tau) - \ \sin\tau + \ \overline{3} \ \cos\tau,$$
$$_3 = 2\exp(\ \overline{3}\tau) - \ \sin\tau - \ \overline{3} \ \cos\tau, \qquad _4 = 4 \ _1 \ _3 - \ _2^2.$$

26. $- \ = -\frac{2}{9} \ + \ ^{-1\ 2}.$

Solution in parametric form:

$$= 3a \ _1^{-2} \ _2^2, \qquad = 2a \ _1^{-2}(\ _2^2 - 2 \ _1 \ _3), \qquad \text{where} \quad A = 16(3a)^{3\ 2}.$$

27. $- \ = -\frac{5}{36} \ + \ ^{-7\ 5}.$

Solution in parametric form:

$$= 48a \ _1^{5\ 2} \ _4^{-5\ 4}, \qquad = 5a \ _1^{-1\ 2} \ _4^{-5\ 4}(8 \ _1^3 - \ _2 \ _4), \qquad \text{where} \quad A = (48a)^{2\ 5}a^2.$$

28. $- \ = -\frac{12}{49} \ + \frac{6}{49} \ (-3 \ ^{1\ 2} + 23 \ + 12 \ ^2 \ ^{-1\ 2}).$

Solution in parametric form:

$$= a \ _2^{-4}(7 \ _1 \ _3 - 2 \ _2^2)^2, \qquad = -7a \ _1 \ _2^{-4}(4 \ _1^2 \ _2 - 4 \ _1 \ _3^2 + \ _2^2 \ _3), \qquad \text{where} \quad A = \ \overline{a} \ 2.$$

29. $\quad - \quad = -\frac{30}{121} \quad + \frac{3}{242} \quad (21 \quad ^{1\ \ 2} + 35 \quad + 6 \quad ^{2\ \ -1\ \ 2}).$

Solution in parametric form:

$$= a \quad _1^{-6}(11 \quad _2 \quad _4 - 64 \quad _1^3)^2, \qquad = -11a \quad _1^{-6} \quad _4(\quad _4^2 - 5 \quad _2^2 \quad _4 + 32 \quad _1^3 \quad _2), \quad \text{where} \quad A = -32 \quad \overline{a}.$$

In the solutions of equations 30 and 31, the following notation is used:

$$_1 = \tanh(\tau + \quad) + \tan\tau, \qquad _2 = \tanh(\tau + \quad) - \tan\tau,$$

$$\theta_1 = \cosh\tau - \sin(\tau + \quad), \quad \theta_2 = \sinh\tau + \cos(\tau + \quad), \quad \theta_3 = \sinh\tau - \cos(\tau + \quad).$$

30. $\quad - \quad = -\frac{3}{16} \quad + \quad ^{-5\ \ 3}.$

1 . Solution in parametric form with $A < 0$:

$$= 8a \quad _1^{-3\ \ 2}, \qquad = 3a \quad _1^{-3\ \ 2}(2 - \quad _1 \quad _2), \qquad \text{where} \quad A = -12a^8 \quad ^3.$$

2 . Solution in parametric form with $A > 0$:

$$= 4a\theta_1^{3\ \ 2}\theta_2^{-3\ \ 2}, \qquad = 3a\theta_1^{-1\ \ 2}\theta_2^{-3\ \ 2}(\theta_1^2 - \theta_2\theta_3), \qquad \text{where} \quad A = 3a^2(4a)^{2\ \ 3}.$$

31. $\quad - \quad = -\frac{12}{49} \quad + \frac{4}{49} \quad (-10 \quad ^{1\ \ 2} + 27 \quad + 10 \quad ^{2\ \ -1\ \ 2}).$

1 . Solution in parametric form with $A < 0$:

$$= a(10 - 7 \quad _1 \quad _2)^2, \qquad = 7a \quad _1(\quad _1^3 + 3 \quad _1 \quad _2^2 - 4 \quad _2), \qquad \text{where} \quad A = -2 \quad \overline{a}.$$

2 . Solution in parametric form with $A > 0$:

$$= a\theta_1^{-4}(7\theta_2\theta_3 - 5\theta_1^2), \qquad = -7a\theta_1^{-4}\theta_2(\theta_2^3 - 3\theta_2\theta_3^2 + 2\theta_1^2\theta_3), \qquad \text{where} \quad A = \quad \overline{a}.$$

In the solutions of equations 32–43, the following notation is used:

$$Z = \begin{cases} \quad _1 \quad (\tau) + \quad _2 \quad (\tau) & \textit{for the upper sign,} \\ \quad _1 \quad (\tau) + \quad _2 \quad (\tau) & \textit{for the lower sign,} \end{cases}$$

$$= \tau(Z \quad)'_\tau + \quad Z \quad, \quad Z = Z_1 \quad _3, \quad _1 = \tau Z'_\tau + \tfrac{1}{3}Z, \quad _2 = \quad _1^2 \quad \tau^2 Z^2, \quad _3 = \tfrac{2}{3}\tau^2 Z^3 - 2 \quad _1 \quad _2,$$

where $\quad (\tau)$ *and* $\quad (\tau)$ *are the Bessel functions, and* $\quad (\tau)$ *and* $\quad (\tau)$ *are the modified Bessel functions.*

The solutions of equations 32–43 contain only the ratio $Z'_\tau \quad Z$, *where the prime denotes differentiation with respect to* τ. *Therefore, for symmetry, function* Z *is defined in terms of two "arbitrary" constants* $_1$ *and* $_2$ *(instead, we can set, for instance,* $_1 = 1$ *and* $_2 = \quad$*).*

32. $\quad - \quad = \quad ^{-1\ \ 2}.$

Solution in parametric form:

$$= a\tau^{-4\ \ 3}Z^{-2} \quad _1^2, \qquad = a\tau^{-4\ \ 3}Z^{-2} \quad _2, \qquad \text{where} \quad A = \mp\tfrac{1}{3}a^{3\ \ 2}.$$

33. $\quad - \quad = \quad ^{-2}.$

Solution in parametric form:

$$= 2a\tau^{4\ \ 3}Z^2 \quad _2^{-1}, \qquad = \quad 3a\tau^{-2\ \ 3}Z^{-1} \quad _2^{-1} \quad _3, \qquad \text{where} \quad A = -36a^3.$$

34. $\quad - \quad = \quad (\quad + 2)[\quad ^{1\ \ 2} + 2(\quad + 2) \quad + (\quad + 1)(\quad + 3) \quad ^{2\ \ -1\ \ 2}].$

Solution in parametric form:

$$= aZ^{-2}[\quad - (\quad + 1)Z \quad]^2, \qquad = aZ^{-2}(\quad ^2 - 2 \quad Z \qquad \tau^2 Z^2), \qquad \text{where } A = \quad \overline{a}, \qquad = \frac{1}{\quad + 2}.$$

35. $\quad - \quad = \quad (\quad + 2)[\quad ^{1\ \ 2} + 2(\quad + 2) \quad + (2 \quad + 3) \quad ^{2\ \ -1\ \ 2}].$

Solution in parametric form:

$$= a \quad ^{-2}[\tau^2 Z \qquad (2 - \quad) \quad]^2, \qquad = \quad a\tau^2 \quad ^{-2}[\quad ^2 + 2(1 - \quad)Z \qquad \tau^2 Z^2],$$

where $A = \mp \quad \overline{a}, \quad = \dfrac{1}{\quad + 2}.$

36. $- \; = \; {}^{1\;2} + 2 \; {}^{2} + B \; {}^{-1\;2}$.

Solution in parametric form:

$$= A^2 Z^{-2}(\tau Z' - Z\;)^2, \qquad = A^2 Z^{-2}[\tau^2 (Z')^2 - (\;{}^2 \mp \tau^2) Z^2],$$

where $B = (1 - \;{}^2)A^3$ and the prime denotes differentiation with respect to τ.

37. $- \; = 2 \; {}^{2} - \; {}^{1\;2}$.

Solution in parametric form:

$$= a(Z_0')^{-2}(\tau Z_0 \quad 2Z_0')^2, \qquad = \quad a\tau(Z_0')^{-2}[\tau(Z_0')^2 + 2Z_0 Z_0' \quad \tau Z_0^2],$$

where $A = \quad \overline{a}$ and the prime denotes differentiation with respect to τ.

38. $- \; = -\frac{12}{49} \; + \frac{6}{49} \; (\; {}^{1\;2} + 8 \; + 5 \; {}^{2\;\;-1\;2})$.

Solution in parametric form:

$$= 3a \;{}_1^{-4}(5 \;{}_1^2 - 7\tau^2 Z^2)^2, \qquad = 28a\tau^2 Z^2 \;{}_1^{-4}(3\tau^2 Z^2 - Z \;{}_1 - 3 \;{}_1^2),$$

where $A = 2 \quad \overline{3a}$; Z and $\;{}_1$ are expressed in term of modified Bessel functions.

39. $- \; = -\frac{6}{25} \; + \frac{6}{25} \; (2 \; {}^{1\;2} + 7 \; + 4 \; {}^{2\;\;-1\;2})$.

Solution in parametric form:

$$= a\tau^{-4} Z^{-6}(\;{}_1\;{}_2 - 2\;{}_3)^2, \qquad = 5a\tau^{-4} Z^{-6}\;{}_2(\;{}_2^2 - \;{}_1\;{}_3), \qquad \text{where} \quad A = - \quad \overline{a} \;\; 2.$$

40. $- \; = -\frac{3}{16} \; + 3 \; {}^{-1\;3} - 12 \; {}^{2\;\;-5\;3}$.

Solution in parametric form:

$$= \frac{a \;{}_{3\;2}^{3\;2}}{\tau^{3\;2} Z_{3\;2}^{3\;2}}, \qquad = \frac{3a}{4}\,\frac{\;{}_{3\;2}^{2} - 2Z_{3\;2}\;{}_{3\;2} - \tau^2 Z_{3\;2}^2}{\tau^{3\;2} Z_{3\;2}^{3\;2}\;{}_{3\;2}^{1\;2}},$$

where $Z_{3\;2}$ and $\;{}_{3\;2}$ are expressed in term of modified Bessel functions; $A = \frac{1}{8}a^{4\;\;3}$.

41. $- \; = \frac{3}{8} \; + \frac{3}{8} \; \overline{\;{}^2 \quad {}^2} \quad \frac{3\;{}^2}{16\;\overline{\;{}^2 \quad {}^2}}$.

Solution in parametric form:

$$= -\tfrac{1}{4} a\tau^{-1} Z^{-3\;2} \;{}_1^{-1\;2} \;{}_2^{-1}(2\tau^2 Z^3 \;{}_1 \mp 3 \;{}_2^2),$$
$$= \mp\tfrac{1}{8} a\tau^{-1} Z^{-3\;2} \;{}_1^{-1\;2} \;{}_2^{-1}(3 \;{}_2^2 - 12 \;{}_1^2 \;{}_2 \quad 4\tau^2 Z^3 \;{}_1),$$

where $\;{}^2 = \frac{3}{2}a^2$.

42. $- \; = \frac{9}{32} \; + \frac{15}{32} \; \overline{\;{}^2 \mp \;{}^2} \mp \frac{3\;{}^2}{64\;\overline{\;{}^2 \mp \;{}^2}}$.

Solution in parametric form:

$$= -\tfrac{1}{2} a\tau^{-1} Z^{-3\;2} \;{}_2^{-3\;2} \;{}_3^{-1\;2}(2\tau^2 Z^3 \;{}_3 \quad 3 \;{}_2^3),$$
$$= \quad \tfrac{1}{4} a\tau^{-1} Z^{-3\;2} \;{}_2^{-3\;2} \;{}_3^{-1\;2}(3 \;{}_2^3 \mp \tau^2 Z^3 \;{}_3 - 3 \;{}_3^2),$$

where $\;{}^2 = 6a^2$.

43. $\quad - \quad = -\dfrac{3}{32} \quad -\dfrac{3}{32} \quad \overline{\quad{}^2 + \quad{}^2} + \dfrac{15 \quad{}^2}{64 \quad \overline{\quad{}^2 + \quad{}^2}}.$

Solution in parametric form:

$$= \tfrac{1}{2}a \quad {}_2^{-3\ 2} \quad {}_3^{-1}(\quad {}_3^2 - \quad {}_2^3), \qquad = \tfrac{1}{24}a \quad {}_2^{-3\ 2} \quad {}_3^{-1}(3 \quad {}_3^2 - 12 \quad {}_2^3 \quad 4\tau^2 Z^3 \quad {}_3).$$

In the solutions of equations 44–52, the following notation is used:

$$_1 = \tau^3 \quad \overline{(4\wp^3 - 1)} + 3\tau^2\wp \mp 1, \qquad {}_2 = \tau^2\wp \mp 1,$$
$$_3 = \quad \overline{(4\wp^3 - 1)} \quad 2\tau\wp^2, \qquad {}_4 = \tau \quad \overline{(4\wp^3 - 1)} + 2\wp.$$

Here, the function $\wp = \wp(\tau)$ *is given implicitly as follows:* $\tau = \dfrac{\wp}{\overline{(4\wp^3 - 1)}} - \quad {}_2$. *The upper sign in this formula corresponds to the classical elliptic Weierstrass function* $\wp = \wp(\tau + \quad {}_2, 0, 1)$.

44. $\quad - \quad = \quad {}^2 - \frac{9}{625} \quad {}^{-1}.$

Solution in parametric form:

$$= 5a\big(\tau^2\wp \mp \tfrac{1}{2} \quad , \qquad = a\tau^2 \quad {}_4, \qquad \text{where} \quad A = \tfrac{6}{125}a^{-1}.$$

45. $\quad - \quad = -\frac{6}{25} \quad + \quad {}^2.$

Solution in parametric form:

$$= 5a\tau^2\wp, \qquad = a\tau^2 \quad {}_4, \qquad \text{where} \quad A = \tfrac{6}{125}a^{-1}.$$

46. $\quad - \quad = \frac{6}{25} \quad + \quad {}^2.$

Solution in parametric form:

$$= 5a \quad {}_2, \qquad = a\tau^2 \quad {}_4, \qquad \text{where} \quad A = \tfrac{6}{125}a^{-1}.$$

47. $\quad - \quad = 12 \quad + \quad {}^{-5\ 2}.$

Solution in parametric form:

$$= a\wp^{-6\ 7} \quad {}_3^{-4\ 7}, \qquad = a\wp^{-6\ 7} \quad {}_3^{-4\ 7}(14\wp^2 \quad {}_4 - 3), \qquad \text{where} \quad A = \mp 147a^{7\ 2}.$$

48. $\quad - \quad = \frac{63}{4} \quad + \quad {}^{-5\ 3}.$

Solution in parametric form:

$$= 2a \quad {}_3^{3\ 2} \quad {}_4^{-9\ 8}, \qquad = a \quad {}_3^{-1\ 2} \quad {}_4^{-9\ 8}(9 \quad {}_3^2 \mp 16\wp \quad {}_4^2), \qquad \text{where} \quad A = -\tfrac{128}{3}a^2(2a)^{2\ 3}.$$

49. $\quad - \quad = 2 \quad + 2 \quad (10 \quad {}^{1\ 2} + 31 \quad + 30 \quad {}^{2\ -1\ 2}).$

Solution in parametric form:

$$= a\wp^{-2}\big[\tau \quad \overline{(4\wp^3 - 1)} - 3\wp\big]^2, \qquad = -2a\tau\wp^{-2}\big[\wp \quad \overline{(4\wp^3 - 1)} \quad 2\tau\wp^3 \quad \tau\big],$$

where $A = \quad \overline{a}$.

50. $\quad - \quad = 2 \quad + 2 \quad (-10 \quad {}^{1\;2} + 19 \quad + 30 \quad {}^{2\;\;-1\;\;2}).$

Solution in parametric form:

$$= a \quad {}_{2}^{-2}(\quad {}_{1} - 6 \quad {}_{2})^{2}, \qquad = -2a \quad {}_{2}^{-2}(\; 6 \quad {}_{2}^{3} - \quad {}_{1}^{2} + 7 \quad {}_{1} \quad {}_{2}), \qquad \text{where} \quad A = - \quad \overline{a}.$$

51. $\quad - \quad = -\frac{28}{121} \quad + \frac{2}{121} \quad (5 \quad {}^{1\;2} + 106 \quad + 15 \quad {}^{2\;\;-1\;\;2}).$

Solution in parametric form:

$$= a(22\wp^{2} \quad {}_{4} - 5)^{2}, \qquad = \quad 44a\wp^{2} \quad {}_{3}(7\wp \quad {}_{3} \mp 2\tau), \qquad \text{where} \quad A = \quad 2 \quad \overline{a}.$$

52. $\quad - \quad = -\frac{12}{49} \quad + \frac{1}{49} \quad (5 \quad {}^{1\;2} + 262 \quad + 65 \quad {}^{2\;\;-1\;\;2}).$

Solution in parametric form:

$$= a \quad {}_{3}^{-4}(28\wp \quad {}_{4}^{2} \mp 15 \quad {}_{3}^{2})^{2}, \qquad = 56a \quad {}_{3}^{-4} \quad {}_{4}^{2}(6\wp \quad {}_{2} + \quad {}_{4}), \qquad \text{where} \quad A = \mp 3 \quad \overline{a}.$$

In the solutions of equations 53–60, the following notation is used:

$$= \quad \frac{\tau \quad \tau}{\overline{\quad (4\tau^{3}-1)}} \quad \textit{(incomplete elliptic integral of the second kind)},$$

$$R = \quad \overline{\quad (4\tau^{3}-1)}, \quad {}_{1} = \tau(2 \quad \mp \tau^{-1}R + \quad), \quad {}_{2} = \tau^{-1}(R \quad {}_{1} - 1), \quad {}_{3} = 4\tau \quad {}_{1}^{2} \mp \quad {}_{2}^{2}.$$

53. $\quad - \quad = -\frac{12}{49} \quad + \quad {}^{1\;2}.$

Solution in parametric form:

$$= 7a\tau^{2}(\quad + \quad)^{-4}, \qquad = -2a(\quad + \quad)^{-4}(\quad R + \quad R - 2\tau^{2}), \qquad \text{where} \quad A = \quad \tfrac{12}{49} \quad \overline{7a}.$$

54. $\quad - \quad = 6 \quad + \quad {}^{-4}.$

Solution in parametric form:

$$= a\tau^{-3} \quad {}^{5} \quad {}_{1}^{-2} \quad {}^{5}, \qquad = a\tau^{-3} \quad {}^{5} \quad {}_{1}^{-2} \quad {}^{5}(5R \quad {}_{1} - 2), \qquad \text{where} \quad A = \mp 150a^{5}.$$

55. $\quad - \quad = 20 \quad + \quad {}^{-1\;2}.$

Solution in parametric form:

$$= a \quad {}_{1}^{-4} \quad {}^{3} \quad {}_{2}^{2}, \qquad = -4a \quad {}_{1}^{-4} \quad {}^{3}(\quad {}_{2}^{2} \mp 9\tau \quad {}_{1}^{2}), \qquad \text{where} \quad A = \quad 108a^{3} \quad {}^{2}.$$

56. $\quad - \quad = \frac{15}{4} \quad + \quad {}^{-7}.$

Solution in parametric form:

$$= a \quad {}_{1}^{1} \quad {}^{2} \quad {}_{3}^{-3} \quad {}^{8}, \qquad = \tfrac{1}{2}a \quad {}_{1}^{-3} \quad {}^{2} \quad {}_{3}^{-3} \quad {}^{8}(\quad {}_{2} \quad {}_{3} - 3 \quad {}_{1}^{2}), \qquad \text{where} \quad A = \quad \tfrac{3}{4}a^{8}.$$

57. $\quad - \quad = -\frac{10}{49} \quad + \frac{2}{49} \quad (4 \quad {}^{1\;2} + 61 \quad + 12 \quad {}^{2\;\;-1\;\;2}).$

Solution in parametric form:

$$= a(7R \quad {}_{1} - 3)^{2}, \qquad = 14a \quad {}_{1}[\quad (10\tau^{3} - 1) \quad {}_{1} - R], \qquad \text{where} \quad A = \quad \overline{a}.$$

58. $\quad - \quad = -\frac{12}{49} \quad + \frac{2}{49} \quad (\quad {}^{1\;2} + 166 \quad + 55 \quad {}^{2\;\;-1\;\;2}).$

Solution in parametric form:

$$= a \quad {}_{2}^{-4}(42\tau \quad {}_{1}^{2} \mp 5 \quad {}_{2}^{2})^{2}, \qquad = \mp 84a \quad {}_{1}^{2} \quad {}_{2}^{-4}(3\tau \quad {}_{2}^{2} + \quad {}_{2} \mp 12\tau^{2} \quad {}_{1}^{2}), \qquad \text{where} \quad A = \quad \overline{a}.$$

59. $\quad - \quad = -\frac{4}{25} \quad + \frac{1}{50} \quad (7 \quad ^{1\ 2} + 49 \quad + 6 \quad ^{2\ \ -1\ 2}).$

Solution in parametric form:

$$= a \ {}_1^{-4}(5 \ {}_2 \ {}_3 - 16 \ {}_1^2)^2, \qquad = -5a \ {}_1^{-4} \ {}_3(\ 3 \ {}_3^2 - \ {}_2^2 \ {}_3 + 8 \ {}_1^2 \ {}_3), \qquad \text{where} \quad A = 8 \quad \overline{a}.$$

60. $\quad - \quad = \frac{15}{4} \quad + 6 \quad ^{-1\ 3} - 3 \quad ^{2\ \ -5\ 3}.$

Solution in parametric form:

$$= 2a\tau^{3\ 2} \ {}_1^{3\ 2} \ {}_2^{-3\ 4}, \qquad = a\tau^{-1\ 2} \ {}_1^{-1\ 2} \ {}_2^{-3\ 4}(2\tau \ {}_2^2 + \ {}_2 - 3\tau^2 \ {}_1^2), \quad \text{where} \ A = -\tfrac{1}{3}a(2a)^{1\ 3}.$$

61. $\quad - \quad = -\frac{3}{16} \quad + \quad ^{-1\ 3} + B \quad ^{-5\ 3}.$

The substitution $\quad = \tau^{-3\ 2}$ leads to an equation

$$'_\tau = -\tfrac{3}{2}\tau^{-5\ 2} \quad + \tfrac{9}{32}\tau^{-4} - \tfrac{3}{2}A\tau^{-2} - \tfrac{3}{2}B,$$

coincident with equation 1.3.3.13 when $\quad = -1 \ \ 2, \quad = 0, \quad = 3 \ \ 4, \quad = 3A \ \ 2, a^2 = -3B.$

62. $\quad - \quad = -\frac{5}{36} \quad + \quad ^{-3\ 5} - B \quad ^{-7\ 5}, \qquad B > 0.$

The transformation $\quad = (\quad - \tfrac{1}{3} \quad \overline{\tau} + A \quad B \quad ^{-5\ 4}, \quad = \tfrac{5}{6} \quad + (\tfrac{5}{3}B \quad ^{1\ 2} \quad \overline{\tau} \quad ^{1\ 5}$ leads to an equation of the form 1.3.1.3:

$$'_\tau - \quad = -\frac{2}{9}\tau + \frac{2A}{3B} + \quad \frac{5}{27B} \quad ^{1\ 2} \frac{1}{\overline{\tau}}.$$

63. $\quad - \quad = \ (\quad ^2 + B \quad + \quad)^{-1\ 2}.$

The transformation $\quad = \dfrac{4(\ {}_2 \quad ^2 + \ {}_1 \quad + \ {}_0)}{4A - \quad ^2}, \quad = \xi + \dfrac{4(\ {}_2 \quad ^2 + \ {}_1 \quad + \ {}_0)}{4A - \quad ^2}$, where parameters ${}_2$, ${}_1$, and ${}_0$ are found from the relations $B = 4A \ {}_2 - \ {}_0$ and $\quad = \ {}_1^2 - 4 \ {}_0 \ {}_2$, leads to a Riccati equation:

$$k \quad ' = (-\tfrac{1}{4}\xi + \ {}_2) \quad ^2 + \ {}_1 \quad + A\xi + \ {}_0.$$

For $\quad > 0$, we can set ${}_2 = 0$, ${}_1 = \quad \overline{}$, and ${}_0 = -B$.

In books by Zaitsev & Polyanin (1993, 1994) it is shown that the original equation is reducible to the degenerate hypergeometric equation.

64. $\quad - \quad = -\frac{12}{49} \quad + 3 \quad (\frac{1}{49} + B) \quad ^{1\ 2} + 3 \quad ^2(\frac{4}{49} - \frac{5}{2}B) + \frac{15}{4} \quad ^3(\frac{1}{49} + \frac{5}{4}B) \quad ^{-1\ 2}.$

The substitution $\quad = (\xi^2 + \tfrac{5}{4}A)^2$ leads to an equation of the form 1.3.3.13 with $\quad = 3, a = 4 \ \ 7, \quad = 0, \quad = A$, and $\quad = 12A(\tfrac{2}{7} - B)$:

$$' = (4\xi^2 + 5A)\xi \quad - \left[\tfrac{48}{49}\xi^4 + 12A\left(\tfrac{2}{7} - B \quad \xi^2 + 3A^2\right]\xi^3.$$

65. $\quad - \quad = -\frac{6}{25} \quad + \frac{4}{75}B^2\left[(2 - \quad) \quad ^{1\ 3} - \frac{3}{2}B(2 \quad + 1) + B^2(1 - 3 \quad) \quad ^{-1\ 3} - \quad B^3 \quad ^{-2\ 3}\right].$

The transformation $\quad = \quad ^{-3}, \quad = \quad \xi + \dfrac{B^2(3 - 2B \quad)}{5 \quad (B \quad + 1)} \quad + \dfrac{1}{B} \quad ^2 \quad ^{-2}$ leads to a Riccati equation:

$$\left(2\xi^2 - \tfrac{2}{5}B^3\xi + \tfrac{4}{25}AB^6 \quad ' = B\xi \quad ^2 + \left(\xi - \tfrac{2}{5}B^3 \quad + \tfrac{3}{5}B^2.\right.\right.$$

66. $\quad - \quad = \frac{3}{4} \quad - \frac{3}{2} \quad ^{1\ 3} + \frac{3}{4} \quad ^{2\ \ -1\ 3} - \frac{27}{625} \quad ^{4\ \ -5\ 3}.$

The transformation

$$= A^{3\ 2} \quad ^{-3\ 2}, \qquad = 3A^{3\ 2}\left(\xi \quad ^2 - \tfrac{3}{25} \quad ^2 - \tfrac{1}{2} \quad + \tfrac{1}{2} \quad ^{-3\ 2}, \qquad \text{where} \qquad = \quad \left(\xi^2 - \tfrac{6}{25}\xi \quad ^{-1},\right.\right.$$

leads to an equation of the form 1.3.1.46: $\quad ' - \quad = \frac{6}{25}\xi - \xi^2.$

67. $- \;\; = -\frac{6}{25} \;\; + \frac{7}{5} \;\;\; {}^{1\ \ 3} + \frac{31}{3} \;\;\; {}^{2\ \ -1\ \ 3} - \frac{100}{3} \;\;\; {}^{4\ \ -5\ \ 3}.$

Denote $A = \frac{7}{100}a$ and perform the transformation

$$= \xi^{3\ \ 2}, \qquad = \tfrac{7}{20}\big(\quad + \tfrac{8}{7}\xi - \tfrac{3}{5}a - \tfrac{7}{50}a^2\xi^{-1} \qquad \overline{\xi}, \qquad \text{where} \quad \xi = z - \tfrac{3}{10}a.$$

As a result we obtain an equation of the form 1.3.4.30 with $\; = \frac{1}{7}, \;\; = -\frac{3}{10}a$:

$$\big[\big(z - \tfrac{3}{10}a \qquad + \tfrac{8}{7}z^2 - \tfrac{9}{7}az + \tfrac{1}{7}a^2\big] \;\; ' = -\tfrac{1}{2} \;\; {}^2 + 2z \quad .$$

68. $- \;\; = -\frac{10}{49} \;\; + \frac{13}{5} \;\;\; {}^{2\ \ -1\ \ 5} - \frac{7}{20} \;\;\; {}^{3\ \ -4\ \ 5}.$

Denote $A = 8a^{-2}$. The transformation

$$\overline{\tau} = a^3\big(\tfrac{5}{112} \;\; {}^{3\ \ 5} - \tfrac{1}{16} \;\;\; {}^{-2\ \ 5} \quad , \qquad = \tfrac{4}{7}\tau + \;\; {}^{-3\ \ 5} - \tfrac{39}{42}a^2$$

leads to an equation of the form 1.3.1.64 with $B = -1 \;\; 49$:

$$\,'_{\tau} - \quad = -\tfrac{12}{49}\tau + \tfrac{39}{98}a^2 - \tfrac{15}{784}a^3\tau^{-1\ \ 2}.$$

69. $- \;\; = -\frac{33}{196} \;\; + \frac{286}{3} \;\;\; {}^{2\ \ -5\ \ 11} - \frac{770}{9} \;\;\; {}^{3\ \ -13\ \ 11}.$

Denote $A = \frac{8}{5}a^{-2}$. The transformation

$$\overline{\tau} = \tfrac{15}{448}a^3\big(\;\; {}^{8\ \ 11} - \tfrac{14}{11} \;\;\; {}^{-3\ \ 11} \quad , \qquad = \tfrac{3}{7}\tau + \;\; {}^{-8\ \ 11} - \tfrac{39}{56}a^2$$

leads to an equation of the form 1.3.1.64 with $B = -1 \;\; 49$:

$$\,'_{\tau} - \quad = -\tfrac{12}{49}\tau + \tfrac{39}{98}a^2 - \tfrac{15}{784}a^3\tau^{-1\ \ 2}.$$

70. $- \;\; = -\frac{21}{100} \;\; + \frac{7}{9} \;\; {}^{2}(123 \;\; {}^{-1\ \ 7} + 280 \;\;\; {}^{-5\ \ 7} - 400 \;\; {}^{2\ \ -9\ \ 7}).$

Denote $A = 1 \;\; a$. The transformation

$$= \xi^{-7\ \ 4}, \qquad = \tfrac{35}{3}a^{-2}\big(\quad + 4\xi + \tfrac{7}{5}a + \tfrac{3}{50}a^2\xi^{-1} \;\; \xi^{-3\ \ 4}, \qquad \text{where} \quad \xi = z - \tfrac{21}{20}a,$$

leads to an equation of the form 1.3.4.30 with $\; = 3, \;\; = -\frac{21}{20}a$:

$$\big[\big(z - \tfrac{21}{20}a \qquad + 4z^2 - 7az + 3a^2\big] \;\; ' = \tfrac{3}{4} \;\; {}^2 + 2z \quad .$$

71. $- \;\; = \;\; + \qquad .$

1 . For $\; \neq 3$, the transformation

$$\tau = B^2 \;\; (\quad - 3) - + 1 \;\; {}^2, \qquad = 2(\quad - 3)B^2 \qquad {}^{-1} - \frac{\;\;{}^2}{2} + - + a$$

leads to an equation

$$\,'_{\tau} - \quad = \frac{2(\quad -1)}{(\quad -3)^2} \;\; \tau - \quad B\tau^{1\ \ 2} + [2 \quad - 3 - a(\quad - 3)^2]B^2 + [2 - \quad + a(\quad - 3)^2]B^3\tau^{-1\ \ 2} \quad .$$

2 . Let $\; \neq 1$ and $a > -1 \;\; 4$. Denote

$$a = -\frac{(\quad + 2)(\quad + \quad + 1)}{(2 \quad + \quad + 3)^2}, \qquad \text{where} \qquad = \;\; {}_{1,2} = \frac{1}{2} \qquad \frac{\quad - 1}{1 + 4a} - \qquad - 3 \quad .$$

Then the transformation

$$= \xi^{\frac{+2}{-1}} \quad , \qquad = \frac{\quad - 1}{2 \quad + \quad + 3}\xi^{\frac{+2}{-1}} \;\; \xi \;\; ' + \frac{\quad + 2}{\quad - 1} \qquad , \qquad\qquad = \;\; {}_{1,2}$$

reduce the original equation to the classical Emden–Fowler equation $\;\; '' = A\xi \qquad$, where $A = \frac{2 \quad + \quad + 3}{\quad - 1} \;\; {}^2$, which is discussed below in Section 2.3.

72. $\quad - \quad = -\dfrac{+1}{(\quad +2)^2} \quad + \quad {}^{2} \quad {}^{+1} + B \quad {}^{3} \quad {}^{+1}.$

Denote $A = -\dfrac{a}{2(\quad +2)^2 \quad {}^2}$, $B = -\dfrac{2}{2(\quad +2)^3 \quad {}^2}$. The transformation

$$\tau = -\frac{(\quad +2)^2}{\quad} \quad - \quad {}^{-1} + \frac{+2}{\quad} \quad - \quad , \qquad = \frac{2(\quad +1)}{+2}\tau + \quad + \frac{+2}{\quad}a$$

leads to the equation

$$\tau{}' - \quad = \frac{2 \quad (\quad +1)}{(\quad +2)^2}\tau + a + \quad \tau^{-1} \quad {}^2$$

(see Table 7 with $\quad = 0$ in Subsection 1.3.1).

73. $\quad - \quad = \quad {}^{2} \quad {}^{2\lambda} \quad - (\quad +1) \quad {}^{\lambda} \quad + \quad .$

Particular solution: $\quad {}_0 = ae^{\lambda} \quad - \quad .$

74. $\quad - \quad = \quad {}^{2} \quad {}^{2\lambda} \quad + \quad {}^{\lambda} \quad + \quad {}^{\lambda} \quad .$

Particular solution: $\quad {}_0 = ae^{\lambda} \quad + \quad + \dfrac{\quad}{a\lambda}.$

75. $\quad - \quad = 2 \quad {}^{2} \quad \sin(2 \quad) + 2 \quad \sin(\quad).$

Particular solution: $\quad {}_0 = -2a\sin(\lambda \quad).$

76. $\quad - \quad = \quad {}^{2} f \quad f \quad - \dfrac{(f + \quad)^2}{(f \quad)^3} f \quad , \qquad f = f(\quad).$

Particular solutions: $\quad {}_1 = a \quad {}' + \dfrac{+}{\quad {}'}, \quad {}_2 = -a \quad {}' + \dfrac{+}{\quad {}'}.$

1.3.2. Equations of the Form $\quad = (\quad) \quad + 1$

1. $\quad = (\quad + \quad) \quad + 1.$

The substitution $\xi = \quad - \frac{1}{2}a \quad {}^2 - \quad$ leads to a Riccati equation with respect to $\quad = (\xi)$: $\quad {}' = \frac{1}{2}a \quad {}^2 + \quad + \xi.$

2. $\quad = (\quad + \quad)^{-2} \quad + 1.$

The substitution $a\xi = -(a \quad + \quad)^{-1}$ leads to an equation of the form 1.3.1.33: $\quad {}' = \quad + (a\xi)^{-2}.$

3. $\quad = \quad - \dfrac{1}{\quad} \quad + 1.$

The substitution $\xi = \quad - a \quad$ leads to a Bernoulli equation: $\xi \quad {}' + a\xi \quad + a^2 \quad {}^2 = 0.$

4. $\quad = (\quad + \quad)^{-1 \quad 2} \quad + 1.$

The substitution $z = \dfrac{2}{a}(a \quad + \quad)^{1 \quad 2}$ leads to an equation of the form 1.3.1.2: $\quad {}' = \quad + \frac{1}{2}az.$

5. $\quad = 3(\quad {}^{3 \quad 2} + 8 \quad)^{-1 \quad 2} \quad + 1.$

The substitution $z = 12a^{-1}(a \quad {}^{1 \quad 2} + 8)^{1 \quad 2}$ leads to an equation of the form 1.3.1.10 with $\quad = 3$: $\quad {}' = \quad - \frac{2}{9}z + \frac{1}{5184}a^2z^3.$

6. $= (\quad {}^{-2\ 3} - \tfrac{2}{3} \quad {}^{-1} \quad {}^{-1\ 3}) \quad + 1.$

The transformation $\quad = a^3 \ {}^2 \ {}^3, \quad = \xi - \ {}^2$ leads to a Riccati equation: $3a^3 \ {}^2\xi \ {}' = \xi - \ {}^2$.

7. $= \quad {}^{\lambda} \quad + 1.$

The substitution $\xi = \dfrac{a}{\lambda} e^{\lambda}$ leads to an equation of the form 1.3.1.16: $\ {}' = \ + (\lambda\xi)^{-1}$.

8. $= (\quad {}^{\lambda} \ + \quad {}^{-\lambda} \ ) \ + 1.$

The transformation $\xi = \ + \dfrac{}{\lambda} e^{-\lambda} \ - \dfrac{a}{\lambda} e^{\lambda} \ , \quad = e^{\lambda}$ leads to a Riccati equation: $\ {}' = a \ {}^2 + \lambda\xi \ - \ .$

9. $= \quad \cosh \quad + 1.$

This is a special case of equation 1.3.3.75 with $\ = 0$ and $\ = 1$.

10. $= \quad \sinh \quad + 1.$

This is a special case of equation 1.3.3.76 with $\ = 0$ and $\ = 1$.

11. $= \quad \cos(\quad) \quad + 1.$

The transformation $\quad = -\dfrac{2}{\lambda} \arctan \dfrac{4}{\lambda}, \quad = \tau - \dfrac{8a}{16 \ {}^2 + \lambda^2}$ leads to a Riccati equation: ${}'_{\tau} = -2\tau \ {}^2 + a \ - \tfrac{1}{8}\lambda^2\tau$.

12. $= \quad \sin(\quad) \quad + 1.$

The substitution $\ = \xi + \dfrac{}{2\lambda}$ leads to an equation of the form 1.3.2.11: $\ {}' = a \cos(\lambda\xi) \ + 1$.

1.3.3. Equations of the Form $\quad = \ {}_1(\ ) \ + \ {}_0(\ )$

1.3.3-1. Preliminary remarks.

With the aid of the substitution $\xi = \quad {}_1(\ )$, these equations are reducible to the form:

$$ {}' = \ + \ (\xi), \qquad \text{where} \qquad (\xi) = \ {}_0(\ ) \ {}_1(\ ), \tag{1}$$

and by means of the substitution $z = \quad {}_0(\ )$ they can be reduced to the form:

$$ {}' = g(z) \ + 1, \qquad \text{where} \quad g(z) = \ {}_1(\ ) \ {}_0(\ ). \tag{2}$$

Specific equations of the form (1) and (2) are outlined in Subsection 1.3.1 and 1.3.2, respectively.

1.3.3-2. Solvable equations and their solutions.

1. $= (\quad + 3 \ ) \ + \quad {}^3 - \quad {}^2 - 2 \ {}^2 \ .$

The substitution $\ = \ {}^2 + \ $ leads to a linear equation with respect to $\ = (\ )$: $(-2 \ {}^2 + a + \ ) \ {}' = \ + \ .$

2. $= (3 \quad + \ ) \ - \ {}^2 \ {}^3 - \quad {}^2 + \quad .$

The substitution $\ = \quad + a \ {}^2$ leads to a Bernoulli equation with respect to $\ = (\ )$: $(- \ {}^2 + \quad + \ ) \ {}'_w = \quad + a \ {}^2$.

3. $\mathbf{2 \quad = (7 \quad + 5\ ) \ - 3\ ^{2}\ ^{3} - 2 \quad ^{2} - 3\ ^{2}\ .}$

This is a special case of equation 1.3.3.11 with $\ = 3\ \ 2$, $k = 1\ \ 2$.

4. $\mathbf{\quad = [(3 - \quad) \ - 1] \ \ + (\quad - 1)(\ ^{3} - \ ^{2} - \quad).}$

The transformation $\ = \quad z$, $\ = -z \quad ^{-1} + \ ^{2} - \quad - a$ leads to an equation $\quad ' = \quad + az + z$ whose solvable cases are outlined in Subsection 1.3.1 (see Table 5).

5. $\mathbf{\quad + \ (\quad ^{2} + \ ) \ + \quad = 0.}$

The substitution $z = -\frac{1}{2} \quad ^{2}$ leads to an equation of the form 1.3.2.1 with respect to $\ = \ (z)$: $\ ' = (-2az + \ ) \ + 1$.

6. $\mathbf{\quad + \ (1 - \quad ^{-1}) \ = \ ^{2}.}$

Solution in parametric form:

$$= -(\tau + e^{-\tau} \quad ^{-1} + \ln \quad), \qquad = -a(\tau + e^{-\tau} \quad ^{-1}), \qquad \text{where} \qquad = \frac{e^{-\tau} \quad \tau}{\tau} + \quad .$$

7. $\mathbf{\quad - \ (1 - \quad ^{-1}) \ = \ ^{2}\ .}$

Solution in parametric form:

$$= \tfrac{1}{2} \ \exp(\mp\tau^2) \quad ^{-1}, \qquad = \mp\tfrac{1}{2}a \quad ^{-1}\left[2\tau^2 \quad \exp(\mp\tau^2)\right], \qquad \text{where} \qquad = \ \exp(\mp\tau^2)\frac{\tau}{\tau} + \quad .$$

8. $\mathbf{\quad = \quad ^{-1}[(1 + 2\ \ ) \ + \quad] \ - \quad ^{2}\ (\ + \ ).}$

The transformation $\ = \frac{}{z}$, $\ = -\frac{1}{z} + \quad ^{+1} + a$ leads to a separable equation: $\ ' = \quad ^{-} \ - a$.

9. $\mathbf{\quad = \ (\ - \ ) \quad ^{-1} \ + \ [\ ^{2} - (2 \quad + 1) \quad + \ (\ + 1) \ ^{2}] \ ^{2}\ ^{-1}.}$

The substitution $\xi = a \quad \frac{}{+1} - \quad$ leads to an Abel equation of the form 1.3.1.2: $\ ' = \quad + (\ + 1) \ a^{-2}\xi$.

10. $\mathbf{\quad = [\ (2 \quad + \ ) \ ^{k} + \ ] \quad ^{-1} \ + (- \ ^{2} \quad ^{2k} - \quad ^{k} + \ ) \ ^{2}\ ^{-1}.}$

The substitution $\ = \quad (\ + a \quad)$ leads to a Bernoulli equation with respect to $\ = \ (\ )$: $(\quad ^{2} - \quad - \ ) \ '_{w} = - \quad - a \quad ^{+1}$.

11. $\mathbf{\quad = [\ (2 \quad + \ ) \ ^{2k} + \ (2 \quad - \ )] \quad ^{-k-1} \ - (\ ^{2} \quad ^{4k} + \quad ^{2k} + \ ^{2} \quad) \ ^{2}\ ^{-2k-1}.}$

The transformation $z = \quad$, $\ = \quad (\ + a \quad + \quad ^{-} \ )$ leads to a Riccati equation with respect to $z = z(\ )$:

$$(- \quad ^{2} + 2a \quad - \ )z' = akz^2 + k \ z + \ \ k. \tag{1}$$

The substitution $z = \frac{\ ^{2} + \ _{0} \quad '}{ak \quad}$, where $\ _{0} = \quad - 2a \quad$, reduces equation (1) to a second-order linear equation:

$$(\quad ^{2} + \ _{0})^{2} \quad '' + (2 \quad + k) \ (\quad ^{2} + \ _{0}) \quad ' + a \ k^{2} \quad = 0. \tag{2}$$

The transformation $\xi = \frac{}{\sqrt{\ ^{2} + (\ _{0} \quad)}}$, $\ = (1 - \xi^2) \quad ^{2} \quad$, where $\ = -\frac{\ + k}{2}$, brings equation (2) to the Legendre equation 2.1.2.226:

$$(1 - \xi^2) \quad '' \ - 2\xi \quad ' + [\ (\ + 1) - \quad ^{2}(1 - \xi^2)^{-1}] \quad = 0,$$

where $\quad$ is a root of the quadratic equation $\ ^{2} + \quad + \frac{\ ^{2} - k^2}{4 \quad ^{2}} - \frac{a \ \ k^2}{\ _{0}} = 0$.

12. $= [(\quad +2 \quad -3) \quad + \quad -2 \quad +3]^{\;-} \quad + [(\quad + \quad -1)^{\;2} + (\quad - \quad -2 \quad +3) \quad - \quad + \quad -2]^{\;1-2}$.

The transformation $= \frac{\xi}{\;}\;{}'$, $= A\xi^{\;-\;+2 \quad +\;-1} - {}^{2-} + {}^{1-}$ leads to the generalized Emden–Fowler equation $'' = A\xi \quad (\;')$, which is discussed in Section 2.5.

13. $= [\;(2 \quad +1)^{\;2} + \quad + \;(2 \quad -1)]^{\;-2} - (\;{}^{2\;4} + \;{}^{3} + \;{}^{2} + \quad + \;{}^{2})^{\;2\;-3}$.

Here, a, , , , and are arbitrary numbers.

The substitution $= \quad +a^{\;+1} + \;{}^{-1}$ leads to a Riccati equation with respect to $= (\;)$: $(- \;{}^{2} + \quad - \quad +2 \;a\;)\;' = a\;{}^{2} + \quad + \;$.

14. $= [\;(\quad -1) \quad + \;(2 \quad + \quad)]^{\;\lambda-1}(\quad + \;)^{-\lambda-2} - [\quad + \;(\quad + \quad)]^{\;2\lambda-1}(\quad + \;)^{-2\lambda-3}$.

The substitution $= \frac{1}{\;} + \frac{1}{(a \;+\;)}\;{}^{\lambda+}\;(a \;+\;)^{-\lambda}$ leads to an equation of the form 1.3.4.5: $(\quad +a^{\;+1} + \quad)\;' = [a \quad + \;(\lambda + \;)^{\;-1}]\;$.

15. $- \;\left[(\quad -1) \quad +1\right]^{\;-1} = \;{}^{2\;-1}(\quad +1)(\quad -1)$.

Solution in parametric form:

$$= \frac{(\quad -1)(\tau^{\;+1}+1)}{\tau} + \ln\frac{\tau^{\;+1}+1}{\tau}, \qquad = a\left[1 + (\;\tau \quad - \tau^{-1})\;\right],$$

where $= \dfrac{\tau}{\tau^{\;+1}+1} + \;$.

16. $- \;(1 - \;{}^{-1\;2}) = \;{}^{2\;\;-1\;2}$.

Solution in parametric form:

$$= \mp\;{}^{2}\tau^{2}Z^{-2}(Z'_{\tau})^{2}, \qquad = a\;{}^{2}\tau^{2}Z^{-2}\left[(Z'_{\tau})^{2} \quad Z^{2}\right],$$

where

$$Z = \begin{cases} {}_{1}\;{}_{0}(\tau) + {}_{2}\;{}_{0}(\tau) & \text{for the upper sign,}\\ {}_{1}\;{}_{0}(\tau) + {}_{2}\;{}_{0}(\tau) & \text{for the lower sign,}\end{cases}$$

${}_{0}(\tau)$ and ${}_{0}(\tau)$ are the Bessel functions, and ${}_{0}(\tau)$ and ${}_{0}(\tau)$ are the modified Bessel functions.

17. $= 3(\quad + \;)^{-1\;3\;\;-5\;3} + 3(\quad + \;)^{-2\;3\;\;-7\;3}$.

The substitution $= \frac{1}{\;} + \frac{1}{3}\;\frac{a \;+\;}{\;}\;{}^{1\;3}$ leads to a separable equation for $= (\;)$: $' = \;{}^{-1\;3}(a \;+\;)^{-2\;3}(\tfrac{1}{9}a - 3\;{}^{3})$.

18. $3 \quad = (-7 \quad + 6s - 2\;)^{\;-1\;3} + 6(\;s \quad -1)^{\;-2\;3}$
$+ 2(\quad + 5\;)(- \quad + 3s + 4\;)^{\;1\;3}, \qquad = s(3s + 4\;)$.

The transformation $= (\xi + \lambda\;)^{-1}$, $= (\quad + 4\lambda + 3 \quad - A\;)^{\;2\;3}$ leads to an equation of the form 1.3.4.10 with $a = 1\;\;3$: $[(\xi + \lambda\;) \quad + (4\lambda + 3\;)\xi]\;' = \frac{2}{3}\;{}^{2} + 2(3\lambda + \;) \quad + 2\xi$.

In the solutions of equations 19 and 20, the following notation is used:

$$= \sqrt{\tau(\tau+1)} - \ln\left(\sqrt{\tau} + \sqrt{\tau+1}\right), \qquad g = 1 - \sqrt{\frac{\tau+1}{\tau}}\ln\left(\sqrt{\tau} + \sqrt{\tau+1}\right).$$

19. $+ \frac{1}{2}\;(6 \quad -1)^{\;-1} = -\frac{1}{2}\;{}^{2}(\quad -1)(4 \quad -1)^{\;-1}$.

Solution in parametric form:

$$= \tau^{\;-2}g^{2}, \qquad = a(1 - \tau^{\;-2}g^{2} - \tau^{2\;\;-2}g).$$

20. $\quad -\frac{1}{2}\ (1+2\quad{}^{-2})\ =\ \frac{1}{16}\ {}^{2}(3\quad +4\quad {}^{-1}).$

Solution in parametric form:

$$=\ \tau^{-1}\ \ g^{-1\ 2},\qquad =-\tfrac{1}{4}a\ \tau^{-1}\ \ {}^{-1}g^{-1\ 2}(\ {}^{2}-4\tau g^{2}),\qquad =-\ {}^{2}.$$

In the solutions of equations 21–23, the following notation is used:

$$=\exp(3\tau),\quad {}_{1}=\quad+\quad\sin(\ \overline{3}\tau),\quad {}_{2}=2\quad-\quad\sin(\ \overline{3}\tau)+\quad\overline{3}\quad\cos(\ \overline{3}\tau),$$
$${}_{3}=2\ \ {}_{1}(\ {}_{2})'_{\tau}-(\ {}_{1})'_{\tau}\ \ {}_{2}-\ \ {}_{1}\ \ {}_{2},\qquad {}_{4}=2\ \ {}_{1}(\ {}_{3})'_{\tau}-5(\ {}_{1})'_{\tau}\ \ {}_{3}+\ \ {}_{1}\ \ {}_{3}.$$

21. $\quad +\frac{1}{14}\ (13\quad -20)\ \ {}^{-9\ \ 7}\ =-\frac{3}{14}\ \ {}^{2}(\quad -1)(\quad -8)\ \ {}^{-11\ \ 7}.$

Solution in parametric form:

$$=64\ \ {}^{3}_{1}\ \ {}_{2}\ \ {}^{-2}_{3},\qquad =a(4\ \ {}_{1})^{-6\ \ 7}\ \ {}^{-2\ \ 7}_{2}\ \ {}^{-10\ \ 7}_{3}(\ {}^{2}_{3}-64\ \ {}^{3}_{1}\ \ {}_{2}+7\ \ {}^{2}_{2}\ \ {}_{3}).$$

22. $\quad +\frac{5}{56}\ (23\quad -16)\ \ {}^{-9\ \ 7}\ =-\frac{3}{56}\ \ {}^{2}(\quad -1)(25\quad -32)\ \ {}^{-11\ \ 7}.$

Solution in parametric form:

$$=-\tfrac{256}{25}\ \ {}^{3}_{1}\ \ {}^{-3}_{3}\ \ {}_{4},\qquad =a\big(\tfrac{256}{25}\ \ {}^{3}_{1}\ \ {}_{4}\quad{}^{-2\ \ 7}\ \ {}^{-15\ \ 7}_{3}\big(\ {}^{3}_{3}+7\ \ {}^{2}_{4}+\tfrac{256}{25}\ \ {}^{3}_{1}\ \ {}_{4}\ \ .$$

23. $\quad +\frac{1}{26}\ (19\quad +85)\ \ {}^{-18\ \ 13}\ =-\frac{3}{26}\ \ {}^{2}(\quad -1)(\quad +25)\ \ {}^{-23\ \ 13}.$

Solution in parametric form:

$$=-25\ \ {}^{3}_{3}\ \ {}^{-2}_{4},\qquad =a(25\ \ {}^{3}_{3})^{-5\ \ 13}\ \ {}^{-16\ \ 13}_{4}(\ {}^{2}_{4}+25\ \ {}^{3}_{3}-208\ \ {}^{3}_{1}\ \ {}_{4}).$$

In the solutions of equations 24–27, the following notation is used:

$${}_{1}=\cosh(\tau+\quad)\cos\tau,\qquad {}_{2}=\tanh(\tau+\quad)+\tan\tau,\qquad {}_{3}=\tanh(\tau+\quad)-\tan\tau,\qquad {}_{4}=3\ \ {}_{2}\ \ {}_{3},$$
$$\theta_1=\cosh\tau-\sin(\tau+\quad),\quad \theta_2=\sinh\tau+\cos(\tau+\quad),\quad \theta_3=\sinh\tau-\cos(\tau+\quad),\quad \theta_4=3\theta_2\theta_3-2\theta_1^2.$$

24. $\quad +\frac{1}{15}\ (13\quad -18)\ \ {}^{-7\ \ 5}\ =-\frac{4}{15}\ \ {}^{2}(\quad -1)(\quad -6)\ \ {}^{-9\ \ 5}.$

1 . Solution in parametric form with $A<0$:

$$=-12\ \ {}^{-3}_{1}\ \ {}_{2},\qquad =a(12\ \ {}_{2})^{-2\ \ 5}\ \ {}^{-9\ \ 5}_{1}(\ {}^{3}_{1}-5\ \ {}_{1}\ \ {}^{2}_{2}+12\ \ {}_{2}).$$

2 . Solution in parametric form with $A>0$:

$$=6\theta_1^2\theta_2^{-3}\theta_3,\qquad =a(6\theta_1^2\theta_3)^{-2\ \ 5}\theta_2^{-9\ \ 5}(\theta_2^3+5\theta_2\theta_3^2-6\theta_1^2\theta_3).$$

25. $\quad +\frac{1}{2}\ (5\quad +1)\ \ {}^{-1\ \ 2}\ =\ \ {}^{2}(1-\ \ {}^{2}).$

Solution in parametric form:

$$=\ \ {}^{2}_{1}\ \ {}^{-2}_{2},\qquad =-a\ \ {}^{-3}_{2}(\ {}^{3}_{1}-\ \ {}_{1}\ \ {}^{2}_{2}+4\ \ {}_{2}).$$

26. $\quad +\frac{3}{35}\ (19\quad -14)\ \ {}^{7\ \ 5}\ =-\frac{4}{35}\ \ {}^{2}(\quad -1)(9\quad -14)\ \ {}^{9\ \ 5}.$

1 . Solution in parametric form with $A<0$:

$$=-\tfrac{28}{9}\ \ {}^{4}_{1}\ \ {}_{3},\qquad =a\big(\tfrac{28}{9}\ \ {}^{6}_{1}\ \ {}_{3}\quad{}^{-2\ \ 5}\big(\ {}^{4}_{1}-\tfrac{5}{9}\ \ {}^{2}_{3}+\tfrac{28}{9}\ \ {}_{3}\ \ .$$

2 . Solution in parametric form with $A>0$:

$$=\tfrac{14}{9}\theta_1^2\theta_2^4\theta_4,\qquad =\big(\tfrac{14}{9}\theta_1^2\theta_4\quad{}^{-2\ \ 5}\big(\theta_2^4+\tfrac{5}{9}\theta_4^2-\tfrac{14}{9}\theta_1^2\theta_4\ \ .$$

27. $+ \frac{3}{10}\ (3\ \ +7)\ ^{-13\ \ 10}\ = -\frac{1}{5}\ ^{2}(\ \ -1)(\ \ +9)\ ^{-8\ \ 5}.$

1 . Solution in parametric form with $A < 0$:

$$= 9\ {}^{4}_{1}\ {}^{-2}_{3}, \qquad = a(9\ {}^{4}_{1})^{-3\ \ 10}\ {}^{-7\ \ 5}_{3}(\ {}^{2}_{3} - 20\ {}_{3} - 9\ {}^{4}_{1}).$$

2 . Solution in parametric form with $A > 0$:

$$= -\tfrac{9}{2}\theta_2^4\theta_4^2, \qquad = -4a\left(\tfrac{9}{2}\theta_2^4\ ^{-3\ \ 10}\theta_4^{-7\ \ 5}\left(\theta_4^2 - 5\theta_1^2\theta_4 + \tfrac{9}{2}\theta_2^4\ .\right.\right.$$

In the solutions of equations 28–30, the following notation is used:

$$_2 = \ (\tau^2 - 1), \qquad _3 = \tau^3 - 3\tau + \quad , \qquad _4 = \ (\tau^4 - 6\tau^2 + 4\quad \tau - 3),$$
$$_6 = \ (\tau^6 - 15\tau^4 + 20\quad \tau^3 - 45\tau^2 + 12\quad \tau - 8\quad ^2 + 27).$$

28. $+ \frac{1}{10}\ (7\ \ -12)\ ^{-7\ \ 5}\ = -\frac{1}{10}\ ^{2}(\ \ -1)(\ \ -16)\ ^{-9\ \ 5}.$

Solution in parametric form:

$$= \ 16\ {}_2\ {}^{2}_{3}\ {}^{-2}_{4}, \qquad = a(16\ {}_2\ {}^{2}_{3}\ {}^{3}_{4})^{-2\ \ 5}(\ {}^{2}_{4}\quad 15\ {}^{2}_{2}\ {}_4 \mp 16\ {}_2\ {}^{2}_{3}).$$

29. $+ \frac{3}{20}\ (13\ \ -8)\ ^{-7\ \ 5}\ = -\frac{1}{20}\ ^{2}(\ \ -1)(27\ \ -32)\ ^{-9\ \ 5}.$

Solution in parametric form:

$$= \tfrac{32}{27}\ {}^{2}_{3}\ {}^{-3}_{4}\ {}_6, \qquad = \ a(3\ {}_4)^{-9\ \ 5}\ {}^{-4\ \ 5}_{3}\ {}^{-2\ \ 5}_{6}\left(\tfrac{5}{4}\ {}^{2}_{6} \mp 8\ {}^{2}_{3}\ {}_6\quad \tfrac{27}{4}\ {}^{3}_{4}\ .\right.$$

30. $+ \frac{3}{14}\ (3\ \ +11)\ ^{-10\ \ 7}\ = -\frac{1}{14}\ ^{2}(\ \ -1)(\ \ -27)\ ^{-13\ \ 7}.$

Solution in parametric form:

$$= \mp 27\ {}^{3}_{4}\ {}^{-2}_{6}, \qquad = \mp a(3\ {}_4)^{-9\ \ 7}\ {}^{-8\ \ 7}_{6}(\ {}^{2}_{6} \mp 28\ {}^{2}_{3}\ {}_6\quad 27\ {}^{3}_{4}).$$

In the solutions of equations 31–38, the following notation is used:

$$= \frac{\tau\ \ \tau}{\overline{\overline{\ (4\tau^3 - 1)}}} \ \text{(incomplete elliptic integral of the second kind)},$$

$$R = \ \overline{(4\tau^3 - 1)}, \qquad _1 = \tau(2\ \ \mp\ \ ^{-1}R + \ \ ), \qquad _2 = \tau^{-1}(R\ {}_1 - 1),$$
$$_3 = 4\tau\ {}^{2}_{1} \mp\ {}^{2}_{2}, \qquad _4 = \ {}_2\ {}_3 - 8\ {}^{2}_{1}, \qquad _5 = 2R(\ \ +\ \ ) - \tau^2.$$

31. $-\frac{1}{2}\ (\ \ +1)\ ^{-7\ \ 4}\ = \frac{1}{4}\ ^{2}(\ \ -1)(3\ \ +5)\ ^{-5\ \ 2}.$

Solution in parametric form:

$$= \tfrac{1}{6}\tau^{-3}\ {}^{-1}_{1}R, \qquad = \tfrac{1}{3}a\left(\tfrac{1}{6}\tau R\ ^{-3\ \ 4}\ {}^{-1\ \ 4}_{1}\left[(11\tau^3 - 2)\ {}_1 - \tfrac{1}{2}R\right].\right.$$

The lower sign is taken in the notation adopted.

32. $-\frac{1}{2}\ (\ \ +1)\ ^{-7\ \ 4}\ = \frac{1}{4}\ ^{2}(\ \ -1)(\ \ +5)\ ^{-5\ \ 2}.$

Solution in parametric form:

$$= -\tfrac{1}{3}\tau^{-2}\ {}^{2}_{1}\ {}_2, \qquad = -\tfrac{1}{3}a(\tau\ {}_1)^{-1\ \ 2}\left(\tfrac{1}{3}\ {}_2\ ^{-3\ \ 4}(2\tau\ {}^{2}_{2} + \ {}_2 - 3\tau^2\ {}^{2}_{1}).\right.$$

The lower sign is taken in the notation adopted.

33. $-\frac{1}{14}\ (4\ \ +3)^{-8\ 7} = -\frac{1}{14}\ ^{2}(\ \ -1)(16\ \ +5)^{-9\ 7}.$

Solution in parametric form:

$$= \ \tfrac{3}{16}\ {}_{1}^{-2}\ {}_{2}^{-1}\ {}_{3}^{2}, \qquad = -\tfrac{1}{16}a\big(\tfrac{13}{16}\ {}_{1}^{12}\ {}_{2}^{6}\ {}_{3}^{2}\ \ {}^{-1\ 7}(3\ {}_{3}^{2}\ \ \ 7\ {}_{2}^{2}\ {}_{3} \mp 16\ {}_{1}^{2}\ {}_{2}).$$

34. $+\frac{1}{6}\ (13\ \ -3)^{-2\ 3} = -\frac{1}{6}\ ^{2}(\ \ -1)(5\ \ -1)^{-1\ 3}.$

Solution in parametric form:

$$= \mp\ {}_{3}^{3}\ {}_{4}^{-2}, \qquad = -a\ {}_{3}\ {}_{4}^{-8\ 3}(\ {}_{3}^{3}\ \ \ \ {}_{4}^{2}\ \ \ 4\ {}_{1}^{2}\ {}_{4}).$$

35. $-\frac{1}{28}\ (8\ \ -1)^{-8\ 7} = \frac{1}{28}\ ^{2}(\ \ -1)(32\ \ +3)^{-9\ 7}.$

Solution in parametric form:

$$= \mp\ {}_{1}^{-2}\ {}_{3}^{3}\ {}_{4}^{-1}, \qquad = -\tfrac{1}{32}a\big(\tfrac{3}{32}\ {}_{1}^{12}\ {}_{3}^{3}\ {}_{4}^{6}\ \ {}^{-1\ 7}(3\ {}_{3}^{3}\ \ \ 7\ {}_{4}^{2}\ \ \ 32\ {}_{1}^{2}\ {}_{4}).$$

36. $-\ \ (5\ \ -4)^{-4} = \ ^{2}(\ \ -1)(3\ \ -1)^{-7}.$

Solution in parametric form:

$$= \ \tfrac{1}{6}\tau^{-1}(\ +\ )^{-1}R, \qquad = 36a(\ +\ )^{2}R^{-3}[(1\ \ \ 2\tau^{3})(\ +\ ) - \tau^{2}R].$$

37. $-\frac{2}{5}\ (3\ \ -10)^{-4} = \frac{1}{5}\ ^{2}(\ \ -1)(8\ \ -5)^{-7}.$

Solution in parametric form:

$$= \ \tfrac{5}{24}\tau^{-1}(\ +\ )^{-2}\ {}_{5}, \qquad = \ \tfrac{576}{125}a(\ +\ )^{4}\ {}_{5}^{-3}[\ {}_{5}^{2} + 5\tau^{2}\ {}_{5} \mp \tau^{3}(\ +\ )^{2}].$$

38. $+\frac{1}{42}\ (39\ \ -4)^{-9\ 7} = -\frac{1}{42}\ ^{2}(\ \ -1)(9\ \ -16)^{-11\ 7}.$

Solution in parametric form:

$$= \ 16\tau^{3}(\ +\ )^{2}\ {}_{5}^{-2}, \qquad = \tfrac{1}{3}a[16\tau^{3}(\ +\ )^{2}\ {}_{5}^{5}]^{-2\ 7}[3\ {}_{5}^{2} + 7\tau^{2}\ {}_{5} \mp 48\tau^{3}(\ +\ )^{2}].$$

In the solutions of equations 39–41, the following notation is used:

$$= \ \ \exp(\mp\tau^{2})\ \ \tau + \ \ , \qquad g = 2\tau\ \ \ \exp(\mp\tau^{2}).$$

39. $+\ (\ \ -2)^{-1} = 2\ ^{2}(\ \ -1)^{-1}.$

Solution in parametric form:

$$= \tau\exp(\mp\tau^{2})\ \ , \qquad = a\big[1 \mp 2\tau^{2} - \tau\exp(\mp\tau^{2})\ \ ^{-1}\big].$$

40. $+\ (3\ \ -2)^{-1} = -2\ ^{2}(\ \ -1)^{2\ \ -1}.$

Solution in parametric form:

$$= \tfrac{1}{2}\exp(\mp\tau^{2})\ \ ^{-2}g, \qquad = \tfrac{1}{2}a\big[2 - \exp(\mp\tau^{2})\ \ ^{-2}g \mp\ \ ^{-2}g^{-2}\big].$$

41. $+\ (1-\ \ ^{-2}) = \ ^{2}\ \ ^{-1}.$

Solution in parametric form:

$$= \frac{1}{\ \mp 2}\ \ g^{-1}, \qquad = \frac{a}{\ \mp 8}\ \ ^{-1}g^{-1}\big[g\exp(\mp\tau^{2}) - 2\ \ ^{2}\big].$$

In the solutions of equations 42–52, the following notation is used:

$$_1 = \tau^3 \ \overline{(4\wp^3-1)} + 3\tau^2\wp \mp 1, \qquad _2 = \tau^2\wp \mp 1, \qquad _3 = \ \overline{(4\wp^3-1)} \ \ 2\tau\wp^2,$$

$$_4 = \tau \ \overline{(4\wp^3-1)} + 2\wp, \qquad _5 = \tau^3 \ \overline{(4\wp^3-1)} - 4\tau^2\wp \ \ 6, \qquad _6 = \tau \ \overline{(4\wp^3-1)} - \wp.$$

Here, the function $\wp = \wp(\tau)$ *is defined implicitly as follows:* $\tau = \dfrac{\wp}{(4\wp^3-1)} -$ *. The upper sign in the above relations corresponds to the classical Weierstrass elliptic function* $\wp = \wp(\tau + \ , 0, 1)$.

42. $\quad - \frac{1}{4} \ (3 \ \ - 4) \ ^{-5 \ 2} \ = \frac{1}{4} \ ^2(\ \ - 1)(\ \ + 2) \ ^{-4}$.

Solution in parametric form:

$$= \tfrac{2}{3}\tau^{-2}\wp^{-2} \ \ _6, \qquad = -\tfrac{1}{2}a\tau\big(\tfrac{2}{3} \ \ ^3_6 \ ^{-1 \ 2}(\ ^2_6 + 2\wp \ \ _6 - 3\tau^2\wp^3).$$

The upper sign is taken in the notation adopted.

43. $\quad + \frac{1}{30} \ (33 \ \ + 2) \ ^{-6 \ 5} \ = -\frac{1}{30} \ ^2(\ \ - 1)(9 \ \ - 4) \ ^{-7 \ 5}$.

Solution in parametric form:

$$= 4\tau^2\wp^3 \ \ ^{-2}_6, \qquad = \ \tfrac{1}{3}a(4\tau^2\wp^3 \ \ ^8_6)^{-1 \ 5}(3 \ \ ^2_6 + 5\wp \ \ _6 \mp 12\tau^2\wp^3).$$

44. $\quad - \frac{1}{8} \ (\ \ - 8) \ ^{-5 \ 2} \ = -\frac{1}{8} \ ^2(\ \ - 1)(3 \ \ - 4) \ ^{-4}$.

Solution in parametric form:

$$= \tfrac{4}{3} \ \ _1 \ \ ^{-2}_2, \qquad = \tfrac{1}{4}a\big(\tfrac{4}{3} \ \ ^3_1 \ ^{-1 \ 2}(\ ^2_1 - 4 \ \ _1 \ \ _2 + 3 \ \ ^3_2).$$

The lower sign is taken in the notation adopted.

45. $\quad + \frac{1}{30} \ (17 \ \ + 18) \ ^{-22 \ 15} \ = -\frac{1}{30} \ ^2(\ \ - 1)(\ \ + 4) \ ^{-29 \ 15}$.

Solution in parametric form:

$$= \ 4 \ \ ^{-2}_1 \ \ ^3_2, \qquad = \ a \ \ ^{-16 \ 15}_1(4 \ \ _2)^{-7 \ 15}(\ ^2_1 - 5 \ \ _1 \ \ _2 \mp 4 \ \ ^3_2).$$

46. $\quad - \frac{1}{13} \ (6 \ \ - 13) \ ^{-5 \ 2} \ = -\frac{1}{26} \ ^2(\ \ - 1)(\ \ - 13) \ ^{-4}$.

Solution in parametric form:

$$= \tfrac{13}{6} \ \ ^{-2}_2 \ \ _5, \qquad = -\tfrac{1}{13}a\big(\tfrac{13}{6} \ \ ^3_5 \ ^{-1 \ 2}(2 \ \ ^2_5 \ \ 13 \ \ _2 \ \ _5 - 6 \ \ ^3_2).$$

The upper sign is taken in the notation adopted.

47. $\quad + \frac{1}{30} \ (24 \ \ + 11) \ ^{27 \ 20} \ = -\frac{1}{60} \ ^2(\ \ - 1)(9 \ \ + 1) \ ^{-17 \ 10}$.

Solution in parametric form:

$$= 4 \ \ ^3_2 \ \ ^{-2}_5, \qquad = \tfrac{1}{3}a(4 \ \ ^3_2)^{-7 \ 20} \ \ ^{-13 \ 10}_5(3 \ \ ^2_5 + 20 \ \ _2 \ \ _5 - 12 \ \ ^3_2).$$

The upper sign is taken in the notation adopted.

48. $\quad -\frac{2}{5}\ \ (3\ \ +2)\ ^{-8\ 5}\ =\frac{1}{5}\ ^{2}(\ \ -1)(8\ \ +1)\ ^{-11\ 5}.$

Solution in parametric form:

$$=\mp\tfrac{1}{3}\wp^{-1}\ \ {}_3^{-2}\ \ {}_4,\qquad =\mp a(3\wp\ \ {}_3^{2})^{-2\ 5}\ \ {}_4^{-3\ 5}(3\wp\ \ {}_3^{2}\mp\wp^{2}\ \ {}_4^{2}\qquad {}_4).$$

49. $\quad -\frac{6}{5}\ \ (4\ \ +1)\ ^{-7\ 5}\ =\frac{1}{5}\ ^{2}(\ \ -1)(27\ \ +8)\ ^{-9\ 5}.$

Solution in parametric form:

$$=-\ \ {}_2\ \ {}_3^{2}\ \ {}_4^{-3},\qquad =a\ \ {}_2^{-2\ 5}\ \ {}_3^{-3\ 5}\ \ {}_4^{-9\ 5}(\ \ {}_4^{3}+\ \ {}_2\ \ {}_3^{2}-10\ \ {}_2^{2}).$$

50. $\quad +\frac{3}{10}\ \ (13\ \ -3)\ ^{-4\ 5}\ =-\frac{1}{10}\ ^{2}(\ \ -1)(27\ \ -7)\ ^{-3\ 5}.$

Solution in parametric form:

$$=2\ \ {}_2^{2}\ \ {}_4^{-3},\qquad =a(4\ \ {}_2^{-1}\ \ {}_4^{9})^{-2\ 5}(2\ \ {}_4^{3}-5\ \ {}_2\ \ {}_3^{2}+4\ \ {}_2^{2}).$$

51. $\quad -\frac{1}{5}\ \ (\ \ +4)\ ^{-8\ 5}\ =\frac{1}{5}\ ^{2}(\ \ -1)(3\ \ +7)\ ^{-11\ 5}.$

Solution in parametric form:

$$=\tfrac{1}{3}\ \ {}_3^{-1}\wp^{-3}\ \ \overline{(4\wp^3-1)},$$

$$=-\tfrac{1}{6}a\ \ {}_3^{-2\ 5}\ \ \tfrac{1}{3}\wp^{2}\ \ \overline{(4\wp^3-1)}\ \ ^{-3\ 5}\ \ 14\wp^{3}\ \ {}_3+2\ \ \overline{(4\wp^3-1)}\ \ .$$

52. $\quad -\ \ (2\ \ -1)\ ^{-5\ 2}\ =\frac{1}{2}\ ^{2}(\ \ -1)(3\ \ +1)\ ^{-4}.$

Solution in parametric form:

$$=\tfrac{1}{6}\tau^{-1}\wp^{-2}\ \ \overline{4\wp^3-1},\qquad =-a\big[6\tau\big(\ \ \overline{4\wp^3-1}\ \ ^{-3}\big]^{1\ 2}\big(\wp\ \ \overline{4\wp^3-1}+2\tau\wp^{3}-2\tau\ \ .$$

The upper sign is taken in the notation adopted.

In the solutions of equations 53–55, the following notation is used:

$$Q_1=\ \ \tau^2+\ \ \tau-1,\quad Q_2=\tau^2\ \ \ 1,\quad Q_3=\tau^3\ \ \ 3\tau+\ \ \ .$$

53. $\quad +\frac{1}{5}\ \ (\ \ -6)\ ^{-7\ 5}\ =\frac{2}{5}\ ^{2}(\ \ -1)(\ \ +4)\ ^{-9\ 5}.$

Solution in parametric form:

$$=\ \ 3\tau Q_2^2Q_3^{-1},\qquad =a(3\tau Q_2^2)^{-2\ 5}Q_3^{-3\ 5}\big[(1\ \ \ 5\tau^2)Q_3\mp 3\tau Q_2^2\big].$$

54. $\quad +\frac{1}{5}\ \ (21\ \ +19)\ ^{-7\ 5}\ =-\frac{2}{5}\ ^{2}(\ \ -1)(9\ \ -4)\ ^{-9\ 5}.$

Solution in parametric form:

$$=\ \ Q_1Q_2^2Q_3^{-2},\qquad =\ \ aQ_1^{-2\ 5}Q_2^{-4\ 5}Q_3^{-6\ 5}(Q_3^2\mp Q_1Q_2^2\ \ \ Q_1^2).$$

55. $\quad -3\ \ \ ^{-7\ 4}\ =\frac{1}{4}\ ^{2}(\ \ -1)(\ \ -9)\ ^{-5\ 2}.$

Solution in parametric form:

$$=Q_1^{-2}Q_3^2,\qquad =-aQ_1^{-1\ 2}Q_3^{-3\ 2}(Q_3^2+Q_1Q_2^2-Q_1^2).$$

The lower sign is taken in the notation adopted.

In the solutions of equations 56 and 57, the following notation is used:

$$=\ \ \ \tau\ \overline{\ \ ^{-2}}\ \exp(\mp\tau^2)\ \ \tau+\ \ \ .$$

56. $\quad -\ \ \big[(\ \ +1)\ \ -1\big]\ ^{-2}\ =\ \ ^{2}(\ \ +1)(\ \ -1)\ ^{-2}.$

Solution in parametric form:

$$=\frac{2}{k+1}\tau^{-\frac{2}{\ \ +1}}\exp(\ \ \tau^2)\ \ \ ,\qquad =a\ \ \frac{k+1}{2}\tau^{\frac{2}{\ \ +1}}\exp(\mp\tau^2)\ \ ^{-1}\ \ \ (k+1)\tau^2-1\ \ ,$$

where $\beta=\dfrac{k-1}{k}$.

57. $\quad - \; [(\; -2) \; +2 \; -3] \; ^{-k} \; = \; ^{2}(\; -2)(\; -1)^{2} \; ^{1-2k}.$

Solution in parametric form:

$$=\mp 2\tau^{\frac{2}{2-}} \exp(\; \tau^2) \quad , \qquad = a(\mp 2 \quad)^{1-} \; \exp\left[\mp(k-2)\tau^2\right] \; \tau^{\frac{2(1-\;)}{2-}} \exp(\mp\tau^2) + \frac{2}{2-k} \; 4\tau^2 \qquad .$$

58. $\quad - \tfrac{1}{2} \; [(4 \; -7) \; -4 \; +5] \; ^{-k} \; = \tfrac{1}{2} \; ^{2}(2 \; -3)(\; -1)^{2} \; ^{1-2k}.$

Solution in parametric form:

$$= (\tau Z)^{2 \quad -2}, \qquad = \tfrac{1}{2} a (\tau Z)^{-3 \; 5 \quad -7 \; 5}(2 \quad ^2 + 5Z \quad - \tau^2 Z^2),$$

where $Z = \; _1 \; (\tau) + \; _2 \; (\tau)$, $\; = \tau Z'_\tau + \; Z$, $\; = \dfrac{1-k}{3-2k}$; $\; (\tau)$ and $\; (\tau)$ are the modified Bessel functions.

In the solutions of equations 59 and 60, the following notation is used:

$$N \; = \; \frac{\tau}{\tau \quad + a} + \quad .$$

59. $\quad - \left[(2 \; -1) \; - \quad \right] \; ^{- \; -1} \; = \; (\; - \;) \; ^{-2} \; .$

1 . Solution in parametric form:

$$= \tau N^{-1}, \qquad = \tau^{-} \; N^{\; -1}[(\tau \quad + a)N \quad - \tau].$$

2 . Particular solution: $\; _0 = (a - \quad) ^{-} \; .$

60. $\quad - \left[(\; +1) \; - \quad \right] \; ^{-1}(\; - \;)^{- \; -2} \; = \; ^{2} \; (\; - \;)^{-2 \; -3}.$

1 . Solution in parametric form:

$$= \tau N^{-1}, \qquad = (\tau - aN \;)^{- \; -1}[\tau - (\tau \quad + a)N \;].$$

2 . Particular solution: $\; _0 = - \quad (\; - a)^{- \; -1}.$

61. $\quad - \; \left[(2 \; -3) \; +1\right] \; ^{-k} \; = \; ^{2}(\; -2)[(\; -1) \; +1] \; ^{2(1-k)}.$

Solution in parametric form:

$$= \frac{\tau+1}{(1-k)(2-k)} \quad ^{-2} \; (1-k)(2k-3)(\tau+1)^{\frac{1}{1-}} + (2-k) \qquad ,$$

$$= \frac{a}{1-k} \; \tau \quad - (1-k)(\tau+1)^{\frac{2-}{1-}} \quad ,$$

where $\quad = \; \dfrac{1}{\tau}(\tau+1)^{\frac{1}{1-}} \quad \tau + \quad .$

In the solutions of equations 62–66, the following notation is used:

$$R \; = \; \overline{1 \quad \tau \; ^{+1}}, \qquad = \; \left(1 \quad \tau \; ^{+1} \quad ^{-1 \;\; 2} \quad \tau + \quad , \qquad = R \qquad - \tau, \qquad = \; \left(1 \quad \tau \; ^{+1} \quad ^{\frac{1}{-2}} \quad \tau + \quad .$$

62. $\quad - \; \left[(\; +2 \; -3) \; +3-2 \; \right] \; ^{-k} \; = \; ^{2}\left[(\; + \; -1) \; ^{2} - (\; +2 \; -3) \; + \; -2\right] \; ^{1-2k}.$

Solution in parametric form:

$$= \tau^{-1} \qquad R^{\frac{2}{2-}}, \qquad = a\tau^{\; -2 \quad 1-} \qquad \frac{+1}{2-k}\tau^{\; +1} \qquad - \qquad R^2 + \tau R^{\frac{2(1-\;)}{2-}}\Big).$$

63. $- - [(\ + 2) \ - 2] \ ^{-\frac{2 \ +1}{}} \ = \frac{2}{} [(\ + 1) \ ^2 - 2 \ - \ + 1] \ ^{-\frac{3 \ +2}{}}.$

Solution in parametric form:

$$= \ 2 \ ^{-1} R \ , \qquad = \ 2^{-\frac{+1}{}} a \ ^{\frac{1}{}} R^{-\frac{+1}{}} (\qquad 2R \ ).$$

64. $- - \ \frac{+4}{+2} \ - 2 \ ^{-\frac{2 \ +1}{}} \ = \frac{2}{(\ + 2)} [2 \ ^2 + (\ ^2 + \ - 4) \ - (\ - 1)(\ + 2)] \ ^{-\frac{3 \ +2}{}}.$

Solution in parametric form:

$$= 2 \ \tau^{- \quad -2} \quad , \qquad = a(2 \ )^{-\frac{1}{}} \ ^{\frac{2}{}} \ ^{-\frac{1+}{}} \quad - \frac{}{+2} \ ^2 - \tau \quad + \frac{1}{} \tau \ ^{+1} \ ^2 \quad ,$$

where $= \frac{+2}{+1}.$

65. $+ \frac{}{+3} \ \frac{3 \ +5}{2} \ + \frac{-1}{+1} \ ^{-\frac{+4}{+3}}$

$$= - \frac{2}{2(\ + 3)} \ (\ + 1) \ ^2 - \frac{^2 + 2 \ + 5}{+1} \ + \frac{4}{+1} \ ^{-\frac{+5}{+3}}.$$

Solution in parametric form:

$$= \tau \ ^{+1} \ ^2 \ ^{-2}, \qquad = a\tau^{-\frac{+1}{+3}} \ ^{-\frac{2}{+3}} \ ^{-\frac{2(\ +2)}{+3}} \ ^2 - \tau \ ^{+1} \ ^2 + (\ 1)^{\frac{1}{+3}} \frac{+3}{+1} \tau \qquad .$$

66. $- \ \frac{+2}{} + \qquad = - \frac{2}{} \ \frac{+1}{} + \qquad .$

Solution in parametric form:

$$= \ \tau \ ^{\frac{1}{}} \ _1 , \qquad = a \ ^{\frac{1}{}} \ _1 \qquad _1 R^2 + \frac{1}{} \tau \ .$$

67. $= (\qquad + \ ) \ + \ ^2 - \qquad - \ ^2.$

The transformation $= \ln \ , \quad = \quad +$ leads to a linear equation: $(- \ ^2 + a \ + \ ) \ ' = \quad + \ .$

68. $= [\ (2 \ + \ ) \ ^\lambda \ + \ ] \qquad + (- \ ^2 \ ^{2\lambda} \ - \qquad ^\lambda \ + \ ) \ ^2 \quad .$

The substitution $z = e$ leads to an equation of the form 1.3.3.10: $' = [a(2 \ + \lambda) z^\lambda + \ ] z \ ^{-1} + (-a^2 \ z^{2\lambda} - a \ z^\lambda + \ ) z^2 \ ^{-1}.$

69. $= (\quad ^\lambda \ + \ ) \ + \ [\ ^2 \ ^{2\lambda} \ + \ (\quad + 1) \ ^\lambda \ + \ ^2 \quad].$

The substitution $\xi = \frac{a}{\lambda} e^\lambda \ +$ leads to an equation of the form 1.3.1.2: $' = \quad + \ \lambda\xi.$

70. $= \ ^\lambda \ (2 \qquad + \ + \ ) \ - \ ^{2\lambda} \ (\ ^2 \quad ^2 + \qquad + \ ).$

The substitution $= e^\lambda \ (\xi + a \ )$ leads to a linear equation with respect to $= \ (\xi)$: $(-\lambda\xi^2 + \ \xi - \ ) \ ' = a \ + \xi.$

71. $= \qquad (2 \quad ^2 + 2 \ + \ ) \ + \ ^2 \ (- \quad ^4 - \quad ^2 + \ ).$

The substitution $= e \quad (\xi + \quad ^2)$ leads to a Riccati equation with respect to $= \ (\xi)$: $(-a\xi^2 + \ \xi + \ ) \ ' = \ ^2 + \xi.$

72. $\quad + \ (1+2 \ ) \quad = - \ ^{2} \ \ ^{2} \ ^{2}$ **.**

Solution in parametric form:

$$= \frac{2}{\ } \exp(\ \tau^2) \ , \qquad = -\frac{a}{2} \tau^{-2} \left[2\tau^2 \exp(\ \tau^2) \quad 1 \right] \exp[2 \exp(\ \tau^2) \ ],$$

where $\ = \ \ \tau^{-1} \exp(\mp\tau^2) \ \tau + \ .$

73. $\quad - \ \left[1+2 \ \ +2 \ (\ \ +1) \ \right] \ ^{(\ +1)} \quad = - \ ^{2} \ (\ \ +1)(1+ \ \ ) \ \ ^{2(\ +1)}$ **.**

Solution in parametric form:

$$= \ 2 \ \tau \qquad + \frac{1}{\ +1} \ \exp\left[(\ \ +1)\tau \ \ \right], \qquad = a\tau \ \ \frac{\tau}{1 \ \ \tau} + \qquad \exp\left[(\ \ +1)\tau \ \ \right],$$

where $\ = \ \ (1 \ \ \tau \ ^{+1})^{-1 \ 2} \ \tau + \ .$

74. $\quad + \ (1+2 \ \ ^{1 \ 2}) \exp(2 \ \ ^{1 \ 2}) \ \ = - \ ^{2} \ \ ^{3 \ 2} \exp(4 \ \ ^{1 \ 2})$**.**

Solution in parametric form:

$$= \ \tau^{-4} Z^{-2} \ \ ^{2}, \qquad = -a \ \tau^{-4} Z^{-2} (\ ^{2} \ \ \tau^2 Z^2) \exp(-2 \ \tau^{-2} Z^{-1} \ \ ).$$

Here,

$$= (\mp \ )^{-1 \ 2}, \qquad = \tau Z'_\tau + Z, \quad Z = \begin{cases} \ _1 \ _1(\tau) + \ _2 \ _1(\tau) & \text{for the upper sign,} \\ \ _1 \ _1(\tau) + \ _2 \ _1(\tau) & \text{for the lower sign,} \end{cases}$$

where $\ _1(\tau)$ and $\ _1(\tau)$ are the Bessel functions, and $\ _1(\tau)$ and $\ _1(\tau)$ are the modified Bessel functions.

75. $\quad = (\ \ \cosh \ \ + \ ) \ - \ \ \sinh \ \ + \ .$

The transformation $\ = -a \sinh \ $, $\xi = e \ $ leads to a Riccati equation: $2(\ \ + \ )\xi' = a\xi^2 + 2 \ \xi - a.$

76. $\quad = (\ \ \sinh \ \ + \ ) \ - \ \ \cosh \ \ + \ .$

The transformation $\ = -a \cosh \ $, $\xi = e \ $ leads to a Riccati equation: $2(\ \ + \ )\xi' = a\xi^2 + 2 \ \xi + a.$

77. $\quad = (2 \ln \ \ + \ \ +1) \ + \ (-\ln^2 \ \ - \ \ln \ \ + \ ).$

The transformation $\ = e^w$, $\ = (\xi + \ )e^w$ leads to a linear equation: $(-\xi^2 + a\xi + \ ) \ ' = \ \ + \xi.$

78. $\quad = (2 \ln^2 \ \ + 2 \ln \ \ + \ ) \ + \ (-\ln^4 \ \ - \ \ln^2 \ \ + \ ).$

The transformation $\ = e^w$, $\ = (z + \ ^2)e^w$ leads to a Riccati equation: $(-z^2 + az + \ ) \ ' = \ ^2 + z.$

79. $\quad = \ \ \cos(\ \ ^2) \ + \ .$

The substitution $z = \frac{1}{2} \ \ ^2$ leads to an Abel equation of the form 1.3.2.11: $\ ' = a \cos(2\lambda z) \ + 1.$

80. $\quad = \ \ \sin(\ \ ^2) \ + \ .$

The substitution $z = \frac{1}{2} \ \ ^2$ leads to an Abel equation of the form 1.3.2.12: $\ ' = a \sin(2\lambda z) \ + 1.$

1.3.4. Equations of the Form $[g_1(\ )\ + g_0(\ )]\quad = {}_2(\ )\ {}^2 + {}_1(\ )\ + {}_0(\ )$

1.3.4-1. Preliminary remarks.

With the aid of the substitution

$$= \quad + \frac{g_0}{g_1}\quad , \qquad \text{where} \qquad = \exp\ - \quad \frac{\ {}_2}{g_1}\quad , \tag{1}$$

these equations are reducible to a simpler form:

$$' = {}_1(\ )\quad + {}_0(\ ), \tag{2}$$

where

$${}_1 = \quad — \quad \frac{g_0}{g_1}\Big) + \frac{\ {}_1}{g_1} - 2\frac{g_0\ {}_2}{g_1^2}\quad , \qquad {}_0 = \quad \frac{\ {}_0}{g_1} - \frac{g_0\ {}_1}{g_1^2} + \frac{g_0^2\ {}_2}{g_1^3}\Big)\quad {}^2.$$

Specific Abel equations of the form (2) are outlined in 1.3.1–1.3.3. In the degenerate cases with ${}_0 \equiv 0$ or ${}_1 \equiv 0$, the variables in equation (2) are separable.

1.3.4-2. Solvable equations and their solutions.

1. $(\quad + B\quad + \ )\quad + B\quad + \quad + \quad = 0.$

Solution: $A\ {}^2 + k\ {}^2 + 2(B\quad + a\quad + \quad) = \quad .$

2. $(\ + \quad + \ )\quad = \quad + \quad + \gamma.$

The substitution $\ = \quad - a\ - \ $ leads to the equation

$$' = (a + \ )\ + (\beta - a\ )\ + \ - $$

which is separable with $a = -\ $. For $a \neq -\ $, the substitution $\ = (a + \ )\ $ leads to an equation of the form 1.3.1.1 or 1.3.1.2:

$$' = \quad + \Delta^{-2}(\beta - a\ )\ + \Delta^{-2}(\ -\ ), \qquad \text{where} \quad \Delta = a + \ .$$

3. $(\ + \quad {}^2 + \quad + \ )\quad = -\quad {}^2 + 2\quad + \quad + (\ + \ - \ )\ + s.$

The substitution $\ = z + k\ $ leads to a Riccati equation with respect to $\ = \ (z)$:

$$[-az^2 + (\ - k)z + \ - \ k]\ ' = ak\ {}^2 + (\ + k)\ + z + \ .$$

4. $(\ + \quad + \ )\ + \quad {}^{-1} + \quad + \ = 0.$

Solution: $\ {}^2 + \dfrac{2k}{\ + 1}\ {}^{+1} + 2(A\quad + a\ + \ ) = \quad .$

5. $(\ + \quad {}^{+1} + \quad)\quad = (\quad + \quad {}^{-1})\ .$

The substitution $\ = \ (\ -\ )$ leads to a Bernoulli equation with respect to $\ = \ (\ )$: $[-\ {}^2 + (\ + \ )\ - \ ]\ '_w = \quad + a\ {}^2.$

6. $\quad = \quad {}^2 + \quad + \quad + s.$

The transformation $\xi = \ {}^-\ ,\quad = -\dfrac{a}{\ }\ {}^-$ leads to an equation $\ ' = \quad + A\xi + B\xi\ $, where $A = -a\ {}^{-2}\ ,\ B = -a\ {}^{-2}\ ,\quad = (a - \ )\ a$ (see Subsection 1.3.1).

7. $\quad = -\quad {}^2 + (2\quad + 1)\quad + \quad - \ {}^2\ {}^2 - \quad + \ .$

The substitution $\ = \quad + a\ $ leads to a Bernoulli equation with respect to $\ = \ (\ )$: $(-\ {}^2 + \quad + \ )\ '_w = \quad + a\ {}^2.$

8. $\mathbf{2 \quad = (1- \;)\;^2 + [\;(2\; +1)\; +2\; -1]\; -\;^2\;\;^2 -\; -\;.}$

The transformation $\;=\xi^2,\;\; = \xi\; + a\xi^2 + 1$ leads to a Riccati equation: $(-\;\;^2 + 2a\; -\;)\xi' = a\xi^2 + \;\xi + 1.$

9. $\mathbf{(\quad - \quad + B\; - B\;)\quad = \quad ^2 + \quad + (B - \quad)\;.}$

The transformation $\;= \;+k,\;\; = \xi$ leads to a linear equation with respect to $\; = \;(\;)$: $[(\; - A)\xi^2 + \;\xi]\;' = A\xi\; + B.$

10. $\mathbf{[(3\quad + \;s)\; + (4\; + 3s)\;]\quad = 2\quad ^2 + 2(3\; + s)\; + 2\;.}$

The substitution $\; = a\;^2 + (3\lambda + \;)\; +$ leads to an Abel equation of the form 1.3.3.3: $2\quad ' = (7a\; + 5\;)\; - 3a^2\;^3 - 2\;\;^2 - 3\;^2$, where $\; = \; + 2\lambda,\;\; = \frac{1}{2}a(13\lambda + 6\;).$

11. $\mathbf{[(4\quad + \;s)\; + (4\; + 3s)\;]\quad = \frac{3}{2}\quad ^2 + 2(3\; + s)\; + 2\;.}$

The substitution $\; = \frac{3}{4}a\;^2 + (3\lambda + \;)\; +$ leads to an Abel equation of the form 1.3.3.3: $2\quad ' = (7a\; + 5\;)\; - 3a^2\;^3 - 2\;\;^2 - 3\;^2$, where $\; = \; + 2\lambda,\;\; = \frac{1}{8}a(60\lambda + 25\;).$

12. $\mathbf{(2\quad + \quad + \quad + \;)\quad = \quad ^2 + \quad ^2\;^2 + \quad + \;(\quad + \; - \;)\; + s.}$

The substitution $\; = z + k$ leads to a Riccati equation with respect to $\; = \;(z)$:

$$[Az^2 + (\; - ak)z + \; - \;k]\;' = 2Ak\;^2 + (2Az + ak + \;)\; + az + \;.$$

13. $\mathbf{2\quad + (1 - \;)\quad - \frac{2(\quad + 1)}{+3}\quad = \frac{1-}{2}\;^2 + \frac{-1}{+3}\; + \;.}$

The substitution $\; = \dfrac{1-}{2}\;^2 + \dfrac{-1}{+3}\; +$ leads to an equation of the form 1.3.3.4:

$$' = [(3 - \;)\; - 1]\; + (\; - 1)(\;^3 - \;^2 - a\;),\qquad \text{where}\quad a = A - 2(\; + 1)(\; + 3)^{-2}.$$

14. $\mathbf{(2\quad + \;)\quad = (2 - \;)\;^2 + (1 - \;)\quad + \quad ^2 + \quad ^{+2}.}$

The transformation $z = \quad ,\;\; = -A\quad + a\;\;z^2 + \quad z -$ leads to a separable equation: $\;' = \;\;(2az + \;)(a\;\;z^2 + \quad z - \;).$

15. $\mathbf{(\quad + \;^2 + \;)\quad = \;^2 + \quad + \;.}$

Solution: $(\; + \;)^2 + a + \; = \;(\; - a\;)^2.$

16. $\mathbf{(2\quad + B\;^2 + \;)\quad = \quad ^2 + \;(\quad + B)\;^2 + \;.}$

The substitution $\; = z + k$ leads to a Riccati equation with respect to $\; = \;(z)$: $(Az^2 + \; - \;k)\;' = (2Ak + B)\;^2 + 2Az\; + \;.$

17. $\mathbf{(\quad + B\;^2 + \quad)\quad = \quad ^2 + E\quad + \quad ^2 + \quad .}$

The substitution $\; = \;z$ leads to a linear equation with respect to $\; = \;(z)$:

$$[(\; - A)z^2 + (\; - B)z + \;]\;' = (Az + B)\; + k$$

18. $\mathbf{(\quad + B\;^2 + \quad)\quad = \quad ^2 + B\quad + (\quad + \;)\; + B\quad + \quad .}$

This is a special case of equation 1.3.4.22. Solutions: $\; = \quad -$ and $A\; + B\; + k = 0.$

19. $\mathbf{(2\quad + B\;^2 + \quad)\quad = \quad ^2 + \quad + \quad ^2 + \quad - \quad - \quad ^2 - \quad .}$

The substitution $\; = \xi\; + \beta$ leads to a linear equation with respect to $\; = \;(\xi)$: $[-A\xi^2 + (\; - B)\xi + \;]\;' = (2A\xi + B)\; + 2A\beta + k.$

20. $(\quad +B \quad ^2+ \quad) \quad = \quad ^2+ \quad + \quad ^2+(\quad - \quad) - \quad - \quad .$

The substitution $=\xi \; +\beta$ leads to a linear equation with respect to $=(\xi)$: $[(\; -B)\xi+ \;] \; ' =$ $(A\xi+B) \; +A\beta+k.$

21. $(\quad + \quad +B \quad ^2+B \quad) \quad = \quad ^2+ \quad + \; (\; -B) \; .$

The transformation $= \; -k, \quad =\xi$ leads to a linear equation with respect to $= (\xi)$: $[(\; -A)\xi^2+(\; -B)\xi] \; ' =(A\xi+B) \; -kB.$

22. $(\quad +B \quad ^2+ \; _1 \; + \; _1 \; + \; _1) \quad = \quad ^2+B \quad + \; _2 \; + \; _2 \; + \; _2.$

Jacobi equation.

1 . With the help of the transformation $= \bar{} + \; , \quad = \bar{} +\beta$, where and β are the parameters which are determined by solving the algebraic system

$$A \;\; \beta+B \;\; ^2+a_1 \; + \; _1\beta+ \; _1=0, \qquad A\beta^2+B \;\; \beta+a_2 \; + \; _2\beta+ \; _2=0,$$

we obtain the equation

$$(A^{--}+B^{-2}+\bar{a}_1{}^{-}+\bar{}_1{}^{-})^{-\prime} = A^{-2}+B^{--}+\bar{a}_2{}^{-}+\bar{}_2{}^{-},$$

where $\bar{a}_1=2B \; +A\beta+a_1$, $\bar{a}_2=B\beta+a_2$, $\bar{}_1=A \; + \; _1$, $_2=2A\beta+B \; + \; _2$. The transformation $z=\bar{} \; \bar{}, \quad =1 \; \bar{}$ leads to a linear equation:

$$[\bar{}_1z^2+(\bar{a}_1-\bar{}_2)z-\bar{a}_2] \; ' =(\bar{}_1z+\bar{a}_1) \; +Az+B.$$

2 . The original equation can be also rewritten in the form

$$(\;\; ' - \;)(\; _3 \; + \; _3 \; +k_3)- \; '(\; _1 \; + \; _1 \; +k_1)+ \; _2 \; + \; _2 \; +k_2=0.$$

The solution of this equation in parametric form can be obtained from the solution of the following system of constant coefficient linear differential equations:

$$\begin{aligned} (\; _1)' &= \; _1 \; _1+ \; _1 \; _2+k_1 \; _3, \\ (\; _2)' &= \; _2 \; _1+ \; _2 \; _2+k_2 \; _3, \\ (\; _3)' &= \; _3 \; _1+ \; _3 \; _2+k_3 \; _3, \end{aligned}$$

using the formulas $(\;)= \; _1 \; _3$ and $(\;)= \; _2 \; _3$.

23. $(\quad +B \quad ^2+ \quad + \quad + \;) \quad = \quad + \; B \quad ^2+ \quad + \; (\quad + \; - \quad) \; +s.$

The substitution $=z+k$ leads to a Riccati equation with respect to $= (z)$:

$$[(\; -ak)z+ \; - \; k] \; ' =(Ak+B) \; ^2+(Az+ak+ \;) \; +az+ \; .$$

24. $(2 \quad +B \quad ^2+ \quad + \quad + \;) \quad = \quad ^2+ \; (\quad +B) \quad ^2+ \quad + \quad +s.$

The substitution $=z+k$ leads to a Riccati equation with respect to $= (z)$:

$$(Az^2+ \; - \; k) \; ' =(2Ak+B) \; ^2+(2Az+ak+ \;) \; +az+ \; .$$

25. $(2 \quad - \quad ^2+ \quad + \quad + \;) \quad = \quad ^2+ \quad + \; (\quad + \; - \quad) \; +s.$

The substitution $=z+k$ leads to a Riccati equation with respect to $= (z)$:

$$[Az^2+(\; -ak)z+ \; - \; k] \; ' =Ak \;\; ^2+(2Az+ak+ \;) \; +az+ \; .$$

26. $(2 \quad +B \quad ^2+ \quad - \quad + \;) \quad = \quad ^2+ \; (\quad +B) \quad ^2+ \quad - \quad +s.$

The substitution $=z+k$ leads to a Riccati equation with respect to $= (z)$:

$$[Az^2+(\; -ak)z+ \; - \; k] \; ' =(2Ak+B) \; ^2+2Az \; +az+ \; .$$

27. $(2 \quad + B \;^2 + \quad + \quad + \;) \quad = \quad ^2 + \;(\quad + B)\;^2 + \quad + \quad ^2 \quad + s.$

The substitution $\; = z + k$ leads to a Riccati equation with respect to $\; = \;(z)$:

$$[Az^2 + (\; - ak)z + \; - \; k]\;' = (2Ak + B)\;^2 + (2Az + ak + \;)\; + az + \; .$$

28. $[\quad + B\;^2 + (\; - 1) \quad - (\quad + B\;)\;] \quad = \quad ^2 + B \quad - (\quad + B \quad) + (\; - 1)B \quad .$

This is a special case of equation 1.3.4.22. Solution in parametric form:

$$= \frac{a \; + A}{\; + \;}, \qquad = \frac{\; - B}{\; + \;}.$$

The solution can be presented in implicit form as well:

$$(A \; + B\;)\; + [A(\; - \;) + B(a - \;)]\;^{-1}(a \; - \;) = 0.$$

29. $[(\; + \;)\; + (1 - \;)\;^2 + (2 \; - 1)\; - \;] \quad = 2 \;^2 + 2 \; .$

The substitution $\; = a \; + \;$ leads to an equation of the form 1.3.4.8:

$$2 \quad ' = (1 - \;)\;^2 + [a(2 \; + 1)\; + 2 \; - 1]\; - a^2 \;^2 - \; - \;, \qquad \text{where} \quad = (2 \; - 1)a - \; .$$

30. $[(\; + \;)\; + (\; + 1)\;^2 - \;(2 \; + 1)\; + \;^2\;] \quad = \dfrac{2}{3 \; - 1}\;^2 + 2 \; .$

The transformation $z = \dfrac{3 \; - 1}{\; - 1}\dfrac{1}{\;}$, $\; = \dfrac{3 \; - 1}{\; - 1}\dfrac{\;}{\;} + \dfrac{\;}{\; - 1}$ leads to an equation of the form 1.3.4.8:

$$2z \quad ' = (1 - \;)\;^2 + [a(2 \; + 1)z + 2 \; - 1]\; - a^2 \; z^2 - \; z - \;, \qquad = \frac{(3 \; - 1)\; + a \;(2 \; + 1)}{\; - 1}.$$

31. $(2 \quad + \;) \quad = - \;(\; + 3)\;^2 - \;(\; + 2)\; + \; .$

The transformation $z = \;$, $\; = - \;^{+1} + a(\; + 1)\;^2\;^2 + \;(\; + 1)\;$ leads to a separable equation: $\;' = (\; + 1)^2(2az + \;)(az^2 + \; z)$.

32. $[(\;_2 \;^2 + \;_1 \; + \;_0)\; + \;_2 \;^2 + \;_1 \; + \;_0] \quad = \;_2 \;^2 + \;_1 \; + \;_0.$

This is a Riccati equation with respect to $\; = \;(\;)$. The substitution $\; = - \dfrac{\;_2 \;^2 + \;_1 \; + \;_0}{a_2 \; + \;_2}\dfrac{\;'}{\;}$ leads to a second-order linear equation:

$$\;_2 \;'' - [(\;_2)' + \;_1 \;_2]\;' + \;_0 \;_2^2 \; = 0, \qquad \text{where} \quad = \frac{a \; + \;}{\;_2 \;^2 + \;_1 \; + \;_0}; \qquad = 1,\; 2,\; 3.$$

33. $[(12 \;^2 \;^2 - 7 \quad + 1)\; + 4 \;^2 - 5 \;] \quad = -2 \;(3 \;^2 \;^2 + 2 \quad + 3 \;^2).$

The substitution $\; = \;(3a^2 \;^2 + 2 \quad + 3 \;^2)$ leads to an Abel equation of the form 1.3.3.3: $2 \quad ' = (7a \; + 5 \;)\; - 3a^2 \;^3 - 2 \;^2 - 3 \;^2 \; .$

34. $[(\; - 1)(\; + B)\; + \;(\;^2 + E \; + \;)]$

$$= [\;(1 - \;)\; - B \;]\;^2 + [\;(2 - \;)\;^2 + E(1 - \;)\; - \;]\; .$$

Solution: $A \; + \;^2 + \; + B \; + \; = \; .$

35. $\;^2(2 \quad + \;) \quad = -4 \;^2 \;^2 - 3 \quad + \;^2 + \; .$

The transformation $z = \;$, $\; = 2a \;^2 \;^2 + 2 \quad - \;^2 - k$ leads to a separable equation: $\;' = 2(2az + \;)(2az^2 + 2 \; z - k)$.

36. $(\quad + \quad + \quad ^2) \quad = \quad ^2 + \quad + \quad .$

The transformation $\quad = \quad ,\ z = \quad ^{-2}$ leads to a linear equation: $(\quad - a \quad)z' = (\quad -2)(az + \quad + \quad)$.

37. $(2 \quad + \quad) \quad = -\ (3 \quad + \quad) \quad ^2 -\ (2 \quad + \quad) \quad + \quad + \quad ^- .$

The transformation $z = \quad ,\quad = -A \quad ^+ \quad + (\quad + \quad)(az^2 + \quad z) - \quad$ leads to a separable equation: $\quad ' = (\quad + \quad)^2 (2az + \quad) \quad az^2 + \quad z - \frac{\quad}{\quad + \quad} \quad .$

38. $\quad = - \quad ^2 + \quad (2 \quad + 1) \quad + \quad - \quad ^2 \quad ^2 - \quad + \quad .$

The transformation $\quad = \ln \xi, \quad = \quad + a\xi$ leads to a Bernoulli equation with respect to $\xi = \xi(\quad)$: $(- \quad ^2 + \quad + \quad)\xi'_w = \quad \xi + a\xi^2$.

1.3.5. Some Types of First- and Second-Order Equations Reducible to Abel Equations of the Second Kind

Notation: , g, , , , , , , and are arbitrary functions of their arguments.

1.3.5-1. Quasi-homogeneous equations.

1 . Let us consider a quasi-homogeneous equation of the form

$$(\quad) \quad ^{+1} \quad ' + g(\quad) + A \quad ^\lambda = 0.$$

In the special case $\lambda = 0$ this equation is homogeneous.

The transformation $z = \quad , \quad = A \quad ^\lambda + g(z) - \quad z \quad (z)$ leads to an Abel equation:

$$\quad ' = [-(\lambda + \quad) \quad + g' - \quad z \quad '] \quad + \lambda \quad (g - \quad z \quad).$$

2 . A quasi-homogeneous equation of the form

$$(\quad) \quad ^{+1} \quad ' + g(\quad) + \quad ^\lambda [\quad (\quad) \quad ^{+1} \quad ' + \quad (\quad)] = 0$$

can be reduced by the transformation $z = \quad , \quad = \quad ^{-\lambda}$ to an Abel equation:

$$[g(z) - \quad z \quad (z)] \quad + \quad (z) - \quad z \quad (z) \quad ' = \lambda \quad (z) \quad ^2 + \lambda \quad (z) \quad .$$

1.3.5-2. Equations of the theory of chemical reactors and the combustion theory.

In the theory of chemical reactors and the combustion theory, one encounters equations of the form

$$\quad '' \quad - a \quad ' = \quad (\quad).$$

The substitution $\quad (\quad) = \quad ' \quad a$ leads to the Abel equation $\quad ' - \quad = a^{-2} \quad (\quad)$, whose solvable cases are given in Subsection 1.3.1.

1.3.5-3. Equations of the theory of nonlinear oscillations.

1 . Let us consider equations of the theory of nonlinear oscillations of the form

$$\quad '' \quad + \quad (\quad) \quad ' + \quad = 0.$$

The substitution $z(\quad) = \quad '$ leads to the Abel equation

$$zz' + \quad (\quad)z + \quad = 0, \tag{1}$$

which is reduced, with the aid of the substitution $\tau = \frac{1}{2}(a - \quad ^2)$, to the following form:

$$zz'_\tau = g(\tau)z + 1, \qquad \text{where} \quad g(\tau) = \frac{(\quad \overline{a - 2\tau}}{\overline{a - 2\tau}}. \tag{2}$$

Specific cases of equation (2) are outlined in Subsection 1.3.2.

2 . An equation of the theory of nonlinear oscillations of the form

$$'' + \quad (') + \quad = 0$$

can be reduced by the transformation $z = \ '$, $\quad = - \ - \ (')$ to an Abel equation of the form (1):

$$' + \ '(z) \quad + z = 0.$$

1.3.5-4. Second-order homogeneous equations of various types.

1 . A homogeneous equation with respect to the independent variable has the form

$$^{2} \ '' = g(\ ) \ ' + \ (\ ).$$

The substitution $(\ ) = \ '$ leads to an Abel equation: $' = [g(\ ) + 1] \quad + \ (\ )$.

2 . A generalized homogeneous equation

$$'' = g(\quad) \ ' + \ ^{- \ -1} \ (\quad)$$

can be reduced by the transformation $= \quad , \quad = \quad (\ ' + k \ )$ to an Abel equation:

$$' = [g(\ ) + 2k + 1] \quad + \ (\ ) - k \ g(\ ) - k(k + 1) \ .$$

To the Emden–Fowler equation, discussed in Section 2.3, there correspond $g(\ ) = 0$, $(\ ) = A$, and $k = \frac{+2}{-1}$.

3 . A generalized homogeneous equation

$$'' = \qquad - \ ' + \ ^{-2} \qquad - \ '$$

can be reduced by the transformation $= - \ '$, $= \ ^{+2} \ ^{-1}$ to an Abel equation:

$$[\ (\ ) \ + \ (\ ) + \ - \ ^{2}] \ '_{\eta} = [(\beta - 1) \ + \ + 2] \ .$$

To the generalized Emden–Fowler equation, discussed in Section 2.5, there correspond $= \ - l$, $\beta = \ + l$, $(\ ) = A$, and $(\ ) = 0$.

1.3.5-5. Second-order equations invariant under some transformations.

1 . An equation invariant under "dilatation–translation" transformation has the form

$$'' = \ e \ (\ ') + \ ^{-2} g(\ ').$$

The transformation $= \ '$, $= \ ^{+2} e$ leads to an Abel equation:

$$[\ (\ ) \ + g(\ ) + \ ] \ ' = (\beta \ + \ + 2) \ .$$

2 . An equation invariant under "translation–dilatation" transformation has the form

$$'' = e \qquad \frac{'}{} + g \ \frac{'}{} \ .$$

The transformation $\xi = \ ' \quad , \quad = e \quad ^{-1}$ leads to an Abel equation:

$$[\ (\xi) \ + g(\xi) - \xi^{2}] \ ' = [(\beta - 1)\xi + \ ] \ .$$

1.4. Equations Containing Polynomial Functions of y

1.4.1. Abel Equations of the First Kind $= {}_3(\ )\ {}^3 + {}_2(\ )\ {}^2 + {}_1(\ )\ + {}_0(\ )$

1.4.1-1. Preliminary remarks.

1 . If ${}_0 = {}_0(\ )$ is a particular solution of the equation in question, the substitution $- {}_0 = 1$ reduces it to an Abel equation of the second kind:

$$' = -(3\ {}_3\ {}_0^2 + 2\ {}_2\ {}_0 + {}_1)\ {}^2 - (3\ {}_3\ {}_0 + {}_2)\ -\ {}_3,$$

which is discussed in Section 1.3. For ${}_0(\ ) \equiv 0$, we can choose ${}_0 \equiv 0$ as a particular solution.

2 . The transformation

$$\xi = \quad {}_3\ {}^2 \quad, \qquad = \quad + \frac{{}_2}{3\ {}_3}\quad {}^{-1}, \qquad \text{where} \qquad = \exp \quad {}_1 - \frac{{}_2^2}{3\ {}_3} \qquad ,$$

brings the original equation to the normal form:

$$' = \ {}^3 + \ (\xi), \qquad \text{where} \qquad = \frac{1}{\ {}_3\ {}^3}\quad {}_0 + \frac{1}{3} \text{—} \quad \frac{{}_2}{{}_3} \quad - \frac{{}_1\ {}_2}{3\ {}_3} + \frac{2\ {}_2^3}{27\ {}_3^2} \quad .$$

1.4.1-2. Solvable equations and their solutions.

1. $= \quad {}^3 + \quad {}^{-3}\ {}^2.$

This is a special case of equation 1.4.1.9 with $= -1\ \ 2$.

2. $= -\ {}^3 + 3\ {}^2\ {}^2\ \ - 2\ {}^3\ {}^3 + \ .$

The substitution $= 1\ \ + a$ leads to an Abel equation of the form 1.3.2.1: $' = 3a\ \ + 1$.

3. $= -\ {}^3 + (\quad + \ )\ {}^2.$

The substitution $= -1$ leads to an Abel equation of the form 1.3.2.1: $' = (a\ \ + \ )\ \ + 1$.

4. $= -\ {}^3 + (\quad + \ )^{-2}\ {}^2.$

The substitution $= -1$ leads to an Abel equation of the form 1.3.2.2: $' = (a\ \ + \ )^{-2}\ \ + 1$.

5. $= -\ {}^3 + (\quad + \ )^{-1\ \ 2}\ {}^2.$

The substitution $= -1$ leads to an Abel equation of the form 1.3.2.4: $' = (a\ \ + \ )^{-1\ \ 2}\ \ + 1$.

6. $= \quad {}^3 + 3\quad {}^2 - \ \ - 2\ \ {}^3\ {}^3.$

This is a special case of equation 1.4.1.10 with $= 0$ and $= 1$.

7. $= \quad {}^3 + \quad {}^2.$

The substitution $=$ leads to a separable equation: $' = a\ {}^3 + \ {}^2 + \ .$

8. $= \quad {}^3 + 3\quad {}^2\ {}^2 - \ \ - 2\ \ {}^3\ {}^4.$

This is a special case of equation 1.4.1.10 with $= \ = 1$.

9. $= \;^{2} \;^{+1} \;^{3} + \; - \;^{-2}$.

The substitution $= \;^{+1}$ leads to a separable equation: $' = a \;^{3} + (\;+1)\; + \;$.

For $a = -\frac{1}{3}(\;+1)A^{-2}$ and $= \frac{2}{3}A(\;+1)$, the solution is written in parametric form:

$$= \exp \frac{}{+1}, \quad = -A\;1 + \frac{1}{\tau}\;\exp(-\;), \quad \text{where} \quad = \tau - \tfrac{1}{3}\ln|\tau + \tfrac{1}{3}| + \;.$$

10. $= \;^{3} + 3 \;^{+} \;^{2} - \;^{-1} - 2\;^{3}\;^{+3}$.

The substitution $= \; + \;$ leads to a Bernoulli equation: $' = a\;^{3} - 3a\;^{2}\;^{+2}$.

11. $= \;^{3} + 3\;^{+}\;^{2} + \;^{k} - 2\;^{3}\;^{+3} + \;^{+k} - \;^{-1}$.

The substitution $= \; + \;$ leads to a Bernoulli equation: $' = a\;^{3} + (\; - 3a\;^{2}\;^{+2}\;)\;$.

12. $9 \; = - \;(\;^{1-} + \;)^{2\lambda+1}\;^{3} - \;^{-2}\;(9\; + 2 + 9\;^{-1})(\;^{1-} + \;)^{-\lambda-2}$.

For $\lambda = \dfrac{1}{3a(1-\;)}$, the substitution $= \dfrac{3}{} + \dfrac{1}{a\; + \;}\left(a\;^{1-} + \;^{-\lambda}\right.$ leads to the Abel equation $' = \; + a\; + \;$, which is discussed in Subsection 1.3.1.

13. $= \;^{4}\;^{3} + (\;^{2} - 1)\; + \;$.

The substitution $= \;$ leads to a separable equation: $' = \;(a\;^{3} + \; + \;)$.

14. $= \;^{3} + 3\;^{2} - \; - 2\;^{3}\;^{3}$.

The substitution $= \; + \;$ leads to a Bernoulli equation: $' = a\;^{-1}\;^{3} - 3a\;^{2}\;^{2}\;^{-1}$.

15. $= \;^{2}\;^{+1}\;^{3} + (\; - \;)\; + \;^{1-}$.

The substitution $= \;$ leads to a separable equation: $' = a\;^{3} + \; + \;$.

16. $= \;^{+2}\;^{3} + (\; - 1)\; + \;^{-1}$.

The substitution $= \;$ leads to a separable equation: $' = \;^{-1}(a\;^{3} + \; + \;)$.

17. $^{2}\; = \;^{3} - 3\;^{2}\;^{4} + 2\;^{3}\;^{6} + 2\;^{3}$.

The transformation $= 1\;\xi, \; = a\;^{2} + 1$ leads to an equation of the form 1.3.2.2: $' = 3a\xi^{-2} + 1$.

18. $= -(\; + \;)\;^{3} + \;^{2}$.

The substitution $= -1$ leads to an equation $' = \; + a\; + \;$, which is discussed in Subsection 1.3.1.

19. $= (\;^{2} + B\; + \;)^{-1}\;^{2}\;^{3} + \;^{2}$.

The substitution $= -1$ leads to an Abel equation of the form 1.3.1.63: $' = \; - (A\;^{2} + B\; + \;)^{-1}\;^{2}$.

20. $= - \;^{-16}\;^{9}\left(\; - \tfrac{6}{25}\;^{34}\;^{9}\;^{3} + \tfrac{2}{27}\right(9\; - \tfrac{2}{25}\;^{-11}\;^{18}\left(\; - \tfrac{6}{25}\;^{-61}\;^{18}\right.$.

Solution in parametric form:

$$= \frac{6}{25a}\tau^{2}\wp, \quad = \mp\frac{125a}{108}(a\;_{1})^{-25\;18}\tau^{7\;9}\wp^{7\;18}\;_{1}^{-1}\;_{2}^{-1}(18\wp\;_{1}\;5\;_{2}),$$

where

$$_{1} = \tau^{2}\wp \mp 1, \quad _{2} = \tau\;\overline{(4\wp^{3}-1)} + 2\wp.$$

The function $\wp = \wp(\tau)$ is defined implicitly by $\tau = \dfrac{\wp}{\overline{(4\wp^{3}-1)}} - \;$. The upper sign in the formulas corresponds to the classical Weierstrass elliptic function $\wp = \wp(\tau + \;, 0, 1)$.

21. $= - \ ^3 + \ \ \lambda \ \ ^2.$

The substitution $= -1$ leads to an Abel equation of the form 1.3.2.7: $' = ae^{\lambda} \ \ + 1.$

22. $= - \ ^3 + 3 \ ^2 \ ^{2\lambda} \ \ - 2 \ ^3 \ ^{3\lambda} \ + \ \ \ ^{\lambda} \ .$

The substitution $= \frac{1}{\ } + ae^{\lambda}$ leads to an Abel equation of the form 1.3.2.7: $' = 3ae^{\lambda} \ \ + 1.$

23. $= -\frac{1}{3} \ ^{-1} \ ^{2\lambda} \ \ ^3 + \frac{2}{3} \ ^2 \ ^{-\lambda} \ .$

Solution in parametric form:

$$= \frac{\ }{\lambda}, \qquad = -\lambda \ \ 1 + \frac{1}{\tau} \ \ e^{-F}, \qquad \text{where} \qquad = \tau - \tfrac{1}{3} \ln|\tau + \tfrac{1}{3}| + \ \ .$$

24. $= \ ^{2\lambda} \ \ ^3 + \ \ ^{\lambda} \ \ ^2 + \ \ + \ \ ^{-\lambda} \ .$

The substitution $= e^{-\lambda}$ leads to a separable equation: $' = a \ ^3 + \ \ ^2 + (\ + \lambda) \ + \ .$

25. $= \ ^{\lambda} \ \ ^3 + 3 \ \ ^{\lambda} \ \ ^2 + \ \ - 2 \ \ ^3 \ ^{\lambda} \ + \ \ .$

The substitution $= \ + \ $ leads to a Bernoulli equation: $' = ae^{\lambda} \ \ ^3 + (\ - 3a \ ^2 e^{\lambda} \ ) \ .$

26. $= \ ^{\lambda} \ \ ^3 + 3 \ \ ^{(\lambda + \ )} \ \ ^2 - 2 \ \ ^3 \ ^{(\lambda+3 \ )} \ - \ \ \ .$

The substitution $= \ + \ e$ leads to a Bernoulli equation: $' = ae^{\lambda} \ \ ^3 - 3a \ ^2 e^{(\lambda+2 \ )} \ \ .$

27. $= \ ^{\lambda} \ \ ^3 + 3 \ \ ^{(\lambda + \ )} \ \ ^2 + 2 \ \ ^2 \ ^{(\lambda+2 \ )} \ - \ \ \ .$

The substitution $= \ + \ e$ leads to a Bernoulli equation: $' = ae^{\lambda} \ \ ^3 - a \ ^2 e^{(\lambda+2 \ )} \ \ .$

28. $= \ ^{\lambda} \ \ ^3 + 3 \ \ ^{(\lambda + \ )} \ \ ^2 + \ \ - 2 \ \ ^3 \ ^{(\lambda+3 \ )} \ .$

The substitution $= \ + \ e$ leads to a Bernoulli equation: $' = ae^{\lambda} \ \ ^3 + \big[\ - 3a \ ^2 e^{(\lambda+2 \ )} \ \big] \ .$

29. $= \ ^{\lambda} \ \ ^3 + 3 \ \ ^{(\lambda + \ )} \ \ ^2 + [(3 \ \ ^2 + \ ) \ ^{(\lambda+2 \ )} \ + s] \ + \ (\ \ ^2 + \ ) \ ^{(\lambda+3 \ )} \ + \ (s - \ ) \ \ .$

The substitution $= \ + \ e$ leads to a Bernoulli equation: $' = ae^{\lambda} \ \ ^3 + [\ e^{(\lambda+2 \ )} \ + \ ] \ .$

30. $= [\ + \ \exp(2 \ \ )] \ ^3 + \ ^2.$

The substitution $= -1$ leads to an equation of the form 1.3.1.8: $' = \ -a - \ \exp(2 \ \ a).$

31. $= -\frac{2}{3} \ ^{-1} \exp(2 \ \ ^2) \ ^3 + (1 - \frac{4}{3} \ \ ^2) \exp(- \ \ ^2).$

The substitution $= \frac{1}{2a} + \ \exp(-a \ ^2)$ leads to an equation of the form 1.3.1.16: $' = \ + (6a \ )^{-1}.$

32. $= - \ \exp(2 \ \ ^3) \ ^3 + (1 - 2 \ \ ^3) \exp(- \ \ ^3).$

The transformation $\xi = \ ^2$, $= \frac{2}{3a} + \ \exp(-a \ ^3)$ leads to an equation of the form 1.3.1.32: $' = \ 2(9a)^{-1} \xi^{-1} \ ^2.$

33. $= - \ ^{-2} \exp(2 \ \ ^3) \ ^3 + 2 \ (1 - \ \ ^3) \exp(- \ \ ^3).$

The substitution $= \frac{1}{3a} + \ ^2 \ \exp(-a \ ^3)$ leads to an equation of the form 1.3.1.33: $' = \ + (9a)^{-1} \ ^{-2}.$

34. $= -\frac{2}{9}\ {}^{-1}\ {}^{-1}\ {}^{2}\exp(2\ \ {}^{3}\ {}^{2})\ {}^{3} + \frac{3}{4}\ \ {}^{-1}\ {}^{2}(2\ \ {}^{3}\ {}^{2} - 1)\exp(-\ \ {}^{3}\ {}^{2})$**.**

Solution in parametric form:

$$= \ \tau^{-4}\ {}^{3}Z^{-2}\ \ {}^{2}_{1}, \qquad = -\frac{1}{\ }\tau^{-2}\ {}^{3}Z^{-1}\ \ {}^{-1}_{2}(\tau^{2}Z^{3} \mp \ {}_{1}\ \ {}_{2})\exp\bigl(\ \tfrac{2}{3}\tau^{-2}Z^{-3}\ \ {}^{3}_{1}\ ,$$

where
$$a = \mp\tfrac{2}{3}\ {}^{-3}\ {}^{2}, \qquad {}_{1} = \tau Z'_{\tau} + \tfrac{1}{3}Z, \qquad {}_{2} = \ {}^{2}_{1}\ \ \tau^{2}Z^{2},$$
$$Z = \begin{matrix} {}_{1}\ {}_{1}\ {}_{3}(\tau) + \ {}_{2}\ {}_{1}\ {}_{3}(\tau) & \text{for the upper sign,} \\ {}_{1}\ {}_{1}\ {}_{3}(\tau) + \ {}_{2}\ {}_{1}\ {}_{3}(\tau) & \text{for the lower sign,} \end{matrix}$$

${}_{1}\ {}_{3}(\tau)$ and ${}_{1}\ {}_{3}(\tau)$ are the Bessel functions, and ${}_{1}\ {}_{3}(\tau)$ and ${}_{1}\ {}_{3}(\tau)$ are the modified Bessel functions.

35. $= -\ \ {}^{-3}(\ {}^{2} - \ )^{4\ \ 3}\exp\ \frac{2}{3}\ \ {}^{3} + \frac{1}{27\ \ {}^{2}}\ {}^{2}(\ {}^{2} - \ )^{-13\ \ 6}(2\ \ {}^{4} - 9\ \ \ {}^{2} + 27\ \ {}^{2})\exp\ -\frac{2}{6}$ **.**

Solution in parametric form:

$$= \frac{\overline{2a}}{\tau}, \qquad = -\frac{\overline{2}}{3a^{2}\ {}^{3}\tau^{2}\ {}^{3}(2\ \ -\tau^{2})^{7\ \ 6}}\ \frac{4(\tau+1)\ \ {}^{2} + 4\tau^{2}\ \ - 3\tau^{4}}{2(\tau+1)\ \ - \tau^{2}}\exp\ -\frac{\ }{3\tau^{2}}\ \ ,$$

where $\ = \tau - \ln|1+\tau| + \ \ .$

36. $= \ \ {}^{3} + \ \ \cosh(\ \ )\ \ {}^{2}$**.**

The transformation $\ = \frac{1}{\ } + \frac{\ }{\lambda}\sinh(\lambda\ \ ), \quad = e^{\lambda}$ leads to a Riccati equation: $2a\ \ ' = \ {}^{2} - 2\lambda\ \ - \ .$

37. $= \ \ {}^{3} + \ \ \sinh(\ \ )\ \ {}^{2}$**.**

The transformation $\ = \frac{1}{\ } + \frac{\ }{\lambda}\cosh(\lambda\ \ ), \quad = e^{\lambda}$ leads to a Riccati equation: $2a\ \ ' = \ {}^{2} - 2\lambda\ \ + \ .$

38. $= -\ \ {}^{3} + 3\ \ {}^{2}\cosh^{2}\ \ \ - 2\ \ {}^{3}\cosh^{3}\ \ + \ \ \sinh\ \ $**.**

The substitution $\ = a\cosh\ \ + 1$ leads to an Abel equation of the form 1.3.2.9: $\ ' = 3a\cosh\ \ \ + 1.$

39. $= -\ \ {}^{3} + 3\ \ {}^{2}\sinh^{2}\ \ \ - 2\ \ {}^{3}\sinh^{3}\ \ + \ \ \cosh\ \ $**.**

The substitution $\ = a\sinh\ \ + 1$ leads to an Abel equation of the form 1.3.2.10: $\ ' = 3a\sinh\ \ \ + 1.$

40. $= -\ \ {}^{3} + \ \ \cos(\ \ )\ \ {}^{2}$**.**

The substitution $\ = -1$ leads to an Abel equation of the form 1.3.2.11: $\ ' = a\cos(\lambda\ \ )\ \ + 1.$

41. $= -\ \ {}^{3} + \ \ \sin(\ \ )\ \ {}^{2}$**.**

The substitution $\ = -1$ leads to an Abel equation of the form 1.3.2.12: $\ ' = a\sin(\lambda\ \ )\ \ + 1.$

42. $= -\ \ {}^{3} + 3\ \ {}^{2}\cos^{2}(\ \ )\ + \ \ \ \sin(\ \ ) + 2\ \ {}^{3}\cos^{3}(\ \ )$**.**

The substitution $\ = -a\cos(\lambda\ \ ) + 1$ leads to an Abel equation of the form 1.3.2.11: $\ ' = -3a\cos(\lambda\ \ )\ \ + 1.$

43. $= -\ \ {}^{3} + 3\ \ {}^{2}\sin^{2}(\ \ )\ + \ \ \ \cos(\ \ ) - 2\ \ {}^{3}\sin^{3}(\ \ )$**.**

The substitution $\ = a\sin(\lambda\ \ ) + 1$ leads to an Abel equation of the form 1.3.2.12: $\ ' = 3a\sin(\lambda\ \ )\ \ + 1.$

In equations 44–47, the following notation is used: $\boldsymbol{f = f(\ ),\quad = (\ ),\quad = (\ ).}$

44. $\boldsymbol{= f\ ^3 + \ f\ ^2 + — \ + f\ ^3.}$

The substitution $= g$ leads to a separable equation: $' = g^2(a\ ^3 + \ + \ ).$

45. $\boldsymbol{= f\ ^3 + 3f\ ^2 + (\ + 3f\ ^2)\ + f\ ^3 + \ - \ .}$

The substitution $= \ + (\ )$ leads to a Bernoulli equation: $' = g(\ )\ + (\ )\ ^3.$

46. $\boldsymbol{= \dfrac{}{f^2(\ + \ )^3}\ ^3 + \dfrac{f}{f}\ + f\ .}$

Solution: $\dfrac{}{^3 - a\ + 1} + \ = \dfrac{1}{a}\ln|ag + \ |,$ where $= \dfrac{}{(ag + \ )}.$

47. $\boldsymbol{= (\ - f)(\ - \ )\ - \dfrac{f +}{+}\ + \dfrac{-}{f -}f\ + \dfrac{-f}{-f}\ .}$

Solution: $|\ - \ |\ |\ - g|^b\ - \dfrac{a\ + g}{a +}^{- -b} = \exp\dfrac{a}{a +}\ (\ - g)^2\ .$

1.4.2. Equations of the Form $\boldsymbol{(\ _{22}\ ^2 + \ _{12}\ + \ _{11}\ ^2 + \ _0)\ = \ _{22}\ ^2 + \ _{12}\ + \ _{11}\ ^2 + \ _0}$

1.4.2-1. Preliminary remarks. Some transformations.

1 . For $A_{22} = 0$, this is an Abel equation (see Subsection 1.3.4). For $B_{11} = 0$, this is an Abel equation with respect to $= (\ )$.

2 . The transformation $z = \ ,\ = \ ^{-2}$ leads to an Abel equation of the second kind:

$$[(A_0 z - B_0)\ + A_{22}z^3 + (A_{12} - B_{22})z^2 + (A_{11} - B_{12})z - B_{11}]\ ' = 2A_0\ ^2 + 2(A_{22}z^2 + A_{12}z + A_{11})\ .$$

3 . The transformation $= \ ^- + \ ,\ = \ ^- + \beta$, where and β are parameters, which are determined by solving the second-order algebraic system

$$A_{22}\beta^2 + A_{12}\ \beta + A_{11}\ ^2 + A_0 = 0,\quad B_{22}\beta^2 + B_{12}\ \beta + B_{11}\ ^2 + B_0 = 0,$$

leads to the equation

$$[A_{22}\ ^{-2} + A_{12}\ ^{- -} + A_{11}\ ^{-2} + (2A_{22}\beta + A_{12}\ )\ ^- + (2A_{11}\ + A_{12}\beta)\ ^-)]\ ^{-\prime}$$
$$= B_{22}\ ^{-2} + B_{12}\ ^{- -} + B_{11}\ ^{-2} + (2B_{22}\beta + B_{12}\ )\ ^- + (2B_{11}\ + B_{12}\beta)\ ^-.$$

The transformation $\xi = \ ^-\ ^-,\ = 1\ ^-$ reduces this equation to an Abel equation of the second kind:

$$\{[a_2\xi^2 + (a_1 - \ _2)\xi - \ _1]\ + A_{22}\xi^3 + (A_{12} - B_{22})\xi^2 + (A_{11} - B_{12})\xi - B_{11}\}\ '$$
$$= (a_2\xi + a_1)\ ^2 + (A_{22}\xi^2 + A_{12}\xi + A_{11})\ ,$$

where $a_1 = 2A_{11}\ + A_{12}\beta$, $\ _1 = 2B_{11}\ + B_{12}\beta$, $a_2 = 2A_{22}\beta + A_{12}\ $, and $\ _2 = 2B_{22}\beta + B_{12}\ $.

4 . The substitution $= \ + \varepsilon\ $, where parameter ε is determined by solving the cubic equation

$$(A_{22}\varepsilon^2 + A_{12}\varepsilon + A_{11})\varepsilon - B_{22}\varepsilon^2 - B_{12}\varepsilon - B_{11} = 0,$$

leads to an Abel equation of the second kind with respect to $= (\ )$:

$$[Q\ + (B_{22} - A_{22}\varepsilon)\ ^2 + B_0 - A_0\varepsilon]\ ' = (A_{22}\varepsilon^2 + A_{12}\varepsilon + A_{11})\ ^2 + (2A_{22}\varepsilon + A_{12})\ + A_{22}\ ^2 + A_0,$$

where $Q = 2B_{22}\varepsilon + B_{12} - \varepsilon(2A_{22}\varepsilon + A_{12})$.

1.4.2-2. Solvable equations and their solutions.

1. $(\;^{2} + \;^{2}) \quad = -2 \quad + B \;^{2} + \;.$

Solution: $A \;^{3} - B \;^{3} + 3(\;^{2} \; - a \;) = \;.$

2. $(\;^{2} + B \;^{2} - \;^{2}B) \quad = \quad ^{2} + 2B \quad .$

The transformation $\; = \; + a, \quad = \xi$ leads to a linear equation: $(-A\xi^{3} + \quad \xi^{2} + B\xi) \;' =$ $(A\xi^{2} + B) \quad + 2aB.$

3. $(\;^{2} + B \quad + \quad ^{2}) \quad = \quad ^{2} + E \quad + \quad ^{2}.$

Homogeneous equation. The substitution $z = \quad$ leads to a separable equation: $\quad z' =$ $(Az^{2} + Bz + \quad)^{-1}[-Az^{3} + (\quad - B)z^{2} + (\quad - \quad)z + \quad].$

4. $(\;^{2} - 2 \quad + B \quad ^{2}) \quad = -B \;^{2} + 2B \quad - \quad ^{3} \;^{2} + \;.$

The substitution $\; = z + k$ leads to a Riccati equation with respect to $\; = \;(z)$:

$$[-(Ak + B)z^{2} + a] \;' = k(B - Ak) \;^{2} + Az^{2}.$$

5. $(\;^{2} + 2B \quad + \quad ^{2} \;^{2}) \quad = B \;^{2} + 2 \quad ^{2} \quad + B \;^{2} \;^{2} + \;.$

The substitution $\; = z + k$ leads to a Riccati equation with respect to $\; = \;(z)$:

$$[(B - Ak)z^{2} + a] \;' = 2k(Ak + B) \;^{2} + 2(Ak + B)z \quad + Az^{2}.$$

6. $(\;^{2} + B \quad + \quad ^{2} + \;) \quad = \quad ^{2} + B \quad + \quad ^{2} + \;.$

The substitution $\; = z + k$ leads to a Riccati equation with respect to $\; = \;(z)$:

$$(\; - ak) \;' = (Ak^{2} + Bk + \quad) \;^{2} + (2Ak + B)z \quad + Az^{2} + a.$$

7. $(\;^{2} + 2B \quad + \quad ^{2} + \;) \quad = -B \;^{2} - 2 \quad + E \;^{2} + \;.$

Solution: $A \;^{3} - \quad ^{3} + 3(B \quad ^{2} + \quad ^{2} \quad + a \; - \quad) = \;.$

8. $(\;^{2} - 2 \quad + B \;^{2} + \quad - B) \quad = - \quad ^{2} + 2B \quad - B \;^{2} + \quad - B.$

This is a special case of equation 1.4.2.21 with $a = 1$ and $\; = 1$.

9. $(\;^{2} + 2 \quad + B \;^{2} + \quad - B) \quad = \quad ^{2} + 2B \quad + B \;^{2} - \quad + B.$

This is a special case of equation 1.4.2.21 with $a = 1$ and $\; = -1$.

10. $(\;^{2} - 4 \quad + B \;^{2} + 4 \quad - B) \quad = -2 \quad ^{2} + 2B \quad - 2B \;^{2} + 8 \quad - 2B.$

This is a special case of equation 1.4.2.21 with $a = 1$ and $\; = 2$.

11. $(\;^{2} + 4 \quad + B \;^{2} + 4 \quad - B) \quad = 2 \quad ^{2} + 2B \quad + 2B \;^{2} - 8 \quad + 2B.$

This is a special case of equation 1.4.2.21 with $a = 1$ and $\; = -2$.

12. $(\;^{2} - 6 \quad + B \;^{2} + 9 \quad - B) \quad = -3 \quad ^{2} + 2B \quad - 3B \;^{2} + 27 \quad - 3B.$

This is a special case of equation 1.4.2.21 with $a = 1$ and $\; = 3$.

13. $(\;^{2} + 6 \quad + B \;^{2} + 9 \quad - B) \quad = 3 \quad ^{2} + 2B \quad + 3B \;^{2} - 27 \quad + 3B.$

This is a special case of equation 1.4.2.21 with $a = 1$ and $\; = -3$.

14. $2(\quad ^{2} - \quad + B \;^{2} + \quad - 4B) \quad = - \quad ^{2} + 4B \quad - B \;^{2} + \quad - 4B.$

This is a special case of equation 1.4.2.21 with $a = 2$ and $\; = 1$.

15. $2(\ ^2 + \ \ + B \ ^2 + \ \ - 4B) \ = \ ^2 + 4B \ \ + B \ ^2 - \ \ + 4B.$

This is a special case of equation 1.4.2.21 with $a = 2$ and $\ = -1$.

16. $(\ ^2 - 2 \ \ + \ ^2 + \ ^2 - \ ^3) \ = - \ ^2 + 2 \ \ - \ ^2 + \ ^3 - \ ^2 .$

This is a special case of equation 1.4.2.21 with $A = 1$ and $B = 1$.

17. $(\ ^2 - 2 \ \ - \ ^2 + \ ^2 + \ ^3) \ = - \ ^2 - 2 \ \ + \ ^2 + \ ^3 + \ ^2 .$

This is a special case of equation 1.4.2.21 with $A = 1$ and $B = -1$.

18. $(\ ^2 - 2 \ \ + 2 \ ^2 + \ ^2 - 2 \ ^3) \ = - \ ^2 + 4 \ \ - 2 \ ^2 + \ ^3 - 2 \ ^2 .$

This is a special case of equation 1.4.2.21 with $A = 1$ and $B = 2$.

19. $(\ ^2 - 2 \ \ - 2 \ ^2 + \ ^2 + 2 \ ^3) \ = - \ ^2 - 4 \ \ + 2 \ ^2 + \ ^3 + 2 \ ^2 .$

This is a special case of equation 1.4.2.21 with $A = 1$ and $B = -2$.

20. $(\ ^2 + B \ \ + \ ^2 + \ ) \ = \ ^2 + \ (2 \ \ + B - 2 \ ) \ \ + (- \ ^2 + \ \ + \ ) \ ^2 + \ .$

The substitution $\ = z + k \ $ leads to a Riccati equation with respect to $\ = \ (z)$:

$$[(\ - Ak)z^2 + \ - ak] \ ' = (Ak^2 + Bk + \ ) \ ^2 + (2Ak + B)z \ \ + Az^2 + a.$$

21. $(\ ^2 - 2 \ \ + \ B \ ^2 + \ ^2 \ - \ ^3 B) \ = - \ ^2 + 2 \ B \ - \ B \ ^2 + \ ^3 \ - \ ^2 \ B.$

The transformation $\ = \ + a, \ = \xi \ + \ $ leads to a linear equation:

$$(-aA\xi^3 + \ A\xi^2 + aB\xi - \ B) \ ' = (aA\xi^2 - 2 \ A\xi + aB) \ \ + 2a^2 B - 2 \ ^2 A.$$

1.4.3. Equations of the Form $(\ _{22} \ ^2 + \ _{12} \ \ + \ _{11} \ ^2 + \ _2 \ + \ _1 \ ) \ = \ _{22} \ ^2 + \ _{12} \ \ + \ _{11} \ ^2 + \ _2 \ + \ _1$

1.4.3-1. Preliminary remarks.

1 . For $A_{22} = 0$, this is an Abel equation (see Subsection 1.3.4). For $B_{11} = 0$ this is an Abel equation with respect to $\ = \ (\ )$.

2 . The transformation $\xi = \ \ , \ \ = 1 \ \ $ leads to an Abel equation of the second kind:

$$\{[A_2\xi^2 + (A_1 - B_2)\xi - B_1] \ \ + A_{22}\xi^3 + (A_{12} - B_{22})\xi^2 + (A_{11} - B_{12})\xi - B_{11}\} \ ' \\ = (A_2\xi + A_1) \ ^2 + (A_{22}\xi^2 + A_{12}\xi + A_{11}) \ .$$

3 . In Item 3 of Subsection 1.4.4, another transformation is given which reduces the original equation to an Abel equation of the second kind.

4 . Dynamical systems of the second-order

$$\frac{}{} = \ (\ , \ ), \qquad \frac{}{} = Q(\ , \ ), \tag{1}$$

which describe the behavior of simplest Lagrangian and Hamiltonian systems in mechanics, are often reduced to equations of the type in question if

$$\begin{aligned} (\ , \ ) &= \ (\ , \ )(A_{22} \ ^2 + A_{12} \ \ + A_{11} \ ^2 + A_2 \ \ + A_1 \ ), \\ Q(\ , \ ) &= \ (\ , \ )(B_{22} \ ^2 + B_{12} \ \ + B_{11} \ ^2 + B_2 \ \ + B_1 \ ), \end{aligned} \tag{2}$$

where $\ = \ (\ , \ )$ is an arbitrary function.

In particular, dynamical systems (1) with functions (2) and $\equiv 1$ arise in analyzing complex equilibrium states. In this case, the functions and Q are substituted by their Taylor-series expansions in the vicinity of the equilibrium state $= \; = 0$ with the first and second order terms retained.

Whenever a solution of the ordinary differential equation

$$(A_{22}\ {}^2 + A_{12}\quad + A_{11}\ {}^2 + A_2\ + A_1\ )\ ' = B_{22}\ {}^2 + B_{12}\quad + B_{11}\ {}^2 + B_2\ + B_1$$

is obtained in parametric form, $= (\ ,\ _1)$, $= (\ ,\ _1)$, the corresponding solution of system (1), (2) is determined by

$$= (\ ,\ _1), \qquad = (\ ,\ _1), \qquad = \frac{'}{(\ (\ ,\ _1),\ (\ ,\ _1)}+\ _2.$$

The last relation defines an implicit dependence of the parameter on , $= (\ ,\ _1,\ _2)$, and makes it possible to establish, with the aid of the first two formulas, the dependence of and on .

1.4.3-2. Solvable equations and their solutions.

1. $(\ ^2 - \ ^2 + \quad)\quad = \ ^2 - \ ^2 + \quad .$

Solution in parametric form:

$$= a + \ |\ |^{-1}e^{4}\ , \qquad = -a + \ |\ |^{-1}e^{4}\ .$$

2. $(\ ^2 - \ ^2 + \quad)\quad = 2\ ^2 - 2\quad + \quad .$

Solution in parametric form:

$$= \ + \ ^2e\quad , \qquad = \ ^2e\quad .$$

3. $(\ ^2 - \ ^2 + \quad - \quad)\quad = \ ^2 - \ ^2 - \quad + \quad .$

Solution in parametric form:

$$= a + \ e^{2}\ , \qquad = -a + \ e^{2}\ .$$

4. $(\ ^2 - \ ^2 + \quad + 2\quad)\quad = \ ^2 - \ ^2 + 2\quad + \quad .$

Solution in parametric form:

$$= -a + \ |\ |^{3}e^{4}\ , \qquad = a + \ |\ |^{3}e^{4}\ .$$

5. $(\ ^2 - \ ^2 + \quad + 2\quad)\quad = 2\quad - 2\ ^2 + \quad + 2\quad .$

Solution in parametric form:

$$= \ + \ ^{-2}e^{-}\quad , \qquad = -2\ + \ ^{-2}e^{-}\quad .$$

6. $(\ ^2 - \ ^2 + \quad - 2\quad)\quad = 4\ ^2 - 6\quad + 2\ ^2 + \quad - 2\quad .$

Solution in parametric form:

$$= \tfrac{1}{3}\ + \ |\ |^{2\ 3}e\quad , \qquad = \tfrac{2}{3}\ + \ |\ |^{2\ 3}e\quad .$$

7. $(\ ^2 - \ ^2 + \quad + 3\quad)\quad = -\ ^2 + 4\quad - 3\ ^2 + \quad + 3\quad .$

Solution in parametric form:

$$= \tfrac{1}{2}\ + \ |\ |^{-1}e^{-}\quad , \qquad = -\tfrac{3}{2}\ + \ |\ |^{-1}e^{-}\quad .$$

8. $(\ ^2 - \quad + \quad + \quad) \quad = \quad - \ ^2 + \quad + \quad .$

Solution in parametric form:

$$= - \ + \ |\ |^{-1} e \quad , \qquad = \ + \ |\ |^{-1} e \quad .$$

9. $(\ ^2 - \quad + \quad + \quad) \quad = \ ^2 - \quad + 2 \quad .$

Solution in parametric form:

$$= -a \ + \quad ^2 e \ , \qquad = \quad ^2 e \ .$$

10. $(\ ^2 - \quad + \quad - 2 \quad) \quad = 3\ ^2 - 5 \quad + 2\ ^2 + \quad - 2 \quad .$

Solution in parametric form:

$$= \tfrac{1}{2} \ + \ |\ |^{1\ \ 2} e \quad , \qquad = \ + \ |\ |^{1\ \ 2} e \quad .$$

11. $(\ ^2 + \quad - 2\ ^2 + \quad + \quad) \quad = \ ^2 + \quad - 2\ ^2 + 2 \quad .$

Solution in parametric form:

$$= a \ + \quad ^{-2} e^{9} \ , \qquad = -2a \ + \quad ^{-2} e^{9} \ .$$

12. $(\ ^2 + \quad - 2\ ^2 + \quad + \quad) \quad = 2\ ^2 - \quad - \ ^2 + \quad + \quad .$

Solution in parametric form:

$$= \ + \ |\ |^{3} e \quad , \qquad = - \ + \ |\ |^{3} e \quad .$$

13. $(\ ^2 + \quad - 2\ ^2 + \quad - \quad) \quad = \ ^2 + \quad - 2\ ^2 - 2 \quad + 2 \quad .$

Solution in parametric form:

$$= a \ + \ e^{3} \ , \qquad = -2a \ + \ e^{3} \ .$$

14. $(\ ^2 + \quad - 2\ ^2 + \quad - 2 \quad) \quad = 5\ ^2 - 7 \quad + 2\ ^2 + \quad - 2 \quad .$

Solution in parametric form:

$$= \tfrac{1}{4} \ + \ |\ |^{3\ \ 4} e \quad , \qquad = \tfrac{1}{2} \ + \ |\ |^{3\ \ 4} e \quad .$$

15. $(\ ^2 - 2 \quad + \ ^2 + \quad) \quad = \quad .$

Solution: $\quad = \ + \dfrac{a}{\ - \ln|\ |}.$

16. $(\ ^2 - 2 \quad + \ ^2 + \quad + \quad) \quad = - \ ^2 + 2 \quad - \ ^2 + \quad + \quad .$

Solution in parametric form:

$$= -\frac{a}{2\ln|\ |} + \quad , \qquad = \frac{a}{2\ln|\ |} + \quad .$$

17. $(\ ^2 - 2 \quad + \ ^2 + \quad + 2 \quad) \quad = -2(\ ^2 - 2 \quad + \ ^2) + \quad + 2 \quad .$

Solution in parametric form:

$$= -\frac{a}{3\ln|\ |} + \quad , \qquad = \frac{2a}{3\ln|\ |} + \quad .$$

18. $(\ ^2 - 2 \quad + \ ^2 + \quad - 2 \quad) \quad = 2(\ ^2 - 2 \quad + \ ^2) + \quad - 2 \quad .$

Solution in parametric form:

$$= \frac{a}{\ln|\ |} + \quad , \qquad = \frac{2a}{\ln|\ |} + \quad .$$

19. $(\ {}^2+2\ \ +\ {}^2+\ \ +2\ \ )\ \ =-\ {}^2-2\ \ -\ {}^2+2\ \ +\ \ .$

Solution in parametric form:

$$=\ {}^2\ \ {}^{1}\ {}^{3}+\frac{4\ {}^2}{5a}\ \ +\ \ ,\qquad =-\ {}^2\ \ {}^{1}\ {}^{3}+\frac{4\ {}^2}{5a}\ \ +\ \ ,\qquad a\neq 0.$$

20. $(\ {}^2+2\ \ +\ {}^2+\ \ -\ \ )\ \ =-\ {}^2-2\ \ -\ {}^2+\ \ -\ \ .$

Solution in parametric form:

$$=\ \sqrt[3]{1-\frac{4\ {}^3}{3a}+\ \ {}^2}\ ,\qquad =-\ \sqrt[3]{1-\frac{4\ {}^3}{3a}+\ \ {}^2}\ ,\qquad a\neq 0.$$

21. $(\ {}^2+2\ \ +\ {}^2+\ \ -2\ \ )\ \ =-\ {}^2-2\ \ -\ {}^2-2\ \ +\ \ .$

Solution in parametric form:

$$=\ {}^2\ \ {}^{3}+\frac{4\ {}^2}{a}\ \ +\ \ ,\qquad =-\ {}^2\ \ {}^{3}+\frac{4\ {}^2}{a}\ \ +\ \ ,\qquad a\neq 0.$$

22. $(\ {}^2+2\ \ -3\ {}^2+\ \ +\ \ )\ \ =3\ {}^2-2\ \ -\ {}^2+\ \ +\ \ .$

Solution in parametric form:

$$=\tfrac{1}{2}\ \ +\ \ {}^2e\ \ ,\qquad =-\tfrac{1}{2}\ \ +\ \ {}^2e\ \ .$$

23. $(\ {}^2+2\ \ -3\ {}^2+\ \ +\ \ )\ \ =\ {}^2+2\ \ -3\ {}^2-\ \ +3\ \ .$

Solution in parametric form:

$$=a\ \ +\ \ |\ |^{-1}e^{8}\ ,\qquad =-3a\ \ +\ \ |\ |^{-1}e^{8}\ .$$

24. $(\ {}^2+2\ \ -3\ {}^2+\ \ +2\ \ )\ \ =\ {}^2+2\ \ -3\ {}^2+3\ \ .$

Solution in parametric form:

$$=a\ \ +\ \ |\ |^{-3}e^{16}\ ,\qquad =-3a\ \ +\ \ |\ |^{-3}e^{16}\ .$$

25. $(\ {}^2-\ {}^2+\ \ +\ \ )\ \ =\ {}^2-\ {}^2+\ \ +\ \ .$

Solution in parametric form:

$$=(a-\ \ )\ +\ \ |\ |^{-\frac{+b}{-b}}e^{4}\ ,\qquad =(\ \ -a)\ +\ \ |\ |^{-\frac{+b}{-b}}e^{4}\ ,\qquad a\neq\ \ .$$

26. $(\ {}^2-\ \ +\ \ +\ \ )\ \ =\ {}^2-\ \ +(\ \ +\ \ )\ .$

Solution in parametric form:

$$=-\ \ +\ \ |\ |^{\frac{+b}{b}}e\ ,\qquad =\ \ |\ |^{\frac{+b}{b}}e\ ,\qquad \neq 0.$$

27. $(\ {}^2+\ \ -2\ {}^2+\ \ +\ \ )\ \ =\ {}^2+\ \ -2\ {}^2+(\ \ -\ \ )\ \ +2\ \ .$

Solution in parametric form:

$$=(2a-\ \ )\ +\ \ |\ |^{-\frac{+b}{2\ -b}}e^{9}\ ,\qquad =2(\ \ -2a)\ +\ \ |\ |^{-\frac{+b}{2\ -b}}e^{9}\ ,\qquad \neq 2a.$$

28. $(\ {}^2-2\ \ +\ {}^2+\ \ -\ \ )\ \ =(\ {}^2-2\ \ +\ {}^2)+\ \ -\ \ .$

Solution in parametric form:

$$=\frac{a}{\ \ -1}\frac{1}{\ln|\ |}+\ \ ,\qquad =\frac{a}{\ \ -1}\frac{1}{\ln|\ |}+\ \ ,\qquad \neq 1.$$

29. $(\ {}^2 + 2\quad - 3\ {}^2 + \quad + \quad)\quad = \ {}^2 + 2\quad - 3\ {}^2 + (\ - 2\ )\ + 3\quad.$

Solution in parametric form:

$$= (3a - \ ) + \ |\ |^{-\frac{+b}{3\ -b}} e^{16}\ , \qquad = 3(\ - 3a) + \ |\ |^{-\frac{+b}{3\ -b}} e^{16}\ , \qquad \neq 3a.$$

30. $(\ {}^2 - 3\quad + 2\ {}^2 + \quad + \quad)\quad = \ {}^2 - 3\quad + 2\ {}^2 + (3\quad + \ )\ - 2\quad.$

Solution in parametric form:

$$= (2a + \ ) + \ |\ |^{\frac{+b}{2\ +b}} e^{-}\ , \qquad = 2(2a + \ ) + \ |\ |^{\frac{+b}{2\ +b}} e^{-}\ , \qquad \neq -2a.$$

31. $(\ {}^2 + 3\quad - 4\ {}^2 + \quad + \quad)\quad = \ {}^2 + 3\quad - 4\ {}^2 + (\ - 3\ )\ + 4\quad.$

Solution in parametric form:

$$= (4a - \ ) + \ |\ |^{-\frac{+b}{4\ -b}} e^{25}\ , \qquad = 4(\ - 4a) + \ |\ |^{-\frac{+b}{4\ -b}} e^{25}\ , \qquad \neq 4a.$$

32. $[\ {}^2 + \quad - (\ + 1)\ {}^2 + \quad - 2\quad]\quad = (\ + 4)\ {}^2 - (\ + 6)\quad + 2\ {}^2 + \quad - 2\quad.$

Solution in parametric form:

$$= \frac{}{A + 3} + \ |\ |^{\frac{+2}{+3}} e^{b}\ , \qquad = \frac{2}{A + 3} + \ |\ |^{\frac{+2}{+3}} e^{b}\ , \qquad A \neq -3.$$

33. $(\ {}^2 - 2\quad + \ {}^2\ {}^2 + \quad - \quad)\quad = \ {}^2 - 2\ {}^2\quad + \ {}^3\ {}^2 + \quad - \quad.$

Solution in parametric form:

$$= \sqrt[3]{1 + \frac{2(A - 1)}{3}\ {}^3 + \ {}^2}\ , \qquad = A\sqrt[3]{1 + \frac{2(A - 1)}{3}\ {}^3 + \ {}^2}\ , \qquad \neq 0.$$

34. $[\ {}^2 - 2\quad + (2\quad - 1)\ {}^2 + \quad - \quad]\quad = (2 - \ )\ {}^2 - 2\quad + \ {}^2 + \quad - \quad.$

Solution in parametric form:

$$= \frac{}{1 - A} + \ {}^2 e^{b}\ , \qquad = \frac{A}{1 - A} + \ {}^2 e^{b}\ , \qquad A \neq 1.$$

35. $(\ {}^2 - 2\quad + \ {}^2\ {}^2 + \quad + \quad)\quad = (\ {}^2 - 2\quad + \ {}^2\ {}^2) + (\quad + \quad + \ )\ - \quad.$

Solution in parametric form:

$$= \ {}^2\ \frac{+b}{+b} + \frac{(1 - A)^2}{(2 - A)a + }\ {}^2 + \quad, \qquad = A\ {}^2\ \frac{+b}{+b} + \frac{(1 - A)^2}{(2 - A)a + }\ {}^2 + \quad,$$

where $a + \ \neq 0$ and $(2 - A)a + \ \neq 0$.

36. $[\ {}^2 - (\ + 2)\quad + (\ + 1)\ {}^2 + \quad - \quad]\quad = - \quad + \ {}^2 + \quad - \quad.$

Solution in parametric form:

$$= \frac{}{1 - A} + \ |\ |\ e^{(\ -1)b}\ , \qquad = \frac{A}{1 - A} + \ |\ |\ e^{(\ -1)b}\ , \qquad A \neq 1.$$

37. $[\quad {}^2 + \quad - (\ + 1)\ {}^2 + \quad + \quad]\quad = (\ + 1)\ {}^2 - \quad - \quad {}^2 + \quad + \quad.$

Solution in parametric form:

$$= \ + \ |\ |^{2\ +1} e^{b}\ , \qquad = - \ + \ |\ |^{2\ +1} e^{b}\ .$$

38. $(\quad {}^2 + B\quad + \quad {}^2 + \quad)\quad = \quad {}^2 + E\quad + \quad {}^2 + \quad.$

The substitution $\ = \ z$ leads to a linear equation with respect to $\ = \ (z)$:

$$[-Az^3 + (\ - B)z^2 + (\ - \ )z + \ ]\ ' = (Az^2 + Bz + \ )\ + k.$$

39. $(\quad {}^2 + B\quad + \quad {}^2 - \ B\ - \quad)\quad = \quad {}^2 + E\quad + (\ - E)\ .$

The transformation $\ = \ + \ , \ = \xi$ leads to a linear equation:

$$[-A\xi^3 + (\ - B)\xi^2 + (\ - \ )\xi]\ ' = (A\xi^2 + B\xi + \ )\ + \quad.$$

40. $(\quad {}^2 + 2B\quad + \quad {}^2\ {}^2 + \quad + \quad)\quad = B\ {}^2 + 2\quad {}^2\quad + B\ {}^2\ {}^2 + \quad + \quad {}^2\ .$

This is a special case of equation 1.4.3.57 with $\ = Ak^2$.

41. $(\quad ^2 + 2B \quad + \quad ^2\,^2 + \quad + \quad) \quad = B\,^2 + 2 \quad ^2 \quad + B\,^2\,^2 + \quad + \quad .$

This is a special case of equation 1.4.3.62 with $\quad = Ak^2$.

42. $(\quad ^2 + 2B \quad + \quad ^2\,^2 + \quad - \quad) \quad = B\,^2 + 2 \quad ^2 \quad + B\,^2\,^2 + \quad - \quad .$

This is a special case of equation 1.4.3.61 with $\quad = Ak^2$.

43. $(\quad ^2 + 2B \quad - B \quad ^2 + \quad + \quad) \quad = B\,^2 + 2 \quad ^2 \quad - \quad ^3\,^2 + \quad + \quad ^2 .$

This is a special case of equation 1.4.3.58 with $\quad = \quad$.

44. $(\quad ^2 + 2B \quad - B \quad ^2 + \quad + \quad) \quad = B\,^2 + 2 \quad ^2 \quad - \quad ^3\,^2 + \quad + \quad .$

This is a special case of equation 1.4.3.62 with $\quad = -Bk$.

45. $(\quad ^2 + 2B \quad - B \quad ^2 + \quad - \quad) \quad = B\,^2 + 2 \quad ^2 \quad - \quad ^3\,^2 + \quad - \quad .$

This is a special case of equation 1.4.3.61 with $\quad = -Bk$.

46. $(\quad ^2 + 2 \quad + \quad ^2 + \quad + \quad) \quad = \quad ^2 + 2 \quad ^2 \quad + \quad ^2 + \quad + \quad ^2 .$

This is a special case of equation 1.4.3.57 with $B = Ak$.

47. $(\quad ^2 + 2 \quad + \quad ^2 + \quad + \quad) \quad = \quad ^2 + 2 \quad ^2 \quad + \quad ^2 + \quad + \quad .$

This is a special case of equation 1.4.3.62 with $B = Ak$.

48. $(\quad ^2 + 2 \quad + \quad ^2 + \quad - \quad) \quad = \quad ^2 + 2 \quad ^2 \quad + \quad ^2 + \quad - \quad .$

This is a special case of equation 1.4.3.61 with $B = Ak$.

49. $(\quad ^2 - 2 \quad + B \quad ^2 + \quad + \quad) \quad = -B\,^2 + 2B \quad - \quad ^3\,^2 + \quad + \quad ^2 .$

This is a special case of equation 1.4.3.59 with $\quad = \quad$.

50. $(\quad ^2 - 2 \quad + B \quad ^2 + \quad + \quad) \quad = -B\,^2 + 2B \quad - \quad ^3\,^2 + \quad + \quad .$

This is a special case of equation 1.4.3.59 with $\quad = ak$.

51. $[\quad ^2 + 2 \quad + \quad ^2\,^2 + (\quad - 1)B \quad - 2 \quad B \quad]$

$$= - \quad (\quad ^2 + 2 \quad + \quad ^2\,^2) - (\quad ^2 + 1)B \quad + \quad (\quad - 1)B \quad .$$

Solution in parametric form ($A \neq 2,\ B \neq 0$):

$$= \quad ^2 \quad + \frac{A+1}{(A-2)B} \, ^2 + \quad , \qquad = -A \quad ^2 \quad + \frac{A+1}{(A-2)B} \, ^2 + \quad .$$

52. $[\quad ^2 - 2 \quad + \quad ^2\,^2 + (B-1) \quad + (\quad - B) \quad]$

$$= \quad (\quad ^2 - 2 \quad + \quad ^2\,^2) + (\quad B - 1) \quad - \quad (B-1) \quad .$$

Solution in parametric form ($B \neq 2,\ k \neq 0$):

$$= \quad ^2 \quad - \frac{A-1}{(B-2)k} \, ^2 + \quad , \qquad = A \quad ^2 \quad - \frac{A-1}{(B-2)k} \, ^2 + \quad .$$

53. $[2 \quad ^2 - (\quad + 3) \quad + (\quad + 1) \quad ^2 + B \quad - \quad B \quad] \quad = (\quad + 1) \quad ^2 - (3 \quad + 1) \quad + 2 \quad ^2 + B \quad - \quad B \quad .$

Solution in parametric form:

$$= \frac{}{1-A} + \quad | \, |^{-1} e^{-} \quad , \qquad = \frac{A}{1-A} + \quad | \, |^{-1} e^{-} \quad , \qquad A \neq 1.$$

54. $\mathbf{[2\ \ ^2-(3\ \ +1)\ \ \ +(3\ \ -1)\ \ ^2+B\ \ -\ \ B\ \ ]\ \ =(3-\ \ )\ \ ^2-(\ \ +3)\ \ \ +2\ \ \ ^2+B\ \ -\ \ B\ \ .}$

Solution in parametric form:

$$= \frac{}{1-A} + \ \ |\ \ |^3 e \quad , \qquad = \frac{A}{1-A} + \ \ |\ \ |^3 e \quad , \qquad A \neq 1.$$

55. $\mathbf{[\ \ (\ \ ^2-2\ \ \ +\ \ ^2)-\ \ (\ \ -B)\ \ +B(\ \ -B)\ \ ]\ \ =B(\ \ ^2-2\ \ \ +\ \ ^2)-\ \ (\ \ -B)\ \ +B(\ \ -B)\ \ .}$

Solution in parametric form:

$$= \frac{A}{\ln|\ \ |} + \quad , \qquad = \frac{B}{\ln|\ \ |} + \quad .$$

56. $\mathbf{(\ \ \ ^2+B\ \ \ +\ \ \ ^2+\ \ \ +\ \ )\ \ =\ \ \ ^2+B\ \ \ +\ \ \ ^2+\ \ \ +(\ \ \ +\ \ -\ \ )\ .}$

The substitution $\ \ = z + k\ \ $ leads to a Riccati equation with respect to $\ \ =\ \ (z)$:

$$(\ \ - ak)z\ \ ' = (Ak^2 + Bk + \ \ )\ \ ^2 + [(2Ak + B)z + ak + \ \ ]\ \ + Az^2 + az.$$

57. $\mathbf{(\ \ \ ^2+2B\ \ \ +\ \ \ ^2+\ \ \ +\ \ )\ \ =B\ \ ^2+2\ \ \ \ ^2\ \ \ +\ (-\ \ \ ^2+B\ \ +\ \ )\ \ ^2+\ \ \ +\ \ \ ^2\ .}$

The substitution $\ \ = z + k\ \ $ leads to a Riccati equation with respect to $\ \ =\ \ (z)$:

$$[(B - Ak)z + \ \ - ak]z\ \ ' = (Ak^2 + 2Bk + \ \ )\ \ ^2 + [2(Ak + B)z + ak + \ \ ]\ \ + Az^2 + az.$$

58. $\mathbf{(\ \ \ ^2+2B\ \ \ -B\ \ \ ^2+\ \ \ +\ \ )\ \ =B\ \ ^2+2\ \ \ \ ^2\ \ \ -\ \ \ \ ^3\ ^2+\ \ \ +(\ \ \ +\ \ -\ \ )\ .}$

The substitution $\ \ = z + k\ \ $ leads to a Riccati equation with respect to $\ \ =\ \ (z)$:

$$[(B - Ak)z + \ \ \ - ak]z\ \ ' = (Ak^2 + Bk)\ \ ^2 + [2(Ak + B)z + ak + \ \ ]\ \ + Az^2 + az.$$

59. $\mathbf{(\ \ \ ^2-2\ \ \ \ \ +B\ \ \ ^2+\ \ \ +\ \ )\ \ =-B\ \ ^2+2B\ \ \ \ -\ \ \ \ ^3\ ^2+\ \ \ +(\ \ \ +\ \ -\ \ )\ .}$

The substitution $\ \ = z + k\ \ $ leads to a Riccati equation with respect to $\ \ =\ \ (z)$:

$$[-(Ak + B)z + \ \ \ - ak]z\ \ ' = k(B - Ak)\ \ ^2 + (ak + \ \ )\ \ + Az^2 + az.$$

60. $\mathbf{(\ \ \ ^2+2B\ \ \ +\ \ \ ^2\ ^2+\ \ \ +\ \ )\ \ =B\ \ ^2+2\ \ \ \ ^2\ \ \ +B\ \ ^2\ \ ^2+\ \ \ +(\ \ \ +\ \ -\ \ )\ .}$

The substitution $\ \ = z + k\ \ $ leads to a Riccati equation with respect to $\ \ =\ \ (z)$:

$$[(B - Ak)z + \ \ \ - ak]z\ \ ' = 2k(Ak + B)\ \ ^2 + [2(Ak + B)z + ak + \ \ ]\ \ + Az^2 + az.$$

61. $\mathbf{(\ \ \ ^2+2B\ \ \ +\ \ \ ^2+\ \ \ -\ \ \ )\ \ =B\ \ ^2+2\ \ \ \ ^2\ \ \ +\ (-\ \ \ ^2+B\ \ +\ \ )\ \ ^2+\ \ \ -\ \ \ \ .}$

The substitution $\ \ = z + k\ \ $ leads to a Riccati equation with respect to $\ \ =\ \ (z)$:

$$[(B - Ak)z^2 + \ \ \ - ak]z\ \ ' = (Ak^2 + 2Bk + \ \ )\ \ ^2 + 2(Ak + B)z\ \ \ + Az^2 + az.$$

62. $\mathbf{(\ \ \ ^2+2B\ \ \ +\ \ \ ^2+\ \ \ +\ \ )\ \ =B\ \ ^2+2\ \ \ \ ^2\ \ \ +\ (-\ \ \ ^2+B\ \ +\ \ )\ \ ^2+\ \ \ \ +\ \ \ \ .}$

The substitution $\ \ = z + k\ \ $ leads to a Riccati equation with respect to $\ \ =\ \ (z)$:

$$(B - Ak)z^2\ \ ' = (Ak^2 + 2Bk + \ \ )\ \ ^2 + [2(Ak + B)z + ak + \ \ ]\ \ + Az^2 + az.$$

63. $\mathbf{\{(\ \ -1)\ \ ^2+[2-\ \ \ (\ \ +1)]\ \ \ +(\ \ \ -1)\ \ ^2+B\ \ -B\ \ \ \}}$

$$\mathbf{=(\ \ -\ \ )\ \ ^2+[2\ \ -\ \ \ (\ \ +1)]\ \ \ +(\ \ -1)\ \ \ ^2+B\ \ -B\ \ \ .}$$

Solution in parametric form:

$$= \frac{}{1-k} + \ \ |\ \ |\ \ e \quad , \qquad = \frac{k}{1-k} + \ \ |\ \ |\ \ e \quad , \qquad k \neq 1.$$

64. $\mathbf{[\ \ (\ \ \ ^2+\ \ \ \ \ +\gamma\ \ ^2)+(2\ \ \ -\ \ \ ^2\ \ )\ \ +(\ \ -\ \ \ B\ \ )\ \ ]}$

$$\mathbf{+B(\ \ \ ^2+\ \ \ \ \ +\gamma\ \ ^2)+(\ \ -\ \ \ B\ \ )\ \ +(2\gamma-B^2\ \ )\ \ =0.}$$

Solution: $\quad\ \ ^2 + \beta\ \ \ +\ \ \ ^2 - A\sigma\ \ - B\sigma\ \ + \sigma =\ \ \exp(-A\ \ \ - B\ \ ).$

65. $(\;_{22}\;^2 + \;_{12}\; + \;_{11}\;^2 + \;_2\; + \;_1\;)\; = B_{22}\;^2 + (2\;_{22}\; + \;_{12} - 2B_{22})$
$+ (-\;_{22}\;^2 + B_{22}\; + \;_{11})\;^2 + B_2\; + (\;_2\; + \;_1 - B_2)\;.$

The substitution $\; = z + k\;$ leads to a Riccati equation with respect to $\; = \;(z)$:

$$[(B_{22} - A_{22}k)z + B_2 - A_2k]z\;' = (A_{22}k^2 + A_{12}k + A_{11})\;^2$$
$$+ [(2A_{22}k + A_{12})z + A_2k + A_1]\; + A_{22}z^2 + A_2z.$$

In equations 66–70, the following notation is used: $\; = \; - \;B \neq 0,\quad \; = \; + \;B.$

66. $(\;^2\;^2 - 2\;\; + \;^2\;^2 - \;\; + \;B\;)\;$
$= \;^2B\;^2 - 2\;B\;\; + B\;^2\;^2 - \;\; + \;B\;.$

Solution in parametric form:

$$\; = \frac{A}{\ln|\;|} + a\;,\qquad \; = \frac{B}{\ln|\;|} + \;.$$

67. $[\;^2\;^2 - \;\; + \;B\;^2 - \;\; + (\;B - \;)\;]\;$
$= \;^2 - \;\; + \;B\;^2\;^2 - (\; + \;)\; + \;B\;.$

Solution in parametric form:

$$\; = A\; + a\;|\;|^{+1}e\;,\qquad \; = B\; + \;|\;|^{+1}e\;.$$

68. $[\;^2\;^2 - (\; - \;)\; + (\;B - \;)\;^2 + \; - \;B\;]\;$
$= (\;B\; + \;)\;^2 - (\; + \;)\; + \;B\;^2\;^2 + \; - \;B\;.$

Solution in parametric form:

$$\; = A\; + a\;|\;|^{+1}\exp\frac{l}{\Delta}\;,\qquad \; = B\; + \;|\;|^{+1}\exp\frac{l}{\Delta}\;.$$

69. $(\;^3\;^2 - 2\;^2B\; + \;B^2\;^2 - \;^2 + \;)\; = \;^2B\;^2 - 2\;B^2\; + B^3\;^2 - \; + \;^2\;.$

Solution in parametric form:

$$\; = A\;^3\sqrt{\tfrac{2}{3}k\Delta\;^3 + 1} + a\;^2,\qquad \; = B\;^3\sqrt{\tfrac{2}{3}k\Delta\;^3 + 1} + \;^2.$$

70. $[\;^3\;^2 - 2\;^2B\; + \;B^2\;^2 + \; - (\; + \;)\;]\;$
$= \;^2B\;^2 - 2\;B^2\; + B^3\;^2 + (\;B - \;)\; - \;B\;.$

Solution in parametric form ($l \neq 1$):

$$\; = A\;^2\;^{+1} + \frac{k\Delta}{l-1}\;^2\; + a\;,\qquad \; = B\;^2\;^{+1} + \frac{k\Delta}{l-1}\;^2\; + \;.$$

1.4.4. Equations of the Form $(\;_{22}\;^2 + \;_{12}\; + \;_{11}\;^2 + \;_2\; + \;_1\; + \;_0)\; = \;_{22}\;^2 + \;_{12}\; + \;_{11}\;^2 + \;_2\; + \;_1\; + \;_0$

1.4.4-1. Preliminary remarks. Some transformations.

1 . With $A_{22} = 0$, this is an Abel equation (see Subsection 1.3.4). With $B_{11} = 0$, this is an Abel equation with respect to $\; = \;(\;)$.

See Subsection 1.4.2 for the case $A_2 = A_1 = B_2 = B_1 = 0$.

See Subsection 1.4.3 for the case $A_0 = B_0 = 0$.

2 . The transformation $= {}^{-} + \ ,\quad = {}^{-} + \beta$, where and β are parameters, which are determined by solving the second-order algebraic system

$$A_{22}\beta^2 + A_{12}\quad \beta + A_{11}\quad {}^2 + A_2\beta + A_1\quad + A_0 = 0,$$
$$B_{22}\beta^2 + B_{12}\quad \beta + B_{11}\quad {}^2 + B_2\beta + B_1\quad + B_0 = 0,$$

leads to the equation

$$(A_{22}\ {}^{-2} + A_{12}\ {}^{-\,-} + A_{11}\ {}^{-2} + a_2\ {}^{-} + a_1\ {}^{-})\ {}^{-\prime}_{-} = B_{22}\ {}^{-2} + B_{12}\ {}^{-\,-} + B_{11}\ {}^{-2} + \ _2\ {}^{-} + \ _1\ {}^{-}, \qquad (1)$$

where

$$a_2 = 2A_{22}\beta + A_{12}\quad + A_2, \qquad a_1 = 2A_{11}\quad + A_{12}\beta + A_1,$$
$$_2 = 2B_{22}\beta + B_{12}\quad + B_2, \qquad _1 = 2B_{11}\quad + B_{12}\beta + B_1.$$

The transformation $\xi = {}^{-}\ {}^{-},\quad = 1\ \ {}^{-}$ reduces equation (1) to an Abel equation of the second kind:

$$\{[a_2\xi^2 + (a_1 - \ _2)\xi - \ _1]\quad + A_{22}\xi^3 + (A_{12} - B_{22})\xi^2 + (A_{11} - B_{12})\xi - B_{11}\}\ \ '$$
$$= (a_2\xi + a_1)\quad {}^2 + (A_{22}\xi^2 + A_{12}\xi + A_{11})\quad .$$

3 . The substitution $= z + \varepsilon$, where the parameter ε is determined by solving the cubic equation

$$(A_{22}\varepsilon^2 + A_{12}\varepsilon + A_{11})\varepsilon - B_{22}\varepsilon^2 - B_{12}\varepsilon - B_{11} = 0,$$

leads to an Abel equation of the second kind with respect to $= \ (z)$:

$$[(Qz + R)\quad + (B_{22} - A_{22}\varepsilon)z^2 + (B_2 - A_2\varepsilon)z + B_0 - A_0\varepsilon]\ \ '$$
$$= (A_{22}\varepsilon^2 + A_{12}\varepsilon + A_{11})\quad {}^2 + [(2A_{22}\varepsilon + A_{12})z + A_2\varepsilon + A_1]\quad + A_{22}z^2 + A_2 z + A_0,$$

where $Q = 2B_{22}\varepsilon + B_{12} - \varepsilon(2A_{22}\varepsilon + A_{12}),\ R = B_2\varepsilon + B_1 - \varepsilon(A_2\varepsilon + A_1)$.

1.4.4-2. Solvable equations and their solutions.

1. $\boldsymbol{(\quad + \quad + \ )^2 \quad = (\quad + \quad + \gamma)^2.}$

This is a special case of equation 1.7.1.6 with $(z) = z^{-2}$.

2. $\boldsymbol{(\quad {}^2 + B \quad - \ B \quad + \quad - \quad) \quad = \quad {}^2 + \quad + (\ - \quad)\ .}$

The transformation $= \ + \ ,\quad = \ \xi$ leads to a linear equation:

$$[-A\xi^3 + (\quad - B)\xi^2 + \quad \xi]\ \ ' = (A\xi^2 + B\xi)\quad + k.$$

3. $\boldsymbol{(\quad {}^2 + 2 \quad + B\ {}^2 + \quad - B) \quad = \quad {}^2 + 2B \quad + \quad {}^2 + 2(B - \quad) \ + \quad - \ .}$

The transformation $= \ + 1,\quad = \xi\ \ - 1$ leads to a linear equation:

$$(-A\xi^3 - A\xi^2 + B\xi + \quad)\ \ ' = (A\xi^2 + 2A\xi + B)\quad + 2(B - A).$$

4. $\boldsymbol{(\quad {}^2 - 2 \quad + B\ {}^2 + \quad - B) \quad = - \quad {}^2 + 2B \quad + \quad {}^2 + 2(B + \quad) \ + \quad + \ .}$

The transformation $= \ - 1,\quad = \xi\ \ - 1$ leads to a linear equation:

$$(-A\xi^3 + A\xi^2 + B\xi + \quad)\ \ ' = (A\xi^2 - 2A\xi + B)\quad + 2(A - B).$$

5. $\boldsymbol{(\quad {}^2 + 2 \quad + B\ {}^2 + \quad - B) \quad = \quad {}^2 + 2B \quad + \quad {}^2 + 2(\quad - B) \ - \quad + \ .}$

The transformation $= \ - 1,\quad = \xi\ \ + 1$ leads to a linear equation:

$$(-A\xi^3 - A\xi^2 + B\xi + \quad)\ \ ' = (A\xi^2 + 2A\xi + B)\quad + 2(A - B).$$

6. $\boldsymbol{(\quad {}^2 - 2 \quad + B\ {}^2 + \quad - B) \quad = - \quad {}^2 + 2B \quad + \quad {}^2 - 2(B + \quad) \ + \quad + \ .}$

The transformation $= \ + 1,\quad = \xi\ \ + 1$ leads to a linear equation:

$$(-A\xi^3 + A\xi^2 + B\xi + \quad)\ \ ' = (A\xi^2 - 2A\xi + B)\quad + 2(B - A).$$

7. $\boldsymbol{(\quad {}^2 - 2 \quad + B\ {}^2 + \ - B) \quad = \quad {}^2 + 2B \quad + \quad {}^2 - 2(\ + \ ) \ - 2(B + \ ) \ + 2 \ + \ + \ .}$

The transformation $= \ + 1,\quad = \xi\ \ + 1$ leads to a linear equation:

$$[-A\xi^3 + (2A + \quad)\xi^2 + B\xi + \quad]\ \ ' = (A\xi^2 - 2A\xi + B)\quad + 2(B - A).$$

8. $\boldsymbol{(2\quad {}^2 - 2 \quad + B\ {}^2 + 2 \ - 4B) \quad = - \quad {}^2 + 2B \quad + \quad {}^2 - 2(B + 2\quad) \ + \quad + 4\ .}$

This is a special case of equation 1.4.4.34 with $= 2,\ \beta = 1$, and $= -A$.

9. $\left({}^{2} + 4 \quad + B \ {}^{2} + 4 \quad - B \right) \quad = 2 \quad {}^{2} + 2B \quad + \quad {}^{2} - 2(\quad - 2B) \ + \quad - 8 \ .$

The transformation $= \ + 1, \quad = \xi \quad - 2$ leads to a linear equation:

$$(-A\xi^3 - 2A\xi^2 + B\xi + \quad)\ ' = (A\xi^2 + 4A\xi + B) \quad + 2B - 8A.$$

10. $\left({}^{2} - 4 \quad + B \ {}^{2} + 4 \quad - B \right) \quad = -2 \quad {}^{2} + 2B \quad + \quad {}^{2} - 2(2B + \quad) \ + 8 \quad + \quad .$

The transformation $= \ + 1, \quad = \xi \quad + 2$ leads to a linear equation:

$$(-A\xi^3 + 2A\xi^2 + B\xi + \quad)\ ' = (A\xi^2 - 4A\xi + B) \quad + 2B - 8A.$$

11. $\left({}^{2} + 4 \quad + B \ {}^{2} + 4 \quad - B \right) \quad = \quad {}^{2} + 2B \quad + 2B \ {}^{2} + 4(\quad - 2 \quad) \ + 2B + 4 \quad - 16 \ .$

The transformation $= \ + 1, \quad = \xi \quad - 2$ leads to a linear equation:

$$[-A\xi^3 + (\quad - 4A)\xi^2 + B\xi + 2B]\ ' = (A\xi^2 + 4A\xi + B) \quad + 2B - 8A.$$

12. $\left(2 \quad {}^{2} + 2 \quad + B \ {}^{2} + 2 \quad - 4B \right) \quad = \quad {}^{2} + 2B \quad + \quad {}^{2} + 2(B - 2 \quad) \ + 4 \quad - \quad .$

This is a special case of equation 1.4.4.34 with $= 2$, $\beta = -1$, and $= A$.

13. $\left(2 \quad {}^{2} + 2 \quad - B \ {}^{2} + 2 \quad + 4B \right) \quad = \quad {}^{2} - 2B \quad - \quad {}^{2} + 2(B - 2 \quad) \ - \quad - 4 \quad .$

This is a special case of equation 1.4.4.34 with $= -2$, $\beta = 1$, and $= -A$.

14. $\left({}^{2} + 2B \quad + \quad {}^{2}\ {}^{2} + \quad + \quad + \quad \right) \quad = B \ {}^{2} + 2 \quad {}^{2} \quad + B \ {}^{2}\ {}^{2} + \quad + \quad {}^{2} \quad + s.$

This is a special case of equation 1.4.4.27 with $= Ak^2$.

15. $\left({}^{2} + 2B \quad + \quad {}^{2}\ {}^{2} + \quad + \quad + \quad \right) \quad = B \ {}^{2} + 2 \quad {}^{2} \quad + B \ {}^{2}\ {}^{2} + \quad + \quad + s.$

This is a special case of equation 1.4.4.32 with $= Ak^2$.

16. $\left({}^{2} + 2B \quad + \quad {}^{2}\ {}^{2} + \quad - \quad + \ \right) \quad = B \ {}^{2} + 2 \quad {}^{2} \quad + B \ {}^{2}\ {}^{2} + \quad - \quad + s.$

This is a special case of equation 1.4.4.31 with $= Ak^2$.

17. $\left({}^{2} + 2B \quad - B \quad {}^{2} + \quad + \quad + \ \right) \quad = B \ {}^{2} + 2 \quad {}^{2} \quad - \quad {}^{3}\ {}^{2} + \quad + \quad {}^{2} \quad + s.$

This is a special case of equation 1.4.4.28 with $= \ .$

18. $\left({}^{2} + 2B \quad - B \quad {}^{2} + \quad + \quad + \quad \right) \quad = B \ {}^{2} + 2 \quad {}^{2} \quad - \quad {}^{3}\ {}^{2} + \quad + \quad + s.$

This is a special case of equation 1.4.4.32 with $= -Bk$.

19. $\left({}^{2} + 2B \quad - B \quad {}^{2} + \quad - \quad + \ \right) \quad = B \ {}^{2} + 2 \quad {}^{2} \quad - \quad {}^{3}\ {}^{2} + \quad - \quad + s.$

This is a special case of equation 1.4.4.31 with $= -Bk$.

20. $\left({}^{2} + 2 \quad + \quad {}^{2} + \quad + \quad + \quad \right) \quad = \quad {}^{2} + 2 \quad {}^{2} \quad + \quad {}^{2} + \quad + \quad {}^{2} \quad + s.$

This is a special case of equation 1.4.4.27 with $B = Ak$.

21. $\left({}^{2} + 2 \quad + \quad {}^{2} + \quad + \quad + \quad \right) \quad = \quad {}^{2} + 2 \quad {}^{2} \quad + \quad {}^{2} + \quad + \quad + s.$

This is a special case of equation 1.4.4.32 with $B = Ak$.

22. $\left({}^{2} + 2 \quad + \quad {}^{2} + \quad - \quad + \ \right) \quad = \quad {}^{2} + 2 \quad {}^{2} \quad + \quad {}^{2} + \quad - \quad + s.$

This is a special case of equation 1.4.4.31 with $B = Ak$.

23. $\left({}^{2} - 2 \quad + B \quad {}^{2} + \quad + \quad + \ \right) \quad = -B \ {}^{2} + 2B \quad - \quad {}^{3}\ {}^{2} + \quad + \quad {}^{2} \quad + s.$

This is a special case of equation 1.4.4.29 with $= \ .$

24. $(\quad {}^2 - 2 \qquad + B \quad {}^2 + \quad + \quad + \;) \quad = -B \;{}^2 + 2B \qquad - \quad {}^3\;{}^2 + \quad + \quad + s.$

This is a special case of equation 1.4.4.29 with $\quad = ak$.

25. $(\quad {}^2 + 2B \quad + \quad {}^2 - 2 \quad + \quad + \quad {}^2) \quad = B \;{}^2 + E \quad + \quad {}^2 + \quad - E \quad - B \;{}^2 - \quad .$

The substitution $\quad = \quad - \beta$ leads to an equation of the form 1.4.3.38:

$$(A \quad {}^2 + 2B \quad + \quad {}^2 + \bar{k} \quad)\;' = B \quad {}^2 + \qquad + \quad {}^2 + \bar{k} \quad, \quad \text{where} \quad \bar{k} = k + 2B\beta.$$

26. $(\quad {}^2 + B \quad + \quad {}^2 + \quad + \quad + \;) \quad = \quad {}^2 + B \quad + \quad {}^2 + \quad + (\quad + - \;) + s.$

The substitution $\quad = z + k$ leads to a Riccati equation with respect to $\quad = \quad(z)$:

$$[(\quad - ak)z + \quad - \quad k]\;' = (Ak^2 + Bk + \quad)\;{}^2 + [(2Ak + B)z + ak + \quad] \quad + Az^2 + az + \quad .$$

27. $(\quad {}^2 + 2B \quad + \quad {}^2 + \quad + \quad + \;) \quad = B \;{}^2 + 2 \quad {}^2 \quad + (- \quad {}^2 + B \;+\;) \;{}^2 + \quad + \quad {}^2 \quad + s.$

The substitution $\quad = z + k$ leads to a Riccati equation with respect to $\quad = \quad(z)$:

$$[(B - Ak)z^2 + (\quad - ak)z + \quad - \quad k]\;' \\ = (Ak^2 + 2Bk + \quad)\;{}^2 + [2(Ak + B)z + ak + \quad] \quad + Az^2 + az + \quad .$$

28. $(\quad {}^2 + 2B \quad - B \quad {}^2 + \quad + \quad + \;) \quad = B \;{}^2 + 2 \quad {}^2 \quad - \quad {}^3\;{}^2 + \quad + (\quad + - \;) + s.$

The substitution $\quad = z + k$ leads to a Riccati equation with respect to $\quad = \quad(z)$:

$$[(B - Ak)z^2 + (\quad - ak)z + \quad - \quad k]\;' = (Ak^2 + Bk)\;{}^2 + [2(Ak + B)z + ak + \quad] \quad + Az^2 + az + \quad .$$

29. $(\quad {}^2 - 2 \qquad + B \quad {}^2 + \quad + \quad + \;) \quad = -B \;{}^2 + 2B \qquad - \quad {}^3\;{}^2 + \quad + (\quad + - \;) + s.$

The substitution $\quad = z + k$ leads to a Riccati equation with respect to $\quad = \quad(z)$:

$$[-(Ak + B)z^2 + (\quad - ak)z + \quad - \quad k]\;' = k(B - Ak)\;{}^2 + (ak + \quad) \quad + Az^2 + az + \quad .$$

30. $(\quad {}^2 + 2B \quad + \quad {}^2\;{}^2 + \quad + \quad + \;) \quad = B \;{}^2 + 2 \quad {}^2 \quad + B \;{}^2\;{}^2 + \quad + (\quad + - \;) + s.$

The substitution $\quad = z + k$ leads to a Riccati equation with respect to $\quad = \quad(z)$:

$$[(B - Ak)z^2 + (\quad - ak)z + \quad - \quad k]\;' = 2k(Ak + B)\;{}^2 + [2(Ak + B)z + ak + \quad] \quad + Az^2 + az + \quad .$$

31. $(\quad {}^2 + 2B \quad + \quad {}^2 + \quad - \quad + \;)$
$$= B \;{}^2 + 2 \quad {}^2 \quad + (- \quad {}^2 + B \quad + \quad) \;{}^2 + \quad - \quad + s.$$

The substitution $\quad = z + k$ leads to a Riccati equation with respect to $\quad = \quad(z)$:

$$[(B - Ak)z^2 + (\quad - ak)z + \quad - \quad k]\;' = (Ak^2 + 2Bk + \quad)\;{}^2 + 2(Ak + B)z \quad + Az^2 + az + \quad .$$

32. $(\quad {}^2 + 2B \quad + \quad {}^2 + \quad + \quad + \;) \quad = B \;{}^2 + 2 \quad {}^2 \quad + (- \quad {}^2 + B \;+\;) \;{}^2 + \quad + \quad + s.$

The substitution $\quad = z + k$ leads to a Riccati equation with respect to $\quad = \quad(z)$:

$$[(B - Ak)z^2 + \quad - \quad k]\;' = (Ak^2 + 2Bk + \quad)\;{}^2 + [2(Ak + B)z + ak + \quad] \quad + Az^2 + az + \quad .$$

33. $[\;(\quad {}^2 + \qquad + \gamma \;{}^2) + (\quad + 2 \;) \;+ (\;\varepsilon + \;) \;+ \quad + \;]$
$$+ B(\quad {}^2 + \qquad + \gamma \;{}^2) + (B \quad + \;) \;+ (B\varepsilon + 2\gamma) \;+ B \quad + \varepsilon = 0.$$

Solution: $\quad {}^2 + \beta \quad + \quad {}^2 + \quad + \varepsilon \quad + \sigma = \quad \exp(-A \quad - B \quad).$

34. $(\quad {}^2 - 2 \quad + B \, {}^2 + \quad {}^2 \; - \; {}^2 B) \quad = \quad {}^2 + 2B$
$+ \quad {}^2 - 2 \; (\quad + \quad) \; - 2(\quad + \; B) \; + \; {}^2 \; + \; {}^2(2 \quad + \quad).$

The transformation $= \quad + \quad , \quad = \xi \quad + \beta$ leads to a linear equation:

$$[- \; A\xi^3 + (2\beta A + \quad)\xi^2 + B\xi + \quad] \; ' = (\; A\xi^2 - 2\beta A\xi + B) \quad + 2(\; B - \beta^2 A).$$

35. $(\; {}_{22} \, {}^2 + \; {}_{12} \quad + \; {}_{11} \, {}^2 + \; {}_2 \; + \; {}_1 \; + \; {}_0) \quad = B_{22} \, {}^2 + \; (2 \; {}_{22} \; + \; {}_{12} - 2B_{22})$
$+ \; (- \; {}_{22} \, {}^2 + B_{22} \; + \; {}_{11}) \, {}^2 + B_2 \; + \; (\; {}_2 \; + \; {}_1 - B_2) \; + B_0.$

The substitution $= z + k$ leads to a Riccati equation with respect to $= \; (z)$:

$$[(B_{22} - A_{22}k)z^2 + (B_2 - A_2k)z + B_0 - A_0k] \; '$$
$$= (A_{22}k^2 + A_{12}k + A_{11}) \; {}^2 + [(2A_{22}k + A_{12})z + A_2k + A_1] \; + A_{22}z^2 + A_2z + A_0.$$

36. $(\; {}_{22} \, {}^2 + \; {}_{12} \quad + \; {}_{11} \, {}^2 + \; {}_2 \; + \; {}_1 \; + \; {}_0) \quad = B_{22} \, {}^2 + B_{12} \quad + B_{11} \, {}^2 + B_2 \; + B_1 \; + B_0.$

Here, A , B , and A_1 are arbitrary parameters, and the other parameters are defined by the relations:

$$A_2 = -A_{12} \quad - 2A_{22}\beta,$$
$$A_0 = -A_{11} \; {}^2 + A_{22}\beta^2 - A_1 \quad ,$$
$$B_2 = (2A_{11} - B_{12}) \quad + (A_{12} - 2B_{22})\beta + A_1,$$
$$B_1 = -2B_{11} \quad - B_{12}\beta,$$
$$B_0 = B_{11} \; {}^2 + (B_{12} - 2A_{11}) \; \beta + (B_{22} - A_{12})\beta^2 - A_1\beta$$

$(\; , \beta$ are arbitrary parameters).

The transformation $= \quad + \quad , \quad = \xi \quad + \beta$ leads to a linear equation:

$$[-A_{22}\xi^3 + (B_{22} - A_{12})\xi^2 + (B_{12} - A_{11})\xi + B_{11}] \; ' = (A_{22}\xi^2 + A_{12}\xi + A_{11}) \quad + k,$$

where $k = 2A_{11} \quad + A_{12}\beta + A_1$.

1.4.5. Equations of the Form $(\; {}_3 \, {}^3 + \; {}_2 \quad {}^2 + \; {}_1 \, {}^2 \; + \; {}_0 \, {}^3 + \; {}_1 \; + \; {}_0 \;)$ $= \; {}_3 \, {}^3 + \; {}_2 \quad {}^2 + \; {}_1 \, {}^2 \; + \; {}_0 \, {}^3 + \; {}_1 \; + \; {}_0$

1. $(\, {}^3 - \; {}^2 \; + \quad + \;) \quad = \quad {}^2 - \; {}^3 + \quad + \quad .$

Solution in parametric form $(\; \neq 0)$:

$$= \; {}^{-1} \, | \, |^{\frac{-b}{2b}} e^{-} \; + \tfrac{1}{2} \quad | \, |^{-\frac{-b}{2b}} e \; , \qquad = \; {}^{-1} \, | \, |^{\frac{-b}{2b}} e^{-} \; - \tfrac{1}{2} \quad | \, |^{-\frac{-b}{2b}} e \; .$$

2. $(\, {}^3 - \quad {}^2 - \; {}^2 \; + \; {}^3 + \quad) \quad = - \; {}^3 + \quad {}^2 + \; {}^2 \; - \; {}^3 + \quad .$

Solution in parametric form:

$$= \; {}^{-1} \, \text{sign} \; e^{-1} \quad + \tfrac{1}{8}a \; | \, |e^{1} \quad , \qquad = \; {}^{-1} \, \text{sign} \; e^{-1} \quad - \tfrac{1}{8}a \; | \, |e^{1} \quad .$$

3. $(\, {}^3 + \quad {}^2 - \; {}^2 \; - \; {}^3 + \quad + \quad) \quad = - \; {}^3 - \quad {}^2 + \; {}^2 \; + \; {}^3 + \quad + \quad .$

Solution in parametric form $(a \neq - \;)$:

$$= \; + \; | \, |^{\frac{b-}{b+}} \exp \; -\frac{4 \; {}^2}{a +} \quad , \qquad = \; - \; | \, |^{\frac{b-}{b+}} \exp \; -\frac{4 \; {}^2}{a +} \quad .$$

4. $(\, {}^3 + \quad {}^2 - 2 \, {}^2 \; + 2 \quad + \quad) \quad = - \; {}^3 + \quad {}^2 + 4 \, {}^2 \; - 4 \, {}^3 - \quad + 4 \quad .$

Solution in parametric form:

$$= \; {}^{-1} \, {}^{-1} e^{-1} \quad + \tfrac{1}{3}a \quad {}^2 e^{1} \quad , \qquad = \; {}^{-1} \, {}^{-1} e^{-1} \quad - \tfrac{2}{3}a \quad {}^2 e^{1} \quad .$$

5. $(\;^{3}+ \quad ^{2}-5 \;^{2} \;+3 \;^{3}+ \quad + \quad) \quad =-3 \;^{3}-3 \quad ^{2}+15 \;^{2} \;-9 \;^{3}- \quad +3 \quad .$

Solution in parametric form:

$$= \;^{-1}\,\mathrm{sign}\;\; e^{-1} \;\; + \tfrac{1}{32}a \;\; | \,|e^{1} \;\; , \qquad = \;^{-1}\,\mathrm{sign}\;\; e^{-1} \;\; - \tfrac{3}{32}a \;\; | \,|e^{1} \;\; .$$

6. $(\;^{3}+2 \quad ^{2}- \;^{2} \;-2 \;^{3}+2 \quad + \quad) \quad =2 \quad ^{2}+2 \;^{2} \;-4 \;^{3}- \quad +4 \quad .$

Solution in parametric form:

$$= \;^{-1}\,| \,|e^{-1} \;\; - \tfrac{1}{3}a \;\; | \,|^{-1}e^{1} \;\; , \qquad = \;^{-1}\,| \,|e^{-1} \;\; + \tfrac{2}{3}a \;\; | \,|^{-1}e^{1} \;\; .$$

7. $(\;^{3}-3 \;^{2} \;+2 \;^{3}+2 \quad + \quad) \quad =-2 \;^{3}+6 \;^{2} \;-4 \;^{3}- \quad +4 \quad .$

Solution in parametric form:

$$= \;^{-1}\,\mathrm{sign}\;\; e^{-1} \;\; + \tfrac{1}{9}a \;\; | \,|e^{1} \;\; , \qquad = \;^{-1}\,\mathrm{sign}\;\; e^{-1} \;\; - \tfrac{2}{9}a \;\; | \,|e^{1} \;\; .$$

8. $(\;^{3}+3 \quad ^{2}-4 \;^{3}+ \quad + \quad) \quad =-2 \;^{3}-6 \quad ^{2}+8 \;^{3}+(\; - \;) \; +2 \quad .$

Solution in parametric form $(a \neq - \;)$:

$$= \; + \;\; | \,|^{\frac{b-2}{b+}} \exp -\frac{27 \;^{2}}{2(a+ \;)} \;\; , \qquad = \; -2 \;\; | \,|^{\frac{b-2}{b+}} \exp -\frac{27 \;^{2}}{2(a+ \;)} \;\; .$$

9. $(\;^{3}+3 \quad ^{2}- \;^{2} \;-3 \;^{3}+ \quad + \quad) \quad =- \;^{3}+ \quad ^{2}+9 \;^{2} \;-9 \;^{3}-(2 \; - \;) \; +3 \quad .$

Solution in parametric form $(a \neq \;)$:

$$= \;^{-1}\,| \,|^{-\frac{3 \;-b}{2(\;-b)}} e^{-} - \tfrac{1}{16}(a- \;) \;\; | \,|^{\frac{3 \;-b}{2(\;-b)}} e \;, \qquad = \;^{-1}\,| \,|^{-\frac{3 \;-b}{2(\;-b)}} e^{-} + \tfrac{3}{16}(a- \;) \;\; | \,|^{\frac{3 \;-b}{2(\;-b)}} e \;.$$

10. $(\;^{3}+3 \quad ^{2}+3 \;^{2} \;+ \;^{3}- \quad + \quad) \quad =- \;^{3}-3 \quad ^{2}-3 \;^{2} \;- \;^{3}- \quad + \quad .$

Solution in parametric form:

$$= \quad \frac{^{2}}{\overline{a}} \;\; \overline{2 \;^{4}+1}, \qquad = \;\mp \frac{^{2}}{\overline{a}} \;\; \overline{2 \;^{4}+1}.$$

11. $(\;^{3}+3 \quad ^{2}+3 \;^{2} \;+ \;^{3}+ \quad + \quad) \quad =- \;^{3}-3 \quad ^{2}-3 \;^{2} \;- \;^{3}+ \quad + \quad .$

Solution in parametric form $(\;\neq -2a)$:

$$= \quad + \quad ^{3} \;\; | \,|^{\frac{b-}{+b}} + \frac{4 \;^{3}}{2a+} \;\; , \qquad = \quad - \quad ^{3} \;\; | \,|^{\frac{b-}{+b}} + \frac{4 \;^{3}}{2a+} \;\; .$$

12. $(\;^{3}-4 \quad ^{2}+4 \;^{2} \;+ \quad - \quad) \quad =3 \;^{3}-14 \quad ^{2}+20 \;^{2} \;-8 \;^{3}+2 \quad -2 \quad .$

Solution in parametric form:

$$= \; + \quad ^{2}\exp -\frac{a}{2 \;^{2}} \;\; , \qquad = \; +2 \quad ^{2}\exp -\frac{a}{2 \;^{2}} \;\; .$$

13. $(\;^{3}-4 \quad ^{2}+5 \;^{2} \;-2 \;^{3}+2 \quad -3 \quad) \quad =2 \;^{3}-8 \quad ^{2}+10 \;^{2} \;-4 \;^{3}+3 \quad -4 \quad .$

Solution in parametric form:

$$= \;^{-1}\,\mathrm{sign}\;\; e^{-1} \;\; +a \;\; | \,|e^{1} \;\; , \qquad = \;^{-1}\,\mathrm{sign}\;\; e^{-1} \;\; +2a \;\; | \,|e^{1} \;\; .$$

14. $(\;^{3}-5 \quad ^{2}+7 \;^{2} \;-3 \;^{3}+ \quad -2 \quad) \quad =3 \;^{3}-15 \quad ^{2}+21 \;^{2} \;-9 \;^{3}+2 \quad -3 \quad .$

Solution in parametric form:

$$= \;^{-1}\,\mathrm{sign}\;\; e^{-1} \;\; + \tfrac{1}{8}a \;\; | \,|e^{1} \;\; , \qquad = \;^{-1}\,\mathrm{sign}\;\; e^{-1} \;\; + \tfrac{3}{8}a \;\; | \,|e^{1} \;\; .$$

15. $(\;^3 - 5 \;\;^2 + 8 \;^2 \; - 4 \;^3 + \;\; + \;\;) \;\; = 2 \;^3 - 10 \;\;^2 + 16 \;^2 \; - 8 \;^3 + (3 \; + \;) \; - 2 \;\; .$

Solution in parametric form $(a \neq - \;)$:

$$= \; + \;\; | \;|^{\frac{2\;+b}{\;+b}} \exp \;\frac{\;\;2}{2(a+\;)} \;, \qquad = \; +2 \;\; | \;|^{\frac{2\;+b}{\;+b}} \exp \;\frac{\;\;2}{2(a+\;)} \;.$$

16. $(\;^3 + 5 \;\;^2 + 3 \;^2 \; - 9 \;^3 + \;\; + \;\;) \;\; = -3 \;^3 - 15 \;\;^2 - 9 \;^2 \; + 27 \;^3 + (\; - 2 \;) \; + 3 \;\; .$

Solution in parametric form $(a \neq - \;)$:

$$= \; + \;\; | \;|^{\frac{b-3}{b+}} \exp \; - \frac{32 \;\;^2}{a+} \;, \qquad = \; -3 \;\; | \;|^{\frac{b-3}{b+}} \exp \; - \frac{32 \;\;^2}{a+} \;.$$

17. $(\;^3 - 6 \;\;^2 + 11 \;^2 \; - 6 \;^3 + \;\; + \;\;) \;\; = 2 \;^3 - 11 \;\;^2 + 18 \;^2 \; - 9 \;^3 + (4 \; + \;) \; - 3 \;\; .$

Solution in parametric form $(a \neq -\frac{1}{2} \;)$:

$$= \;\;^{-1} \; | \;|^{-\frac{3\;+b}{4\;+2b}} e^{-} \; - (a + \tfrac{1}{2} \;) \;\; | \;|^{\frac{3\;+b}{4\;+2b}} e \;, \qquad = \;\;^{-1} \; | \;|^{-\frac{3\;+b}{4\;+2b}} e^{-} \; - 3(a + \tfrac{1}{2} \;) \;\; | \;|^{\frac{3\;+b}{4\;+2b}} e \;.$$

18. $(\;^3 - 6 \;\;^2 + 12 \;^2 \; - 8 \;^3 - \;\; + \;\;) \;\; = 2 \;^3 - 12 \;\;^2 + 24 \;^2 \; - 16 \;^3 - \;\; + \;\; .$

Solution in parametric form $(a > 0)$:

$$= \;\; \frac{\;\;2}{2 \;\; \overline{a}} \;\; \overline{2 \;^4 + 1}, \qquad = \;\; \frac{\;\;2}{\overline{a}} \;\; \overline{2 \;^4 + 1}.$$

19. $(2 \;^3 - 3 \;\;^2 + \;^2 \; + \;\; + \;\;) \;\; = \;^3 - \;\;^2 + (\; + \;) \; .$

Solution in parametric form $(a \neq -2 \;)$:

$$= \;\;^{-1} \; | \;|^{-\frac{b}{\;+2b}} e^{-} \; + (a + 2 \;) \;\; | \;|^{\frac{b}{\;+2b}} e \;, \qquad = \;\;^{-1} \; | \;|^{-\frac{b}{\;+2b}} e^{-} \;.$$

20. $(2 \;^3 + 3 \;\;^2 - 3 \;^2 \; - 2 \;^3 + \;\; + \;\;) \;\; = - \;^3 + 3 \;\;^2 + 6 \;^2 \; - 8 \;^3 - (\; - \;) \; + 2 \;\; .$

Solution in parametric form $(a \neq 2 \;)$:

$$= \;\;^{-1} \; | \;|^{-\frac{2\;-b}{\;-2b}} e^{-} \; - \tfrac{1}{27}(a - 2 \;) \;\; | \;|^{\frac{2\;-b}{\;-2b}} e \;, \qquad = \;\;^{-1} \; | \;|^{-\frac{2\;-b}{\;-2b}} e^{-} \; + \tfrac{2}{27}(a - 2 \;) \;\; | \;|^{\frac{2\;-b}{\;-2b}} e \;.$$

21. $(2 \;^3 - 9 \;\;^2 + 13 \;^2 \; - 6 \;^3 + \;\; + \;\;) \;\; = 3 \;^3 - 13 \;\;^2 + 18 \;^2 \; - 8 \;^3 + (3 \; + \;) \; - 2 \;\; .$

Solution in parametric form $(a \neq -\frac{2}{3} \;)$:

$$= \;\;^{-1} \; | \;|^{-\frac{2\;+b}{3\;+2b}} e^{-} \; - (3a + 2 \;) \;\; | \;|^{\frac{2\;+b}{3\;+2b}} e \;, \qquad = \;\;^{-1} \; | \;|^{-\frac{2\;+b}{3\;+2b}} e^{-} \; - 2(3a + 2 \;) \;\; | \;|^{\frac{2\;+b}{3\;+2b}} e \;.$$

22. $(3 \;^3 - \;\;^2 - 3 \;^2 \; + \;^3 + \;\;) \;\; = - \;^3 + 3 \;\;^2 + \;^2 \; - 3 \;^3 + \;\; .$

Solution in parametric form:

$$= \;\;^{-1} \;^{-1} e^{-1} \; + \tfrac{1}{8} a \;\;^2 e^{1} \;, \qquad = \;\;^{-1} \;^{-1} e^{-1} \; - \tfrac{1}{8} a \;\;^2 e^{1} \;.$$

23. $(3 \;^3 + \;\;^2 - 3 \;^2 \; - \;^3 + \;\;) \;\; = \;^3 + 3 \;\;^2 - \;^2 \; - 3 \;^3 + \;\; .$

Solution in parametric form:

$$= \;\;^{-1} \; | \;| e^{-1} \; - \tfrac{1}{8} a \;\; | \;|^{-1} e^{1} \;, \qquad = \;\;^{-1} \; | \;| e^{-1} \; + \tfrac{1}{8} a \;\; | \;|^{-1} e^{1} \;.$$

24. $(\;\;^2 - 2 \;\;^2 \; + \;^2 \;^3 + \;\; - \;\;) \;\; = \;^3 - 2 \;\;^2 + \;^2 \;^2 \; + \;\; - \;\; .$

Solution in parametric form $(k \neq 1)$:

$$= \; + \;\; | \;| \exp \; - \frac{a}{2(k-1) \;^2} \;, \qquad = \; + k \;\; | \;| \exp \; - \frac{a}{2(k-1) \;^2} \;.$$

25. $(\ ^3 - 3 \quad ^2 + 3 \ ^2 \ ^2 \ - \ ^3 \ ^3 - \quad + \quad) \quad = \quad ^3 - 3 \ ^2 \quad ^2 + 3 \ ^3 \ ^2 \ - \ ^4 \ ^3 - \quad + \quad .$

Solution in parametric form $(a > 0,\ k \neq 1)$:

$$= \quad {}^2 \frac{k-1}{2 \quad \overline{a}} \quad \overline{2 \ ^2 + 1}, \qquad = \quad k \ {}^2 \frac{k-1}{2 \quad \overline{a}} \quad \overline{2 \ ^2 + 1}.$$

26. $(\ ^3 - 3 \quad ^2 + 3 \ ^2 \ ^2 \ - \ ^3 \ ^3 + \quad + \quad)$

$= \quad ^3 - 3 \ ^2 \quad ^2 + 3 \ ^3 \ ^2 \ - \ ^4 \ ^3 + [(\quad + 1) \quad + \quad] \ - \quad .$

Solution in parametric form $(\quad \neq \frac{1}{2} a(k-3),\ k \neq 1)$:

$$= \quad + \quad ^3 \ | \ |^{\frac{+b}{+b}} + \frac{(k-1)^3}{(k-3)a - 2} \ ^3 , \qquad = \quad + k \quad ^3 \ | \ |^{\frac{+b}{+b}} + \frac{(k-1)^3}{(k-3)a - 2} \ ^3 .$$

27. $[\ ^3 - (\quad + 2) \quad ^2 + (2 \quad + 1) \ ^2 \ - \quad ^3 + 2 \quad - (\quad + 1) \quad]$

$= \quad ^3 - \ (\quad + 2) \quad ^2 + \ (2 \quad + 1) \ ^2 \ - \ ^2 \ ^3 + (\quad + 1) \quad - 2 \quad .$

Solution in parametric form $(k \neq 1)$:

$$= \quad ^{-1} \operatorname{sign} \ e^{-1} \quad + \frac{a}{(k-1)^2} | \ |e^{1} \quad , \qquad = \quad ^{-1} \operatorname{sign} \ e^{-1} \quad + \frac{ak}{(k-1)^2} | \ |e^{1} \quad .$$

28. $[\ ^3 - (\quad + 2) \quad ^2 - \ (\quad - 4) \ ^2 \quad + \ ^2(\quad - 2) \ ^3 + \quad - \quad]$

$= (2 \quad - 1) \ ^3 - \ (4 \quad - 1) \quad ^2 + \ ^2(2 \quad + 1) \ ^2 \ - \ ^3 \ ^3 + \quad - \quad .$

Solution in parametric form $(k \neq -1)$:

$$= \ + \quad ^2 \exp - \frac{a}{2(k-1)^2 \ ^2} \ , \qquad = \ + k \quad ^2 \exp - \frac{a}{2(k-1)^2 \ ^2} \ .$$

29. $[\ ^3 - (2 \quad + 1) \quad ^2 + \ (\quad + 2) \ ^2 \ - \ ^2 \ ^3 + \quad + \quad]$

$= \quad ^3 - \ (2 \quad + 1) \quad ^2 + \ ^2(\quad + 2) \ ^2 \ - \ ^3 \ ^3 + [(\quad + 1) \quad + \quad] \ - \quad .$

Solution in parametric form $(a \neq - \ ,\ k \neq 1)$:

$$= \ + \ | \ |^{\frac{+b}{+b}} \exp \frac{(k-1)^3}{2(a + \ )} \ ^2 , \qquad = \ + k \ | \ |^{\frac{+b}{+b}} \exp \frac{(k-1)^3}{2(a + \ )} \ ^2 .$$

30. $(\quad ^3 + \quad ^2 - \quad ^2 \ - \ ^3 + \quad + \quad) \quad = \ ^3 + \quad ^2 - \ ^2 \ - \quad ^3 + \quad + \quad .$

1 . Solution in parametric form with $\quad \neq 0$:

$$= \quad ^{-1} \ | \ |^{\frac{-b}{2b}} | \ + 1|^{\frac{b \ -}{2b}} - \tfrac{1}{4} \quad | \ |^{\frac{b-}{2b}} | \ + 1|^{\frac{-b}{2b}} ,$$

$$= \quad ^{-1} \ | \ |^{\frac{-b}{2b}} | \ + 1|^{\frac{b \ -}{2b}} + \tfrac{1}{4} \quad | \ |^{\frac{b-}{2b}} | \ + 1|^{\frac{-b}{2b}} .$$

2 . Solution in parametric form with $\quad = 0$:

$$= \quad ^{-1} \ | \ |^{\frac{-1}{2}} e^{-1} \quad - \tfrac{1}{8} a \quad | \ |^{\frac{1-}{2}} e^{1} \quad , \qquad = \quad ^{-1} \ | \ |^{\frac{-1}{2}} e^{-1} \quad + \tfrac{1}{8} a \quad | \ |^{\frac{1-}{2}} e^{1} \quad .$$

31. $(\ _3 \ ^3 + \ _2 \quad ^2 + \ _1 \ ^2 \quad + \ _0 \ ^3 + \quad) \quad = B_3 \ ^3 + B_2 \quad ^2 + B_1 \ ^2 \quad + B_0 \ ^3 + \quad .$

This is a special case of equation 1.7.1.13 with $R \ (\ , \ ) = a$. The transformation $\ = \quad ,$ $\ = \ ^2$ leads to a linear equation:

$$[\quad (\ ) - \quad (\ )] \ ' = 2 \quad (\ ) \quad + 2a,$$

where $\quad (\ ) = A_3 \ ^3 + A_2 \ ^2 + A_1 \ + A_0$ and $\quad (\ ) = B_3 \ ^3 + B_2 \ ^2 + B_1 \ + B_0$.

32. $[\ \ {}^3 + (\ \ +2)\ \ {}^2 - (\ \ -4)\ {}^2\ \ -(\ \ -2)\ {}^3 + \ \ - \ \ ]$

$= -(\ \ -2)\ {}^3 - (\ \ -4)\ \ {}^2 + (\ \ +2)\ {}^2\ \ +\ \ \ {}^3 - \ \ +\ \ .$

Solution in parametric form:

$$= \ + \ |\ |^{1-}\ \exp\frac{a}{8\ {}^2}, \qquad = \ - \ |\ |^{1-}\ \exp\frac{a}{8\ {}^2}.$$

33. $[\ \ {}^3 + 3(\ \ +1)\ \ {}^2 + 12\ {}^2\ \ -4(\ \ -3)\ {}^3 + \ \ - \ \ ]$

$= -(2\ \ -3)\ {}^3 - 6(\ \ -2)\ \ {}^2 + 12\ {}^2\ \ +8\ \ \ {}^3 - 2\ \ +2\ \ .$

Solution in parametric form:

$$= \ + \ |\ |^{1-}\ \exp\frac{a}{18\ {}^2}, \qquad = \ -2\ |\ |^{1-}\ \exp\frac{a}{18\ {}^2}.$$

1.5. Equations of the Form $f(\ ,y)y\ \ =\ \ (\ ,y)$ Containing Arbitrary Parameters

1.5.1. Equations Containing Power Functions

1.5.1-1. Equations of the form $\ '=\ (\ ,\ )$.

1. $=\ \ \overline{\ \ } + B\ \ {}^{-1}\ {}^2.$

The substitution $=2A^{-1}\ \overline{\ }$ leads to an Abel equation of the form 1.3.1.32: $\ ' =\ +2BA^{-2}\ \ {}^{-1}\ {}^2$.

2. $=\ \ \overline{\ \ } + B\ \ {}^{-1}.$

Let $A = \ 2a^{-1}\ \ \overline{\ }$, $B = \mp 4$ (>0). Solution in parametric form:

$$= a\ \ (\tau), \qquad = \ [2\tau\ \ \ (\tau)]^2, \qquad \text{where} \quad (\tau) = \exp(\mp\tau^2)\ \ \ \exp(\mp\tau^2)\ \ \tau + \ \ \ {}^{-1}.$$

3. $=\ \ \overline{\ \ } + B\ \ {}^{-2}.$

The substitution $=2A^{-1}\ \overline{\ }$ leads to an Abel equation of the form 1.3.1.33: $\ ' =\ +2BA^{-2}\ \ {}^{-2}$.

4. $=\ \ \overline{\ \ } +\ \ +\ \ .$

The substitution $=2a^{-1}\ \overline{\ }$ leads to the Abel equation $\ ' =\ +2a^{-2}(\ \ +\ \ )$, which is discussed in Subsection 1.3.1 (see Table 5).

5. $=\ \ +\ \ \overline{1-}\ .$

Solution: $\dfrac{}{a\ \ +\frac{1}{1-}\ \ +} = \ln|\ \ | + \ ,$ where $=\ \ \frac{1}{-1}.$

6. $=\ \ {}^s - B\ \ {}^k.$

The transformation $=(\ \ ')^1\ \ ,\ \ =\lambda(\ \ z)^1\ \ $, where $\lambda = (B\ \ A)^1$, leads to the generalized Emden–Fowler equation:

$$'' = -\frac{\lambda k}{B} z^{-\frac{1}{}}\ \ \overline{1-}\ (\ \ ')\overline{-1},$$

which is discussed in Section 2.5 (in the classification table, one should search for the equations satisfying the condition $\ +\ \ +1=0$).

7. $= (\quad + \quad + \quad) \; .$

This is a special case of equation 1.7.1.1 with $(\xi) = \xi$.

8. $= \quad {}^{- \; -} \quad + \quad .$

Solution:

$$\frac{}{- \lambda \quad + 1} + \quad = \frac{a}{} {}^{1} \quad \ln |\ | , \quad \text{where} \quad = \frac{a}{} {}^{1} \quad {}^{- \quad -1}, \; \lambda = \frac{+1}{} \; \frac{}{a} {}^{1} \; .$$

9. $= \quad {}^{-1} \quad {}^{+1} + \quad {}^{k-1} \quad {}^{k+1}.$

This is a generalized homogeneous equation of the form 1.7.1.3 with $(\xi) = a\xi + \ \xi$.

10. $= \quad {}^{k} \quad \overline{\ } + \quad + \quad {}^{s} \quad \overline{\ }.$

This is a special case of equation 1.7.1.4 with $(\ ) = a$, $g(\ ) =$, $(\ ) =$, and $= 1 \;\; 2$.

11. $= \quad {}^{k} \quad {}^{1+} \quad + \quad + \quad {}^{s} \quad {}^{1-} \quad .$

This is a special case of equation 1.7.1.4 with $(\ ) = a$, $g(\ ) =$, and $(\ ) =$.

12. $= \quad {}^{-1} \quad {}^{1-} \quad (\quad + \quad)^{k}.$

This is a special case of equation 1.7.1.7 with $(\xi) = \xi$.

$\boxed{\text{1.5.1-2. Other equations.}}$

13. $= \quad + \quad \sqrt{\quad {}^{2} + \quad {}^{2}}.$

The substitution $=$ leads to a separable equation: $' = (a-1) \quad + \quad \sqrt{\quad {}^{2} + \ }$.

14. $= \; + \quad {}^{-} \quad + \quad {}^{-k} \quad {}^{k}.$

The substitution $=$ leads to a separable equation: $' = \quad {}^{-2}(a \quad + \quad)$.

15. $(\quad + \quad) \quad = 1.$

Solution: $= e^{b} \quad + a \quad e^{-b}$.

16. $(\quad + \ ) \quad + \quad = 0.$

Solution: $- a = (\quad {}^{b} + \quad$.

17. $(\quad + \quad) \quad = [\quad {}^{(\lambda-1)} \quad {}^{\lambda} - \ ].$

This is a special case of equation 1.7.1.16 with $(\xi) = a\xi$, $g(\xi) = 1$, $(\xi) = \ \xi^{\lambda}$, and $k =$.

18. $(\quad + \quad {}^{2} + \quad) \quad = \quad + \quad + \quad {}^{2}.$

The transformation $=$, $z = \quad {}^{-2}$ leads to a linear equation with respect to $z = z(\ )$: $(k - a \ )z' = (\quad - 2)(az + \ + \ )$.

19. $(\quad + \quad {}^{2} + \quad) \quad = \quad + \quad + \quad {}^{2}.$

The transformation $=$, $z = \quad {}^{-2}$ leads to a linear equation with respect to $z = z(\ )$: $(k - a \ )z' = (\quad - 2)(a \quad z + \ + \ )$.

20. $(\quad + \quad + \ ) \quad = \quad {}^{k} \quad {}^{-k} + \quad {}^{-} \quad + \ .$

The transformation $=$, $z = \quad {}^{-1}$ leads to a linear equation: $(\quad {}^{-} + \beta \quad {}^{-} \quad - \quad {}^{+1} - a \ )z' = (\quad - 1)(\quad + a)z + \quad - 1$.

21. $(\quad + \quad + \quad ^{2} + B \quad) \quad = \quad ^{k} \quad ^{-k} + \quad \quad ^{-} \quad + \quad \quad + B \quad ^{2}.$

The transformation $\quad = \quad$, $z = \quad ^{-2}$ leads to a linear equation: $(\quad ^{-} + \beta \quad ^{-} \quad - \quad ^{+1} - a \quad) z' =$ $(\quad - 2)(\quad + a) z + (\quad - 2)(B \quad + A)$.

22. $[(\quad + \quad) \quad + \quad] \quad = (\quad + \quad) \quad - \quad .$

This is a special case of equation 1.7.1.14 with $(\xi) = \xi \quad$, $g(\xi) = 1$, and $(\xi) = \quad \xi \quad$.

23. $[(\quad + \quad) \quad + \quad] \quad = (\quad + \quad) \quad - \quad .$

This is a special case of equation 1.7.1.15 with $(\xi) = \xi \quad$, $g(\xi) = 1$, and $(\xi) = \quad \xi \quad$.

24. $(\quad + \quad + \gamma) \quad = (\quad + \quad + \quad) \quad .$

This is a special case of equation 1.7.1.6 with $(\xi) = \xi \quad$.

25. $(\quad + \quad) \quad = \quad ^{-1} \quad ^{1-} \quad .$

This is a special case of equation 1.7.1.7 with $(\xi) = 1 \quad \xi$.

26. $(\quad + \quad + s) \quad + \quad ^{k} + \quad \quad ^{-1} \quad + \quad = 0.$

Solution:

$$a \quad (\quad) + \quad (\quad) + \quad \quad + \quad + \beta \quad = \quad ,$$

where $(\quad) = \begin{cases} \dfrac{ \quad ^{+1}}{ \quad + 1} & \text{if } \quad \neq -1, \\ \ln | \quad | & \text{if } \quad = -1, \end{cases} \qquad (\quad) = \begin{cases} \dfrac{ \quad ^{+1}}{k + 1} & \text{if } k \neq -1, \\ \ln | \quad | & \text{if } k = -1. \end{cases}$

27. $(\quad ^{2} \quad + \quad \quad + \quad ^{k}) \quad = \quad \quad + \quad ^{q} + \gamma.$

This is a Riccati equation with respect to $\quad = \quad (\quad)$.

28. $(\quad \quad + \quad) \quad = \quad ^{k} \quad + \quad ^{-k} + \quad .$

The transformation $\quad = \quad$, $z = \quad ^{+} \quad ^{-1}$ leads to a linear equation: $(\quad ^{-} \quad - a \quad) z' =$ $(\quad + \quad - 1)(a \quad z + 1)$.

29. $(\quad \quad + \quad) \quad + (\quad \quad + \quad) = 0.$

Solution: $\dfrac{(\quad ^{b})}{A} + \dfrac{(\quad)}{B} = \quad$, where $A = \dfrac{\beta - }{a\beta - }$, $B = \dfrac{ - \quad a}{a\beta - }$.

30. $(\quad ^{k} \quad ^{+k} + s) \quad + (\quad \quad ^{+k} \quad ^{k} + s) = 0.$

Solution: $ak \quad + k \quad - (\quad)^{-} = \quad .$

31. $(\quad \quad + \quad ^{2} + B \quad) \quad = \quad ^{k} \quad + \quad ^{-k} + \quad \quad + B \quad ^{2}.$

The transformation $\quad = \quad$, $z = \quad ^{+} \quad ^{-2}$ leads to a linear equation: $(\quad ^{-} \quad - a \quad) z' =$ $(\quad + \quad - 2)(a \quad z + B \quad + A)$.

32. $(\quad \quad ^{-1} + \quad ^{k}) \quad + \quad \quad ^{-1} \quad + \quad ^{s} = 0.$

This is a special case of equation 1.7.1.19 with $(\quad) = \quad$ and $g(\quad) = \quad$.

33. $(\quad \quad + \quad ^{k}) \quad = \quad ^{s} + \quad .$

This is a Bernoulli equation with respect to $\quad = \quad (\quad)$ (see Subsection 1.1.5).

34. $(\quad ^{-k} \quad + \quad) \quad = (\quad ^{\lambda} \quad ^{-k} \quad ^{\lambda} \quad - \quad).$

This is a special case of equation 1.7.1.16 with $(\xi) = a\xi$, $g(\xi) = 1$, and $(\xi) = \quad \xi^{\lambda}$.

35. $(\quad {}^{-k} + \quad) \quad = \quad (\quad {}^{\lambda} \quad {}^{\lambda} \quad {}^{-k} - \quad).$

This is a special case of equation 1.7.1.17 with $(\xi) = a\xi$, $g(\xi) = 1$, and $(\xi) = \ \xi^{\lambda}$.

36. $(\quad {}^{+1} \quad {}^{-1} + \quad {}^{k+1} \quad {}^{k-1}) \quad = \quad {}^{s} \quad {}^{s}.$

This is a special case of equation 1.7.1.3 with $(\xi) = \ \xi\ (a\xi + \ \xi\ )^{-1}$.

37. $(\quad + \quad)^{k} \quad = \quad {}^{-1} \quad {}^{1-} \quad .$

This is a special case of equation 1.7.1.7 with $(\xi) = \ \xi^{-}\ $.

38. $$ {}_1\frac{+}{{}^{3}_{1}} + {}_2\frac{-}{{}^{3}_{2}}\Big) \quad - \quad \frac{1}{{}^{3}_{1}} + \frac{2}{{}^{3}_{2}}\Big) = 0,$$

where ${}^{2}_{1} = (\ + \ )^{2} + \ {}^{2}, \quad {}^{2}_{2} = (\ - \ )^{2} + \ {}^{2}.$

This is the equation of force lines corresponding to the Coulomb law in electricity.

Solution: $e_1 \frac{+a}{1} + e_2 \frac{-a}{2} = \quad .$

39. $\quad - \quad = (\quad {}^{k} + \quad {}^{k})(\quad + \quad).$

This is a special case of equation 1.7.1.24 with $(\ ) = \quad {}^{2}$ and $g(v, \ ) = av \ + \quad .$

40. $\quad - \quad = (\quad {}^{k} + \quad {}^{k})(\quad - \quad).$

This is a special case of equation 1.7.1.25 with $(\ ) = \quad {}^{2}$ and $g(v, \ ) = av \ + \quad .$

41. $\quad + \quad = (\quad {}^{k} + \quad {}^{k})(\quad - \quad).$

This is a special case of equation 1.7.1.24 with $(\ ) = \ {}^{-} \ {}^{2}$ and $g(v, \ ) = (av \ + \quad)^{-1}$.

1.5.2. Equations Containing Exponential Functions

1.5.2-1. Equations with exponential functions.

1. $\quad = \quad {}^{\lambda} \quad + \quad .$

Solution: $\quad = -\frac{1}{\lambda} \ln \quad e^{-b\lambda} \quad - \frac{a}{} \quad .$

2. $\quad = \quad + \quad .$

Solution: $\quad = \ e \quad - \ln \quad - a \quad \exp(\ e \ ) \quad .$

3. $\quad = \quad {}^{+} \quad - \quad .$

This is a special case of equation 1.7.1.2 with $(\xi) = Ae\ $, $\ = 1$, and $\ = 0$.

4. $\quad = \quad {}^{\nu} \ {}^{+\lambda} \quad + \quad .$

This is a special case of equation 1.7.2.5 with $(\ ) = ae \quad$ and $g(\ ) = \ e \quad .$

5. $\quad = \quad {}^{\nu} \ {}^{+\lambda} \quad + \quad {}^{-\lambda} \ .$

This is a special case of equation 1.7.2.8 with $(\ ) = ae \quad$, $g(\ ) = 0$, and $(\ ) = \ e \quad .$

6. $\quad = \quad {}^{2} \ {}^{-} \quad + \quad + \quad {}^{-} \quad .$

This is a special case of equation 1.7.2.9 with $(\xi) = \xi + \ .$

7. $\quad = \quad {}^{+\lambda} \quad + \quad + \quad {}^{-\lambda} \ .$

The substitution $\quad = e^{\lambda}\ $ leads to a Riccati equation: $\ ' = a\lambda e \quad {}^{2} + \ \lambda e \quad + \ \lambda e \quad .$

8. $(\quad + \quad) \quad = 1.$

Solution: $\quad = ae \quad - \ln \quad - \quad \exp(ae \quad) \quad .$

9. $(\quad + \;) \quad = \quad ^{+} \; - \quad .$

This is a special case of equation 1.7.2.13 with $(\xi) = \;$, $g(\xi) = 1$, and $(\xi) = e \;$.

10. $(\; ^{+} \; + \;) \quad = \; .$

The substitution $\;(\;) = \; + \beta$ leads to a separable equation: $\;' = \beta + \; (ae^{w} + \;)^{-1}$.

11. $(\quad + \quad) \quad = \quad ^{-} \; .$

This is a special case of equation 1.7.2.9 with $(\xi) = \xi^{-1}$.

12. $(\; ^{+} \; + \quad) \; + \; ^{\nu \; +} \; + \quad = 0.$

This is a special case of equation 1.7.2.15 with $(\;) = e \;$ and $g(\;) = \; e \;$.

1.5.2-2. Equations with power and exponential functions.

13. $\quad = \quad + \quad .$

1 . Solution in parametric form with $\; \neq -1$:

$$= \tau^{\frac{1}{\;+1}}, \qquad = \frac{}{\;+1}\tau - \ln \quad - \frac{a}{\;+1} \quad \tau^{-\frac{}{\;+1}} \exp \frac{\tau}{\;+1} \quad \tau \; .$$

2 . Solution in parametric form with $\; = -1$, $\; \neq -1$:

$$= e^{\tau}, \qquad = -\ln \quad e^{-b\tau} - \frac{a}{\;+1} e^{\tau} \quad .$$

3 . Solution in parametric form with $\; = -1$, $\; = -1$:

$$= e^{\tau}, \qquad = -\tau - \ln(\; - a\tau).$$

14. $\quad = \quad ^{-1} + \quad .$

Solution in parametric form:

$$= \ln(A \quad ^{-1}) \mp \tau^2, \qquad = B[2 \quad \exp(\mp\tau^2) \quad ^{-1}],$$

where $a = \mp 2B^2$, $\; = \; A^{-1}B$, $\; = \; \exp(\mp\tau^2) \; \tau + \;$.

15. $\quad = \quad ^{\nu \; + \lambda} \; + \quad .$

This is a special case of equation 1.7.2.5 with $(\;) = ae \;$ and $g(\;) = \quad$.

16. $\quad = \quad ^{\lambda} \; + \quad ^{\nu} \; .$

This is a special case of equation 1.7.2.5 with $(\;) = a \;$ and $g(\;) = \; e \;$.

17. $\quad = \quad ^{\lambda} \; + \quad .$

This is a special case of equation 1.7.2.5 with $(\;) = a \;$ and $g(\;) = \quad$.

18. $\quad = \quad ^{\lambda} \; + \quad ^{-\lambda} \; .$

This is a special case of equation 1.7.2.8 with $(\;) = a \;$, $g(\;) = 0$, and $(\;) = \quad$.

19. $= (+ {}^{\lambda} +) - {}^{\lambda}.$

This is a special case of equation 1.7.2.10 with $(\xi) = \ \xi$.

20. $= (+ {}^{-k})^{1\ k}.$

Solution in parametric form:

$$= \exp\ \tau - \frac{1}{k}\big[\ (\tau) + \ \big]\ , \qquad = \ (\tau) + \ , \qquad \text{where} \qquad (\tau) = \frac{k\ \ \tau}{k(\ + ae^{\ \tau})^{-1}\ \ + 1}.$$

21. $= ({}^{k} + \ )^{1\ k}.$

Solution in parametric form:

$$= \ (\tau) + \ , \qquad = \exp\ \tau + \frac{1}{k}\big[\ (\tau) + \ \big]\ , \qquad \text{where} \qquad (\tau) = \frac{k\ \ \tau}{k(a + \ e^{-\ \tau})^{1}\ \ - 1}.$$

22. $= \ {}^{-1\ \lambda} + \ {}^{-1\ \lambda}\ .$

This is a special case of equation 1.7.2.2 with $(\xi) = a\xi^{\ -1} + \ \xi^{\ -1}$.

23. $= \ {}^{-1} + \ {}^{-1}\ .$

This is a special case of equation 1.7.2.4 with $(\xi) = a\xi + \ \xi$.

24. $= {}^{\lambda}\ {}^{+1} + {}^{-\lambda}.$

This is a special case of equation 1.7.2.1 with $(\xi) = a\xi^{\ +1} + \ $.

25. $= \ {}^{+1} + \ {}^{+1}.$

This is a special case of equation 1.7.2.3 with $(\xi) = a\xi + \ \xi$.

26. $= {}^{\lambda}\ {}^{+1} + {}^{\lambda}\ {}^{+1}.$

This is a special case of equation 1.7.2.1 with $(\xi) = a\xi^{\ +1} + \ \xi^{\ +1}$.

27. $= \ {}^{k} + \ {}^{k+1} - \ .$

This is a special case of equation 1.7.2.7 with $(\ ) = \ $, $g(\xi) = a + \ \xi$, and $\ = 1$.

28. $= {}^{\lambda}\ {}^{1+} + \ + {}^{\nu}\ {}^{1-}.$

This is a special case of equation 1.7.1.4 with $(\ ) = ae^{\lambda}$, $g(\ ) = \ e$, and $(\ ) = \ e$.

29. $= {}^{\lambda}\ {}^{1+} + \ + \ {}^{1-}.$

This is a special case of equation 1.7.1.4 with $(\ ) = ae^{\lambda}$, $g(\ ) = \ e$, and $(\ ) = \ $.

30. $= {}^{k}\ {}^{1+} + {}^{\lambda} + \ {}^{1-}.$

This is a special case of equation 1.7.1.4 with $(\ ) = a$, $g(\ ) = \ e^{\lambda}$, and $(\ ) = \ $.

31. $= {}^{\lambda}\ {}^{1+} + \ + \ {}^{1-}.$

This is a special case of equation 1.7.1.4 with $(\ ) = ae^{\lambda}$, $g(\ ) = \ $, and $(\ ) = \ e$.

32. $= {}^{\lambda}\ {}^{1+} + \ + {}^{k}\ {}^{1-}.$

This is a special case of equation 1.7.1.4 with $(\ ) = ae^{\lambda}$, $g(\ ) = \ $, and $(\ ) = \ $.

33. $= \ {}^{+k} + \ {}^{+k} - \ .$

This is a special case of equation 1.7.2.6 with $(\ ) = \ {}^{-1}$ and $g(\xi) = a\xi + \ \xi$.

34. **(+) = $^{+}$ − .**

This is a special case of equation 1.7.1.15 with $(\xi) = \lambda$, $g(\xi) = 1$, and $(\xi) = \ e \ $.

35. **= − .**

This is a special case of equation 1.7.2.11 with $(\xi) = a\xi$ and $= 1$.

36. **^{2} = − ^{2}.**

This is a special case of equation 1.7.2.12 with $(\xi) = a\xi$ and $= 1$.

37. **(+) = 1.**

1 . Solution in parametric form with $\neq -1$:

$$= \frac{a}{+1}\tau - \ln \quad - \frac{}{+1} \quad \tau^{-\frac{}{+1}} \exp \frac{a\tau}{+1} \quad \tau \ , \qquad = \tau^{\frac{1}{+1}}.$$

2 . Solution in parametric form with $= -1$, $a \neq -1$:

$$= -\ln \quad e^{-\ \tau} - \frac{}{a+1} e^{\tau} \ , \qquad = e^{\tau}.$$

3 . Solution in parametric form with $= -1$, $a = -1$:

$$= -\tau - \ln(\ - \ \tau), \qquad = e^{\tau}.$$

38. **(+) = 1.**

Solution in implicit form: $= \begin{cases} e^{b} \ + \frac{a}{1-} e & \text{if } \neq 1, \\ e \ (\ + a \ ) & \text{if } = 1. \end{cases}$

39. **(+ ^{2}) = 1.**

Solutions in parametric form:

$$= -\frac{1}{2}\tau(\ln Z)'_{\tau}, \qquad = \ln \ \frac{\tau^2}{4a} \ , \qquad Z = \ _1 \ _0(\tau) + \ _2 \ _0(\tau)$$

and

$$= -\frac{1}{2}\tau(\ln Z)'_{\tau}, \qquad = \ln \ -\frac{\tau^2}{4a} \ , \qquad Z = \ _1 \ _0(\tau) + \ _2 \ _0(\tau),$$

where $_0(\tau)$ and $_0(\tau)$ are the Bessel functions, and $_0(\tau)$ and $_0(\tau)$ are the modified Bessel functions.

40. **(+ $^{-1}$) = 1.**

Let $a = \ A \ B$, $= \mp 2A^2$. Solution in parametric form:

$$= A\big[2\tau \ \exp(\mp\tau^2) \ (\tau)\big], \qquad = \ln\big[B \ (\tau)\big] \mp \tau^2, \qquad \text{where} \quad (\tau) = \quad \exp(\mp\tau^2) \ \tau + \quad ^{-1}.$$

41. **($^{+}$ +) = $^{+}$ − .**

This is a special case of equation 1.7.1.14 with $(\xi) = e$, $g(\xi) = 1$, and $(\xi) = \ e \ $.

42. **($^{+}$ +) = $^{+}$ − .**

This is a special case of equation 1.7.1.15 with $(\xi) = e$, $g(\xi) = 1$, and $(\xi) = \ e \ $.

43. **(+) = .**

This is a special case of equation 1.7.2.3 with $(\xi) = (a\xi + \ )^{-1}$.

44. (+) + $^{\nu\ +}$ + **= 0.**

This is a special case of equation 1.7.2.15 with ( ) = and g( ) = e .

45. (+) + + **= 0.**

This is a special case of equation 1.7.2.15 with ( ) = and g( ) = .

46. (+) = (−).

This is a special case of equation 1.7.2.17 with (ξ) = ξ, $g(\xi) = 1$, and (ξ) = ξ .

47. (+) = − .

This is a special case of equation 1.7.2.16 with (ξ) = ξ, $g(\xi) = 1$, and (ξ) = ξ .

48. ($^{\lambda}$ +) = $^{\nu}$.

This is a Bernoulli equation with respect to = ( ) (see Subsection 1.1.5).

49. ($^{\lambda}$ +) = .

This is a Bernoulli equation with respect to = ( ).

50. (+ $^{\lambda}$) = ^{k}.

This is a Bernoulli equation with respect to = ( ).

51. (+ ^{k}) = $^{\lambda}$.

This is a Bernoulli equation with respect to = ( ).

52. ($^{-1}$ +) + $^{-1}$ + $^{\lambda}$ **= 0.**

This is a special case of equation 1.7.1.19 with ( ) = and g( ) = e^{λ} .

53. ($^{-1}$ + $^{\lambda}$) + $^{-1}$ + **= 0.**

This is a special case of equation 1.7.1.19 with ( ) = e^{λ} and g( ) = .

54. ($^{-1}$ + ^{k}) + $^{-1}$ + $^{\lambda}$ **= 0.**

This is a special case of equation 1.7.1.19 with ( ) = and g( ) = e^{λ} .

55. [(+) +] = (+) − .

This is a special case of equation 1.7.2.14 with (ξ) = ξ , g( ) = 1, and (ξ) = ξ .

56. [(+) +] = (+) − .

This is a special case of equation 1.7.2.13 with (ξ) = ξ , g( ) = 1, and (ξ) = ξ .

1.5.3. Equations Containing Hyperbolic Functions

1. = **cosh**( ) + **cosh**( ).

This is a special case of equation 1.7.2.18 with ( ) = 0, g( ) = a, and ( ) = cosh( ).

2. = **sinh**( ) + **sinh**( ).

This is a special case of equation 1.7.2.18 with ( ) = a, g( ) = 0, and ( ) = sinh( ).

3. = **cosh**( ) + .

This is a special case of equation 1.7.2.18 with ( ) = 0, g( ) = a , and ( ) = .

4. = **sinh(** **) +** **.**

This is a special case of equation 1.7.2.18 with ( ) = a , g( ) = 0, and ( ) = .

5. = $^{1+}$ **+** **+ sinh(** **)** $^{1-}$ **.**

This is a special case of equation 1.7.1.4 with ( ) = a, g( ) = , and ( ) = sinh(λ).

6. = $^{1+}$ **+ sinh(** **) +** $^{1-}$ **.**

This is a special case of equation 1.7.1.4 with ( ) = a, g( ) = sinh(λ), and ( ) = .

7. = **cosh** **(** **sinh** $^{-1}$ **+** **).**

This is a special case of equation 1.7.2.22 with (ξ) = $a\xi$ + ξ.

8. = **sinh** **(** **cosh** $^{-1}$ **+** **).**

This is a special case of equation 1.7.2.24 with (ξ) = $a\xi$ + ξ.

9. **= (** **cosh** **+** **) coth** **.**

This is a special case of equation 1.7.2.25 with (ξ) = $a\xi$ + .

10. **= (** **sinh** **+** **) tanh** **.**

This is a special case of equation 1.7.2.23 with (ξ) = $a\xi$ + .

11. **(** **cosh** **+** **)** **=** $^{+1}$ **sinh** **.**

This is a special case of equation 1.7.2.24 with (ξ) = $\xi(a\xi +\ )^{-1}$.

12. **(** **sinh** **+** **)** **=** $^{+1}$ **cosh** **.**

This is a special case of equation 1.7.2.22 with (ξ) = $\xi(a\xi +\ )^{-1}$.

13. **(** **+** **cosh** **)** **=** ^{k}**.**

This is a Bernoulli equation with respect to = ( ) (see Subsection 1.1.5).

14. **(** **+** **tanh** **)** **=** ^{k}**.**

This is a Bernoulli equation with respect to = ( ).

15. **(** **+** **cosh** **)** **= cosh**^{k}**(** **).**

This is a Bernoulli equation with respect to = ( ).

16. **(** **+** **tanh** **)** **= tanh**^{k}**(** **).**

This is a Bernoulli equation with respect to = ( ).

17. **(** $^{-1}$ **+** **)** **+** $^{-1}$ **+** **sinh**^{k}**(** **) = 0.**

This is a special case of equation 1.7.1.19 with ( ) = and g( ) = sinh (λ).

18. **(** $^{-1}$ **+** **)** **+** $^{-1}$ **+** **tanh**^{k}**(** **) = 0.**

This is a special case of equation 1.7.1.19 with ( ) = and g( ) = tanh (λ).

19. **(** **+** **)** **= cosh**^{k}**(** **).**

This is a Bernoulli equation with respect to = ( ).

20. **(** **+** **)** **= tanh**^{k}**(** **).**

This is a Bernoulli equation with respect to = ( ).

21. **(cosh +) = sinh^{k}( ).**

This is a Bernoulli equation with respect to = ( ).

22. **(tanh +) = ^{k}.**

This is a Bernoulli equation with respect to = ( ).

23. **($^{-1}$ + sinh^{k}) + $^{-1}$ + = 0.**

This is a special case of equation 1.7.1.19 with ( ) = sinh and $g($ $)$ = .

24. **($^{-1}$ + tanh^{k}) + $^{-1}$ + = 0.**

This is a special case of equation 1.7.1.19 with ( ) = tanh and $g($ $)$ = .

1.5.4. Equations Containing Logarithmic Functions

1. **= (+ ln +).**

This is a special case of equation 1.7.2.3 with $(\xi) = \ln\xi + \beta$.

2. **= ^{k} $^{-1}$ ^{k} $^{+1}$(ln + ln).**

This is a special case of equation 1.7.1.3 with $(\xi) = a\xi$ $\ln\xi$.

3. **= ln^{2} + ln + ^{k} .**

This is a special case of equation 1.7.3.1 with ( ) = a , $g($ $)$ = , and ( ) = .

4. **= (+ ln) + .**

This is a special case of equation 1.7.2.4 with $(\xi) = \ln$ $\xi + \beta$.

5. **= (ln + ln).**

This is a special case of equation 1.7.1.3 with $(\xi) = \ln\xi$.

6. **= ^{s} ^{k}(ln + ln) − .**

This is a special case of equation 1.7.1.5 with ( ) = $\frac{a}{\ }$ $^{-1}$ and $g(\xi) = \ln\xi$.

7. **(+) = $^{-1}$ + (ln − ln).**

This is a special case of equation 1.7.1.12 with (ξ) = , $g(\xi)$ = $\ln\xi$, and $(\xi) = 1$.

8. **(+) = ln + (−) .**

This is a special case of equation 1.7.2.16 with $(\xi) = \beta$, $g(\xi) = 1$, and $(\xi) = \ln\xi$.

9. **(+ ^{k}) = (ln + ln − ^{k}).**

This is a special case of equation 1.7.1.16 with $(\xi) = a$, $g(\xi) = 1$, and (ξ) = $\ln\xi$.

10. **(+ ^{k}) = (ln + ln − ^{k}).**

This is a special case of equation 1.7.1.17 with $(\xi) = a$, $g(\xi) = 1$, and (ξ) = $\ln\xi$.

11. **($^{-1}$ +) + $^{-1}$ + ln^{k}( ) = 0.**

This is a special case of equation 1.7.1.19 with ( ) = and $g($ $)$ = ln (λ).

12. **(ln +) = 1.**

Solution: $= e^{b}\ \frac{a}{\ }\ \frac{e^{-b}}{\ }\ + \Big) - \frac{a}{\ }\ln$.

13. **(ln) = (^{k} ^{k} +) − ln .**

This is a special case of equation 1.7.3.7 with $(\xi) = a\xi \;\; + \;\; \xi$ and $= 1$.

14. **(+ ln) = (− ln +).**

This is a special case of equation 1.7.3.9 with $(\xi) = a$, $g(\xi) = 1$, and $(\xi) = \;\xi + \;$.

15. **(+ ln) = ln^{k}( ).**

This is a Bernoulli equation with respect to $= \;(\;)$.

16. **(+ ln) = (^{k} ^{k} − ln).**

This is a special case of equation 1.7.3.10 with $(\xi) = a\xi$, $g(\xi) = 1$, and $(\xi) = \;\xi\;$.

17. **(+ ln) = (^{k} ^{k} − ln).**

This is a special case of equation 1.7.3.9 with $(\xi) = a\xi$, $g(\xi) = 1$, and $(\xi) = \;\xi\;$.

18. **($^{-1}$ + ln^{k}) + $^{-1}$ + = 0.**

This is a special case of equation 1.7.1.19 with $(\;) = \;\ln\;$ and $g(\;) = \;$.

19. **(ln +) = ln^{k}( ).**

This is a Bernoulli equation with respect to $= \;(\;)$.

20. **(ln + ln^{k}) = ^{s}.**

This is a Bernoulli equation with respect to $= \;(\;)$.

1.5.5. Equations Containing Trigonometric Functions

1. **= cos( ) + cos( ).**

This is a special case of equation 1.7.4.11 with $(\;) = \;$, $g(\;) = 0$, and $(\;) = \beta\cos(\;)$.

2. **= sin( ) cos( ) + cos( ) sin( ).**

This is a special case of equation 1.7.1.1 with $(\xi) = \sin\xi$ and $= 0$.

3. **= tan( ).**

The solution is given by the relation:

$$\int_0^w \exp\bigl(\tfrac{1}{2}\;^2\;\cos(\;\overline{a}\;\; = \;\exp\bigl(\tfrac{1}{2}a\;\;^2, \qquad \text{where} \qquad = \sqrt{\frac{}{a}}.$$

4. **= cos( ) + .**

This is a special case of equation 1.7.4.11 with $(\;) = \;$, $g(\;) = 0$, and $(\;) = \;$.

5. **= sin( ) + .**

This is a special case of equation 1.7.4.11 with $(\;) = 0$, $g(\;) = \;$, and $(\;) = \;$.

6. **= cos (sin $^{-1}$ +).**

This is a special case of equation 1.7.4.4 with $(\xi) = a\xi \;\; + \;\; \xi$.

7. **= sin (cos $^{-1}$ +).**

This is a special case of equation 1.7.4.3 with $(\xi) = a\xi \;\; + \;\; \xi$.

8. $= \dfrac{\sin^2}{\cos^2} + \dfrac{\cos^2}{\sin^2}.$

This is a special case of equation 1.7.4.14 with $(\xi) = a\xi + \xi^{-1}$.

9. $= {}^{1+} + + \sin(\ ) {}^{1-}.$

This is a special case of equation 1.7.1.4 with $(\ ) = a$, $g(\ ) =$, and $(\ ) = \sin(\lambda \ )$.

10. $= {}^{1+} + \sin(\ ) + {}^{1-}.$

This is a special case of equation 1.7.1.4 with $(\ ) = a$, $g(\ ) = \sin(\lambda \ )$, and $(\ ) =$.

11. $+ \sin(\ + \ ) = 0.$

The substitution $= \tan \dfrac{+}{2}$ leads to a Riccati equation of the form 1.2.2.35 with $= 2$: $2 \ ' - {}^2 + 2(a \ - 1) \ - {}^2 = 0$.

12. $= {}^2 \tan(\ ) + .$

The substitution $=$ leads to an equation of the form 1.5.5.3: $' = a\tan(\ )$.

13. $= \cos^2 + \cos \ \sin .$

This is a special case of equation 1.7.4.8 with $(\xi) = \frac{1}{2}(a\xi + \ )$.

14. $= \sin^2 + \cos \ \sin .$

This is a special case of equation 1.7.4.7 with $(\xi) = \frac{1}{2}(a\xi + \ )$.

15. $= \sin^k \ \cos^{2-k} - \sin 2 .$

This is a special case of equation 1.7.4.18 with $(\ ) = a \ {}^{-2} \ {}^{-1}$ and $g(\xi) = \xi$.

16. $(1 + \tan^2 \ ) = \tan \ {}^{+1} + \tan + \tan^{1-} .$

This is a special case of equation 1.7.4.19 with $(\ ) = a$, $g(\ ) =$, and $(\ ) =$.

17. $(\ {}^{-1} + \ ) + \ {}^{-1} + \sin^k(\ ) = 0.$

This is a special case of equation 1.7.1.19 with $(\ ) =$ and $g(\ ) = \sin \ (\lambda \ )$.

18. $(\ {}^{-1} + \ ) + \ {}^{-1} + \tan^k(\ ) = 0.$

This is a special case of equation 1.7.1.19 with $(\ ) =$ and $g(\ ) = \tan \ (\lambda \ )$.

19. $(\ + \ ) = \cos^k(\ ).$

This is a Bernoulli equation with respect to $= (\ )$ (see Subsection 1.1.5).

20. $(\ + \ ) = \tan^k(\ ).$

This is a Bernoulli equation with respect to $= (\ )$.

21. $(\ \cos + \ ) = {}^{+1} \sin .$

This is a special case of equation 1.7.4.3 with $(\xi) = \xi(a\xi + \ )^{-1}$.

22. $(\ \sin + \ ) = {}^{+1} \cos .$

This is a special case of equation 1.7.4.4 with $(\xi) = \xi(a\xi + \ )^{-1}$.

23. $(\ + \ \cos \ ) = {}^k.$

This is a Bernoulli equation with respect to $= (\ )$.

24. (+ cos) = cos^k( ).

This is a Bernoulli equation with respect to = ( ).

25. ($^{-1}$ + cos^k) + $^{-1}$ + = 0.

This is a special case of equation 1.7.1.19 with ( ) = cos and g( ) = .

26. (cos +) = cos^k( ).

This is a Bernoulli equation with respect to = ( ).

27. (+ tan) = ^{k}.

This is a Bernoulli equation with respect to = ( ).

28. (+ tan) = tan^k( ).

This is a Bernoulli equation with respect to = ( ).

29. ($^{-1}$ + tan^k) + $^{-1}$ + = 0.

This is a special case of equation 1.7.1.19 with ( ) = tan and g( ) = .

30. (tan +) = tan^k( ).

This is a Bernoulli equation with respect to = ( ).

1.5.6. Equations Containing Combinations of Exponential, Hyperbolic, Logarithmic, and Trigonometric Functions

1. = $^\lambda$ + ln .

This is a special case of equation 1.7.2.5 with ( ) = a and g( ) = ln .

2. = ln ( ) $^\lambda$ + .

This is a special case of equation 1.7.2.5 with ( ) = a ln ( ) and g( ) = .

3. = $^\lambda$ (+ ln) .

This is a special case of equation 1.7.2.2 with (ξ) = a ln ξ.

4. = $^{-\lambda}$ (+ ln) .

This is a special case of equation 1.7.2.1 with (ξ) = a ln ξ.

5. = ln^2 + ln + $^\lambda$.

This is a special case of equation 1.7.3.1 with ( ) = a, g( ) = , and ( ) = e^λ .

6. = ln^2 + $^\lambda$ ln + .

This is a special case of equation 1.7.3.1 with ( ) = a, g( ) = e^λ , and ( ) = .

7. = sin + tan .

This is a special case of equation 1.7.5.6 with (ξ) = $a\xi$ + .

8. = (sin +) tan .

This is a special case of equation 1.7.5.4 with (ξ) = $a\xi$ + .

9. = sin^2 + $^-$ cos^2 .

This is a special case of equation 1.7.5.8 with (ξ) = $\frac{1}{2}$($a\xi$ + ξ).

10. $= \cos(\ )^{\lambda} + \ .$

This is a special case of equation 1.7.2.5 with $(\ ) = a\cos(\ )$ and $g(\ ) = \ .$

11. $= \ ^{\lambda} + \cos(\ ).$

This is a special case of equation 1.7.2.5 with $(\ ) = a$ and $g(\ ) = \cos(\ ).$

12. $= \ ^{\lambda} + \tan(\ ).$

This is a special case of equation 1.7.2.5 with $(\ ) = a$ and $g(\ ) = \tan(\ ).$

13. $= \tan(\ )^{\lambda} + \ .$

This is a special case of equation 1.7.2.5 with $(\ ) = a\tan(\ )$ and $g(\ ) = \ .$

14. $= \ ^{\lambda}\cos(\ ) + B\sin(\ ) + \ ^{\lambda}.$

The substitution $= \tan(\frac{1}{2}a\ )$ leads to a linear equation: $' = aBe + aAe^{\lambda}.$

15. $= \sin(\ )\sinh(\ ) + \cos(\ )\cosh(\ ).$

This is a special case of equation 1.7.2.18 with $(\ ) = a\sin(\ )$, $g(\ ) = \cos(\ )$, and $(\ ) = 0.$

16. $= \ln^2 + \ln + \sin(\ ).$

This is a special case of equation 1.7.3.1 with $(\ ) = a$, $g(\ ) = \ $, and $(\ ) = \sin(\lambda\ ).$

17. $(1 + \tan^2\ ) = \tan^{1+} + \tan + \ ^{\lambda}\tan^{1-}\ .$

This is a special case of equation 1.7.4.19 with $(\ ) = a$, $g(\ ) = \ $, and $(\ ) = e^{\lambda}.$

18. $(\ \cos + \ ) = \cot\ .$

This is a special case of equation 1.7.5.5 with $(\xi) = (a\xi + \ )^{-1}.$

19. $(\ \sin + \ ) = \tan\ .$

This is a special case of equation 1.7.5.4 with $(\xi) = (a\xi + \ )^{-1}.$

20. $(\ \cos + \ ) = \tan\ .$

This is a special case of equation 1.7.5.6 with $(\xi) = (a\xi + \ )^{-1}.$

21. $(\ \sin + \ ) = \cot\ .$

This is a special case of equation 1.7.5.7 with $(\xi) = (a\xi + \ )^{-1}.$

22. $(\ + \ ) + \ln + \ = 0.$

This is a special case of equation 1.7.2.15 with $(\ ) = \ $ and $g(\ ) = \ln\ .$

23. $(\ + \ ) + \cos + \ = 0.$

This is a special case of equation 1.7.2.15 with $(\ ) = \ $ and $g(\ ) = \cos\ .$

24. $(\ \cos + \ ) + \cos(\ ) + \ = 0.$

This is a special case of equation 1.7.2.15 with $(\ ) = \cos$ and $g(\ ) = \cos(\lambda\ ).$

1.6. Equations of the Form $\;(\;,y,y\;)=0$ Containing Arbitrary Parameters

1.6.1. Equations of the Second Degree in

1.6.1-1. Equations of the form $\;(\;,\;)(\;')^2 = g(\;,\;)$.

1. $(\quad)^2 = \quad + \quad ^2$.

See equation 1.6.3.43.

2. $(\quad)^2 = \quad + \quad ^2 + \quad + \;$.

The substitution $\quad = 2\sqrt{\quad + a\quad ^2 + \quad + \quad}$ leads to an Abel equation of the form 1.3.1.2: $\;' - \quad = 4a\quad + 2\;$.

3. $(\quad)^2 = \quad ^3 + \quad + \;$.

Solution: $\quad = \quad (a\quad ^3 + \quad + \;)^{-1\;2}\quad$.

4. $(\quad)^2 = \quad + \quad \overline{\quad}$.

See equation 1.6.3.26.

5. $(\quad)^2 = \quad + \quad \overline{\quad} + \;,\qquad \neq 0$.

The substitution $a\quad = 2\sqrt{a\;+\quad\overline{\quad}+\quad}$ leads to an Abel equation of the form 1.3.1.32: $\;' - \quad = \quad a^{-2}\quad ^{-1\;2}$.

6. $(\quad)^2 = \quad + \quad ^{+1} - \dfrac{+1}{2(\quad+3)^2}\quad ^2 + \;$.

The substitution $\quad = 2\Big(\quad + a\quad ^{+1} - \dfrac{+1}{2(\quad+3)^2}\quad ^2 + \quad\Big)^{1\;2}$ leads to an Abel equation of the form 1.3.1.10: $\;' - \quad = -\dfrac{2(\quad+1)}{(\quad+3)^2}\quad + 2a(\quad+1)\quad$.

7. $(\quad)^2 = \quad + \quad ^2 + \quad ^{+1} + \;$.

For $\lambda \neq 0$, the substitution $\lambda\quad = 2(\lambda\quad + a\quad ^2 + \quad ^{+1} + \;)^{1\;2}$ leads to the Abel equation $\;' - \quad = 4a\lambda^{-2}\quad + 2\;\lambda^{-2}(\quad+1)\quad$, which is outlined in Subsection 1.3.1 (see Table 5).

Special cases of the original equation are equations 1.6.1.1–1.6.1.6.

8. $(\quad)^2 = \quad + \;$.

See equation 1.6.3.32.

9. $(\quad)^2 = \quad + \quad + \;,\qquad \neq 0$.

The substitution $a\quad = 2\sqrt{a\;+\quad+\quad ^{-1}}$ leads to an Abel equation of the form 1.3.1.33: $\;' - \quad = -2\;a^{-2}\quad ^{-2}$.

10. $^2(\quad)^2 = \quad ^2\quad ^2 + \;$.

See equation 1.6.3.34.

11. $^2(\quad)^2 = \quad ^{-2}\quad ^5\quad ^2 + \;$.

See equation 1.6.3.28.

12. $(\ \ ^2 + \ \ )(\ \ )^2 = 1.$

See equation 1.6.3.44.

13. $(\ \ ^2 + \ \ )(\ \ )^2 = \ \ ^2\ \ .$

See equation 1.6.3.46.

14. $(\ \ \ \ + \ )(\ \ )^2 = \ \ .$

See equation 1.6.3.33.

15. $\ \ ^2(\ \ )^2 = \ \ ^2 + \ \ .$

See equation 1.6.3.45.

16. $(\ \ ^2\ ^2 + \ )(\ \ )^2 = \ ^2.$

See equation 1.6.3.35.

17. $(\ \ \ \overline{\ } + \ \ )(\ \ )^2 = 1.$

See equation 1.6.3.27.

18. $(\ \ ^2\ ^3\ ^5 + \ \ )(\ \ )^2 = \ \ ^2\ \ .$

See equation 1.6.3.29.

19. $(\ \ )^2 = \ \ \ \ + \ .$

See equation 1.6.3.3 with $k = 2$.

20. $(\ \ )^2 = \ \ + \ \ \ .$

See equation 1.6.3.4 with $k = 2$.

21. $(\ \ )^2 = \ \ ^2 + \ \ \ .$

See equation 1.6.3.8 with $k = 2$.

22. $\ ^2(\ \ )^2 = \ \ ^2\ \ + \ .$

See equation 1.6.3.9 with $k = 2$.

23. $(\ \ \ + \ \ ^2)(\ \ )^2 = 1.$

See equation 1.6.3.9 with $k = -2$.

24. $(\ \ \ \ ^2 + \ )(\ \ )^2 = \ ^2.$

See equation 1.6.3.8 with $k = -2$.

25. $(\ \ )^2 = \ \ \ + \ \ln\ \ .$

See equation 1.6.3.13.

26. $(\ \ )^2 = \ \ \ + \ \ln\ \ + \ \ ,\ \ \ \ \ \ \neq 0.$

The substitution $\lambda\ \ = 2\ \overline{\lambda\ \ + a\ln\ \ +\ \ }$ leads to an Abel equation of the form 1.3.1.16: $\ \ '\ -\ \ = 2a\lambda^{-2}\ ^{-1}.$

27. $(\ \ln\ \ + \ \ )(\ \ )^2 = 1.$

See equation 1.6.3.14.

1.6.1-2. Equations of the form $\ (\ ,\ )(\ ')^2 = g(\ ,\ )\ ' +\ (\ ,\ ).$

28. $(\quad)^2 + \quad + \quad = 0.$

Solution in parametric form:

$$= -2 \ - a \ln \ + \ , \qquad = - \ ^2 - a \ .$$

29. $(\quad)^2 + \quad = \quad + \ .$

We differentiate the equation with respect to , take as the independent variable, and assume $\xi = \ '$ to obtain a linear equation with respect to $= (\xi)$: $(a\xi^2 - \ )\ ' + a\xi \ + 2\xi^2 = 0$.

30. $(\quad)^2 + \quad + \quad + \ ^2 = 0.$

The transformation $= e\ ,\ = \ ^2$ leads to an autonomous equation: $\left(\ ' + 2 \ + \tfrac{1}{2}a\ \right)^2 = \tfrac{1}{4}a^2 - \ - \ $. Having extracted the root and carried over the terms $2 \ + \tfrac{1}{2}a$ from the left-hand side to the right-hand side, we obtain a separable equation of the form 1.1.2.

31. $= \quad + \ ^2 + (\quad)^2 + \quad + \ , \qquad \neq 0.$

Differentiating with respect to and changing to new variables $= \ '$ and $(\ ) = -2a\ $, we arrive at an Abel equation of the form 1.3.1.2: $\ ' - \ = -4a \ - 2a\ $.

32. $(\quad)^2 + (\quad + \ )\quad - \quad + \ = 0, \qquad \neq 0.$

Solutions: $= (a \ + \ )\ + a \ ^2 + \ a^{-1}$ and $4a \ = 4 \ - (a \ + \ )^2$.

33. $(\quad)^2 + (\quad + \quad)\quad + \quad = 0.$

This equation can be factorized: $(\ ' + a \ )(\ ' + \ ) = 0$. Therefore, the solutions are: $= \ e^{-}$ and $= -\tfrac{1}{2}\ ^2 + \ $.

34. $(\quad)^2 + \ ^2 \ + \quad = 0.$

The transformation $z = \ln\ ,\ = \ ^{-3}$ leads to an equation independent implicitly of z: $(\ ')^2 + (a + 6\ )\ ' + (3a + \ + 9\ )\ = 0$. Rewriting the latter equation to solve for $\ '$, we obtain a separable equation of the form 1.1.2.

35. $(\quad)^2 - \quad - \quad = 0.$

Solution in parametric form:

$$= \frac{}{\sqrt{\ ^2 + 1}}\left[\ + a \ln\left(\ + \sqrt{\ ^2 + 1}\ \right)\right], \qquad = a \ - \frac{}{}.$$

36. $(\quad)^2 - \quad + \quad = 0.$

1 . For $a \neq 1$, the solution in parametric form is written as:

$$= \ + \frac{}{2a - 1}\ ^2, \quad a \ = \ ^2 + \ , \qquad \text{where} \quad k = \frac{1}{a - 1}.$$

2 . For $a = 1$, the solution is: $= \ + \ $.

37. $(\quad)^2 + \quad + \quad = 0.$

1 . For $a \neq -1$, the solution in parametric form is written as:

$$= \ |(a+1)\ ^2 + \ |^{-\frac{+2}{2(\ +1)}}, \qquad = -\frac{}{a}(\ ^2 + \ ).$$

There are two singular solutions: $= \ \sqrt{- \ (a+1)}$.

2 . For $a=-1$, the solution in parametric form is written as:

$$= \quad \exp\, -\frac{\ \ ^2}{2}\ , \qquad = \qquad + - \ .$$

38. $(\quad)^2 - \qquad + \qquad = 0.$

Solution in parametric form:

$$= \ (\ \ - a)\exp(-\ \ a), \qquad = \ \ ^2 \exp(-\ \ a).$$

There is a singular solution: $\quad = 0.$

39. $(\quad)^2 - \qquad + \quad ^2 \quad + \qquad + \quad = 0, \qquad \neq 0.$

We divide the equation by $\ '$ and differentiate with respect to . Passing to the new variables $=\ '$ and $(\ ) = -2a$, we arrive at an Abel equation of the form 1.3.1.33: $\ ' - \quad = a \quad ^{-2}.$

40. $(\quad)^2 + \qquad + \qquad = 0.$

Solution in parametric form:

$$a \quad + \ (\ + \ ^2) = 0, \qquad (\ ^2 + a + \ \ ) \ = \ ^{b\ (\ +b)}, \qquad \text{where} \qquad = \frac{a+2}{2(a+\ \ )}.$$

There two singular solutions $\quad = \qquad \overline{-a-}$ corresponding to the limit . In addition, $= 0$ is also a singular solution.

41. $(\quad)^2 + (\quad - \quad) \quad + \quad = 0.$

Solutions: $(\quad - \quad + a) + \quad = 0$ and $(\quad - a)^2 = 4 \quad .$

42. $(\quad)^2 + (\quad - \quad + \ ) \quad - \quad = 0.$

Solution: $\quad = \qquad + \dfrac{k}{a\quad +}$. In addition, there is a singular solution which can be written in parametric form as:

$$= -\frac{k}{(a\ \ +\ \ )^2}, \qquad = \qquad + \frac{k}{a\ \ +}.$$

43. $(\quad)^2 - (\quad + \quad - \quad - \ ) \quad + \quad = 0.$

Differentiating with respect to and factorizing, we obtain $(2a \quad ' - a \quad - \quad + a + \ \ )\ \ '' \ \ = 0.$ Equating both factors to zero and integrating, we arrive at the solutions:

$$= \qquad + \frac{(a+\ \ )}{a\quad -} \quad \text{and} \quad (a \quad + \quad - a - \ \ )^2 - 4a \qquad = 0.$$

44. $(\quad)^2 + \qquad + \qquad = 0.$

The substitution $\ = e \ $ leads to an equation of the form 1.6.1.68: $(\ ')^2 + a \quad ' + \ e^{(\ \ +1)} \qquad = 0.$

45. $\ ^2(\quad)^2 - (2 \quad + \ ) \quad + \ ^2 = 0.$

Solutions: $\quad = a \quad ^2 \quad + a$ and $\quad = -\frac{1}{4} a \quad ^{-1}.$

46. $\ ^2(\quad)^2 - 2 \qquad + \ ^2 - \ (\quad - 1) \ ^2 = 0.$

Solutions: $\overline{\ ^2 + a \ ^2} = \quad ^{1+}$, where $k = \quad \overline{(a-1)\ \ a}.$

47. $\mathbf{(\ ^2-1)\ ^2(\ \ )^2+2\qquad -\ ^2+\ \ ^2\ \ ^2=0.}$

Solution in parametric form:

$$=\ (\ ^2+1)^{-1}\ ^2\big(\ +\ \ \overline{\ ^2+1}\ \ ^{-1}\ ,\qquad =\ \ +a\ \ \ \overline{\ ^2+1}.$$

48. $\mathbf{\ ^2(\ \ )^2+(\ \ \ ^2\ \ ^3+\ )\ \ +\ \ \ ^3=0.}$

The equation can be factorized: $(\ \ '+a\ \ ^3)(\ ^2\ '+\ \ )=0$. Equating each of the factors to zero, we obtain the solutions: $\ =\ (2a\ \ +\ \ )^{-1}\ ^2$ and $\ =\ \ \ +\ \ .$

49. $\mathbf{(\ ^2-\ \ )(\ \ )^2-2\qquad -\ ^2=0.}$

Solving for , differentiating with respect to , and setting $(\ )=\ '$, we obtain a factorized equation: $(\ \ '-\ \ )(\ ^2\ \ ^2+\ \ ^2-a\ \ ^2)=0$. Equating each of the factors to zero, we arrive at the solutions:

$$=\frac{1}{2}(\ ^2-a-\ \ ^2)\quad\text{and}\quad\ ^2+\ \ ^2=a\ \ (\ \ne 0).$$

50. $\mathbf{(\ ^2-\ \ ^2)(\ \ )^2+2\qquad +\ \ ^2=0.}$

The equation can be factorized: $(\ \ '+a\ '+\ \ )(\ \ '-a\ '+\ \ )=0$. Equating each of the factors to zero, we obtain the solutions: $(\ +a)\ =\ \ $ and $(\ -a)\ =\ \ .$

51. $\mathbf{(\ ^2+\ \ )(\ \ )^2-2\qquad +\ \ ^2+\ \ =0.}$

Differentiating with respect to , we obtain a factorized equation: $[(\ ^2+a)\ '-\ \ ]\ ''\ =0$. Therefore, the solutions of the original equation are:

$$=\ \ _1\ +\ \ _2,\quad\text{where }a\ \ _1^2+\ \ _2^2+\ \ =0;\qquad\ ^2+a\ \ ^2+a\ \ =0.$$

52. $\mathbf{\ ^3(\ \ )^2+\ \ ^2\qquad +\ \ =0.}$

Solutions: $\qquad=\ \ ^2\ \ +a$ and $\ ^2=4a.$

53. $\mathbf{\ \ (\ \ )^2-(\ \ \ ^2+\ \ \ ^2+\ \ )\ \ +\ \qquad=0.}$

This differential equation represents an equation of curvature lines of a surface defined by the relation $A\ \ ^2+B\ \ ^2+\ \ z^2=1$, where $a=AB(\ \ -B)$, $\ =AB(A-\ \ )$, $k=\ \ (B-A)$.

Solutions:

$$(a\ \ -\ )\ ^2=\ \ (a\ \ -\ )\ \ ^2-k\qquad\text{and}\quad a\ \ ^2=\ \ \ ^2\ \ 2\ \ \ \overline{\ -\ k}-k.$$

54. $\mathbf{\ ^2(\ \ )^2+2\qquad\ \ +(1-\ \ )\ \ ^2+\ \ \ ^2+(\ \ -1)\ \ =0.}$

Solutions:

$$\ ^2+a\ \ ^2-\ \ =(a-1)(\ \ +\ \ )^2\quad\text{and}\quad\ ^2+a\ \ ^2-\ \ =0.$$

55. $\mathbf{(\ -\ )\ \ ^2(\ \ )^2-2\qquad +\ \ \ ^2-\ \ \ ^2-\ \ \ =0.}$

Solutions:

$$\ ^2+\ \ ^2=\ \ \ +\ -\frac{a-}{4a}\ \ ^2\quad\text{and}\quad(a-\ \ )\ \ ^2-\ \ \ ^2=(a-\ \ )\ .$$

56. $\mathbf{(\ \ -\ \ ^2)(\ \ )^2+2\qquad -\ \ ^2=0.}$

Solution: $(\ \ \ +\ \ )^2=4a\ \ .$

57. $\mathbf{(\;^2 - \;^2\;^2)(\;)^2 + 2 \qquad + (1 - \;^2)\;^2 = 0.}$

Solution in parametric form:

$$= \frac{}{\;^2+1}, \qquad = a \quad - \frac{}{\;^2+1}.$$

58. $\mathbf{(\quad - \quad)^2[\;^2(\quad)^2 + \;^2] - \;^2(\qquad + \;)^2 = 0.}$

We solve the equation for $a \quad -$ and differentiate with respect to . Setting $(\;) = \;'$, we obtain a factorized equation with respect to $(\;)$: $(a \quad - \;)[(a^2 \;^2 + \;^2)^{3\;2} \quad a\;k\;'] = 0$. Equating each of the factors to zero and integrating, we arrive at the solutions:

$$(\quad - \quad)^2 + (a \quad - \quad)^2 = k^2 \quad \text{and} \quad a \quad - \quad = k \quad \overline{2}.$$

59. $\mathbf{(\qquad + \;)^2 - 2 \qquad + \;^2 = 0, \qquad \neq 0.}$

The substitution $2a \;-\;^2 = \;^2$ leads to the equation $\;' - a(\; - a) + \;^2 = 0$. Further assuming $- a = \quad (\;)$, we obtain $(\quad + a)\;' + \;^2 + 1 = 0$. Taking to be the independent variable, we arrive at a linear equation whose solution is: $= (\;^2+1)^{-1\;2}\big[\quad - a\ln\big(\quad + \quad \overline{\;^2+1}\;\big]$.

60. $\mathbf{(\qquad + \qquad)^2 + \qquad ^{+1} \qquad + \quad = 0.}$

Solution: $= \dfrac{\;^2\;^2 +}{a} \;^{-} + \;.$

61. $\mathbf{(\qquad + \qquad)^2 - \quad ^{2\;+2}(\quad)^2 - \quad = 0.}$

Solutions: $= \;^{-} \quad \overline{a\;^2 + \;^{-2}}.$

62. $\mathbf{(\qquad + \;)^2 = \;^2(\;^2 + \;^2)^k[(\quad)^2 + 1].}$

This equation splits into two equations of the form 1.8.1.4 with $(\;) = \quad a \qquad ^2$:

$$' + \quad = a(\;^2 + \;^2) \quad 2 \quad \overline{(\;')^2 + 1}.$$

63. $\mathbf{\;^2\;(\qquad + \qquad)^2 + \;^2\;(\qquad + \qquad)^2 = 1.}$

Solution: $= \;_1\;^{-} + \;_2\;^{-}$. Here, the constants $_1$ and $_2$ are related by the constraint $(a\;^2_1 + \;^2_2)(\quad - \quad)^2 = 1$.

64. $\mathbf{(\qquad - \;)^2 = \;^2(\;^2 + \;^2)^k(\qquad + \;)^2.}$

This equation splits into two equations of the form 1.7.1.20 with $(\;) = \quad a \qquad ^2$:

$$' - \quad = a(\;^2 + \;^2) \quad ^2(\quad ' + \;).$$

65. $\mathbf{(\qquad - \;)^2 = \;^2(\;^2 + \;^2)^k[(\quad)^2 + 1].}$

This equation splits into two equations of the form 1.8.1.3 with $(\;) = \quad a \qquad ^2$:

$$' - \quad = a(\;^2 + \;^2) \quad 2 \quad \overline{(\;')^2 + 1}.$$

66. $\mathbf{(\qquad + \quad + 2 \qquad)^2 = 4(\qquad + \qquad ^2 + \;).}$

The substitution $= \qquad + a\;^2 +$ leads to a separable equation: $' = \quad 2 \quad \overline{\;}$.

67. $\mathbf{(\;_2 \quad + \;_2 \quad + \;_2)(\quad)^2 + (\;_1 \quad + \;_1 \quad + \;_1) \qquad + \;_0 \quad + \;_0 \quad + \;_0 = 0.}$

The Legendre transformation $= \;'$, $= \;' - \quad (\;' = \;)$ leads to a linear equation:

$$[\;(\;) + g(\;)]\;' = g(\;) \quad + \;(\;),$$

where $(\;) = a_2\;^2 + a_1\; + a_0$, $g(\;) = \;_2\;^2 + \;_1\; + \;_0$, and $(\;) = -\;_2\;^2 - \;_1\; - \;_0$.

68. $(\ \ )^2 + \quad + \quad {}^{\lambda} \quad = 0.$

1 . With $\quad \neq 2$, solving for $\ '$ and performing the substitution $\quad = e^{\lambda} \quad {}^{-2}$, we arrive at a separable equation: $\ ' = \lambda \quad + \dfrac{2 -}{2}\bigl(a \quad \overline{a^2 - 4} \quad$ (see Subsection 1.1.2).

2 . With $\quad = 2$, solving the original equation for $\ '$, we obtain a separable equation: $2\ ' = \bigl(-a \quad \overline{a^2 - 4\ e^{\lambda}} \quad .$

69. $\quad + \quad = \quad {}^{3}\ (\quad - \quad)^2.$

Solution: $\quad = \frac{1}{2}a \quad {}^2 e \quad - \frac{1}{2} \quad e^{-} \quad .$

70. $\quad = (\quad {}^2 - \quad {}^2)(\quad + \quad {}^{-}\ ).$

This is a special case of equation 1.8.1.56 with $\ (\ ) = \quad .$

71. $\quad {}^{2\lambda}\ (\quad + \quad)^2 + \quad {}^2\ (\quad + \quad)^2 = 1.$

Solution: $\quad = \quad {}_1 e^{-\lambda} \quad + \quad {}_2 e^{-} \quad$. Here, the constants $\ {}_1$ and $\ {}_2$ are related by the constraint $(a \quad {}^2_1 + \quad {}^2_2)(\beta - \lambda)^2 = 1.$

72. $\quad = \ (\quad {}^2 - \quad {}^2) \cosh \quad .$

This is a special case of equation 1.8.1.58 with $\ (\ ) = a \quad .$

73. $\quad = \ (\quad {}^2 - \quad {}^2) \sinh \quad .$

This is a special case of equation 1.8.1.59 with $\ (\ ) = a \quad .$

74. $(\quad \cosh \quad - \quad \sinh \quad)^2 + (\quad \sinh \quad - \quad \cosh \quad)^2 = 1.$

Solution: $\quad = \quad {}_1 \sinh \quad + \quad {}_2 \cosh \quad$. Here, the constants $\ {}_1$ and $\ {}_2$ are related by the constraint $a \quad {}^2_1 + \quad {}^2_2 = 1.$

75. $(\ \ )^2 - \quad + \quad {}^2 \ln(\quad) = 0.$

Solutions: $a \quad = \exp(\quad - \quad {}^2)$ and $a \quad = \exp(\frac{1}{4} \quad {}^2).$

76. $\quad = \ (\quad {}^2 + \quad {}^2) \cos \quad .$

This is a special case of equation 1.8.1.64 with $\ (\ ) = a \quad .$

77. $\quad = \ (\quad {}^2 + \quad {}^2) \sin \quad .$

This is a special case of equation 1.8.1.65 with $\ (\ ) = \quad$ and $a = 0.$

78. $(\quad \cos \quad + \quad \sin \quad)^2 + (\quad \sin \quad - \quad \cos \quad)^2 = 1.$

Solution: $\quad = \quad {}_1 \sin \quad + \quad {}_2 \cos \quad$. Here, the constants $\ {}_1$ and $\ {}_2$ are related by the constraint $a \quad {}^2_1 + \quad {}^2_2 = 1.$

79. $(\quad \cosh \quad - \quad \sinh \quad)^2 + \ [(\quad)^2 - \quad {}^2] + \quad = 0.$

Solution: $\quad = \quad {}_1 \sinh \quad + \quad {}_2 \cosh \quad$. Here, the constants $\ {}_1$ and $\ {}_2$ are related by the constraint $a \quad {}^2_1 + \ (\quad {}^2_1 - \quad {}^2_2) + \quad = 0.$

80. $(\quad \cos \quad + \quad \sin \quad)^2 + \ [(\quad)^2 + \quad {}^2] + \quad = 0.$

Solution: $\quad = \quad {}_1 \sin \quad + \quad {}_2 \cos \quad$. Here, the constants $\ {}_1$ and $\ {}_2$ are related by the constraint $a \quad {}^2_1 + \ (\quad {}^2_1 + \quad {}^2_2) + \quad = 0.$

1.6.2. Equations of the Third Degree in

1.6.2-1. Equations of the form $(\ ,\ )(\ ')^3 = g(\ ,\ )\ ' + (\ ,\ )$.

1. $(\quad)^3 + \quad + \quad + \quad = 0.$

This is a special case of equation 1.8.1.9 with $(\ ) = \ ^3$.

2. $(\quad + \quad + \ )(\quad)^3 = \quad + \quad + \gamma.$

Dividing both sides by $a \ + \quad +$ and raising to the power 1/3, we finally arrive at an equation of the form 1.7.1.6 with $(\ ) = \ ^{-1\ 3}$.

3. $(\quad)^3 + \quad = \ .$

This is a special case of equation 1.8.1.7 with $(\ ) = a \ ^3 + \quad .$

4. $(\quad)^3 + \quad = \ .$

This is a special case of equation 1.8.1.8 with $(\ ) = a \ ^3 + \quad .$

5. $(\quad)^3 + \quad = \ .$

This is a special case of equation 1.8.1.10 with $(\ ) = a \ ^3$.

6. $(\quad)^3 + \quad = \ .$

This is a special case of equation 1.8.1.11 with $(\ ) = \quad$ and $g(\ ) = a \ ^3$.

7. $(\quad)^3 - \quad + \ ^3 = 0, \qquad \neq 0.$

Solution in parametric form:

$$= \frac{a}{\ ^3 + 1}, \qquad = \quad + \frac{a^2}{6}\frac{4\ ^3 + 1}{(\ ^3 + 1)^2}.$$

8. $(\quad)^3 - \quad + 2 \ ^2 = 0.$

Differentiating with respect to and eliminating , we obtain a factorized equation with respect to $(\ ) = \ '$: $[2(\ ')^2 - a \ ' + a\ ](9 \ - a\ ^2) = 0$. Equating each of the factors to zero and integrating, we find the solutions: $= \frac{1}{4}a\ (\ -\ )^2$ and $= \frac{1}{27}a\ ^3$.

9. $(\quad)^3 + \quad = \ .$

This is a special case of equation 1.8.1.11 with $(\ ) = a \ ^3$ and $g(\ ) = \quad .$

10. $\ ^3\ ^2(\quad)^3 + 2 \quad = \ .$

Solution: $= 2 \ \overline{\ } + a \ ^3$.

11. $(\quad)^3 + \quad = \ .$

This is a special case of equation 1.8.1.15 with $(\ ) = a \ ^3$.

12. $\ ^3\ (\quad)^3 + (\quad + \ ) + \quad = 0.$

Solution: $= \ e^{-}\ + (a\ ^3 - \ )\ ^{-1}$.

1.6.2-2. Equations of the form $(\ ,\ )(\ ')^3 = g(\ ,\ )(\ ')^2 + (\ ,\ )\ ' + (\ ,\ )$.

13. $(\quad)^3 + (\quad)^2 = \ .$

This is a special case of equation 1.8.1.7 with $(\ ) = a \ ^3 + \ ^2$.

14. $(\ \ )^3 + (\ \ )^2 = \ \ .$

This is a special case of equation 1.8.1.8 with $(\ \ ) = a\ \ ^3 + \ \ ^2$.

15. $(\ \ )^3 + (\ \ )^2 + \ \ + \ \ + \ \ = 0.$

Solution in parametric form:

$$2\ \ = -3\ \ ^2 + 2a\ \ - 2a^2 \ln(\ \ + a) + \ \ , \qquad = -a\ \ - \ \ ^3 - a\ \ ^2 - \ \ .$$

In addition, there is a singular solution: $= -a\ \ - \ \ .$

16. $(\ \ )^3 + (\ \ )^2 + \ \ = \ \ + \ \ .$

Solution in parametric form:

$$= \ \ + \tfrac{3}{2}a\ \ ^2 + 2\ \ + \ \ \ln|\ \ |, \qquad = a\ \ ^3 + \ \ ^2 + \ \ - \ \ .$$

17. $(\ \ )^3 + (\ \ )^2 = \ \ .$

This is a special case of equation 1.8.1.11 with $(\ \ ) = \ \ ^2$ and $g(\ \ ) = a\ \ ^3$.

18. $(\ \ )^3 + (\ \ )^2 = \ \ .$

This is a special case of equation 1.8.1.11 with $(\ \ ) = a\ \ ^3$ and $g(\ \ ) = \ \ ^2$.

19. $(\ \ ^2 - \ \ ^2)(\ \ )^3 + (\ \ ^2 - \ \ ^2)(\ \ )^2 + \ \ + \ \ = 0.$

The equation can be factorized: $(\ \ ' + \ \ )[(\ \ ')^2(\ \ ^2 - a^2) + 1] = 0$, whence we find the solutions: $= -\frac{1}{2}\ \ ^2 + \ \ $ and $= \ \ \arcsin(\ \ a) + \ \ .$

20. $(\ \ + \ \ )^3 + \ \ ^3(\ \ )^3 + \ \ = 0.$

Solution: $= \ \ _1 - \ \ _2 e^{-}$. Here, the constants $_1$ and $_2$ are related by the constraint $a\ \ _1^3 + \ \ _2^3 + \ \ = 0$.

21. $(\ \ - \ \ )^3 + \ \ + \ \ = 0.$

This is a special case of equation 1.8.1.16 with $(\ \ ) = 1$, $g(\ \ ) = a$, $(\ \ ) = \ \ $, and $= 3$.

22. $(\ \ - \ \ )^3 + \ \ + \ \ = 0.$

This is a special case of equation 1.8.1.16 with $(\ \ ) = 1$, $g(\ \ ) = a\ \ $, $(\ \ ) = \ \ $, and $= 3$.

23. $(\ \ - \ \ )^3 + \ \ + \ \ = 0.$

This is a special case of equation 1.8.1.16 with $(\ \ ) = 1$, $g(\ \ ) = \ \ $, $(\ \ ) = a\ \ $, and $= 3$.

24. $(\ \ \cosh\ \ - \ \ \sinh\ \ )^3 + (\ \ \sinh\ \ - \ \ \cosh\ \ )^3 + \ \ = 0.$

Solution: $= \ \ _1 \sinh\ \ - \ \ _2 \cosh\ \ $. Here, the constants $_1$ and $_2$ are related by the constraint $a\ \ _1^3 + \ \ _2^3 + \ \ = 0$.

25. $(\ \ \cosh\ \ - \ \ \sinh\ \ )^3 + (\ \ ^2 - \ \ ^2) + \ \ = 0.$

Solution: $= \ \ _1 \sinh\ \ + \ \ _2 \cosh\ \ $. Here, the constants $_1$ and $_2$ are related by the constraint $a\ \ _1^3 + (\ \ _1^2 - \ \ _2^2) + \ \ = 0$.

26. $(\ \ \cosh\ \ - \ \ \sinh\ \ )^3 = (\ \ ^2 - \ \ ^2)\cosh\ \ .$

This is a special case of equation 1.6.4.21 with $= 3$ and $= 1$.

27. $(\ \ \sinh\ \ - \ \ \cosh\ \ )^3 = (\ \ ^2 - \ \ ^2)\sinh\ \ .$

This is a special case of equation 1.6.4.22 with $= 3$ and $= 1$.

28. $(\quad \cos \quad + \quad \sin \quad)^3 + (\quad \sin \quad - \quad \cos \quad)^3 + \quad = 0.$

Solution: $\quad = \quad_1 \sin \quad - \quad_2 \cos \quad$. Here, the constants $\quad_1$ and $\quad_2$ are related by the constraint $a \quad_1^3 + \quad_2^3 + \quad = 0$.

29. $(\quad \cos \quad + \quad \sin \quad)^3 + (\quad ^2 + \quad ^2) + \quad = 0.$

This is a special case of equation 1.6.4.24 with $\quad = 3$ and $k = 1$.

30. $(\quad \cos \quad + \quad \sin \quad)^3 = (\quad ^2 + \quad ^2) \cos \quad .$

This is a special case of equation 1.6.4.25 with $\quad = 3$ and $\quad = 1$.

31. $(\quad \sin \quad - \quad \cos \quad)^3 = (\quad ^2 + \quad ^2) \sin \quad .$

This is a special case of equation 1.6.4.26 with $\quad = 3$ and $\quad = 1$.

1.6.3. Equations of the Form $(\quad) = (\quad) + g(\quad)$

1.6.3-1. Some transformations.

1 . In the general case, the equation

$$(\quad ') = (\quad) + g(\quad) \tag{1}$$

can be reduced with the aid of the transformation $\quad = [g(\quad)]^1 \quad , \quad = [\ (\quad)]^{-1}$ to the same form

$$(\quad ') = (\quad) + (\quad),$$

where functions $\quad = (\quad)$ and $\quad = (\quad)$ are defined parametrically by the following formulas:

$$(\quad) = \frac{1}{(\quad)}, \quad = [\ (\quad)]^{-1} \quad ,$$

$$(\quad) = \frac{1}{g(\quad)}, \quad = [g(\quad)]^1 \quad .$$

2 . Taking as the independent variable, we obtain from equation (1) an equation of the same class for $\quad = (\quad)$:

$$(\quad ')^{-} = g(\quad) + (\quad).$$

3 . The equation

$$' = a \quad ^{-} + g(\quad) \qquad (k = 1, \quad = a \quad ^{-}$$

can be reduced with the aid of the substitution $(\quad) = 2a^{-1} \quad ^{-}$ to the Abel equation $\quad ' - \quad = 2a^{-2} g(\quad)$, which is outlined in Subsection 1.3.1.

4 . The equation

$$' = \quad ^{-1} + g(\quad) \qquad (k = 1, \quad = \quad ^{-1})$$

is an alternative form of representation of the Abel equation $\quad ' = g(\quad) \quad + 1$, which is outlined in Subsection 1.3.2.

5 . The equation

$$' = a \quad + g(\quad) \qquad (k = 1, \quad = a$$

can be reduced, with the aid of the substitution $a \quad = -\int g(\quad)$ followed by raising both sides of the equation to the power of 1 , to an equation of the class in question:

$$(\quad ')^1 = a \quad + \quad g(\quad) \quad .$$

6 . The equation

$$(\ \ ')^2 = a \ \ + g(\ ) \qquad (k = 2, \quad = a \ , \ a \neq 0)$$

can be reduced with the aid of the substitution $a \quad = 2 \ \overline{a \ \ + g(\ )}$ to an Abel equation of the second kind:

$$' = \quad + \ (\ ), \qquad \text{where} \qquad = 2a^{-2} g'(\ ),$$

which is outlined in Subsection 1.3.1.

7 . The equation

$$(\ \ ')^{1 \ \ 2} = a \ \ + g(\ ) \qquad (k = 1 \ \ 2, \quad = a \ \ )$$

can be reduced by squaring both sides and performing the substitution $z = a \ \ + g(\ )$ to the Riccati equation:

$$z' \ = az^2 + g' \ .$$

For some specific functions $g = g(\ )$, the solutions of the latter equation are given in Section 1.2.

8 . The equation

$$(\ \ ')^{1 \ \ 2} = a \ \ ^{1 \ \ 2} + g(\ ) \qquad (k = 1 \ \ 2, \quad = a \ \ ^{1 \ \ 2})$$

can be reduced by squaring both sides and performing the substitution $\quad = \exp(a^2 \ \ )\xi^2$ to an Abel equation of the second kind:

$$\xi\xi' \ = a \exp(-\tfrac{1}{2}a^2 \ \ )g\xi + \tfrac{1}{2}\exp(-a^2 \ \ )g^2$$

(see Subsection 1.3.3).

9 . The equation

$$(\ \ ')^{-1 \ \ 2} = \ (\ ) + a \qquad (k = -1 \ \ 2, \ \ g = a \ \ )$$

can be reduced by squaring both sides and performing the substitution $v = \ (\ ) + a$ to a Riccati equation:

$$v' \ = av^2 + \ \ '.$$

For some specific functions $\ = \ (\ )$, the solutions of the latter equation are given in Section 1.2.

1.6.3-2. Classification tables and exact solutions.

For the sake of convenience, Tables 9–13 given below list all the equations outlined in Subsection 1.6.3. The five tables classify the equations in which functions and g are of the same form. The right-most column of a table indicates the numbers of the equations where the corresponding solutions are given. After the tables follow the equations—they are arranged into groups so that the solutions of the equations within each group are expressed in terms of the functions indicated before the groups as a notation list.

1. $(\ \ )^k = \quad ^s + B.$

Solution: $\quad = \ (A \quad + B)^{-1} \qquad + \ .$

2. $(\ \ )^k = \quad + B \quad .$

Solution: $\quad = \ (A + B \quad)^1 \qquad + \ .$

3. $(\ \ )^k = \qquad + B.$

Solution: $\quad = \ (Ae \ \ + B)^{-1} \qquad + \ .$

4. $(\ \ )^k = \quad + B \quad .$

Solution: $\quad = \ (A + Be \ \ )^1 \qquad + \ .$

TABLE 9

Solvable equations of the form $(\ \ ')\ \ = A \quad + B$

k			Equation	k			Equation
arbitrary	arbitrary ($\neq k$)	$\frac{k}{k-}$	1.6.3.7	-1	1	-1	1.6.3.23
arbitrary	arbitrary	0	1.6.3.1	-1	1	1 2	1.6.3.42
arbitrary ($k \neq -1$, 1)	$\frac{k}{1-k}$	$-\frac{k}{1+k}$	1.6.3.6	-1 2	arbitrary ($\neq -1$, 0)	1	1.6.3.17
arbitrary	0	arbitrary	1.6.3.2	-1 2	-1	1	1.6.3.38
arbitrary	1	1	1.6.3.5	1 2	1	arbitrary ($\neq -1$, 0)	1.6.3.16
-2	-1	-2	1.6.3.46	1 2	1	-1	1.6.3.37
-2	-1	1	1.6.3.33	1	-1	-2	1.6.3.20
-2	-2 5	-2	1.6.3.29	1	-1	-1 2	1.6.3.39
-2	1 2	1	1.6.3.27	1	-1	1	1.6.3.22
-2	2	-2	1.6.3.35	1	1 2	-2	1.6.3.30
-2	2	1	1.6.3.44	1	1 2	-1	1.6.3.11
-1	arbitrary ($\neq 0$)	1	1.6.3.10	1	1 2	-1 2	1.6.3.24
-1	arbitrary ($\neq -2$, 0)	2	1.6.3.15	1	1 2	1	1.6.3.41
-1	-2	-1	1.6.3.21	2	-2	-1	1.6.3.45
-1	-2	1 2	1.6.3.31	2	-2	-2 5	1.6.3.28
-1	-2	2	1.6.3.36	2	-2	2	1.6.3.34
-1	-1	1 2	1.6.3.12	2	1	-1	1.6.3.32
-1	-1 2	-1	1.6.3.40	2	1	1 2	1.6.3.26
-1	-1 2	1 2	1.6.3.25	2	1	2	1.6.3.43

5. $(\quad)^k = \quad + B \quad .$

Solution in parametric form:

$$= \quad (A\tau^1 \quad + B)^{-1} \quad \tau + \quad , \qquad = \frac{1}{A} \quad \tau - B \quad (A\tau^1 \quad + B)^{-1} \quad \tau - B \quad .$$

6. $(\quad)^k = \quad^{\frac{k}{1-k}} + B \quad^{-\frac{k}{1+k}}, \qquad |\quad| \neq 1.$

Solution in parametric form:

$$= a \qquad \frac{\tau}{\tau^1 \quad + \beta} + \quad \Big)^{\overline{\quad +1}}, \qquad = \quad \tau - \beta \qquad \frac{\tau}{\tau^1 \quad + \beta} - \beta \quad \Big)^{\overline{\quad -1}},$$

where $A = a^{-\overline{\quad 1+\quad}} \quad^{-\overline{\quad 1-\quad}} \beta B,\ B = a^{\overline{\quad 1+\quad}} \quad \frac{(k-1)}{a(k+1)} \qquad .$

TABLE 10
Solvable equations of the form
$(\;'\;) \;\; = Ae \;\; + B$

k		Equation
arbitrary	$-k$	1.6.3.9
arbitrary	0	1.6.3.3
-1	-1	1.5.2.40
-1	1	1.5.2.38
-1	2	1.5.2.39
$-1\;\;2$	1	1.6.3.19
1	arbitrary	1.5.2.13

TABLE 11
Solvable equations of the form
$(\;'\;) \;\; = A \;\;\; + Be$

k		Equation
arbitrary	k	1.6.3.8
arbitrary	0	1.6.3.4
-1	arbitrary	1.5.2.37
$1\;\;2$	1	1.6.3.18
1	-1	1.5.2.14

TABLE 12
Solvable equations of the form
$(\;'\;) \;\; = Ae \;\; + Be$

k	Equation
-1	1.5.2.8
1	1.5.2.2

TABLE 13
Solvable equations containing logarithmic functions

Form of equation	Equation
$(\;'\;)^{-2} = A\ln \;\; + B$	1.6.3.14
$(\;'\;)^{-1} = A\ln \;\; + B$	1.5.4.12
$(\;'\;)^{2} = A \;\; + B\ln$	1.6.3.13

7. $(\quad)^{k} = \quad {}^{s} + B \quad \frac{ks}{k-s}, \qquad \neq s.$

Solution in parameteric form:

$$= \quad {}^{-} \exp \quad \frac{k-}{} \quad A + B\tau^{\overline{\;-\;}} \quad {}^{\frac{1}{}} - \tau \quad {}^{-1} \quad \tau \quad ,$$

$$= \quad \tau \exp \quad \frac{k-}{} \quad A + B\tau^{\overline{\;-\;}} \quad {}^{\frac{1}{}} - \tau \quad {}^{-1} \quad \tau \quad {}^{\overline{\;-\;}} \quad .$$

8. $(\quad)^{k} = \quad {}^{k} + B \quad .$

Solution in parametric form:

$$= \quad (A + Be^{-\;\;\tau})^{1} \quad - \frac{1}{k} \quad {}^{-1} \quad \tau + \quad ,$$

$$= \exp \quad \tau + \frac{1}{k} \quad (A + Be^{-\;\;\tau})^{1} \quad - \frac{1}{k} \quad {}^{-1} \quad \tau + \frac{}{k} \quad .$$

9. $(\quad)^{k} = \quad + B \quad {}^{-k}.$

Solution in parametric form:

$$= \exp \quad \tau - \frac{1}{k} \quad (B + Ae^{\;\;\tau})^{-1} \quad + \frac{1}{k} \quad {}^{-1} \quad \tau - \frac{}{k} \quad ,$$

$$= \quad (B + Ae^{\;\;\tau})^{-1} \quad + \frac{1}{k} \quad {}^{-1} \quad \tau + \quad .$$

10. $(\quad)^{-1} = \quad {}^{s} + B \quad .$

Solution: $\quad = e \quad A \quad e^{-} \quad + \quad .$

In the solutions of equations 11–14, the following notation is used:

$$= \quad \exp(\mp\tau^2) \quad \tau + \quad ^{-1}.$$

11. $\quad = \quad ^{1\ 2} + B \quad ^{-1}.$

Solution in parametric form:

$$= a \quad \exp(\mp\tau^2), \qquad = \quad [2\tau \qquad \exp(\mp\tau^2)]^2, \qquad \text{where} \quad A = \quad 2a^{-1\ \ 1\ 2}, \ \ B = \mp 4 \ .$$

12. $(\quad)^{-1} = \quad ^{-1} + B \quad ^{1\ 2}.$

Solution in parametric form:

$$= a[2\tau \qquad \exp(\mp\tau^2)]^2, \qquad = \qquad \exp(\mp\tau^2), \qquad \text{where} \quad A = \mp 4a, \ \ B = \quad 2a^{1\ 2\ \ -1}.$$

13. $(\quad)^2 = \qquad + B \ln \quad .$

Solution in parametric form:

$$= a \quad \exp(\mp\tau^2), \qquad = \quad [2\tau \qquad \exp(\mp\tau^2)]^2 \quad 4\ln(a \quad) - 4\tau^2 \ ,$$

where $A = 4a^{-2}\ ,\ B = \mp 4 \ A.$

14. $(\quad)^{-2} = \quad \ln \quad + B \quad .$

Solution in parametric form:

$$= a \ [2\tau \qquad \exp(\mp\tau^2)]^2 \quad 4\ln(\quad) - 4\tau^2 \ , \qquad = \qquad \exp(\mp\tau^2),$$

where $A = \mp 4aB,\ B = 4a \ ^{-2}.$

In the solutions of equations 15–19, the following notation is used:

$$Z = \begin{cases} \ _1 \quad (\tau) + \ _2 \quad (\tau) & \textit{for the upper sign,} \\ \ _1 \quad (\tau) + \ _2 \quad (\tau) & \textit{for the lower sign,} \end{cases}$$

where (τ) *and* (τ) *are the Bessel functions, and* (τ) *and* (τ) *are the modified Bessel functions.*

The solutions of equations 15–19 contain only the ratio $Z'_\tau \ Z = (\ln Z)'_\tau$*. Therefore, for the sake of symmetric appearance, two "arbitrary" constants* $_1$ *and* $_2$ *are indicated in the definition of function* Z *(instead, we can set, for instance,* $_1 = 1$ *and* $_2 =$ *).*

15. $(\quad)^{-1} = \quad ^s + B \quad ^2, \qquad s \neq -2,\ s \neq 0.$

Solution in parametric form:

$$= a\tau^{-2} \ \left[\tau(\ln Z)'_\tau + \quad \right], \qquad = \ \tau^2 \ ,$$

where $\quad = \dfrac{1}{\ + 2},\ A = \mp\dfrac{\ + 2}{2}a \ ^{-1-} \ ,\ B = -\dfrac{\ + 2}{2}a^{-1\ \ -1}.$

16. $(\quad)^{1\ 2} = \qquad + B \quad , \qquad \neq -1, \quad \neq 0.$

Solution in parametric form:

$$= a\tau^2 \ , \qquad = \ \tau^2 \quad \tau(\ln Z)'_\tau + \qquad \frac{\ + 1}{2}\tau^2 \ ,$$

where $\quad = \dfrac{1}{\ + 1},\ A = \ ^{-1} \ -\dfrac{(\ + 1)}{2a} \ ^{1\ 2},\ B = \mp\dfrac{\ + 1}{2}a^{-} \quad A.$

17. $(\quad)^{-1\ 2} = \quad {}^{s} + B \quad, \qquad s \neq -1,\ s \neq 0.$

Solution in parametric form:

$$= a\tau^2 \quad \tau(\ln Z)'_\tau + \qquad \frac{+1}{2}\tau^2 \quad , \qquad = \quad \tau^2 \quad ,$$

where $\quad = \frac{1}{+1},\ A = \mp\frac{+1}{2}a \quad {}^{-}\ B,\ B = a^{-1} \quad -\frac{(\ +1)a}{2} \quad {}^{1\ 2}.$

18. $(\quad)^{1\ 2} = \quad + B \quad .$

Solution in parametric form:

$$= \ln(a\tau^2), \qquad = \quad \big[\tau(\ln Z)'_\tau \quad \tfrac{1}{2}\tau^2\big],$$

where $\quad = 0,\ A = \quad {}^{-1}\big(-\tfrac{1}{2} \quad {}^{1\ 2},\ B = \mp\tfrac{1}{2}a^{-1}\ A.$

19. $(\quad)^{-1\ 2} = \quad + B \quad .$

Solution in parametric form:

$$= a\big[\tau(\ln Z)'_\tau \quad \tfrac{1}{2}\tau^2\big], \qquad = \ln(\ \tau^2),$$

where $\quad = 0,\ A = \mp\tfrac{1}{2}a \quad {}^{-1}B,\ B = a^{-1}\big(-\tfrac{1}{2}a \quad {}^{1\ 2}.$

In the solutions of equations 20–35, the following notation is used:

$$Z = \begin{cases} \ _1\ \ _{1\ 3}(\tau) + \ \ _2\ \ _{1\ 3}(\tau) & \textit{for the upper sign,} \\ \ _1\ \ _{1\ 3}(\tau) + \ \ _2\ \ _{1\ 3}(\tau) & \textit{for the lower sign,} \end{cases}$$

where $\ _{1\ 3}(\tau)$ *and* $\ _{1\ 3}(\tau)$ *are the Bessel functions, and* $\ _{1\ 3}(\tau)$ *and* $\ _{1\ 3}(\tau)$ *are the modified Bessel functions;*

$$\ _1 = \tau Z'_\tau + \tfrac{1}{3}Z, \qquad \ _2 = \ \ {}^2_1 \quad \tau^2 Z^2, \qquad \ _3 = \ \ \tfrac{2}{3}\tau^2 Z^3 - 2\ \ _1\ \ _2.$$

The solutions of equations 20–35 contain only the ratio $Z'_\tau \ \ Z = (\ln Z)'_\tau$*. Therefore, for the sake of symmetric appearance, two "arbitrary" constants* $\ _1$ *and* $\ _2$ *are indicated in the definition of function Z (instead, we can set, for instance,* $\ _1 = 1$ *and* $\ _2 = \quad$*).*

20. $\quad = \quad {}^{-1} + B \quad {}^{-2}.$

Solution in parametric form:

$$= a\tau^{-4\ 3}Z^{-2} \quad _2, \qquad = \quad \tau^{-2\ 3}Z^{-1} \ \ {}^{-1}_{2} \quad _3, \qquad \text{where} \quad A = 2a^{-1\ 2},\ B = \mp\tfrac{2}{3}a \ .$$

21. $(\quad)^{-1} = \quad {}^{-2} + B \quad {}^{-1}.$

Solution in parametric form:

$$= a\tau^{-2\ 3}Z^{-1} \ \ {}^{-1}_{2} \quad _3, \qquad = \quad \tau^{-4\ 3}Z^{-2} \quad _2, \qquad \text{where} \quad A = \mp\tfrac{2}{3}a \ ,\ B = 2a^2 \ {}^{-1}.$$

22. $\quad = \quad {}^{-1} + B \quad .$

Solution in parametric form:

$$= a\tau^{-2\ 3}Z^{-1} \quad _1, \qquad = \quad \tau^{-4\ 3}Z^{-2} \quad _2, \qquad \text{where} \quad A = \mp\tfrac{2}{3}a^{-1\ 2},\ B = 2a^{-2} \ .$$

23. $(\quad)^{-1} = \quad + B \quad {}^{-1}.$

Solution in parametric form:

$$= a\tau^{-4\ 3}Z^{-2} \quad _2, \qquad = \quad \tau^{-2\ 3}Z^{-1} \quad _1, \qquad \text{where} \quad A = 2a^{-2} \ ,\ B = \mp\tfrac{2}{3}a^2 \ {}^{-1}.$$

24. $= \quad {}^{1\ 2} + B \quad {}^{-1\ 2}.$

Solution in parametric form:

$$= a\tau^{-4\ 3}Z^{-2}\ \ {}^{2}_{1}, \qquad = \ \tau^{-8\ 3}Z^{-4}\ \ {}^{2}_{2}, \qquad \text{where} \quad A = 2a^{-1\ \ 1\ 2},\ \ B = \mp\tfrac{2}{3}a^{-1\ \ 2}\ \ .$$

25. $(\quad)^{-1} = \quad {}^{-1\ 2} + B \quad {}^{1\ 2}.$

Solution in parametric form:

$$= a\tau^{-8\ 3}Z^{-4}\ \ {}^{2}_{2}, \qquad = \ \tau^{-4\ 3}Z^{-2}\ \ {}^{2}_{1}, \qquad \text{where} \quad A = \mp\tfrac{2}{3}a\ \ {}^{-1\ 2},\ \ B = 2a^{1\ \ 2\ \ -1}.$$

26. $(\quad)^{2} = \quad + B \quad {}^{1\ 2}.$

Solution in parametric form:

$$= a\tau^{-4\ 3}Z^{-2}\ \ {}^{2}_{1}, \qquad = \ \tau^{-8\ 3}Z^{-4}(\ \ {}^{2}_{2}\ \ \tfrac{4}{3}\tau^{2}Z^{3}\ \ {}_{1}\ , \qquad \text{where} \quad A = 4a^{-2}\ ,\ \ B = \mp\tfrac{4}{3}a^{-1\ \ 2}\ A.$$

27. $(\quad)^{-2} = \quad {}^{1\ 2} + B \quad .$

Solution in parametric form:

$$= a\tau^{-8\ 3}Z^{-4}(\ \ {}^{2}_{2}\ \ \tfrac{4}{3}\tau^{2}Z^{3}\ \ {}_{1}), \qquad = \ \tau^{-4\ 3}Z^{-2}\ \ {}^{2}_{1},$$

where $A = \mp\tfrac{4}{3}a\ \ {}^{-1\ \ 2}B,\ B = 4a\ \ {}^{-2}.$

28. $(\quad)^{2} = \quad {}^{-2} + B \quad {}^{-2\ 5}.$

Solution in parametric form:

$$= a\tau^{-5\ 3}Z^{-5\ 2}\ \ {}^{5\ 2}_{1}, \qquad = \ \tau^{-4\ 3}Z^{-2}(\ \ {}^{2}_{2}\ \ \tfrac{4}{3}\tau^{2}Z^{3}\ \ {}_{1})^{1\ 2},$$

where $A = \mp\tfrac{4}{3}a^{-2\ \ 5\ \ 2}B,\ B = \tfrac{16}{25}a^{-8\ \ 5\ \ 2}.$

29. $(\quad)^{-2} = \quad {}^{-2\ 5} + B \quad {}^{-2}.$

Solution in parametric form:

$$= a\tau^{-4\ 3}Z^{-2}(\ \ {}^{2}_{2}\ \ \tfrac{4}{3}\tau^{2}Z^{3}\ \ {}_{1})^{1\ 2}, \qquad = \ \tau^{-5\ 3}Z^{-5\ 2}\ \ {}^{5\ 2}_{1},$$

where $A = \tfrac{16}{25}a^{2\ \ -8\ \ 5},\ B = \mp\tfrac{4}{3}a^{2\ \ -2\ \ 5}A.$

30. $= \quad {}^{1\ 2} + B \quad {}^{-2}.$

Solution in parametric form:

$$= a\tau^{4\ 3}Z^{2}\ \ {}^{-1}_{2}, \qquad = \ \tau^{-4\ 3}Z^{-2}\ \ {}^{-2}_{2}\ \ {}^{2}_{3}, \qquad \text{where} \quad A = \ \ \tfrac{4}{3}a^{-1\ \ 1\ 2},\ \ B = -4a\ \ .$$

31. $(\quad)^{-1} = \quad {}^{-2} + B \quad {}^{1\ 2}.$

Solution in parametric form:

$$= a\tau^{-4\ 3}Z^{-2}\ \ {}^{-2}_{2}\ \ {}^{2}_{3}, \qquad = \ \tau^{4\ 3}Z^{2}\ \ {}^{-1}_{2}, \qquad \text{where} \quad A = -4a\ ,\ \ B = \ \ \tfrac{4}{3}a^{1\ \ 2\ \ -1}.$$

32. $(\quad)^{2} = \quad + B \quad {}^{-1}.$

Solution in parametric form:

$$= a\tau^{4\ 3}Z^{2}\ \ {}^{-1}_{2}, \qquad = \ \tau^{-4\ 3}Z^{-2}\ \ {}^{-2}_{2}(\ \ {}^{2}_{3} - 4\ \ {}^{3}_{2}), \qquad \text{where} \quad A = \tfrac{16}{9}a^{-2}\ ,\ \ B = 4a\ \ A.$$

33. $(\quad)^{-2} = \quad{}^{-1} + B \quad .$

Solution in parametric form:

$$= a\tau^{-4\ 3}Z^{-2}\ {}_2^{-2}(\ {}_3^2 - 4\ {}_2^3), \qquad = \tau^{4\ 3}Z^2\ {}_2^{-1}, \qquad \text{where}\quad A = 4a\ B,\ \ B = \tfrac{16}{9}a\ {}^{-2}.$$

34. $(\quad)^2 = \quad{}^{-2} + B\ {}^2.$

Solution in parametric form:

$$= a\tau^{2\ 3}Z\ {}_2^{-1\ 2}, \qquad = \tau^{-2\ 3}Z^{-1}\ {}_2^{-1}(\ {}_3^2 - 4\ {}_2^3)^{1\ 2}, \qquad \text{where}\quad A = 4a^{2\ 2}B,\ \ B = \tfrac{16}{9}a^{-4\ 2}.$$

35. $(\quad)^{-2} = \quad{}^2 + B\ {}^{-2}.$

Solution in parametric form:

$$= a\tau^{-2\ 3}Z^{-1}\ {}_2^{-1}(\ {}_3^2 - 4\ {}_2^3)^{1\ 2}, \qquad = \tau^{2\ 3}Z\ {}_2^{-1\ 2}, \qquad \text{where}\quad A = \tfrac{16}{9}a^{2\ -4},\ \ B = 4a^{2\ 2}A.$$

In the solutions of equations 36–46, the following notation is used:

$$R = \begin{cases} {}_1\tau \ + \ {}_2\tau^{-} & \textit{for the upper sign,} \\ {}_1\sin(\ \ln\tau) + \ {}_2\cos(\ \ln\tau) & \textit{for the lower sign,} \\ {}_1\ln\tau + \ {}_2 & \textit{for}\ \ = 0, \end{cases}$$

$$Q = \begin{cases} (1+\ )\ {}_1\tau \ + (1-\ )\ {}_2\tau^{-} & \textit{for the upper sign,} \\ (\ {}_1 - \ {}_2)\sin(\ \ln\tau) + (\ {}_2 + \ {}_1)\cos(\ \ln\tau) & \textit{for the lower sign,} \\ {}_1\ln\tau + \ {}_1 + \ {}_2 & \textit{for}\ \ = 0. \end{cases}$$

The expressions of R and Q contain two "arbitrary" constants ${}_1$ an ${}_2$. One of them can be fixed to set it equal to any nonzero number (for example, we can set ${}_2 = 1$), while the other constant remains arbitrary.

36. $(\quad)^{-1} = \quad{}^{-2} + B\ {}^2.$

Solution in parametric form:

$$= a\tau^{-2}R^{-1}Q, \qquad = \tau^2, \qquad = \overline{|1 - 4AB|},$$

where $A = -\dfrac{1 \mp\ {}^2}{2}a\ ,\ \ B = -\tfrac{1}{2}a^{-1\ -1}.$

37. $(\quad)^{1\ 2} = \quad + B\ {}^{-1}.$

Solution in parametric form:

$$= a\tau^2, \qquad = \tau^{-2}\ \ R^{-1}Q - \frac{1 \mp\ {}^2}{2}\ , \qquad \text{where}\quad A = {}^{-1}\ -\frac{}{2a}\ {}^{1\ 2},\ \ B = \frac{1 \mp\ {}^2}{2}a\ A.$$

38. $(\quad)^{-1\ 2} = \quad{}^{-1} + B \quad .$

Solution in parametric form:

$$= a\tau^{-2}\ \ R^{-1}Q - \frac{1 \mp\ {}^2}{2}\ , \qquad = \tau^2, \qquad \text{where}\quad A = \frac{1 \mp\ {}^2}{2}a\ B,\ \ B = a^{-1}\ -\frac{a}{2}\ {}^{1\ 2}.$$

39. $\quad = \quad{}^{-1} + B\ {}^{-1\ 2}.$

Solution in parametric form:

$$= a\tau^2R^2, \qquad = \tau Q, \qquad \text{where}\quad A = (-1\ \ {}^2)\frac{{}^2}{2a},\qquad B = a^{-1\ 2}\ .$$

40. $(\quad)^{-1} = \quad^{-1\ 2} + B\quad^{-1}.$

Solution in parametric form:

$$= a\tau Q, \qquad = \tau^2 R^2, \qquad \text{where} \quad A = a\ ^{-1\ 2}, \quad B = (-1 \qquad ^2)\frac{2}{2a}.$$

41. $= \quad^{1\ 2} + B\quad.$

Solution in parametric form:

$$= a\tau R, \qquad = \tau^2 Q^2, \qquad \text{where} \quad A = 2(-1 \qquad ^2)a^{-1\ 1\ 2}, \quad B = 4a^{-2}\ .$$

42. $(\quad)^{-1} = \quad + B\quad^{1\ 2}.$

Solution in parametric form:

$$= a\tau^2 Q^2, \qquad = \tau R, \qquad \text{where} \quad A = 4a\ ^{-2}, \quad B = 2(-1 \qquad ^2)a^{1\ 2\ -1}.$$

43. $(\quad)^2 = \quad + B\quad^2.$

Solution in parametric form:

$$= a\tau R, \qquad = \tau^2[Q^2 - (-1 \qquad ^2)R^2], \qquad \text{where} \quad A = 16a^{-2}\ , \quad B = (-1 \qquad ^2)a^{-2}\ A.$$

44. $(\quad)^{-2} = \quad^2 + B\quad.$

Solution in parametric form:

$$= a\tau^2[Q^2 - (-1 \qquad ^2)R^2], \qquad = \tau R, \qquad \text{where} \quad A = (-1 \qquad ^2)a\ ^{-2}B, \quad B = 16a\ ^{-2}.$$

45. $(\quad)^2 = \quad^{-2} + B\quad^{-1}.$

Solution in parametric form:

$$= a\tau^2 R^2, \qquad = \tau[Q^2 - (-1 \qquad ^2)R^2]^{1\ 2}, \qquad \text{where} \quad A = (-1 \qquad ^2)a^{-1\ 2}B, \quad B = a^{-1\ 2}.$$

46. $(\quad)^{-2} = \quad^{-1} + B\quad^{-2}.$

Solution in parametric form:

$$= a\tau[Q^2 - (-1 \qquad ^2)R^2]^{1\ 2}, \qquad = \tau^2 R^2, \qquad \text{where} \quad A = a^2\ ^{-1}, \quad B = (-1 \qquad ^2)a^2\ ^{-1}A.$$

1.6.4. Other Equations

1.6.4-1. Equations containing algebraic and power functions with respect to $\ '$.

1. $= \quad + \quad^2 + \quad\overline{\quad} + \ , \qquad \neq 0.$

Differentiating the equation with respect to and changing to new variables $= \ '$ and $(\ ) = -2a\ $, we arrive at an Abel equation of the form 1.3.1.32: $\ ' - \quad = -a\ ^{-1\ 2}$.

2. $\quad - \quad = (\quad + \quad)\ \overline{(\quad)^2 + 1}.$

This is a special case of equation 1.8.1.5 with $(\ , v) = a\ + \ v$.

3. $\quad + \quad = (\quad + \quad)\ \overline{(\quad)^2 + 1}.$

This is a special case of equation 1.8.1.6 with $(\ , v) = a\ + \ v$.

4. $= \quad + (\quad)\ .$

Solution: $= \quad + a\quad$. In addition, there is a singular solution: $= A\ \overline{\quad}^{-1}$, where $aA\ ^{-1} \quad = -(\ - 1)\ ^{-1}, \quad \neq 1$.

5. $= \quad + \quad (\ ) \ .$

This is a special case of equation 1.8.1.15 with $(\ ) = a$.

6. $= \quad (\ )^2 \ + 2 \quad .$

This is a special case of equation 1.8.1.51 with $(\ ) = a$.

7. $= \quad + \quad {}^2 + \ (\ \ )^2 + \ (\ \ ) \ {}^{+1} + \ , \qquad \neq 0.$

Differentiating the equation with respect to and passing to the new variables $= \ '$ and $(\ ) = -2a$, we arrive at the Abel equation $' - \ = -4a \ \ -2a\ (\ \ +1)$, whose solvable case are outlined in Subsection 1.3.1.

8. $(\ \ ) + (\ \ ) \ = \ .$

1 . Solution in parametric form with $\neq -1, \quad \neq -1$:

$$= a \quad + \quad , \qquad = \quad + \frac{a}{\ +1} \ {}^{+1} + \frac{}{\ +1} \ {}^{+1}.$$

2 . Solution in parametric form with $= -1, \quad \neq -1$:

$$= \frac{a}{} + \quad , \qquad = \quad + a \ln|\ | + \frac{}{\ +1} \ {}^{+1}.$$

9. $(\ \ ) + (\ \ ) \ = \ .$

1 . Solution in parametric form with $\neq 1, \quad \neq 1$:

$$= \quad + \frac{a}{\ -1} \ {}^{-1} + \frac{}{\ -1} \ {}^{-1}, \qquad = a \quad + \quad .$$

2 . Solution in parametric form with $= 1, \quad \neq 1$:

$$= \quad + a \ln|\ | + \frac{}{\ -1} \ {}^{-1}, \qquad = a \ + \quad .$$

10. $(\ \ ) + \quad (\ \ ) \ + (\ \ )^k = 0.$

This is a special case of equation 1.8.1.12 with $(\ ) = a$, $g(\ ) =$, and $(\ ) =$.

11. $= \quad (\quad - \ ) \ .$

The Legendre transformation $= \ ', \quad = \ ' - \quad (\ ' = \ )$ leads to a separable equation: $= a \quad (\ ')\ .$

1 . Solution in parametric form with $\neq - \ , \quad \neq -1$:

$$= \ \frac{}{a}^{\frac{1}{}} \ \frac{+}{+1} \ \frac{}{a}^{\frac{1}{}} + \quad {}^{-\frac{}{+}}, \qquad = \frac{1-}{1+} \ \frac{}{a}^{\frac{1}{}} - \quad \frac{+}{+1} \ \frac{}{a}^{\frac{1}{}} + \quad {}^{-\frac{}{+}}.$$

2 . Solution in parametric form with $= - \ , \quad \neq -1$:

$$= \quad \frac{}{a}^{\frac{1}{}} \exp \frac{}{+1} \ \frac{}{a}^{\frac{1}{}} , \qquad = \quad \frac{}{a}^{\frac{1}{}} - 1 \ \exp \frac{}{+1} \ \frac{}{a}^{\frac{1}{}} .$$

3 . Solution in parametric form with $\neq - \ , \quad = -1$:

$$= \frac{a}{} \left[a(1- \quad)\ln|\ | + \quad \right]^{\frac{}{1-}}, \qquad = \quad - \left[a(1- \quad)\ln|\ | + \quad \right]^{\frac{1}{1-}}.$$

4 . Solution with $= 1, \quad = -1$: $= \quad {}^{\overline{-1}}.$

12. $\quad + \quad = \quad ^{k} \; (\quad - \quad)^{k-1}.$

Solution: $\quad = \tfrac{1}{2}a \quad ^{-1}e \quad - \tfrac{1}{2} \quad e^{-} \quad .$

13. $\quad ^{k} \quad ^{+k}(\quad)^{k} + \; (\qquad + \qquad) \quad + \quad = 0.$

Solution: $\quad = - \quad _{1} \quad ^{-} \quad + \quad _{2}$. Here, the constants $_{1}$ and $_{2}$ are related by the constraint $a(\quad _{1} \quad) \quad + \; (\quad _{2} \quad) \quad + \quad = 0.$

14. $\quad _{1} \quad ^{1} \quad ^{1}(\qquad + \quad _{2} \quad) \quad ^{1} + \quad _{2} \quad ^{2} \quad ^{2}(\qquad + \quad _{1} \quad) \quad ^{2} + \quad = 0, \qquad _{1} \neq \quad _{2}.$

Solution: $\quad = \dfrac{\quad _{1}}{\quad _{2} - \quad _{1}} \quad ^{-} \quad ^{1} - \dfrac{\quad _{2}}{\quad _{2} - \quad _{1}} \quad ^{-} \quad ^{2}$. Here, the constants $_{1}$ and $_{2}$ are related by the constraint $a_{1} \quad _{1} \quad ^{1} + a_{2} \quad _{2} \quad ^{2} + \quad = 0.$

15. $\quad (\quad + \qquad) \quad + \; (\quad ^{2} + 2 \qquad) \quad + \quad = 0.$

The contact transformation $X = \quad ' + a \quad , \quad = (\quad ')^{2} + 2a \quad , \quad ' = 2 \quad '$, where $\quad = \quad (X)$, leads to a separable equation: $\quad ' = -2X^{-} \; (\qquad + \quad).$

The inverse contact transformation: $\quad = \tfrac{1}{2}a^{-1}(2X - \quad '), \quad = \tfrac{1}{8}a^{-1}[4 \quad - (\quad ')^{2}], \quad ' = \tfrac{1}{2} \quad ' .$

16. $\quad (\quad + \quad) \quad + \quad ^{k} \; (\quad)^{k} + \quad = 0.$

Solution: $\quad = \quad _{1} - \quad _{2}e^{-} \quad .$ Here, the constants $_{1}$ and $_{2}$ are related by the constraint $a \quad _{1} + \quad _{2} + \quad = 0.$

17. $\quad ^{k} \; (\quad - \quad)^{k} + \; [(\quad)^{2} - \quad ^{2}] \quad + \quad = 0.$

Solution: $\quad = \tfrac{1}{2} \quad _{1}e \quad - \tfrac{1}{2} \quad _{2}e^{-} \quad .$ Here, the constants $_{1}$ and $_{2}$ are related by the constraint $a \quad _{2} + \; (\quad _{1} \quad _{2}) \quad + \quad = 0.$

18. $\quad ^{k} \quad (\quad + \gamma \quad)^{k} + \qquad (\quad + \quad) \quad + \quad = 0, \qquad \neq \gamma.$

Solution: $\quad = \dfrac{\quad _{1}}{\quad - \beta}e^{-} \quad - \dfrac{\quad _{2}}{\quad - \beta}e^{-} \quad .$ Here, the constants $_{1}$ and $_{2}$ are related by the constraint $a \quad _{1} + \quad _{2} + \quad = 0.$

19. $\quad (\quad \cosh \quad - \quad \sinh \quad) \quad + \; (\quad \sinh \quad - \quad \cosh \quad)^{k} + \quad = 0.$

Solution: $\quad = \quad _{1}\sinh \quad - \quad _{2}\cosh \quad .$ Here, the constants $_{1}$ and $_{2}$ are related by the constraint $a \quad _{1} + \quad _{2} + \quad = 0.$

20. $\quad (\quad \cosh \quad - \quad \sinh \quad) \quad + \; (\quad ^{2} - \quad ^{2})^{k} + \quad = 0.$

Solution: $\quad = \quad _{1}\sinh \quad + \quad _{2}\cosh \quad .$ Here, the constants $_{1}$ and $_{2}$ are related by the constraint $a \quad _{1} + \; (\quad ^{2}_{1} - \quad ^{2}_{2}) \quad + \quad = 0.$

21. $\quad (\quad \cosh \quad - \quad \sinh \quad) \quad = \; (\quad ^{2} - \quad ^{2}) \quad \cosh \quad .$

The contact transformation $X = (\quad ')^{2} - \quad ^{2}, \quad = \quad ' \cosh \quad - \quad \sinh \quad , \quad ' = \tfrac{1}{2}\cosh \quad (\quad ')^{-1}$ leads to a separable equation: $2 \quad X \qquad ' = \qquad .$

22. $\quad (\quad \sinh \quad - \quad \cosh \quad) \quad = \; (\quad ^{2} - \quad ^{2}) \quad \sinh \quad .$

The contact transformation $X = (\quad ')^{2} - \quad ^{2}, \quad = \quad ' \sinh \quad - \quad \cosh \quad , \quad ' = \tfrac{1}{2}\sinh \quad (\quad ')^{-1}$ leads to a separable equation: $2 \quad X \qquad ' = \qquad .$

23. $\quad (\quad \cos \quad + \quad \sin \quad) \quad + \; (\quad \sin \quad - \quad \cos \quad)^{k} + \quad = 0.$

Solution: $\quad = \quad _{1}\sin \quad - \quad _{2}\cos \quad .$ Here, the constants $_{1}$ and $_{2}$ are related by the constraint $a \quad _{1} + \quad _{2} + \quad = 0.$

24. $(\ \cos\ +\ \sin\ )\ +\ (\ ^2 +\ ^2)^k +\ = 0.$

Solution: $=\ _1 \sin\ +\ _2 \cos\ $. Here, the constants $_1$ and $_2$ are related by the constraint $a\ _1 + (\ _1^2 +\ _2^2)\ +\ = 0$.

25. $(\ \cos\ +\ \sin\ )\ =\ (\ ^2 +\ ^2)\ \cos\ .$

The contact transformation $X = (\ ')^2 +\ ^2,\ \ =\ '\cos\ +\ \sin\ ,\ \ ' = \frac{1}{2}\cos\ (\ ')^{-1}$ leads to a separable equation: $2\ X\ \ ' =\ \ .$

26. $(\ \sin\ -\ \cos\ )\ =\ (\ ^2 +\ ^2)\ \sin\ .$

The contact transformation $X = (\ ')^2 +\ ^2,\ \ =\ '\sin\ -\ \cos\ ,\ \ ' = \frac{1}{2}\sin\ (\ ')^{-1}$ leads to a separable equation: $2\ X\ \ ' =\ \ .$

1.6.4-2. Equations containing exponential, logarithmic, and other functions with respect to $'$.

27. $=\ \exp(\ \ ) +\ \exp(\ \ ).$

This is a special case of equation 1.8.1.7 with $(\ ) = a\exp(\lambda\ ) +\ \exp(\ \ )$.

28. $=\ \exp(\ \ ) +\ \exp(\ \ ).$

This is a special case of equation 1.8.1.8 with $(\ ) = a\exp(\lambda\ ) +\ \exp(\ \ )$.

29. $=\ \ +\ \ \exp(\ \ ).$

This is a special case of equation 1.8.1.15 with $(\ ) = a\exp(\lambda\ )$.

30. $=\ \ \exp(\ \ ) +\ \exp(\ \ ).$

This is a special case of equation 1.8.1.11 with $(\ ) = a\exp(\lambda\ )$ and $g(\ ) =\ \exp(\ \ )$.

31. $=\ \sinh(\ \ ) +\ \sinh(\ \ ).$

This is a special case of equation 1.8.1.7 with $(\ ) = a\sinh(\lambda\ ) +\ \sinh(\ \ )$.

32. $=\ \sinh(\ \ ) +\ \sinh(\ \ ).$

This is a special case of equation 1.8.1.8 with $(\ ) = a\sinh(\lambda\ ) +\ \sinh(\ \ )$.

33. $=\ \ +\ \ \sinh\ (\ \ ).$

This is a special case of equation 1.8.1.15 with $(\ ) = a\sinh\ (\lambda\ )$.

34. $=\ \ \sinh(\ \ ) +\ \sinh(\ \ ).$

This is a special case of equation 1.8.1.11 with $(\ ) = a\sinh(\lambda\ )$ and $g(\ ) =\ \sinh(\ \ )$.

35. $=\ \cosh(\ \ ) +\ \cosh(\ \ ).$

This is a special case of equation 1.8.1.7 with $(\ ) = a\cosh(\lambda\ ) +\ \cosh(\ \ )$.

36. $=\ \cosh(\ \ ) +\ \cosh(\ \ ).$

This is a special case of equation 1.8.1.8 with $(\ ) = a\cosh(\lambda\ ) +\ \cosh(\ \ )$.

37. $=\ \ +\ \ \cosh\ (\ \ ).$

This is a special case of equation 1.8.1.15 with $(\ ) = a\cosh\ (\lambda\ )$.

38. $=\ \ \cosh(\ \ ) +\ \cosh(\ \ ).$

This is a special case of equation 1.8.1.11 with $(\ ) = a\cosh(\lambda\ )$ and $g(\ ) =\ \cosh(\ \ )$.

39. **ln $\quad$ + $\quad$ + $\quad$ + $\quad$ = 0.**

1 . Solution in parametric form with $a \neq 0,\ a \neq -1$:

$$= \frac{1}{a} + \quad {}^{-\frac{1}{+1}}, \qquad = -\frac{1}{a}(\quad + \ln \quad + \quad).$$

2 . Solution in parametric form with $a = 0$:

$$= -\frac{\ln \quad + \quad}{}, \qquad = \quad + (\quad - 1) \ln \quad + \frac{1}{2}(\ln \quad)^2.$$

3 . Solutions with $a = -1$:

$$= \quad + \ln \quad + \quad \text{and} \quad = \ln(-1 \quad) + \quad - 1.$$

40. **+ ln $\quad$ + $\quad$ ($\quad$ + $\quad$)^k + $\quad$ = 0.**

This is a special case of equation 1.8.1.41 with $(\ , \ ) = \ln \quad + a \quad + \ .$

41. **= $\quad$ + $\quad$ ^{2} + $\quad$ ln $\quad$ + $\quad$, $\quad$ ≠ 0.**

Differentiating the equation with respect to and changing to new variables $= \ '$ and $(\ ) = -2a$, we arrive at an Abel equation of the form 1.3.1.16: $' - \quad = -2a \quad {}^{-1}$.

42. **= $\quad$ + $\quad$ ln $\quad$ ($\quad$).**

This is a special case of equation 1.8.1.15 with $(\ ) = a \ln \quad (\lambda \quad)$.

43. **= $\quad$ sin($\quad$) + $\quad$ sin($\quad$).**

This is a special case of equation 1.8.1.7 with $(\ ) = a \sin(\lambda \quad) + \quad \sin(\quad)$.

44. **= $\quad$ sin($\quad$) + $\quad$ sin($\quad$).**

This is a special case of equation 1.8.1.8 with $(\ ) = a \sin(\lambda \quad) + \quad \sin(\quad)$.

45. **= $\quad$ + $\quad$ sin $\quad$ ($\quad$).**

This is a special case of equation 1.8.1.15 with $(\ ) = a \sin \quad (\lambda \quad)$.

46. **= $\quad$ sin($\quad$) + $\quad$ sin($\quad$).**

This is a special case of equation 1.8.1.11 with $(\ ) = a \sin(\lambda \quad)$ and $g(\ ) = \quad \sin(\quad)$.

47. **= $\quad$ cos($\quad$) + $\quad$ cos($\quad$).**

This is a special case of equation 1.8.1.7 with $(\ ) = a \cos(\lambda \quad) + \quad \cos(\quad)$.

48. **= $\quad$ cos($\quad$) + $\quad$ cos($\quad$).**

This is a special case of equation 1.8.1.8 with $(\ ) = a \cos(\lambda \quad) + \quad \cos(\quad)$.

49. **= $\quad$ + $\quad$ cos $\quad$ ($\quad$).**

This is a special case of equation 1.8.1.15 with $(\ ) = a \cos \quad (\lambda \quad)$.

50. **= $\quad$ cos($\quad$) + $\quad$ cos($\quad$).**

This is a special case of equation 1.8.1.11 with $(\ ) = a \cos(\lambda \quad)$ and $g(\ ) = \quad \cos(\quad)$.

51. **= $\quad$ + $\quad$ tan $\quad$ ($\quad$).**

This is a special case of equation 1.8.1.15 with $(\ ) = a \tan \quad (\lambda \quad)$.

1.7. Equations of the Form $f(\ ,y)y\ =\ (\ ,y)$ Containing Arbitrary Functions

Notation: , *g, and* *are arbitrary composite functions whose argument, indicated after the function name, can depend on both* *and* .

1.7.1. Equations Containing Power Functions

1. $= f(\quad + \quad + \ ).$

In the case $= 0$, we have an equation of the form 1.1.1. If $\neq 0$, the substitution $(\ ) = a\ +\ +$ leads to a separable equation: $' = (\ ) + a$.

2. $= f(\ + \quad + \ ) - \quad ^{-1}.$

The substitution $= \ + a\ +$ leads to a separable equation: $' = (\ )$.

3. $= -f(\quad)$.

Generalized homogeneous equation. The substitution $z =$ leads to a separable equation: $z' = \ z + \ z\ (z)$.

4. $= f(\ )\ ^{1+} + (\ )\ + (\ )\ ^{1-}$.

The substitution $=$ leads to a Riccati equation: $' = (\ )\ ^2 + g(\ )\ + (\ )$.

5. $= - \text{——} + \ ^{k} f(\ )\ (\quad)$.

The substitution $z =$ leads to a separable equation: $z' = \frac{-}{}\ (\ ) z^{\frac{+\ -1}{}} g(z)$.

6. $= f\left(\frac{+\quad +}{+\quad +\gamma}\right)$.

1 . For $\Delta = a\beta -\ \neq 0$, the transformation $= \ + \frac{-\ \beta}{\Delta}$, $= v(\ ) + \frac{-a}{\Delta}$ leads to an equation:

$$v' = \frac{a\ +\ v}{+\beta v}.$$

Dividing both the numerator and denominator of the fraction on the right-hand side by , we obtain a homogeneous equation of the form 1.1.6.

2 . For $\Delta = 0$ and $\neq 0$, the substitution $v(\ ) = a\ +\ +$ leads to a separable equation of the form 1.1.2:

$$v' = a + \frac{v}{\beta v + \ -\ \beta}.$$

3 . For $\Delta = 0$ and $\beta \neq 0$, the substitution $v(\ ) = \ + \beta\ +$ also leads to a separable equation:

$$v' = \ + \beta \frac{v + \ \beta -}{\beta v}.$$

7. $= \ ^{-1}\ ^{1-} f(\quad + \quad)$.

The substitution $= a\ +$ leads to a separable equation: $' = \ ^{-1}[a\ +\ (\ )]$.

8. $+ \quad + (\ ) f(\ ^{+1} + \quad ^{+1}) = 0.$

The substitution $= \ ^{+1} + a\ ^{+1}$ leads to a separable equation: $' + (\ + 1) g(\ )\ (\ ) = 0$.

9. $[\quad f(\ )+\quad(\ )]\quad=\ (\ ).$

This is a Bernoulli equation with respect to $\ =\ (\ )$ (see Subsection 1.1.5).

10. $[\ ^{2}+\ f(\ )+\ (\ )]\quad=\ (\ ).$

This is a Riccati equation with respect to $\ =\ (\ )$ (see Section 1.2).

11. $\quad=[f(\ )\ +\ (\ )]\ \overline{(\ -\ )(\ -\ )}.$

The substitution $\ ^{2}=(\ -a)\ (\ -\ )$ leads to a Riccati equation: $2\ '=[\ (\ )+g(\ )]\ ^{2}-a\ (\ )-g(\ ).$

12. $f\ —\ +\quad —\quad=\ —\ +\quad ^{-1}\ —\ .$

The substitution $\ =\ $ leads to a Bernoulli equation with respect to $\ =\ (\ )$: $[g(\ )-\ (\ )]\ '=\ (\ )\ +\ (\ )\ ^{+1}.$

13. $[\ (\ ,\ )+\quad(\ ,\ )]\quad=Q\ (\ ,\ )+\quad(\ ,\ ).$

Darboux equation. Here, and Q are homogeneous polynomials of order , and R is a homogeneous polynomial of order . Dividing the Darboux equation by leads to an equation of the form 1.7.1.12.

14. $[f(\quad+\ )+\quad(\quad+\ )]\quad=\ (\quad+\ )-\quad(\quad+\ ).$

The substitution $\ =a\ +\ $ leads to a linear equation with respect to $\ =\ (\ )$: $[a\ (\ )+\ (\ )]\ '=\ g(\ )\ +\ (\ ).$

15. $[f(\quad+\ )+\quad(\quad+\ )]\quad=\ (\quad+\ )-\quad(\quad+\ ).$

The substitution $\ =a\ +\ $ leads to a linear equation with respect to $\ =\ (\ )$: $[a\ (\ )+\ (\ )]\ '=-ag(\ )\ +\ (\ ).$

16. $[f(\quad)+\quad ^{k}\ (\quad)]\quad=\ [\ (\quad)-\quad ^{k}\ (\quad)].$

The transformation $\ =\ ,\ z=\ ^{-}$ leads to a linear equation with respect to $z=z(\ )$: $[\ (\ )+\ (\ )]z'=-k\ (\ )z-k\ g(\ ).$

17. $[f(\quad)+\quad ^{k}\ (\quad)]\quad=\ [\ (\quad)-\quad ^{k}\ (\quad)].$

The transformation $\ =\ ,\ z=\ ^{-}$ leads to a linear equation with respect to $z=z(\ )$: $[\ (\ )+\ (\ )]z'=-k\ (\ )z+k\ g(\ ).$

18. $[sf(\quad)-\quad(\ ^{k}\ ^{s})]\quad=\ [\ (\ ^{k}\ ^{s})-\ f(\quad)].$

The transformation $\ =\ ,\ =\ $ leads to a separable equation: $\ (\ )\ '=\ g(\ ).$

19. $[f(\ )+\quad ^{-1}]\ +\ (\ )+\quad ^{-1}\quad=0.$

Solution: $\ (\ )\ +\ g(\ )\ +a\quad=\ .$

20. $\ -\ =f(\ ^{2}+\ ^{2})(\quad+\ ).$

Setting $\ =\ (\ )\cos\ ,\ =\ (\ )\sin\ $ and integrating, we obtain a solution in implicit form: $\ =\ ^{-1}\ (\ ^{2})\ +\ .$

Reference: G. W. Bluman, J. D. Cole (1974, p. 100).

21. $- = f(\ ^2 - \ ^2)(\ \ -\ )$.

Setting $= (\ ) \cosh\ ,\ = (\ ) \sinh$ and integrating, we obtain a solution in implicit form: $= -\ ^{-1}\ (\ ^2)\ +\ .$

22. $[\ f(\ ^2 + \ ^2) + \ (\ ^2 + \ ^2) + \ (\ ^2 + \ ^2)](\ \ +\ ) = \ \ -\ .$

The transformation $= \cos\ ,\ = \sin$ leads to an equation of the form 1.7.4.11 with respect to $= (\ )$: $\ ' = \ (\ ^2) \cos\ + g(\ ^2) \sin\ + \ ^{-1}\ (\ ^2)$.

23. $[\ f(\ ^2 - \ ^2) + \ (\ ^2 - \ ^2) + \ (\ ^2 - \ ^2)](\ \ -\ ) = \ \ -\ .$

1 . For $>$, the transformation $= \cosh\ ,\ = \sinh$ leads to an equation of the form 1.7.2.18 with respect to $= (\ )$: $\ ' = -\ (\ ^2) \cosh\ - g(\ ^2) \sinh\ - \ ^{-1}\ (\ ^2)$.

2 . For $<$, the transformation $= z \sinh\ ,\ = z \cosh$ leads to an equation of the form 1.7.2.18 with respect to $= (z)$: $\ ' = -\ (-z^2) \sinh\ - g(-z^2) \cosh\ - z^{-1}\ (-z^2)$.

24. $- = f\Bigl(\ ^2 + \ ^2\ \ \frac{}{\overline{\ ^2 + \ ^2}}, \frac{}{\overline{\ ^2 + \ ^2}}\Bigr)(\ \ +\ )$.

Setting $= (\ ) \cos\ ,\ = (\ ) \sin$ and integrating, we obtain the solution:

$$\int \frac{}{g(\cos\ , \sin\ )} = \int \frac{(\ ^2)}{}\ \ +\ .$$

25. $- = f\Bigl(\ ^2 - \ ^2\ \ \frac{}{\overline{\ ^2 - \ ^2}}, \frac{}{\overline{\ ^2 - \ ^2}}\Bigr)(\ \ -\ )$.

Setting $= (\ ) \cosh\ ,\ = (\ ) \sinh$ and integrating, we obtain the solution:

$$\int \frac{}{g(\cosh\ , \sinh\ )} + \int \frac{(\ ^2)}{}\ \ =\ .$$

26. $f(\ ,\ )\ \ + \ (\ ,\ ) = 0,$ **where** $\frac{f}{} = \frac{}{}$.

Exact differential equation.

Solution: $\int_0 (\ _0,\ )\ \ + \int_0 g(\ ,\ )\ \ = \ ,$ where $_0$ and $_0$ are arbitrary numbers.

1.7.2. Equations Containing Exponential and Hyperbolic Functions

1. $= \ ^{-\lambda}\ f(\ ^{\lambda}\ )$.

The substitution $= e^{\lambda}$ leads to a separable equation: $' = (\ ) + \lambda\ .$

2. $= \ ^{\lambda}\ f(\ ^{\lambda}\ )$.

The substitution $= e^{\lambda}$ leads to a separable equation: $\ ' = \lambda\ ^2\ (\ ) +\ .$

3. $= \ f(\ \ \ )$.

This equation is invariant under “translation–dilatation” transformation. The substitution $z = e$ leads to a separable equation: $z' = \ z + \ z\ (z)$.

4. $= \frac{1}{} f(\ \ \ )$.

This equation is invariant under “dilatation–translation” transformation. The substitution $z = \ e$ leads to a separable equation: $z' = \ z + \ z\ (z)$.

5. $= f(\ )^{\lambda} + (\ ).$

The substitution $= e^{-\lambda}$ leads to a linear equation: $' = -\lambda g(\ ) - \lambda (\ ).$

6. $= -- + f(\ )(\ ).$

The substitution $z = \ e$ leads to a separable equation: $z' = (\ ) z g(z).$

7. $= -- + {}^{k} f(\ )(\ ).$

The substitution $z = e$ leads to a separable equation:

$$z' = \exp -(1-k) \quad (\ ) z^{\frac{+\ -1}{}} g(z).$$

8. $= f(\ )^{\lambda} + (\ ) + (\ )^{-\lambda}.$

The substitution $= e^{\lambda}$ leads to a Riccati equation: $' = \lambda (\ )^{2} + \lambda g(\ ) + \lambda (\ ).$

9. $= \ ^{-} f(\ + \ ).$

The substitution $= ae + e$ leads to a separable equation: $' = e [a + \beta (\ )].$

10. $= f(\ + \ ^{\lambda} + \ ) - \ ^{\lambda}.$

The substitution $= + ae^{\lambda} +$ leads to a separable equation: $' = (\ ).$

11. $= -- + \frac{f(\ )}{}.$

The substitution $= \ e$ leads to a linear equation with respect to $= (\ )$: $^{2} (\ )' = - + (\ ).$

12. $= -- + \frac{f(\ )}{2}.$

The substitution $= \ e$ leads to a Riccati equation: $^{2} (\ )' = - \ ^{2} + (\ ).$

13. $[f(\ + \ ) + (\ + \ )] = (\ + \ ) - (\ + \ ).$

The transformation $= a +$, $z = e^{-}$ leads to a linear equation with respect to $z = z(\ )$: $[a (\ ) + (\ )] z' = - (\ ) z + a g(\ ).$

14. $[f(\ + \ ) + (\ + \ )] = (\ + \ ) - (\ + \ ).$

The transformation $= a +$, $z = e^{-}$ leads to a linear equation with respect to $z = z(\ )$: $[a (\ ) + (\ )] z' = - (\ ) z - g(\ ).$

15. $[\ f(\ ) + \ ] + (\ ) + = 0.$

Solution: $e^{-} (\ ) + e^{-} g(\ ) - ae^{-}\ ^{-} = .$

16. $[f(\ ) + (\ )] = (\ ) - (\ ).$

The substitution $= \ e$ leads to a linear equation with respect to $= (\ )$: $[\ (\ ) + (\ )]' = - g(\ ) + (\ ).$

17. $[f(\ ) + (\ )] = [(\ ) - (\ )].$

The substitution $= e$ leads to a linear equation with respect to $= (\ )$: $[\ (\ ) + (\ )]' = g(\ ) + (\ ).$

18. $= f(\ )\sinh(\ ) + (\ )\cosh(\ ) + (\ ).$

The substitution $= e^{\lambda}$ leads to a Riccati equation: $2\ ' = \lambda(\ + g)\ ^2 + 2\lambda\ + \lambda(g - \ ).$

19. $= f(\ )\sinh^2(\ ) + (\ )\cosh^2(\ ) + (\ )\sinh(2\ ) + s(\ ).$

The substitution $= \tanh(\lambda\ )$ leads to a Riccati equation: $' = \lambda(\ + \ )\ ^2 + 2\lambda\ + \lambda(g - \ ).$

20. $= f(\ )\sinh(\ + (\ )\ .$

The substitution $= \lambda\ + g(\ )$ leads to an equation of the form 1.7.2.18: $' = \lambda\ (\ )\sinh\ + g'(\ ).$

21. $= f(\ )\cosh(\ + (\ )\ .$

The substitution $= \lambda\ + g(\ )$ leads to an equation of the form 1.7.2.18: $' = \lambda\ (\ )\cosh\ + g'(\ ).$

22. $= \coth\ f(\ \sinh\ ).$

The transformation $= \sinh\ ,\ z =$ leads to an equation of the form 1.7.1.3: $z' = \ z\ (\ z).$

23. $= {}^{-1}\tanh\ f(\ \sinh\ ).$

The transformation $= \ ,\ z = \sinh$ leads to an equation of the form 1.7.1.3: $z' = z\ (\ z).$

24. $= \tanh\ f(\ \cosh\ ).$

The substitution $= \cosh$ leads to an equation of the form 1.7.1.3: $' = \ (\ ).$

25. $= {}^{-1}\coth\ f(\ \cosh\ ).$

The substitution $z = \cosh$ leads to an equation of the form 1.7.1.3: $z' = z\ (\ z).$

1.7.3. Equations Containing Logarithmic Functions

1. $= f(\ )\ \ln^2\ + (\ )\ \ln\ + (\ )\ .$

The substitution $= \ln$ leads to a Riccati equation: $' = (\ )\ ^2 + g(\ )\ + (\ ).$

2. $= {}^{-1}\ {}^{+1}f(\ \ln\ ).$

The substitution $= \ln$ leads to an equation of the form 1.7.1.3: $' = -[\ \ (\ )].$

3. $= {}^{-\ -1}\ f(\ \ln\ ).$

The substitution $z = \ln$ leads to an equation of the form 1.7.1.3: $z' = \dfrac{z\ (\ z)}{z}.$

4. $= {}^{-1}\ f(\ \ln\ ).$

The substitution $= \ln$ leads to an equation of the form 1.7.2.4: $' = \dfrac{1}{\ }[\ e\ \ (\ e\ )].$

5. $= {}^{-}\ f(\ \ln\ ).$

The substitution $z = \ln$ leads to an equation of the form 1.7.2.3: $z' = z\dfrac{(e\ z)}{e\ z}.$

6. $= -\ {}^{-1}\ \ln\ + f(\ )\ (\ \ln\ ).$

The substitution $(\ ) = \ \ln$ leads to a separable equation: $' = \ (\ )g(\ ).$

7. $= -\!\!—\!\!— + \dfrac{f(\quad)}{\ln}.$

The transformation $=$, $z = \ln$ leads to a linear equation with respect to $z = z(\ )$: $^2\ (\ )z' = -\ z + \ (\ )$.

8. $= -\!\!—\!\!— + \dfrac{f(\quad)}{(\ln\ )^2}.$

The transformation $=$, $z = \ln$ leads to a Riccati equation: $^2\ (\ )z' = -\ z^2 + \ (\ )$.

9. $[f(\quad) + \ \ln\ (\quad)]\ = [\ (\quad) - \ \ln\ (\quad)].$

The transformation $=$, $z = \ln$ leads to a linear equation with respect to $z = z(\ )$: $[\ (\ ) + \ (\ )]z' = -\ g(\ )z + \ (\ )$.

10. $[f(\quad) + \ \ln\ (\quad)]\ = [\ (\quad) - \ \ln\ (\quad)].$

The transformation $=$, $z = \ln$ leads to a linear equation with respect to $z = z(\ )$: $[\ (\ ) + \ (\ )]z' = \ g(\ )z + \ (\ )$.

1.7.4. Equations Containing Trigonometric Functions

1. $= \ ^{+1} \sin\ (\ \cos\ ).$

This is an equation of the type 1.7.4.3 with $(\xi) = \xi\ (\xi)$.

2. $= \ ^{+1} \cos\ (\ \sin\ ).$

This is an equation of the type 1.7.4.4 with $(\xi) = \xi\ (\xi)$.

3. $= \tan\ f(\ \cos\ ).$

The substitution $= \cos$ leads to an equation of the form 1.7.1.3: $' = -\ (\ )$.

4. $= \cot\ f(\ \sin\ ).$

The substitution $= \sin$ leads to an equation of the form 1.7.1.3: $' = \ (\ )$.

5. $= \ ^{-1} \tan\ f(\ \sin\ ).$

The transformation $=$, $z = \sin$ leads to an equation of the form 1.7.1.3: $z' = z\ (\ z)$.

6. $= \ ^{-1} \cot\ f(\ \cos\ ).$

The transformation $=$, $z = \cos$ leads to an equation of the form 1.7.1.3: $z' = -z\ (\ z)$.

7. $= \ ^{-1} \sin 2\ f(\ \tan\ ).$

The transformation $=$, $z = \tan$ leads to an equation of the form 1.7.1.3: $z' = 2z\ (\ z)$.

8. $= \ ^{-1} \sin 2\ f(\ \cot\ ).$

The transformation $=$, $z = \cot$ leads to an equation of the form 1.7.1.3: $z' = -2z\ (\ z)$.

9. $= \dfrac{}{\sin 2} f(\ \tan\ ).$

The substitution $= \tan$ leads to an equation of the form 1.7.1.3: $2\ ' = \ (\ )$.

10. $= \dfrac{}{\sin 2} f(\ \cot\ ).$

The substitution $= \cot$ leads to an equation of the form 1.7.1.3: $2\ ' = -\ (\ )$.

11. $\quad = f(\ )\cos(\ \ ) + (\ )\sin(\ \ ) + (\ ).$

The substitution $= \tan(a\ \ 2)$ leads to a Riccati equation: $2\ ' = a(\ -\ )\ ^2 + 2ag\ + a(\ +\ ).$

12. $\quad = f(\ )\cos^2(\ \ ) + (\ )\sin^2(\ \ ) + (\ )\sin(2\ \ ) + s(\ ).$

The substitution $= \tan(a\ )$ leads to a Riccati equation: $' = a(g +\ )\ ^2 + 2a\ \ + a(\ +\ ).$

13. $\quad = f(\ +\ \tan\ ) - \ \tan^2\ .$

The substitution $= \ + a\tan\ $ leads to a separable equation: $' = a + (\ ).$

14. $$= \frac{\sin 2}{\sin 2} f(\tan\ \ \tan\ ).$$

The transformation $= \tan\ ,\ z = \tan\ $ leads to an equation of the form 1.7.1.3: $z' = z\ (\ z).$

15. $\quad = \cot\ \ \tan\ \ f(\sin\ \ \sin\ ).$

The transformation $= \sin\ ,\ z = \sin\ $ leads to an equation of the form 1.7.1.3: $z' = z\ (\ z).$

16. $$= -\cot\ \ \tan\ \ + \frac{f(\ )}{\cos}\ (\sin\ \ \sin\ ).$$

The substitution $(\ ) = \sin\ \ \sin\ $ leads to a separable equation: $' = \sin\ \ (\ )g(\ ).$

17. $$= -\frac{\sin 2}{\sin 2} + \cos^2\ \ f(\ )\ (\tan\ \ \tan\ ).$$

The substitution $(\ ) = \tan\ \ \tan\ $ leads to a separable equation: $' = \tan\ \ (\ )g(\ ).$

18. $\quad = -\ \ ^{-1}\sin 2\ + \cos^2\ \ f(\ )\ (\ ^2\ \ \tan\ ).$

The substitution $(\ ) = \ ^2\ \tan\ $ leads to a separable equation: $' = \ ^2\ \ (\ )g(\ ).$

19. $\quad (1 + \tan^2\ )\ \ = f(\ )\tan\ ^{+1}\ + (\ )\tan\ + (\ )\tan^{1-}\ .$

The substitution $= \tan\ $ leads to a Riccati equation: $' = \ (\ )\ ^2 + \ g(\ )\ + \ (\ ).$

20. $\quad = f(\ )\sin\big(\ \ + (\ )\ \big).$

The substitution $= \lambda\ + g(\ )$ leads to an equation of the form 1.7.4.11: $' = \lambda\ (\ )\sin\ \ + g'(\ ).$

21. $\quad = f(\ )\cos\big(\ \ + (\ )\ \big).$

The substitution $= \lambda\ + g(\ )$ leads to an equation of the form 1.7.4.11: $' = \lambda\ (\ )\cos\ \ + g'(\ ).$

22. $\quad = f(\ )\sin^2\big(\ \ + (\ )\ \big).$

The substitution $= \lambda\ + g(\ )$ leads to an equation of the form 1.7.4.12: $' = \lambda\ (\ )\sin^2\ \ + g'(\ ).$

23. $\quad = f(\ )\cos^2\big(\ \ + (\ )\ \big).$

The substitution $= \lambda\ + g(\ )$ leads to an equation of the form 1.7.4.12: $' = \lambda\ (\ )\cos^2\ \ + g'(\ ).$

1.7.5. Equations Containing Combinations of Exponential, Logarithmic, and Trigonometric Functions

1. $\quad = -\sin 2\ + \cos^2\ \ f(\ )\ (\ ^2\ \ \tan\ ).$

The substitution $(\ ) = e^2\ \tan\ $ leads to a separable equation: $' = e^2\ \ (\ )g(\ ).$

2. $= \dfrac{(\quad \cos \quad)}{\sin}$.

This is an equation of the type 1.7.5.5 with $(\xi) = \quad (\xi)\ \xi$.

3. $= \quad \cos \quad (\quad \sin \quad)$.

This is an equation of the type 1.7.5.7 with $(\xi) = \xi \quad (\xi)$.

4. $= \tan \quad f(\quad \sin \quad)$.

The substitution $z = \sin$ leads to an equation of the form 1.7.2.3: $z' = z \ (e \ z)$.

5. $= \cot \quad f(\quad \cos \quad)$.

The substitution $z = \cos$ leads to an equation of the form 1.7.2.3: $z' = -z \ (e \ z)$.

6. $= \tan \quad f(\quad \cos \quad)$.

The substitution $= \cos$ leads to an equation of the form 1.7.2.4: $' = - \ (\ e \ )$.

7. $= \cot \quad f(\quad \sin \quad)$.

The substitution $= \sin$ leads to an equation of the form 1.7.2.4: $' = \ (\ e \ )$.

8. $= \sin 2 \quad f(\quad \tan \quad)$.

The substitution $z = \tan$ leads to an equation of the form 1.7.2.3: $z' = 2z \ (e \ z)$.

9. $= \sin 2 \quad f(\quad \cot \quad)$.

The substitution $z = \cot$ leads to an equation of the form 1.7.2.3: $z' = -2z \ (e \ z)$.

10. $= \dfrac{(\quad \sin \quad)}{\cos}$.

This is an equation of the type 1.7.5.4 with $(\xi) = \quad (\xi)\ \xi$.

11. $= \quad \sin \quad (\quad \cos \quad)$.

This is an equation of the type 1.7.5.6 with $(\xi) = \xi \quad (\xi)$.

12. $= \dfrac{f(\quad \tan \quad)}{\sin 2}$.

The substitution $= \tan$ leads to an equation of the form 1.7.2.4: $2 \quad ' = \ (\ e \ )$.

13. $= \dfrac{f(\quad \cot \quad)}{\sin 2}$.

The substitution $= \cot$ leads to an equation of the form 1.7.2.4: $2 \quad ' = - \ (\ e \ )$.

14. $= \ ^{-\lambda} \ f(\quad + \ln \quad)$.

The substitution $= \lambda \quad + \ln$ leads to a separable equation: $' = e^{-} \quad (\ ) + \lambda$.

15. $= \ ^{\lambda} \ f(\quad + \ln \quad)$.

The substitution $= \lambda \quad + \ln$ leads to a separable equation: $' = \lambda e \quad (\ ) + 1$.

1.8. Equations of the Form $(\; , y, y \;) = 0$ Containing Arbitrary Functions

1.8.1. Some Equations

1.8.1-1. Arguments of arbitrary functions depend on and .

1. $(\quad)^2 + [f(\;) + \; (\;)] \quad + f(\;) \; (\;) = 0.$

The equation can be factorized: $[\; ' + \; (\;)][\; ' + g(\;)] = 0$, i.e., it falls into two simpler equations $\; ' + \; (\;) = 0$ and $\; ' + g(\;) = 0$. Therefore, the solutions are:

$$+ \quad (\;) \quad = \quad \text{and} \quad + \quad g(\;) \quad = \quad .$$

2. $(\quad)^2 + 2f \qquad + \quad ^2 = (\quad - f^2) \exp \; -2 \qquad f \qquad .$

Here, $\; = \; (\;), g = g(\;)$. Solution:

$$= \begin{cases} \exp \; - \qquad \sin \qquad \sqrt{g - \;^2} \qquad + & \text{if } g > \;^2, \\ \quad \exp \; - & \text{if } g \equiv \;^2, \\ \exp \; - \qquad \cosh \qquad \sqrt{\;^2 - g} \qquad + & \text{if } g < \;^2. \end{cases}$$

3. $\quad - \quad = f(\;^2 + \;^2) \; \sqrt{(\quad)^2 + 1}.$

Raising the equation to the second power and applying the transformation $\; = \; (\;) \cos \;$, $\; = \; (\;) \sin \;$, one arrives at the relation $\;^4 = \;^2(\;^2)[(\; ')^2 + \;^2]$. Solving it for $\; '$ yields a separable equation: $\; (\;^2) \; ' = \; \sqrt{\;^2 - \;^2(\;^2)}$.

4. $\quad + \quad = f(\;^2 + \;^2) \; \sqrt{(\quad)^2 + 1}.$

Raising the equation to the second power and applying the transformation $\; = \; (\;) \cos \;$, $\; = \; (\;) \sin \;$, one arrives at the relation $\;^2(\; ')^2 = \;^2(\;^2)[(\; ')^2 + \;^2]$. Solving it for $\; '$ yields a separable equation: $\; ' = \dfrac{\; (\;^2)}{\sqrt{\;^2 - \;^2(\;^2)}}$.

Reference: G. W. Bluman, J. D. Cole (1974, p. 100).

5. $\quad - \quad = \sqrt{\;^2 + \;^2} \, f \left(\dfrac{\quad}{\sqrt{\;^2 + \;^2}}, \dfrac{\quad}{\sqrt{\;^2 + \;^2}} \right) \sqrt{(\quad)^2 + 1}.$

Raising the equation to the second power and applying the transformation $\; = \; (\;) \cos \;$, $\; = \; (\;) \sin \;$, one arrives at the relation $\;^2 = \;^2(\cos \; , \sin \;)[(\; ')^2 + \;^2]$. Solving it for $\; '$ yields a separable equation: $\; (\cos \; , \sin \;) \; ' = \; \sqrt{1 - \;^2(\cos \; , \sin \;)}$.

6. $\quad + \quad = \sqrt{\;^2 + \;^2} \, f \left(\dfrac{\quad}{\sqrt{\;^2 + \;^2}}, \dfrac{\quad}{\sqrt{\;^2 + \;^2}} \right) \sqrt{(\quad)^2 + 1}.$

Raising the equation to the second power and applying the transformation $\; = \; (\;) \cos \;$, $\; = \; (\;) \sin \;$, one arrives at the relation $\;^2(\; ')^2 = \;^2(\cos \; , \sin \;)[(\; ')^2 + \;^2]$. Solving it for $\; '$ yields a separable equation: $\sqrt{1 - \;^2(\cos \; , \sin \;)} \; ' = \; (\cos \; , \sin \;)$.

1.8.1-2. Argument of arbitrary functions is $'$.

7. $= f(\;)$.

Solution in parametric form:

$$= (\;), \qquad = \quad '(\;) \quad + \quad .$$

8. $= f(\;)$.

Solution in parametric form:

$$= \quad '(\;) — + \quad , \qquad = (\;).$$

9. $f(\;) + \quad + \quad + s = 0$.

Solution in parametric form:

$$= \quad - \quad \frac{'(\;)}{a +}, \qquad = -a \quad - \quad - \quad (\;).$$

In addition, there is a particular solution $= \quad + \beta$, where and β are determined by solving the system of two algebraic equations:

$$a + \quad = 0, \qquad (\;) + \quad \beta + \quad = 0.$$

10. $= \quad + f(\;)$.

The Clairaut equation. Solution: $= \quad + \quad (\;)$.

In addition, there is a singular solution, which may be written in the parametric form as:

$$= - \quad '(\;), \qquad = - \quad '(\;) + \quad (\;).$$

11. $= \quad f(\;) + \quad (\;)$.

The Lagrange–d'Alembert equation. For the case $(\;) =$, see equation 1.8.1.10. Having differentiated with respect to , we arrive at a linear equation with respect to $= \quad (\;)$, where $= \quad '$: $[\quad - \quad (\;)] \quad ' = \quad '(\;) \quad + g'(\;)$. See also 1.8.1.12.

12. $f(\;) + \quad (\;) + \quad (\;) = 0$.

The Legendre transformation $X = \quad '$, $= \quad ' - \quad$, $' =$ leads to a linear equation: $[\quad (X) + X g(X)] \quad ' - g(X) \quad + \quad (X) = 0$.

Inverse transformation: $= \quad '$, $= X \quad ' - \quad$, $' = X$.

13. $= \quad {}^2 f(\;) + \quad (\;) + \quad (\;)$.

Having differentiated with respect to , we arrive at an Abel equation with respect to $= \quad (\;)$, where $= \quad '$:

$$\bigl[2 \quad (\;) \quad + g(\;) - \quad \bigr] \quad ' = - \quad '(\;) \quad {}^2 - g'(\;) \quad - \quad '(\;)$$

(see Subsection 1.3.4).

14. $= \quad {}^2 f(\;) + \quad (\;) + \quad (\;)$.

Having differentiated with respect to , we arrive at an Abel equation with respect to $= \quad (\;)$, where $= \quad '$:

$$\bigl[2 \quad (\;) \quad + \quad g(\;) - 1\bigr] \quad ' = - \quad '(\;) \quad {}^2 - \quad g'(\;) \quad - \quad '(\;)$$

(see Subsection 1.3.4).

15. $= \quad f(\;) + \quad .$

Differentiating with respect to and denoting $= \quad '$, we obtain a Bernoulli equation for $= \quad (\;)$: $(\;) \quad ' - \quad '(\;) \quad - \quad {}^{2-} \quad = 0$.

16. $(\quad - \quad) \quad f(\;) + \quad (\;) + \quad (\;) = 0$.

The Legendre transformation $= \quad '$, $= \quad ' - \quad$ $(\quad ' = \quad)$ leads to a Bernoulli equation: $[\quad g(\;) + \quad (\;)] \quad ' = g(\;) \quad + \quad (\;) \quad .$

1.8.1-3. Arguments of arbitrary functions are linear with respect to $'$.

17. $= (\quad)^2 + f(\quad - 2\quad).$

Solution: $= (\ ) + \dfrac{1}{4a}(\ -\ )^2$. In addition, there is a singular solution, which can be represented in parametric form as:

$$= \ + 2a\ '(\ ), \qquad = (\ ) + a[\ '(\ )]^2.$$

18. $= f(\quad + \quad) + (\quad + \quad)(\ ^2 + 2\quad).$

The contact transformation $X = \ ' + a\ ,\ = \tfrac{1}{2}(\ ')^2 + a\ ,\ ' = \ '$, where $= (X)$, leads to a linear equation: $' = 2g(X)\ + (X)$.

Inverse transformation: $= a^{-1}(X - \ '),\ = \tfrac{1}{2}a^{-1}[2\ - (\ ')^2],\ ' = \ '$.

19. $= f(\quad + \quad)(\ ^2 + 2\quad) + (\quad + \quad)(\ ^2 + 2\quad)^k.$

The contact transformation $X = \ ' + a\ ,\ = \tfrac{1}{2}(\ ')^2 + a\ ,\ ' = \ '$, where $= (X)$, leads to a Bernoulli equation: $' = 2\ (X)\ + 2\ g(X)\ $.

Inverse transformation: $= a^{-1}(X - \ '),\ = \tfrac{1}{2}a^{-1}[2\ - (\ ')^2],\ ' = \ '$.

20. $= f(\quad).$

The substitution $= \ln$ leads to an equation of the form 1.8.1.32: $= (\ '\ )$.

21. $= f(\ )\ (\quad - \ ).$

The Legendre transformation $X = \ ',\ = \ ' - \ ,\ ' =$ leads to a separable equation: $' = (X)g(\ )$.

Inverse transformation: $= \ ',\ = X\ ' - \ ,\ ' = X$.

22. $f(\quad - \ ) + (\quad - \ )(\ )^k = \ .$

The modified Legendre transformation $X = \ ' - \ ,\ = \ ',\ ' = 1$ leads to a Bernoulli equation: $a\ ' = (X)\ + g(X)\ $.

Inverse transformation: $= (\ ')^{-1},\ = (\ ')^{-1} - X,\ ' = \ $.

23. $= f(\quad + \ )\ (\ ^2\ ).$

The contact transformation $X = \ ' + \ ,\ = \ ^2\ ',\ ' = \ $, where $= (X)$, leads to a separable equation: $' = (X)g(\ )$.

Inverse transformation: $= \ ',\ = X - (\ ')^{-1},\ ' = (\ ')^{-2}$.

24. $= f(\quad + \ ) + \ ^2\ (\quad + \ ).$

The contact transformation $X = \ ' + \ ,\ = \ ^2\ ',\ ' = \ $, where $= (X)$, leads to a linear equation: $' = g(X)\ + (X)$.

Inverse transformation: $= \ ',\ = X - (\ ')^{-1},\ ' = (\ ')^{-2}$.

25. $f(\quad + \ ) + \ ^3(\ )^2\ (\quad + \ ) = \ .$

The contact transformation $X = \ ' + \ ,\ = \ ^2\ ',\ ' = \ $, where $= (X)$, leads to a Bernoulli equation: $a\ ' = (X)\ + g(X)\ ^2$.

Inverse transformation: $= \ ',\ = X - (\ ')^{-1},\ ' = (\ ')^{-2}$.

26. $f(\quad + \quad) = \ ^2(\ ^2 + 1).$

Setting $(\ ) = \ ' +$ and differentiating with respect to , we obtain

$$'[\ '(\ ) - 2 \ + 2 \ ] = 0. \qquad (1)$$

Equating the first factor to zero, after integrating we find $^2 = -(\ -\ )^2 + B$. Substituting the latter into the original equation yields $B = \ (\ )$. As a result we obtain the solution: $^2 = \ (\ ) - (\ -\ )^2$.

There is also a singular solution that corresponds to equating the second factor of (1) to zero. This solution in parametric form is written as:

$$= \ -\tfrac{1}{2}\ '(\ ), \qquad ^2 = \ (\ ) - \tfrac{1}{4}[\ '(\ )]^2.$$

27. $= f(\quad - \quad) + \ ^2(\ ^2 - 1)\ (\quad - \quad).$

The contact transformation $= \ ' - X$, $= [(\ ')^2 - 1]^{1\ 2}$, $' = \ '[(\ ')^2 - 1]^{-1\ 2}$, where $= (X)$, leads to the equation $' = (X) + \ ^2 g(X)$, which is linear in $= \ ^2$.

Inverse transformation: $X = \ ' - \ ,\ = -\ [(\ ')^2 - 1]^{1\ 2},\ ' = -\ '[(\ ')^2 - 1]^{-1\ 2}$.

28. $= \dfrac{1}{\ ^2} f \quad + - \ + \quad - - \quad + - \ .$

The contact transformation $X = \ ' + \ ,\ = \ ^2(\ ')^2 - \ ^2,\ ' = 2\ ^2\ '$ leads to a linear equation: $X\ ' = 2g(X)\ + 2X\ (X)$.

Inverse transformation:

$$= \frac{1}{X}\ \overline{X\ ' - \ }, \qquad = \frac{X\ ' - 2}{2\ \overline{X\ ' - \ }}, \qquad ' = \frac{X^2\ '}{2(X\ ' - \ )}.$$

29. $^{-1} f \quad - - + \quad - - \quad - - = \ .$

For $a \neq 1$, the contact transformation $X = \ ' - a\ ,\ = \ ^{1-}\ ' - \ ^{-}\ ,\ ' = \ ^{1-}$ leads to a linear equation: $' = g(X)\ + \ (X)$.

Inverse transformation:

$$= (\ ')^{\frac{1}{1-}}, \qquad = \frac{1}{1-a}(X\ ' - \ )(\ ')^{\overline{1-}}, \qquad ' = \frac{X\ ' - a}{(1-a)\ '}.$$

30. $^{+1} f \quad + - = (\ ^{+1} \ - \ ).$

For $a \neq -1$, the contact transformation $X = \ ' + a\ ,\ = \ ^{+1}\ ' - \ ,\ ' = \ ^{+1}$ leads to a separable equation: $(X)\ ' = g(\ )$.

Inverse transformation:

$$= (\ ')^{\frac{1}{+1}}, \qquad = \frac{1}{a+1}(X\ ' - \ )(\ ')^{-\overline{+1}}, \qquad ' = \frac{X\ ' + a}{(a+1)\ '}.$$

31. $= f(\quad).$

The substitution $= \ln$ leads to an equation of the form 1.8.1.32: $= (\ '\ )$.

32. $= f(\quad).$

We pass to a new variable $(\ ) = \ '$, divide both sides of the equation by , and differentiate with respect to . As a result we arrive at a separable equation: $'_w(\ )\ ' = (\quad + \ )\ (\ )$.

Solution in parametric form:

$$\ln|\ | = \frac{'_w(\ )}{(\quad + \ )\ (\ )} + \ , \qquad = (\ ).$$

In addition, there are singular solutions $= A\ ^{-}$, where A are roots of the algebraic equation $A\ - (-\ ) = 0$.

33. $= f(\quad) - \ .$

The contact transformation $X = e \ {}'$, $= {}' + \ ,$ ${}' = e^{-}$, where $= (X)$, leads to a separable equation: ${}' = (X)$.

Inverse transformation: $= -\ln\ {}'$, $= -X\ {}'$, ${}' = X\ {}'$.

34. $f(\quad) - (\quad + \ ) = 0.$

The contact transformation $X = e \ {}'$, $= {}' + \ ,$ ${}' = e^{-}$, where $= (X)$, leads to a separable equation: $g(\ )\ {}' = -\ (X)$.

Inverse transformation: $= -\ln\ {}'$, $= -X\ {}'$, ${}' = X\ {}'$.

35. $f(\quad) + (\quad)(\quad + \ ) = \ ^{-}\ .$

The contact transformation $X = e \ {}'$, $= {}' + \ ,$ ${}' = e^{-}$, where $= (X)$, leads to a linear equation: $a\ {}' = g(X)\ + \ (X)$.

Inverse transformation: $= -\ln\ {}'$, $= -X\ {}'$, ${}' = X\ {}'$.

36. $f(\quad) - (\quad + \ ) = \ .$

This equation can be rewritten in the form 1.8.1.34:

$$e\ (e\ {}') - (\ {}' + \ ) = 0, \qquad \text{where} \quad (\ ) = -\ln\ (\ ), \quad g(v) = -\ln\ (v).$$

37. $(\ \sin\ - \ \cos\ ) = f(\ \cos\ + \ \sin\ ).$

The contact transformation

$$X = \frac{1}{(\ {}')^2 + \ ^2}, \qquad = \frac{{}' \cos\ + \ \sin}{(\ {}')^2 + \ ^2}, \qquad {}' = \frac{}{{}'}(\ {}' \sin\ - \ \cos\ )$$

leads to the homogeneous equation: ${}' = (\ X)$.

38. $(\ ^{+1}\ ,\ \ + \ ) = 0.$

Solution: $= {}_1\ ^{-} + {}_2$. Here, the constants ${}_1$ and ${}_2$ are related by the constraint $(-\ {}_1\ ,\ {}_2\ ) = 0$.

39. $(\ ^2\ + 2\ ,\ ^3\ + \ ^2\ ) = 0.$

Solution: $= {}_1\ ^{-1} + {}_2\ ^{-2}$. Here, the constants ${}_1$ and ${}_2$ are related by the constraint $(\ {}_1, -\ {}_2) = 0$.

The singular solution can be represented in parametric form as:

$$(\ , v) = 0, \qquad (\ , v) + \ (\ , v) = 0, \qquad \text{where} \quad = \ ^2\ + 2\ , \quad v = \ ^3\ + \ ^2\ .$$

The subscripts and v denote the respective partial derivatives, and is the parameter.

40. $(\ ^{+1}\ + \ ,\ ^{+1}\ + \ ) = 0.$

Solution: $= {}_1\ ^{-} + {}_2\ ^{-}$. Here, the constants ${}_1$ and ${}_2$ are related by the constraint $\big(\ {}_1(\ -\ ),\ {}_2(\ -\ )\ \big) = 0$.

41. $(\ ,\ + \ ) = 0.$

Solution: $= {}_1 e^{-}\ + {}_2$. Here, the constants ${}_1$ and ${}_2$ are related by the constraint $(-\ {}_1,\ {}_2) = 0$.

42. $(\ + \ ,\ + \ ) = 0.$

Solution: $= {}_1 e^{-}\ + {}_2 e^{-}$. Here, the constants ${}_1$ and ${}_2$ are related by the constraint $\big(\ {}_1(\beta - \ ),\ {}_2(\ - \beta)\ \big) = 0$.

43. **(cosh − sinh , sinh − cosh) = 0.**

Solution: $=\ _1 \sinh\ +\ _2 \cosh$. Here, the constants $_1$ and $_2$ are related by the constraint $(\ _1, -\ _2) = 0$.

44. **(, − ln) = 0.**

Solution: $=\ _1 \ln\ +\ _2$. Here, the constants $_1$ and $_2$ are related by the constraint $(\ _1,\ _2) = 0$.

45. **—, ln − — = 0.**

Solution: $=\ _1 \exp(\ _2\ )$. Here, the constants $_1$ and $_2$ are related by the constraint $(\ _2, \ln\ _1) = 0$.

46. **——, ln − —— ln = 0.**

Solution: $=\ _1\ ^2$. Here, the constants $_1$ and $_2$ are related by the constraint $(\ _2, \ln\ _1) = 0$.

47. **(cos + sin , sin − cos) = 0.**

Solution: $=\ _1 \sin\ +\ _2 \cos$. Here, the constants $_1$ and $_2$ are related by the constraint $(\ _1, -\ _2) = 0$.

48. **—, − — = 0, = ( ).**

Solution: $=\ _1\ (\ ) +\ _2$. Here, the constants $_1$ and $_2$ are related by the constraint $(\ _1,\ _2) = 0$.

49. $$\frac{-}{-}, \frac{-}{-} = 0, \quad = (\ ), \quad = (\ ).$$

Solution: $=\ _1\ (\ ) +\ _2\ (\ )$. Here, the constants $_1$ and $_2$ are related by the constraint $(\ _1, -\ _2) = 0$.

The singular solution can be represented in parametric form as:

$$(\ , v) = 0, \qquad (\ , v) + \quad (\ , v) = 0, \qquad \text{where} \quad = \frac{'\ \_}{'\ \_\ '}, \quad v = \frac{'\ \_}{'\ \_\ '}.$$

The subscripts and v denote the respective partial derivatives, and is the parameter.

50. **(+ , − (+) = 0.**

Here, $= (\ ,\ )$, $= —$, $= —$. Differentiating with respect to , we obtain

$$(\ +\ \ ')'(\ -\ ) = 0,$$

where $= \frac{F}{}$ and $= \frac{F}{}$ are partial derivatives of function $(\ , v)$. Equating the first factor to zero, we find the solution:

$$(\ ,\ ) = \ + A, \qquad \text{where} \quad (\ , A) = 0.$$

It remains to be checked whether the equation $-\ = 0$ possesses any solutions and which of them satisfy the original equation.

1.8.1-4. Arguments of arbitrary functions are nonlinear with respect to $\ '$.

51. $\ = f(\ \ ^2) + 2\ \ .$

Solution: $[\ -\ (\ )]^2 = 4\ \ .$

52. $\ = 2\ (\ \ )^3 + f(\ -3\ \ ^2).$

This is a special case of equation 1.8.1.72 with $\ = 3$.

Solution: $\ = (\ )+2a\ \dfrac{\ -\ }{3a}\ ^{3\ 2}$. In addition, there is the following singular solution written in parametric form:

$$= \ + 3a[\ '(\ )]^2, \qquad = \ (\ ) + 2a[\ '(\ )]^3.$$

53. $\ = f(\ \ ^2 -\ \ ) + (2\ \ ^3 - 3\ \ )\ (\ \ ^2 -\ \ ), \qquad \neq 0.$

The contact transformation $X = a(\ ')^2 -\ \ ,\quad = 2a(\ ')^3 - 3\ \ ,\quad ' = 3\ '$ leads to a linear equation: $\ ' = 3g(X)\ \ + 3\ (X)$.

Inverse transformation: $\ = \frac{1}{9}\ ^{-1}[a(\ ')^2 - 9X],\quad = \frac{1}{81}\ ^{-1}[2a(\ ')^3 - 27\ \ ],\quad ' = \frac{1}{3}\ '$.

54. $\ = f(\ \ +\ \ )\ (\frac{1}{2}\ \ ^2 +\ \ ).$

The contact transformation $X = \ ' + a\ ,\quad = \frac{1}{2}(\ ')^2 + a\ ,\quad ' = \ '$, where $\ = \ (X)$, leads to a separable equation: $\ ' = \ (X)g(\ )$.

Inverse transformation: $\ = a^{-1}(X -\ '),\quad = \frac{1}{2}a^{-1}[2\ \ - (\ ')^2],\quad ' = \ '$.

55. $\ (\ \ -\ \ ) = f(\ \ ^2 -\ ^2).$

The contact transformation $X = (\ ')^2 -\ ^2,\quad = e\ (\ ' -\ ),\quad ' = \frac{1}{2}e\ (\ ')^{-1}$ leads to a linear (separable) equation: $2\ (X)\ ' = \ $.

56. $\ = f(\ ^2 -\ ^2)(\ \ +\ ^{-}\ ).$

The contact transformation

$$X = (\ ')^2 -\ ^2, \qquad = \ '(ae\ + e^{-}\ ) -\ (ae\ - e^{-}\ ), \qquad ' = \tfrac{1}{2}(ae\ + e^{-}\ )(\ ')^{-1}$$

leads to a separable equation: $2\ (X)\ ' = 1$.

57. $\ = \ f(\ ^2 -\ ^2)\ (\ \ -\ \ ).$

The contact transformation $X = (\ ')^2 -\ ^2,\quad = e\ (\ ' -\ ),\quad ' = \frac{1}{2}e\ (\ ')^{-1}$ leads to a separable equation: $2\ (X)g(\ )\ ' = 1$.

58. $\ = f(\ ^2 -\ ^2)\cosh\ \ .$

The contact transformation $X = (\ ')^2 -\ ^2,\quad = \ '\cosh\ -\ \sinh\ ,\quad ' = \frac{1}{2}\cosh\ (\ ')^{-1}$ leads to a separable equation: $2\ (X)\ ' = 1$.

59. $\ = f(\ ^2 -\ ^2)\sinh\ \ .$

The contact transformation $X = (\ ')^2 -\ ^2,\quad = \ '\sinh\ -\ \cosh\ ,\quad ' = \frac{1}{2}\sinh\ (\ ')^{-1}$ leads to a separable equation: $2\ (X)\ ' = 1$.

60. $\ = f(\ ^2 -\ ^2)(\ \cosh\ +\ \sinh\ ).$

The contact transformation

$$X = (\ ')^2 -\ ^2, \qquad = \ '(a\cosh\ +\ \sinh\ ) -\ (a\sinh\ +\ \cosh\ ), \qquad ' = \frac{a\cosh\ +\ \sinh\ }{2\ '}$$

leads to a separable equation: $2\ (X)\ ' = 1$.

61. $= f(\ ^2 - \ ^2)\ (\quad \cosh \quad - \quad \sinh \quad) \cosh \quad .$

The contact transformation $X = (\ ')^2 - \ ^2, \quad = \ ' \cosh \quad - \quad \sinh \quad, \quad ' = \frac{1}{2} \cosh \quad (\ ')^{-1}$ leads to a separable equation: $2\ (X)g(\ )\ ' = 1$.

62. $= f(\ ^2 - \ ^2)\ (\quad \sinh \quad - \quad \cosh \quad) \sinh \quad .$

The contact transformation $X = (\ ')^2 - \ ^2, \quad = \ ' \sinh \quad - \quad \cosh \quad, \quad ' = \frac{1}{2} \sinh \quad (\ ')^{-1}$ leads to a separable equation: $2\ (X)g(\ )\ ' = 1$.

63. $f(\ ^2 - \ ^2) + \quad ^2 \cosh \quad - \quad \sinh \quad = \cosh \quad .$

The contact transformation $X = (\ ')^2 - \ ^2, \quad = \ ' \cosh \quad - \quad \sinh \quad, \quad ' = \frac{1}{2} \cosh \quad (\ ')^{-1}$ leads to a linear equation: $2\ ' = a \quad + \ (X)$.

64. $= f(\ ^2 + \ ^2) \cos \quad .$

This is a special case of equation 1.8.1.65 with $a = 1$ and $\ = 0$.

65. $= f(\ ^2 + \ ^2)(\ \cos \quad + \ \sin \quad).$

The contact transformation

$$X = (\ ')^2 + \ ^2, \qquad = \ '(a \cos \quad + \ \sin \quad) + \ (a \sin \quad - \ \cos \quad), \qquad ' = \tfrac{1}{2}(a \cos \quad + \ \sin \quad)(\ ')^{-1}$$

leads to a separable equation: $2\ (X)\ ' = 1$.

66. $= f(\ ^2 + \ ^2)\ (\quad \cos \quad + \quad \sin \quad) \cos \quad .$

The contact transformation $X = (\ ')^2 + \ ^2, \quad = \ ' \cos \quad + \ \sin \quad, \quad ' = \frac{1}{2} \cos \quad (\ ')^{-1}$ leads to a separable equation: $2\ (X)g(\ )\ ' = 1$.

67. $f(\ ^2 + \ ^2) + \quad ^2 \cos \quad + \quad \sin \quad = \cos \quad .$

The contact transformation $X = (\ ')^2 + \ ^2, \quad = \ ' \cos \quad + \ \sin \quad, \quad ' = \frac{1}{2} \cos \quad (\ ')^{-1}$ leads to a linear equation: $2\ ' = a \quad + \ (X)$.

68. $= f(\ ^2 + \ ^2)\ (\quad \sin \quad - \quad \cos \quad) \sin \quad .$

The contact transformation $X = (\ ')^2 + \ ^2, \quad = \ ' \sin \quad - \ \cos \quad, \quad ' = \frac{1}{2} \sin \quad (\ ')^{-1}$ leads to a separable equation: $2\ (X)g(\ )\ ' = 1$.

69. $^2 \quad = f \quad + — \quad (\ ^2\ ^2 - \ ^2).$

The contact transformation $X = \ ' + \quad, \quad = \ ^2(\ ')^2 - \ ^2, \quad ' = 2\ ^2\ '$ leads to a separable equation: $\ ' = 2\ (X)g(\ )$.

Inverse transformation:

$$= \frac{1}{X}\ \overline{X\ ' - \ }, \qquad = \frac{X\ ' - 2}{2\ \overline{X\ ' - \ }}, \qquad ' = \frac{X^2\ '}{2(X\ ' - \ )}.$$

70. $= f(\ ^2 - \quad)\ (2\quad ^3 - 3\quad), \qquad \neq 0.$

The contact transformation $X = a(\ ')^2 - \quad, \quad = 2a(\ ')^3 - 3\quad, \quad ' = 3\ '$ leads to a separable equation: $\ ' = 3\ (X)g(\ )$.

Inverse transformation: $= \frac{1}{9}\ ^{-1}[a(\ ')^2 - 9X], \quad = \frac{1}{81}\ ^{-1}[2a(\ ')^3 - 27\ ], \quad ' = \frac{1}{3}\ '$.

71. $= f(\quad) + \frac{}{-1} \quad .$

Solution: $= (\quad) + \frac{}{-1}\ ^{-1}$.

72. $= \;(\;\;-1)(\;\;)\;\;+f\big(\;\;-\;\;\;(\;\;)\;\;\big)^{-1}.$

Differentiating with respect to , we obtain a factorized equation:

$$\big[1-a\;\;(\;\;-1)(\;\;')\;^{-2}\;\;''\;\big]\big[\;\;'-\;\;'(\;\;)\big]=0, \tag{1}$$

where $\;\;=\;\;-a\;\;(\;\;')\;^{-1}$. Equate the first factor to zero and integrate the obtained equation. Substituting the expression obtained into the original equation, we find the solution:

$$=\;(\;\;)+a(\;\;-1)\;\frac{\;-\;}{a}\;\;\overline{\;\;\;^{-1}}.$$

Equating the second factor in (1) to zero, we have another solution that can be written in parametric form as:

$$=\;\;+a\;\;[\;\;'(\;)]\;^{-1},\qquad =\;(\;)+a(\;\;-1)[\;\;'(\;)]\;\;.$$

73. $= f\big(\;(\;\;)^{k}-\;\;\;\;\big(\;\;(\;\;)^{k+1}-\;(\;\;+1)\;\;\big).$

The contact transformation ($a\;\;\neq 0,\ k\neq -1$)

$$X=a(\;\;')\;\;-\;\;\;,\qquad =ak(\;\;')\;^{+1}-\;(k+1)\;\;,\qquad '\;=(k+1)\;\;'$$

leads to a separable equation: $\;\;'\;=(k+1)\;\;(X)g(\;\;)$.

Inverse transformation:

$$=\frac{a(\;\;')\;\;}{(k+1)}-\frac{X}{\;\;},\qquad =\frac{ak(\;\;')\;^{+1}}{(k+1)\;^{+2}}-\frac{\;\;}{(k+1)},\qquad '\;=\frac{\;'}{k+1}.$$

74. $= f\big(\;(\;\;)^{k}-\;\;\;\;+\big(\;\;(\;\;)^{k+1}-\;(\;\;+1)\;\;\big)\;\;\big(\;(\;\;)^{k}-\;\;\;\big).$

The contact transformation ($a\;\;\neq 0,\ k\neq -1$)

$$X=a(\;\;')\;\;-\;\;\;,\qquad =ak(\;\;')\;^{+1}-\;(k+1)\;\;,\qquad '\;=(k+1)\;\;'$$

leads to a linear equation: $\;\;'\;=(k+1)g(X)\;\;+(k+1)\;\;(X)$.

Inverse transformation:

$$=\frac{a(\;\;')\;\;}{(k+1)}-\frac{X}{\;\;},\qquad =\frac{ak(\;\;')\;^{+1}}{(k+1)\;^{+2}}-\frac{\;\;}{(k+1)},\qquad '\;=\frac{\;'}{k+1}.$$

75. $\big(\;\;\;+\;\;\;,\;\;\overline{\;^{2}+\;\;\;}\;\big)=0.$

Solution: $\;^{2}=-a\;^{2}+2\;\;_{1}\;+\;\;_{2}$. Here, the constants $\;_{1}$ and $\;_{2}$ are related by the constraint

$$\big(\;_{1},\;\;\overline{\;_{1}^{2}+a\;\;_{2}}\;\big)=0\quad\text{if}\quad >0,$$
$$\big(\;_{1},-\;\;\overline{\;_{1}^{2}+a\;\;_{2}}\;\big)=0\quad\text{if}\quad <0.$$

76. $(\;\;\;-\;\;\;,\;\;^{2}-\;\;^{2})=0.$

Solution: $\;=\;_{1}e\;\;+\;_{2}e^{-}\;$. Here, the constants $\;_{1}$ and $\;_{2}$ are related by the constraint $(-2\;_{2},-4\;_{1}\;_{2})=0$.

77. $(\;\;\cosh\;\;-\;\;\sinh\;\;,\;\;^{2}-\;\;^{2})=0.$

Solution: $\;=\;_{1}\sinh\;\;+\;_{2}\cosh\;\;$. Here, the constants $\;_{1}$ and $\;_{2}$ are related by the constraint $(\;_{1},\;_{1}^{2}-\;_{2}^{2})=0$.

78. $(\;\;\cos\;\;+\;\;\sin\;\;,\;\;^{2}+\;\;^{2})=0.$

Solution: $\;=\;_{1}\sin\;\;+\;_{2}\cos\;\;$. Here, the constants $\;_{1}$ and $\;_{2}$ are related by the constraint $(\;_{1},\;_{1}^{2}+\;_{2}^{2})=0$.

1.8.2. Some Transformations

1. $= f(\, , \,)$.

Substituting $\; = \; '$ and differentiating both sides of the equation with respect to $\;$, we obtain an equation with respect to $\; = \; (\;)$:

$$[1 - \;\; (\, , \,)] \; ' = \;\; (\, , \,), \qquad \text{where} \qquad = —, \qquad = —.$$

If $\; = \; (\;)$ is the solution of the latter equation, the solution of the original equation can be represented in parametric form as:

$$= \; (\; (\;), \;), \qquad = \; (\;).$$

2. $= f(\, , \,)$.

Differentiating with respect to $\;$ and setting $\; = \; '$, we obtain an equation with respect to $\; = \; (\;)$:

$$[\; - \;\; (\, , \,)] \; ' = \;\; (\, , \,), \qquad \text{where} \qquad = —, \qquad = —.$$

If $\; = \; (\;)$ is the solution of the latter equation, the solution of the original equation can be represented in parametric form as:

$$= \; (\;), \qquad = \; (\; (\;), \;).$$

3. $= f \quad {}^{k} \; {}^{s}, \; —— \; .$

Set $z =$ and $= \dfrac{'}{\;}$. Divide both sides of the equation by and differentiate with respect to $\;$. As a result we arrive at the following equation with respect to $\; = \; (z)$:

$$z(\;\; + k)(\;\; + \; {}_{w} \; ') = (\;\; + \;) \; , \qquad \text{where} \qquad = \; (z, \;),$$

which is usually simpler than the original equation, since it is readily solved for the derivative. If $\; = \; (z)$ is the solution of the equation obtained, the solution of the original equation is written in parametric form as:

$$= z, \qquad = \; (z, \; (z)).$$

4. $= f(\qquad , \qquad)$.

The substitution $\; = \ln \;$ leads to an equation of the form 1.8.2.3: $\; = \; \big(\qquad , \quad ' \quad .$

5. $= f(\qquad , \qquad)$.

The substitution $\; = \ln \;$ leads to an equation of the form 1.8.2.3: $\; = \; \big(\qquad , \quad ' \quad .$

6. $f(\, , \qquad - \, , \quad) = 0$.

The Legendre transformation $\; = \; ', \;\; = \; ' - \;\; (\; ' = \;)$, where $\; = \; (\;)$, leads to the equation $\; (\; ', \; , \;) = 0$. Inverse transformation: $\; = \; ', \;\; = \; ' - \; , \;\; ' = \; .$

7. $(\;\;)^2 = \qquad + f(\;)$.

For $\lambda \neq 0$, the transformation $\lambda \;\; = 2 \; \overline{\lambda \;\; + \; (\;)}$ leads to an Abel equation of the second kind,

$$' = \;\; + \; (\;), \qquad \text{where} \qquad = 2\lambda^{-2} \; '(\;),$$

which is outlined in Subsection 1.3.1 for specific functions $\;$.

8. $= \qquad + \quad {}^{2} + f(\;), \qquad \neq 0.$

Differentiating the equation with respect to $\;$ and changing to new variables $\; = \; '$ and $\; = -2a \;$, we arrive at an Abel equation of the second kind,

$$' = \;\; + \; (\;), \qquad \text{where} \qquad = -2a \; '(\;),$$

which is outlined in Subsection 1.3.1 for specific functions $\;$.

For information about contact transformations, see Subsection 0.1.8.

Chapter 2

Second-Order Differential Equations

2.1. Linear Equations

2.1.1. Representation of the General Solution Through a Particular Solution

1 . A homogeneous linear equation of the second order has the general form

$$ {}_2(\ )\ '' + {}_1(\ )\ ' + {}_0(\ )\ = 0. \qquad (1)$$

Let ${}_0 = {}_0(\ )$ be a nontrivial particular solution $(\ {}_0\ \ 0)$ of this equation. Then the general solution of equation (1) can be found from the formula:

$$ = {}_0 \quad {}_1 + {}_2 \quad \frac{e^{-F}}{{}^2_0} \quad , \quad \text{where} \quad = \frac{{}_1}{{}_2} \quad . \qquad (2)$$

For specific equations described below in 2.1.2–2.1.9, often only particular solutions are given, while the general solutions can be obtained with formula (2) (see also Paragraph 0.2.1-1, Item 3).

Only homogeneous equations are considered in Subsections 2.1.2 through 2.1.8; the solutions of the corresponding nonhomogeneous equations can be obtained using relations (7) and (8) of Subsection 0.2.1.

2 . Suppose a particular solution of a homogeneous linear equation is obtained in the closed form $= [\ (\ )]$, with this formula valid for $(\ ) \ge 0$. If the equation makes sense in a range of where $(\ ) < 0$, then the function $= |\ (\ )|$ will be a particular solution of the equation in that range.

3 . Suppose $= {}_1\ (\ )[g(\ )]\ + {}_2\ (\ )[g(\ )]^b$ is the general solution of the homogeneous linear equation with $a \ne$, where a and are free parameters. Then the function $= {}_1\ (\ )[g(\ )]\ + {}_2\ (\ )[g(\ )]\ \ln g(\ )$ will be the general solution of this equation with $a =$.

2.1.2. Equations Containing Power Functions

2.1.2-1. Equations of the form $'' + (\ )\ = 0.$

1. $+ \quad = 0.$

Equation of free oscillations.

Solution: $= \begin{cases} {}_1 \sinh(\ \sqrt{|a|}\,) + {}_2 \cosh(\ \sqrt{|a|}\,) & \text{if } a < 0, \\ {}_1 + {}_2 & \text{if } a = 0, \\ {}_1 \sin(\ \sqrt{a}\,) + {}_2 \cos(\ \sqrt{a}\,) & \text{if } a > 0. \end{cases}$

2. $- (\ + \ ) \ = 0, \qquad \ne 0.$

The substitution $\xi = a^{-2\ 3}(a\ \ + \ )$ leads to the Airy equation:

$$'' - \xi\ \ = 0, \qquad (1)$$

which often arises in various applications. The solution of equation (1) can be written as:

$$ = {}_1\,\mathrm{Ai}(\xi) + {}_2\,\mathrm{Bi}(\xi),$$

where $\mathrm{Ai}(\xi)$ and $\mathrm{Bi}(\xi)$ are the Airy functions of the first and second kind, respectively.

The Airy functions admit the following integral representation:

$$\mathrm{Ai}(\xi) = \frac{1}{\ }\ {}_0\ \cos\bigl(\tfrac{1}{3}\ {}^3 + \xi\ \ ,\quad \mathrm{Bi}(\xi) = \frac{1}{\ }\ {}_0\ \exp\bigl(-\tfrac{1}{3}\ {}^3 + \xi\ \ + \sin\bigl(\tfrac{1}{3}\ {}^3 + \xi\ \ .$$

The Airy functions can be expressed in terms of the Bessel functions and the modified Bessel functions of order 1 3 by the relations:

$$\mathrm{Ai}(\xi) = \tfrac{1}{3}\ \overline{\xi}\,\bigl[\ {}_{-1\ 3}(z) - {}_{1\ 3}(z)\bigr],\quad \mathrm{Ai}(-\xi) = \tfrac{1}{3}\ \overline{\xi}\,\bigl[\ {}_{-1\ 3}(z) + {}_{1\ 3}(z)\bigr],$$
$$\mathrm{Bi}(\xi) = \ \overline{\tfrac{1}{3}\xi}\,\bigl[\ {}_{-1\ 3}(z) + {}_{1\ 3}(z)\bigr],\quad \mathrm{Bi}(-\xi) = \ \overline{\tfrac{1}{3}\xi}\,\bigl[\ {}_{-1\ 3}(z) - {}_{1\ 3}(z)\bigr],$$

where $z = \tfrac{2}{3}\xi^{3\ 2}$.

For large values of ξ, the leading terms of the asymptotic expansions of the Airy functions are:

$$\mathrm{Ai}(\xi) = \frac{1}{2\ \overline{\ }}\xi^{-1\ 4}\exp(-z),\quad \mathrm{Ai}(-\xi) = \frac{1}{\overline{\ }}\xi^{-1\ 4}\sin\ \ z + \frac{\ }{4}\ \ ,$$
$$\mathrm{Bi}(\xi) = \frac{1}{\overline{\ }}\xi^{-1\ 4}\exp(z),\qquad \mathrm{Bi}(-\xi) = \frac{1}{\overline{\ }}\xi^{-1\ 4}\cos\ \ z + \frac{\ }{4}\ \ .$$

The Airy equation (1) is a special case of equation 2.1.2.7 with $a = \ = 1$.

3. $\ \ -(\ {}^2\ {}^2 + \ )\ \ = 0.$

Particular solution: $\ {}_0 = \exp\bigl(\tfrac{1}{2}a\ {}^2\ $.

4. $\ \ -(\ \ {}^2 + \ )\ \ = 0.$

The Weber equation (two canonical forms of the equation correspond to $a = \ \tfrac{1}{4}$).

1 . The transformation $z = \ {}^2\ \overline{a},\ \ = e\ \ {}^2$ leads to the degenerate hypergeometric equation 2.1.2.70: $z\ \ '' + \ \frac{1}{2} - z\ \ \ ' - \frac{1}{4}\ \ \frac{\ }{\overline{a}} + 1\ \ \ = 0.$

2 . For $a = k^2 > 0,\ \ = -(2\ \ + 1)k$, where $\ = 1,\ 2,\ \ $, there is a solution of the form:

$$ = \exp\bigl(-\tfrac{1}{2}k\ {}^2\ \ \ \ (\ \overline{k}\ \ ,\qquad k > 0,$$

where $\ \ (z) = (-1)\ \ \exp\bigl(z^2\ \ \frac{\ }{z}\ \exp\bigl(-z^2$ is the Hermite polynomial of order .

See also Subsection S.2.10.

References: H. Bateman and A. Erdélyi (1953, Vol. 2), M. Abramowitz and I. A. Stegun (1964).

5. $\ \ + \ {}^3\ (2 - \ \ )\ \ = 0.$

Particular solution: $\ {}_0 = \exp\bigl(-\tfrac{1}{2}a^2\ {}^2 + a\ \ \ $.

6. $\ \ -(\ \ {}^2 + \ \ + \ )\ \ = 0.$

The substitution $\xi = \ + \frac{\ }{2a}$ leads to an equation of the form 2.1.2.4: $\ '' - \ a\xi^2 + \ - \frac{\ {}^2}{4a}\ \ = 0.$

7. $\quad - \quad = 0.$

1 . For $= -2$, this is the Euler equation 2.1.2.123, while for $= -4$, this is the equation 2.1.2.211 (in both cases the solution is expressed in terms of elementary function).

2 . Assume $2\ (\ +2) = 2\ \ +1$, where is an integer. Then the solution is:

$$= \begin{cases} (\ ^{1-2}\ )^{\ +1}\ \ _1 \exp \frac{\overline{a}}{} \Big) + \ _2 \exp -\frac{\overline{a}}{} \Big) & \text{if } \ge 0, \\ (\ ^{1-2}\ )^{-}\ \ _1 \exp \frac{\overline{a}}{} \Big) + \ _2 \exp -\frac{\overline{a}}{} \Big) & \text{if } < 0, \end{cases}$$

where $= \frac{}{},\quad = \frac{\ +2}{2} = \frac{1}{2\ \ +1}.$

3 . For any , the solution is expressed in terms of the Bessel functions and modified Bessel functions of the first or second kind (see 2.1.2.126 and 2.1.2.127):

$$= \begin{cases} \ _1\ -\ \frac{1}{2}\ \frac{\overline{-a}}{} \Big) + \ _2\ -\ \frac{1}{2}\ \frac{\overline{-a}}{} \Big) & \text{if } a < 0, \\ \ _1\ -\ \frac{1}{2}\ \frac{\overline{a}}{} \Big) + \ _2\ -\ \frac{1}{2}\ \frac{\overline{a}}{} \Big) & \text{if } a > 0, \end{cases}$$

where $= \frac{1}{2}(\ +2).$

8. $\quad - (\ ^{2} + \ ^{-1})\ = 0.$

Particular solution: $_0 = \exp \frac{a}{\ +1}\ ^{+1}\ .$

9. $\quad - \ ^{-2}(\ + \ +1)\ = 0.$

Particular solution: $_0 = \ \exp(a\ \ .$

10. $\quad + (\ ^{2} + \ ^{-1})\ = 0.$

The substitution $\xi = \ ^{+1}$ leads to a linear equation of the form 2.1.2.108: $(\ +1)^2 \xi\ '' + (\ +1)\ ' + (a\xi + \ )\ = 0.$

$\boxed{\text{2.1.2-2. Equations of the form } \ '' + (\ )\ ' + g(\ )\ = 0.}$

11. $\quad + \quad + \quad = 0.$

Second-order constant coefficient linear equation. In physics this equation is called an *equation of damped vibrations.*

$$\text{Solution:} \quad = \begin{cases} \exp(-\frac{1}{2}a\ \ [\ _1 \exp(\frac{1}{2}\lambda\ \ + \ _2 \exp(-\frac{1}{2}\lambda\ \ ] & \text{if } \lambda^2 = a^2 - 4\ > 0, \\ \exp(-\frac{1}{2}a\ \ [\ _1 \sin(\frac{1}{2}\lambda\ \ + \ _2 \cos(\frac{1}{2}\lambda\ \ ] & \text{if } \lambda^2 = 4\ - a^2 > 0, \\ \exp(-\frac{1}{2}a\ \ (\ _1 + \ _2 & \text{if } a^2 = 4\ . \end{cases}$$

12. $\quad + \quad + (\ + \ )\ = 0.$

1 . Solution with $\neq 0$:

$$= \exp(-\tfrac{1}{2}a\ \ \overline{\xi}\,[\ _1\ _{1\ 3}(\tfrac{2}{3}\ \overline{\ }\ \xi^{3\ 2} + \ _2\ _{1\ 3}(\tfrac{2}{3}\ \overline{\ }\ \xi^{3\ 2}\ ], \quad \xi = \ + \frac{4\ -a^2}{4},$$

where $_{1\ 3}(z)$ and $_{1\ 3}(z)$ are the Bessel functions.

2 . For $= 0$, see equation 2.1.2.11.

13. $+ \quad - (\quad ^2 + \quad) \quad = 0.$

The substitution $= \exp\left(\frac{1}{2} \quad ^2 \quad - \right.$ leads to a linear equation of the form 2.1.2.108: $'' + (2 \quad - \quad + a) \quad ' + (a \quad - \quad - \quad + \quad -) \quad = 0.$

14. $+ \quad + \ (- \quad ^2 + \quad + 1) \quad = 0.$

Particular solution: $_0 = \exp\left(-\frac{1}{2} \quad ^2\right.$.

15. $+ \quad + \quad (- \quad ^3 + \quad + 2) \quad = 0.$

Particular solution: $_0 = \exp\left(-\frac{1}{3} \quad ^3\right.$.

16. $+ \quad + \ (- \quad ^2 \ + \quad + \quad ^{-1}) \quad = 0.$

Particular solution: $_0 = \exp \ -\dfrac{}{+1} \quad ^{+1}$.

17. $+ \quad + \ (- \quad ^2 \ - \quad + \quad ^{-1}) \quad = 0.$

Particular solution: $_0 = \exp \ -\dfrac{}{+1} \quad ^{+1} - a$.

18. $+ \quad + (\quad + 1) \quad = 0, \qquad = 1,\ 2,\ 3,\ \ldots$

Solution: $= \dfrac{}{} \ \exp\left(-\frac{1}{2} \quad ^2\right. \quad _1 + \quad _2 \quad \exp\left(\frac{1}{2} \quad ^2\right.$.

19. $- 2 \quad + 2 \quad = 0, \qquad = 1,\ 2,\ 3,\ \ldots$

Solution: $= \exp(\ ^2) \dfrac{}{} \ \exp(- \ ^2) \quad _1 + \quad _2 \quad \exp(\ ^2)$.

For $_1 = (-1)$ and $_2 = 0$, this solution defines the Hermite polynomials.

20. $+ \quad + \quad = 0.$

Solution: $= \quad _1 \ \left(\frac{1}{2}a^{-1} \ , \frac{1}{2}, -\frac{1}{2}a \quad ^2\right. + \quad _2 \ \left(\frac{1}{2}a^{-1} \ , \frac{1}{2}, -\frac{1}{2}a \quad ^2\right.$, where $(a, \ ; \ )$ and $(a, \ ; \ )$ are the degenerate hypergeometric functions (see equation 2.1.2.70 and Subsection S.2.7).

21. $+ \quad + \quad = 0.$

Solution: $= e^{-b} \quad \left[\ _1 \ \left(\frac{1}{2}a^{-3} \ ^2, \frac{1}{2}, -\frac{1}{2}a\xi^2\right. + \quad _2 \ \left(\frac{1}{2}a^{-3} \ ^2, \frac{1}{2}, -\frac{1}{2}a\xi^2\right. \right], \ \xi = \quad - 2a^{-2}$, where $(a, \ ; \ )$ and $(a, \ ; \ )$ are the degenerate hypergeometric functions (see equation 2.1.2.70 and Subsection S.2.7).

22. $+ \quad + (\quad + \ ) \quad = 0.$

This is a special case of equation 2.1.2.108 with $a_2 = \ _1 = 0$ and $_2 = 1$.

23. $+ 2 \quad + (\ ^4 + \ ^2 \ ^2 + \quad + \ ) \quad = 0.$

This is a special case of equation 2.1.2.49 with $= 1$ and $= 2$.

24. $+ (\quad + \ ) \quad + \quad = 0.$

Particular solution: $_0 = \exp\left(-\frac{1}{2}a \quad ^2 - \right.$.

25. $+ (\quad + \ ) \quad - \quad = 0.$

Particular solution: $_0 = a \quad + \ $.

26. $+ (\quad + \ ) \quad + \ (\quad + \quad - \ ) \quad = 0.$

Particular solution: $_0 = e^{-c}$.

27. $+ (\quad + 2\ )\quad + (\quad - \quad + {}^2)\ = 0.$

Particular solution: $_0 = \quad e^{-b}\ .$

28. $+ (\quad + \ )\quad + (\quad + \ )\ = 0.$

This is a special case of equation 2.1.2.108 with $a_2 = 0$ and $_2 = 1$.

29. $+ (\quad + \ )\quad + \ [(\ -\ )\ {}^2 + \quad + 1]\ = 0.$

Particular solution: $_0 = \exp\bigl(-\tfrac{1}{2}\quad {}^2\ .$

30. $+ 2(\quad + \ )\quad + (\ {}^2\ {}^2 + 2\quad + \ )\ = 0.$

The substitution $= \exp\bigl(\tfrac{1}{2}a\ {}^2 +$ leads to a constant coefficient linear equation of the form 2.1.2.1: $'' + (\ - a - {}^2)\ = 0.$

31. $+ (\quad + \ )\quad + (\ {}^2 + \quad + \gamma)\ = 0.$

The substitution $= \exp(\ {}^2)$, where is a root of the quadratic equation $4\ {}^2 + 2a\ + \ = 0$, leads to an equation of the form 2.1.2.108: $'' + [(a+4\ )\ + \ ]\ ' + [(\beta + 2\ )\ + \ + 2\ ]\ = 0.$

32. $+ (\quad + \ )\quad + \ (-\ {}^2\ + \quad {}^{+1} + \quad + \quad {}^{-1})\ = 0.$

Particular solution: $_0 = \exp\ -\dfrac{}{+ 1}\quad {}^{+1}\ .$

33. $+ \ (\ {}^2 - {}^2)\quad - \ (\ + \ )\ = 0.$

Particular solution: $_0 = \quad - \ .$

34. $+ (\ {}^2 + \ )\quad + \ (\ {}^2 + \ - \ )\ = 0.$

Particular solution: $_0 = e^{-c}\ .$

35. $+ (\ {}^2 + 2\ )\quad + (\ {}^2 - \quad + {}^2)\ = 0.$

Particular solution: $_0 = \quad e^{-b}\ .$

36. $+ (2\ {}^2 + \ )\quad + (\ {}^4 + \quad {}^2 + 2\ + \ )\ = 0.$

The substitution $= \exp\bigl(\tfrac{1}{3}\ {}^3$ leads to a constant coefficient linear equation of the form 2.1.2.11: $'' + a\ ' + \ = 0.$

37. $+ (\ {}^2 + \quad)\quad + (\ {}^2 + \quad + \gamma)\ = 0.$

1 . This is a special case of equation 2.1.2.146 with $= 1$.

2 . Let $= 0, \beta = 3a, \ = 2$. Particular solution: $_0 = \quad \exp\bigl(-\tfrac{1}{3}a\ {}^3 - \tfrac{1}{2}\quad {}^2\ .$

38. $+ (\quad {}^2 + \quad + 2\ )\quad + \ {}^2(\ {}^2 + 1)\ = 0.$

Particular solution: $_0 = (a\ + 1)e^{-}\ .$

39. $+ (\ {}^2 + \quad + \ )\quad + \ (\quad {}^2 + \quad + 2\ )\ = 0.$

Particular solution: $_0 = \exp\bigl(-\tfrac{1}{3}a\ {}^3 - \quad .$

40. $+ (\ {}^2 + \quad + \ )\quad + (\quad {}^3 + \quad {}^2 + \ )\ = 0.$

Particular solution: $_0 = \exp\bigl(-\tfrac{1}{2}\quad {}^2 - \quad .$

41. $+ (\ {}^3 + 2\ )\quad + (\quad {}^3 - \quad {}^2 + {}^2)\ = 0.$

Particular solution: $_0 = \quad e^{-b}\ .$

42. $\mathbf{+(\quad {}^3+\quad)\quad +2(2\quad {}^2+\ )\quad =0.}$

Particular solution: $\ _0 = \ \exp\left(-\tfrac{1}{4}a\ \ {}^4-\tfrac{1}{2}\ \ {}^2\right.$.

43. $\mathbf{+(\quad {}^3+\quad {}^2+2\ )\quad +\ {}^2(\quad {}^3+1)\quad =0.}$

Particular solution: $\ _0 = (a\quad +1)e^{-}\quad$.

44. $\mathbf{+\qquad\quad =0.}$

This equation is encountered in the theory of diffusion boundary layer.

Solution: $\quad = \ _1 + \ _2\ \ \exp\left(-\frac{a\ \ {}^{+1}}{+1}\right)\quad$.

45. $\mathbf{+\qquad +\quad {}^{-1}\ =0.}$

For $\ =-1$, we obtain the Euler equation 2.1.2.123. For $\ \neq -1$, the substitution $z=\quad {}^{+1}$ leads to an equation of the form 2.1.2.108: $(\ +1)^2 z\ \ ''+(\ +1)(az+\ )\ '+\quad =0.$

46. $\mathbf{+2\qquad +\ (\quad {}^2+\qquad {}^{-1})\quad =0.}$

Particular solution: $\ _0 = \ \exp\left(-\frac{a}{+1}\quad {}^{+1}\right.$.

47. $\mathbf{+\qquad +(\quad {}^2+\qquad {}^{-1})\quad =0.}$

The substitution $\xi = \quad {}^{+1}$ leads to a linear equation of the form 2.1.2.108: $(\ +1)^2\xi\ \ ''+(\ +1)(a\xi+\ )\ '+(\ \xi+\ )\ =0.$

48. $\mathbf{+\qquad -\ (\quad {}^{+}+\quad {}^2+\qquad {}^{-1})\quad =0.}$

Particular solution: $\ _0 = \exp\left(\frac{\quad}{+1}\quad {}^{+1}\right.$.

49. $\mathbf{+2\qquad +(\ {}^2\ {}^2+\quad {}^2+\qquad {}^{-1}+\qquad {}^{-1})\quad =0.}$

The substitution $\quad = \ \exp\left(\frac{a}{+1}\quad {}^{+1}\right.$ leads to a linear equation of the form 2.1.2.10: $\ ''+(\quad {}^2+\qquad {}^{-1})\quad =0.$

50. $\mathbf{+(\qquad +\ )\quad +\ (\qquad +\ -\ )\quad =0.}$

Particular solution: $\ _0 = e^{-c}\quad$.

51. $\mathbf{+(\qquad +2\ )\quad +(\qquad -\qquad {}^{-1}+\ {}^2)\quad =0.}$

Particular solution: $\ _0 = \ \ e^{-b}\quad$.

52. $\mathbf{+(\qquad +\qquad {}^{-1}+2\ )\quad +\ {}^2(\qquad +1)\quad =0.}$

Particular solution: $\ _0 = (a\quad +1)e^{-}\quad$.

53. $\mathbf{+(\qquad +2\qquad {}^{-1}-\quad {}^2\ )\quad +\ (\qquad +\qquad {}^{-1}-\quad {}^2\ )\quad =0.}$

Particular solution: $\ _0 = (a\quad +2)e^{-}\quad$.

54. $\mathbf{+\quad [\quad {}^2+(\quad +\ )\quad +\quad]\quad -\quad (\quad +\ )\quad =0.}$

Particular solution: $\ _0 = \quad +\ $.

55. $\mathbf{+(\qquad +\qquad)\quad -(\qquad {}^{-1}+\qquad {}^{-1})\quad =0.}$

Particular solution: $\ _0 = \quad$.

56. $+ (\quad + \quad) \quad + (\quad ^{-1} + \quad ^{-1}) = 0.$

Integrating yields first-order linear equation: $' + (a \quad + \quad) \; = \quad .$

57. $+ (\quad + \quad) \quad + [\; (\quad + 1) \quad ^{-1} + \; (\quad + 1) \quad ^{-1}] = 0.$

Particular solution: $_0 = \quad \exp\, -\frac{a}{\quad + 1} \quad ^{+1} - \frac{\quad}{\quad + 1} \quad ^{+1} \; .$

58. $+ (\quad + \quad) \quad + \; (\quad + \quad - \;) = 0.$

Particular solution: $_0 = e^{-c} \quad .$

59. $+ (\quad + \quad) \quad + [\quad ^{+} + \; (\quad + 1) \quad ^{-1} - \quad ^{-1}] = 0.$

Particular solution: $_0 = \quad \exp\, -\frac{\quad}{\quad + 1} \quad ^{+1} \; .$

60. $+ (\quad + \quad + \;) \quad + (\quad ^{+} + \quad + \quad ^{-1}) = 0.$

Particular solution: $_0 = \exp\, -\frac{a}{\quad + 1} \quad ^{+1} - \quad .$

2.1.2-3. Equations of the form $(a \quad + \;) \,'' + \; (\;) \,' + g(\;) \; = 0.$

61. $+ \frac{1}{2} \quad + \quad = 0.$

Solution: $= \begin{cases} _1 \cos \quad \overline{4a} \quad + \; _2 \sin \quad \overline{4a} & \text{if } a \quad > 0, \\ _1 \cosh \quad \overline{4|a \quad |} + \; _2 \sinh \quad \overline{4|a \quad |} & \text{if } a \quad < 0. \end{cases}$

62. $+ \quad + \quad = 0.$

1 . The solution is expressed in terms of Bessel functions:

$$= \frac{1-}{2} [\; _1 \quad (2 \quad \overline{\quad} \quad + \; _2 \quad (2 \quad \overline{\quad} \quad], \qquad \text{where} \qquad = |1 - a|.$$

2 . For $a = \frac{1}{2}(2 \quad + 1)$, where $\; = 0,\, 1, \quad$, the solution is:

$$= \begin{cases} _1 \text{——} \cos \quad \overline{4a} \quad + \; _2 \text{——} \sin \quad \overline{4a} & \text{if } a \quad > 0, \\ _1 \text{——} \cosh \quad \overline{4|a \quad |} + \; _2 \text{——} \sinh \quad \overline{4|a \quad |} & \text{if } a \quad < 0. \end{cases}$$

63. $+ \quad + \quad = 0.$

1 . The solution is expressed in terms of Bessel functions:

$$= \frac{1-}{2} [\; _1 \quad (\quad ^{-} \quad + \; _2 \quad (\quad ^{-} \quad], \qquad \text{where} \qquad = \tfrac{1}{2}|1 - a|.$$

2 . For $a = 2 \quad$, where $\; = 1,\, 2, \quad$, the solution is:

$$= \begin{cases} _1 \; \frac{1}{\quad}\text{——} \quad \cos(\quad ^{-} \quad + \; _2 \; \frac{1}{\quad}\text{——} \quad \sin(\quad ^{-} & \text{if} \quad > 0, \\ _1 \; \frac{1}{\quad}\text{——} \quad \cosh(\quad \overline{-} \quad + \; _2 \; \frac{1}{\quad}\text{——} \quad \sinh(\quad \overline{-} & \text{if} \quad < 0. \end{cases}$$

64. $+ \quad + (\quad + \;) = 0.$

This is a special case of equation 2.1.2.108 with $a_2 = 1$ and $a_1 = \; _2 = 0$.

65. $+ \quad + \quad {}^{1-2} \quad = 0.$

For $= 1$, this is the Euler equation 2.1.2.123. For $\neq 1$, the solution is:

$$= \begin{cases} {}_1 \sin\left(\dfrac{-}{-1}\ {}^{1-}\right) + {}_2 \cos\left(\dfrac{-}{-1}\ {}^{1-}\right) & \text{if } > 0, \\ {}_1 \exp\left(\dfrac{-}{-1}\ {}^{1-}\right) + {}_2 \exp\left(\dfrac{-\ -}{-1}\ {}^{1-}\right) & \text{if } < 0. \end{cases}$$

66. $+ (1 - 3\ ) \quad - \quad {}^{2\ \ 2\ \ 2\ \ -1} \quad = 0.$

Solution: $= {}_1(a \quad + 1) \exp(-a \quad) + {}_2(-a \quad + 1) \exp(a \quad).$

67. $+ \quad + \quad = 0.$

If $= -1$ and $= 0$, we have the Euler equation 2.1.2.123. If $\neq -1$ and $\neq 0$, the solution is expressed in terms of Bessel functions:

$$= {}^{\frac{1-}{2}} \quad {}_1 \quad \frac{2\ \ -}{+1} \ {}^{\frac{+1}{2}} + {}_2 \quad \frac{2\ \ -}{+1}\ {}^{\frac{+1}{2}} \quad , \qquad \text{where} \quad = \frac{|1-a|}{+1}.$$

68. $+ \quad + \quad (- \quad {}^{+1} + \quad + \quad) \quad = 0.$

Particular solution: ${}_0 = \exp\left(- \dfrac{}{+1}\ {}^{+1}\right).$

69. $+ \quad + \quad = 0.$

Particular solution: ${}_0 = \quad e^{-} \quad .$

70. $+ (\ -\ ) \quad - \quad = 0.$

The degenerate hypergeometric equation.

1 . If $\neq 0, -1, -2, -3,$, Kummer's series is a particular solution:

$$(a,\ ;\ ) = 1 + \sum_{=1} \frac{(a)}{(\ )} \frac{}{k!},$$

where $(a) \ = a(a+1) \quad (a+k-1)$, $(a)_0 = 1$. If $> a > 0$, this solution can be written in terms of a definite integral:

$$(a,\ ;\ ) = \frac{\Gamma(\ )}{\Gamma(a)\,\Gamma(\ -a)} \int_0^1 e \quad {}^{-1}(1-\ )^{b-\ -1} \quad ,$$

where $\Gamma(z) = \int_0 e^{-} \quad {}^{-1}$ is the gamma function.

If is not an integer, then the general solution has the form:

$$= {}_1 \ (a,\ ;\ ) + {}_2 \ {}^{1-b} \ (a - \ + 1,\ 2 -\ ;\ ).$$

Table 14 gives some special cases where is expressed in terms of simpler functions.

The function possesses the properties:

$$(a,\ ;\ ) = e \quad (\ -a,\ ;\ -\ ); \qquad \frac{}{} \ (a,\ ;\ ) = \frac{(a)}{(\ )} \ (a+\ ,\ +\ ;\ ).$$

The following asymptotic relations hold:

$$(a,\ ;\ ) = \frac{\Gamma(\ )}{\Gamma(a)} e \quad {}^{-b} \ 1 + \frac{1}{|\ |} \qquad \text{if} \quad + \ ,$$

$$(a,\ ;\ ) = \frac{\Gamma(\ )}{\Gamma(\ -a)} (-\ )^{-} \ 1 + \frac{1}{|\ |} \qquad \text{if} \quad - \ .$$

TABLE 14

Special cases of Kummer's function $(a, \ ; z)$

a		z		Conventional notation
a	a		e	
1	2	2	$\frac{1}{\ }e \ \sinh$	
a	$a+1$	$-$	$a^{\ -} \ (a, \ )$	Incomplete gamma function $(a, \ ) = {}_{0} \ e^{-} \ ^{-1}$
$\frac{1}{2}$	$\frac{3}{2}$	$-\ ^{2}$	$\frac{\ }{2}$ erf	Error function erf $= \frac{2}{\ } {}_{0} \exp(-\ ^{2})$
$-$	$\frac{1}{2}$	$\frac{\ ^2}{2}$	$\frac{!}{(2\ )!} \ -\frac{1}{2}^{\ -} \ {}_{2}(\ )$	Hermite polynomials $(\ ) = (-1) \ e^{\ ^2} \frac{\ }{\ } (e^{-\ ^2}$, $= 0, 1, 2, 3,$
$-$	$\frac{3}{2}$	$\frac{\ ^2}{2}$	$\frac{!}{(2\ +1)!} \ -\frac{1}{2}^{\ -} \ {}_{2\ +1}(\ )$	
$-$			$\frac{!}{(\ )} L^{(b-1)}(\ )$	Laguerre polynomials $L^{(\ )}(\ ) = \frac{e^{\ -}}{!} \frac{\ }{\ } (e^{-} \ ^{+}$, $= \ -1,$ $(\ ) = (\ +1) \ (\ + \ -1)$
$+\frac{1}{2}$	$2\ +1$	2	$\Gamma(1+\ )e \ \frac{\ }{2}^{\ -} \ (\ )$	Modified Bessel functions $(\ )$
$+1$	$2\ +2$	2	$\Gamma \ +\frac{3}{2} \ e \ \frac{\ }{2}^{- \ -\frac{1}{2}} \ {}_{+\frac{1}{2}}(\ )$	

2 . The following function is a solution of the degenerate hypergeometric equation:

$$(a, \ ; \ ) = \frac{\Gamma(1-\ )}{\Gamma(a-\ +1)} \ (a, \ ; \ ) + \frac{\Gamma(\ -1)}{\Gamma(a)} \ ^{1-b} \ (a-\ +1, \ 2-\ ; \ ).$$

Calculate the limit as $\quad$ (is an integer) to obtain

$$(a, \ ; \ ) = \frac{(-1)^{\ -1}}{!\,\Gamma(a-\ )} \ (a, \ +1; \ )\ln$$

$$+ \ {}_{=0} \frac{(a)}{(\ +1)} \big[\ (a+\ ) - \ (1+\ ) - \ (1+\ +\ ) \big] \frac{\ }{!}$$

$$+ \frac{(\ -1)!}{\Gamma(a)} \ {}_{=0}^{-1} \frac{(a-\ )}{(1-\ )} \frac{\ ^{-}}{!},$$

where $= 0, 1, 2,$ (the last sum is omitted for $= 0$), $(z) = [\ln\Gamma(z)]'$ is the logarithmic derivative of the gamma function:

$$(1) = -\ , \qquad (\ ) = -\ + \ {}_{=1}^{-1} k^{-1}, \qquad = 0.5772 \quad \text{is the Euler constant.}$$

If is a negative number, then the function can be expressed in terms of the one with positive second argument using the relation

$$(a,\ ;\ ) = \ ^{1-b}\ (a- \ +1,\ 2- \ ;\ ),$$

which holds for any value of .

3 . For $\neq 0, -1, -2, -3,$, the general solution of the degenerate hypergeometric equation can be written in the form:

$$= \ _1\ (a,\ ;\ ) + \ _2\ (a,\ ;\ ),$$

while for $= 0, -1, -2, -3,$, it can be represented as:

$$= \ ^{1-b}\big[\ _1\ (a- \ +1,\ 2- \ ;\ ) + \ _2\ (a- \ +1,\ 2- \ ;\ )\big].$$

The functions and are described in Subsection S.2.7 in more detail; see also the books by Abramowitz & Stegun (1964) and Bateman & Erdélyi (1953, Vol. 1).

71. $+(\ +\ )\ +\ [(\ -\ )\ +\ ]\ = 0.$

Particular solution: $_0 = e^{-c}$.

72. $+(2\ +\ )\ +\ (\ +\ )\ = 0.$

Solution: $= \begin{cases} e^{-}\ (\ _1 + \ _2\ ^{1-b}) & \text{if } \neq 1, \\ e^{-}\ (\ _1 + \ _2 \ln|\ |) & \text{if } = 1. \end{cases}$

73. $+[(\ +\ )\ +\ +\ ]\ +(\ +\ +\ )\ = 0.$

Here, and are positive integers; $a \neq$ or $\neq$.

Solution: $= \ _1 e^{-}\ \dfrac{\ ^{-1}}{\ ^{-1}}\big[\ ^{-}\ e^{(\ -b)}\big] + \ _2 e^{-b}\ \dfrac{\ ^{-1}}{\ ^{-1}}\big[\ ^{-}\ e^{(b-\ )}\big].$

74. $+(\ +\ )\ +(\ +\ )\ = 0.$

This is a special case of equation 2.1.2.108.

75. $-(\ +1)\ -\ ^2(\ +\ )\ = 0.$

Particular solution: $_0 = \exp\big(-\tfrac{1}{2}\ ^2$.

76. $-(2\ +1)\ +(\ ^3 + \ ^2 + \ )\ = 0.$

Solution: $= e\ \ _1 \sin\big(\tfrac{1}{2}\ ^2\ -\ + \ _2 \cos\big(\tfrac{1}{2}\ ^2\ -$.

77. $+(\ +\ )\ +\ (-\ ^2 + \ +\ +1) = 0.$

Particular solution: $_0 = \exp\big(-\tfrac{1}{2}\ ^2$.

78. $-(2\ ^2+1)\ +\ ^3\ = 0.$

Solution: $= \ _1 \exp\big[\tfrac{1}{2}\big(a + \overline{a^2 -}\ ^2\big] + \ _2 \exp\big[\tfrac{1}{2}\big(a - \overline{a^2 -}\ ^2\big].$

79. $+(\ ^2 + \ -5)\ +2\ ^2(\ -2)\ ^3\ = 0.$

Particular solution: $_0 = (a\ ^2+1)\exp(-a\ ^2).$

80. $+(\ ^2+\ )\ -[\ ^2+(\ +\ +\ ^2)\ +\ +2\ ]\ = 0.$

Particular solution: $_0 = \ e^{c}$.

81. $+(\ {}^2 + \quad + 2) \quad + \quad = 0.$

Particular solution: $_0 = a + \quad .$

82. $+(\ {}^2 + \quad + \ ) \quad + (2 \quad + \ ) \quad = 0.$

Integrating, we obtain a first-order linear equation: $' + (a\ {}^2 + \quad + \ - 1) \quad = \quad .$

83. $+(\ {}^2 + \quad + \ ) \quad + (\ - 1)(\quad + \ ) \quad = 0.$

Particular solution: $_0 = \ {}^{1-c}.$

84. $+(\ {}^2 + \quad + \ ) \quad + (\ {}^2 + B \quad + \quad) \quad = 0.$

1 . Let $A = ak$, $B = k(\ - k)$, $\quad = \ k$, where k is an arbitrary number.
Particular solution: $_0 = e^{-}\quad .$

2 . Let $A = a(\ + k)$, $B = a(\ + 1) - k(\ + k)$, $\quad = -\ k$.
Particular solution: $_0 = \exp\bigl(-\frac{1}{2}a\ {}^2 + k \quad .$

3 . Let $A = a(\ + k)$, $B = 2a - \ k - k^2$, $\quad = \ (\ - 1) + k(\ - 2)$.
Particular solution: $_0 = \ {}^{1-c}\exp\bigl(-\frac{1}{2}a\ {}^2 + k \quad .$

4 . Let $A = -ak$, $B = a(\ - 1) - k(\ + k)$, $\quad = \ (\ - 1) + k(\ - 2)$.
Particular solution: $_0 = \ {}^{1-c}e \quad .$

85. $+(\ {}^2 + \quad + 2) \quad + (\ {}^2 + \quad + \ ) \quad = 0.$

The substitution $\quad = \quad$ leads to a linear equation of the form 2.1.2.108: $'' + (a \quad + \ ) \ ' + (\quad + \ - a) \quad = 0.$

86. $+(\ {}^3 + \ ) \quad + \ (\ - 1)\ {}^2 \quad = 0.$

Particular solution: $_0 = \ {}^{1-b}.$

87. $+ \ (\ {}^2 + \ ) \quad + (3 \quad {}^2 + \ ) \quad = 0.$

Particular solution: $_0 = \quad \exp\bigl(-\frac{1}{3}a\ {}^3 - \quad .$

88. $+(\ {}^3 + \ {}^2 + 2) \quad + \quad = 0.$

Particular solution: $_0 = a + \quad .$

89. $+(\ {}^3 + \ {}^2 + \quad - 1) \quad + \ {}^2 \ {}^3 \quad = 0.$

Particular solution: $_0 = (a \quad + 1)e^{-} \quad .$

90. $+(\ {}^3 + \ {}^2 + \quad + \ ) \quad + (\ - 1)(\ {}^2 + \quad + \ ) \quad = 0.$

Particular solution: $_0 = \ {}^{1-} \quad .$

91. $+ \quad + (\quad - \quad {}^{-1} - \ {}^2 \quad + 2 \ ) \quad = 0.$

Particular solution: $_0 = \quad e^{-b} \quad .$

92. $+(\quad + 2) \quad + \quad {}^{-1} \quad = 0.$

Particular solution: $_0 = \ {}^{-1}.$

93. $+(\quad + 1 - \ ) \quad + \quad {}^2 \ {}^{-1} \quad = 0.$

1 . For $\ \neq \frac{1}{4}$, the solution has the form: $\quad = \ _1\exp\bigl(\beta_1 \quad + \ _2\exp\bigl(\beta_2 \quad .$ Here, β_1 and β_2 are roots of the quadratic equation: ${}^2\beta^2 + \ \beta + \ = 0.$

2 . For $\ = \frac{1}{4}$, the solution has the form: $\quad = (\ _1 + \ _2 \quad)\exp\bigl(-\frac{1}{2}\ {}^{-1} \quad .$

94. $+ (\quad + \;)\quad + \quad {}^{-1} = 0.$

Particular solution: $_0 = {}^{1-b}\exp(-a \quad).$

95. $+ (\quad + \;)\quad + \;(\; -1)\quad {}^{-1} = 0.$

Particular solution: $_0 = {}^{1-b}.$

96. $+ (\quad + \;)\quad + \;(\; + \quad -1)\quad {}^{-1} = 0.$

Particular solution: $_0 = \exp(-a \quad).$

97. $+ (\quad + \;)\quad + \;(\quad - \quad + \;) = 0.$

Particular solution: $_0 = e^{-c}\quad .$

98. $+ (\quad + \; -3 \; +1)\quad + \;{}^{2}\;(\; - \;)\;{}^{2}\;{}^{-1} = 0.$

Particular solution: $_0 = (a \quad +1)\exp(-a \quad).$

99. $+ (\quad + \;)\quad + (\;{}^{2}\;{}^{-1} + \quad {}^{-1}) = 0.$

This is a special case of equation 2.1.2.146 with $\quad = 0$.

100. $+ (\quad + \quad {}^{-1} + 2)\quad + \quad {}^{-2} = 0.$

Particular solution: $_0 = a + \quad .$

101. $+ (\quad + \quad)\quad + (\quad + \quad {}^{-1} - \;) = 0.$

Particular solution: $_0 = \quad \exp(-a \quad).$

102. $+ (\quad + \quad {}^{-1} + \quad -1)\quad + \;{}^{2} \quad = 0.$

Particular solution: $_0 = (a \; +1)e^{-}\quad .$

103. $+ (\quad + \quad + \;)\quad + (\; -1)(\quad {}^{-1} + \quad {}^{-1}) = 0.$

Particular solution: $_0 = {}^{1-c}.$

104. $+ (\quad {}^{+} \; + \quad + \quad +1-2\;)\quad + \;{}^{2}\quad {}^{2\;+\;-1} = 0.$

Particular solution: $_0 = (a \quad +1)\exp(-a \quad).$

105. $(\; + \;)\quad + (\quad + \;)\quad + \quad = 0.$

Particular solution: $_0 = \exp\left(-\dfrac{\quad + \quad -1}{\quad + a}\quad\right).$

106. $(\;{}_1 + \;{}_0)\quad + (\;{}_1 + \;{}_0) \quad - \quad {}_1 = 0.$

If $\quad = 1, 2, 3, \quad$, a polynomial of order $\quad$ in $\quad$ is a particular solution of the equation, which can be represented as: $_0 = \sum_{=0} -\dfrac{1}{_1}\quad {}^{-\;-1}[(a_1 \; + a_0)\;{}^{2} + \;_0\;] \quad$, where

$= \text{—}, \quad = \dfrac{+1}{+1}$ with $\quad \neq -1$.

107. $(\quad + \;)\quad + s(\quad + \;)\quad - s^2[(\; + \;) + \quad + \;] = 0.$

Particular solution: $_0 = e \quad .$

TABLE 15

Solutions of equation 2.1.2.108 for different values of the determining parameters

Solution: $= e \quad (z)$, where $z = \frac{-}{\lambda}$					
Constraints	k	λ			Parameters
$a_2 \neq 0$, $a_1^2 \neq 4a_0a_2$	$\frac{\overline{\quad} - a_1}{2a_2}$	$-\frac{a_2}{2a_2k + a_1}$	$-\frac{2}{a_2}$	$(a, \ ; z)$	$a = B(k) \ (2a_2k + a_1)$, $= (a_2 \ _1 - a_1 \ _2)a_2^{-2}$
$a_2 = 0$, $a_1 \neq 0$	$-\frac{a_0}{a_1}$	1	$-\frac{2 \ _2k + \ _1}{a_1}$	$\left(a, \frac{1}{2}; \beta z^2\right.$	$a = B(k) \ (2a_1)$, $\beta = -a_1 \ (2 \ _2)$
$a_2 \neq 0$, $a_1^2 = 4a_0a_2$	$-\frac{a_1}{2a_2}$	a_2	$-\frac{2}{a_2}$	$z \quad ^2Z \ (\beta \quad \overline{z}$	$= 1 - (2 \ _2k + \ _1)a_2^{-1}$, $\beta = 2 \quad \overline{B(k)}$
$a_2 = a_1 = 0$, $a_0 \neq 0$	$-\frac{1}{2 \ _2}$	1	$\frac{^2_1 - 4 \ _0 \ _2}{4a_0 \ _2}$	$z^{1 \ 2}Z_{1 \ 3}(\beta z^{3 \ 2}$ see also 2.1.2.12	$\beta = \frac{2}{3} \ \frac{a_0}{_2} \ ^{1 \ 2}$
Notation: $= a_1^2 - 4a_0a_2$, $B(k) = \ _2k^2 + \ _1k + \ _0$					

108. $(\ _2 \ + \ _2) \quad + (\ _1 \ + \ _1) \quad + (\ _0 \ + \ _0) \quad = 0.$

Let the function $\ (a, \ ; \ )$ be an arbitrary solution of the degenerate hypergeometric equation $\ '' \ + (\ - \ ) \ ' - a \ = 0$ (see 2.1.2.70), and the function $Z \ (\ )$ be an arbitrary solution of the Bessel equation $\ ^2 \ '' \ + \ \ ' + (\ ^2 - \ ^2) \ = 0$ (see 2.1.2.126). The results of solving the original equation are presented in Table 15.

109. $(\ + \ ) \quad + (\quad + \quad + \ ) \quad + (\quad ^{-1} + \quad ^{-1}) \quad = 0.$

Particular solution: $\ _0 = \exp \ - \ \frac{a \quad + \quad + \ -1}{+\lambda} \quad .$

2.1.2-4. Equations of the form $\ ^2 \ '' \ + \ (\ ) \ ' + g(\ ) \ = 0.$

110. $\ ^2 \quad + \quad = 0.$

This is a special case of equation 2.1.2.123. The substitution $\ = e \ $ leads to a constant coefficient linear equation: $\ '' - \ \ ' + a \ = 0.$

111. $\ ^2 \quad + (\quad + \ ) \ = 0.$

This is a special case of equation 2.1.2.132.

112. $\ ^2 \quad + [\ ^2 \ ^2 - \ (\ + 1)] \ = 0, \qquad = 0, 1, 2, \ldots$

Solution: $\ ^{+1} = (\ ^3 \ ) \ \frac{_1 \cos a \ + \ _2 \sin a}{^2 \ ^{-1}}$, where $\ = \text{—}.$

113. $\ ^2 \quad - [\ ^2 \ ^2 + \ (\ + 1)] \ = 0, \qquad = 0, 1, 2, \ldots$

Solution: $\ ^{+1} = (\ ^3 \ ) \ \frac{_1e \ + \ _2e^-}{^2 \ ^{-1}}$, where $\ = \text{—}.$

114. $\ ^2 \quad - (\ ^2 \ ^2 + 2 \quad + \ ^2 - \ ) \ = 0.$

Particular solution: $\ _0 = \ ^be \quad .$

115. $\;^{2}\quad + (\quad ^{2} + \quad + \;)\quad = 0.$

The substitution $\quad = \quad^{\lambda}$, where λ is a root of the quadratic equation $\lambda^2 - \lambda + \quad = 0$, leads to an equation of the form 2.1.2.108: $\quad'' \; + 2\lambda \quad' + (a \quad + \;)\quad = 0.$

For $a = -\frac{1}{4}$, $\quad = k$, and $\quad = \frac{1}{4} - \quad^2$, the original equation is referred to as Whittaker's equation.

116. $\;^{2}\quad - (\quad ^{3} + \frac{5}{16}\quad = 0.$

Particular solution: $\quad_0 = \quad^{-1\;\;4} \exp\bigl(\frac{2}{3}\;\;\overline{a}\;\;^{3\;\;2}\;.$

117. $\;^{2}\quad - [\;^{2}\;^{4} + \;(2\quad - 1)\;^{2} + \;(\quad + 1)]\quad = 0.$

Particular solution: $\quad_0 = \quad^{-b} \exp\bigl(-\frac{1}{2}a\;\;^{2}\;.$

118. $\;^{2}\quad + (\quad\quad + \;)\quad = 0.$

This is a special case of equation 2.1.2.132.

119. $\;^{2}\quad - [\;^{2}\;^{2}\quad + \;(2\quad + \quad - 1)\quad + \;(\quad - 1)]\quad = 0.$

Particular solution: $\quad_0 = \quad^{b} \exp(a\quad\quad).$

120. $\;^{2}\quad + (\quad ^{2}\quad + \quad\quad + \;)\quad = 0.$

This is a special case of equation 2.1.2.146.

121. $\;^{2}\quad + (\quad ^{3}\quad + \quad ^{2}\quad + \frac{1}{4} - \frac{1}{4}\quad ^{2}\quad = 0.$

The transformation $\xi = a\quad + \;,\quad = \quad\frac{-1}{2}$ leads to an equation of the form 2.1.2.7: $\;'' \; + (a\;\;)^{-2}\xi\quad = 0.$

122. $\;^{2}\quad + \bigl[\quad ^{2}\;(\quad\quad + \;)\quad + \frac{1}{4} - \frac{1}{4}\quad ^{2}\bigr]\quad = 0.$

The transformation $\xi = \quad\quad + \;,\quad = \quad\frac{-1}{2}$ leads to an equation of the form 2.1.2.7: $\;'' \; + a(\quad)^{-2}\xi\quad\quad = 0.$

123. $\;^{2}\quad + \quad\quad + \quad = 0.$

The Euler equation. Solution:

$$= \begin{cases} |\;|\;|^{\frac{1-}{2}}\bigl(\;_1|\;|\;+\;_2|\;|^{-} & \text{if } (1-a)^2 > 4\;, \\ |\;|\;|^{\frac{1-}{2}}(\;_1 + \;_2 \ln|\;|) & \text{if } (1-a)^2 = 4\;, \\ |\;|\;|^{\frac{1-}{2}}\bigl[\;_1 \sin(\;\ln|\;|) + \;_2 \cos(\;\ln|\;|)\bigr] & \text{if } (1-a)^2 < 4\;, \end{cases}$$

where $\quad = \frac{1}{2}|(1-a)^2 - 4\;|^{1\;\;2}.$

124. $\;^{2}\quad + \quad\quad + \bigl[\;^{2} - (\quad + \frac{1}{2}\;^{2}\bigr]\quad = 0,\qquad = 0,\ 1,\ 2,\ \ldots$

This is a special case of equation 2.1.2.126.

Solution: $\quad = \quad^{+1\;\;2}\;\frac{1}{\;}\quad\;_1\frac{\sin}{\;} + \;_2\frac{\cos}{\;}\quad.$

125. $\;^{2}\quad + \quad\quad - \bigl[\;^{2} + (\quad + \frac{1}{2}\;^{2}\bigr]\quad = 0,\qquad = 0,\ 1,\ 2,\ \ldots$

This is a special case of equation 2.1.2.127.

Solution: $\quad = \quad^{+1\;\;2}\;\frac{1}{\;}\quad\;_1\frac{e}{\;} + \;_2\frac{e^{-}}{\;}\quad.$

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126. $\;^{2} \quad + \quad + (\;^{2} - \;^{2}) \quad = 0.$

The Bessel equation.

1 . Let be an arbitrary noninteger. Then the general solution is given by:

$$= \;_1 \quad (\;) + \;_2 \quad (\;), \tag{1}$$

where ( ) and ( ) are the Bessel functions of the first and second kind:

$$(\;) = \;_{=0} \frac{(-1) \;(\quad 2)^{\;+2}}{k!\,\Gamma(\;+k+1)}, \qquad (\;) = \frac{(\;)\cos \quad - \;_{-}(\;)}{\sin}. \tag{2}$$

Solution (1) is denoted by $= Z\;(\;)$ which is referred to as the cylindrical function.

The cylindrical functions possess the following properties:

$$2\; Z\;(\;) = \;[Z_{-1}(\;) + Z_{+1}(\;)],$$

$$—[\quad Z\;(\;)] = \quad Z_{-1}(\;), \qquad —[\;^{-}\; Z\;(\;)] = -\;^{-}\; Z_{+1}(\;).$$

The functions ( ) and ( ) can be expressed in terms of definite integrals (with > 0):

$$(\;) = \;_0^{\pi} \cos(\;\sin\theta - \;\theta)\;\;\theta - \sin \quad _0 \;\exp(-\;\sinh\; - \;)\;,$$

$$(\;) = \;_0^{\pi} \sin(\;\sin\theta - \;\theta)\;\;\theta - \;_0\;(e \quad + e^{-}\quad\cos\quad)e^{-\;\sinh}\quad.$$

2 . In the case = $+\frac{1}{2}$, where = 0, 1, 2, , the Bessel functions are expressed in terms of elementary functions:

$$_{+\frac{1}{2}}(\;) = \sqrt{\frac{2}{}} \;^{+\frac{1}{2}}\; -\frac{1}{}— \quad \frac{\sin}{}, \qquad _{-\;-\frac{1}{2}}(\;) = \sqrt{\frac{2}{}} \;^{+\frac{1}{2}}\; \frac{1}{}— \quad \frac{\cos}{},$$

$$_{+\frac{1}{2}}(\;) = (-1)^{\;+1}\;_{-\;-\frac{1}{2}}(\;).$$

3 . Let = be an arbitrary integer. The following relations hold:

$$_{-}(\;) = (-1) \quad (\;), \qquad _{-}(\;) = (-1) \quad (\;).$$

The solution is given by formula (1) in which the function ( ) is obtained by substituting = into formula (2), while ( ) is found by taking the limit as and for nonnegative becomes

$$(\;) = \frac{2}{} \quad (\;)\ln\frac{}{2} - \frac{1}{} \;_{=0}^{-1} \frac{(\; - k - 1)!}{k!} \;\frac{2}{}^{\;-2}$$

$$-\frac{1}{} \;_{=0}\;(-1)\;\;\frac{}{2}^{\;+2}\;\frac{(k+1) + \;(\;+k+1)}{k!\,(\;+k)!},$$

where $(1) = -\mathcal{C}$, $(\;) = -\mathcal{C} + \;_{=1}^{-1} k^{-1}$, $\mathcal{C} = 0.5772$ is the Euler constant, $(\;) = [\ln\Gamma(\;)]'$ is the logarithmic derivative of the gamma function.

For nonnegative integer and large , we can write

$$\overline{\quad}\;_{2}\;(\;) = (-1)\;(\cos\quad + \sin\quad) + \quad(\;^{-2}),$$

$$\overline{\quad}\;_{2\;+1}(\;) = (-1)^{\;+1}(\cos\quad - \sin\quad) + \quad(\;^{-2}).$$

The function ( ) can be expressed in terms of a definite integral:

$$(\;) = \frac{1}{} \;_0^{\pi} \cos(\;\sin\; - \;)\;\;; \qquad = 0, 1, 2,$$

The Bessel functions are described in Subsection S.2.5 in more detail; see also the books by Abramowitz & Stegun (1964) and Bateman & Erdélyi (1953, Vol. 2).

127. $\;^{2}\quad + \quad - (\;^{2} + \;^{2})\quad = 0.$

The modified Bessel equation. It can be reduced to equation 2.1.2.126 by means of the substitution $\;=\;^{-}\;(\;^{2}=-1)$.

Solution:

$$= \;_{1}\;(\;) + \;_{2}\;(\;),$$

where $(\;)$ and $(\;)$ are the modified Bessel functions of the first and second kind:

$$(\;) = \sum_{=0} \frac{(\;2)^{2\;+}}{k!\,\Gamma(\;+k+1)}, \qquad (\;) = \frac{}{2}\,\frac{_{-}(\;)-\;(\;)}{\sin}.$$

The modified Bessel function $(\;)$ can be expressed in terms of the Bessel function:

$$(\;) = e^{-\pi\;\;2}\;(\;e^{\pi\;\;2}),\qquad ^{2}=-1.$$

The case $\;=\;+\frac{1}{2}$, where $\;=0,\,1,\,2,\;$, is given in 2.1.2.125.

If $\;=\;$ is a nonnegative integer, we have

$$(\;) = (-1)^{\;+1}\;(\;)\ln\frac{}{2} + \frac{1}{2}\sum_{=0}^{-1}(-1)\;\frac{}{2}^{\;2\;-}\frac{(\;-\;-1)!}{!}$$

$$+\frac{1}{2}(-1)\sum_{=0}\frac{}{2}^{\;+2}\;\frac{(\;+\;+1)+(\;+1)}{!\,(\;+\;)!},$$

where (z) is the logarithmic derivative of the gamma function (see 2.1.2.126, Item 3); for $=0$, the first sum is omitted.

As $\;+\;$, the leading terms of the asymptotic expansion are:

$$(\;)\;\;\frac{e}{\sqrt{2}},\qquad (\;)\;\;\frac{}{\sqrt{2}}e^{-}.$$

The modified Bessel functions are described in Subsection S.2.6 in more detail; see also the books by Abramowitz & Stegun (1964) and Bateman & Erdélyi (1953, Vol. 2).

128. $\;^{2}\quad + 2\quad - (\;^{2}\;^{2} + 2)\quad = 0.$

Solution: $\;^{2}\; = \;_{1}(a\;-1)e\; + \;_{2}(a\;+1)e^{-}\;.$

129. $\;^{2}\quad - 2\quad + [\;^{2}\;^{2} + \;(\;+1)]\quad = 0.$

Solution: $\;= \begin{cases} |\;|\;(\;_{1}\sin\;+\;_{2}\cos\;) & \text{if } \;\neq 0,\\ \;_{1}|\;|\;+\;_{2}|\;|^{\;+1} & \text{if } \;=0.\end{cases}$

130. $\;^{2}\quad - 2\quad + [-\;^{2}\;^{2} + \;(\;+1)]\quad = 0.$

Solution: $\;= \begin{cases} |\;|\;(\;_{1}e^{b}\;+\;_{2}e^{-b}\;) & \text{if } \;\neq 0,\\ \;_{1}|\;|\;+\;_{2}|\;|^{\;+1} & \text{if } \;=0.\end{cases}$

131. $\;^{2}\quad + \quad + (\;^{2} + \;+\;)\quad = 0.$

The substitution $\;=\;$, where k is a root of the quadratic equation $k^{2}+(\lambda-1)k+\;=0$, leads to an equation of the form 2.1.2.108: $\;''+(\lambda+2k)\;'+(a\;+\;)\;=0$.

132. $\;^{2}\quad + \quad + (\;+\;)\quad = 0,\qquad \neq 0.$

The case $\;=0$ corresponds to the Euler equation 2.1.2.123.

For $\;\neq 0$, the solution is:

$$= \;^{\frac{1-}{2}}\;_{1}\;\frac{2}{}\;^{-}\;\frac{}{2}\; + \;_{2}\;\frac{2}{}\;^{-}\;\frac{}{2}\;,$$

where $\;=\frac{1}{}\sqrt{(1-a)^{2}-4\;}$; (z) and (z) are the Bessel functions of the first and second kind.

133. $\;^{2}\quad+\quad+\quad(\quad+\;)\;=0.$

The substitution $\xi=$ leads to an equation of the form 2.1.2.108: $\;^{2}\xi\;''+\;(\;-1+a)\;'+(\;\xi+\;)\;=0.$

134. $\;^{2}\quad+(\quad+\;)\quad+\quad=0.$

The transformation $\;=z^{-1},\quad=z\;e\quad$, where k is a root of the quadratic equation $k^{2}+(1-a)k+\;=0$, leads to an equation of the form 2.1.2.108:

$$z\;''+[(2-\;)z+2k+2-a]\;'+[(1-\;)z+2k+2-a-\;k]\;=0.$$

135. $\;^{2}\quad+\;^{2}\quad+(\;^{2}+\quad+\;)\;=0.$

The substitution $\;=\;\exp\bigl(-\tfrac{1}{2}a\quad$ leads to a linear equation of the form 2.1.2.115: $\;^{2}\;''+[(\tfrac{1}{4}a^{2}+\;)\;^{2}+\quad+\;]\;=0.$

136. $\;^{2}\quad+(\;^{2}+\;)\quad+\;[(\;-\;)\;^{2}+\;]\;=0.$

Particular solution: $\;_{0}=e^{-c}\;.$

137. $\;^{2}\quad+(\;^{2}+\quad)\quad-\quad=0.$

Particular solution: $\;_{0}=\;^{-b}e^{-}\;.$

138. $\;^{2}\quad+(\;^{2}+\quad)\quad+[\;(\;-\;)\;^{2}+(\quad+\quad-2\quad)\;+\;(\;-\quad-1)]\;=0.$

Particular solution: $\;_{0}=\;^{-}\;e^{-}\;.$

139. $\;_{2}\;^{2}\quad+(\;_{1}\;^{2}+\;_{1}\;)\quad+(\;_{0}\;^{2}+\;_{0}\;+\;_{0})\;=0.$

The substitution $\;=\quad$, where k is a root of the quadratic equation $a_{2}k^{2}+(\;_{1}-a_{2})k+\;_{0}=0$, leads to an equation of the form 2.1.2.108: $a_{2}\;''+(a_{1}\;+2a_{2}k+\;_{1})\;'+(a_{0}\;+a_{1}k+\;_{0})\;=0.$

140. $\;^{2}\quad+[\;^{2}+(\quad-1)\;+\;]\quad+\;^{2}\quad=0.$

Particular solution: $\;_{0}=(a\;+1)e^{-}\;.$

141. $\;^{2}\quad-2\;(\;^{2}-\;)\quad+\{2\;^{2}+[(-1)\;-1]\;\}\;=0.$

For $\;=0,\,1,\,2,\quad$, particular solutions are polynomials:

$$\;_{0}=\quad(\;),\quad\text{where}\quad_{0}(\;)=1,\quad_{1}(\;)=\;,\quad_{2}(\;)=2\;^{2}-1-2a,\quad_{3}(\;)=2\;^{3}-(3+2a)\;,$$

The polynomials contain only even powers of for even and only odd powers of for odd .

142. $\;^{2}\quad+\;(\;^{2}+\quad+\;)\quad+(\;^{3}+B\;^{2}+\quad+\;)\;=0.$

1 . The substitution $\;=\quad$, where k is a root of the quadratic equation $k^{2}+(\;-1)k+\;=0$, leads to an equation of the form 2.1.2.84 (see also 2.1.2.80–2.1.2.83):

$$''+(a\;^{2}+\quad+\;+2k)\;'+[A\;^{2}+(B+ak)\;+\quad+\;k]\;=0.$$

2 . Let and be arbitrary parameters.

For $A=a\;,\;B=a\;+\quad-\;^{2},\quad=\quad+\quad-2\quad,\quad=\;(\;-\quad-1)$, a particular solution is: $\;_{0}=\;^{-}\;e^{-}\;.$

For $A=a(\;-\;),\;B=a(\;-\;+1)+\;(\;-\;),\quad=\quad+\quad-2\quad,\quad=\;(\;-\quad-1)$, a particular solution is: $\;_{0}=\;^{-}\;\exp(-\tfrac{1}{2}a\;^{2}-\quad).$

143. $\;^{2}\quad+\quad-(\quad+\quad^{-1}+\;^{2}\;^{2}+2\quad+\;^{2}-\;)\;=0.$

Particular solution: $\;_{0}=\;^{c}e^{b}\;.$

144. $^{2} \quad + \quad + (\quad ^{+2} \quad - \; ^{2} \; ^{4} \; ^{+2} + \quad ^{-1} - \; ^{2} - \quad) \quad = 0.$

Particular solution: $\quad _0 = \; ^{-} \quad \exp \; -\frac{}{2 \quad +1} \; ^{2 \quad +1}$.

145. $^{2} \quad + \; (\quad + \;) \quad + \; (\quad - 1) \quad = 0.$

Particular solution: $\quad _0 = \; ^{-b}.$

146. $^{2} \quad + \; (\quad + \;) \quad + (\quad ^{2} \; + \quad + \gamma) \quad = 0.$

The transformation $z = \quad , \quad = \; z^{-}$, where k is a root of the quadratic equation $^{2}k^{2} + \; (\; -1)k + \quad = 0$, leads to a linear equation of the form 2.1.2.108: $^{2}z \;'' +$ $[\; az + 2k \; ^{2} + \; (\; -1 + \;)] \;' + (\; z + k \; a + \beta) \; = 0.$

147. $^{2} \quad + \; (2 \quad + \;) \quad + [\; ^{2} \; ^{2} \; + \; (\; + \; -1) \quad + \; ^{2} \; + \quad + \gamma] \quad = 0.$

The substitution $\quad = \; \exp(a \quad)$ leads to a linear equation of the form 2.1.2.146: $^{2} \;'' + \quad ' + (\; ^{2} \; + \beta \quad + \;) \; = 0.$

148. $^{2} \quad + (\quad ^{+2} + \; ^{2} + \;) \quad + (\quad ^{+1} + \quad + \;) \quad = 0.$

Particular solution: $\quad _0 = \exp \; -\frac{a}{+1} \; ^{+1} -$.

2.1.2-5. Equations of the form $(a \; ^{2} + \quad + \;) \;'' \; + \; (\;) \;' + g(\;) \; = 0.$

149. $(1 - \; ^{2}) \quad + \; (\; - 1) \quad = 0, \qquad = 0, 1, 2, \ldots$

This equation is encountered in hydrodynamics when describing axially symmetric Stokes flows.

1 . For $\quad \geq 2$, the solution is given by:

$$= \; _1 \quad (\;) + \; _2 \quad (\;),$$

where $\quad (\;)$ and $\quad (\;)$ are the Gegenbauer functions which can be expressed in terms of the Legendre functions of the first and second kind (see 2.1.2.153) as follows:

$$(\;) = \frac{_{-2}(\;) - \quad (\;)}{2 \quad -1}, \qquad (\;) = \frac{Q \; _{-2}(\;) - Q \; (\;)}{2 \quad -1}.$$

2 . For $\quad = 0$ and $\quad = 1$, the solution is: $\quad = \; _1 + \; _2$.

150. $(\; ^{2} - \; ^{2}) \quad + \quad - 6 \quad = 0.$

Particular solution: $\quad _0 = (4 \; - \;) | \; + a|^{\frac{2 \; +b}{2}} | \; - a|^{\frac{2 \; -b}{2}}.$

151. $(\; ^{2} - 1) \quad + \quad + \quad = 0.$

1 . For $a = k^{2} > 0$, the solution is:

$$= \begin{cases} _1 \cos(k \operatorname{arccosh} | \; |) + \; _2 \sin(k \operatorname{arccosh} | \; |) & \text{if } | \; | > 1, \\ _1 \exp(k \arccos \;) + \; _2 \exp(-k \arccos \;) & \text{if } | \; | < 1, \end{cases}$$

where $\operatorname{arccosh} \quad = \ln\big(\; + \sqrt{\; ^{2} - 1}\big)$.

2 . For $a = -k^{2} < 0$, the solution is:

$$= \begin{cases} _1 \exp(k \operatorname{arccosh} | \; |) + \; _2 \exp(-k \operatorname{arccosh} | \; |) & \text{if } | \; | > 1, \\ _1 \cos(k \arccos \;) + \; _2 \sin(k \arccos \;) & \text{if } | \; | < 1. \end{cases}$$

3 . For $a = - \; ^{2}$, where $\quad$ is a nonnegative integer, particular solutions are the Chebyshev polynomials: $\quad (\;) = \cos(\; \arccos \;).$

152. $\mathbf{(1- \ ^2) \quad - \quad + \ ^2 \ = 0, \qquad = 0,\ 1,\ 2,\ \ldots}$

This is a special case of equation 2.1.2.151 with $a = - \ ^2$. Particular solution:

$$_0 = \quad (\ ) = \cos(\ \arccos\ ) = \frac{(-1)}{2\ (\frac{1}{2}} \quad \overline{1- \ ^2} \ \text{——} \left[(1- \ ^2)^{\ -\frac{1}{2}}\right]$$

$$= \frac{}{2} \sum_{=0}^{[\ \ 2]} (-1) \ \frac{(\ - \ \ -1)!}{!(\ -2\ \ )!}(2\ )^{\ -2} \ ,$$

where $(\ )$ is the Chebyshev polynomial of the first kind, $(a) \ = a(a+1) \quad (a+ \ -1)$, and $[\ ]$ stands for the integer part of a number .

153. $\mathbf{(1- \ ^2) \quad -2 \quad + \ (\ +1) \ = 0, \qquad = 0,\ 1,\ 2,\ \ldots}$

The Legendre equation.

The solution is given by:

$$= \ _1 \quad (\ ) + \ _2 Q \ (\ ),$$

where the Legendre polynomials $(\ )$ and the Legendre functions of the second kind $Q \ (\ )$ are given by the formulas:

$$(\ ) = \frac{1}{!\,2} \text{——} (\ ^2 - 1)\ , \quad Q\ (\ ) = \frac{1}{2} \quad (\ ) \ln \frac{1+}{1-} - \sum_{=1} \frac{1}{} \quad _{-1}(\ ) \quad _{-}(\ ).$$

The functions $= \quad (\ )$ can be conveniently calculated by the recurrence relations:

$$_0(\ ) = 1, \quad _1(\ ) = \ , \quad _2(\ ) = \frac{1}{2}(3\ \ ^2 - 1), \quad , \quad _{+1}(\ ) = \frac{2\ +1}{+1} \quad (\ ) - \frac{}{+1} \quad _{-1}(\ ).$$

Three leading functions $Q \ = Q \ (\ )$ are:

$$Q_0(\ ) = \frac{1}{2}\ln\frac{1+}{1-}, \quad Q_1(\ ) = \frac{}{2}\ln\frac{1+}{1-} - 1, \quad Q_2(\ ) = \frac{3\ \ ^2-1}{4}\ln\frac{1+}{1-} - \frac{3}{2}\ .$$

All zeros of the polynomial $(\ )$ are real and lie on the interval $-1 < \ < 1$; the functions $(\ )$ form an orthogonal system on the closed interval $-1 \leq \ \leq 1$, with the following relations taking place:

$$\int_{-1}^{1} \ (\ ) \quad (\ ) \quad = \begin{cases} 0 & \text{if } \ \neq \ , \\ \dfrac{2}{2\ \ +1} & \text{if } \ = \ . \end{cases}$$

154. $\mathbf{(1- \ ^2) \quad -2 \quad + \ (\ +1) \ = 0.}$

The Legendre equation; is an arbitrary number. The case $=$ where is a nonnegative integer is considered in 2.1.2.153.

The substitution $z = \ ^2$ leads to the hypergeometric equation. Therefore, with $|\ | < 1$ the solution can be written as:

$$= \ _1 \quad \left(-\frac{}{2}, \ \frac{1+}{2}, \ \frac{1}{2}; \quad \right) + \ _2 \quad \left(\frac{1-}{2}, \ 1+\frac{}{2}, \ \frac{3}{2}; \quad \right),$$

where $(\ , \beta, \ ; \ )$ is the hypergeometric series (see 2.1.2.171).

The Legendre equation is discussed in the books by Abramowitz & Stegun (1964), Bateman & Erdélyi (1953, Vol. 1), and Kamke (1977) in more detail (see also Subsection S.2.9).

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155. $\mathbf{(1 - \ ^2) \quad - 3 \quad + \ (\ + 2) \ = 0, \qquad = 1,\ 2,\ 3,\ \ldots}$

Particular solution:

$$_0 = \quad (\ ) = \frac{\sin[(\ +1)\arccos\ ]}{\sqrt{1-\ ^2}} = \frac{(-1)\ (\ +1)}{2\ ^{+1}\left(\frac{1}{2}\right)_{+1}} \frac{1}{\sqrt{1-\ ^2}} \frac{}{}\left[(1-\ ^2)^{\ +\frac{1}{2}}\right]$$

$$= \sum_{=0}^{[\ 2]} (-1)\ \frac{(\ -\ )!}{\ !(\ -2\ )!}(2\ )^{\ -2\ },$$

where $(\ )$ is the Chebyshev polynomial of the second kind, $(a)\ = a(a+1)\ \ (a+\ -1)$, and $[\ ]$ stands for the integer part of a number .

156. $\mathbf{(\ ^2 - 1) \quad + 2(\ + 1) \quad - (\ + \ + 1)(\ - \ ) \ = 0, \qquad = 1,\ 2,\ 3,\ \ldots}$

Solution: $= \frac{}{} (\ )$, where $(\ )$ is the general solution of the Legendre equation 2.1.2.154.

157. $\mathbf{(\ ^2 - 1) \quad - 2(\ - 1) \quad - (\ - \ + 1)(\ + \ ) \ = 0, \qquad = 1,\ 2,\ 3,\ \ldots}$

Solution: $= |\ ^2 - 1|\ \frac{}{} (\ )$, where $(\ )$ is the general solution of the Legendre equation 2.1.2.154.

158. $\mathbf{(\ ^2 - 1) \quad + (2\ + 1) \quad - \ (2\ + \ ) \ = 0.}$

1 . Particular solution:

$$_0 = \frac{\Gamma(2a+\ )}{\Gamma(\ +1)\Gamma(2a)}\ \left(2a+\ ,-\ ,a+\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{2}\ \right), \tag{1}$$

where $(\ ,\beta,\ ;z)$ is the hypergeometric function (see equation 2.1.2.171 and Subsection S.2.8).

2 . For $=\ $, where $= 0,\ 1,\ $, the right-hand side of (1) defines the Gegenbauer polynomials,

$$^{(\ )}(\ ) = \frac{\Gamma(2a+\ )}{\Gamma(\ +1)\Gamma(2a)}\ \left(2a+\ ,-\ ,a+\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{2}\ \right) = \sum_{=0} \frac{\Gamma(a+k)\Gamma(2a+\ +k)(\ -1)}{k!(\ -k)!\,2\ \Gamma(a)\Gamma(2a+2k)}.$$

159. $\mathbf{(1 - \ ^2) \quad + (2\ - 3) \quad + (\ + 1)(\ + 2\ - 1) \ = 0, \qquad = 0,\ 1,\ 2,\ \ldots}$

Particular solution:

$$_0(\ ) = (1-\ ^2)^{\ -1\ 2}\ ^{(\ )}(\ ) = (1-\ ^2)^{\ -1\ 2} \sum_{=0} \frac{\Gamma(a+k)\Gamma(2a+\ +k)(\ -1)}{k!(\ -k)!\,2\ \Gamma(a)\Gamma(2a+2k)},$$

where $^{(\ )}(\ )$ are the Gegenbauer polynomials.

160. $\mathbf{(1 - \ ^2) \quad + \big[\ - \ - (\ + \ + 2)\ \big] \quad + \ (\ + \ + \ + 1) \ = 0, \qquad = 0,\ 1,\ 2,\ \ldots}$

Particular solution:

$$_0(\ ) = \ ^{,}(\ ) = \frac{(-1)}{2\ \ !}(1-\ )^{-}\ (1+\ )^{-}\ \frac{}{}\ (1-\ )^{\ +}\ (1+\ )^{\ +}$$

$$= 2^{-} \sum_{=0} \ _{+} \ ^{-}_{+}\ (\ -1)^{\ -}\ (\ +1)\ ,$$

where $^{,}(\ )$ are the Jacobi polynomials and $_{b}$ are binomial coefficients.

161. $(1-\ ^2)\quad + \left[\ -\ +(\ +\ -2)\ \right]\quad +(\ +1)(\ +\ +\ )\ = 0,\qquad = 0,\ 1,\ 2,\ \ldots$

Particular solution: $_0(\ )=(1-\ )\ (1+\ )\quad {}^{,}(\ )$, where $\ {}^{,}(\ )$ are the Jacobi polynomials (see 2.1.2.160).

162. $(\ \ ^2+\ )\quad +\quad +\quad = 0.$

The substitution $z = \dfrac{}{\sqrt{a\ ^2+\ }}$ leads to a constant coefficient linear equation: $\ ''+\ \ =0$.

163. $(\ ^2+\ )\quad +2\quad +2(\ -1)\ \ = 0.$

Particular solution: $\ _0=|\ ^2+a|^{1-b}$.

164. $(\ ^2-\ ^2)\quad +2\quad +\ (\ -1)\ \ = 0.$

Solution: $\ =\ _1|\ -a|^{1-b}+\ _2|\ +a|^{1-b}$.

165. $(\ ^2+\ ^2)\quad +2\quad +\ (\ -1)\ \ = 0.$

Solution: $\ =\ _1(\ ^2+a^2\ )^{\frac{1-b}{2}}\sin\ +\ _2(\ ^2+a^2\ )^{\frac{1-b}{2}}\cos\ $, where $\ =(1-\ )\arctan(a\ \ )$.

166. $(\ \ ^2+\ )\quad +(2\ \ +1)\quad +\quad = 0,\qquad = 1,\ 2,\ 3,\ \ldots$

This equation can be obtained by -fold differentiation of an equation of the form 2.1.2.162: $(a\ \ ^2+\ )\ ''\ +a\ \ '\ +(\ -a\ \ ^2)\ \ =0$.

Solution: $\ =\ ^{(\ )}$.

167. $(1-\ ^2)\quad -\quad +(2\ \ ^2+\ )\ \ = 0.$

This is an algebraic form of the Mathieu equation. The substitution $\ =\cos z$ leads to the Mathieu equation 2.1.6.29: $\ ''\ +(a+\ \ +a\cos 2z)\ \ =0$.

168. $(1-\ ^2)\quad +(\ \ +\ )\quad +\quad = 0.$

1 . The substitution $2z=1+\ $ leads to the hypergeometric equation 2.1.2.171: $z(1-z)\ ''+[az+\frac{1}{2}(\ -a)]\ '+\ \ =0$.

2 . For $a=-2\ \ -3$, $\ =0$, and $\ =\lambda$, the Gegenbauer functions are solutions of the equation.

3 . In the special case $a=-\ \ -\beta-2$, $\ =\beta-\ $, and $\ =\ (\ +\ +\beta+1)$, solutions of the equation are the Jacobi polynomials:

$$\ ^{(\ ,\ )}(\ )=2^{-}\ \sum_{=0}\ \ _{+}\ \ _{+}^{-}\ (\ -1)^{-}\ (\ +1)\ ,$$

where $\ _b$ are binomial coefficients (see Paragraph S.2.1-2).

169. $(\ \ ^2+\ )\quad +(\ \ ^2+\ )\quad +\ [(\ -\ \ )\ ^2+\ -\ \ ]\ \ = 0.$

Particular solution: $\ _0=e^{-\lambda}\ $.

170. $(\ \ ^2+\ )\quad +[\ (\ +\ )\ ^2+(\ -\ )\ +2\ \ ]\quad +\ ^2(\ \ ^2+\ )\ \ = 0.$

Particular solution: $\ _0=(\lambda\ \ +1)e^{-\lambda}\ $.

171. $\mathbf{(\quad -1)\qquad +[(\quad +\quad +1)\quad -\gamma]\quad +\qquad = 0.}$

The Gaussian hypergeometric equation. For $\quad \neq 0,\ -1,\ -2,\ -3,\quad$, a solution can be expressed in terms of the hypergeometric series:

$$(\ ,\beta,\ ;\ ) = 1 + \sum_{=1} \frac{(\ )\ (\beta)}{(\ )}\,\frac{}{k!},\qquad (\ )\ =\ (\ \ +1)\quad (\ \ +k-1),$$

which, *a fortiori*, is convergent for $|\ \ | < 1$.

For $\ > \beta > 0$, this solution can be expressed in terms of a definite integral:

$$(\ ,\beta,\ ;\ ) = \frac{\Gamma(\ )}{\Gamma(\beta)\,\Gamma(\ -\beta)}\int_0^1 \ ^{-1}(1-\ )^{\ -\ -1}(1-\quad)^{-}\quad,$$

where $\Gamma(\beta)$ is the gamma function.

If $\ $ is not an integer, the general solution of the hypergeometric equation has the form:

$$= \ _1\ (\ ,\beta,\ ;\ ) + \ _2\ ^{1-}\ (\ -\ \ +1,\ \beta-\ \ +1,\ 2-\ ;\ ).$$

In the degenerate cases $\ = 0,\ -1,\ -2,\ -3,\quad$, a particular solution of the hypergeometric equation corresponds to $\ _1 = 0$ and $\ _2 = 1$. If $\ $ is a positive integer, another particular solution corresponds to $\ _1 = 1$ and $\ _2 = 0$. In both these cases, the general solution can be constructed by means of the formula given in 2.1.1.

Table 16 presents some special cases where $\ $ is expressed in terms of elementary functions.

Table 17 gives the general solutions of the hypergeometric equation for some values of the determining parameters.

The function $\ $ possesses the following properties:

$$\begin{aligned}
&(\ ,\beta,\ ;\ ) = \ (\beta,\ ,\ ;\ ),\\
&(\ ,\beta,\ ;\ ) = (1-\ )^{\ -\ -}\ (\ -\ ,\ \ -\beta,\ ;\ ),\\
&(\ ,\beta,\ ;\ ) = (1-\ )^{-}\ (\ ,\ \ -\beta,\ ;\ \frac{}{-1}),\\
\overline{\qquad}\ &(\ ,\beta,\ ;\ ) = \frac{(\ )\ (\beta)}{(\ )}\ (\ +\ ,\ \beta+\ ,\ \ +\ ;\ ).
\end{aligned}$$

The hypergeometric functions are discussed in the books by Abramowitz & Stegun (1964) and Bateman & Erdélyi (1953, Vol. 1) in more detail; see also Subsection S.2.8.

172. $\mathbf{(\ +\ )\qquad +(\ \ +\ )\quad +\qquad = 0.}$

The substitution $\ = -az$ leads to the hypergeometric equation 2.1.2.171: $z(1-z)\ '' + [(\ \ a)-\ \ z]\ ' -\ \ = 0$.

173. $\mathbf{2\ (\ -1)\qquad +(2\ \ -1)\qquad +(\ \ +\ )\ = 0.}$

The substitution $\ = \cos^2\xi$ leads to the Mathieu equation 2.1.6.29: $\ '' - (a+2\ \ +a\cos 2\xi)\ = 0$.

174. $\mathbf{(\ ^2 + 2\quad +\ )\qquad +(\ +\ )\quad -\quad ^2\ = 0.}$

Solution: $\ = \ _1\big(\ \ +a+\sqrt{\ ^2+2a\ \ +\ }\ \big)\ +\ _2\big(\ \ +a+\sqrt{\ ^2+2a\ \ +\ }\ \big)^{-}.$

175. $\mathbf{(\quad ^2 +\quad +\ )\qquad +(\quad +\ )\quad +(\ -2\ )\ = 0.}$

Integrating yields a first-order linear equation: $(a\ \ ^2 +\quad +\ )\ ' + [(\ \ -2a)\ \ +k-\ ]\ =\ $.

176. $\mathbf{(\quad ^2 +\quad +\ )\qquad +(\quad +\ )\quad -\qquad = 0.}$

Particular solution: $\ _0 = k\ \ +\ $.

TABLE 16
Some special cases in which the hypergeometric function
(, β, ; z) is expressed in terms of elementary functions

	β		z	
$-$	β			$\sum_{=0} \frac{(-\ )\ (\beta)}{(\ )} \frac{}{k!}$, where $=1,2,3,$
$-$	β	$-\ -$		$\sum_{=0} \frac{(-\ )\ (\beta)}{(-\ -\ )} \frac{}{k!}$, where $=1,2,3,$
	β	β		$(1-\ )^{-}$
	$+\frac{1}{2}$	$\frac{1}{2}$	^{2}	$\frac{1}{2}\left[(1+\ )^{-2}\ +(1-\ )^{-2}\ \right]$
	$+\frac{1}{2}$	$\frac{3}{2}$	^{2}	$\frac{(1+\ )^{1-2}\ -(1-\ )^{1-2}}{2\ (1-2\ )}$
	$-$	$\frac{1}{2}$	$-\ ^{2}$	$\frac{1}{2}\left[(\ \overline{1+\ ^{2}}+\ \ ^{2}\ +(\ \overline{1+\ ^{2}}-\ \ ^{2}\ \right]$
	$1-$	$\frac{1}{2}$	$-\ ^{2}$	$\frac{(\ \overline{1+\ ^{2}}+\ \ ^{2\ -1}+(\ \overline{1+\ ^{2}}-\ \ ^{2\ -1}}{2\ \overline{1+\ ^{2}}}$
	$-\frac{1}{2}$	$2\ -1$		$2^{2\ -2}(1+\ \overline{1-\ }\ ^{2-2}$
	$-$	$\frac{1}{2}$	$\sin$	$\cos(2\ \ )$
	$1-$	$\frac{3}{2}$	$\sin^{2}$	$\frac{\sin[(2\ -1)\ ]}{(\ -1)\sin(2\ )}$
	$2-$	$\frac{3}{2}$	$\sin^{2}$	$\frac{\sin[(2\ -2)\ ]}{(\ -1)\sin(2\ )}$
	$1-$	$\frac{1}{2}$	$\sin^{2}$	$\frac{\cos[(2\ -1)\ ]}{\cos}$
	$+\frac{1}{2}$	$\frac{1}{2}$	$-\tan^{2}$	$\cos^{2}\ \ \cos(2\ \ )$
	$+1$	$\frac{1}{2}$		$(1+\ )(1-\ )^{-\ -1}$
	$+\frac{1}{2}$	$2\ +1$		$\left(\frac{1}{2}+\frac{1}{2}\ \overline{1-\ }\ \right)^{-2}$
	$+\frac{1}{2}$	2		$\left(1-\ \ ^{-1\ 2}\left(\frac{1}{2}+\frac{1}{2}\ \overline{1-\ }\ \right)^{1-2}\right.$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	^{2}	$^{-1}\arcsin$
$\frac{1}{2}$	1	$\frac{3}{2}$	$-\ ^{2}$	$^{-1}\arctan$
1	1	2	$-$	$^{-1}\ln(\ +1)$
$\frac{1}{2}$	1	$\frac{3}{2}$	^{2}	$\frac{1}{2}\ln\frac{1+}{1-}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$-\ ^{2}$	$^{-1}\ln(\ +\ \overline{1+\ ^{2}}$
$+1$	$+\ \ +1$	$+\ \ +k+2$		$\frac{(-1)\ \ (\ +\ \ +k+1)!}{!\ k!\ (\ +\ \ )!\ (\ \ +k)!}\frac{+}{+}\ \ (1-\ )\ ^{+}\ \overline{\ \ \ }\ ,$ $=-\ ^{-1}\ln(1-\ ),\ \ ,\ \ ,k=0,1,2,3,$

TABLE 17

General solutions of the hypergeometric equation for some values of the determining parameters

	β		Solution: $= ( )$
0	β		${}_1 + {}_2 \; \lvert \; \rvert^{-} \lvert \; \rvert \; -1\rvert^{- \; -1}$
	$+\frac{1}{2}$	$2 \; +1$	${}_1\bigl(1+ \; \overline{1-} \; {}^{-2} + {}_2 \; {}^{-2} \; \bigl(1+ \; \overline{1-} \; {}^{2}$
	$-\frac{1}{2}$	$\frac{1}{2}$	${}_1\bigl(1+ \; {}^{- \; 1-2} + {}_2\bigl(1- \; {}^{- \; 1-2}$
	$+\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{\overline{}} \; {}_1\bigl(1+ \; {}^{- \; 1-2} + {}_2\bigl(1- \; {}^{- \; 1-2}$
1	β		$\lvert \; \rvert^{1-} \lvert \; -1\rvert^{- \; -1} \; {}_1 + {}_2 \; \lvert \; \rvert^{-2} \lvert \; -1\rvert^{-}$
	β		$\lvert \; -1\rvert^{-} \; {}_1 + {}_2 \; \lvert \; \rvert^{-} \lvert \; -1\rvert^{-1}$
	β	$+1$	$\lvert \; \rvert^{-} \; {}_1 + {}_2 \; \lvert \; \rvert^{-1} \lvert \; -1\rvert^{-}$

177. $(\; ^2 + 2 \; + \;) \; + (\; + \;) \; + \; = 0.$

The substitution $\xi = \dfrac{}{\sqrt{a \; ^2 + 2 \; +}}$ leads to a constant coefficient linear equation of the form 2.1.2.1: $'' + \; = 0.$

178. $(\; ^2 + 2 \; + \;) \; + 3(\; + \;) \; + \; = 0.$

The substitution $= \sqrt{|a \; ^2 + 2 \; + \; |}$ leads to a linear equation of the form 2.1.2.177: $(a \; ^2 + 2 \; + \;) \; '' + (a \; + \;) \; ' + (\; - a) \; = 0.$

179. $(\; _2 \; ^2 + \; _2 \; + \; _2) \; + (\; _1 \; + \; _1) \; + \; _0 \; = 0.$

Let λ_1 and λ_2 are roots of the quadratic equation $a_2\lambda^2 + \; _2\lambda + \; _2 = 0.$

1 . For $\lambda_1 \neq \lambda_2$, the substitution $z = \dfrac{-\lambda_1}{\lambda_2 - \lambda_1}$ leads to the hypergeometric equation 2.1.2.171: $z(1-z) \; '' - (A \; + B) \; ' - \; = 0$, where $A = \dfrac{_1}{a_2}$, $B = \dfrac{_1\lambda_1 + \; _1}{a_2(\lambda_2 - \lambda_1)}$, $= \dfrac{_0}{a_2}$.

2 . For $\lambda_1 = \lambda_2 = \lambda$, the transformation $= \lambda + \xi^{-1}$, $= \xi$, where k is a root of the quadratic equation $a_2k^2 + (a_2 - \; _1)k + \; _0 = 0$, leads to a linear equation of the form 2.1.2.108: $a_2\xi \; '' - [(\; _1 + \lambda \; _1)\xi + \; _1 - 2a_2(k+1)] \; ' - k(\; _1 + \lambda \; _1) \; = 0.$

3 . Let $_0 = -a_2 \; (\; -1) - \; _1$, where is a positive integer. Then, among solutions there exists a polynomial of degree $\leq$.

180. $(\; ^2 + \; + \;) \; - (\; ^2 - \; ^2) \; + (\; + \;) \; = 0.$

Particular solution: $_0 = \; - k.$

181. $(\; ^2 + \; + \;) \; + (\; ^3 + \; ^3) \; - (\; ^2 - \; + \; ^2) \; = 0.$

Particular solution: $_0 = \; + k.$

2.1.2-6. Equations of the form $(a_3\ {}^3+a_2\ {}^2+a_1\ +a_0)\ ''\ +\ (\ )\ '+g(\ )\ =0.$

182. $^3\ \ +(\ \ +\ )\ =0.$

This is a special case of equation 2.1.2.132 with $=-1$.

183. $^3\ \ +(\ {}^2+\ )\ \ +\ \ =0.$

The substitution $=1\ z$ leads to an equation of the form 2.1.2.139: $z^2\ ''+z(2-a-\ z)\ '+\ =0$.

184. $^3\ \ +(\ {}^2+\ \ )\ \ +\ \ =0.$

Particular solution: $_0=a-2+$.

185. $^3\ \ +(\ {}^2+\ \ )\ \ +\ \ =0.$

The substitution $=1\ z$ leads to an equation of the form 2.1.2.108: $z\ ''+(2-a-\ z)\ '+\ =0$.

186. $^3\ \ +(\ {}^2+\ \ )\ \ +(\ \ +\ )\ =0.$

1 . The substitution $=$, where $k=-$, leads to a linear equation of the form 2.1.2.134: $^2\ ''\ +[(a+2k)\ +\ ]\ '+[k(a+k-1)+\ ]\ =0$.

2 . If $=0$ and $=\ (a-2)$, a particular solution is: $_0=e^b$.

187. $^3\ \ +(\ {}^3-\ {}^2+\ \ \ +\ )\ \ +\ {}^2\ \ \ =0.$

Particular solution: $_0=(a\ +1)e^-$.

188. $^3\ \ +\ (\ \ \ +\ )\ \ -(\ \ \ -\ \ \ {}^{-1}+\ )\ =0.$

Particular solution: $_0=\ \exp(\ \ )$.

189. $(\ {}^2+\ )\ \ \ +2(\ {}^2+\ )\ \ -2\ \ \ =0.$

Particular solution: $_0=a\ +$.

190. $(\ {}^2+\ )\ \ \ +(\ {}^2+\ )\ \ +s\ \ =0.$

The substitution $az=-\ {}^2$ leads to the hypergeometric equation 2.1.2.171: $z(1-z)\ ''+\frac{1}{2}\left[1+\ a^{-1}-(1+\ )z\right]\ '-\frac{1}{4}\ \ =0$.

191. $^2(\ \ +\ )\ \ \ +[\ {}^2+(2\ +\ \ \ )\ +\ \ ]\ \ +\ (\ -2\ )\ =0.$

Particular solution: $_0=\exp(\lambda\ \ )$.

192. $^2(\ \ +\ )\ \ \ -2\ (\ \ +2\ )\ \ +2(\ \ +3\ )\ =0.$

Solution: $=\dfrac{_1\ {}^2+\ _2\ {}^3}{a\ +}$.

193. $^2(\ \ +\ )\ \ \ +[\ (2-\ \ -\ \ )\ {}^2-\ (\ \ +\ \ )\ ]\ \ +[\ \ \ (\ -1)\ +\ \ \ (\ \ +1)]\ =0.$

Solution: $=\begin{cases}\dfrac{_1|\ |\ +\ _2|\ |\ {}^{+1}}{a\ +} & \text{if}\ \ \neq\ -1,\\[2mm] \dfrac{|\ |\ (\ _1+\ _2\ln|\ |)}{a\ +} & \text{if}\ \ =\ -1.\end{cases}$

194. $^2(\ +\ _2)\ \ \ +\ (\ _1\ +\ _1)\ \ +(\ _0\ +\ _0)\ =0.$

The substitution $=$, where k is a root of the quadratic equation $a_2k^2+k(a_1-a_2)+a_0=0$, leads to a linear equation of the form 2.1.2.172: $(\ +a_2)\ ''\ +[(2k+\ _1)\ +2ka_2+a_1]\ '+[k^2+k(\ _1-1)+\ _0]\ =0$.

195. (^{3} + ^{2} +) + (^{2} + + 2) + (− 2) = 0.

Particular solution: $_0 = 2a -\ + (2\ - \beta)\ ^{-1}$.

196. (^{3} + ^{2} +) + (^{2} + + 2) − (+ 2 −) = 0.

Particular solution: $_0 = \ + 2(\beta -\ ) + \frac{\lambda}{}$, where $\lambda = \frac{+ (\ - \beta)(2\ - \beta)}{- a}$.

197. (^{3} + ^{2} +) + [−2 ^{2} − (+ 1) +] + 2(+ 1) = 0.

Particular solution: $_0 = (ak +\ - 1)\ ^2 + (\ + k)(2\ - k)$.

198. (^{3} + ^{2} +) + (^{2} + +) + (− 1)[(−) + −] = 0.

Particular solution: $_0 = \ ^{1-}$.

199. (^{3} + ^{2} +) + [(−) ^{2} + (2 − 1) −] + (−2 + 1) = 0.

Particular solution: $_0 = (a +\ )\ ^2 + (2\ + 4\ - 1)(\ +\ )$.

200. (^{3} + ^{2} +) + (^{2} + +) + [−2(+) + 1] = 0.

With the constraint

$$2(2a +\ )(\ + k) + (2\ + 2\ + 1)[\ + 1 + 2k(a +\ )] = 0,$$

a particular solution has the form: $_0 = (2a +\ )\ ^2 + (2\ + 2\ + 1)(\ - k)$.

201. (^{3} + ^{2} +) + ^{2} (^{2} −) − ^{3} = 0.

Particular solution: $_0 = (a\ + 2)e^{-}$.

202. 2 (^{2} + +) + (^{2} −) + ^{2} = 0.

The substitution $\xi = \frac{}{a\ ^2 +\ +}\ ^{1\ 2}$ leads to a constant coefficient linear equation: $2\ '' + \lambda\ = 0$.

203. (^{2} + + 1) + (^{2} + + γ) + (+) = 0.

The substitution $= \ ^{1-}$ leads to an equation of the same form:

$$(a\ ^2 +\ + 1)\ '' + [(\ + 2a - 2a\ )\ ^2 + (\beta + 2\ - 2\ )\ + 2 -\ ]\ '$$
$$+ [\ + (1 -\ )(\ - a\ )]\ +\ + (1 -\ )(\beta -\ )\ = 0.$$

204. (− 1)(−) + (+ + 1) ^{2} − [+ + 1 + (γ +) −] + γ + (− *q*) = 0.

Heun's equation.

1 . For $|a| \geq 1$ and $\neq 0, -1, -2, -3,$, a solution can be represented as the power series:

$$(a,\ ;\ ,\beta,\ ,\ ,\ ) = \sum_{=0} ,$$

where the coefficients are determined by the recurrence formulas:

$$_0 = 1, \quad a\ _1 = \ ,$$
$$a(\ + 1)(\ +\ )\ _{+1} = a(\ +\ +\ - 1) +\ + \beta -\ +\ + -$$
$$- \big[(\ - 1)(\ - 2) + (\ - 1)(\ + \beta + 1) +\ \beta\big]\ _{-1}.$$

A fortiori, the series is convergent for $|\ | \leq 1$.

TABLE 18

Some transformations preserving the form of Heun's equation

No	New variables	Parameters of transformed equation for $=\ (\xi)$					
1*	$\xi = \ ,\quad =$	a			β		
2	$\xi = 1 - \ ,\quad =$	$1-a$	$\beta -$		β		
3	$\xi = \ ,\quad = \lvert\ \rvert^{\ -1}$	a	$_1$	$- \ +1$	$\beta - \ +1$	$2-$	
4	$\xi = \frac{\ }{a},\quad = \lvert\ \rvert^{\ -1}$	$\frac{1}{a}$	$\frac{_2}{a}$	$- \ +1$	$\beta - \ +1$	$2-$	$+\beta -$ $- \ +1$
5	$\xi = \frac{1}{\ },\quad = \lvert\ \rvert$	$\frac{1}{a}$	$_3$		$- \ +1$	$-\beta+1$	
6	$\xi = \frac{\ }{a},\quad =$	$\frac{1}{a}$			β		$+\beta -$ $- \ +1$
7	$\xi = 1 - \frac{\ }{a},\quad =$	$1-\frac{1}{a}$			β	$+\beta -$ $- \ +1$	
8	$\xi = \frac{\ }{a},\quad = \lvert\ \rvert$	a		$- \ +1$	$+ \ -1$	$-\beta+1$	$+\beta -$ $- \ +1$
9	$\xi = \frac{\ -1}{\ },\quad = \lvert\ \rvert$	$1-\frac{1}{a}$			$- \ +1$		$-\beta+1$
10	$\xi = \frac{a(\ -1)}{(a-1)},\quad = \lvert\ \rvert$	$\frac{a}{a-1}$			$- \ +1$		$+\beta -$ $- \ +1$
11	$\xi = \frac{\ }{\ -1},\quad = \lvert\ -1\rvert$	$\frac{a}{a-1}$			$- \ +1$		$-\beta+1$
12	$\xi = \frac{(a-1)}{a(\ -1)},\quad = \lvert\ -1\rvert$	$1-\frac{1}{a}$			$- \ +1$		$+\beta -$ $- \ +1$
Notation: $_1 = \ +(\ - \ +1)(\beta - \ +1) - \ \beta + \ (\ -1),\quad _2 = \ _1 + a\ (1- \ ),$ $_3 = \ a^{-1} + \ (\ - \ +1) + \ a^{-1}(\ -\beta) - a\ .$							

* This row corresponds to the original equation, while the others refer to the transformed equation for $=\ (\xi)$.

2 . If is not an integer, the general solution of Heun's equation can be presented as follows:

$$= \ _1\ (a,\ ;\ ,\beta,\ ,\ ,\ ) + \ _2\lvert\ \rvert^{1-}\ (a,\ _1;\ - \ +1, \beta - \ +1, 2-\ ,\ ,\ ),$$

where $_1 = \ +(\ - \ +1)(\beta - \ +1) - \ \beta + \ (\ -1)$.

Table 18 lists some transformations preserving the form of Heun's equation. (Whenever at least one of the indicated equations is integrable by quadrature with some values of parameters, all the other equations are also integrable for those values of the parameters.)

References: H. Bateman and A. Erdélyi (1955, Vol. 3), E. Kamke (1977), S. Yu. Slavyanov, W. Lay, and A. Seeger (1955), A. Ronveaux (1995).

205. $(\ ^3 + \ ^2 + \ + \ )\ - (\ ^2 - \ ^2)\ + (\ + \ )\ = 0.$

Particular solution: $_0 = \ - \lambda$.

206. $2(\ ^3 + \ ^2 + \ + \ )\ + (3\ ^2 + 2\ + \ )\ + \ = 0.$

The substitution $\xi = \int \frac{\ }{\sqrt{a\ ^3 + \ ^2 + \ + \ }}$ leads to a constant coefficient linear equation: $2\ '' + \lambda\ = 0.$

207. $\mathbf{2(\quad{}^3 + \quad{}^2 + \quad + \ ) \qquad + 3(3 \quad{}^2 + 2 \quad + \ ) \quad + (6 \quad + 2 \ + \ ) \quad = 0.}$

This equation is obtained by differentiating the equation 2.1.2.206.

208. $\mathbf{(\quad{}^3 + \quad{}^2 + \quad + \ ) \qquad + [\quad{}^2 + (\ \gamma + \ ) \ + \ \gamma] \quad - (\quad + \ ) \quad = 0.}$

Particular solution: $\ _0 = \ + \ .$

209. $\mathbf{(\quad{}^3 + \quad{}^2 + \quad + \ ) \qquad + (\ {}^3 + \ {}^3) \quad - (\ {}^2 - \quad + \ {}^2) \quad = 0.}$

Particular solution: $\ _0 = \ + \lambda.$

210. $\mathbf{2\ (\quad{}^2 + \quad + \ ) \qquad + [\ (2 - \ )\ {}^2 + \ (1 - \ ) \ - \quad] \quad + \qquad {}^{k+1} \quad = 0.}$

The substitution $\xi = \qquad {}^2(a\ {}^2 + \ + \ )^{-1\ 2}$ leads to a constant coefficient linear equation: $2\ '' + \lambda \ = 0.$

2.1.2-7. Equations of the form $(a_4\ {}^4 + a_3\ {}^3 + a_2\ {}^2 + a_1\ + a_0)\ '' + \ (\ )\ ' + g(\ )\ = 0.$

211. $\mathbf{\ {}^4 \qquad + \qquad = 0.}$

The transformation $z = 1 \quad , \quad = \qquad$ leads to a constant coefficient linear equation of the form 2.1.2.1: $\ '' + a \ = 0.$

212. $\mathbf{\ {}^4 \qquad + (\quad {}^2 + \quad + \ ) \quad = 0.}$

The transformation $z = 1 \quad , \quad = \qquad$ leads to a linear equation of the form 2.1.2.115: $z^2\ '' + (\ z^2 + \ z + a)\ = 0.$

213. $\mathbf{\ {}^4 \qquad - (\ + \ )\ {}^2 \qquad + [(\ + \ ) \ + \quad] \quad = 0.}$

Solution: $\ = \begin{cases} \ _1\ e^{-} \quad + \ _2\ e^{-b} & \text{if } a \neq \ , \\ (\ _1 \ + \ _2)e^{-} & \text{if } a = \ . \end{cases}$

214. $\mathbf{\ {}^4 \qquad + 2\ {}^2(\ + \ ) \quad + \qquad = 0.}$

The substitution $z = 1$ leads to a constant coefficient linear equation: $\ '' - 2a\ ' + \quad = 0.$

215. $\mathbf{\ {}^4 \qquad + \qquad - (\quad {}^{-1} + \qquad {}^{-2} + \ {}^2) \quad = 0.}$

Particular solution: $\ _0 = \ e^{-b} \quad .$

216. $\mathbf{\ {}^2(\ - \ )^2 \qquad + \qquad = 0.}$

Solution: $\ = \ _1|\ |\ |\ - a|^{1-} \ + \ _2|\ |^{1-}\ |\ - a|$, where is a root of the quadratic equation $(\ - 1)a^2 = - \ .$

217. $\mathbf{\ {}^2(\ - \ )^2 \qquad + \qquad = \ {}^2(\ - \ )^2.}$

Solution:

$$= |\ |\ |\ - a|^{1-} \qquad _1 + \frac{}{a(2 \quad - 1)} \quad |\ |^{1-}\ |\ - a|$$
$$+ |\ |^{1-}\ |\ - a| \qquad _2 - \frac{}{a(2 \quad - 1)} \quad |\ |\ |\ - a|^{1-} \qquad ,$$

where is a root of the quadratic equation $(\ - 1)a^2 = - \ .$

218. $\mathbf{\ {}^2(\ - 1)^2 \qquad + (\ {}^2 + \quad + \ ) \quad = 0.}$

Let and be roots of the quadratic equations

$$a\ (\ - 1) + \ = 0, \qquad a\ (\ - 1) + \ + \ + \ = 0.$$

The substitution $\ = \ (\ - 1)$ leads to the hypergeometric equation of the form 2.1.2.171: $a\ (\ - 1)\ '' + 2a[(\ + \ )\ - \ ]\ ' + (2a \quad - \ - 2\ )\ = 0.$

219. $\;^{2}(\;^{2}+\;)\quad+(\;^{2}+\;)\quad+\quad=0.$

The substitution $\xi=\;^{2}$ leads to a linear equation of the form 2.1.2.194: $4\xi^{2}(\xi+a)\;''+2\xi[(\;+1)\xi+a+\;]\;'+\quad=0.$

220. $(\;^{2}+1)^{2}\quad+\quad=0.$

The Halm equation. Solution:

$$=\begin{cases}\sqrt{\;^{2}+1}\,\big[\;_{1}\cos(\beta\arctan\;)+\;_{2}\sin(\beta\arctan\;)\big] & \text{if } a+1=\beta^{2}>0,\\ \sqrt{\;^{2}+1}\,\big[\;_{1}\cosh(\beta\arctan\;)+\;_{2}\sinh(\beta\arctan\;)\big] & \text{if } a+1=-\beta^{2}<0,\\ \sqrt{\;^{2}+1}\,(\;_{1}+\;_{2}\arctan\;) & \text{if } a=-1.\end{cases}$$

221. $(\;^{2}-1)^{2}\quad+\quad=0.$

Solution:

$$=\begin{cases}\sqrt{|\;^{2}-1|}\,\big[\;_{1}\cos\big(\beta\ln|z|\big)+\;_{2}\sin\big(\beta\ln|z|\big)\big] & \text{if } a-1=4\beta^{2}>0,\\ (\;+1)\big(\;_{1}|z|^{(2\;-1)\;2}+\;_{2}|z|^{-(2\;+1)\;2}\big) & \text{if } a-1=-4\beta^{2}<0,\\ \sqrt{|\;^{2}-1|}\,\big(\;_{1}+\;_{2}\ln|z|\big) & \text{if } a=1,\end{cases}$$

where $z=(\;+1)\;(\;-1).$

222. $(\;^{2}\quad^{2})^{2}\quad+\;^{2}\quad=0.$

This is the equation of bending of a double-walled compressed bar with a parabolic cross-section.

1 . For the upper sign (constricted bar), the solution is as follows:

$$=\sqrt{\;^{2}+a^{2}}\,(\;_{1}\cos\quad+\;_{2}\sin\quad),\qquad\text{where}\quad=\sqrt{1+(\;a)^{2}}\,\arctan(\;a).$$

2 . For the lower sign (bar with salients), the solution is given by:

$$=\sqrt{a^{2}-\;^{2}}\,(\;_{1}\cos\quad+\;_{2}\sin\quad),\qquad\text{where}\quad=\frac{\sqrt{\;^{2}-a^{2}}}{2a}\ln\frac{a+}{a-};\quad|\;|<a.$$

223. $4(\;^{2}+1)^{2}\quad+(\;^{2}+\;-3)\quad=0.$

Solution:

$$=\begin{cases}(\;^{2}+1)^{1\;4}(\;_{1}\cos\xi+\;_{2}\sin\xi) & \text{if } a>1,\\ (\;^{2}+1)^{1\;4}(\;_{1}\cosh\xi+\;_{2}\sinh\xi) & \text{if } a<1,\end{cases}$$

where $\xi=\frac{1}{2}\sqrt{|a-1|}\,\ln\big(\;+\sqrt{|\;^{2}+1|}\big).$

224. $(\;^{2}+\;)^{2}\quad+2\;(\;^{2}+\;)\quad+\quad=0.$

The substitution $\xi=\dfrac{}{a\;^{2}+}$ leads to a constant coefficient linear equation: $\;''+\quad=0.$

225. $(\;^{2}-1)^{2}\quad+2\;(\;^{2}-1)\quad-[\;(\;+1)(\;^{2}-1)+\;^{2}]\quad=0.$

Here, is an arbitrary number and is a nonnegative integer. This is a special case of equation 2.1.2.226.

If $\;=0$, this equation coincides with the Legendre equation 2.1.2.154. Denote its general solution by $\;(\;)$. If $\;=1,\ 2,\ 3,\ \;$, the general solution of the original equation is given by the formula: $\;=|\;^{2}-1|\;^{2}\frac{}{}\;(\;).$

226. $\mathbf{(1-\ \ ^2)^2 \quad -2\ \ (1-\ \ ^2) \quad +[\ \ (\ \ +1)(1-\ \ ^2)-\ \ ^2]\ \ =0.}$

The Legendre equation, and are arbitrary parameters.

The transformation $= 1-2\xi, \quad =|\ \ ^2-1|\ \ ^2$ leads to the hypergeometric equation 2.1.2.171:

$$\xi(\xi-1)\ \ '' + (\ \ +1)(1-2\xi)\ \ ' + (\ \ -\ \ )(\ \ +\ \ +1)\ \ = 0$$

with parameters $= \ \ -\ \ ,\ \beta = \ \ +\ \ +1, \quad = \ \ +1.$

In particular, the original equation is integrable by quadrature if $=$ or $= -\ \ -1.$

In Subsection S.2.9, the Legendre equation are discussed in more detail. See also the books by Abramowitz & Stegun (1964) and Bateman & Erdélyi (1953, Vol. 1).

227. $\mathbf{(\ \ ^2-1)^2 \quad + \quad (\ \ ^2-1) \quad +(\ \ ^2+\ \ \ +\ \ )\ \ =0.}$

The transformation $\xi = \frac{1}{2}(\ \ +1), \quad = |\ \ +1|^{-}\ |\ \ -1|^{-}$, where and are parameters that are determined by solving the second-order algebraic system

$$4a\ \ (\ \ -1)+2\ \ +\ \ +\ \ +e=0, \qquad (\ \ -\ \ )[2a(\ \ +\ \ -1)+\ \ ]=\ \ ,$$

leads to the hypergeometric equation 2.1.2.171 with respect to $= \ \ (\xi).$

228. $\mathbf{(\ \ \ ^2+\ \ )^2 \quad +(2\ \ \ +\ \ )(\ \ \ ^2+\ \ ) \quad + \quad =0.}$

The substitution $\xi = \dfrac{}{a\ \ ^2+}$ leads to a constant coefficient linear equation of the form 2.1.2.11: $''+\ \ '+k\ \ =0.$

229. $\mathbf{(\ \ \ ^2+\ \ )^2 \quad +(\ \ \ ^2+\ \ )(\ \ \ ^2+\ \ ) \quad +2(\ \ -\ \ ) \quad =0.}$

Particular solution: $_0 = \exp\left(-\dfrac{^2+}{a\ \ ^2+}\right).$

230. $\mathbf{(\ \ ^2+\ \ )^2 \quad + \quad (\ \ ^2+\ \ ) \quad -(\ \ \ ^{+1}+\ \ )\ \ =0.}$

Particular solution: $_0 = \sqrt{\ \ ^2+a}.$

231. $\mathbf{(\ \ ^2+\ \ )^2 \quad + \quad (\ \ ^2+\ \ ) \quad - \quad [\ \ \ ^{+1}+(\ \ -1)\ \ ^2+\ \ ]\ \ =0.}$

Particular solution: $_0 = (\ \ ^2+a)\ \ ^2.$

232. $\mathbf{(\ \ -\ \ )^2(\ \ -\ \ )^2 \quad - \quad =0, \qquad \neq\ \ .}$

The transformation $\xi = \ln\dfrac{-a}{-}, \quad = (\ \ -\ \ )$ leads to a constant coefficient linear equation: $(a-\ \ )^2(\ \ ''-\ \ ')-\ \ =0.$ Therefore, the solution is as follows:

$$= \ _1|\ \ -a|^{(1+\lambda)\ \ 2}|\ \ -\ \ |^{(1-\lambda)\ \ 2} + \ _2|\ \ -a|^{(1-\lambda)\ \ 2}|\ \ -\ \ |^{(1+\lambda)\ \ 2},$$

where $\lambda^2 = 4\ \ (a-\ \ )^{-2}+1 \neq 0.$

233. $\mathbf{(\ \ -\ \ )^2(\ \ -\ \ )^2 \quad +(\ \ -\ \ )(\ \ -\ \ )(2\ \ +\ \ ) \quad + \quad =0.}$

Let k_1 and k_2 be roots of the quadratic equation $(a-\ \ )^2k^2+(a-\ \ )(a+\ \ +\lambda)k+\ \ =0.$

Solution:

$$= \begin{cases} _1|z|^{\ \ 1} + \ _2|z|^{\ \ 2} & \text{if } k_1 \neq k_2, \\ |z|\ \ \left(\ _1 + \ _2\ln|z|\right) & \text{if } k_1 = k_2 = k, \end{cases}$$

where $z = (\ \ -a)\ \ (\ \ -\ \ ).$

234. $\mathbf{(\ \ \ ^2+\ \ \ +\ \ )^2 \quad + \quad =0.}$

The transformation $\xi = \dfrac{}{a\ \ ^2+\ \ \ +}, \quad = \sqrt{|a\ \ ^2+\ \ \ +\ \ |}$ leads to a constant coefficient linear equation of the form 2.1.2.1: $''+(A+a\ \ -\frac{1}{4}\ \ ^2)\ \ =0.$

235. $\mathbf{(\;^{2} - 1)^{2} \quad + 2 \;(\;^{2} - 1) \quad + [(\;^{2} - 1)(\;^{2} \;^{2} - \;) - \;^{2}] \; = 0.}$

Equation for prolate spheroidal wave functions, $= 0, 1,$ It arises when separating variables in the wave equation written in the system of prolate spheroidal coordinates.

1 . In applications, one usually looks for eigenvalues $\lambda = \lambda$ and eigenfunctions $=\quad(\;)$ that assume finite values at $= \;\; 1$. The following functions are solutions of the eigenvalue problem:

$$^{(1)}\,(a, \;) = \sum_{=0,1} \;(a)\; _{+}(\;) \quad \text{(prolate angular functions of the first kind)},$$

$$^{(2)}\,(a, \;) = \sum_{=-} \;(a) Q\; _{+}(\;) \quad \text{(prolate angular functions of the second kind)},$$

where $(\;)$ and $Q\;(\;)$ are the associated Legendre functions of the first and second kind. For $-1 \le \;\; \le 1$, we have $(\;) = (1 - \;^{2})\;\;^{2}\frac{m}{m}\;(\;)$. The summation is performed over either even or odd values of , depending on whether $|\; - \;|$ is even or odd, respectively.

2 . The following recurrence relations for the coefficients $= \;(a)$ hold:

$$_{+2} + (\beta \; - \lambda \;\;) \;\; + \;\; _{-2} = 0,$$

where

$$= \frac{a^{2}(2 \;\; + k + 1)(2 \;\; + k + 2)}{(2 \;\; + 2k + 3)(2 \;\; + 2k + 5)},$$

$$\beta \; = (\;\; + k)(\;\; + k + 1) + a^{2}\frac{2(\;\; + k)(\;\; + k + 1) - 2 \;\;^{2} - 1}{(2 \;\; + 2k - 1)(2 \;\; + 2k + 3)},$$

$$= \frac{a^{2}k(k - 1)}{(2 \;\; + 2k - 3)(2 \;\; + 2k - 1)}.$$

3 . For $a \quad 0$, the eigenvalues are defined by:

$$\lambda \quad = \;(\;\; + 1) + \frac{1}{2}\; 1 - \frac{(2 \;\; - 1)(2 \;\; + 1)}{(2 \;\; - 1)(2 \;\; + 3)} \; a^{2} + \;\;\big(a^{4}\;.$$

4 . For a , we have:

$$\lambda \quad = a \;\; + \;\;^{2} - \tfrac{1}{8}(\;^{2} + 5) - \tfrac{1}{64}\;(\;^{2} + 11 - 32 \;\;^{2})a^{-1} + \;\;\big(a^{-2}\;, \qquad = 2(\;\; - \;\;) + 1.$$

References: H. Bateman and A. Erdélyi (1955, Vol. 3), M. Abramowitz and I. A. Stegun (1964).

236. $\mathbf{(\;^{2} + 1)^{2} \quad + 2 \;(\;^{2} + 1) \quad + [(\;^{2} + 1)(\;^{2} \;^{2} - \;) + \;^{2}] \; = 0.}$

Equation of oblate spheroidal wave functions, $= 0, 1,$ The transformations $= \quad ,$ $a = \mp\, a$ lead to equation 2.1.2.235.

See the books by Bateman & Erdélyi (1955, Vol. 3) and Abramowitz & Stegun (1964) for more information on this equation.

237. $\mathbf{(\quad ^{2} + \quad + \;)^{2} \quad + (2 \quad + \;)(\quad ^{2} + \quad + \;) \quad + \quad = 0.}$

The substitution $\xi = \dfrac{\quad}{a\;^{2} + \quad +}$ leads to a constant coefficient linear equation of the form 2.1.2.11: $'' \; + (k - \;)\;' + \quad = 0$.

2.1.2-8. Other equations.

238. $^{6} \quad - {}^{5} \quad + \quad = 0.$

The transformation $\xi = {}^{-2}$, $\quad = \quad {}^{-2}$ leads to a constant coefficient linear equation of the form 2.1.2.1: $4 \quad '' + a \quad = 0$.

239. $^{6} \quad + (3 \, {}^{2} + \quad) \, {}^{3} \quad + \quad = 0.$

The substitution $\xi = {}^{-2}$ leads to a constant coefficient linear equation: $4 \quad '' - 2a \quad ' + \quad = 0$.

240. $$+ \sum_{=1}^{3} \frac{(1 - \quad - \quad)}{-} - \frac{}{(\ _1 - \ _1)(\ _2 - \ _2)(\ _3 - \ _3)} \sum_{=1}^{3} \frac{{}^{-1}}{-} = 0,$$

where $\sum_{=1}^{3} (\quad + \beta \ ) = 1$, $|a \ | + | \quad | > 0$, $\Delta \quad = a \quad {}_{+1} - a \ {}_{+1} \quad \neq 0$, $a \ {}_{+3} = a \ $, $\quad {}_{+3} = \quad$.

Riemann's equation. Denote this equation by:

$$\begin{matrix} a_1 & a_2 & a_3 & {}_1 & {}_2 & {}_3 \\ {}_1 & {}_2 & {}_3 & \beta_1 & \beta_2 & \beta_3 \end{matrix} \quad = 0. \tag{1}$$

For $a_1 = \ _2 = 0$, $a_3 = \ _3 = 1$, $\ _1 = \ _3 = 0$, $\ _2 = \quad$, $\beta_1 = 1 - \quad$, $\beta_2 = \beta$, and $\beta_3 = \quad - \quad - \beta$, equation (1) transforms into the hypergeometric equation 2.1.2.171.

The transformation

$$\xi = \frac{A \quad + B}{+}, \qquad = \frac{| \ _1 \quad - a_1| \ | \ _3 \quad - a_3|}{| \ _2 \quad - a_2| \ ^{+}} \quad , \tag{2}$$

where $A \quad - B \quad \neq 0$, brings the original equation into an equation of similar form:

$$\begin{matrix} A_1 & A_2 & A_3 & {}_1 + & {}_2 - \ - & {}_3 + \\ B_1 & B_2 & B_3 & \beta_1 + & \beta_2 - \ - & \beta_3 + \end{matrix} \quad \xi \quad = 0, \tag{3}$$

where $A \quad = Aa \quad + B \quad$, $B \quad = \quad a \quad + \quad$.

In (2), assume $\quad = - \ _1$, $\quad = - \ _3$, $A = \ _1 \ \Delta_3$, $B = -a_1 \ \Delta_3$, $\quad = - \ _2 \ \Delta_2$, and $\quad = a_2 \ \Delta_2$ to obtain the hypergeometric equation (3).

241. $\quad + \ (\quad + \ ) \ {}^{-4} \quad = 0.$

The transformation $\xi = \dfrac{}{a \ +}$, $\quad = \dfrac{}{a \ +}$ leads to an equation of the form 2.1.2.7: $'' + \quad {}^{-2} \xi^{-} \quad = 0$.

242. $\quad + \quad - (\ {}^{2} \quad + 2 \quad {}^{-1} + \quad + \ ) \quad = 0.$

Particular solution: $_0 = \ e^{b}$.

243. $\quad + (\quad + \ ) \quad - \quad = 0.$

Particular solution: $_0 = a \quad + \ $.

244. $\quad + (\quad {}^{-1} + \quad) \quad + (\quad - 1) \quad = 0.$

Particular solution: $_0 = \ {}^{1-}$.

245. $\quad + (2 \quad {}^{-1} + \quad {}^{2} + \quad) \quad + \quad = 0.$

Particular solution: $_0 = a + \quad$.

246. $+(\quad+\ )\quad+\ [(\ -\ )\quad+\ ]\ =0.$

Particular solution: $_0=e^{-c}\ .$

247. $+(\quad-\quad^{-1}+\quad+\ )\quad+\ ^2\quad=0.$

Particular solution: $_0=(a\quad+1)e^{-}\ .$

248. $+(\quad^{+}\ +1)\quad+\quad(1+\quad^{-1})\ =0.$

Particular solution: $_0=\exp\ -\dfrac{a}{+1}\quad^{+1}\ .$

249. $(\quad+\ )\quad+(\quad+\ )\quad+\ [(\ -\quad)\quad+\ -\quad]\ =0.$

Particular solution: $_0=e^{-\lambda}\ .$

250. $(\quad+\quad+\ )\quad=\quad(\ -1)\quad^{-2}\ .$

Particular solution: $_0=a\quad+\quad+\ .$

251. $(\quad+1)\quad+[(\ -\ )\quad+\ -\ ]\quad+\ (1-\ )\quad^{-1}\ =0.$

Particular solution: $_0=(\quad+1)^b\ .$

252. $(\ ^2\ +\ )\quad+(\ ^2\ +\ -\quad)\quad-\ ^2\ ^2\ ^{-1}\ =0.$

Solution: $=\ _1(\quad+\ \overline{\ ^2\ +a}\ ^b\quad+\ _2(\quad+\ \overline{\ ^2\ +a}\ ^{-b}\ .$

253. $^2(\ ^2\ ^2\ -1)\quad+\ [\ ^2(\ +1)\ ^2\ +\ -1]\quad-\ (\ +1)\ ^2\ ^2\ ^2\quad=0.$

Solution: $=\quad(a\quad)$, where $(\ )$ is the general solution of the Legendre equation 2.1.2.154.

254. $^2(\quad-1)\quad+\ (\ p\quad+q)\quad+(\quad+s)\ =0.$

Find the roots A_1, A_2 and B_1, B_2 of the quadratic equations

$$A^2-(\ +1)A-\ =0,\quad B^2-(\ -1)B+\ =0$$

and define parameters $,\ ,\beta$, and by the relations

$$=A_1,\quad =(A_1+B_1)\ ^{-1},\quad \beta=(A_1+B_2)\ ^{-1},\quad =1+(A_1-A_2)\ ^{-1}.$$

Then the solution of the original equation has the form $=\ ^c\ (a\quad)$, where $=\ (z)$ is the general solution of the hypergeometric equation 2.1.2.171: $z(z-1)\ ''+[(\ +\beta+1)z-\ ]\ '+\ \beta\ =0.$

255. $(\quad+\ )^2\quad-\quad^{-2}[(\ -1)\quad+\ (\ -1)]\ =0.$

Particular solution: $_0=|\quad+a|^b\ .$

256. $(\quad+\ )^2\quad+(\quad+\ )(\quad+\ )\quad+\ (\ -\quad)\quad^{-1}\ =0.$

Particular solution: $_0=\exp\ -\dfrac{+}{a\quad+}\ .$

257. $(\quad+\ )^2\quad+\quad(\ +\ )\quad-\quad^{-2}(\quad^{+1}+\quad-\ )\ =0.$

Particular solution: $_0=(\quad+a)^1\ .$

258. $(\quad+\ )^2\quad+\quad(\quad+\ )\quad+(\quad-\quad^{-1}-1)\ =0.$

Particular solution: $_0=\exp\ -\dfrac{}{a\quad+}\ .$

259. $\ {}^2(\quad + \ )^2 \quad + (\ \ + 1)\ (\ {}^2\ {}^2 - \ {}^2) \quad + \quad = 0.$

The substitution $\xi = \frac{1}{\ }\ln\frac{a}{a\ \ +\ }$ leads to a constant coefficient linear equation of the form 2.1.2.11: $\ '' - (\ \ +2)\ ' + \ \ = 0.$

260. $(\quad {}^{+1} + \quad + \ )\quad + (\quad + \quad {}^{-1} + \gamma)\quad + [\ (\ - \ - \ )\quad {}^{-1} + (\ -1)(\ - \ )\quad {}^{-2}]\ = 0.$

Particular solution: $\ _0 = \exp\left[\frac{(a\ \ + a - \ )\quad + (\ \ - \beta)\quad {}^{-1} - }{a\quad {}^{+1} + \quad + }\right].$

261. $(\quad + \quad + \ )\quad + (\ - \ )\quad + \quad = 0.$

Particular solution: $\ _0 = \ \ - \lambda.$

262. $(\quad + \quad + \ )\quad + (\ {}^2 - \ {}^2)\quad + (\ + \ )\quad = 0.$

Particular solution: $\ _0 = \ \ - \lambda.$

263. $2(\quad + \quad + \ )\quad + (\quad {}^{-1} + \quad {}^{-1})\quad + \quad = 0.$

The substitution $\xi = \int\frac{\ }{\sqrt{a\ \ + \quad + \ }}$ leads to a constant coefficient linear equation: $2\ '' + \quad = 0.$

264. $(\quad + \ )\quad {}^{+1}\quad + (\quad + \ )\quad - \quad {}^{-1}\quad = 0.$

Particular solution: $\ _0 = \exp\left(-\int\frac{\ }{(a\ \ + \ )}\right).$

265. $\ (\ )\quad + [2\ \ (\ ) + (\ {}^2 + \ )Q_{\ -2}(\ )]\quad + \ Q_{\ -2}(\ )\quad = 0.$

Here, $\ (\ )$ and $Q_{\ -2}(\ )$ are arbitrary polynomials of degrees $\ $ and $\ -2$, respectively.

Particular solution: $\ _0 = a + \ .$

2.1.3. Equations Containing Exponential Functions

2.1.3-1. Equations with exponential functions.

1. $\quad + \quad {}^{\lambda}\quad = 0, \quad \neq 0.$

Solution: $\ = \ _1\ _0(z) + \ _2\ _0(z)$, where $z = 2\lambda^{-1}\sqrt{\ a}\,e^{\lambda\ \ /2}$; $\ _0(z)$ and $\ _0(z)$ are the Bessel functions.

2. $\quad + (\quad - \ )\quad = 0.$

Solution: $\ = \ _1\ _{2\ \sqrt{\ }}\big(2\sqrt{\ a}\,e^{\ \ /2}\big) + \ _2\ _{2\ \sqrt{\ }}\big(2\sqrt{\ a}\,e^{\ \ /2}\big)$, where $\ (z)$ and $\ (z)$ are the Bessel functions.

3. $\quad + \ (\quad {}^{\lambda} - \quad {}^{2\lambda}\ )\quad = 0.$

Particular solution: $\ _0 = \exp\left(-\frac{a}{\lambda}e^{\lambda\ }\right).$

4. $\quad - [\ {}^2\ {}^2 + \ (2\ \ + 1)\quad + \ {}^2]\quad = 0.$

Particular solution: $\ _0 = \exp(ae^{\ } + \ \ ).$

5. $\quad - (\quad {}^{2\lambda} + \quad {}^{\lambda} + \ )\quad = 0.$

The transformation $z = e^{\lambda\ }$, $\ = z^{-\ }\ $, where $k = \ \ \bar{}\ \lambda$, leads to an equation of the form 2.1.2.108: $\lambda^2 z\ '' + \lambda^2(2k+1)\ ' - (az + \ )\ = 0.$

6. $+ (\quad ^{4\lambda} \quad + \quad ^{3\lambda} \quad + \quad ^{2\lambda} \quad - \frac{1}{4} \quad ^{2}) \quad = 0.$

The transformation $\xi = e^{\lambda}$, $\quad = \quad e^{\lambda} \quad ^{2}$ leads to a linear equation of the form 2.1.2.6: $'' + \lambda^{-2}(a\xi^{2} + \quad \xi + \quad) \quad = 0.$

7. $+ [\quad ^{2\lambda} (\quad ^{\lambda} + \quad) - \frac{1}{4} \quad ^{2}] \quad = 0.$

The transformation $\xi = \quad e^{\lambda} \quad + \quad ,$ $\quad = \quad e^{\lambda} \quad ^{2}$ leads to an equation of the form 2.1.2.7: $'' + a(\quad \lambda)^{-2}\xi \quad = 0.$

8. $+ \quad + \quad ^{2} \quad = 0.$

The transformation $\xi = e$, $\quad = \quad e$ leads to a constant coefficient linear equation of the form 2.1.2.1: $'' + \quad a^{-2} \quad = 0.$

9. $- \quad + \quad ^{2} \quad = 0.$

The substitution $\xi = e$ leads to a constant coefficient linear equation of the form 2.1.2.1: $'' + \quad a^{-2} \quad = 0.$

10. $+ \quad + (\quad ^{\lambda} + \quad) \quad = 0.$

Solution: $\quad = e^{-} \quad ^{2} [\quad _{1} \quad (2\lambda^{-1} \quad ^{-} e^{\lambda} \quad ^{2} + \quad _{2} \quad (2\lambda^{-1} \quad ^{-} e^{\lambda} \quad ^{2}]$, where $\quad = \lambda^{-1} \quad \overline{a^{2} - 4}$; (z) and (z) are the Bessel functions.

11. $- \quad + (\quad ^{3\lambda} + \quad ^{2\lambda} + \frac{1}{4} - \frac{1}{4} \quad ^{2} \quad = 0.$

The substitution $z = e$ leads to a second-order linear equation of the form 2.1.2.121: $z^{2} \quad '' + (az^{3\lambda} + \quad z^{2\lambda} + \frac{1}{4} - \frac{1}{4}\lambda^{2} \quad = 0.$

12. $- \quad + [\quad ^{2\lambda} (\quad ^{\lambda} + \quad) + \frac{1}{4} - \frac{1}{4} \quad ^{2}] \quad = 0.$

The substitution $z = e$ leads to a second-order linear equation of the form 2.1.2.122: $z^{2} \quad '' + [az^{2\lambda}(\quad z^{\lambda} + \quad) + \frac{1}{4} - \frac{1}{4}\lambda^{2}] \quad = 0.$

13. $+ 2 \quad ^{\lambda} \quad + \quad ^{\lambda} (\quad ^{\lambda} + \quad) \quad = 0.$

Solution: $\quad = \exp -\frac{a}{\lambda}e^{\lambda} \quad (\quad _{1} + \quad _{2} \quad).$

14. $+ (\quad + \quad) \quad ^{\lambda} \quad + \quad ^{\lambda} (\quad ^{\lambda} + \quad) \quad = 0.$

Particular solution: $_{0} = \exp -\frac{a}{\lambda}e^{\lambda} \quad .$

15. $+ \quad ^{\lambda} \quad - \quad (\quad ^{\lambda} + \quad + \quad) \quad = 0.$

Particular solution: $_{0} = \exp \quad -e \quad .$

16. $+ 2 \quad + (\quad ^{2\lambda} + \quad ^{\lambda} + \quad ^{2} \quad ^{2} + \quad + \quad) \quad = 0.$

The substitution $\quad = \exp \frac{k}{\quad}e$ leads to a linear equation of the form 2.1.3.5: $'' + (ae^{2\lambda} + \quad e^{\lambda} + \quad) \quad = 0.$

17. $- (\quad + 2 \quad) \quad + \quad ^{2} \quad ^{2} \quad = 0.$

Particular solution: $_{0} = \exp \frac{\quad}{a}e \quad .$

18. **+ ($^{2\lambda}$ +) − $^{2\lambda}$ = 0.**

Particular solution: $_0 = ae^{\lambda} + \lambda e^{-\lambda}$.

19. **+ ($^{\lambda}$ −) + $^{2\lambda}$ = 0.**

The substitution $\xi = e^{\lambda}$ leads to a constant coefficient linear equation: $\lambda^2 \;'' + a\lambda \;' + \; = 0.$

20. **+ ($^{\lambda}$ +) + ($^{\lambda}$ + −) = 0.**

Particular solution: $_0 = e^{-c}$.

21. **+ (+ $^{2\lambda}$) + (− − $^{2\lambda}$) = 0.**

Particular solution: $_0 = \; e^{\lambda} + ae^{-\lambda}$.

22. **+ ($^{\lambda}$ + − 3) + ^{2} (−) $^{2\lambda}$ = 0.**

Particular solution: $_0 = (ae^{\lambda} + 1)\exp(-ae^{\lambda}).$

23. **+ (2 $^{\lambda}$ −) + (^{2} $^{2\lambda}$ +) = 0.**

This is a special case of equation 2.1.3.28 with $\; = k = 0.$

24. **+ (2 $^{\lambda}$ +) + [^{2} $^{2\lambda}$ + (+) $^{\lambda}$ +] = 0.**

The substitution $\; = \exp \dfrac{a}{\lambda} e^{\lambda}$ leads to a constant coefficient linear equation of the form 2.1.2.11: $\;'' + \;\;' + \; = 0.$

25. **+ ($^{\lambda}$ + 2 −) + ($^{2\lambda}$ + $^{\lambda}$ + ^{2} −) = 0.**

The transformation $\xi = e^{\lambda} \;\lambda, \quad = e^{b}$ leads to a constant coefficient linear equation: $\;'' + a \;' + \; = 0.$

26. **+ (+) + [(−) ^{2} + (+ + − 2) + (−)] = 0.**

Particular solution: $_0 = \exp(- \; e \; - k \;).$

27. **+ ($^{\lambda}$ +) + ($^{2\lambda}$ + $^{\lambda}$ + γ) = 0.**

The substitution $\xi = e$ leads to an equation of the form 2.1.2.146: $\xi^2 \;'' + (a\xi^{\lambda} + \; + 1)\xi \;' + (\; \xi^{2\lambda} + \beta\xi^{\lambda} + \;) \; = 0.$

28. **+ (2 $^{\lambda}$ −) + (^{2} $^{2\lambda}$ + ^{2} + +) = 0.**

The substitution $\; = \exp \dfrac{a}{\lambda} e^{\lambda} - \dfrac{\lambda}{2}$ leads to a linear equation of the form 2.1.3.5: $\;'' + (\; e^{2} \; + \; e \; + k - \tfrac{1}{4}\lambda^2 \;\; = 0.$

29. **+ (2 $^{\lambda}$ + −) + (^{2} $^{2\lambda}$ + $^{\lambda}$ + ^{2} + +) = 0.**

The substitution $\; = \exp \dfrac{a}{\lambda} e^{\lambda} + \dfrac{-\lambda}{2}$ leads to an equation of the form 2.1.3.5: $\;'' + \big[\; e^{2} \; + \; e \; + k - \tfrac{1}{4}(\; - \lambda)^2\big] \; = 0.$

30. **+ ($^{\lambda}$ +) + $^{\lambda}$ (+) = 0.**

Particular solution: $_0 = \exp\; -\dfrac{a}{\lambda} e^{\lambda}$.

31. **+ $^{\lambda}$ (^{2} +) + [$^{\lambda}$ (− ^{2}) −] = 0.**

Particular solution: $_0 = ae \; + \; e^{-}$.

32. + ($^{\lambda}$ + +) + ($^{\lambda}$ +) = 0.

Particular solution: $_0 = \exp\ -\frac{a}{\lambda}e^{\lambda}\ - - e\ \ -$.

33. + ($^{\lambda}$ + +) + [$^{(\lambda+\ )}$ + $^{\lambda}$ +] = 0.

Particular solution: $_0 = \exp\ - - e\ \ -$.

34. + ($^{\lambda}$ + 2 −) + [$^{(\lambda+\ )}$ + $^{2\lambda}$ + $^{2\ 2}$ + (−)] = 0.

1 . If $\lambda = 0$, the equation transforms into 2.1.3.24, and if = 0, into 2.1.3.25.

2 . For $\lambda \neq 0$, the transformation $\xi = \frac{1}{\lambda}e^{\lambda}$, = exp $-e$ leads to a constant coefficient linear equation: $'' + a\ ' + \ = 0$.

35. + [$^{(\lambda+\ )}$ + $^{\lambda}$ + − 2] + ^{2} $^{(2\lambda+\ )}$ = 0.

Particular solution: $_0 = (ae^{\lambda} + 1)\exp(-ae^{\lambda})$.

36. + exp( ) + [exp( ) −] = 0.

Particular solution: $_0 = e^{-c}$.

37. ($^{\lambda}$ +) − $^{2\ \lambda}$ = 0.

Particular solution: $_0 = ae^{\lambda} +$.

38. ($^{2\ 2\lambda}$ +) − − $^{2\ 2\ 2\ 2\lambda}$ = 0.

Solution: $= {}_1\big(ae^{\lambda} + \overline{a^2e^{2\lambda} +}\ \big) + {}_2\big(ae^{\lambda} + \overline{a^2e^{2\lambda} +}\ \big)^{-}$.

39. 2($^{\lambda}$ +) + $^{\lambda}$ + = 0.

The substitution $\xi = \ (ae^{\lambda} + \ )^{-1\ 2}$ leads to a constant coefficient linear equation of the form 2.1.2.1: $2\ '' + \ = 0$.

40. ($^{\lambda}$ +) + ($^{\lambda}$ +) + [(−) $^{\lambda}$ + −] = 0.

Particular solution: $_0 = e^{-}$.

41. ($^{\lambda}$ +) + ($^{\lambda}$ +) + ($^{\lambda}$ +) = 0.

For the case $a = 0$, see equation 2.1.3.27. For $a \neq 0$, the transformation $\xi = ae^{\lambda}$, $= \xi^{-}$, where k is a root of the quadratic equation $\lambda^2k^2 + \lambda k + \ = 0$, leads to an equation of the form 2.1.2.172: $a\lambda^2\xi(\xi+\ )\ '' + \lambda[(2ak\lambda + a\lambda +\ )\xi + a(2\ k\lambda + \lambda +\ )]\ ' + (ak^2\lambda^2 + \ k\lambda +\ )\ = 0$.

42. (+) + ($^{\lambda}$ + +) + ($^{\lambda}$ + −) = 0.

Integrating yields a first-order linear equation: $(e\ + k)\ ' + (ae^{\lambda} + e\ - e\ + \ ) = \ $.

43. ($^{\lambda}$ +)^2 + $^{\lambda}$ (− $^{\lambda}$) = 0.

Particular solution: $_0 = (ae^{\lambda} + \ )$, where $k = -\frac{}{a\lambda}$.

44. ($^{\lambda}$ +)^2 + ($^{\lambda}$ +) + $^{\lambda}$ (+ − $^{\lambda}$) = 0.

Particular solution: $_0 = (ae^{\lambda} + \ )$, where $k = -\frac{}{a\lambda}$.

45. $(\ \ ^{\lambda}\ +\ )^2\ \ \ +(\ \ ^{\lambda}\ +\ )(\ \ ^{\lambda}\ +\ )\ \ +\ \ \ = 0.$

The substitution $\xi = \dfrac{}{ae^{\lambda}\ +}$ leads to a constant coefficient linear equation of the form 2.1.2.11: $''+\ \ '+\ \ =0.$

46. $(\ \ ^{\lambda}\ +\ )^2\ \ +\ \ (\ \ ^{\lambda}\ +\ )\ \ +\ ^{\lambda}(\ \ -\ \ ^{\lambda}\ +\ \ )\ = 0.$

Particular solution: $_0 = (ae^{\lambda}\ +\ )$, where $k = -\dfrac{}{a\lambda}$.

47. $4(\ \ ^{\lambda}\ +\ )\ \ \ +[\ \ ^{2\lambda}(\ \ ^{\lambda}\ +\ )^{-4} -\ ^{2}(\ \ ^{\lambda}\ +\ )\ ]\ = 0.$

The transformation $\xi = \dfrac{ae^{\lambda}\ +}{e^{\lambda}\ +}$, $\ = \dfrac{e^{\lambda}\ \ ^{2}}{e^{\lambda}\ +}$ leads to an equation of the form 2.1.2.7: $4\ ''+k(\Delta\lambda)^{-2}\xi^{-}\ \ =0$, where $\Delta = a\ \ -\ \ .$

2.1.3-2. Equations with power and exponential functions.

48. $\ +\ \ ^{\lambda}\ \ +(\ \ ^{\lambda}\ -\ \ ^{2}\ +\ \ ^{-1})\ = 0.$

Particular solution: $_0 = \exp\left(-\dfrac{}{+1}\ \ ^{+1}\right).$

49. $\ +2\ \ ^{\lambda}\ \ +(\ ^{2}\ ^{2\lambda}\ +\ \ ^{\lambda}\ +\ \ ^{2}\ +\ \ ^{-1})\ = 0.$

The substitution $\ = \exp\left(\dfrac{a}{\lambda}e^{\lambda}\right)$ leads to a linear equation of the form 2.1.2.10: $''+(\ ^{2}\ +\ \ ^{-1})\ =0.$

50. $\ +(\ \ +\ )\ ^{\lambda}\ \ -\ \ ^{\lambda}\ \ = 0.$

Particular solution: $_0 = a\ \ +\ .$

51. $\ +(\ \ ^{\lambda}\ +2\ )\ \ +(\ \ ^{\lambda}\ -\ \ ^{\lambda}\ +\ ^{2})\ = 0.$

Particular solution: $_0 =\ \ e^{-b}\ .$

52. $\ +\ (\ \ ^{\lambda}\ +\ \ \ )\ \ -(\ \ ^{\lambda}\ +\ \ \ )\ \ = 0.$

Particular solution: $_0 =\ .$

53. $\ +(\ \ +\ \ ^{\lambda})\ \ +(\ \ ^{\lambda}\ +\ \ ^{-1})\ = 0.$

Particular solution: $_0 = \exp\left(-\dfrac{a}{+1}\ \ ^{+1}\right).$

54. $\ +(\ \ +\ )\exp(\ \ )\ \ -\ \exp(\ \ )\ = 0.$

Particular solution: $_0 = a\ \ +\ .$

55. $\ +\ \ \exp(\ \ )\ \ -\ \ ^{-1}\exp(\ \ )\ = 0.$

Particular solution: $_0 =\ .$

56. $\ -(2\ \ ^{2}+1)\ \ +4\ \ ^{3}\exp(2\ \ ^{2})\ = 0.$

Solution: $\ =\exp\left(\tfrac{1}{2}a\ \ ^{2}\right)\left[\ _1\ \ _{\frac{}{2\lambda}}(z)+\ _2\ \ _{\frac{}{2\lambda}}(z)\right]$, where $z=\lambda^{-1}\sqrt{\ }\exp(\lambda\ \ ^{2})$; $\ (z)$ and $\ (z)$ are the Bessel functions.

57. $\ +\ \ ^{\lambda}\ \ +\ \ ^{\lambda}(1+\ \ )\ = 0.$

Particular solution: $_0 =\ \exp\left(-\dfrac{a}{\lambda}e^{\lambda}\ \right).$

58. $+ \quad \lambda \quad -[\ (\quad +1)\ \lambda\ +\ (\quad +2)] \quad = 0.$

Particular solution: $_0 = \quad e^{b}\ .$

59. $+(\quad \lambda\ +\ )\quad +\ (\ -1)\ \lambda \quad = 0.$

Particular solution: $_0 = \quad ^{1-b}.$

60. $+[\ (\quad +1)\ \lambda\ +\quad -1]\quad +\ ^{2}\ \lambda \quad = 0.$

Particular solution: $_0 = (\quad +1)e^{-b}\ .$

61. $+[(\ ^{2}+\quad)\ \lambda\ +2]\quad +\ \lambda \quad = 0.$

Particular solution: $_0 = a + \quad .$

62. $+(\quad +\ \lambda\ )\quad +\quad ^{-1}(\ \lambda\ +\ -1)\quad = 0.$

Particular solution: $_0 = \exp(-a \quad).$

63. $+(\ \lambda\ +\quad)\quad +[\ (\quad -1)\ \lambda\ +\quad ^{-1}]\quad = 0.$

Particular solution: $_0 = \quad \exp(- \quad).$

64. $+[(\quad +1)\ \lambda\ +\quad +1-2\ ]\quad +\ ^{2}\quad ^{2\ -1}\ \lambda \quad = 0.$

Particular solution: $_0 = (a \quad +1)\exp(-a \quad).$

65. $+(\ \lambda\ +\quad)\quad +(\ \lambda\ +\quad)\quad = 0.$

Integrating, we obtain a first-order linear equation: $\ ' + (ae^{\lambda}\ +\ e \quad -1) \ = \ .$

66. $+[\quad \exp(\quad)+\ ]\quad +\ (\ -1)\quad ^{-1}\exp(\quad)\quad = 0.$

Particular solution: $_0 = \quad ^{1-c}.$

67. $(\ +\ )\quad +(\ \lambda\ +\ )\quad +\ \lambda \quad = 0.$

Particular solution: $_0 = \exp \dfrac{1 - \quad - \ e^{\lambda}}{+a}\quad .$

68. $4\ ^{2}\quad +[\ ^{2}\ \exp(\quad)+1-\ ^{2}]\quad = 0.$

The transformation $\xi = \quad ,\quad = \quad \frac{-1}{2}$ leads to a linear equation of the form 2.1.3.1: $4\ '' + a(\ )^{-2}e \quad = 0.$

69. $^{2}\quad +2\quad +[(\ ^{2\ 2}\ -\ ^{2})\ ^{2\ 2}+\ (\ -1)]\quad = 0.$

Solution: $\ = \ ^{-}\big[\ _1\ (\ e^{c}\ ) + \ _2\ (\ e^{c}\ )\big]$, where (z) and (z) are the Bessel functions.

70. $^{2}\quad +\ \lambda\quad +\ (\ \lambda\ -\ -1)\quad = 0.$

Particular solution: $_0 = \quad ^{-b}.$

71. $^{2}\quad +\ (\ \lambda\ +2\ )\quad +[\ (\quad +\ )\ \lambda\ -\ ^{2\ 2}+\ (\ -1)]\quad = 0.$

Particular solution: $_0 = \quad ^{-b}e^{-c}\ .$

72. $^{4}\quad +(\ ^{2}\ -\ ^{2})\quad = 0.$

Solution: $\ = \big[\ _1\ (e^{1}\ ) + \ _2\ (e^{1}\ )\big]$, where (z) and (z) are the Bessel functions.

73. $^{4}\quad +[\ \exp(2\quad)+\ \exp(\quad)+\ ]\quad = 0.$

The transformation $\xi = 1 \quad ,\quad = \quad$ leads to a linear equation of the form 2.1.3.5: $\ '' + (ae^{2\lambda}\ +\ e^{\lambda}\ +\ )\ = 0.$

74. $\quad {}^{4} \quad + \quad {}^{2}\,\lambda \quad + [\ (\ -\ )\ \lambda\ -\ {}^{2}] \quad = 0.$

Particular solution: $\ {}_{0} = \quad \exp(\quad).$

75. $(\ {}^{2} +\ )^{2} \quad + \quad \lambda\ (\ {}^{2} +\ ) \quad - (\quad \lambda \quad +\ ) \quad = 0.$

Particular solution: $\ {}_{0} = \sqrt{\ {}^{2} + a}.$

76. $(\quad +\ )^{2} \quad + \ (\quad +\ )\ \lambda \quad - \quad {}^{-2}(\quad \lambda \quad + \quad -\ ) \quad = 0.$

Particular solution: $\ {}_{0} = (\quad + a)^{1} \quad .$

77. $(\quad +\ )^{2} \quad + \ (\quad +\ )\ \lambda \quad + (\ \lambda\ - \quad {}^{-1} - 1) \quad = 0.$

Particular solution: $\ {}_{0} = \exp\ -\ \dfrac{\quad}{a \quad +} \quad .$

78. $(\ \lambda\ + \quad +\ ) \quad - \quad {}^{2}\,\lambda \quad = 0.$

Particular solution: $\ {}_{0} = ae^{\lambda} \ + \quad + \ .$

79. $[(\quad +\ )\ \lambda\ +\ ] \quad - \quad {}^{2} \quad = 0.$

Particular solution: $\ {}_{0} = \ e^{-\lambda} \ + a \ + \ .$

2.1.4. Equations Containing Hyperbolic Functions

$\boxed{\text{2.1.4-1. Equations with hyperbolic sine.}}$

1. $\quad + (\ \sinh^{2} \quad +\ ) \quad = 0.$

Applying the formula $\sinh^{2} \quad = \frac{1}{2}\cosh 2 \quad - \frac{1}{2}$, we obtain the modified Mathieu equation 2.1.4.9: $\ '' \ + \big(\ - \frac{1}{2}a + \frac{1}{2}a\cosh 2 \quad = 0.$

2. $\quad + \ \sinh(\quad) \quad + [\ \sinh(\quad) -\ ] \quad = 0.$

Particular solution: $\ {}_{0} = e^{-b} \ .$

3. $\quad + [\ \sinh\ (\quad) +\ ] \quad + \quad \sinh\ (\quad) \quad = 0.$

Particular solution: $\ {}_{0} = e^{-b} \ .$

4. $\quad + (\quad +\ ) \sinh\ (\quad) \quad - \quad \sinh\ (\quad) \quad = 0.$

Particular solution: $\ {}_{0} = a \ + \ .$

5. $\quad + (\ \sinh \quad + \quad {}^{+1}) \quad + \quad (\ \sinh \quad + \ ) \quad = 0.$

Particular solution: $\ {}_{0} = \exp\ -\dfrac{\quad}{\ + 1} \quad {}^{+1} \quad .$

6. $\sinh^{2}(\quad) \quad - \quad = 0.$

The substitution $a \ = \ \ln \dfrac{z}{\sqrt{z^{2}+1}}$ $(z > 0)$ leads to a linear equation of the form 2.1.2.190: $z(z^{2}+1)\ '' + (3z^{2}+1)\ ' - 4a^{-2}\ z \ = 0.$

7. $\quad \sinh^{2} \quad - [\ {}^{2}\sinh^{2} \quad + \ (\quad - 1)] \quad = 0, \qquad \neq 0; \qquad = 1,\ 2,\ 3,\ \ldots$

Solution: $\ = \sinh \quad \dfrac{1}{\sinh} \text{—} \quad (\ {}_{1}e \quad + \ {}_{2}e^{-} \ ).$

8. $[\ \sinh(\quad) + \quad +\ ] \quad - \quad {}^{2}\sinh(\quad) \quad = 0.$

Particular solution: $\ {}_{0} = a\sinh(\lambda\ ) + \quad + \ .$

2.1.4-2. Equations with hyperbolic cosine.

9. $- (\quad - 2q \cosh 2 \quad) \quad = 0.$

The modified Mathieu equation. The substitution $= \xi$ leads to the Mathieu equation 2.1.6.29:

$$'' + (a - 2 \quad \cos 2\xi) \quad = 0.$$

For eigenvalues $a = a \ (\ )$ and $a = \ (\ )$, the corresponding solutions of the modified Mathieu equation are:

$$\mathrm{Ce}_{2\ +}(\ ,\ ) = \mathrm{ce}_{2\ +}(\quad ,\ ) = \sum_{=0} A^{2\ +}_{2\ +} \cosh[(2k + \ )\ ],$$

$$\mathrm{Se}_{2\ +}(\ ,\ ) = -\ \mathrm{se}_{2\ +}(\quad ,\ ) = \sum_{=0} B^{2\ +}_{2\ +} \sinh[(2k + \ )\ ],$$

where can be either 0 or 1, and the coefficients $A^{2\ +}_{2\ +}$ and $B^{2\ +}_{2\ +}$ are specified in 2.1.6.29.

The modified Mathieu equation is discussed in the books by Abramowitz & Stegun (1964), Bateman & Erdélyi (1955, vol. 3), and McLachlan (1947) in more detail.

10. $+ (\quad \cosh^2 \quad + \ ) \quad = 0.$

Applying the formula $\cosh 2 \quad = 2\cosh^2 \quad - 1$, we obtain the modified Mathieu equation 2.1.4.9: $'' + \left(\frac{1}{2}a + \quad + \frac{1}{2}a \cosh 2 \quad \right) = 0.$

11. $+ \quad \cosh(\quad) \quad + [\quad \cosh(\quad) - \quad] \quad = 0.$

Particular solution: $_0 = e^{-b}$.

12. $+ [\quad \cosh\ (\quad) + \ ] \quad + \quad \cos\ (\quad) \quad = 0.$

Particular solution: $_0 = e^{-b}$.

13. $+ (\quad + \ ) \cosh\ (\quad) \quad - \quad \cosh\ (\quad) \quad = 0.$

Particular solution: $_0 = a \quad + \ .$

14. $+ \quad \cosh\ (\quad) \quad - \quad ^{-1} \cosh\ (\quad) \quad = 0.$

Particular solution: $_0 = \ .$

15. $+ \quad \cosh\ (\quad) \quad - [\ (\quad + 1) \cosh\ (\quad) + \ (\quad + 2)] \quad = 0.$

Particular solution: $_0 = \quad e^{b}$.

16. $+ [\quad \cosh\ (\quad) + \ ] \quad + \ (\ - 1) \cosh\ (\quad) \quad = 0.$

Particular solution: $_0 = \quad ^{1-b}.$

17. $+ [(\quad ^2 + \quad) \cosh\ (\quad) + 2] \quad + \quad \cosh\ (\quad) \quad = 0.$

Particular solution: $_0 = a + \quad .$

18. $^2 \quad + \quad \cosh\ (\quad) \quad + [\ \cosh\ (\quad) - \quad - 1] \quad = 0.$

Particular solution: $_0 = \quad ^{-b}.$

19. $(\ \cosh \quad + \ ) \quad + (\ \cosh \quad + \ ) \quad + [(\ - \quad) \cosh \quad + \quad - \quad] \quad = 0.$

Particular solution: $_0 = e^{-\lambda}$.

20. $\mathbf{\cosh^2(\quad) \quad - \quad = 0.}$

The substitution $= \dfrac{1}{2a} \ln \dfrac{z}{1-z}$ $(0 < z < 1)$ leads to the hypergeometric equation 2.1.2.171: $z(z-1) \; '' + (2z-1) \; ' + a^{-2} \quad = 0$.

21. $\mathbf{[\quad \cosh(\quad) + \quad + \;] \quad - \quad {}^2 \cosh(\quad) \quad = 0.}$

Particular solution: $_0 = a \cosh(\lambda \;\,) + \quad + \;$.

2.1.4-3. Equations with hyperbolic tangent.

22. $\mathbf{+ [\quad \tanh(\quad) + \;] \quad = 0.}$

The transformation $z = \dfrac{1 - \tanh(\lambda \;\,)}{1 + \tanh(\lambda \;\,)}$, $\quad = z^{- \quad \lambda}$, where k is a root of the quadratic equation $4k^2 + \quad - a = 0$, leads to a linear equation of the form 2.1.2.172: $4\lambda^2 z(z+1) \; '' + 4\lambda(2k + \lambda)(z+1) \; ' + (4k^2 + a + \;) \quad = 0$.

23. $\mathbf{- 4 \; {}^2 \tanh^2(3 \quad) \quad = 0.}$

Particular solution: $_0 = \sinh(3a \;\,)[\cosh(3a \;\,)]^{-1 \;\; 3}$.

24. $\mathbf{+ [\quad - \; (\; + \;) \tanh^2(\quad)] \quad = 0.}$

Particular solution: $_0 = [\cosh(\lambda \;\,)]^{- \quad \lambda}$.

25. $\mathbf{+ [3 \quad - \; {}^2 - \; (\; + \;) \tanh^2(\quad)] \quad = 0.}$

Particular solution: $_0 = \sinh(\lambda \;\,)[\cosh(\lambda \;\,)]^{- \quad \lambda}$.

26. $\mathbf{+ \quad - \; [\; + \quad \tanh(\quad)] \quad = 0.}$

Particular solution: $_0 = \cosh(\lambda \;\,)$.

27. $\mathbf{+ 2 \tanh \quad + \quad = 0.}$

Solution: $\cosh \quad = \begin{cases} {}_1 \cos(\quad) + \; {}_2 \sin(\quad) & \text{if } a - 1 = \; {}^2 > 0, \\ {}_1 \cosh(\quad) + \; {}_2 \sinh(\quad) & \text{if } a - 1 = - \; {}^2 < 0. \end{cases}$

28. $\mathbf{+ \quad \tanh(\quad) \quad + \; [\quad \tanh(\quad) - \;] \quad = 0.}$

Particular solution: $_0 = e^{-b} \;$.

29. $\mathbf{+ 2 \tanh \quad + (\quad {}^2 + \quad + \;) \quad = 0.}$

The substitution $\quad = \cosh$ leads to a second-order linear equation of the form 2.1.2.6: $'' + (a \;\, {}^2 + \quad + \; - 1) \quad = 0$.

30. $\mathbf{+ 2 \tanh \quad + (\quad + 1) \quad = 0.}$

The substitution $\quad = \cosh$ leads to a linear equation of the form 2.1.2.7: $'' + a \qquad = 0$.

31. $\mathbf{+ 2 \tanh \quad + (\quad {}^2 \; + \quad {}^{-1} + 1) \quad = 0.}$

The substitution $\quad = \cosh$ leads to a second-order linear equation of the form 2.1.2.10: $'' + (a \;\, {}^2 \; + \quad {}^{-1}) \quad = 0$.

32. $\mathbf{+ (2 \tanh \quad + \;) \quad + (\quad \tanh \quad + \;) \quad = 0.}$

The substitution $\quad = \cosh$ leads to a constant coefficient linear equation: $'' + a \; ' + (\; - 1) \quad = 0$.

33. $+\ \tanh(\ )\ -\ [\ +\ \tanh^{+1}(\ )]\ =0.$

Particular solution: $_0=\cosh(\lambda\ )$.

34. $+[\ \tanh(\ )+\ ]\ +\ \tan(\ )\ =0.$

Particular solution: $_0=e^{-b}$.

35. $+(\ +\ )\tanh(\ )\ -\ \tanh(\ )\ =0.$

Particular solution: $_0=a\ +\ $.

36. $+\ \tanh(\ )\ -\ ^{-1}\tanh(\ )\ =0.$

Particular solution: $_0=\ $.

37. $+\ \tanh(\ )\ -[\ (\ +1)\tanh(\ )+\ (\ +2)]\ =0.$

Particular solution: $_0=\ e^{b}$.

38. $+[\ \tanh(\ )+\ ]\ +\ (\ -1)\tanh(\ )\ =0.$

Particular solution: $_0=\ ^{1-b}$.

39. $+[(\ ^2+\ )\tanh(\ )+2]\ +\ \tanh(\ )\ =0.$

Particular solution: $_0=a+\ $.

40. $+(\ \tanh\ +\ ^{+1})\ +\ (\ \tanh\ +\ )\ =0.$

Particular solution: $_0=\exp\ -\dfrac{}{+1}\ ^{+1}$.

41. $^2\ +\ \tanh(\ )\ +\ [\ \tanh(\ )-\ -1]\ =0.$

Particular solution: $_0=\ ^{-b}$.

42. $(\ \tanh\ +\ )\ +(\ \tanh\ +\ )\ +\ [(\ -\ )\tanh\ +\ -\ ]\ =0.$

Particular solution: $_0=e^{-\lambda}$.

43. $[\ \tanh(\ )+\ ]\ +[\ \tanh(\ )+\ ]\ +[\ \tanh(\ )+\ ]\ =0.$

The transformation $z=\dfrac{1+\tanh(\lambda\ )}{1-\tanh(\lambda\ )}$, $=z^{-\ \lambda}$, where k is a root of the quadratic equation $4(a-\ )k^2+2(\ -\ )k+\ -\ =0$, leads to a linear equation of the form 2.1.2.172:

$$4\lambda^2 z[(a+\ )z+\ -a]\ ''+2\lambda\{[2(2k+\lambda)(a+\ )+\ +\ ]z+2(2k+\lambda)(\ -a)+\ -\ \}\ '$$
$$+[4(a+\ )k^2+2(\ +\ )k+\ +\ ]\ =0.$$

$\boxed{\text{2.1.4-4. Equations with hyperbolic cotangent.}}$

44. $+[\ \coth(\ )+\ ]\ =0.$

The transformation $z=\dfrac{1-\tanh(\lambda\ )}{1+\tanh(\lambda\ )}$, $=z^{-\ \lambda}$, where k is a root of the quadratic equation $4k^2+\ -a=0$, leads to an equation of the form 2.1.2.172: $4\lambda^2 z(z-1)\ ''+4\lambda(2k+\lambda)(z-1)\ '+(4k^2+a+\ )\ =0$.

45. $-4\ ^2\coth^2(3\ )\ =0.$

Particular solution: $_0=\cosh(3a\ )[\sinh(3a\ )]^{-1\ 3}$.

46. $+[\quad - \; (\quad + \quad)\coth^2(\quad)] \quad = 0.$

Particular solution: $_0 = [\sinh(\lambda \quad)]^{-\quad\lambda}.$

47. $+[3 \quad - \quad ^2 - \quad (\quad + \quad)\coth^2(\quad)] \quad = 0.$

Particular solution: $_0 = \cosh(\lambda \quad)[\sinh(\lambda \quad)]^{-\quad\lambda}.$

48. $+ \quad \coth(\quad) \quad + \; [\quad \coth(\quad) - \quad] \quad = 0.$

Particular solution: $_0 = e^{-b} \quad .$

49. $+[\quad \coth \; (\quad) + \quad] \quad + \quad \coth \; (\quad) \quad = 0.$

Particular solution: $_0 = e^{-b} \quad .$

50. $+(\quad + \quad)\coth \; (\quad) \quad - \quad \coth \; (\quad) \quad = 0.$

Particular solution: $_0 = a \quad + \quad .$

51. $+2 \quad \coth \quad + (\quad ^2 - \quad ^2) \quad = 0, \qquad = 1, \; 2, \; 3, \; \ldots$

Solution: $= \dfrac{1}{\sinh} \text{——} \quad (\;_1 e \quad + \quad _2 e^{-} \quad).$

52. $[\quad + \quad \coth(\quad)] \quad + [\quad + \quad \coth(\quad)] \quad + [\quad + \quad \coth(\quad)] \quad = 0.$

Multiply this equation by $\tanh(\lambda \quad)$ to obtain equation 2.1.4.43.

2.1.4-5. Equations containing combinations of hyperbolic functions.

53. $- \quad [\quad \cosh^2(\quad) + \quad \sinh(\quad)] \quad = 0.$

Particular solution: $_0 = \exp \; \dfrac{a}{} \sinh(\quad) \; .$

54. $- \quad [\quad \sinh^2(\quad) + \quad \cosh(\quad)] \quad = 0.$

Particular solution: $_0 = \exp \; \dfrac{a}{} \cosh(\quad) \; .$

55. $+(\quad \cosh^2 \quad + \quad \sinh^2 \quad + \quad) \quad = 0.$

Apply the formulas $2\sinh^2 \quad = \cosh(2 \quad) - 1$ and $2\cosh^2 \quad = \cosh(2 \quad) + 1$ to obtain an equation of the form 2.1.4.9: $'' \; + \; \dfrac{a -}{2} + \quad + \dfrac{a +}{2} \cosh(2 \quad) \quad = 0.$

56. $+ \quad \sinh(\quad) \quad - \; [\quad + \cosh(\quad)] \quad = 0.$

Particular solution: $_0 = \sinh(\lambda \quad).$

57. $+ \quad \cosh(\quad) \quad - \; [\quad + \sinh(\quad)] \quad = 0.$

Particular solution: $_0 = \cosh(\lambda \quad).$

58. $- \quad \tanh(\quad) \quad - \quad ^2 \cosh^2(\quad) \quad = 0.$

Solution: $= \; _1 \exp \; \dfrac{a}{\lambda} \sinh(\lambda \quad) \; + \; _2 \exp \; -\dfrac{a}{\lambda} \sinh(\lambda \quad) \; .$

59. $- \tanh \quad + \quad ^2 \coth^2 \quad (\sinh \quad)^2 \quad ^{-2} \quad = 0.$

Solution: $= \overline{\sinh} \quad _1 \; _{\frac{1}{2}} \; \dfrac{a}{} \sinh \quad + \; _2 \; _{\frac{1}{2}} \; \dfrac{a}{} \sinh$, where (z) and (z) are the Bessel functions.

60. $\sinh(\quad)\qquad + [\ \cosh^{-4}(\quad) - {}^2\sinh(\quad)]\quad = 0.$

The transformation $\xi = \tanh(\lambda\ \ )$, $\ \ = \dfrac{}{\cosh(\lambda\ \ )}$ leads to an equation of the form 2.1.2.7: $''\ + a\lambda^{-2}\xi^{-}\qquad = 0.$

61. $\cosh(\quad)\qquad + [\ \sinh^{-4}(\quad) - {}^2\cosh(\quad)]\quad = 0.$

The transformation $\xi = \coth(\lambda\ \ )$, $\ \ = \dfrac{}{\sinh(\lambda\ \ )}$ leads to an equation of the form 2.1.2.7: $''\ + a\lambda^{-2}\xi^{-}\qquad = 0.$

2.1.5. Equations Containing Logarithmic Functions

2.1.5-1. Equations of the form $(\ )\ ''\ + g(\ )\ = 0.$

1. $\qquad - (\ ^2\ ^2\ln^2\quad + \quad\ln\quad + \quad)\quad = 0.$

Particular solution: $_0 = e^{-\ \ ^2\ 4\quad ^2\ 2}.$

2. $\qquad - (\ ^2\ ^2\ \ln^2\quad + \qquad ^{-1}\ln\quad + \qquad ^{-1})\quad = 0.$

Particular solution: $_0 = e^{-F\ \ (\ \ +1)F}$, where $\ \ = \dfrac{a\quad ^{+1}}{(\ \ +1)^2}.$

3. $\qquad - (\ ^2\ \ln^2\quad + \quad)\quad = 0.$

Particular solution: $_0 = e^{-}\qquad .$

4. $\qquad - [\ ^2\ \ln^2(\quad) + \quad\ln^{-1}(\quad)]\quad = 0.$

Particular solution: $_0 = \exp\ a\quad\ln(\quad)\qquad .$

5. $\ ^2\qquad + (\ \ln\quad + \quad)\quad = 0.$

The transformation $\xi = a\ln\quad + \quad - \frac{1}{4}$, $\quad = \quad ^{-1\ \ 2}$ leads to an equation of the form 2.1.2.2: $''\ + a^{-2}\xi\quad = 0.$

6. $\ ^2\qquad + (\ \ln^2\quad + \ \ln\quad + \quad)\quad = 0.$

The transformation $\xi = \ln\ \ ,\quad = \quad ^{-1\ \ 2}$ leads to an equation of the form 2.1.2.6: $''\ + (a\xi^2 + \ \ \xi + \quad - \frac{1}{4})\quad = 0.$

7. $\ ^2\qquad + \big[\ (\ \ln\quad + \quad)\quad + \frac{1}{4}\big]\quad = 0.$

The transformation $\xi = \ \ \ln\quad + \ \ ,\quad = \quad ^{-1\ \ 2}$ leads to an equation of the form 2.1.2.7: $''\ + a\ \ ^{-2}\xi\qquad = 0.$

8. $\ ^2\ln(\quad)\qquad + \quad = 0.$

Solution: $\quad = \ _1\ln(a\ \ ) + \ _2\ln(a\ \ )\ [\ln(a\ \ )]^{-2}\qquad .$

9. $(\ \ \ln\quad + \qquad + \quad)\qquad - \qquad = 0.$

Particular solution: $_0 = a\ \ \ln\quad + \qquad + \ \ .$

10. $\ ^2(\ \ln\quad + \qquad + \quad)\qquad + \qquad = 0.$

Particular solution: $_0 = a\ln\quad + \qquad + \ \ .$

2.1.5-2. Equations of the form $(\ )''+g(\ )'+(\ )=0.$

11. $\quad + \ \ln(\ ) \ + [\ \ln(\ ) - \ ] \ = 0.$

Particular solution: $_0 = e^{-c}\ .$

12. $\quad + [\ \ln(\ ) + \ ] \ + \ \ln(\ ) \ = 0.$

Particular solution: $_0 = e^{-c}\ .$

13. $\quad + (\ + \ )\ln(\ ) \ - \ \ln(\ ) \ = 0.$

Particular solution: $_0 = a\ + \ .$

14. $\quad + \ \ln(\ ) \ - \ ^{-1}\ln(\ ) \ = 0.$

Particular solution: $_0 = \ .$

15. $\quad + \ \ln \ + \ (\ln \ + 1) \ = 0.$

Particular solution: $_0 = e^{\ 1-}\ .$

16. $\quad + (\ \ln \ + \ ) \ + (\ \ln \ + \ ) \ = 0.$

Particular solution: $_0 = e^{\ -}\ .$

17. $\quad + (2\ \ln \ + 1) \ + (\ ^2 \ln^2 \ + \ \ln \ + \ ) \ = 0.$

Solution: $= e^{\ -}\ (\ _1 + \ _2 \ln \ ).$

18. $\quad + \ln \ (\ + \ ) \ + \ (\ \ln^2 \ + 1) \ = 0.$

Particular solution: $_0 = e^{\ -}\ .$

19. $\quad + \ \ln(\ ) \ + \ \ln^{-1}(\ ) \ = 0.$

Particular solution: $_0 = \exp\ -a\ \ln(\ )\ .$

20. $\quad + \ \ln \ + (\ \ln \ + \ \ln^{-1} \ ) \ = 0.$

Particular solution: $_0 = \ \exp\ -a\ \ln\ .$

21. $\quad + (\ \ln \ + 1) \ - \ ^{-1} \ = 0.$

Particular solution: $_0 = \ln\ .$

22. $\quad + (\ \ln \ + 1) \ - \ \ln^{-1} \ = 0.$

Particular solution: $_0 = \ln\ .$

23. $\quad + (\ \ln \ + \ ) \ + \ (\ - 1)\ln \ = 0.$

Particular solution: $_0 = \ ^{1-b}.$

24. $\quad + [(\ ^2 + \ )\ln(\ ) + 2] \ + \ \ln(\ ) \ = 0.$

Particular solution: $_0 = a + \ .$

25. $\quad + (\ + \ \ln \ ) \ + \ ^{-1}(\ \ln \ + \ - 1) \ = 0.$

Particular solution: $_0 = \exp(-a\ ).$

26. $\quad + (\ + \ \ln \ ) \ + [\ (\ - 1)\ln \ + \ ^{-1}] \ = 0.$

Particular solution: $_0 = \ \exp(-a\ ).$

27. $^{2} \quad + \quad + \ \ln(\quad) = 0.$

Solution: $= \overline{\ln(\quad)} \ {}_1\,{}_{\frac{1}{2}}\Big(\frac{\bar{a}}{}\ln(\quad)\Big) + {}_2\,{}_{\frac{1}{2}}\Big(\frac{\bar{a}}{}\ln(\quad)\Big)\,, \quad = \frac{1}{2}(\ +2),$
where (z) and (z) are the Bessel functions.

28. $^{2} \quad + \quad + (\ \ln^2 \quad + \ \ln^{-1}\ ) = 0.$

The substitution $\xi = \ln$ leads to an equation of the form 2.1.2.10: $'' + (a\xi^2 + \xi^{-1}) = 0.$

29. $^{2} \quad + (2\ \ln \ + 1) \quad + (\ ^2 + \ ^2\ln^2 \ + \ ) = 0.$

The substitution $= \exp(-\frac{1}{2}a\ln^2\ )$ leads to the Bessel equation 2.1.2.126: $^2 \ '' + \ ' +$ $(\ ^2 + \ - a) = 0.$

30. $^{2} \quad + (2\ln \ + \ + 1) \quad + (\ln^2 \ + \ \ln \ + \ ) = 0.$

The transformation $\xi = \ln$, $= \exp(\frac{1}{2}\ln^2\ )$ leads to a constant coefficient linear equation: $'' + a\ ' + (\ - 1) = 0.$

31. $^{2} \quad + (2\ln \ + \ ) \quad + [\ln^2 \ + (\ - 1)\ln \ + \quad + \ ] = 0.$

The substitution $= \exp(\frac{1}{2}\ln^2\ )$ leads to a linear equation of the form 2.1.2.132: $^2 \ '' + a\ ' + (\quad + \ - 1) = 0.$

32. $^{2} \quad + \ \ln(\quad) \quad + [\ \ln(\quad) - \ - 1] = 0.$

Particular solution: $_0 = \ ^{-c}.$

33. $^{2} \quad + (\quad + \ \ln\ ) \quad + (\quad \ln \ - \ln \ + 1) = 0.$

Particular solution: $_0 = \exp\big(-\frac{1}{2}\ \ln^2\ \big).$

34. $(\ + \ ) \quad + (\ \ln \ + \ ) \quad + \quad = 0.$

Particular solution: $_0 = \exp\Big(-\dfrac{\ln \ + \ - 1}{+a}\Big).$

35. $^{4} \quad + \ ^2\ln(\quad) \quad + [\ (\ - \ )\ln(\quad) - \ ^2] = 0.$

Particular solution: $_0 = \ \exp(\quad).$

36. $(\ \ln \ + \ ) \quad + (\ \ln \ + \ ) \quad + [(\ - \quad)\ln \ + \ - \quad] = 0.$

Particular solution: $_0 = e^{-\lambda}\ .$

37. $\ln \quad - \quad - \ ^2 (\ln \ )^{2\ +1} = 0.$

Solution: $= \ _1e^{\ F} + \ _2e^{-\ F},$ where $= \ \ln \quad .$

38. $\ln(\quad) \quad - [\ \ln(\quad) + \ ] \quad - \ ^2\ ^{2\ +1}\ln^{2\ +1}(\quad) = 0.$

Solution: $= \ _1e^{bF} + \ _2e^{-bF},$ where $= \quad \ln(a\ )\quad .$

39. $\ln^2 \quad + (\quad + 1)\ln \quad + \quad = 0.$

The substitution $\xi = \dfrac{}{\ln}$ leads to a constant coefficient linear equation: $'' + a\ ' + \ = 0.$

40. $\ln(\quad) \quad + (\ ^2 - \ ^2) \quad + (\ + \ ) = 0.$

Particular solution: $_0 = \ - \ .$

2.1.6. Equations Containing Trigonometric Functions

2.1.6-1. Equations with sine.

1. $+ \;^{2} \;=\; \sin(\;\;).$

Equation of forced oscillations.

Solution: $= \begin{cases} {}_{1}\sin(a\;\;) + \;{}_{2}\cos(a\;\;) + \dfrac{}{a^2-\lambda^2}\sin(\lambda\;\;) & \text{if } a \neq \lambda, \\ {}_{1}\sin(a\;\;) + \;{}_{2}\cos(a\;\;) - \dfrac{}{2a}\;\cos(a\;\;) & \text{if } a = \lambda. \end{cases}$

2. $+ [\;\sin(\;\;) + \;]\; = 0.$

Applying the substitution $\lambda\;\; = 2\xi + \frac{1}{2}\;\;$, we obtain the Mathieu equation 2.1.6.29: $'' + (4a\lambda^{-2}\cos 2\xi + 4\;\lambda^{-2})\;\; = 0.$

3. $+ (\;\sin^2\;\; + \;)\;\; = 0.$

Applying the formula $2\sin^2\;\; = 1 - \cos 2\;\;$, we obtain the Mathieu equation 2.1.6.29: $'' + (\frac{1}{2}a + \;\; - \frac{1}{2}a\cos 2\;\;)\;\; = 0.$

4. $+ \;\sin(\;\;)\;\; + [\;\;\sin(\;\;) - \;\;^{2} + \;\;^{-1}]\;\; = 0.$

Particular solution: ${}_{0} = \exp\left(-\dfrac{}{+1}\;\;^{+1}\right).$

5. $+ \;\sin\;(\;\;)\;\; + [\;\sin\;(\;\;) - \;]\;\; = 0.$

Particular solution: ${}_{0} = e^{-c}\;\;.$

6. $+ [\;\sin\;(\;\;) + \;]\;\; + \;\;\sin\;(\;\;)\;\; = 0.$

Particular solution: ${}_{0} = e^{-c}\;\;.$

7. $+ (\;\; + \;)\sin\;(\;\;)\;\; - \;\sin\;(\;\;)\;\; = 0.$

Particular solution: ${}_{0} = a\;\; + \;.$

8. $+ \;\;\sin\;(\;\;)\;\; - \;\;^{-1}\sin\;(\;\;)\;\; = 0.$

Particular solution: ${}_{0} = \;.$

9. $+ \;\;\sin\;(\;\;)\;\; + [\;\;^{+k}\sin\;(\;\;) - \;\;^{2k} + \;\;^{k-1}]\;\; = 0.$

Particular solution: ${}_{0} = \exp\left(-\dfrac{}{k+1}\;\;^{+1}\right).$

10. $+ [(\;^{2} + \;\;)\sin\;(\;\;) + 2]\;\; + \;\sin\;(\;\;)\;\; = 0.$

Particular solution: ${}_{0} = a + \;\;.$

11. $+ (\;\;^{+1} + \;\sin\;\;)\;\; + \;\;(\;\sin\;\; + \;)\;\; = 0.$

Particular solution: ${}_{0} = \exp\left(-\dfrac{a}{+1}\;\;^{+1}\right).$

12. $+ (\;\; + \;\sin\;\;)\;\; + [\;(\;\; - 1)\sin\;\; + \;\;^{-1}]\;\; = 0.$

Particular solution: ${}_{0} = \;\exp(-a\;\;).$

13. $+ \;\;\sin\;(\;\;)\;\; - [\;(\;\; + 1)\;\;^{-1}\sin\;(\;\;) + \;^{2}\; + 2\;]\;\; = 0.$

Particular solution: ${}_{0} = \;e^{c}\;\;.$

14. $+[\quad \sin(\quad)+\;]\quad+\;(\;-1)\quad{}^{-1}\sin(\quad)\;=0.$

Particular solution: $_0 = {}^{1-c}.$

15. ${}^{2}\quad+\;(\;\sin\quad+1)\quad+\;(\;\sin\quad-\;)\;=0.$

Particular solution: $_0 = {}^{-b}.$

16. ${}^{2}\quad+\;(\;\sin\quad+\;)\quad+\;(\;\sin\quad-1)\;=0.$

Particular solution: $_0 = {}^{-b}.$

17. ${}^{2}\quad+\quad\sin(\quad)\quad+\;[\quad{}^{-1}\sin(\quad)-\;-1]\;=0.$

Particular solution: $_0 = {}^{-c}.$

18. ${}^{4}\quad+[\;\sin(\quad)+\;]\;=0.$

The transformation $\xi = 1\quad,\quad=\quad$ leads to a linear equation of the form 2.1.6.2: $'' + [a\sin(\lambda\xi) + \;]\; = 0.$

19. ${}^{4}\quad+\quad{}^{2}\sin(\quad)\quad+[\;(\;-\;)\sin(\quad)-\;{}^{2}]\;=0.$

Particular solution: $_0 = \;\exp(\quad).$

20. $\sin(2\;)\quad-\quad+2\;{}^{2}\sin^{2}\quad=0.$

Solution: $=\;_1\sin(a\;)+\;_2\cos(a\;)$, where $=\quad\overline{\tan\quad}\;.$

21. $\sin^{2}\quad+\quad=0.$

This is a special case of equation 2.1.6.23.

22. $\sin^{2}\quad-[\;\sin^{2}\quad+\;(\;-1)]\;=0,\qquad=1,\ 2,\ 3,\ \ldots$

Solution: $=\sin\quad\dfrac{1}{\sin}\;\text{—}\;\Big(\;_1e^{\;-}\;+\;_2e^{-\;-}\;.$

23. $\sin^{2}\quad+(\;\sin^{2}\quad+\;)\;=0.$

Set $\;=2\xi$. Applying the trigonometric formulas $\sin 2\xi = 2\sin\xi\cos\xi$ and $\;=\;(\sin^2\xi + \cos^2\xi)^2$ and dividing both sides of the equation by $\sin^2\;$, we arrive at an equation of the form 2.1.6.131: $'' + (\;\tan^2\xi + \;\cot^2\xi + 4a + 2\;)\; = 0.$

24. $\sin^{2}\quad-\{[(\;{}^{2}\;{}^{2}-(\;+1)^{2}]\sin^{2}\quad+\;(\;+1)\;\sin 2\quad+\;(\;-1)\}\;=0.$

Particular solution: $_0 = e^{\;b}\;\sin\quad(\cos\;+\;\sin\;).$

25. $[\;\sin(\quad)+\quad+\;]\quad+\quad{}^{2}\sin(\quad)\;=0.$

Particular solution: $_0 = a\sin(\lambda\;) + \quad+\;.$

26. $\sin(\quad)\quad+(\;{}^{2}-\;{}^{2})\quad-(\;+\;)\;=0.$

Particular solution: $_0 = \;-\;.$

27. $(\;\sin\quad+\;)\quad+(\;\sin\quad+\;)\quad+\;[(\;-\quad)\sin\quad+\;-\quad]\;=0.$

Particular solution: $_0 = e^{-\lambda}\;.$

TABLE 19

The general solution of the Mathieu equation 2.1.6.29 expressed in terms of auxiliary periodical functions $_1(\ )$ and $_2(\ )$

Constraint	General solution $=(\ )$	Period of $_1$ and $_2$	Index
$_1(\ )>1$	$_1e^{2}\ \ _1(\ )+\ _2e^{-2}\ \ _2(\ )$		is a real number
$_1(\ )<-1$	$_1e^{2}\ \ _1(\ )+\ _2e^{-2}\ \ _2(\ )$	2	$=\ +\frac{1}{2}\ ,\ \ ^2=-1$, is the real part of
$\mid\ _1(\ )\mid<1$	$(\ _1\cos\ \ +\ _2\sin\ \ )\ _1(\ )+$ $+(\ _1\cos\ \ -\ _2\sin\ \ )\ _2(\ )$		$=$ is a pure imaginary number, $\cos(2\ \ )=\ _1(\ )$
$_1(\ )=\ 1$	$_1\ \ _1(\ )+\ _2\ \ _2(\ )$		$=0$

2.1.6-2. Equations with cosine.

28. $+\ ^2\ =\ \cos(\ \ ).$

Equation of forced oscillations.

$$\text{Solution:}\quad = \begin{cases} _1\sin(a\ \ )+\ _2\cos(a\ \ )+\dfrac{}{a^2-\lambda^2}\cos(\lambda\ \ ) & \text{if } a\neq\lambda, \\ _1\sin(a\ \ )+\ _2\cos(a\ \ )+\dfrac{}{2a}\ \sin(a\ \ ) & \text{if } a=\lambda. \end{cases}$$

29. $+(\ \ -2q\cos 2\ \ )\ \ =0.$

The Mathieu equation.

1 . Given numbers a and , there exists a general solution $(\ )$ and a characteristic index such that

$$(\ +\ \ )=e^{2\pi}\ \ (\ ).$$

For small values of , an approximate value of can be found from the equation:

$$\cosh(\ \ )=1+2\sin^2\bigl(\tfrac{1}{2}\ \ \overline{a}\ \ +\frac{2}{(1-a)\ \ \overline{a}}\sin\bigl(\ \ \overline{a}\ \ +\ (\ ^4).$$

If $_1(\ )$ is the solution of the Mathieu equation satisfying the initial conditions $_1(0)=1$ and $'_1(0)=0$, the characteristic index can be determined from the relation:

$$\cosh(2\ \ )=\ _1(\ ).$$

The solution $_1(\ )$, and hence , can be determined with any degree of accuracy by means of numerical or approximate methods.

The general solution differs depending on the value of $_1(\ )$ and can be expressed in terms of two auxiliary periodical functions $_1(\ )$ and $_2(\ )$ (see Table 19).

2 . In applications, of major interest are periodical solutions of the Mathieu equation that exist for certain values of the parameters a and (those values of a are referred to as eigenvalues). The most important solutions are listed in Table 20.

The Mathieu functions possess the following properties:

$$\mathrm{ce}_2\ (\ ,-\ )=(-1)\ \ \mathrm{ce}_2\ \ \frac{}{2}-\ \ ,\ \ ,\qquad \mathrm{ce}_{2\ +1}(\ ,-\ )=(-1)\ \ \mathrm{se}_{2\ +1}\ \frac{}{2}-\ \ ,\ \ ,$$

$$\mathrm{se}_2\ (\ ,-\ )=(-1)\ ^{-1}\mathrm{se}_2\ \ \frac{}{2}-\ \ ,\ \ ,\qquad \mathrm{se}_{2\ +1}(\ ,-\ )=(-1)\ \ \mathrm{ce}_{2\ +1}\ \frac{}{2}-\ \ ,\ \ .$$

TABLE 20

Periodical solutions of the Mathieu equation ce $\;=$ ce $(\;,\;)$ and se $\;=$ se $(\;,\;)$ (for odd , the functions ce and se are 2 -periodical, and for even , they are -periodical); certain eigenvalues $a = a\;(\;)$ and $\;=\;\;(\;)$ correspond to each value of the parameter ; $\;= 0,\ 1,\ 2,$

Mathieu functions	Recurrence relations for coefficients	Normalization conditions
$\mathrm{ce}_{2}\;(\;,\;)= \underset{=0}{\;} A_{2}^{2}\;\cos(2\quad)$	$A_{2}^{2}\;=a_{2}\;A_{0}^{2}\;$; $A_{4}^{2}\;=(a_{2}\;-4)A_{2}^{2}\;-2\;A_{0}^{2}\;$; $A_{2\;+2}^{2}=(a_{2}\;-4\;\;^{2})A_{2}^{2}\;-\;A_{2\;-2}^{2}, \quad\geq 2$	$(A_{0}^{2}\;)^{2}+\underset{=0}{\;}(A_{2}^{2}\quad{}^{2} = \begin{cases} 2 & \text{if } =0 \\ 1 & \text{if } \geq 1 \end{cases}$
$\mathrm{ce}_{2\;+1}(\;,\;)= \underset{=0}{\;} A_{2\;+1}^{2\;+1}\cos\big[(2\;+1)\;\big]$	$A_{3}^{2\;+1}=(a_{2\;+1}-1-\;)A_{1}^{2\;+1}$; $A_{2\;+3}^{2\;+1}=[a_{2\;+1}-(2\;+1)^{2}]\times A_{2\;+1}^{2\;+1}-\;A_{2\;-1}^{2\;+1}, \quad\geq 1$	$\underset{=0}{\;}(A_{2\;+1}^{2\;+1}\;{}^{2}=1$
$\mathrm{se}_{2}\;(\;,\;)= \underset{=0}{\;} B_{2}^{2}\;\sin(2\quad)$, $\mathrm{se}_{0}=0$	$B_{4}^{2}\;=(\;_{2}\;-4)B_{2}^{2}\;$; $B_{2\;+2}^{2}=(\;_{2}\;-4\;\;^{2})B_{2}^{2}\;-\;B_{2\;-2}^{2}, \quad\geq 2$	$\underset{=0}{\;}(B_{2}^{2}\quad{}^{2}=1$
$\mathrm{se}_{2\;+1}(\;,\;)= \underset{=0}{\;} B_{2\;+1}^{2\;+1}\sin\big[(2\;+1)\;\big]$	$B_{3}^{2\;+1}=(\;_{2\;+1}-1-\;)B_{1}^{2\;+1}$; $B_{2\;+3}^{2\;+1}=[\;_{2\;+1}-(2\;+1)^{2}]\times B_{2\;+1}^{2\;+1}-\;B_{2\;-1}^{2\;+1}, \quad\geq 1$	$\underset{=0}{\;}(B_{2\;+1}^{2\;+1}\;{}^{2}=1$

Selecting a sufficiently large and omitting the term with the maximum number in the recurrence relations (indicated in Table 20), we can obtain approximate relations for the eigenvalues a (or) with respect to parameter . Then, equating the determinant of the corresponding homogeneous linear system of equations for coefficients A (or B) to zero, we obtain an algebraic equation for finding $a\;(\;)$ (or $\;(\;)$).

For fixed real $\;\neq 0$, the eigenvalues a and are all real and different, while:

$$\text{if}\quad >0\quad\text{then}\quad a_{0}<\;_{1}<a_{1}<\;_{2}<a_{2}<\quad;$$
$$\text{if}\quad <0\quad\text{then}\quad a_{0}<a_{1}<\;_{1}<\;_{2}<a_{2}<a_{3}<\;_{3}<\;_{4}<$$

The eigenvalues possess the following properties:

$$a_{2}\;(-\;)=a_{2}\;(\;),\qquad _{2}\;(-\;)=\;_{2}\;(\;),\qquad a_{2\;+1}(-\;)=\;_{2\;+1}(\;).$$

The solution of the Mathieu equation corresponding to eigenvalue a (or) has zeros on the interval $0\leq\;<\;$ (is a real number).

Listed below are two leading terms of asymptotic expansions of the Mathieu functions ce $(\;,\;)$ and se $(\;,\;)$, as well as of the corresponding eigenvalues $a\;(\;)$ and $\;(\;)$, as $\quad 0$:

$$\mathrm{ce}_{0}(\;,\;)=\frac{1}{\sqrt{2}}\;1-\frac{}{2}\cos 2\quad,\qquad a_{0}(\;)=-\frac{\;^{2}}{2}+\frac{7\;^{4}}{128};$$

$$\mathrm{ce}_{1}(\;,\;)=\cos\;-\frac{}{8}\cos 3\;,\qquad a_{1}(\;)=1+\;;$$

$$\mathrm{ce}_{2}(\;,\;)=\cos 2\;+\frac{}{4}\;1-\frac{\cos 4}{3}\quad,\qquad a_{2}(\;)=4+\frac{5\;^{2}}{12};$$

$$\mathrm{ce}\;(\;,\;)=\cos\;+\frac{}{4}\;\frac{\cos(\;+2)}{\;+1}-\frac{\cos(\;-2)}{\;-1}\quad,\qquad a\;(\;)=\;^{2}+\frac{\;^{2}}{2(\;^{2}-1)}\qquad(\;\geq 3);$$

$\mathrm{se}_1(\ ,\ ) = \sin\ \ - \frac{}{8}\sin 3\ ,\qquad {}_1(\ ) = 1 -\ ;$

$\mathrm{se}_2(\ ,\ ) = \sin 2\ \ - \frac{}{12}\sin 4\ ,\qquad {}_2(\ ) = 4 - \frac{2}{12};$

$\mathrm{se}\ (\ ,\ ) = \sin\ \ - \frac{}{4}\left[\frac{\sin(\ +2)}{+1} - \frac{\sin(\ -2)}{-1}\right],\qquad (\ ) = {}^2 + \frac{2}{2(\ ^2-1)}\qquad (\ \geq 3).$

The Mathieu functions are discussed in the books by McLachlan (1947), Whittaker & Watson (1952), Bateman & Erdélyi (1955, vol. 3), and Abramowitz & Stegun (1964) in more detail.

30. $\ \ + (\ \cos^2\ \ +\ )\ = 0.$

Applying the formula $2\cos^2\ = 1 + \cos 2\ $, we obtain the Mathieu equation 2.1.6.29: $'' + (\frac{1}{2}a + \ + \frac{1}{2}a\cos 2\ )\ = 0.$

31. $\ + \cos(\ )\ + [\ \cos(\ ) - \ ]\ = 0.$

Particular solution: $_0 = e^{-c}\ .$

32. $\ + [\ \cos(\ ) + \ ]\ + \ \cos(\ )\ = 0.$

Particular solution: $_0 = e^{-c}\ .$

33. $\ + (\ +\ )\cos(\ )\ - \ \cos(\ )\ = 0.$

Particular solution: $_0 = a\ +\ .$

34. $\ + \ \cos(\ )\ - \ ^{-1}\cos(\ )\ = 0.$

Particular solution: $_0 = \ .$

35. $\ + \ \cos(\ )\ + [\ ^{+k}\cos(\ ) - \ ^{2k} + \ ^{k-1}]\ = 0.$

Particular solution: $_0 = \exp\left(-\frac{}{k+1}\ ^{+1}\right).$

36. $\ + \ \cos(\ )\ - [\ (\ +1)\cos(\ ) + \ ^2 + 2\ ]\ = 0.$

Particular solution: $_0 = \ e^{c}\ .$

37. $\ + [(\ ^2 + \ )\cos(\ ) + 2]\ + \ \cos(\ )\ = 0.$

Particular solution: $_0 = a + \ .$

38. $\ + (\ ^{+1} + \ \cos\ \ )\ + \ (\ \cos\ \ +\ )\ = 0.$

Particular solution: $_0 = \exp\left(-\frac{a}{+1}\ ^{+1}\right).$

39. $\ + [\ \cos(\ ) + \ ]\ + (\ -1)\ ^{-1}\cos(\ )\ = 0.$

Particular solution: $_0 = \ ^{1-c}.$

40. $\ + (\ +\ \cos\ \ )\ + [\ (\ -1)\cos\ \ + \ ^{-1}]\ = 0.$

Particular solution: $_0 = \ \exp(-a\ \ ).$

41. $\ ^2\ + (\ \cos\ \ + 1)\ + (\ \cos\ \ -\ )\ = 0.$

Particular solution: $_0 = \ ^{-b}.$

42. $\ ^2\ + (\ \cos\ \ +\ )\ + (\ \cos\ \ - 1)\ = 0.$

Particular solution: $_0 = \ ^{-b}.$

43. $\;^{2}\quad + \quad \cos\;(\;)\quad + [\quad ^{-1}\cos\;(\;) - \;-1]\; = 0.$

Particular solution: $\;_0 = \;^{-c}.$

44. $\;^{4}\quad + \quad ^{2}\cos\;(\;)\quad + [\;(\;-\;)\cos\;(\;) - \;^{2}]\; = 0.$

Particular solution: $\;_0 = \;\exp(\quad).$

45. $\cos^2\quad - [\;\cos^2\; + \;(\;-1)]\; = 0, \qquad = 1,\ 2,\ 3,\ \ldots$

Solution: $\; = \cos\quad \dfrac{1}{\cos} \text{—} \quad \big(\;_1 e^{\;-} + \;_2 e^{-\;-}\big).$

46. $\cos^2\quad + (\;\cos^2\; + \;)\; = 0.$

The substitution $\; = \xi + \frac{1}{2}\;$ leads to a linear equation of the form 2.1.6.23: $\sin^2\xi\;'' + (a\sin^2\xi + \;)\; = 0.$

47. $[\;\cos(\;) + \; + \;]\quad + \quad ^{2}\cos(\;)\; = 0.$

Particular solution: $\;_0 = a\cos(\lambda\;) + \; + \;.$

48. $\cos\;(\;)\quad + (\;^{2} - \;^{2})\quad - (\; + \;)\; = 0.$

Particular solution: $\;_0 = \; - \;.$

49. $(\;\cos\; + \;)\quad + (\;\cos\; + \;)\quad + [(\;-\;)\cos\; + \; - \;]\; = 0.$

Particular solution: $\;_0 = e^{-\lambda}\;.$

2.1.6-3. Equations with tangent.

50. $\quad + [\; + (\;-\;)\tan^2(\;)]\; = 0.$

Particular solution: $\;_0 = [\cos(\lambda\;)]\;^{\lambda}.$

51. $\quad + (\;\tan^2\; + \;)\; = 0.$

The transformation $z = \sin^2\;,\quad = \;\cos\;$, where k is a root of the quadratic equation $k^2 + k + a = 0$, leads to the hypergeometric equation 2.1.2.171: $z(z-1)\;'' + [(1-k)z - \frac{1}{2}]\;' - \frac{1}{4}(k + \;)\; = 0.$

52. $\quad + (\;-\;)\tan(\;)\quad + \quad = 0.$

Particular solution: $\;_0 = [\cos(\lambda\;)]\;^{\lambda}.$

53. $\quad + \;\tan\quad + \quad = 0.$

1 . The substitution $\xi = \sin\;$ leads to a linear equation of the form 2.1.2.168: $(\xi^2 - 1)\;'' + (1-a)\xi\;' - \; = 0.$

2 . Solution for $a = -2$:

$$\cos\; = \begin{cases} \;_1\sin(k\;) + \;_2\cos(k\;) & \text{if } \; + 1 = k^2 > 0, \\ \;_1\sinh(k\;) + \;_2\cosh(k\;) & \text{if } \; + 1 = -k^2 < 0. \end{cases}$$

3 . Solution for $a = 2$ and $\; = 3$: $\; = \;_1\cos^3\; + \;_2\sin\;(1 + 2\cos^2\;).$

54. $\quad + \;\tan\quad + (\;\tan^2\; + \;)\; = 0.$

This is a special case of equation 2.1.6.131.

55. $\mathbf{-2\ \ \tan(\ \ )\ \ +(\ \ ^2+\ \ +\ )\ \ =0.}$

The substitution $=\cos(\lambda\ )$ leads to a second-order linear equation of the form 2.1.2.6: $''+(a\ ^2+\ \ +\ +\lambda^2)\ \ =0$.

56. $\mathbf{-2\ \ \tan(\ \ )\ \ +(\ \ ^2+\ \ ^{-1}-\ ^2)\ \ =0.}$

The substitution $=\cos(\lambda\ )$ leads to a second-order linear equation of the form 2.1.2.10: $''+(a\ ^2+\ \ ^{-1})\ \ =0$.

57. $\mathbf{+\ \ \tan\ (\ \ )\ \ +[\ \ \tan\ (\ \ )-\ ]\ \ =0.}$

Particular solution: $_0=e^{-c}\ .$

58. $\mathbf{+\ \ \tan\ (\ \ )\ \ +[\ \ \tan\ ^{+1}(\ \ )+(\ -\ )\tan^2(\ \ )+\ ]\ \ =0.}$

Particular solution: $_0=[\cos(\lambda\ \ )]^{b\ \ \lambda}.$

59. $\mathbf{+\ \ \tan\ \ \ \ +(\ \tan\ ^{+1}\ -\ \tan\ ^{-1}\ \ +4)\ \ =0.}$

Particular solution: $_0=\sin\ \ \cos\ \ .$

60. $\mathbf{+[\ \tan\ (\ \ )+\ ]\ \ +\ \ \tan\ (\ \ )\ \ =0.}$

Particular solution: $_0=e^{-c}\ .$

61. $\mathbf{+\tan\ \ (\ \tan\ \ +\ -1)\ \ +(\ \ \tan\ ^{+2}\ -\ \tan\ \ +2\ \ +2)\ \ =0.}$

Particular solution: $_0=\sin\ \ \cos^b\ \ .$

62. $\mathbf{+(\ \ +\ )\tan\ (\ \ )\ \ -\ \tan\ (\ \ )\ \ =0.}$

Particular solution: $_0=a\ \ +\ .$

63. $\mathbf{+\ \ \ \tan\ (\ \ )\ \ -\ \ \ ^{-1}\tan\ (\ \ )\ \ =0.}$

Particular solution: $_0=\ \ .$

64. $\mathbf{+\ \ \ \tan\ (\ \ )\ \ +[\ \ \ ^{+k}\tan\ (\ \ )-\ \ ^{2k}+\ \ ^{k-1}]\ \ =0.}$

Particular solution: $_0=\exp\ -\frac{\ }{k+1}\ \ ^{+1}\ .$

65. $\mathbf{-2\ \ \ \tan(\ \ )\ \ +(\ \ +\ )\ \ =0.}$

The substitution $=\cos(\lambda\ )$ leads to a second-order linear equation of the form 2.1.2.64: $''+[(a+\lambda^2)\ \ +\ ]\ \ =0$.

66. $\mathbf{+\ \ \ \tan\ (\ \ )\ \ -[\ (\ \ +1)\tan\ (\ \ )+\ \ ^2\ \ +2\ ]\ \ =0.}$

Particular solution: $_0=\ \ e^{c}\ .$

67. $\mathbf{+[(\ \ ^2+\ \ )\tan\ (\ \ )+2]\ \ +\ \ \tan\ (\ \ )\ \ =0.}$

Particular solution: $_0=a+\ \ \ .$

68. $\mathbf{+(\ \ \ ^{+1}+\ \ \tan\ \ \ )\ \ +\ \ \ (\ \tan\ \ \ +\ )\ \ =0.}$

Particular solution: $_0=\exp\ -\frac{a}{\ +1}\ \ ^{+1}\ .$

69. $\mathbf{+(\ \ \ +\ \ \tan\ \ \ )\ \ +[\ (\ \ \ -1)\tan\ \ \ +\ \ \ ^{-1}]\ \ =0.}$

Particular solution: $_0=\ \ \exp(-a\ \ \ \ ).$

70. $\;\; + [\;\;\; \tan\;(\;\;) + \;]\;\;\; + \;(\; - 1)\;\;\; ^{-1} \tan\;(\;\;)\;\; = 0.$

Particular solution: $\;_{0} = \;^{1-c}.$

71. $^{2}\;\;\; - 2\;\;\; ^{2} \tan(\;\;\;)\;\;\; + (\;\;\; ^{2} + \;\;\; + \;)\;\;\; = 0.$

The substitution $\;\; = \;\; \cos(\lambda\;\;)$ leads to a second-order linear equation of the form 2.1.2.115: $^{2}\;\;''\;\; + [(a + \lambda^{2})\;\; ^{2} + \;\;\; + \;]\;\;\; = 0.$

72. $^{2}\;\;\; + \;(1 - 2\;\;\tan\;\;)\;\;\; - (\;\;\tan\;\;\; + \;\; ^{2})\;\;\; = 0.$

Solution: $\;\;\cos\;\; = \;\;_{1}\;\;(\;\;) + \;\;_{2}\;\;(\;\;),$ where $\;\;(\;\;)$ and $\;\;(\;\;)$ are the Bessel functions.

73. $^{2}\;\;\; - \;(2\;\;\tan\;\;\; + \;\;)\;\;\; + (\;\;\; ^{2} + \;\;\; + \;\; + \;\;\;\tan\;\;)\;\;\; = 0.$

The substitution $\;\; = \;\;\cos\;\;$ leads to a second-order linear equation of the form 2.1.2.131: $^{2}\;\;''\;\; - k\;\;\;' + [(a + 1)\;\; ^{2} + \;\;\; + \;]\;\;\; = 0.$

74. $^{2}\;\;\; + \;(\;\;\tan\;\;\; + 1)\;\;\; + \;(\;\;\tan\;\;\; - \;\;)\;\;\; = 0.$

Particular solution: $\;_{0} = \;^{-b}.$

75. $^{2}\;\;\; + \;(\;\;\tan\;\;\; + \;\;)\;\;\; + \;(\;\;\tan\;\;\; - 1)\;\;\; = 0.$

Particular solution: $\;_{0} = \;^{-b}.$

76. $^{2}\;\;\; + \;\;\;\; \tan\;(\;\;)\;\;\; + \;[\;\;\;\; ^{-1} \tan\;(\;\;) - \;\; - 1]\;\;\; = 0.$

Particular solution: $\;_{0} = \;^{-c}.$

77. $^{4}\;\;\; + \;\;\; ^{2} \tan\;(\;\;)\;\;\; + [\;(\;\; - \;\;) \tan\;(\;\;) - \;\; ^{2}]\;\;\; = 0.$

Particular solution: $\;_{0} = \;\;\exp(\;\;\;).$

78. $(\;\;\tan\;\;\; + \;\;)\;\;\; + (\;\;\; + \;\;)\;\;\; - \;\;\;\; = 0.$

Particular solution: $\;_{0} = \;\;\; + \;\;.$

79. $(\;\;\tan\;\;\; + \;\;)\;\;\; + (\;\;\tan\;\;\; + \;\;)\;\;\; + \;[(\;\; - \;\;\;)\tan\;\;\; + \;\; - \;\;\;]\;\;\; = 0.$

Particular solution: $\;_{0} = e^{-\lambda}\;\;.$

2.1.6-4. Equations with cotangent.

80. $\;\; + \;[\;\; + (\;\; - \;\;)\cot^{2}(\;\;\;)]\;\;\; = 0.$

Particular solution: $\;_{0} = [\sin(\lambda\;\;)]\;\;^{\lambda}.$

81. $\;\; + (\;\;\cot^{2}\;\;\; + \;\;)\;\;\; = 0.$

The substitution $\;\; = \xi + \frac{\pi}{2}$ leads to an equation of the form 2.1.6.51: $\;\;''\;\; + (a \tan^{2}\xi + \;\;)\;\;\; = 0.$

82. $\;\; + \cot\;\;\;\; + \;(\;\; + 1)\;\;\; = 0.$

The substitution $\xi = \cos\;\;$ leads to the Legendre equation 2.1.2.154: $(\xi^{2} - 1)\;\;''\;\; + 2\xi\;\;' - (\;\; + 1)\;\;\; = 0.$

83. $\;\; + 2\;\;\cot(\;\;\;)\;\;\; + (\;\; ^{2} - \;\; ^{2})\;\;\; = 0.$

Particular solution: $\;_{0} = \dfrac{\cos(\;\;\;)}{\sin(a\;\;)}.$

84. $\;\; + (\;\; - \;\;)\cot(\;\;\;)\;\;\; + \;\;\;\;\; = 0.$

Particular solution: $\;_{0} = [\sin(\lambda\;\;)]\;\;^{\lambda}.$

85. $+ \ \cot(\ ) \ + \ = 0.$

The substitution $z = \lambda \ + \frac{\pi}{2}$ leads to a second-order linear equation of the form 2.1.6.53: $'' - a\lambda^{-1} \tan z \ ' + \ \lambda^{-2} \ = 0.$

86. $+ \ \cot(\ ) \ + [\ + (\ - \ - \ ) \cot^2(\ )] \ = 0.$

Particular solution: $_0 = [\sin(\lambda \ )]^b \ \lambda$.

87. $+ \ \cot \ + (\ \cot^2 \ + \ ) \ = 0.$

This is a special case of equation 2.1.6.131.

88. $+ 2 \ \cot(\ ) \ + (\ ^2 + \ + \ ) \ = 0.$

The substitution $= \ \sin(\lambda \ )$ leads to a second-order linear equation of the form 2.1.2.6: $'' + (a \ ^2 + \ + \ + \lambda^2) \ = 0.$

89. $+ 2 \ \cot(\ ) \ + (\ ^2 + \ ^{-1} - \ ^2) \ = 0.$

The substitution $= \ \sin(\lambda \ )$ leads to a second-order linear equation of the form 2.1.2.10: $'' + (a \ ^2 + \ ^{-1}) \ = 0.$

90. $+ \ \cot \ (\ ) \ + [\ \cot \ (\ ) - \ ] \ = 0.$

Particular solution: $_0 = e^{-c} \ .$

91. $+ [\ \cot \ (\ ) + \ ] \ + \ \cot \ (\ ) \ = 0.$

Particular solution: $_0 = e^{-c} \ .$

92. $+ (\ + \ ) \cot \ (\ ) \ - \ \cot \ (\ ) \ = 0.$

Particular solution: $_0 = a \ + \ .$

93. $+ \ \cot \ (\ ) \ - \ ^{-1} \cot \ (\ ) \ = 0.$

Particular solution: $_0 = \ .$

94. $+ \ \cot \ (\ ) \ + [\ ^{+k} \cot \ (\ ) - \ ^{2k} + \ ^{k-1}] \ = 0.$

Particular solution: $_0 = \exp \ -\frac{}{k+1} \ ^{+1} \ .$

95. $+ 2 \ \cot(\ ) \ + (\ + \ ) \ = 0.$

The substitution $= \ \sin(\lambda \ )$ leads to a second-order linear equation of the form 2.1.2.64: $'' + [(a + \lambda^2) \ + \ ] \ = 0.$

96. $+ \ \cot \ (\ ) \ - [\ (\ + 1) \cot \ (\ ) + \ ^2 \ + 2 \ ] \ = 0.$

Particular solution: $_0 = \ e^c \ .$

97. $+ (\ ^{+1} + \ \cot \ ) \ + \ (\ \cot \ + \ ) \ = 0.$

Particular solution: $_0 = \exp \ -\frac{a}{\ + 1} \ ^{+1} \ .$

98. $+ (\ + \ \cot \ ) \ + [\ (\ - 1) \cot \ + \ ^{-1}] \ = 0.$

Particular solution: $_0 = \ \exp(-a \ ).$

99. $+ [\ \cot \ (\ ) + \ ] \ + \ (\ - 1) \ ^{-1} \cot \ (\ ) \ = 0.$

Particular solution: $_0 = \ ^{1-c}.$

100. $^{2} \quad + 2 \quad ^{2}\cot(\quad) \quad + (\quad ^{2} + \quad + \quad) \quad = 0.$

The substitution $\quad = \quad \sin(\lambda \quad)$ leads to a second-order linear equation of the form 2.1.2.115: $^{2} \quad '' \quad + [(a+\lambda^{2}) \quad ^{2} + \quad + \quad] \quad = 0.$

101. $^{2} \quad + \quad (2 \quad \cot \quad + \quad) \quad + (\quad ^{2} + \quad + \quad + \quad \cot \quad) \quad = 0.$

The substitution $\quad = \quad \sin \quad$ leads to a second-order linear equation of the form 2.1.2.131: $^{2} \quad '' \quad + k \quad ' + [(a+1) \quad ^{2} + \quad + \quad] \quad = 0.$

102. $^{2} \quad + \quad (\quad \cot \quad + 1) \quad + \quad (\quad \cot \quad - \quad) \quad = 0.$

Particular solution: $_{0} = \quad ^{-b}.$

103. $^{2} \quad + \quad (\quad \cot \quad + \quad) \quad + \quad (\quad \cot \quad - 1) \quad = 0.$

Particular solution: $_{0} = \quad ^{-b}.$

104. $^{2} \quad + \quad \cot \quad (\quad) \quad + \quad [\quad ^{-1}\cot \quad (\quad) - \quad - 1] \quad = 0.$

Particular solution: $_{0} = \quad ^{-c}.$

105. $^{4} \quad + \quad ^{2}\cot \quad (\quad) \quad + [\quad (\quad - \quad)\cot \quad (\quad) - \quad ^{2}] \quad = 0.$

Particular solution: $_{0} = \quad \exp(\quad).$

106. $(\quad \cot \quad + \quad) \quad + (\quad \cot \quad + \quad) \quad + \quad [(\quad - \quad)\cot \quad + \quad - \quad] \quad = 0.$

Particular solution: $_{0} = e^{-\lambda} \quad .$

2.1.6-5. Equations containing combinations of trigonometric functions.

107. $\quad - \quad [\quad \sin^{2}(\quad) + \quad \cos(\quad)] \quad = 0.$

Particular solution: $_{0} = \exp \; -\frac{a}{\quad} \cos(\quad) \; .$

108. $\quad - \quad [\quad \cos^{2}(\quad) + \quad \sin(\quad)] \quad = 0.$

Particular solution: $_{0} = \exp \; -\frac{a}{\quad} \sin(\quad) \; .$

109. $\quad + (\quad \sin \quad + \quad) \quad + \quad (\quad \sin \quad + \cos \quad) \quad = 0.$

Particular solution: $_{0} = \exp(a \cos \quad).$

110. $\quad + (\quad \sin \quad + \quad \sin \quad) \quad + \quad (\quad \sin \quad ^{+1} \quad + \cos \quad) \quad = 0.$

Particular solution: $_{0} = \exp(\quad \cos \quad).$

111. $\quad + (\quad \cos \quad + \quad) \quad + \quad (\quad \cos \quad - \sin \quad) \quad = 0.$

Particular solution: $_{0} = \exp(-a \sin \quad).$

112. $\quad + (\quad \cos \quad + \quad \cos \quad) \quad + \quad (\quad \cos \quad ^{+1} \quad - \sin \quad) \quad = 0.$

Particular solution: $_{0} = \exp(- \quad \sin \quad).$

113. $\sin \quad + \cos \quad + \quad (\quad + 1)\sin \quad = 0.$

The substitution $\xi = \cos \quad$ leads to the Legendre equation 2.1.2.154: $(1-\xi^{2}) \quad '' \quad - 2\xi \quad ' + (\quad + 1) \quad = 0.$

114. $\mathbf{\sin \quad + (2 \quad + 1) \cos \quad + (\quad - \quad)(\quad + \quad + 1) \sin \quad = 0.}$

Here, is an arbitrary number and is a positive integer. The substitution $\xi = \cos$ leads to an equation of the form 2.1.2.156: $(\xi^2 - 1) \quad '' + 2(\quad + 1)\xi \quad ' + (\quad - \quad)(\quad + \quad + 1) \quad = 0$.

115. $\mathbf{\sin^2 \quad + \sin \quad \cos \quad + [\quad (\quad + 1) \sin^2 \quad - \quad ^2] \quad = 0.}$

Here, is an arbitrary number and is a nonnegative integer.

The transformation $\xi = \cos \quad , \quad = \quad \sin$ leads to an equation of the form 2.1.2.156: $(\xi^2 - 1) \quad '' + 2(\quad + 1)\xi \quad ' + (\quad - \quad)(\quad + \quad + 1) \quad = 0$.

116. $\mathbf{\sin^2 \quad + \sin \quad (\quad \cos \quad + \quad) \quad + (\quad \cos^2 \quad + \quad \cos \quad + \gamma) \quad = 0.}$

Set $\quad = 2\xi$. Applying the trigonometric formulas

$$\sin(2\xi) = 2 \sin\xi \cos\xi, \quad \cos(2\xi) = \cos^2\xi - \sin^2\xi, \qquad = \quad (\sin^2\xi + \cos^2\xi),$$
$$\beta = \beta(\sin^2\xi + \cos^2\xi), \qquad = \quad (\sin^2\xi + \cos^2\xi)^2,$$

and dividing all the terms by $\sin^2$, we arrive at an equation of the form 2.1.6.131:

$$'' + [(\quad - a) \tan\xi + (\quad + a) \cot\xi] \quad ' + [(\quad - \beta + \quad) \tan^2\xi + (\quad + \beta + \quad) \cot^2\xi + 2 \quad - 2 \quad] \quad = 0.$$

117. $\mathbf{\cos^2 \quad + \quad \sin(2 \quad) \quad + [\quad \cos(2 \quad) + \quad] \quad = 0.}$

Dividing the equation by $\cos^2$ and applying the formulas

$$\sin(2 \quad) = 2 \sin \quad \cos \quad , \quad \cos(2 \quad) = \cos^2 \quad - \sin^2 \quad , \qquad = \quad (\sin^2 \quad + \cos^2 \quad),$$

we obtain an equation of the form 2.1.6.131: $\quad '' + 2a \tan \quad ' + [(\quad - \quad) \tan^2 \quad + \quad + \quad] \quad = 0$.

118. $\mathbf{\cos^2(\quad) \quad + (\quad - 1) \quad \sin(2 \quad) \quad + \quad ^2[(\quad - 1) \sin^2(\quad) + \cos^2(\quad)] \quad = 0.}$

Particular solution: $\quad _0 = \cos \quad (a \quad)$.

119. $\mathbf{\cos^2 \quad + \cos \quad (\quad \sin \quad + \quad) \quad + (\quad \sin^2 \quad + \quad \sin \quad + \gamma) \quad = 0.}$

The substitution $\quad = \xi + \frac{\pi}{2}$ leads to a second-order linear equation of the form 2.1.6.116: $\sin^2\xi \quad '' - \sin\xi\,(a\cos\xi + \quad) \quad ' + (\quad \cos^2\xi + \beta\cos\xi + \quad) \quad = 0$.

120. $\mathbf{\sin \quad \cos^2 \quad + \cos \quad (\quad \sin^2 \quad + \quad) \quad + \quad \sin \quad = 0.}$

1 . Dividing the equation by $\sin \quad \cos^2$ and assuming $\quad = \quad (\sin^2 \quad + \cos^2 \quad), \quad = \quad (\sin^2 \quad + \cos^2 \quad)$, we obtain equation 2.1.6.131: $\quad '' + [(a + \quad) \tan \quad + \quad \cot \quad] \quad ' + \quad (\tan^2 \quad + 1) \quad = 0$.

2 . Particular solutions:

$$\begin{aligned} &_0 = \cos && \text{if} \quad = a(\quad + 1),\\ &_0 = \tan^{1-b} && \text{if} \quad = (a + 2)(\quad - 1),\\ &_0 = \sin^{1-b} \quad \cos^{\quad + b - 1} && \text{if} \quad = 2(a + \quad - 1). \end{aligned}$$

121. $\mathbf{\sin \quad \cos^2 \quad + \cos \quad (\quad \sin^2 \quad - 1) \quad + \quad \sin^3 \quad = 0.}$

Solution: $\quad = \quad _1(\cos \quad)^{\ _1} + \quad _2(\cos \quad)^{\ _2}$, where k_1 and k_2 are roots of the quadratic equation $k^2 - ak + \quad = 0$.

122. $\mathbf{\sin^2 \quad \cos^2 \quad + (\quad \sin^2 \quad + \quad \cos^2 \quad + \quad \sin^2 \quad \cos^2 \quad) \quad = 0.}$

Dividing the equation by $\sin^2 \quad \cos^2$ and assuming $a = a(\sin^2 \quad + \cos^2 \quad), \quad = \quad (\sin^2 \quad + \cos^2 \quad)$, we obtain equation 2.1.6.131: $\quad '' + (a \tan^2 \quad + \quad \cot^2 \quad + a + \quad + \quad) \quad = 0$.

123. $\mathbf{\sin(\ \ )\ \ \ + [\ ^2 \sin(\ \ ) + \ \cos^{-4}(\ \ )]\ = 0.}$

The transformation $\xi = \tan(\lambda\ )$, $\ = \dfrac{}{\cos(\lambda\ )}$ leads to an equation of the form 2.1.2.7: $''\ + a\lambda^{-2}\xi^{-}\ \ = 0$.

124. $\mathbf{\cos(\ \ )\ \ \ + [\ ^2 \cos(\ \ ) + \ \sin^{-4}(\ \ )]\ = 0.}$

The substitution $\lambda\ \ = \frac{\pi}{2} - \lambda\xi$ leads to an equation of the form 2.1.6.123.

125. $\mathbf{\ \ + \ \tan\ \ \ + \ ^2(\cos\ \ )^2\ \ = 0.}$

Solution: $\ = \ _1 \sin(a\ \ ) + \ _2 \cos(a\ \ )$, where $\ = \ \cos\ \ $.

126. $\mathbf{\ + \tan\ \ \ + \ ^2 \cos^2\ (\sin\ \ )^{2\ -2}\ \ = 0.}$

Solution: $\ = \overline{\sin\ \ }\ _1\ _{\frac{1}{2}}\ \frac{a}{}\sin\ \ + \ _2\ _{\frac{1}{2}}\ \frac{a}{}\sin\ \ $, where (z) and (z) are the Bessel functions.

127. $\mathbf{\ + \tan\ \ \ - (\ \ - 1)\cot^2\ \ \ = 0.}$

Solution: $\ = \begin{cases} _1|\sin\ |\ + \ _2|\sin\ \ |^{1-} & \text{if } a \neq \frac{1}{2}, \\ \overline{|\sin\ \ |}(\ _1 + \ _2 \ln|\sin\ \ |) & \text{if } a = \frac{1}{2}. \end{cases}$

128. $\mathbf{\ - 2\ \cot(2\ \ )\ \ - \ ^2 \sin^2(2\ \ )\ = 0.}$

Solution: $\ = \ _1 \exp\ \frac{}{a}\sin^2(a\ \ )\ + \ _2 \exp -\frac{}{a}\sin^2(a\ \ )$.

129. $\mathbf{\ - \ \cot\ \ \ + \ ^2(\sin\ \ )^2\ \ = 0.}$

Solution: $\ = \ _1 \sin(a\ \ ) + \ _2 \cos(a\ \ )$, where $\ = \ \sin\ \ $.

130. $\mathbf{\ - 2\cot(2\ \ )\ \ + \ \tan^2\ \ \ = 0.}$

The substitution $\xi = \cos\ \ $ leads to the Euler equation 2.1.2.123: $\xi^2\ '' - \xi\ ' + a\ \ = 0$.

131. $\mathbf{\ + (\ \tan\ + \ \cot\ )\ \ + (\ \tan^2\ + \ \cot^2\ + \gamma)\ = 0.}$

The transformation $\xi = \sin^2\ $, $\ = \ \sin\ \ \cos\ \ $, where and are roots of the quadratic equations

$$^2 + (\ - 1)\ + \beta = 0, \qquad ^2 - (a+1)\ \ + \ \ = 0,$$

leads to the hypergeometric equation 2.1.2.171:

$$4\xi(\xi - 1)\ '' + 2[(2\ + 2\ \ + 2 + \ - a)\xi - 2\ - \ - 1]\ ' + (2\ \ + \ + \ + \ \ - a - \ )\ = 0.$$

132. $\mathbf{\sin(2\ \ )\ \ - 2\ \ \ + 2\ ^2 \sin^2\ (\tan\ \ )^{2\ -1}\ = 0.}$

Solution: $\ = \ _1 \sin(a\ \ ) + \ _2 \cos(a\ \ )$, where $\ = \ \tan\ \ $.

2.1.7. Equations Containing Inverse Trigonometric Functions

2.1.7-1. Equations with arcsine.

1. $\mathbf{\ + (\ \ + \ + \ \arcsin\ )\ \ + [\ (\ \ + \ )\arcsin\ + \ ]\ = 0.}$

Particular solution: $\ _0 = \exp(-\frac{1}{2}a\ \ ^2 - \ \ )$.

2. $+ \ (\arcsin \ ) \quad + \ [\ (\arcsin \ ) \ - \ ] \ = 0.$

Particular solution: $_0 = e^{-c}$.

3. $+ \ (\arcsin \ ) \quad + \ [\quad (\arcsin \ ) \ - \quad ^2 \ + \quad ^{-1}] \ = 0.$

Particular solution: $_0 = \exp\ -\dfrac{a}{+1} \quad ^{+1}$.

4. $+ (\quad + \ )(\arcsin \ ) \quad - \ (\arcsin \ ) \quad = 0.$

Particular solution: $_0 = a \ + \ .$

5. $+ \quad (\arcsin \ ) \quad - \quad ^{-1}(\arcsin \ ) \quad = 0.$

Particular solution: $_0 = \ .$

6. $+ \quad \arcsin \quad - [\ (\quad + 1) \arcsin \quad + \ (\quad + 2)] \ = 0.$

Particular solution: $_0 = \ e^b$.

7. $+ [\ (\quad + 1) \arcsin \quad + \quad - 1] \quad + \quad ^2 \ \arcsin \quad = 0.$

Particular solution: $_0 = (\quad + 1)e^{-b}$.

8. $+ [(\quad ^2 + \quad) \arcsin \quad + 2] \quad + \ \arcsin \quad = 0.$

Particular solution: $_0 = a + \quad .$

9. $+ [\quad (\arcsin \ ) \ + \ ] \quad + \ (\ - 1)(\arcsin \ ) \quad = 0.$

Particular solution: $_0 = \ ^{1-b}.$

10. $+ (\quad ^{+1} + \ \arcsin \ ) \quad + \quad (\ \arcsin \quad + \ ) \ = 0.$

Particular solution: $_0 = \exp\ -\dfrac{a}{+1} \quad ^{+1}$.

11. $+ (\quad + \quad \arcsin \ ) \quad + [\ (\quad - 1) \arcsin \quad + \quad ^{-1}] \ = 0.$

Particular solution: $_0 = \ \exp(-a \quad).$

12. $^2 \quad + \quad \arcsin \quad + \ (\ \arcsin \quad - \quad - 1) \ = 0.$

Particular solution: $_0 = \ ^{-}$.

13. $^2 \quad + \ (\ \arcsin \quad + 2) \quad + [\ (\quad + 1) \arcsin \quad - \ ^2 \ ^2] \ = 0.$

Particular solution: $_0 = \ ^{-1}e^{-}$.

14. $(\quad ^2 + \ ) \quad + \ (\quad ^2 + \ )(\arcsin \ ) \quad - 2 \ [\quad (\arcsin \ ) \ + 1] \ = 0.$

Particular solution: $_0 = a \ ^2 + \ .$

15. $^4 \quad + \quad ^2 \arcsin \quad + [\ (\ - \ ) \arcsin \quad - \ ^2] \ = 0.$

Particular solution: $_0 = \ \exp(\quad).$

16. $(\quad ^2 + \ )^2 \quad + (\quad + \ )(\arcsin \ ) \quad - \ (\arcsin \ ) \quad = 0.$

Particular solution: $_0 = \quad + \ .$

17. $(\ ^2 + \ )^2 \quad + \ (\ ^2 + \ )(\arcsin \ ) \quad - [\quad (\arcsin \ ) \ + \ ] \ = 0.$

Particular solution: $_0 = \ \sqrt{\ ^2 + a}.$

18. $(\quad ^2 + \ )^2 \quad + \ (\quad ^2 + \ )(\arcsin \ ) \quad + [\ (\arcsin \ ) \ - 2 \quad - 1] \ = 0.$

Particular solution: $_0 = \exp\ - \ \dfrac{}{a \ ^2 + }$.

2.1.7-2. Equations with arccosine.

19. **+ (+ + arccos) + [(+) arccos +] = 0.**

Particular solution: $_0 = \exp(-\frac{1}{2}a\ ^2 -\ )$.

20. **+ (arccos) + [(arccos) −] = 0.**

Particular solution: $_0 = e^{-c}$.

21. **+ (arccos) + [(arccos) − ^{2} + $^{-1}$] = 0.**

Particular solution: $_0 = \exp\ -\frac{a}{+1}\ ^{+1}$.

22. **+ (+)(arccos) − (arccos) = 0.**

Particular solution: $_0 = a\ +\ $.

23. **+ (arccos) − $^{-1}$(arccos) = 0.**

Particular solution: $_0 =$.

24. **+ arccos − [(+ 1) arccos + (+ 2)] = 0.**

Particular solution: $_0 = e^b$.

25. **+ [(+ 1) arccos + − 1] + ^{2} arccos = 0.**

Particular solution: $_0 = (\ +1)e^{-b}$.

26. **+ [(^{2} +) arccos + 2] + arccos = 0.**

Particular solution: $_0 = a +$.

27. **+ [(arccos) +] + (− 1)(arccos) = 0.**

Particular solution: $_0 =\ ^{1-b}$.

28. **+ ($^{+1}$ + arccos) + (arccos +) = 0.**

Particular solution: $_0 = \exp\ -\frac{a}{+1}\ ^{+1}$.

29. **+ (+ arccos) + [(− 1) arccos + $^{-1}$] = 0.**

Particular solution: $_0 = \exp(-a\ )$.

30. **^{2} + arccos + (arccos − − 1) = 0.**

Particular solution: $_0 =\ ^-$.

31. **^{2} + (arccos + 2) + [(+ 1) arccos − ^{2} ^{2}] = 0.**

Particular solution: $_0 =\ ^{-1}e^{-}$.

32. **(^{2} +) + (^{2} +)(arccos) − 2 [(arccos) + 1] = 0.**

Particular solution: $_0 = a\ ^2 +$.

33. **^{4} + ^{2} arccos + [(−) arccos − ^{2}] = 0.**

Particular solution: $_0 = \exp(\ )$.

34. **(^{2} +)^2 + (+)(arccos) − (arccos) = 0.**

Particular solution: $_0 =\ +\ $.

35. $(\ ^2 + \ )^2 \quad + \ (\ ^2 + \ )(\arccos\ ) \quad - [\quad (\arccos\ ) \ + \ ] \ = 0.$

Particular solution: $_0 = \sqrt{\ ^2 + a}.$

36. $(\ ^2 + \ )^2 \quad + \ (\ ^2 + \ )(\arccos\ ) \quad + [\ (\arccos\ ) \ - 2 \quad - 1] \ = 0.$

Particular solution: $_0 = \exp\Big(- \dfrac{\quad}{a\ ^2 + \ }\Big).$

2.1.7-3. Equations with arctangent.

37. $+ (\quad + \ + \ \arctan\ ) \quad + [\ (\quad + \ )\arctan\ + \ ] \ = 0.$

Particular solution: $_0 = \exp(-\tfrac{1}{2}a\ ^2 - \quad).$

38. $+ \ (\arctan\ ) \quad + \ [\ (\arctan\ ) \ - \ ] \ = 0.$

Particular solution: $_0 = e^{-c}\ .$

39. $+ \ (\arctan\ ) \quad + \ [\quad (\arctan\ ) \ - \quad ^2 \ + \quad ^{-1}] \ = 0.$

Particular solution: $_0 = \exp\Big(-\dfrac{a}{\ +1}\quad ^{+1}\Big).$

40. $+ (\quad + \ )(\arctan\ ) \quad - \ (\arctan\ ) \quad = 0.$

Particular solution: $_0 = a \ + \ .$

41. $+ \quad (\arctan\ ) \quad - \quad ^{-1}(\arctan\ ) \quad = 0.$

Particular solution: $_0 = \ .$

42. $+ \quad \arctan \quad - [\ (\quad + 1)\arctan\ + \ (\quad + 2)] \ = 0.$

Particular solution: $_0 = \ e^{b}\ .$

43. $+ [\ (\quad + 1)\arctan\ + \quad - 1] \quad + \quad ^2 \ \arctan \quad = 0.$

Particular solution: $_0 = (\quad + 1)e^{-b}\ .$

44. $+ [(\ ^2 + \quad)\arctan\ + 2] \quad + \ \arctan \quad = 0.$

Particular solution: $_0 = a + \quad .$

45. $+ [\quad (\arctan\ ) \ + \ ] \quad + \ (\ - 1)(\arctan\ ) \quad = 0.$

Particular solution: $_0 = \ ^{1-b}.$

46. $+ (\quad ^{+1} + \ \arctan\ ) \quad + \quad (\ \arctan\ + \ ) \ = 0.$

Particular solution: $_0 = \exp\Big(-\dfrac{a}{\ +1}\quad ^{+1}\Big).$

47. $+ (\quad + \quad \arctan\ ) \quad + [\ (\quad - 1)\arctan\ + \quad ^{-1}] \ = 0.$

Particular solution: $_0 = \ \exp(-a \quad).$

48. $+ \quad (\arctan\ \ + \ ) \quad - \ (\arctan\ \ + \ ) \ = 0.$

Particular solution: $_0 = \ .$

49. $+ \ \arctan \quad + \ (\ \arctan\ \ - \quad) \ = 0.$

Particular solution: $_0 = e^{-}\ .$

50. $+\ (\text{arctan}\ +\ )\ +\ \text{arctan}\ = 0.$

Particular solution: $_0 = e^{-b}$.

51. $+\ \text{arctan}\ +\ (\ \text{arctan}\ -\ ^2 + 1)\ = 0.$

Particular solution: $_0 = \exp\bigl(-\tfrac{1}{2}a\ ^2$.

52. $^2\ +\ \text{arctan}\ +\ (\ \text{arctan}\ -\ -1)\ = 0.$

Particular solution: $_0 = \ ^-$.

53. $^2\ +\ (\ \text{arctan}\ +2)\ +[\ (\ +1)\,\text{arctan}\ -\ ^2\ ^2]\ = 0.$

Particular solution: $_0 = \ ^{-1}e^-$.

54. $^2\ +\ (\text{arctan}\ +\ )\ -\ (\text{arctan}\ +\ )\ = 0.$

Particular solution: $_0 =$.

55. $^2\ +\ \text{arctan}\ +\ (\ \text{arctan}\ -\ ^2)\ = 0.$

Particular solution: $_0 = e^-$.

56. $^2\ +\ (\text{arctan}\ +\ ^2)\ +\ \text{arctan}\ = 0.$

Particular solution: $_0 = e^{-b}$.

57. $^2\ +\ [(\ +\ )\,\text{arctan}\ +2]\ +\ \text{arctan}\ = 0.$

Particular solution: $_0 = a +$.

58. $(\ ^2+1)\ -[\ ^2(\ ^2+1)(\text{arctan}\ )^2 +\ ]\ = 0.$

Particular solution: $_0 = (\ ^2+1)^-\ ^2 \exp(a\ \ \text{arctan}\ )$.

59. $(\ ^2+\ )\ +\ (\ ^2+\ )(\text{arctan}\ )\ -2\ [\ (\text{arctan}\ )\ +1]\ = 0.$

Particular solution: $_0 = a\ ^2 +$.

60. $^4\ +\ ^2\,\text{arctan}\ +[\ (\ -\ )\,\text{arctan}\ -\ ^2]\ = 0.$

Particular solution: $_0 = \ \exp(\ )$.

61. $(\ ^2+1)^2\ +[\ (\text{arctan}\ )^2 +\ \text{arctan}\ +\ ]\ = 0.$

The transformation $\xi = \text{arctan}\ ,\ \ = \dfrac{}{\ ^2+1}$ leads to an equation of the form 2.1.2.6: $''\ + (a\xi^2 +\ \xi +\ +1)\ = 0.$

62. $(\ ^2+1)^2\ +[\ (\text{arctan}\ )\ -1]\ = 0.$

The transformation $\xi = \text{arctan}\ ,\ \ = \dfrac{}{\ ^2+1}$ leads to an equation of the form 2.1.2.7: $''\ +\ \xi\ \ = 0.$

63. $(\ ^2+\ )^2\ +(\ +\ )(\text{arctan}\ )\ -\ (\text{arctan}\ )\ = 0.$

Particular solution: $_0 = \ +$.

64. $(\ ^2+\ )^2\ +\ (\ ^2+\ )(\text{arctan}\ )\ -[\ (\text{arctan}\ )\ +\ ]\ = 0.$

Particular solution: $_0 = \sqrt{\ ^2 + a}.$

65. $(\ ^2+\ )^2\ +\ (\ ^2+\ )(\text{arctan}\ )\ +[\ (\text{arctan}\ )\ -2\ \ -1]\ = 0.$

Particular solution: $_0 = \exp\ -\ \dfrac{}{a\ ^2+}$.

2.1.7-4. Equations with arccotangent.

66. $+(\quad+\quad+\ \mathbf{arccot}\ )\quad+[\ (\quad+\ )\,\mathbf{arccot}\quad+\ ]\ =\mathbf{0.}$

Particular solution: $_0=\exp(-\tfrac{1}{2}a\ ^2-\quad)$.

67. $+\ (\mathbf{arccot}\ )\quad+\ [\ (\mathbf{arccot}\ )\ -\ ]\ =\mathbf{0.}$

Particular solution: $_0=e^{-c}\ $.

68. $+\ (\mathbf{arccot}\ )\quad+\ [\quad(\mathbf{arccot}\ )\ -\quad^{2}\ +\quad^{-1}]\ =\mathbf{0.}$

Particular solution: $_0=\exp\ -\dfrac{a}{+1}\quad^{+1}\ $.

69. $+(\quad+\ )(\mathbf{arccot}\ )\quad-\ (\mathbf{arccot}\ )\quad=\mathbf{0.}$

Particular solution: $_0=a\ +\ $.

70. $+\quad(\mathbf{arccot}\ )\quad-\quad^{-1}(\mathbf{arccot}\ )\quad=\mathbf{0.}$

Particular solution: $_0=\ $.

71. $+\quad\mathbf{arccot}\quad-[\ (\quad+1)\,\mathbf{arccot}\quad+\ (\quad+2)]\ =\mathbf{0.}$

Particular solution: $_0=\ e^{b}\ $.

72. $+[\ (\quad+1)\,\mathbf{arccot}\quad+\quad-1]\quad+\quad^{2}\ \mathbf{arccot}\quad=\mathbf{0.}$

Particular solution: $_0=(\quad+1)e^{-b}\ $.

73. $+[(\quad^{2}+\quad)\,\mathbf{arccot}\quad+2]\quad+\ \mathbf{arccot}\quad=\mathbf{0.}$

Particular solution: $_0=a+\quad$.

74. $+[\quad(\mathbf{arccot}\ )\ +\ ]\quad+\ (\ -1)(\mathbf{arccot}\ )\quad=\mathbf{0.}$

Particular solution: $_0=\ ^{1-b}$.

75. $+(\quad^{+1}+\ \mathbf{arccot}\ )\quad+\quad(\ \mathbf{arccot}\quad+\ )\ =\mathbf{0.}$

Particular solution: $_0=\exp\ -\dfrac{a}{+1}\quad^{+1}\ $.

76. $+(\quad+\quad\mathbf{arccot}\ )\quad+[\ (\quad-1)\,\mathbf{arccot}\quad+\quad^{-1}]\ =\mathbf{0.}$

Particular solution: $_0=\ \exp(-a\quad)$.

77. $^{2}\quad+\quad\mathbf{arccot}\quad+\ (\ \mathbf{arccot}\ -\ -1)\ =\mathbf{0.}$

Particular solution: $_0=\ ^{-}\ $.

78. $^{2}\quad+\ (\ \mathbf{arccot}\quad+2)\quad+[\ (\quad+1)\,\mathbf{arccot}\quad-\ ^{2}\ ^{2}]\ =\mathbf{0.}$

Particular solution: $_0=\ ^{-1}e^{-}\ $.

79. $(\quad^{2}+\ )\quad+\ (\quad^{2}+\ )(\mathbf{arccot}\ )\quad-2\ [\quad(\mathbf{arccot}\ )\ +1]\ =\mathbf{0.}$

Particular solution: $_0=a\ ^{2}+\ $.

80. $^{4}\quad+\quad^{2}\,\mathbf{arccot}\quad+[\ (\ -\ )\,\mathbf{arccot}\quad-\ ^{2}]\ =\mathbf{0.}$

Particular solution: $_0=\ \exp(\quad)$.

81. $(\ {}^2+1)^2\qquad +[\ (\mathrm{arccot}\ )^2+\ \ \mathrm{arccot}\ \ +\ ]\ \ =0.$

The transformation $\xi=\mathrm{arccot}\ $, $\ =\dfrac{}{\ ^2+1}$ leads to an equation of the form 2.1.2.6: $''\ +(a\xi^2+\ \xi+\ \ +1)\ \ =0.$

82. $(\ {}^2+1)^2\qquad +[\ (\mathrm{arccot}\ \ )\ \ -1]\ \ =0.$

The transformation $\xi=\mathrm{arccot}\ $, $\ =\dfrac{}{\ ^2+1}$ leads to an equation of the form 2.1.2.7: $''\ +\ \xi\ \ \ =0.$

83. $(\ \ {}^2+\ \ )^2\qquad +(\ \ \ +\ \ )(\mathrm{arccot}\ \ )\qquad -\ (\mathrm{arccot}\ \ )\qquad =0.$

Particular solution: $_0=\ \ \ +\ \ .$

84. $(\ {}^2+\ \ )^2\qquad +\ (\ {}^2+\ \ )(\mathrm{arccot}\ \ )\qquad -[\ \ \ (\mathrm{arccot}\ \ )\ \ +\ \ ]\ \ =0.$

Particular solution: $_0=\ \sqrt{\ ^2+a}.$

85. $(\ \ {}^2+\ \ )^2\qquad +\ (\ \ {}^2+\ \ )(\mathrm{arccot}\ \ )\qquad +[\ (\mathrm{arccot}\ \ )\ \ -2\ \ \ -1]\ \ =0.$

Particular solution: $_0=\exp\left(-\dfrac{}{a\ \ ^2+\ }\right).$

2.1.8. Equations Containing Combinations of Exponential, Logarithmic, Trigonometric, and Other Functions

1. $\ \ +\ \ {}^{\lambda}\ \ \ +\ [\ +\ \ {}^{\lambda}\ \tan(\ \ )]\ \ =0.$

Particular solution: $_0=\cos(\ \ ).$

2. $\ \ +\ \ {}^{\lambda}\ \ \ +\ [\ -\ \ {}^{\lambda}\ \cot(\ \ )]\ \ =0.$

Particular solution: $_0=\sin(\ \ ).$

3. $\ \ +\ \ \cosh\ (\ \ )\ \ \ +\ [\ +\ \ \cosh\ (\ \ )\tan(\ \ )]\ \ =0.$

Particular solution: $_0=\cos(\ \ ).$

4. $\ \ +\ \ \cosh\ (\ \ )\ \ \ +\ [\ -\ \ \cosh\ (\ \ )\cot(\ \ )]\ \ =0.$

Particular solution: $_0=\sin(\ \ ).$

5. $\ \ +\ \ \cosh\ (\ \ )\ \ \ +\ \ {}^{\lambda}\ [\ \cosh\ (\ \ )-\ \ {}^{\lambda}\ +\ \ ]\ \ =0.$

Particular solution: $_0=\exp\left(-\dfrac{}{\lambda}e^{\lambda}\ \right).$

6. $\ \ +\ \ \sinh\ (\ \ )\ \ \ +\ [\ +\ \ \sinh\ (\ \ )\tan(\ \ )]\ \ =0.$

Particular solution: $_0=\cos(\ \ ).$

7. $\ \ +\ \ \sinh\ (\ \ )\ \ \ +\ [\ -\ \ \sinh\ (\ \ )\cot(\ \ )]\ \ =0.$

Particular solution: $_0=\sin(\ \ ).$

8. $\ \ +\ \ \sinh\ (\ \ )\ \ \ +\ \ {}^{\lambda}\ [\ \sinh\ (\ \ )-\ \ {}^{\lambda}\ +\ \ ]\ \ =0.$

Particular solution: $_0=\exp\left(-\dfrac{}{\lambda}e^{\lambda}\ \right).$

9. $\ \ +\ \ \tanh\ (\ \ )\ \ \ +\ [\ +\ \ \tanh\ (\ \ )\tan(\ \ )]\ \ =0.$

Particular solution: $_0=\cos(\ \ ).$

10. $+ \ \tanh(\ ) \ + [\ - \ \tanh(\ )\cot(\ )] \ = 0.$

Particular solution: $_0 = \sin(\ ).$

11. $+ \ \tanh(\ ) \ + \ ^{\lambda} [\ \tanh(\ ) - \ ^{\lambda} + \ ] \ = 0.$

Particular solution: $_0 = \exp\ -\frac{}{\lambda}e^{\lambda}\ .$

12. $+ \ \coth(\ ) \ + [\ + \ \coth(\ )\tan(\ )] \ = 0.$

Particular solution: $_0 = \cos(\ ).$

13. $+ \ \coth(\ ) \ + [\ - \ \coth(\ )\cot(\ )] \ = 0.$

Particular solution: $_0 = \sin(\ ).$

14. $+ \ \coth(\ ) \ + \ ^{\lambda} [\ \coth(\ ) - \ ^{\lambda} + \ ] \ = 0.$

Particular solution: $_0 = \exp\ -\frac{}{\lambda}e^{\lambda}\ .$

15. $+ \ \ln(\ ) \ + [\ + \ \ln(\ )\tan(\ )] \ = 0.$

Particular solution: $_0 = \cos(\ ).$

16. $+ \ \ln(\ ) \ + [\ - \ \ln(\ )\cot(\ )] \ = 0.$

Particular solution: $_0 = \sin(\ ).$

17. $+ \ \ln(\ ) \ + \ ^{\lambda} [\ \ln(\ ) - \ ^{\lambda} + \ ] \ = 0.$

Particular solution: $_0 = \exp\ -\frac{}{\lambda}e^{\lambda}\ .$

18. $+ \ \cos(\ ) \ + \ ^{\lambda} [\ \cos(\ ) - \ ^{\lambda} + \ ] \ = 0.$

Particular solution: $_0 = \exp\ -\frac{}{\lambda}e^{\lambda}\ .$

19. $+ \ \sin(\ ) \ + \ ^{\lambda} [\ \sin(\ ) - \ ^{\lambda} + \ ] \ = 0.$

Particular solution: $_0 = \exp\ -\frac{}{\lambda}e^{\lambda}\ .$

20. $+ \ \tan(\ ) \ + \ ^{\lambda} [\ \tan(\ ) - \ ^{\lambda} + \ ] \ = 0.$

Particular solution: $_0 = \exp\ -\frac{}{\lambda}e^{\lambda}\ .$

21. $+ \ \cot(\ ) \ + \ ^{\lambda} [\ \cot(\ ) - \ ^{\lambda} + \ ] \ = 0.$

Particular solution: $_0 = \exp\ -\frac{}{\lambda}e^{\lambda}\ .$

22. $+ (\ ^{\lambda} + \ \ln\ ) \ + \ ^{\lambda} (\ \ln \ + \ ) \ = 0.$

Particular solution: $_0 = \exp\ -\frac{a}{\lambda}e^{\lambda}\ .$

23. $+ (\ ^{\lambda} + \ \cos\ ) \ + (\ ^{\lambda} \ \cos \ - \sin\ ) \ = 0.$

Particular solution: $_0 = \exp(-\ \sin\ ).$

24. $+ (\ ^{\lambda} + \ \cos\ ) \ + \ ^{\lambda} (\ \cos \ + \ ) \ = 0.$

Particular solution: $_0 = \exp\ -\frac{a}{\lambda}e^{\lambda}\ .$

25. $+(\ {}^{\lambda} + \cos\ ) + \cos^{-1}(\ {}^{\lambda}\cos - \sin\ ) = 0.$

Particular solution: $_0 = \exp\ -\ \cos\ .$

26. $+(\ {}^{\lambda} + \sin\ ) + (\ {}^{\lambda}\sin + \cos\ ) = 0.$

Particular solution: $_0 = \exp(\ \cos\ ).$

27. $+(\ {}^{\lambda} + \sin\ ) + {}^{\lambda}(\ \sin + \ ) = 0.$

Particular solution: $_0 = \exp\ -\frac{a}{\lambda}e^{\lambda}\ .$

28. $+(\ {}^{\lambda} + \sin\ ) + \sin^{-1}(\ {}^{\lambda}\sin + \cos\ ) = 0.$

Particular solution: $_0 = \exp\ -\ \sin\ .$

29. $+(\ {}^{\lambda} + \tan\ ) + (\ +1)(\ {}^{\lambda}\tan + 1) = 0.$

Particular solution: $_0 = \cos^{b+1}\ .$

30. $+(\ {}^{\lambda} + \tan\ ) + {}^{\lambda}(\ \tan + \ ) = 0.$

Particular solution: $_0 = \exp\ -\frac{a}{\lambda}e^{\lambda}\ .$

31. $+(\ {}^{\lambda} + \cot\ ) + (\ -1)(\ {}^{\lambda}\cot - 1) = 0.$

Particular solution: $_0 = \sin^{1-b}\ .$

32. $+(\ {}^{\lambda} + \cot\ ) + {}^{\lambda}(\ \cot + \ ) = 0.$

Particular solution: $_0 = \exp\ -\frac{a}{\lambda}e^{\lambda}\ .$

33. $+(\ \cosh + \cos\ ) + (\ \cosh\ \cos - \sin\ ) = 0.$

Particular solution: $_0 = \exp(-\ \sin\ ).$

34. $+(\ \cosh + \cos\ ) + \cos^{-1}(\ \cosh\ \cos - \sin\ ) = 0.$

Particular solution: $_0 = \exp\ -\ \cos\ .$

35. $+(\ \cosh + \sin\ ) + (\ \cosh\ \sin + \cos\ ) = 0.$

Particular solution: $_0 = \exp(\ \cos\ ).$

36. $+(\ \cosh + \sin\ ) + \sin^{-1}(\ \cosh\ \sin + \cos\ ) = 0.$

Particular solution: $_0 = \exp\ -\ \sin\ .$

37. $+(\ \cosh + \tan\ ) + (\ +1)(\ \cosh\ \tan + 1) = 0.$

Particular solution: $_0 = \cos^{b+1}\ .$

38. $+(\ \cosh + \cot\ ) + (\ -1)(\ \cosh\ \cot - 1) = 0.$

Particular solution: $_0 = \sin^{1-b}\ .$

39. $+(\ \sinh + \cos\ ) + (\ \sinh\ \cos - \sin\ ) = 0.$

Particular solution: $_0 = \exp(-\ \sin\ ).$

40. $+ (\sinh \quad + \cos \quad) \quad + \cos^{-1} (\sinh \quad \cos - \quad \sin) = 0.$

Particular solution: $_0 = \exp - \quad \cos \quad .$

41. $+ (\sinh \quad + \sin) \quad + (\sinh \quad \sin \quad + \cos) = 0.$

Particular solution: $_0 = \exp(\cos).$

42. $+ (\sinh \quad + \sin \quad) \quad + \sin^{-1} (\sinh \quad \sin + \quad \cos) = 0.$

Particular solution: $_0 = \exp - \quad \sin \quad .$

43. $+ (\sinh \quad + \tan) \quad + (+ 1)(\sinh \quad \tan \quad + 1) = 0.$

Particular solution: $_0 = \cos^{b+1} \quad .$

44. $+ (\sinh \quad + \cot) \quad + (- 1)(\sinh \quad \cot \quad - 1) = 0.$

Particular solution: $_0 = \sin^{1-b} \quad .$

45. $+ (\tanh \quad + \cos) \quad + (\tanh \quad \cos \quad - \sin) = 0.$

Particular solution: $_0 = \exp(- \sin).$

46. $+ (\tanh \quad + \cos \quad) \quad + \cos^{-1} (\tanh \quad \cos - \quad \sin) = 0.$

Particular solution: $_0 = \exp - \quad \cos \quad .$

47. $+ (\tanh \quad + \sin) \quad + (\tanh \quad \sin \quad + \cos) = 0.$

Particular solution: $_0 = \exp(\cos).$

48. $+ (\tanh \quad + \sin \quad) \quad + \sin^{-1} (\tanh \quad \sin + \quad \cos) = 0.$

Particular solution: $_0 = \exp - \quad \sin \quad .$

49. $+ (\tanh \quad + \tan) \quad + (+ 1)(\tanh \quad \tan \quad + 1) = 0.$

Particular solution: $_0 = \cos^{b+1} \quad .$

50. $+ (\tanh \quad + \cot) \quad + (- 1)(\tanh \quad \cot \quad - 1) = 0.$

Particular solution: $_0 = \sin^{1-b} \quad .$

51. $+ (\coth \quad + \cos) \quad + (\coth \quad \cos \quad - \sin) = 0.$

Particular solution: $_0 = \exp(- \sin).$

52. $+ (\coth \quad + \cos \quad) \quad + \cos^{-1} (\coth \quad \cos - \quad \sin) = 0.$

Particular solution: $_0 = \exp - \quad \cos \quad .$

53. $+ (\coth \quad + \sin) \quad + (\coth \quad \sin \quad + \cos) = 0.$

Particular solution: $_0 = \exp(\cos).$

54. $+ (\coth \quad + \sin \quad) \quad + \sin^{-1} (\coth \quad \sin + \quad \cos) = 0.$

Particular solution: $_0 = \exp - \quad \sin \quad .$

55. $+ (\coth \quad + \tan) \quad + (+ 1)(\coth \quad \tan \quad + 1) = 0.$

Particular solution: $_0 = \cos^{b+1} \quad .$

56. + (coth + cot) + (− 1)(coth cot − 1) = 0.

Particular solution: $_0 = \sin^{1-b}$.

57. + (ln + cos) + (ln cos − sin) = 0.

Particular solution: $_0 = \exp(-\ \sin\ )$.

58. + (ln + cos) + cos $^{-1}$ (ln cos − sin) = 0.

Particular solution: $_0 = \exp\ -\ \cos$.

59. + (ln + sin) + (ln sin + cos) = 0.

Particular solution: $_0 = \exp(\ \cos\ )$.

60. + (ln + sin) + sin $^{-1}$ (ln sin + cos) = 0.

Particular solution: $_0 = \exp\ -\ \sin$.

61. + (ln + tan) + (+ 1)(ln tan + 1) = 0.

Particular solution: $_0 = \cos^{b+1}$.

62. + (ln + cot) + (− 1)(ln cot − 1) = 0.

Particular solution: $_0 = \sin^{1-b}$.

63. + λ cos( ) + [+ λ sin( )] = 0.

Particular solution: $_0 = \cos(\ )$.

64. + λ sin( ) + [− λ cos( )] = 0.

Particular solution: $_0 = \sin(\ )$.

65. + cosh( ) ln ( ) − [+ sinh( ) ln ( )] = 0.

Particular solution: $_0 = \cosh(\ )$.

66. + cosh( ) cos ( ) − [+ sinh( ) cos ( )] = 0.

Particular solution: $_0 = \cosh(\ )$.

67. + cosh ( ) cos( ) + [+ cosh ( ) sin( )] = 0.

Particular solution: $_0 = \cos(\ )$.

68. + cosh( ) sin ( ) − [+ sinh( ) sin ( )] = 0.

Particular solution: $_0 = \cosh(\ )$.

69. + cosh ( ) sin( ) + [− cosh ( ) cos( )] = 0.

Particular solution: $_0 = \sin(\ )$.

70. + cosh( ) tan ( ) − [+ sinh( ) tan ( )] = 0.

Particular solution: $_0 = \cosh(\ )$.

71. + cosh( ) cot ( ) − [+ sinh( ) cot ( )] = 0.

Particular solution: $_0 = \cosh(\ )$.

72. + sinh( ) ln ( ) − [+ cosh( ) ln ( )] = 0.

Particular solution: $_0$ = sinh( ).

73. + sinh( ) cos ( ) − [+ cosh( ) cos ( )] = 0.

Particular solution: $_0$ = sinh( ).

74. + sinh ( ) cos( ) + [+ sinh ( ) sin( )] = 0.

Particular solution: $_0$ = cos( ).

75. + sinh( ) sin ( ) − [+ cosh( ) sin ( )] = 0.

Particular solution: $_0$ = sinh( ).

76. + sinh ( ) sin( ) + [− sinh ( ) cos( )] = 0.

Particular solution: $_0$ = sin( ).

77. + sinh( ) tan ( ) − [+ cosh( ) tan ( )] = 0.

Particular solution: $_0$ = sinh( ).

78. + sinh( ) cot ( ) − [+ cosh( ) cot ( )] = 0.

Particular solution: $_0$ = sinh( ).

79. + tanh ( ) cos( ) + [+ tanh ( ) sin( )] = 0.

Particular solution: $_0$ = cos( ).

80. + tanh ( ) sin( ) + [− tanh ( ) cos( )] = 0.

Particular solution: $_0$ = sin( ).

81. + coth ( ) cos( ) + [+ coth ( ) sin( )] = 0.

Particular solution: $_0$ = cos( ).

82. + coth ( ) sin( ) + [− coth ( ) cos( )] = 0.

Particular solution: $_0$ = sin( ).

83. + ln ( ) cos( ) + [+ ln ( ) sin( )] = 0.

Particular solution: $_0$ = cos( ).

84. + ln ( ) sin( ) + [− ln ( ) cos( )] = 0.

Particular solution: $_0$ = sin( ).

85. + (+ $^{2\lambda}$) ln ( ) + [(− $^{2\lambda}$) ln ( ) −] = 0.

Particular solution: $_0 = e^{\lambda} + ae^{-\lambda}$.

86. + (+ $^{2\lambda}$) cos ( ) + [(− $^{2\lambda}$) cos ( ) −] = 0.

Particular solution: $_0 = e^{\lambda} + ae^{-\lambda}$.

87. + (+ $^{2\lambda}$) sin ( ) + [(− $^{2\lambda}$) sin ( ) −] = 0.

Particular solution: $_0 = e^{\lambda} + ae^{-\lambda}$.

88. + (+ $^{2\lambda}$) tan ( ) + [(− $^{2\lambda}$) tan ( ) −] = 0.

Particular solution: $_0 = e^{\lambda} + ae^{-\lambda}$.

89. $+(\ \ +\ \ ^{2\lambda})\cot(\ \ )\ \ +\ [(\ \ -\ \ ^{2\lambda})\cot(\ \ )-\ \ ]\ \ = 0.$

Particular solution: $_0 = e^{\lambda} + ae^{-\lambda}$.

90. $+(\ \ \mathrm{sn}^2\ \ +\ \ )\ \ = 0.$

The Lamé equation in the form of Jacobi, sn is the Jacobi elliptic function. See the books by Whittaker & Watson (1952), Bateman & Erdélyi (1955, Vol. 3), and Kamke (1977) for information on this equation.

91. $+[\ \ (\ \ )+B]\ \ = 0.$

The Lamé equation in the form of Weierstrass, $\wp(\ \ )$ is the Weierstrass function. See the books by Whittaker & Watson (1952), Bateman & Erdélyi (1955, Vol. 3), and Kamke (1977) for information on this equation.

92. $+(\ \ \ln\ \ +\ \ ^{\lambda})\ \ +\ (\ \ ^{\lambda}\ln\ \ +1)\ \ = 0.$

Particular solution: $_0 = e^{-}$.

93. $+(1-\ \ ^{\lambda}\ln\ \ )\ \ +\ \ ^{\lambda}\ \ = 0.$

Particular solution: $_0 = \ln$.

94. $+(\ \ \ln\ \ +\ \cosh\ \ )\ \ +\ (\ \cosh\ \ \ln\ \ +1)\ \ = 0.$

Particular solution: $_0 = e^{-}$.

95. $+(1-\ \ \cosh\ \ \ln\ \ )\ \ +\ \cosh\ \ \ = 0.$

Particular solution: $_0 = \ln$.

96. $+(\ \ \ln\ \ +\ \sinh\ \ )\ \ +\ (\ \sinh\ \ \ln\ \ +1)\ \ = 0.$

Particular solution: $_0 = e^{-}$.

97. $+(1-\ \ \sinh\ \ \ln\ \ )\ \ +\ \sinh\ \ \ = 0.$

Particular solution: $_0 = \ln$.

98. $+(\ \ \ln\ \ +\ \tanh\ \ )\ \ +\ (\ \tanh\ \ \ln\ \ +1)\ \ = 0.$

Particular solution: $_0 = e^{-}$.

99. $+(1-\ \ \tanh\ \ \ln\ \ )\ \ +\ \tanh\ \ \ = 0.$

Particular solution: $_0 = \ln$.

100. $+(\ \ \ln\ \ +\ \coth\ \ )\ \ +\ (\ \coth\ \ \ln\ \ +1)\ \ = 0.$

Particular solution: $_0 = e^{-}$.

101. $+(1-\ \ \coth\ \ \ln\ \ )\ \ +\ \coth\ \ \ = 0.$

Particular solution: $_0 = \ln$.

102. $+(\ \ \ln\ \ +\ \cos\ \ )\ \ +\ (\ \cos\ \ \ln\ \ +1)\ \ = 0.$

Particular solution: $_0 = e^{-}$.

103. $+(1-\ \ \cos\ \ \ln\ \ )\ \ +\ \cos\ \ \ = 0.$

Particular solution: $_0 = \ln$.

104. $+(\ \ \ln\ \ +\ \sin\ \ )\ \ +\ (\ \sin\ \ \ln\ \ +1)\ \ = 0.$

Particular solution: $_0 = e^{-}$.

105. **+ (1 − sin ln) + sin = 0.**

Particular solution: $_0 = \ln$.

106. **+ (ln + tan) + (tan ln + 1) = 0.**

Particular solution: $_0 = e^{-}$.

107. **+ (1 − tan ln) + tan = 0.**

Particular solution: $_0 = \ln$.

108. **+ (ln + cot) + (cot ln + 1) = 0.**

Particular solution: $_0 = e^{-}$.

109. **+ (1 − cot ln) + cot = 0.**

Particular solution: $_0 = \ln$.

110. $^{\mathbf{2}}$ **+ (ln +** $^{\lambda}$ **) + (** $^{\lambda}$ **ln − ln + 1) = 0.**

Particular solution: $_0 = \exp(-\frac{1}{2} a \ln^2 \;)$.

111. $^{\mathbf{2}}$ **+ (ln + cosh) + (cosh ln − ln + 1) = 0.**

Particular solution: $_0 = \exp(-\frac{1}{2} a \ln^2 \;)$.

112. $^{\mathbf{2}}$ **+ (ln + sinh) + (sinh ln − ln + 1) = 0.**

Particular solution: $_0 = \exp(-\frac{1}{2} a \ln^2 \;)$.

113. $^{\mathbf{2}}$ **+ (ln + tanh) + (tanh ln − ln + 1) = 0.**

Particular solution: $_0 = \exp(-\frac{1}{2} a \ln^2 \;)$.

114. $^{\mathbf{2}}$ **+ (ln + coth) + (coth ln − ln + 1) = 0.**

Particular solution: $_0 = \exp(-\frac{1}{2} a \ln^2 \;)$.

115. $^{\mathbf{2}}$ **+ (ln + cos) + (cos ln − ln + 1) = 0.**

Particular solution: $_0 = \exp(-\frac{1}{2} a \ln^2 \;)$.

116. $^{\mathbf{2}}$ **+ (ln + sin) + (sin ln − ln + 1) = 0.**

Particular solution: $_0 = \exp(-\frac{1}{2} a \ln^2 \;)$.

117. $^{\mathbf{2}}$ **+ (ln + tan) + (tan ln − ln + 1) = 0.**

Particular solution: $_0 = \exp(-\frac{1}{2} a \ln^2 \;)$.

118. $^{\mathbf{2}}$ **+ (ln + cot) + (cot ln − ln + 1) = 0.**

Particular solution: $_0 = \exp(-\frac{1}{2} a \ln^2 \;)$.

119. **sin$^{\mathbf{2}}$ + sin (+** $^{\lambda}$ **) + (** $^{\lambda}$ **− cos) = 0.**

Particular solution: $_0 = \cot\big(\frac{1}{2}$.

120. **sin$^{\mathbf{2}}$ + sin (+ cosh) + (cosh − cos) = 0.**

Particular solution: $_0 = \cot\big(\frac{1}{2}$.

121. $\sin^2 \quad + \sin \quad (\quad + \quad \sinh \quad) \quad + \quad (\quad \sinh \quad - \cos \quad) \quad = 0.$

Particular solution: $\quad_0 = \cot\bigl(\tfrac{1}{2} \quad .$

122. $\sin^2 \quad + \sin \quad (\quad + \quad \tanh \quad) \quad + \quad (\quad \tanh \quad - \cos \quad) \quad = 0.$

Particular solution: $\quad_0 = \cot\bigl(\tfrac{1}{2} \quad .$

123. $\sin^2 \quad + \sin \quad (\quad + \quad \coth \quad) \quad + \quad (\quad \coth \quad - \cos \quad) \quad = 0.$

Particular solution: $\quad_0 = \cot\bigl(\tfrac{1}{2} \quad .$

124. $\sin^2 \quad + \sin \quad (\quad + \quad \ln \quad) \quad + \quad (\quad \ln \quad - \cos \quad) \quad = 0.$

Particular solution: $\quad_0 = \cot\bigl(\tfrac{1}{2} \quad .$

125. $\cos^2 \quad + \cos \quad (\quad + \quad ^{\lambda} \;) \quad + \quad (\quad ^{\lambda} \; + \sin \quad) \quad = 0.$

Particular solution: $\quad_0 = \cot\bigl(\tfrac{1}{2} \quad + \tfrac{1}{4} \quad .$

126. $\cos^2 \quad + \cos \quad (\quad + \quad \cosh \quad) \quad + \quad (\quad \cosh \quad + \sin \quad) \quad = 0.$

Particular solution: $\quad_0 = \cot\bigl(\tfrac{1}{2} \quad + \tfrac{1}{4} \quad .$

127. $\cos^2 \quad + \cos \quad (\quad + \quad \sinh \quad) \quad + \quad (\quad \sinh \quad + \sin \quad) \quad = 0.$

Particular solution: $\quad_0 = \cot\bigl(\tfrac{1}{2} \quad + \tfrac{1}{4} \quad .$

128. $\cos^2 \quad + \cos \quad (\quad + \quad \tanh \quad) \quad + \quad (\quad \tanh \quad + \sin \quad) \quad = 0.$

Particular solution: $\quad_0 = \cot\bigl(\tfrac{1}{2} \quad + \tfrac{1}{4} \quad .$

129. $\cos^2 \quad + \cos \quad (\quad + \quad \coth \quad) \quad + \quad (\quad \coth \quad + \sin \quad) \quad = 0.$

Particular solution: $\quad_0 = \cot\bigl(\tfrac{1}{2} \quad + \tfrac{1}{4} \quad .$

130. $\cos^2 \quad + \cos \quad (\quad + \quad \ln \quad) \quad + \quad (\quad \ln \quad + \sin \quad) \quad = 0.$

Particular solution: $\quad_0 = \cot\bigl(\tfrac{1}{2} \quad + \tfrac{1}{4} \quad .$

2.1.9. Equations with Arbitrary Functions

Notation: $\quad = \quad (\quad)$ *and* $g = g(\quad)$ *are arbitrary functions;* $a, \; , \; , \; , \; , \; , k, \lambda, \; , \beta,$ *and* $\quad$ *are arbitrary parameters.*

2.1.9-1. Equations containing arbitrary functions (but not containing their derivatives).

1. $\quad + \quad = f.$

Equation of forced oscillations without friction.

Solution:

$$= \begin{cases} \;_1 \cos(k \quad) + \;_2 \sin(k \quad) + k^{-1} \int_0 \quad (\xi) \sin[k(\quad - \xi)] \; \xi & \text{if } a = k^2 > 0, \\ \;_1 \cosh(k \quad) + \;_2 \sinh(k \quad) + k^{-1} \int_0 \quad (\xi) \sinh[k(\quad - \xi)] \; \xi & \text{if } a = -k^2 < 0, \\ \;_1 \quad + \;_2 + \int_0 (\quad - \xi) \; (\xi) \; \xi & \text{if } a = 0, \end{cases}$$

where $\quad_0$ is an arbitrary number.

2. $+ \quad + \quad = f.$

Equation of forced oscillations with friction. The substitution $= \quad \exp(-\tfrac{1}{2}a \;)$ leads to an equation of the form 2.1.9.1: $'' \; + (\; - \tfrac{1}{4}a^2) \; = \; \exp(\tfrac{1}{2}a \;)$.

3. $+ f \quad = \; .$

Solution: $= \; _1 + \; e^{-F} \; _2 + \; e^F g$, where $=$.

4. $+ f \quad + \; (f - \;) \; = 0.$

Particular solution: $_0 = e^{-}$.

5. $+ f \quad + \; (\; f - \; ^2 + \; ^{-1}) \; = 0.$

Particular solution: $_0 = \exp\left(-\dfrac{a}{+1} \; ^{+1}\right)$.

6. $+ \; f \quad - f \quad = 0.$

Particular solution: $_0 = \;$.

7. $+ (f + \quad + \;) \quad + [(\quad + \;)f + \quad ^{-1}] \; = 0.$

Particular solution: $_0 = \exp\left(-\dfrac{a}{+1} \; ^{+1} - \right)$.

8. $+ \; f \quad - [(\quad + 1)f + \; (\quad + 2)] \; = 0.$

Particular solution: $_0 = \; e$.

9. $+ (\; f + \;) \quad + (\; - 1)f \quad = 0.$

Particular solution: $_0 = \; ^{1-}$.

10. $+ [(\quad + 1)f + \quad - 1] \quad + \; ^2 \; f \quad = 0.$

Particular solution: $_0 = (a \; + 1)e^{-}$.

11. $+ [(\; ^2 + \;)f + 2] \quad + \; f \quad = 0.$

Particular solution: $_0 = a + \;$.

12. $+ (f + \quad ^{+1}) \quad + \quad (f + \;) \; = 0.$

Particular solution: $_0 = \exp\left(-\dfrac{a}{+1} \; ^{+1}\right)$.

13. $+ (\; f + \quad) \quad + [(\quad - 1)f + \quad ^{-1}] \; = 0.$

Particular solution: $_0 = \; \exp(-a \quad)$.

14. $+ [(\quad + 1)f + \quad + 1 - 2 \;] \quad + \; ^2 \quad ^2 \; ^{-1} f \quad = 0.$

Particular solution: $_0 = (a \quad + 1)\exp(-a \quad)$.

15. $^2 \quad + \quad + \quad = f.$

The nonhomogeneous Euler equation. The substitution $= e$ leads to an equation of the form 2.1.9.2: $'' + (\; - 1) \; ' + \beta \; = \; (e \;)$.

16. $^2 \quad + \quad + (\; ^2 - \; ^2) \; = f.$

The nonhomogeneous Bessel equation. The general solution is expressed in terms of Bessel functions: $= \; _1 \; (\;) + \; _2 \; (\;) + \tfrac{1}{2} \quad (\;) \quad (\;) \; (\;) \quad - \tfrac{1}{2} \quad (\;) \quad (\;) \; (\;)$.

17. $^{2} \quad + \;\; f \quad + \;\; (f - \quad - 1) \quad = 0.$

Particular solution: $_{0} = \;{}^{-}\;.$

18. $^{2} \quad + \;\; (f + 2\;\;) \quad + [(\quad + \;\;)f - \;{}^{2}\;{}^{2} + \;\;(\;\; - 1)] \quad = 0.$

Particular solution: $_{0} = \;{}^{-}\; e^{-b}\;.$

19. $^{2} \quad + \;\; f \quad + [(\quad {}^{2}\;\;{}^{+1} + \;\;)f - \;{}^{2}\;\;{}^{4}\;\;{}^{+2} - \;\;{}^{2} - \;\;] \quad = 0.$

Particular solution: $_{0} = \;{}^{-}\;\exp\;-\dfrac{a}{2\;\;+1}\;\;{}^{2\;\;+1}\;.$

20. $(\quad {}^{2} + \quad + \;\;) \quad + (\;\; + \;\;)f \quad - f \quad = 0.$

Particular solution: $_{0} = \quad + k.$

21. $^{4} \quad + \;{}^{2}f \quad + [(\;\; - \;\;)f - \;{}^{2}] \quad = 0.$

Particular solution: $_{0} = \;\;\exp(\lambda \quad).$

22. $^{2}(\quad {}^{2} + \;\;) \quad + \;\;(\quad {}^{2} + \;\;)f \quad - [(\quad {}^{2} - \;\;)f + 2\;\;] \quad = 0.$

Particular solution: $_{0} = a \quad + \quad .$

23. $(\;{}^{2} + \;\;)^{2} \quad + (\;{}^{2} + \;\;)f \quad - (\;\; f + \;\;) \quad = 0.$

Particular solution: $_{0} = \overline{\quad {}^{2} + a}.$

24. $(\;{}^{2} + \;\;)^{2} \quad + (\;{}^{2} + \;\;)f \quad - \quad [\;\; f + (\quad - 1)\;\;{}^{2} + \;\;] \quad = 0.$

Particular solution: $_{0} = (\;{}^{2} + a)\quad {}^{2}.$

25. $(\quad + \;\;) \quad + (\quad + \;\;)f \quad - \quad {}^{-2}(\;\; f + \quad - 1) \quad = 0.$

Particular solution: $_{0} = a \quad + \;\;.$

26. $(\quad + \quad) \quad + (\quad + \quad)f \quad - [(\quad {}^{-1} + \;\;)f + \quad (\;\; - 1)\quad {}^{-2}] \quad = 0.$

Particular solution: $_{0} = a \quad + \quad .$

27. $(\quad + \;\;)^{2} \quad + (\quad + \;\;)f \quad - \quad {}^{-2}(\;\; f + \quad - \;\;) \quad = 0.$

Particular solution: $_{0} = (\quad + a)^{1} \quad .$

28. $(\quad + \;\;)^{2} \quad + (\quad + \;\;)f \quad + (f - \quad {}^{-1} - 1) \quad = 0.$

Particular solution: $_{0} = \exp\;-\;\dfrac{\quad}{a \quad + \quad}\;.$

29. $f(\;\;) \quad + [\quad {}^{2} + (\quad + \;\;) \;\; + \quad] \quad - (\quad + \;\;) \quad = 0.$

Particular solution: $_{0} = \quad + \;\;.$

30. $\quad + f \quad + \quad {}^{\lambda}\;(f - \quad {}^{\lambda} \quad + \;\;) \quad = 0.$

Particular solution: $_{0} = \exp\;-\dfrac{a}{\lambda}e^{\lambda} \quad .$

31. $\quad + (f + \quad {}^{\lambda}\;) \quad + \quad {}^{\lambda}\;(f + \;\;) \quad = 0.$

Particular solution: $_{0} = \exp\;-\dfrac{a}{\lambda}e^{\lambda} \quad .$

32. $\quad + (\;\; + \quad {}^{2\lambda}\;)f \quad + \;\;[(\;\; - \quad {}^{2\lambda}\;)f - \;\;] \quad = 0.$

Particular solution: $_{0} = \;\; e^{\lambda} \quad + ae^{-\lambda}\;.$

33. $(\quad{}^{\lambda} + \;)^2 \quad + (\quad{}^{\lambda} + \;)f \quad + \quad{}^{\lambda}(f - \quad{}^{\lambda} + \quad) \; = 0.$

Particular solution: $\;{}_0 = \big(ae^{\lambda} \; + \quad{}^{-\frac{c}{\lambda}}.$

34. $\quad + f\,\sinh(\quad) \quad - [\quad + f\cosh(\quad)] \; = 0.$

Particular solution: $\;{}_0 = \sinh(a\;).$

35. $\quad + f\cosh(\quad) \quad - [\quad + f\sinh(\quad)] \; = 0.$

Particular solution: $\;{}_0 = \cosh(a\;).$

36. $\quad + (1 - f \quad \ln \;) \quad + f \quad = 0.$

Particular solution: $\;{}_0 = \ln \; .$

37. $\quad + (f + \quad \ln \;) \quad + \;(f \ln \quad + 1) \; = 0.$

Particular solution: $\;{}_0 = e^{\quad -\quad} .$

38. ${}^{2} \quad + 2\;(\ln \quad + \;)f \quad + \big[\tfrac{1}{4} - (\ln \quad + \quad + 2)f\big] \; = 0.$

Particular solution: $\;{}_0 = \;{}^{-}(\ln \; + a).$

39. ${}^{2} \quad + \;(f + \;\ln \;) \quad + \;(f \ln \; - \ln \; + 1) \; = 0.$

Particular solution: $\;{}_0 = \exp(-\tfrac{1}{2}a\ln^2 \;).$

40. $\quad + f\sin(\quad) \quad + [\quad - f\cos(\quad)] \; = 0.$

Particular solution: $\;{}_0 = \sin(a\;).$

41. $\quad + f\cos(\quad) \quad + [\quad + f\sin(\quad)] \; = 0.$

Particular solution: $\;{}_0 = \cos(a\;).$

42. $\quad + (f + \;\sin \;) \quad + \;(f \sin \; + \cos \;) \; = 0.$

Particular solution: $\;{}_0 = \exp(a\cos \;).$

43. $\quad + (f + \;\cos \;) \quad + \;(f \cos \; - \sin \;) \; = 0.$

Particular solution: $\;{}_0 = \exp(-a\sin \;).$

44. $\quad + (f + \;\cos \quad) \quad + \;\cos \;{}^{-1} \;(f \cos \quad - \quad \sin \;) \; = 0.$

Particular solution: $\;{}_0 = \exp \; -a \quad \cos \qquad .$

45. $\quad + (f + \;\sin \quad) \quad + \;\sin \;{}^{-1} \;(f \sin \quad + \quad \cos \;) \; = 0.$

Particular solution: $\;{}_0 = \exp \; -a \quad \sin \qquad .$

46. $\sin^2 \qquad + \sin \;(f + \;) \quad + \;(f - \cos \;) \; = 0.$

Particular solution: $\;{}_0 = \cot\big(\tfrac{1}{2} \quad .$

47. $\cos^2 \qquad + \cos \;(\; + f) \quad + \;(f + \sin \;) \; = 0.$

Particular solution: $\;{}_0 = \cot\big(\tfrac{1}{2} \; + \tfrac{1}{4} \quad .$

48. $\quad + f \quad + [\quad + f\tan(\quad) + (\; - \;)\tan^2(\quad)] \; = 0.$

Particular solution: $\;{}_0 = [\cos(\lambda\;)]^{\;\lambda}.$

49. $+ (f + \quad \tan \quad) \qquad + (\quad + 1)(f \tan \quad + 1) \quad = 0.$

Particular solution: $_0 = \cos^{\quad +1} \quad .$

50. $+ \tan \quad (f + \quad - 1) \qquad + [(\quad \tan^2 \quad - 1)f + 2 \quad + 2] \quad = 0.$

Particular solution: $_0 = \sin \quad \cos \quad .$

51. $+ f \qquad + \quad [\quad - f \cot(\quad) + (\quad - \quad) \cot^2(\quad)] \quad = 0.$

Particular solution: $_0 = [\sin(\lambda \quad)]^{\quad \lambda}.$

52. $+ (f + \quad \cot \quad) \qquad + (\quad - 1)(f \cot \quad - 1) \quad = 0.$

Particular solution: $_0 = \sin^{1-} \quad .$

2.1.9-2. Equations containing arbitrary functions and their derivatives.

53. $- (f^2 + f \quad) \quad = 0.$

Particular solution: $_0 = \exp \qquad .$

54. $+ f \qquad - [\quad (\quad + 1) f^2 + \quad f \quad] \quad = 0.$

Particular solution: $_0 = \exp \; a \qquad .$

55. $+ 2f \qquad + (f^2 + f \quad) \quad = 0.$

Solution: $= (\quad _2 + \quad _1) \exp \; - \qquad .$

56. $+ (1 - \quad) f \qquad - \quad (f^2 + f \quad) \quad = 0.$

Particular solution: $_0 = \exp \; a \qquad .$

57. $+ f \qquad + (f \quad - \quad ^2 + \quad \quad) \quad = 0.$

Particular solution: $_0 = \exp \; - \quad g \qquad .$

58. $+ 2f \qquad + (f^2 + f \quad + \quad) \quad = 0.$

The substitution $= \exp$ leads to a constant coefficient linear equation: $'' + a \quad = 0.$

59. $+ 2f \qquad + (f^2 + f \quad + \quad ^2 + \quad ^{-1}) \quad = 0.$

The substitution $= \exp$ leads to a linear equation of the form 2.1.2.10: $'' + a(\quad ^2 + \quad ^{-1}) \quad = 0.$

60. $+ (2f + \quad) \qquad + (f^2 + \quad f + f \quad + \quad) \quad = 0.$

The substitution $= \exp$ leads to a constant coefficient linear equation: $'' + a \quad ' + \quad = 0.$

61. $+ (f + \quad) \qquad + (f \quad + f \quad) \quad = 0.$

Particular solution: $_0 = \exp \; - \qquad .$

62. $+ \;\; f \;\; + (f + \;\; f\,) \;\; = 0.$

Particular solution: $\;_0 = \;\; \exp \; - \qquad .$

63. $+ (\;\; f + \;\;) \;\; + (\;\; f + \;\; f\,) \;\; = 0.$

Particular solution: $\;_0 = \exp \; - \qquad .$

64. $(\;\; + \;\;) \;\; + (f + \;\;) \;\; + f \;\; = 0.$

Particular solution: $\;_0 = \exp \; \dfrac{1 - \;\; -}{+ a} \quad .$

65. $^2 \;\; + \;\; f \;\; + [\;\; f \; + \;\; f - \;\; (\;\; + 1)] \;\; = 0.$

Particular solution: $\;_0 = \;\; ^{+1} \exp \; - \;\; ^{-1} \qquad .$

66. $^2 \;\; + 2 \;\; f \;\; + (\;\; f \;\; + f^2 - f + \;\; ^2 + \;\; + \;\;) \;\; = 0.$

The transformation $= \exp \;\; ^{-1}$ leads to an equation of the form 2.1.2.115: $^2 \;\; '' + (a \;\; ^2 + \;\; + \;\;) \;\; = 0.$

67. $^2 \;\; + \;\; (2f + 1) \;\; + (f^2 + \;\; f \;\; + \;\; ^2 - \;\;) \;\; = 0.$

The substitution $= \exp \; - \;\; ^{-1}$ leads to the Bessel equation 2.1.2.126: $^2 \;\; '' + \;\; ' + (\;\; ^2 - a) \;\; = 0.$

68. $^2 \;\; + \;\; (2f + \;\;) \;\; + [f^2 + (\;\; - 1)f + \;\; f \;\; + \;\; + \;\;] \;\; = 0.$

The substitution $= \exp \;\; ^{-1}$ leads to a linear equation of the form 2.1.2.132: $^2 \;\; '' + a \;\; ' + (\;\; + \;\;) \;\; = 0.$

69. $^2 \;\; + 2 \;\; ^2 f \;\; + [\;\; ^2(f \;\; + f^2 + \;\;) + \;\;] \;\; = 0.$

The transformation $= \exp$ leads to a linear equation of the form 2.1.2.115: $^2 \;\; '' + (a \;\; ^2 + \;\;) \;\; = 0.$

70. $^2 \;\; + \;\; (2f + \;\; + \;\;) \;\; + [f^2 + (\;\; + \;\; - 1)f + \;\; f \;\; + \;\; ^2 + \;\; + \gamma] \;\; = 0.$

The substitution $= \exp \;\; ^{-1}$ leads to a linear equation of the form 2.1.2.146: $^2 \;\; '' + (a \;\; + \;\;) \;\; ' + (\;\; ^2 + \beta \;\; + \;\;) \;\; = 0.$

71. $2f \;\; + f \;\; + \;\; = 0.$

The substitution $\xi = \;\; ^{-1} \; ^2$ leads to a constant coefficient linear equation: $2 \;\; '' + a \;\; = 0.$

72. $f \;\; - f \;\; - \;\; f^3 \;\; = 0.$

Solution: $= \;_1 e \;\; + \;_2 e^{-} \;\;$, where $= \overline{a} \qquad .$

73. $f \quad - f \quad - f^{2 \ +1} \ = 0.$

Solution: $= {}_1 e + {}_2 e^{-}$, where $= \overline{}$.

74. $f \quad -(f + f^2) \quad + f^3 \ = 0.$

Solution: $= {}_1 \exp\ \lambda_1 \quad + {}_2 \exp\ \lambda_2$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 - a\lambda + \ = 0$.

75. $f \quad -(f + f) \quad - f^2(+ f) \ = 0.$

Particular solution: ${}_0 = \exp\ -$.

76. $f \quad -(f + 2\ f) \quad + (\ f + {}^2 f - {}^2 f^3) \ = 0.$

Particular solution: ${}_0 = e \quad \exp$.

77. $f^2 \quad + f(f + \ ) \quad + \quad = 0.$

The substitution $\xi = \quad ^{-1}$ leads to a constant coefficient linear equation: $'' + a\ ' + \ = 0$.

78. $f^2 \quad + f(f + 2 + \ ) \quad + (f \quad + {}^2 + \quad + \ ) \ = 0.$

The transformation $\xi = \quad ^{-1}$, $= \exp \quad ^{-1} g$ leads to a constant coefficient linear equation: $'' + a\ ' + \ = 0$.

79. $f \quad -(\ f + f \ ) \quad - f^{2 \ +1 \ 2 \ +1} \ = 0.$

Solution: $= {}_1 e + {}_2 e^{-}$, where $= \overline{\lambda} \quad g^b$.

80. $\quad + 2f \quad + (f^2 + f + \quad {}^{2\lambda} + \quad {}^{\lambda} + \ ) \ = 0.$

The substitution $= \exp$ leads to a linear equation of the form 2.1.3.5: $'' + (ae^{2\lambda} + \ e^{\lambda} + \ ) \ = 0$.

81. $\quad - f \quad + {}^2 \ {}^{2f} \ = 0.$

Solution: $= {}_1 \sin\ a \quad e \quad + {}_2 \cos\ a \quad e$.

82. $\quad - f \quad - {}^2 \ {}^{2f} \ = 0.$

Solution: $= {}_1 \exp\ a \quad e \quad + {}_2 \exp\ -a \quad e$.

83. $f \quad - f \quad = 0.$

Solution: $= {}_1 + {}_2 \quad ^{-2}$.

84. $4f^2 \quad - [2ff \quad - (f\ )^2 + \ ] \ = 0.$

Solution:

$$= \begin{cases} {}_1 \overline{} \exp\ \tfrac{1}{2} \ \overline{a} \quad ^{-1} + {}_2 \overline{} \exp\ -\tfrac{1}{2} \ \overline{a} \quad ^{-1} & \text{if } a > 0, \\ {}_1 \overline{} \cos\ \tfrac{1}{2} \ \overline{|a|} \quad ^{-1} + {}_2 \overline{} \sin\ \tfrac{1}{2} \ \overline{|a|} \quad ^{-1} & \text{if } a < 0, \\ {}_1 \overline{} + {}_2 \overline{} \quad ^{-1} & \text{if } a = 0. \end{cases}$$

85. $-\frac{f}{f} \quad + \quad {}^{2}(f \quad)^{2} f^{2 \quad -2} \quad = 0.$

Solution: $= \quad - \quad {}_{1} \, {}_{\frac{1}{2}} \, \frac{a}{} \quad + \, {}_{2} \, {}_{\frac{1}{2}} \, \frac{a}{}$, where (z) and (z) are the Bessel functions.

86. $+ \quad \frac{f f}{f^{2} +} - \frac{f}{f} \Big) \quad - \frac{{}^{2}(f \quad)^{2}}{f^{2} +} \quad = 0.$

Solution: $= \, {}_{1}(\quad + \quad \overline{\; {}^{2} + a \;} \; {}^{b} + \, {}_{2}(\quad + \quad \overline{\; {}^{2} + a \;} \; {}^{-b}.$

87. $- \quad \frac{f}{f} + (2 \quad - 1) \frac{f}{f} \quad + \quad (\; {}^{2} - \; {}^{2}) \quad \frac{f}{f} \Big)^{2} + (f \quad)^{2} \quad = 0.$

Solution: $= \quad [\; {}_{1} \; {}_{b}(\;) + \; {}_{2} \; {}_{b}(\;)]$, where ${}_{b}(\;)$ and ${}_{b}(\;)$ are the Bessel functions.

88. $+ \quad \frac{1}{2} \frac{f}{f} - \frac{3}{4} \quad \frac{f}{f} \Big)^{2} + \quad \frac{1}{4} - \; {}^{2} \Big) \quad \frac{f}{f} \Big)^{2} + (f \quad)^{2} \quad = 0.$

Solution: $= \quad \overline{\quad\quad}' \, [\; {}_{1} \quad (\;) + \; {}_{2} \quad (\;)]$, where ${}_{b}(\;)$ and ${}_{b}(\;)$ are the Bessel functions.

89. $+ \frac{f}{f} \quad + \quad \frac{3}{4} \quad \frac{f}{f} \Big)^{2} - \frac{1}{2} \frac{f}{f} - \frac{3}{4} \quad \frac{}{} \Big)^{2} + \frac{1}{2} \frac{}{} + \quad \frac{1}{4} - \; {}^{2} \Big) \quad \frac{}{} \Big)^{2} + (\quad)^{2} \quad = 0.$

Solution: $= \quad \overline{\; g \;\; g' \;} \, [\; {}_{1} \quad (g) + \; {}_{2} \quad (g)]$, where ${}_{b}(g)$ and ${}_{b}(g)$ are the Bessel functions.

90. $- \quad 2 \frac{f}{f} + \frac{}{} - \frac{}{} \Big) \quad + \quad \frac{f}{f} \quad 2 \frac{f}{f} + \frac{}{} - \frac{}{} \Big) - \frac{f}{f} - \quad {}^{2} \quad \frac{}{} \Big)^{2} + (\quad)^{2} \quad = 0.$

Solution: $= \quad [\; {}_{1} \quad (g) + \; {}_{2} \quad (g)]$, where ${}_{b}(g)$ and ${}_{b}(g)$ are the Bessel functions.

91. $- \quad \frac{}{} + (2 \quad - 1) \frac{}{} + 2 \frac{}{} \Big) \quad + \quad \frac{}{} \quad \frac{}{} + (2 \quad - 1) \frac{}{} + 2 \frac{}{} \Big) - \frac{}{} + (\quad)^{2} \quad = 0.$

Solution: $= \quad g \quad [\; {}_{1} \quad (g) + \; {}_{2} \quad (g)]$, where ${}_{b}(g)$ and ${}_{b}(g)$ are the Bessel functions.

2.1.10. Some Transformations

Notation: , *g, and* *are arbitrary composite functions of their arguments, which are written in parentheses following the name of a function (the argument is a function of*).

1. $+ \quad {}^{-4} f(1 \quad) \quad = 0.$

The transformation $\xi = 1 \quad , \quad = \quad$ leads to the equation $'' + \quad (\xi) \quad = 0.$

2. $+ (\quad + \quad)^{-4} f \; \frac{+}{+} \quad = 0.$

The transformation $\xi = \frac{a \quad +}{+}, \quad = \frac{}{+}$ leads to a simpler equation: $'' + \Delta^{-2} \quad (\xi) \quad = 0,$ where $\Delta = a \quad - \quad .$

3. ${}^{2} \quad + \big[\; {}^{2} \; f(\quad + \quad) + \tfrac{1}{4} - \tfrac{1}{4} \; {}^{2} \big] \quad = 0.$

The transformation $\xi = a \quad + \, , \quad = \quad \frac{-1}{2}$ leads to a simpler equation: $'' + (a \quad)^{-2} \quad (\xi) \quad = 0.$

4. $\;{}^{2}\quad + \;(\;f + \;)\quad + (\quad + \;)\; = 0, \qquad f = f(\;), \quad = \;(\;).$

The substitution $=$, where k is a root of the quadratic equation $k^2 + (a-1)k + \;= 0$, leads to the equation $''\; + (\quad + a + 2k)\;\;'\; + (g + k\;\;)\;\; = 0$.

5. $(\;)\qquad + Q\;(\;)\quad + \quad{}_{-1}(\;)\; = 0,$

$$(\;) = \sum_{=0}\qquad, \quad Q\;(\;) = \sum_{=0}\qquad, \quad {}_{-1}(\;) = \sum_{=0}^{-1}\qquad.$$

The substitution $=$, where $k = 1 - \;{}_0\; a_0$, leads to an equation of the same form:

$$(\;)\;\;'' + [Q\;(\;) + 2k\quad(\;)]\;\;' + [R\;{}_{-1}(\;) + \quad{}_{-1}(\;)]\;\; = 0,$$

where $\;{}_{-1}(\;) = k\;\;{}^{-1}[Q\;(\;) + (k-1)\quad(\;)]$.

6. $(\;\;-1)\quad{}_{-1}(\;)\qquad + Q\;(\;)\quad + \quad{}_{-1}(\;)\; = 0,$

$$_{-1}(\;) = \sum_{=0}^{-1}\qquad, \quad Q\;(\;) = \sum_{=0}\qquad, \quad {}_{-1}(\;) = \sum_{=0}^{-1}\qquad.$$

The transformation $\xi = \dfrac{}{\;-1}$, $\;= |\;\;-1|^{-}$, where k is a root of the quadratic equation $a\;{}_{-1}k^2 + (\quad - a\;{}_{-1})k + \quad{}_{-1} = 0$, leads to an equation of the same form:

$$\xi(\xi-1)\quad{}_{-1}(\xi)\;\;'' + [2(1-k)\xi\quad{}_{-1}(\xi) - Q\;(\xi)]\;\;' + [k(k-1)\quad{}_{-1}(\xi) + \quad{}_{-1}(\xi)]\;\; = 0,$$

where

$$_{-1}(\xi) = \sum_{=0}^{-1} a\;\;\xi\;\;(\xi-1)^{\;-\;\;-1}, \quad Q\;(\xi) = \sum_{=0}\quad\xi\;\;(\xi-1)^{\;-}\;,$$

$$R\;{}_{-1}(\xi) = \sum_{=0}^{-1}\quad\xi\;\;(\xi-1)^{\;-\;\;-1}, \qquad {}_{-1}(\xi) = \frac{R\;{}_{-1}(\xi) + kQ\;(\xi) + k(k-1)\quad{}_{-1}(\xi)}{\xi - 1}.$$

7. $\qquad + \left[\;{}^{2\lambda}\; f(\quad{}^{\lambda}\; + \;) - \tfrac{1}{4}\;{}^{2}\right]\; = 0.$

The transformation $\xi = ae^{\lambda}\; + \;,\quad = e^{\lambda}\;\;{}^{2}$ leads to the equation $\;''\; + (a\lambda)^{-2}\;(\xi)\;\; = 0$.

8. $\qquad + f(\;{}^{\lambda}\;)\quad + \;(\;{}^{\lambda}\;)\; = 0.$

The substitution $z = e^{\lambda}$ leads to the equation $\lambda^2 z^2\;\;'' + \lambda z[\;(z) + \lambda]\;\;' + g(z)\;\; = 0$.

9. $\qquad + \left[-\;{}^{2} + \sinh^{-4}(\quad) f\left(\coth(\quad)\right)\right]\; = 0.$

The transformation $\xi = \coth(\lambda\;)$, $\;= \dfrac{}{\sinh(\lambda\;)}$ leads to the equation $\;''\; + \lambda^{-2}\;(\xi)\;\; = 0$.

10. $\qquad + \left[-\;{}^{2} + \cosh^{-4}(\quad) f\left(\tanh(\quad)\right)\right]\; = 0.$

The transformation $\xi = \tanh(\lambda\;)$, $\;= \dfrac{}{\cosh(\lambda\;)}$ leads to the equation $\;''\; + \lambda^{-2}\;(\xi)\;\; = 0$.

11. $\qquad + \left[-\dfrac{1}{4}\;{}^{2} + \dfrac{{}^{2\lambda}}{(\;{}^{\lambda}\; + \;)^4} f\left(\dfrac{\;{}^{\lambda}\; + \;}{\;{}^{\lambda}\; + \;}\right)\right]\; = 0.$

The transformation $\xi = \dfrac{ae^{\lambda}\; + \;}{e^{\lambda}\; + \;}$, $\;= \dfrac{e^{\lambda}\;\;{}^{2}}{e^{\lambda}\; + \;}$ leads to a simpler equation: $\;''\; + (\Delta\lambda)^{-2}\;(\xi)\;\; = 0$, where $\Delta = a\;\; - \;\;$.

12. $f\quad + (2f\tanh\quad + \;)\quad + (\;\tanh\quad + \;)\; = 0, \qquad f = f(\;), \quad = \;(\;), \quad = \;(\;).$

The substitution $\;= \;\cosh$ leads to a simpler equation: $\;''\; + g\;\;'\; + (\;\; - \;\;)\;\; = 0$.

13. $f \quad + (2f \coth \quad + \;) \quad + (\;\coth \quad + \;) \;= 0, \qquad f = f(\;), \quad = (\;), \quad = (\;).$

The substitution $\;=\;\sinh\;$ leads to a simpler equation: $\;'' + g\;' + (\;-\;)\;= 0.$

14. $\;^{2}\quad + \left[f(\;\ln \;+\;) + \tfrac{1}{4}\right]\; = 0.$

The transformation $\xi = a\ln \;+\;,\quad = \;^{-1}\;^{2}$ leads to a simpler equation: $\;'' + a^{-2}\;(\xi)\; = 0.$

15. $(\;^{2} - 1)^{2}\quad + f\;\ln \dfrac{\;-\;}{\;+1}\quad = 0.$

The transformation $\xi = \ln \dfrac{a\;\;-a}{\;+1},\quad = \dfrac{}{\sqrt{|\;^{2}-1|}}$ leads to a simpler equation: $4\;'' + [\;(\xi) - 1]\; = 0.$

16. $\;^{2} f(\ln\;)\quad + \;(\ln\;)\quad + \;(\ln\;)\; = 0.$

The substitution $\xi = \ln\;$ leads to the equation $\;(\xi)\;'' + [g(\xi) - \;(\xi)]\;' + \;(\xi)\; = 0.$

17. $\quad + [\;^{2} + \sin^{-4}(\;)f(\cot(\;))]\; = 0.$

The transformation $\xi = \cot(\lambda\;),\quad = \dfrac{}{\sin(\lambda\;)}$ leads to a simpler equation: $\;'' + \lambda^{-2}\;(\xi)\; = 0.$

18. $\quad + \left[\;^{2} + \cos^{-4}(\;)f\left(\tan(\;)\right)\right]\; = 0.$

The transformation $\xi = \tan(\lambda\;),\quad = \dfrac{}{\cos(\lambda\;)}$ leads to a simpler equation: $\;'' + \lambda^{-2}\;(\xi)\; = 0.$

19. $\quad + \;^{2} + \dfrac{1}{\sin^{4}(\;+\;)} f\;\dfrac{\sin(\;+\;)}{\sin(\;+\;)}\quad = 0.$

The transformation $\xi = \dfrac{\sin(\lambda\;+a)}{\sin(\lambda\;+\;)},\quad = \dfrac{}{\sin(\lambda\;+\;)}$ leads to a simpler equation: $\;'' + [\lambda\sin(\;-a)]^{-2}\;(\xi)\; = 0.$

20. $f \quad + (\;-2f\tan\;)\quad + (\;-\;\tan\;)\; = 0, \qquad f = f(\;), \quad = (\;), \quad = (\;).$

The substitution $\;=\;\cos\;$ leads to a simpler equation: $\;'' + g\;' + (\;+\;)\; = 0.$

21. $f \quad + (\;+2f\cot\;)\quad + (\;+\;\cot\;)\; = 0, \qquad f = f(\;), \quad = (\;), \quad = (\;).$

The substitution $\;=\;\sin\;$ leads to a simpler equation: $\;'' + g\;' + (\;+\;)\; = 0.$

22. $(\;^{2} + 1)^{2}\quad + f(\arctan\;+\;)\; = 0.$

The transformation $\xi = \arctan\;+\;,\quad = \dfrac{}{\sqrt{\;^{2}+1}}$ leads to a simpler equation: $\;'' + [\;(\xi) + 1]\; = 0.$

23. $(\;^{2} + 1)^{2}\quad + f(\operatorname{arccot}\;+\;)\; = 0.$

The transformation $\xi = \operatorname{arccot}\;+\;,\quad = \dfrac{}{\sqrt{\;^{2}+1}}$ leads to a simpler equation: $\;'' + [\;(\xi) + 1]\; = 0.$

24. $\quad + f(\;)\; = 0.$

The transformation $\;=\;(z),\quad = \sqrt{|\;'|}$ leads to an equation of the same form: $\;'' + \;(z)\; = 0,$ where $\;(z) = \dfrac{1}{2}\dfrac{\;'''}{\;'} - \dfrac{3}{4}\left(\dfrac{\;''}{\;'}\right)^{2} + (\;')^{2}\;(\;).$

2.2. Autonomous Equations $y \quad = \quad (y, y \ )$

Preliminary remarks. Equations of this type often arise in different areas of mechanics, applied mathematics, physics, and chemical engineering science.

1 . The substitution ( ) = ′ leads to a first-order equation:

$$' = \ ^{-1} \ (\ , \ ). \tag{1}$$

2 . The solution of the original autonomous equation can be represented in implicit form:

$$= \quad \frac{\qquad}{(\ , \ _1)} + \ _2, \tag{2}$$

where = (, $_1$) is the solution of the first-order equation (1).

3 . The solution of the original autonomous equation can be written in parametric form:

$$= \quad \frac{{}'_\tau(\tau, \ _1)}{(\tau, \ _1)} \ \tau + \ _2, \qquad = (\tau, \ _1), \tag{3}$$

where = (τ, $_1$), = (τ, $_1$) is a parametric form of the solution of the first-order equation (1). Formula (2) is a special case of formula (3) with = τ.

4 . For the special cases = ( ) and = (′), see equations 2.9.1.1 and 2.9.4.35.

2.2.1. Equations of the Form – = ( )

Preliminary remarks. Equations of this type arise in the theory of combustion and the theory of chemical reactors.

1 . The substitution ( ) = ′ leads to the Abel equation ′ – = ( ), which is considered in Subsection 1.3.1 for some specific functions .

2 . The solution of the original autonomous equation can be written in the parametric form (3), where = (τ, $_1$), = (τ, $_1$) is a parametric form of the solution to an Abel equation of the second kind ′ – = ( ).

1. $\quad - \quad = -\frac{2(\ +1)}{(\ +3)^2} \quad \frac{+1}{2 \ ^2} \quad , \qquad \neq 1, \qquad \neq -3.$

Solution in parametric form:

$$= \frac{+3}{-1} \ln \ a \ {}^{1-}_{1} \ \frac{-1}{+3} \ \frac{\tau}{1 \ \tau \ ^{+1}} + \ _2 \ ,$$

$$= {}^2_1\tau \ a \ {}^{1-}_{1} \ \frac{-1}{+3} \ \frac{\tau}{1 \ \tau \ ^{+1}} + \ _2 \ ^{\frac{2}{-1}}.$$

2. $\quad - \quad = 2 \ ^2 \ ^{-1}.$

Solution in parametric form:

$$= -\ln \ _1 \ \exp(\ \tau^2) \ \tau + \ _2 \ , \qquad = a \ _1 \exp(\ \tau^2) \ _1 \ \exp(\ \tau^2) \ \tau + \ _2 \ ^{-1}.$$

3. $\quad - \quad = -\frac{2}{9} \quad + \frac{16}{9} \quad {}^{3\ 2} \quad {}^{-1\ 2}.$

Solution in parametric form:

$$= -3\ln \quad {}_1 \exp(-\tau)\big[\exp(3\tau) + \quad {}_2 \sin(\quad \overline{3}\,\tau \quad \big] \quad ,$$

$$= a\exp(2\tau)\frac{\big[2\exp(3\tau) - \quad {}_2 \sin(\quad \overline{3}\,\tau \quad + \quad \overline{3} \quad {}_2 \cos(\quad \overline{3}\,\tau \quad \big]^2}{\big[\exp(3\tau) + \quad {}_2 \sin(\quad \overline{3}\,\tau \quad \big]^2}.$$

4. $\quad - \quad = -\frac{9}{100} \quad \frac{9}{100} \quad {}^{8\ 3} \quad {}^{-5\ 3}.$

Solution in parametric form:

$$= -\tfrac{5}{4}\ln\big[\quad (\tau^4 - 6\tau^2 + 4 \quad {}_1\tau - 3)\big] + \quad {}_2,$$

$$= a(\tau^3 - 3\tau + \quad {}_1)^{3\ 2}\big[\quad (\tau^4 - 6\tau^2 + 4 \quad {}_1\tau - 3)\big]^{-9\ 8}.$$

5. $\quad - \quad = -\frac{3}{16} \quad - \frac{3}{64} \quad {}^{8\ 3} \quad {}^{-5\ 3}.$

Solution in parametric form:

$$= \quad {}_1 - 2\ln[\sin\tau\cosh(\tau + \quad {}_2) + \cos\tau\sinh(\tau + \quad {}_2)], \qquad = a[\tan\tau + \tanh(\tau + \quad {}_2)]^{-3\ 2}.$$

In the solutions of equations 6–9, the following notation is used:

$$Z = \begin{cases} {}_1 \quad (\tau) + \quad (\tau) & \textit{for the upper sign,} \\ {}_1 \quad (\tau) + \quad (\tau) & \textit{for the lower sign,} \end{cases}$$

where (τ) and (τ) are the Bessel functions, and (τ) and (τ) are the modified Bessel functions.

6. $\quad - \quad = \quad {}^{-1\ 2}.$

Solution in parametric form:

$$= -2 \quad \tau^{-1}Z^{-1}(\tau Z'_\tau + \tfrac{1}{3}Z) \quad \tau + \quad {}_2, \qquad = a\tau^{-4\ 3}Z^{-2}[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \quad \tau^2 Z^2],$$

where $\quad = \frac{1}{3}$, $A = \mp\frac{1}{3}a^{3\ 2}$.

7. $\quad - \quad = \quad {}^{-2}.$

Solution in parametric form:

$$= \mp\tfrac{2}{3} \quad \tau Z^2\big[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \quad \tau^2 Z^2\big]^{-1} \quad \tau + \quad {}_2, \qquad = 2a\tau^{4\ 3}Z^2\big[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \quad \tau^2 Z^2\big]^{-1},$$

where $\quad = \frac{1}{3}$, $A = -36a^3$.

8. $\quad - \quad = 2 \quad {}^2 - \quad {}^{1\ 2}.$

Solution in parametric form:

$$= \quad 2 \quad \tau^{-1}(Z'_\tau)^{-1}(\tau Z \quad 2Z'_\tau) \quad \tau + \quad {}_2, \qquad = a(Z'_\tau)^{-2}(\tau Z \quad 2Z'_\tau)^2,$$

where $\quad = 0$, $A = a^{1\ 2}$.

9. $\quad - \quad = \quad {}^{-1\ 2} + 2B^2 + B \quad {}^{1\ 2}.$

Solution in parametric form:

$$= -2 \quad \tau^{-1}Z^{-1}(\tau Z'_\tau - Z) \quad \tau + \quad {}_2, \qquad = B^2 Z^{-2}(\tau Z'_\tau - Z)^2,$$

where $A = (1 - \quad {}^2)B^3$.

In the solutions of equations 10–14, the function = ( ) is defined in implicit form:

$$\tau = \frac{\wp}{\overline{(4\wp^3-1)}} - {}_1.$$

The upper sign in this formula corresponds to the classical elliptic Weierstrass function $\wp = \wp(\tau + {}_1, 0, 1)$.

10. **– = ^{2} – $\frac{9}{625}$ $^{-1}$.**

Solution in parametric form:

$$= 5\ln\tau + {}_2, \qquad = 5a(\tau^2\wp \mp \tfrac{1}{2}), \qquad \text{where} \quad A = \tfrac{6}{125}a^{-1}.$$

11. **– = ^{2} – $\frac{6}{25}$.**

Solution in parametric form:

$$= 5\ln\tau + {}_2, \qquad = 5a\tau^2\wp, \qquad \text{where} \quad A = \tfrac{6}{125}a^{-1}.$$

12. **– = ^{2} + $\frac{6}{25}$.**

Solution in parametric form:

$$= 5\ln\tau + {}_2, \qquad = 5a(\tau^2\wp \mp 1), \qquad \text{where} \quad A = \tfrac{6}{125}a^{-1}.$$

13. **– = 12 + $^{-5\ 2}$.**

Solution in parametric form:

$$= \mp\tfrac{2}{7} \quad \wp^{-1}(\quad 2\tau\wp^2 \ ^{-1} \quad \tau + {}_2, \qquad = a\wp^{-6\ 7}(\quad 2\tau\wp^2 \ ^{-4\ 7},$$

where $= \overline{(4\wp^3-1)}$, $A = \mp 147a^{7\ 2}$.

14. **– = $\frac{63}{4}$ + $^{-5\ 3}$.**

Solution in parametric form:

$$= -\tfrac{3}{4} \quad (\quad 2\tau\wp^2)(\tau \ +2\wp)^{-1} \quad \tau + {}_2, \qquad = 2a(\quad 2\tau\wp^2 \ ^{3\ 2}(\tau \ +2\wp)^{-9\ 8},$$

where $= \overline{(4\wp^3-1)}$, $A = -\tfrac{128}{3}a^2(2a)^{2\ 3}$.

In the solutions of equations 15–18, the following notation is used:

$$= \frac{\tau \quad \tau}{\overline{(4\tau^3-1)}} + {}_1 \quad (\textit{incomplete elliptic integral of the second kind}),$$

$$R = \overline{(4\tau^3-1)}, \qquad {}_1 = 2\tau \ \mp R, \qquad {}_2 = \tau^{-1}(2\tau R \ \mp R^2 - 1).$$

15. **– = $^{1\ 2}$ – $\frac{12}{49}$.**

Solution in parametric form:

$$= -7 \quad \tau R^{-1} \ ^{-1} \quad \tau + {}_2, \qquad = 7a\tau^{2\ -4}, \qquad \text{where} \quad A = \tfrac{12}{49}(7a)^{1\ 2}.$$

16. **– = 6 + $^{-4}$.**

Solution in parametric form:

$$= -\tfrac{1}{5} \quad \tau^{-1}R^{-1} \ {}_1^{-1} \quad \tau + {}_2, \qquad = a\tau^{-3\ 5} \ {}_1^{-2\ 5}, \qquad \text{where} \quad A = \mp 150a^5.$$

17. $\quad - \quad = 20 \quad + \quad {}^{-1\ 2}.$

Solution in parametric form:

$$= \tfrac{1}{3} \quad R^{-1} \ {}_{1}^{-1} \ {}_{2} \ \ \tau + \ \ {}_{2}, \qquad = a \ {}_{1}^{-4\ 3} \ {}_{2}^{2}, \qquad \text{where} \quad A = \ \ 108a^{3}\ {}^{2}.$$

18. $\quad - \quad = \tfrac{15}{4} \quad + \quad {}^{-7}.$

Solution in parametric form:

$$= \quad R^{-1} \ {}_{1}\bigl(4\tau \ {}_{1}^{2} \mp \ {}_{2}^{2} \ {}^{-1} \ \ \tau + \ \ {}_{2}, \qquad = a \ {}_{1}^{1\ 2}\bigl(4\tau \ {}_{1}^{2} \mp \ {}_{2}^{2} \ {}^{-3\ 8}, \qquad \text{where} \quad A = \ \tfrac{3}{4}a^{8}.$$

19. $\quad - \quad = \quad + B \ {}^{-1} - B^{2} \ {}^{-3}.$

The substitution $\ (\ ) = \ '$ leads to an Abel equation of the form 1.3.1.5:

$$' - \quad = A \ \ + B \ {}^{-1} - B^{2} \ {}^{-3}.$$

20. $\quad - \quad = -\tfrac{3}{16} \quad + \quad {}^{-1\ 3} + B \ {}^{-5\ 3}.$

The substitution $\ (\ ) = \ '$ leads to an Abel equation of the form 1.3.1.61:

$$' - \quad = -\tfrac{3}{16} \ \ + A \ {}^{-1\ 3} + B \ {}^{-5\ 3}.$$

21. $\quad - \quad = -\tfrac{5}{36} \quad + \quad {}^{-3\ 5} + B \ {}^{-7\ 5}.$

The substitution $\ (\ ) = \ '$ leads to an Abel equation of the form 1.3.1.62:

$$' - \quad = -\tfrac{5}{36} \ \ + A \ {}^{-3\ 5} + B \ {}^{-7\ 5}.$$

22. $\quad - \quad = \tfrac{4}{9} \quad + 2 \quad {}^{2} + 2 \quad {}^{2\ 3}.$

The substitution $\ (\ ) = \ '$ leads to an Abel equation of the form 1.3.1.14:

$$' - \quad = \tfrac{4}{9} \ \ + 2A \ {}^{2} + 2A^{2} \ {}^{3}.$$

23. $\quad - \quad = \quad {}^{k-1} - \quad B \ {}^{k} + \quad B^{2} \ {}^{2k-1}.$

The substitution $\ (\ ) = \ '$ leads to an Abel equation of the form 1.3.1.6:

$$' - \quad = A \quad {}^{-1} - kB \quad + kB^{2} \ {}^{2\ -1}.$$

24. $\quad - \quad = \dfrac{2 \quad {}^{2}}{\overline{\ {}^{2} \quad 8 \ {}^{2}}}.$

Solution in parametric form:

$$= \mp \quad {}^{-1} \ {}^{-1}(\ {}^{2} \quad 2 \ \ {}^{2}) \ \ \tau + \ \ {}_{2}, \qquad = \ \ a \ \ {}^{-1} \ {}^{-1}(\ {}^{2} \mp 2 \ \ {}^{2}),$$

where $\quad = \quad \exp(\mp\tau^{2}) \ \ \tau + \ \ {}_{1}, \qquad = 2\tau \qquad \exp(\mp\tau^{2}).$

25. $\quad - \quad = \quad + B \exp(-2 \qquad).$

The substitution $\ (\ ) = \ '$ leads to an Abel equation of the form 1.3.1.8:

$$' - \quad = A + B \exp(-2 \quad A).$$

26. $\quad - \quad = \ {}^{2} \quad {}^{2\lambda} - (\quad + 1) \ {}^{\lambda} \ + \ .$

The substitution $\ (\ ) = \ '$ leads to an Abel equation of the form 1.3.1.73:

$$' - \quad = a^{2}\lambda e^{2\lambda} \ \ - a(\ \lambda + 1)e^{\lambda} \ \ + \ \ .$$

27. $\quad - \quad = \ {}^{2} \quad {}^{2\lambda} + \qquad {}^{\lambda} + \quad {}^{\lambda} \ .$

The substitution $\ (\ ) = \ '$ leads to an Abel equation of the form 1.3.1.74:

$$' - \quad = a^{2}\lambda e^{2\lambda} \ \ + a\lambda \ \ e^{\lambda} \ \ + \ \ e^{\lambda} \ \ .$$

2.2.2. Equations of the Form $\quad + \quad (\ ) \quad + \quad = 0$

2.2.2-1. Preliminary remarks.

Equation of this form are often encountered in the theory of nonlinear oscillations, where plays the role of time.

1 . The transformation

$$z = -\tfrac{1}{2}\ ^2 + a, \qquad = \ '$$

leads to an Abel equation:

$$' = g(z) \quad + 1, \qquad \text{where} \quad g(z) = \quad (\ ) \quad , \qquad = \quad \overline{2(a-z)},$$

whose special cases are outlined in Subsection 1.3.2.

2 . For oscillatory systems with a weak nonlinearity

$$'' \quad + \varepsilon \quad (\ ) \ ' + \quad = 0,$$

two leading terms of the asymptotic solution, as ε 0, are described by the formula

$$= A \cos(\quad + B),$$

where the functions $A = A(\xi)$ and $B = B(\xi)$ depend on the slow variable $\xi = \varepsilon$; they are determined from the autonomous system of first-order differential equations:

$$A' = -\frac{A}{2} \int_0^{2\pi} (A \cos \quad) \sin^2 \qquad , \qquad B' = -\frac{1}{2} \int_0^{2\pi} (A \cos \quad) \sin \quad \cos \qquad .$$

The right-hand sides of these equations depend only on A. The system is solved consecutively starting from the first equation.

2.2.2-2. Solvable equations and their solutions.

1. $\quad + \quad + \quad = 0.$

Solution in parametric form:

$$= -A \quad \frac{\tau}{\tau(\ _1 + 2A^2 \ln|\tau| - 2A\tau)^{1 \ 2}} + \ _2, \qquad = (\ _1 + 2A^2 \ln|\tau| - 2A\tau)^{1 \ 2}, \qquad \text{where} \quad A = \frac{1}{a}.$$

2. $\quad - \varepsilon(1 - \ ^2) \quad + \quad = 0.$

Van der Pol oscillator.

1 . Solution, as ε 0:

$$= a \cos(\quad - \theta) - \tfrac{1}{32}\varepsilon a^3 \sin[3(\quad - \theta)] + \quad (\varepsilon^2),$$

where

$$a^2 = \frac{4}{1 + (4 \ _1^{-2} - 1)e^{-}}, \qquad \theta = \frac{1}{8}\varepsilon \ln a - \frac{7}{64}\varepsilon a^2 + \frac{1}{16}\varepsilon^2 \quad + \ _2.$$

In applications, plays the role of time, $_1$ is the initial oscillation amplitude, and $_2$ is the initial phase with $\varepsilon = 0$.

2 . As ε + , the periodic solution of the Van der Pol equation consists of intervals with fast and slow oscillations and describes damping oscillations with period $= (3 - 2\ln 2)\varepsilon +$ $(\varepsilon^{-1 \ 3})$.

3. $\quad + \ (\quad ^{2} + \ ) \quad + \quad = 0.$

1 . The transformation $z = -\frac{1}{2}\ {}^{2}$, $\quad = \ '$ leads to an Abel equation of the form 1.3.2.1: $' = (-2az + \ ) \quad + 1.$

2 . Solution in parametric form with $a < 0$:

$$= \mp\frac{2}{3}k \quad \tau^{-1 \ 3} \quad \frac{4}{3}k^{2}\tau^{-2 \ 3}Z^{-1} \ \ \tau Z'_{\tau} + \frac{1}{3}Z \ \ - \frac{}{a} \quad ^{-1 \ 2} \quad \tau + \quad {}_{2},$$

$$= \quad \frac{4}{3}k^{2}\tau^{-2 \ 3}Z^{-1} \ \ \tau Z'_{\tau} + \frac{1}{3}Z \ \ - \frac{}{a} \quad ^{1 \ 2}, \qquad a = -\frac{9}{4}k^{-3},$$

where

$$Z = \begin{matrix} {}_{1 \ 1 \ 3}(\tau) + \quad {}_{2 \ 1 \ 3}(\tau) & \text{for the upper sign,} \\ {}_{1 \ 1 \ 3}(\tau) + \quad {}_{2 \ \ 1 \ 3}(\tau) & \text{for the lower sign,} \end{matrix}$$

${}_{1 \ 3}(\tau)$ and ${}_{1 \ 3}(\tau)$ are the Bessel functions, and ${}_{1 \ 3}(\tau)$ and ${}_{1 \ 3}(\tau)$ are the modified Bessel functions.

4. $\quad + \ (\quad ^{2} + \ )^{-2} \quad + \quad = 0.$

The transformation $z = -\frac{1}{2}\ {}^{2}$, $\quad = \ '$ leads to an Abel equation of the form 1.3.2.2: $' = (-2az + \ )^{-2} \quad + 1.$

5. $\quad + \ (\quad ^{2} + \ )^{-1 \ 2} \quad + \quad = 0.$

1 . The transformation $z = -\frac{1}{2}\ {}^{2}$, $\quad = \ '$ leads to an Abel equation of the form 1.3.2.4: $' = (-2az + \ )^{-1 \ 2} \quad + 1.$

2 . Solution in parametric form:

$$= -a \ {}_{1} \quad a \ {}^{2}_{1} \ \ {}^{2} - \frac{}{a} \quad ^{-1 \ 2} \ \frac{\tau}{\tau^{2} - \tau + a} + \quad {}_{2}, \qquad = \ a \ {}^{2}_{1} \ \ {}^{2} - \frac{}{a} \quad ^{1 \ 2},$$

where $\quad = \exp \ - \quad \frac{\tau \quad \tau}{\tau^{2} - \tau + a} \quad .$

6. $\quad - \quad 2 \ + \frac{1}{\ {}^{2} + \ } \quad + \quad = 0.$

The transformation $z = -\frac{1}{2}\ {}^{2} - \frac{}{2a}$, $\quad = \ '$ leads to an Abel equation of the form 1.3.2.3:

$$' = \ A - \frac{1}{Az} \quad + 1, \qquad \text{where} \quad A = -2a.$$

7. $\quad + \quad \exp(\quad ^{2}) \quad + \quad = 0.$

The transformation $z = -\frac{1}{2}\ {}^{2}$, $\quad = \ '$ leads to an Abel equation of the form 1.3.2.7: $' = a\exp(-2\lambda z) \quad + 1.$

8. $\quad + \ [\ \exp(\quad ^{2}) + \ \exp(- \quad ^{2})] \quad + \quad = 0.$

The transformation $z = -\frac{1}{2}\ {}^{2}$, $\quad = \ '$ leads to an Abel equation of the form 1.3.2.8: $' = [\ \exp(2\lambda z) + a\exp(-2\lambda z)] \quad + 1.$

9. $\quad + 2 \quad \exp[\ (\ - \quad ^{2})] \quad + \quad = 0.$

Solution in parametric form:

$$= \mp 2k \quad (\ \ - 4k^{2}\tau^{2} - \ln|k \quad ^{-1}| \ \ ^{-1 \ 2} \quad \tau + \quad {}_{2}, \qquad = (\ \ - 4k^{2}\tau^{2} - \ln|k \quad ^{-1}| \ \ ^{1 \ 2},$$

where $a = \mp\frac{1}{4}k^{-2}$, $\quad = \quad \exp(\mp\tau^{2}) \ \ \tau + \quad {}_{1}.$

10. $\quad + \quad \cosh(\quad^{2}) \quad + \quad = 0.$

This is a special case of equation 2.2.2.8 with $a = \quad = \frac{1}{2}A$.

11. $\quad + \quad \sinh(\quad^{2}) \quad + \quad = 0.$

This is a special case of equation 2.2.2.8 with $a = - \quad = \frac{1}{2}A$.

12. $\quad + 2 \qquad \overline{\sinh^{2}[\quad (B - \quad^{2})] + 2 \quad^{-1}} + \quad = 0.$

Solution in parametric form:

$$= 2a \quad (\quad^{2} + 2 \quad^{2}) \quad^{-1} Q^{-1} \quad \tau + \quad_{2}, \qquad = Q; \qquad A = \tfrac{1}{4}a^{-2},$$

where

$$= \quad \exp(-\tau^{2}) \quad \tau + \quad_{1}, \qquad = 2\tau \quad + \exp(-\tau^{2}), \qquad = \quad \overline{\quad^{2} - 2 \quad^{2} + 8 \quad^{2} \quad^{2}},$$

$$Q = \quad \overline{B - 4a^{2} \operatorname{arcsinh}[a \quad^{-1} \quad^{-1}(\quad^{2} - 2 \quad^{2})]}, \qquad \operatorname{arcsinh} z = \ln\bigl(z + \quad \overline{z^{2} + 1} \quad\bigr).$$

13. $\quad - 2 \qquad \overline{\cosh^{2}[\quad (\quad^{2} - B)] - 2 \quad^{-1}} + \quad = 0.$

Solution in parametric form:

$$= 2a \quad (\quad^{2} - 2 \quad^{2}) \quad^{-1} Q^{-1} \quad \tau + \quad_{2}, \qquad = Q; \qquad A = \tfrac{1}{4}a^{-2},$$

where

$$= \quad \exp(\tau^{2}) \quad \tau + \quad_{1}, \qquad = 2\tau \quad - \exp(\tau^{2}), \qquad = \quad \overline{\quad^{2} + 2 \quad^{2} - 8 \quad^{2} \quad^{2}},$$

$$Q = \quad \overline{B + 4a^{2} \operatorname{arccosh}[a \quad^{-1} \quad^{-1}(\quad^{2} + 2 \quad^{2})]}, \qquad \operatorname{arccosh} z = \quad \ln\bigl(z + \quad \overline{z^{2} - 1} \quad\bigr).$$

14. $\quad + \quad \cos(\quad^{2}) \quad + \quad = 0.$

The transformation $z = -\frac{1}{2} \quad^{2}, \quad = \quad'$ leads to an Abel equation of the form 1.3.2.11: $\quad' = A\cos(2\lambda z) \quad + 1$.

15. $\quad + \quad \sin(\quad^{2}) \quad + \quad = 0.$

The transformation $z = -\frac{1}{2} \quad^{2}, \quad = \quad'$ leads to an Abel equation of the form 1.3.2.12: $\quad' = -A\sin(2\lambda z) \quad + 1$.

2.2.3. Lienard Equations $\quad + \quad (\quad) \quad + g(\quad) = 0$

2.2.3-1. Preliminary remarks.

Equations of this form are encountered in various fields of applied mathematics, mechanics, and physics.

1 . For $\quad (\quad) = 0$, see equation 2.9.1.1.

2 . The substitution $\quad (\quad) = \quad'$ leads to an Abel equation of the second kind:

$$\quad' + \quad (\quad) \quad + g(\quad) = 0,$$

whose special cases are outlined in Subsection 1.3.3.

3 . The transformation $\quad (z) = \quad', \ z = - \quad (\quad)$ leads to an Abel equation of the second kind:

$$\quad' - \quad = \quad (z), \qquad \text{where} \qquad (z) = g(\quad) \quad (\quad),$$

whose special cases are outlined in Subsection 1.3.1.

2.2.3-2. Solvable equations and their solutions.

1. $\quad + \quad + \quad ^3 = 0.$

Duffing equation. This is a special case of equation 2.9.1.1 with $(\ ) = - \ - a \ ^3$.

1 . Solution:

$$= \quad \Big(\ _1 - \ ^2 - \tfrac{1}{2} a \ ^4 \ ^{-1\ 2} \quad + \ _2.$$

The period of oscillations with amplitude is expressed in terms of the complete elliptic integral of the first kind:

$$= \frac{4}{\overline{\overline{1 + a \ ^2}}} \quad \frac{a \ ^2}{2 + 2a \ ^2}, \qquad \text{where} \quad (\ ) = \int_0^{\pi\ 2} \frac{}{\overline{\overline{1 - \ \sin^2}}}.$$

2 . The asymptotic solution, as $a \quad 0$, has the form:

$$= \ _1 \cos[(1 + \tfrac{3}{8} a \ _1^2) \ + \ _2] + \tfrac{1}{32} a \ _1^3 \cos[3(1 + \tfrac{3}{8} a \ _1^2) \ + 3 \ _2] + \ (a^2),$$

where $_1$ and $_2$ are arbitrary constants. The corresponding asymptotics for the period of oscillations with amplitude is described by the formula: $= 2 \ (1 - \tfrac{3}{8} a \ ^2) + \ (a^2)$.

2. $\quad + \quad + \ ^3 + \quad = 0.$

The transformation $(z) = \ '$, $z = -\tfrac{1}{2} a \ ^2$ leads to an Abel equation of the form 1.3.1.2: $' - \ = -\dfrac{2}{a^2} z + \dfrac{}{a}$.

3. $\quad = (\ + 3\ ) \ + \ ^3 - \ ^2 - 2 \ ^2 \ .$

The substitution $(\ ) = \ '$ leads to an Abel equation of the form 1.3.3.1:

$$' = (a \ + 3\ ) \ + \ ^3 - a \ ^2 - 2 \ ^2 \ .$$

4. $\quad = (3 \ + \ ) \ - \ ^2 \ ^3 - \ ^2 + \ .$

The substitution $(\ ) = \ '$ leads to an Abel equation of the form 1.3.3.2:

$$' = (3a \ + \ ) \ - a^2 \ ^3 - a \ ^2 + \ .$$

5. $2 \quad = (7 \ + 5\ ) \ - 3 \ ^2 \ ^3 - 2 \ ^2 - 3 \ ^2 \ .$

The substitution $(\ ) = \ '$ leads to an Abel equation of the form 1.3.3.3:

$$2 \quad ' = (7a \ + 5\ ) \ - 3a^2 \ ^3 - 2 \ ^2 - 3 \ ^2 \ .$$

6. $\quad = \ ^{-1}[(1 + 2\ ) \ + \ ] \ - \ ^2 (\ + \ ).$

The substitution $(\ ) = \ '$ leads to an Abel equation of the form 1.3.3.8:

$$' = \ ^{-1}[(1 + 2\ ) \ + a \ ] \ - \ ^2 (\ + a).$$

7. $\quad = (\ - \ ) \ ^{-1} \ + [\ ^2 - (2 \ + 1) \ + (\ + 1) \ ^2] \ ^2 \ ^{-1}.$

The substitution $(\ ) = \ '$ leads to an Abel equation of the form 1.3.3.9:

$$' = a(\ - \ ) \ ^{-1} \ + [\ ^2 - (2 \ + 1) \ + (\ + 1) \ ^2] \ ^2 \ ^{-1}.$$

8. $\quad = [\ (2 \ + \ ) \ ^k + \ ] \ ^{-1} \ + (- \ ^2 \ ^{2k} - \ ^k + \ ) \ ^2 \ ^{-1}.$

The substitution $(\ ) = \ '$ leads to an Abel equation of the form 1.3.3.10:

$$' = [a(2 \ + k) \ + \ ] \ ^{-1} \ + (-a^2 \ ^2 - a \ + \ ) \ ^2 \ ^{-1}.$$

9. $= [\ (2\quad +\ )\ ^{2k} + \ (2\quad -\ )]\quad ^{-k-1}\quad - (\ ^{2}\quad ^{4k} + \quad ^{2k} + \ ^{2}\quad)\ ^{2}\ ^{-2k-1}.$

The substitution $(\ ) = \ '$ leads to an Abel equation of the form 1.3.3.11:

$$' = [a(2\quad + k)\ ^{2} + \ (2\quad - k)]\quad ^{-\ -1}\quad - (a^{2}\quad ^{4} + \quad ^{2} + \ ^{2}\quad)\ ^{2}\ ^{-2\ -1}.$$

10. $= \quad ^{\lambda}\quad + \ ^{\lambda}\ .$

Solution in parametric form:

$$= -\frac{A}{\lambda}\quad \tau^{-1}(\ _{1} + A^{2}\ln|\tau| - A\tau\ ^{-1}\quad \tau + \ _{2}, \qquad = \frac{1}{\lambda}\ln\ -\frac{\lambda}{}(\ _{1} + A^{2}\ln|\tau| - A\tau)\ ,$$

where $A = \quad a$.

11. $= (\quad + \ )\quad + \ ^{2} - \quad - \ ^{2}.$

The substitution $(\ ) = \ '$ leads to an Abel equation of the form 1.3.3.67:

$$' = (ae\ + \ )\quad + e^{2}\ - a\ e\ - \ ^{2}.$$

12. $= [\ (2\ + \ )\ ^{\lambda}\ + \ ]\quad + (-\ ^{2}\quad ^{2\lambda} - \quad ^{\lambda} + \ )\ ^{2}\quad .$

The substitution $(\ ) = \ '$ leads to an Abel equation of the form 1.3.3.68:

$$' = [a(2\ + \lambda)e^{\lambda}\ + \ ]e\quad + (-a^{2}\ e^{2\lambda}\ - a\ e^{\lambda}\ + \ )e^{2}\quad .$$

13. $= (\quad ^{\lambda} + \ )\quad + [\ ^{2}\ ^{2\lambda} + \quad (\quad + 1)\ ^{\lambda} + \ ^{2}\quad].$

The substitution $(\ ) = \ '$ leads to an Abel equation of the form 1.3.3.69:

$$' = (ae^{\lambda}\ + \ )\ + [a^{2}e^{2\lambda}\ + a\ (\lambda\ + 1)e^{\lambda}\ + \ ^{2}\lambda\ ].$$

14. $= \ ^{\lambda}(2\quad + \ + \ )\quad - \ ^{2\lambda}(\ ^{2}\quad ^{2} + \quad + \ ).$

The substitution $(\ ) = \ '$ leads to an Abel equation of the form 1.3.3.70:

$$' = e^{\lambda}\ (2a\lambda\ + a + \ )\ - e^{2\lambda}\ (a^{2}\lambda\ ^{2} + a\quad + \ ).$$

15. $= \quad (2\quad ^{2} + 2\ + \ )\quad + \ ^{2}\ (-\quad ^{4} - \quad ^{2} + \ ).$

The substitution $(\ ) = \ '$ leads to an Abel equation of the form 1.3.3.71:

$$' = e\quad (2a\ ^{2} + 2\ + \ )\ + e^{2}\quad (-a\ ^{4} - \quad ^{2} + \ ).$$

16. $= (\ \cosh\ + \ )\quad - \quad \sinh\ + \ .$

The substitution $(\ ) = \ '$ leads to an Abel equation of the form 1.3.3.75:

$$' = (a\cosh\ + \ )\ - a\ \sinh\ + \ .$$

17. $= (\ \sinh\ + \ )\quad - \quad \cosh\ + \ .$

The substitution $(\ ) = \ '$ leads to an Abel equation of the form 1.3.3.76:

$$' = (a\sinh\ + \ )\ - a\ \cosh\ + \ .$$

18. $+ \ \sin\ = 0.$

This is the equation of oscillations of the mathematical pendulum, where the variable plays the role of time, and is the angle of deviation from the equilibrium state.

1 . Solution:

$$= \quad (2a\cos \quad + \quad {}_1)^{-1\ 2} \quad + \quad {}_2.$$

2 . With $a > 0$ and the initial conditions $\ (0) = \quad > 0$ and $\ '(0) = 0$, the oscillations of the mathematical pendulum are described by

$$\sin\frac{}{2} = \quad \mathrm{sn}\big(\quad \overline{a} \quad , \qquad = \sin\frac{}{2}.$$

where sn = sn(z) is the Jacobi elliptic function defined parametrically by the following relations:

$$\mathrm{sn}(z) = \sin\beta, \quad z = {}_0 \quad \frac{\beta}{\overline{1 - \quad {}^2 \sin^2\beta}}.$$

3 . The period of oscillations of the mathematical pendulum is expressed in terms of the complete elliptic integral of the second kind:

$$= \frac{4}{\overline{a}} \quad (\quad), \qquad \text{where} \qquad (\quad) = {}_0^{\pi\ 2} \frac{\beta}{\overline{1 - \quad {}^2 \sin^2\beta}}.$$

At small amplitudes, as $\qquad 0$, the following asymptotic formula holds for the period:

$$= \frac{2}{\overline{a}} \quad 1 + \frac{1}{16} \quad {}^2 \quad + \quad (\quad {}^4), \qquad\qquad 0.$$

19. $\quad$ **+ sin( ) + sin( ) = 0.**

Solution in parametric form:

$$= -A \quad {}^{-1}\big(\ {}^2 - \lambda^2 \quad {}^2\ {}^{-1\ 2} \quad + \quad {}_2, \qquad = \frac{1}{\lambda}\arccos\frac{\lambda}{} \quad ,$$

where $A = \quad a, \quad = A \ - A^2 \ln|\ | + \quad {}_1$.

20. $\quad$ **+ cos( ) + cos( ) = 0.**

The substitution $\lambda \ = \lambda \ + \frac{\pi}{2}$ leads to an equation of the form 2.2.3.19:

$$'' \ - a\sin(\lambda \ ) \ ' - \ \sin(\lambda \ ) = 0.$$

2.2.4. Rayleigh Equations $\quad + \ (\ ) + g(\ ) = 0$

2.2.4-1. Preliminary remarks. Some transformations.

Equations of this form arise in the theory of nonlinear oscillations.

1 . Let us discuss the special case $g(\ ) = \ $, which corresponds to the equation

$$'' \ + \ (\ ') + \ = 0. \qquad (1)$$

Differentiating equation (1) with respect to and substituting $z(\ ) = \ '$, we obtain the equation of nonlinear oscillations:

$$z'' \ + \ (z)z' + z = 0, \qquad \text{where} \qquad (z) = \ '(z), \qquad (2)$$

which is considered in Subsection 2.2.2.

The solution of equation (1) can be written in parametric form:

$$= \ (\tau, \ {}_1, \ {}_2), \qquad = - \ (z) - \frac{z'_\tau}{'_\tau},$$

where $\ = \ (\tau, \ {}_1, \ {}_2)$, $z = z(\tau, \ {}_1, \ {}_2)$ is a parametric representation of the solution of equation (2).

2 . The transformation

$$\xi = -\tfrac{1}{2}(\ \ ')^2 + a, \qquad = - \ - \ (\ \ '),$$

reduces equation (1) to an Abel equation of the second kind:

$$\ ' = \ (\xi) \ \ + 1, \qquad \text{where} \qquad (\xi) = z^{-1} \ \ (z), \quad z = \overline{2(a-\xi)}, \tag{3}$$

where function $\ \ = \ \ (z)$ is defined above in equation (2). Specific equations of the form (3) are outlined in Subsection 1.3.2.

3 . The equation of the special form

$$\ '' \ + a(\ \ ')^2 + g(\ \ ) = 0 \tag{4}$$

is reduced, with the aid of the substitution $\ (\ \ ) = (\ \ ')^2$, to the first-order linear equation $\ ' + 2a \ \ + 2g(\ \ ) = 0$. Therefore, the solution of equation (4) can be written in implicit form:

$$= \ _2 \ \ \Big[\ _1 e^{-2} \ \ - \ (\ \ )\Big]^{-1 \ 2} \ , \qquad \text{where} \qquad (\ \ ) = 2e^{-2} \ \ \ e^{2} \ \ g(\ \ ) \ \ .$$

4 . The equation of the special form

$$\ '' \ + a(\ \ ')^4 + \ (\ \ ')^2 + g(\ \ ) = 0 \tag{5}$$

is reduced, with the aid of the substitution $\ (\ \ ) = (\ \ ')^2$, to the Riccati equation $\ ' + 2a \ \ ^2 + 2 \ \ + 2g(\ \ ) = 0$, which is outlined in Section 1.2.

5 . For the oscillatory systems with a weak nonlinearity

$$\ '' \ + \varepsilon \ \ (\ \ ') + \ \ = 0,$$

two leading terms of the asymptotic solution, as $\varepsilon \ \ 0$, are described by the formula

$$= A\cos(\ \ + B),$$

where the functions $A = A(\xi)$ and $B = B(\xi)$ depend on the slow variable $\xi = \varepsilon \ $ and are determined from the autonomous system of first-order differential equations:

$$A' = \frac{1}{2} \ {}_0^{2\pi} \ (-A\sin \ \ )\sin \ \ \ , \qquad AB' = \frac{1}{2} \ {}_0^{2\pi} \ (-A\sin \ \ )\cos \ \ \ .$$

The right-hand sides of these equations depend only on A. The system is solved consecutively starting from the first equation.

2.2.4-2. Solvable equations and their solutions.

1. $\ \ + \ (\ \ \ )^2 + \ \ = 0.$

This equation describes small oscillations in the case where the drag force is proportional to the speed squared.

Solution in implicit form: $\ = \ _2 \ \ a \ \ \Big[\ _1 a^2 e^{-2} \ \ + (\tfrac{1}{2} - a \ \ )\Big]^{-1 \ 2} \ .$

2. $\ \ + \varepsilon\big[\tfrac{1}{3}(\ \ \ )^3 - \ \ \big] + \ \ = 0.$

Van der Pol equation.

1 . Differentiating the equation with respect to and passing on to the new variable $\ (\ \ ) = \ '$, we arrive at an equation of the form 2.2.2.2: $\ '' \ - \varepsilon(1 - \ ^2) \ ' + \ \ = 0.$

2 . Solution, as $\varepsilon \ \ 0$:

$$= \frac{2 \ \ _1}{\overline{1 - \ _2 e^{-}}} \cos \ \ + \frac{2 \ \overline{1 - \ _1^2}}{\overline{1 - \ _2 e^{-}}} \sin \ \ + \ (\varepsilon^2).$$

3. $\ \ + \ (\ \ \ )^4 + \ (\ \ \ )^2 + \ \ = 0.$

The transformation $\xi = -\tfrac{1}{2}(\ \ ')^2, \quad = - \ - a(\ \ ')^4 - \ (\ \ ')^2$ leads to an Abel equation of the form 1.3.2.1: $\ ' = (-8a\xi + 2 \ \ ) \ \ + 1.$

4. $\quad + (\quad)^2\left[\ (\quad)^2 + \ \right]^{-1} + \quad = 0.$

The transformation $\xi = -\frac{1}{2}(\ ')^2$, $\ = -\ -(\ ')^2[a(\ ')^2 + \ ]^{-1}$ leads to an equation of the form 1.3.2.2: $\ ' = 2\ (\ - 2a\xi)^{-2}\ + 1$.

5. $\quad + \quad \exp[\ (\quad)^2] + B + \quad = 0.$

Differentiating the equation with respect to and passing on to the new variable $(\ ) = \ '$, we arrive at an equation of the form 2.2.2.7: $\ '' + 2A\lambda\ \exp(\lambda\ ^2)\ ' + \ = 0$.

6. $\quad + \quad \cosh[\ (\quad)^2] + B + \quad = 0.$

Differentiating the equation with respect to and passing on to the new variable $(\ ) = \ '$, we arrive at an equation of the form 2.2.2.11: $\ '' + 2A\lambda\ \sinh(\lambda\ ^2)\ ' + \ = 0$.

7. $\quad + \quad \sinh[\ (\quad)^2] + B + \quad = 0.$

Differentiating the equation with respect to and passing on to the new variable $(\ ) = \ '$, we arrive at an equation of the form 2.2.2.10: $\ '' + 2A\lambda\ \cosh(\lambda\ ^2)\ ' + \ = 0$.

8. $\quad + \ (\quad)^2 + \quad \sin \quad = 0.$

This equation describes the oscillations of the mathematical pendulum in the case where the drag force is proportional to the speed squared.

Solution in implicit form: $\quad = \ _2 \quad \ _1 e^{-2} + \dfrac{2}{4a^2+1}(\cos \ - 2a \sin\ )^{-1\ 2}$.

9. $\quad + \quad \cos[\ (\quad)^2] + B + \quad = 0.$

Differentiating the equation with respect to and passing on to the new variable $(\ ) = \ '$, we arrive at an equation of the form 2.2.2.15: $\ '' - 2A\lambda\ \sin(\lambda\ ^2)\ ' + \ = 0$.

10. $\quad + \quad \sin[\ (\quad)^2] + B + \quad = 0.$

Differentiating the equation with respect to and passing on to the new variable $(\ ) = \ '$, we arrive at an equation of the form 2.2.2.14: $\ '' + 2A\lambda\ \cos(\lambda\ ^2)\ ' + \ = 0$.

2.3. Emden–Fowler Equation $y \quad = \quad y$

2.3.1. Exact Solutions

2.3.1-1. Preliminary remarks. Classification table.

In this subsection, the value of the insignificant parameter A is in many cases defined in the form of a function of two (one) auxiliary coefficients a and ,

$$A = \ (a,\ ), \tag{1}$$

and the corresponding solutions are represented in parametric form,

$$= \ _1(\tau,\ _1,\ _2, a), \qquad = \ _2(\tau,\ _1,\ _2,\ ), \tag{2}$$

where τ is the parameter, $_1$ and $_2$ are arbitrary constants, and $_1$ and $_2$ are some functions.

Having fixed the auxiliary coefficient sign $a > 0$ (or $\ > 0$), one should express the coefficient in terms of both A and a with the help of (1). As a result, one obtains:

$$= \ (A, a).$$

Substituting this formula into (2), we find a solution of the equation under consideration (where the specific numerical value of the coefficient a can be chosen arbitrarily). The case $a < 0$ (or

TABLE 21
Solvable cases of the Emden–Fowler equation $'' = A$

No			Equation
	One-parameter families		
1	arbitrary	0	2.3.1.2
2	arbitrary	$- \quad -3$	2.3.1.3
3	arbitrary	$-\frac{1}{2}(\quad +3)$	2.3.1.4
4	0	arbitrary	2.3.1.1
5	1	arbitrary	2.3.1.5
	Isolated points		
6	−7	1	2.3.1.15
7	−7	3	2.3.1.16
8	−5 2	−1 2	2.3.1.22
9	−2	−2	2.3.1.28
10	−2	1	2.3.1.27
11	−5 3	−10 3	2.3.1.10
12	−5 3	−7 3	2.3.1.8

No			Equation
13	−5 3	−5 6	2.3.1.23
14	−5 3	−1 2	2.3.1.24
15	−5 3	1	2.3.1.7
16	−5 3	2	2.3.1.9
17	−7 5	−13 5	2.3.1.14
18	−7 5	1	2.3.1.13
19	−1 2	−7 2	2.3.1.12
20	−1 2	−5 2	2.3.1.6
21	−1 2	−2	2.3.1.26
22	−1 2	−4 3	2.3.1.17
23	−1 2	−7 6	2.3.1.18
24	−1 2	−1 2	2.3.1.25
25	−1 2	1	2.3.1.11
26	2	−5	2.3.1.19
27	2	−20 7	2.3.1.21
28	2	−15 7	2.3.1.20

< 0), which may lead to another branch of the solution or to a different domain of definition of the variables and in (2), should be considered in a similar manner.

One can also use a different approach by setting one of the auxiliary coefficients (e.g., a) equal to 1 in (1) and (2); then the other coefficients will be identically expressed in terms of A by means of (1).

Table 21 presents all solvable Emden–Fowler equations whose solutions are outlined in Subsection 2.3.1. The one-parameter families (in the space of the parameters and) and isolated points are presented in a consecutive fashion. Equations are arranged in accordance with the growth of and the growth of (for identical). The number of the equation sought is indicated in the last column in this table.

2.3.1-2. Solvable equations and their solutions.

1. $\quad = \quad .$

Solution: $$= \begin{cases} \dfrac{A\quad^{+2}}{(\quad+1)(\quad+2)} + {}_1 + {}_2 & \text{if } \neq -1,\ \neq -2; \\ -A\ln|\ | + {}_1 + {}_2 & \text{if } = -2; \\ A\quad \ln|\ | \quad + {}_1 + {}_2 & \text{if } = -1. \end{cases}$$

2. $\quad = \quad .$

Solution: $$= \begin{cases} \left(\dfrac{2A}{\quad+1}\quad^{+1} + {}_1\right)^{-1\ 2} + {}_2 & \text{if } \neq -1, \\ (2A\ln|\ | + {}_1)^{-1\ 2} + {}_2 & \text{if } = -1. \end{cases}$$

Special cases.

1 . In the case $= -1\ \ 2$, the solution can be written in the parametric form:

$$= a\ \ {}^{3}_{1}(\tau^3 - 3\tau + \ \ {}_2), \qquad = \ \ {}^{4}_{1}(\tau^2 - 1)^2, \qquad \text{where} \quad A = \ \ \tfrac{4}{9}a^{-2\ \ 3\ \ 2}.$$

2 . In the case $= -4$, the solution can be written in the parametric form:

$$= a\ \ {}^{5}_{1}\tau^{-1}\ \ 2\tau \quad \frac{\tau\ \ \tau}{R} + \ \ {}_2\tau \mp R\ \ , \qquad = \ \ {}^{2}_{1}\tau^{-1},$$

where $R = \quad \overline{(4\tau^3 - 1)}$, $A = \mp 6a^{-2\ \ 5}$.

3 . In the case $= 2$, the solution can be written in the parametric form:

$$= a\ \ {}^{-1}_{1}\tau, \qquad = \ \ {}^{2}_{1}\wp; \qquad A = \ \ 6a^{-2\ \ -1},$$

the function $\wp$ of the parameter τ is defined in implicit form:

$$\tau = \quad \frac{\wp}{\overline{(4\wp^3 - 1)}} - \ \ {}_2.$$

The upper sign in this formula corresponds to the classical elliptic Weierstrass function $\wp = \wp(\tau + \ \ {}_2, 0, 1)$.

4 . In the case $= -5\ \ 2$, the solution can be written in the parametric form:

$$= a\ \ {}^{7}_{1}\wp^{-2}\big[\quad \overline{(4\wp^3 - 1)}\quad 2\tau\wp^2\big], \qquad = \ \ {}^{4}_{1}\wp^{-2}, \qquad \text{where} \quad A = \mp 3a^{-2\ \ 7\ \ 2}.$$

The function $\wp$ of the parameter τ is defined in the previous case.

3. $\quad = \quad - \ \ {}^{-3} \quad .$

1 . Solution in parametric form with $\neq -1$:

$$= a\ \ {}_1{}^{-1} \quad (1 \quad \tau\ \ {}^{+1})^{-1\ \ 2}\ \ \tau + \ \ {}_2\ \ {}^{-1}, \qquad = \ \ {}_1{}^{+1}\tau \quad (1 \quad \tau\ \ {}^{+1})^{-1\ \ 2}\ \ \tau + \ \ {}_2\ \ {}^{-1},$$

where $A = \dfrac{\ \ +1}{2}a\ \ {}^{+1\ \ 1-}\ \ .$

2 . Solution in parametric form with $= -1$:

$$= \ \ {}_1 \quad \exp(\mp\tau^2)\ \ \tau + \ \ {}_2\ \ {}^{-1}, \qquad = \exp(\mp\tau^2) \quad \exp(\mp\tau^2)\ \ \tau + \ \ {}_2\ \ {}^{-1},$$

where $A = \mp 2\ \ {}^2$.

4. $\quad = \quad -\frac{+3}{2} \quad .$

1 . Solution in parametric form with $\neq -1$:

$$= a\ \ {}^{2}_{2}\exp\ 2 \quad \frac{8}{\ \ +1}\tau\ \ {}^{+1} + \tau^2 + \ \ {}_1\ \ {}^{-1\ \ 2}\ \ \tau\ \ ,$$

$$= \ \ {}_2\tau\exp \quad \frac{8}{\ \ +1}\tau\ \ {}^{+1} + \tau^2 + \ \ {}_1\ \ {}^{-1\ \ 2}\ \ \tau\ \ ,$$

where $A = \dfrac{a}{\ \ 2}\ \ {}^{\frac{-1}{2}}$.

2 . Solution in parametric form with $= -1$:

$$= a\ \ {}^{2}_{2}\exp\ 2 \quad \big(8\ln|\tau| + \tau^2 + \ \ {}_1\ \ {}^{-1\ \ 2}\ \ \tau\ \ , \qquad = \ \ {}_2\tau\exp \quad \big(8\ln|\tau| + \tau^2 + \ \ {}_1\ \ {}^{-1\ \ 2}\ \ \tau\ \ ,$$

where $A = \ \ {}^2\ a$.

5. $=$ $\ .$

For $\neq -2$, see equation 2.1.2.7. For $= -2$, see equation 2.1.2.123.

6. $=$ $^{-5\ 2\ \ -1\ 2}.$

Solution in parametric form:

$$= a\ {}_1^{-3}(\tau^3 - 3\tau + \ {}_2)^{-1}, \qquad = \ {}_1(\tau^2-1)^2(\tau^3 - 3\tau + \ {}_2)^{-1}, \qquad \text{where} \quad A = \tfrac{4}{9}a^{1\ 2\ 3\ 2}.$$

7. $=$ $^{-5\ 3}.$

Solution in parametric form:

$$= \ a\ {}_1^{8}(\tau^4 - 6\tau^2 + 4\ {}_2\tau - 3), \qquad = \ {}_1^{9}(\tau^3 - 3\tau + \ {}_2)^{3\ 2}, \qquad \text{where} \quad A = \tfrac{9}{64}a^{-3\ 8\ 3}.$$

8. $=$ $^{-7\ 3\ \ -5\ 3}.$

Solution in parametric form:

$$= \frac{a\ {}_1^{-8}}{\tau^4 - 6\tau^2 + 4\ {}_2\tau - 3}, \qquad = \frac{{}_1(\tau^3 - 3\tau + \ {}_2)^{3\ 2}}{\tau^4 - 6\tau^2 + 4\ {}_2\tau - 3}, \qquad \text{where} \quad A = \tfrac{9}{64}a^{1\ 3\ 8\ 3}.$$

9. $=$ $^{2\ \ -5\ 3}.$

1 . Solution in parametric form with $A < 0$:

$$= a\ {}_1^{2}\cos\tau\cosh(\tau + \ {}_2)[\tan\tau + \tanh(\tau + \ {}_2)], \qquad = \ {}_1^{3}[\cos\tau\cosh(\tau + \ {}_2)]^{3\ 2},$$

where $A = -\frac{3}{16}a^{-4\ 8\ 3}$.

2 . Solution in parametric form with $A > 0$:

$$= a\ {}_1^{2}[\sinh\tau + \cos(\tau + \ {}_2)], \qquad = \ {}_1^{3}[\cosh\tau - \sin(\tau + \ {}_2)]^{3\ 2},$$

where $A = \frac{3}{4}a^{-4\ 8\ 3}$.

10. $=$ $^{-10\ 3\ \ -5\ 3}.$

1 . Solution in parametric form with $A < 0$:

$$\begin{aligned} &= a\ {}_1^{-2}[\cos\tau\cosh(\tau + \ {}_2)]^{-1}[\tan\tau + \tanh(\tau + \ {}_2)]^{-1}, \\ &= \ {}_1[\cos\tau\cosh(\tau + \ {}_2)]^{1\ 2}[\tan\tau + \tanh(\tau + \ {}_2)]^{-1}, \end{aligned}$$

where $A = -\frac{3}{16}a^{4\ 3\ 8\ 3}$.

2 . Solution in parametric form with $A > 0$:

$$= a\ {}_1^{-2}[\sinh\tau + \cos(\tau + \ {}_2)]^{-1}, \qquad = \ {}_1[\cosh\tau - \sin(\tau + \ {}_2)]^{3\ 2}[\sinh\tau + \cos(\tau + \ {}_2)]^{-1},$$

where $A = \frac{3}{4}a^{4\ 3\ 8\ 3}$.

11. $=$ $^{-1\ 2}.$

Solution in parametric form:

$$\begin{aligned} &= a\ \ {}_1\exp(-\tau)\big[\exp(3\tau) + \ {}_2\sin\big(\ \ 3\tau\ \big]\big], \\ &= \ {}_1^{2}\exp(-2\tau)\big[2\exp(3\tau) - \ {}_2\sin\big(\ \ 3\tau\ \ + \ \ 3\ \ {}_2\cos\big(\ \ 3\tau\ \big]^{2}, \end{aligned}$$

where $A = 16a^{-3\ 3\ 2}$.

In the solutions of equations 12–14, the following notation is used:

$$_1 = \exp(3\tau) + \ _2 \sin\bigl(\ \overline{3}\tau\ \bigr), \qquad _2 = 2\exp(3\tau) - \ _2 \sin\bigl(\ \overline{3}\tau\ \bigr) + \ \overline{3}\ \ _2 \cos\bigl(\ \overline{3}\tau\ \bigr),$$
$$_3 = 2\ \ _1(\ _2)'_\tau - (\ _1)'_\tau\ \ _2 - \ \ _1\ \ _2.$$

12. $= \ ^{-7\ 2\ -1\ 2}.$

Solution in parametric form:

$$= a\ \ _1^{-1}\exp(\tau)\ \ _1^{-1}, \qquad = \ \ _1\exp(-\tau)\ \ _1^{-1}\ \ _2^2, \qquad \text{where} \quad A = 16(a\ )^{3\ 2}.$$

13. $= \ ^{-7\ 5}.$

Solution in parametric form:

$$= a\ \ _1^4\exp(-2\tau)\ \ _3, \qquad = \ \ _1^5\exp\bigl(-\tfrac{5}{2}\tau\ \ \ _1^{5\ 2}, \qquad \text{where} \quad A = \tfrac{5}{1024}a^{-3\ 12\ 5}.$$

14. $= \ ^{-13\ 5\ -7\ 5}.$

Solution in parametric form:

$$= a\ \ _1^{-4}\exp(2\tau)\ \ _3^{-1}, \qquad = \ \ _1\exp\bigl(-\tfrac{1}{2}\tau\ \ \ _1^{5\ 2}\ \ _3^{-1}, \qquad \text{where} \quad A = \tfrac{5}{1024}a^{3\ 5\ 12\ 5}.$$

In the solutions of equations 15–18, the following notation is used:

$$= 2\tau\ (\tau) + \ _2\tau \mp R, \qquad (\tau) = \ \frac{\tau\ \ \tau}{R}, \qquad R = \ \overline{(4\tau^3 - 1)},$$

where (τ) *is the incomplete elliptic integral of the second kind in the form of Weierstrass.*

15. $= \ ^{-7}.$

Solution in parametric form:

$$= a\ \ _1^8[4\tau\ ^2 \mp \tau^{-2}(\ R-1)^2], \qquad = \ \ _1^{3\ \ 1\ 2}, \qquad \text{where} \quad A = \ \tfrac{3}{64}a^{-3\ 8}.$$

16. $= \ ^{3\ -7}.$

Solution in parametric form:

$$= a\ \ _1^8[4\tau\ ^2 \mp \tau^{-2}(\ R-1)^2]^{-1}, \qquad = \ \ _1^{5\ \ 1\ 2}[4\tau\ ^2 \mp \tau^{-2}(\ R-1)^2]^{-1},$$

where $A = \ \tfrac{3}{64}a^{-5\ 8}.$

17. $= \ ^{-4\ 3\ -1\ 2}.$

Solution in parametric form:

$$= a\ \ _1^{9\ \ 3}, \qquad = \ \ _1^4\tau^{-2}(\ R-1)^2, \qquad \text{where} \quad A = \ \tfrac{4}{3}a^{-2\ 3\ 3\ 2}.$$

18. $= \ ^{-7\ 6\ -1\ 2}.$

Solution in parametric form:

$$= a\ \ _1^{9\ \ -3}, \qquad = \ \ _1^5\tau^{-2\ \ -3}(\ R-1)^2, \qquad \text{where} \quad A = \ \tfrac{4}{3}a^{-5\ 6\ 3\ 2}.$$

In the solutions of equations 19–24, the function $=\ (\ )$ ***is defined in implicit form:***

$$\tau = \ \frac{\wp}{\overline{\ (4\wp^3 - 1)}} - \ \ _2.$$

The upper sign in this formula corresponds to the classical elliptic Weierstrass function $\wp = \wp(\tau + \ _2, 0, 1)$.

19. $= \ ^{-5\ \ 2}.$

Solution in parametric form:

$$= a\ \ _1\tau^{-1}, \qquad = \ \ _1^3\tau^{-1}\wp, \qquad \text{where} \quad A = \ 6a^{3\ \ -1}.$$

20. $= \quad ^{-15\ 7\ \ 2}$.

Solution in parametric form:

$$= a \quad {}^{7}_{1}\tau^{7}, \qquad = \quad {}_{1}\tau(\tau^{2}\wp \mp 1), \qquad \text{where} \quad A = \ \tfrac{6}{49}a^{1\ \ 7\ \ -1}.$$

21. $= \quad ^{-20\ 7\ \ 2}$.

Solution in parametric form:

$$= a \quad {}^{7}_{1}\tau^{-7}, \qquad = \quad {}^{6}_{1}\tau^{-6}(\tau^{2}\wp \mp 1), \qquad \text{where} \quad A = \ \tfrac{6}{49}a^{6\ \ 7\ \ -1}.$$

22. $= \quad ^{-1\ 2\ \ -5\ 2}$.

Solution in parametric form:

$$= a \quad {}^{7}_{1}\wp^{2}\Big[\ \overline{\quad(4\wp^{3}-1)}\quad 2\tau\wp^{2}\Big]^{-1}, \qquad = \quad {}^{3}_{1}\Big[\ \overline{\quad(4\wp^{3}-1)}\quad 2\tau\wp^{2}\Big]^{-1},$$

where $A = \mp 3a^{-3\ \ 2\ \ 7\ \ 2}$.

23. $= \quad ^{-5\ 6\ \ -5\ 3}$.

Solution in parametric form:

$$= \frac{a \quad {}^{16}_{1}}{\Big[\tau \quad \overline{\quad(4\wp^{3}-1)}+2\wp\Big]^{2}}, \qquad = \frac{{}^{7}_{1}\Big[\ \overline{\quad(4\wp^{3}-1)}\quad 2\tau\wp^{2}\Big]^{3\ \ 2}}{\Big[\tau \quad \overline{\quad(4\wp^{3}-1)}+2\wp\Big]^{2}}, \quad \text{where } A = -\tfrac{1}{6}a^{-7\ \ 6\ \ 8\ \ 3}.$$

24. $= \quad ^{-1\ 2\ \ -5\ 3}$.

Solution in parametric form:

$$= a \quad {}^{16}_{1}\Big[\tau \quad \overline{\quad(4\wp^{3}-1)}+2\wp\Big]^{2}, \qquad = \quad {}^{9}_{1}\Big[\ \overline{\quad(4\wp^{3}-1)}\quad 2\tau\wp^{2}\Big]^{3\ \ 2}, \quad \text{where } A = -\tfrac{1}{6}a^{-3\ \ 2\ \ 8\ \ 3}.$$

In the solutions of equations 25–28, the following notation is used:

$$Z = \begin{cases} \ {}_{1}\ {}_{1\ \ 3}(\tau) + \quad {}_{2}\ {}_{1\ \ 3}(\tau) & \textit{for the upper sign,} \\ \ {}_{1}\ {}_{1\ \ 3}(\tau) + \quad {}_{2}\ {}_{1\ \ 3}(\tau) & \textit{for the lower sign,} \end{cases}$$

where $_{1\ \ 3}(\tau)$ *and* $_{1\ \ 3}(\tau)$ *are the Bessel functions, and* $_{1\ \ 3}(\tau)$ *and* $_{1\ \ 3}(\tau)$ *are the modified Bessel functions.*

25. $= \quad ^{-1\ 2\ \ -1\ 2}$.

Solution in parametric form:

$$= a\tau^{2\ \ 3}Z^{2}, \qquad = \quad \tau^{-2\ \ 3}\big(\tau Z'_{\tau} + \tfrac{1}{3}Z \quad ^{2}, \qquad \text{where} \quad A = \tfrac{1}{3}(\mp \quad a)^{3\ \ 2}.$$

26. $= \quad ^{-2\ \ -1\ 2}$.

Solution in parametric form:

$$= a\tau^{-2\ \ 3}Z^{-2}, \qquad = \quad \tau^{-4\ \ 3}Z^{-2}\big(\tau Z'_{\tau} + \tfrac{1}{3}Z \quad ^{2}, \qquad \text{where} \quad A = \mp\tfrac{1}{3} \quad ^{3\ \ 2}.$$

27. $= \quad ^{-2}$.

Solution in parametric form:

$$= a\tau^{-2\ \ 3}\Big[\big(\tau Z'_{\tau} + \tfrac{1}{3}Z \quad ^{2} \quad \tau^{2}Z^{2}\Big], \qquad = \quad \tau^{2\ \ 3}Z^{2}, \qquad \text{where} \quad A = -\tfrac{9}{2}(\quad a)^{3}.$$

28. $= \quad ^{-2\ \ -2}$.

Solution in parametric form:

$$= \tau^{2\ \ 3}\Big[\big(\tau Z'_{\tau} + \tfrac{1}{3}Z \quad ^{2} \quad \tau^{2}Z^{2}\Big]^{-1}, \qquad = \quad \tau^{4\ \ 3}Z^{2}\Big[\big(\tau Z'_{\tau} + \tfrac{1}{3}Z \quad ^{2} \quad \tau^{2}Z^{2}\Big]^{-1}, \qquad \text{where} \quad A = -\tfrac{9}{2} \quad ^{3}.$$

2.3.2. First Integrals (Conservation Laws)

In this subsection, first integrals of the form

$$(\;,\;)(\; ')\; = \;, \qquad \text{where} \quad k = 2,\ 3,\ 4,\ 5,$$
$${}_{=0}$$

for the Emden–Fowler equation $'' = A$ are given.

2.3.2-1. First integrals with $k = 2$.

1 . For $= 0$ and arbitrary $(\;\neq -1)$,

$$(\;')^2 - \frac{2A}{\;+1}\;{}^{+1} = \;.$$

2 . For $= -\frac{1}{2}(\;+3)$ and arbitrary $(\;\neq -1)$,

$$(\;')^2 - \;' - \frac{2A}{\;+1}\;{}^{-\frac{+1}{2}}\;{}^{+1} = \;.$$

3 . For $= - \;- 3$ and arbitrary $(\;\neq -1)$,

$$^2(\;')^2 - 2\;\;' + \;^2 - \frac{2A}{\;+1}\;{}^{-\;-1}\;{}^{+1} = \;.$$

4 . For $= -\frac{20}{7}$, $= 2$,

$$\tfrac{343}{24}A\;^{8\;7}(\;')^2 - \bigl(\tfrac{49}{3}A\;^{1\;7} - \;\;' - \tfrac{343}{36}A^2\;^{-12\;7\;3} + \tfrac{7}{6}A\;^{-6\;7\;2} - \; = \;.$$

5 . For $= -\frac{15}{7}$, $= 2$,

$$\tfrac{343}{24}A\;^{6\;7}(\;')^2 - \bigl(\tfrac{49}{4}A\;^{-1\;7} + 1\;\;' - \tfrac{343}{36}A^2\;^{-9\;7\;3} - \tfrac{7}{8}A\;^{-8\;7\;2} = \;.$$

2.3.2-2. First integrals with $k = 3$.

1 . For $= 0$, $= -\frac{1}{2}$,

$$(\;')^3 - 6A\;^{1\;2}\;' + 6A^2\; = \;,$$
$$(\;')^3 - \;(\;')^2 - 6A\;^{1\;2}\;' + \tfrac{16}{3}A\;^{3\;2} + 3A^2\;^2 = \;.$$

2 . For $= 1$, $= -\frac{1}{2}$,

$$(\;')^3 - 6A\;^{1\;2}\;' + 4A\;^{3\;2} + 2A^2\;^3 = \;.$$

3 . For $= -\frac{4}{3}$, $= -\frac{1}{2}$,

$$(\;')^3 - \;(\;')^2 - 6A\;^{-1\;3\;1\;2}\;' - 9A^2\;^{-2\;3} = \;.$$

4 . For $= -\frac{5}{2}$, $= -\frac{1}{2}$,

$$^2(\;')^3 - 2\;(\;')^2 + (\;^2 - 6A\;^{-1\;2\;1\;2})\;' + \tfrac{2}{3}A\;^{-3\;2\;3\;2} - 3A^2\;^{-2} = \;,$$
$$^3(\;')^3 - 3\;^2\;(\;')^2 + 3(\;^2 - 2A\;^{1\;2\;1\;2})\;' - \;^3 + 6A\;^{-1\;2\;3\;2} - 6A^2\;^{-1} = \;.$$

5 . For $= -\frac{7}{6}$, $= -\frac{1}{2}$,

$$^2(\;')^3 - 2\;(\;')^2 + (\;^2 - 6A\;^{5\;6\;1\;2})\;' + 6A\;^{-1\;6\;3\;2} + 9A^2\;^{2\;3} = \;.$$

6 . For $= -\frac{7}{2}$, $= -\frac{1}{2}$,

$$^3(\;')^3 - 3\;^2\;(\;')^2 + 3(\;^2 - 2A\;^{-1\;2\;1\;2})\;' - \;^3 + 2A\;^{-3\;2\;3\;2} - 2A^2\;^{-3} = \;.$$

2.3.2-3. First integrals with $k = 4$.

1 . For $= 1$, $= -\frac{5}{3}$,

$$(\ ')^4 + 6A \quad ^{-2\ 3}(\ ')^2 - 18A \ ^{1\ 3}\ ' + 9A^2\ ^{2\ -4\ 3} = \quad ,$$

$$(\ ')^4 - \ (\ ')^3 + 6A\ ^{2\ -2\ 3}(\ ')^2 - 27A \quad ^{1\ 3}\ ' + \frac{81}{4}A\ ^{4\ 3} + 9A^2\ ^{3\ -4\ 3} = \quad .$$

2 . For $= 2$, $= -\frac{5}{3}$,

$$(\ ')^4 + 6A\ ^{2\ -2\ 3}(\ ')^2 - 36A \quad ^{1\ 3}\ ' + 9A^2\ ^{4\ -4\ 3} = \quad .$$

3 . For $= 0$, $= -\frac{5}{3}$,

$$(\ ')^4 - \ (\ ')^3 + 6A \quad ^{-2\ 3}(\ ')^2 - 9A\ ^{1\ 3}\ ' + 9A^2 \quad ^{-4\ 3} = \quad ,$$

$$^2(\ ')^4 - 2 \quad (\ ')^3 + (\ ^2 + 6A\ ^{2\ -2\ 3})(\ ')^2 - 18A \quad ^{1\ 3}\ ' + 12A\ ^{4\ 3} + 9A^2\ ^{2\ -4\ 3} = \quad .$$

4 . For $= -\frac{1}{2}$, $= -\frac{5}{3}$,

$$(\ ')^4 - \ (\ ')^3 + 6A\ ^{1\ 2\ -2\ 3}(\ ')^2 + 9A^2\ ^{-4\ 3} = \quad .$$

5 . For $= -\frac{4}{3}$, $= -\frac{5}{3}$,

$$^2(\ ')^4 - 2 \quad (\ ')^3 + (\ ^2 + 6A\ ^{2\ 3\ -2\ 3})(\ ')^2 + 6A\ ^{-1\ 3\ 1\ 3}\ ' + 9A^2\ ^{-2\ 3\ -4\ 3} = \quad ,$$

$$^3(\ ')^4 - 3\ ^2\ (\ ')^3 + 3\ (\ ^2 + 2A\ ^{2\ 3\ -2\ 3})(\ ')^2$$
$$- (\ ^3 + 3A\ ^{2\ 3\ 1\ 3})\ ' - 3A\ ^{-1\ 3\ 4\ 3} + 9A^2\ ^{1\ 3\ -4\ 3} = \quad .$$

6 . For $= -\frac{7}{3}$, $= -\frac{5}{3}$,

$$^3(\ ')^4 - 3\ ^2\ (\ ')^3 + 3\ (\ ^2 + 2A\ ^{-1\ 3\ -2\ 3})(\ ')^2$$
$$- (\ ^3 - 15A\ ^{-1\ 3\ 1\ 3})\ ' - \frac{3}{4}A\ ^{-4\ 3\ 4\ 3} + 9A^2\ ^{-5\ 3\ -4\ 3} = \quad ,$$

$$^4(\ ')^4 - 4\ ^3\ (\ ')^3 + 6\ ^2(\ ^2 + A\ ^{-1\ 3\ -2\ 3})(\ ')^2$$
$$- 2\ (2\ ^3 - 3A\ ^{-1\ 3\ 1\ 3})\ ' + \ ^4 - 12A\ ^{-1\ 3\ 4\ 3} + 9A^2\ ^{-2\ 3\ -4\ 3} = \quad .$$

7 . For $= -\frac{5}{6}$, $= -\frac{5}{3}$,

$$^3(\ ')^4 - 3\ ^2\ (\ ')^3 + 3\ (\ ^2 + 2A\ ^{7\ 6\ -2\ 3})(\ ')^2$$
$$- (\ ^3 + 12A\ ^{7\ 6\ 1\ 3})\ ' + 6A\ ^{1\ 6\ 4\ 3} + 9A^2\ ^{4\ 3\ -4\ 3} = \quad .$$

8 . For $= -\frac{10}{3}$, $= -\frac{5}{3}$,

$$^4(\ ')^4 - 4\ ^3\ (\ ')^3 + 6\ ^2(\ ^2 + A\ ^{-4\ 3\ -2\ 3})(\ ')^2$$
$$- 4\ (\ ^3 - 6A\ ^{-4\ 3\ 1\ 3})\ ' + \ ^4 - 30A\ ^{-4\ 3\ 4\ 3} + 9A^2\ ^{-8\ 3\ -4\ 3} = \quad .$$

9 . For $= 1$, $= -7$,

$$(\ ')^4 - \ (\ ')^3 + \frac{2}{3}A\ ^{2\ -6}(\ ')^2 - \frac{1}{3}A \quad ^{-5}\ ' - \frac{1}{12}A\ ^{-4} + \frac{1}{9}A^2\ ^{3\ -12} = \quad .$$

10 . For $= 3$, $= -7$,

$$^3(\ ')^4 - 3\ ^2\ (\ ')^3 + 3\ (\ ^2 + \frac{2}{9}A\ ^{5\ -6})(\ ')^2 - \ (\ ^2 + A\ ^{5\ -6})\ ' + \frac{1}{4}A\ ^{4\ -4} + \frac{1}{9}A^2\ ^{9\ -12} = \quad .$$

In the case $k = 4$ we omitted the first integrals of the form

$$^2 + \beta \quad + \quad = \quad ,$$

where function $= \ (\ ,\ ,\ ')$ is the left-hand side of the above integrals for $k = 2$, and $,\ \beta$, and are some constants.

2.3.2-4. First integrals with $k=5$.

1 . For $\;=0,\quad\;=-\tfrac{2}{3}$,

$$(\;')^5-15A\;^{1\;3}(\;')^3+\tfrac{135}{2}A^2\;^{2\;3}\;'-\tfrac{135}{2}A^3\;\;=\;\;.$$

2 . For $\;=-\tfrac{7}{3},\quad\;=-\tfrac{2}{3}$,

$$\begin{aligned}&{}^5(\;')^5-5\;^4\;(\;')^4+5\;^3\;(2\;-3A\;^{-1\;3\;-2\;3})(\;')^3-5\;^2\;^2(2\;-9A\;^{-1\;3\;-2\;3})(\;')^2\\&\quad+5\;(\;^4-9A\;^{-1\;3\;7\;3}+\tfrac{27}{2}A^2\;^{-2\;3\;2\;3})\;'+15A(\;^{-1\;3\;10\;3}-\tfrac{9}{2}A\;^{-2\;3\;5\;3}-\tfrac{9}{2}A^2\;^{-1})=\;\;.\end{aligned}$$

2.3.3. Some Formulas and Transformations

1 . With $\;\neq 1$, the Emden–Fowler equation has a particular solution:

$$=\lambda\;^{\frac{+2}{1-}},\qquad\text{where }\lambda=\left[\frac{(\;+2)(\;+\;+1)}{A(\;-1)^2}\right]^{\frac{1}{-1}}.$$

2 . The transformation $\;=\;\;,\;\;=1\;$ leads to the Emden–Fowler equation with the independent variable raised to a different power:

$$''=A\;^{-\;-\;-3}\;\;.$$

3 . Some more complicated transformations leading to the Emden–Fowler equation are outlined in Subsection 2.5.3 (see Fig. 3).

4 . With $\;\neq 1$ and $\;\neq -2\;-3$, the transformation

$$\xi=\frac{2\;+\;+3}{-1}\;^{\frac{+2}{-1}},\qquad=\;^{\frac{+2}{-1}}\left(\;'+\frac{+2}{-1}\right)$$

leads to an Abel equation:

$$'-\;=-\frac{(\;+2)(\;+\;+1)}{(2\;+\;+3)^2}\xi+A\left(\frac{-1}{2\;+\;+3}\right)^2\xi\;,$$

whose special cases are given in Subsection 1.3.1.

5 . Some more complicated transformations leading to other Abel equations are outlined in Subsection 2.5.3.

2.4. Equations of the Form $y\;=\;_1\;^1y\;^1+\;_2\;^2y\;^2$

See Section 2.3 for the special cases $A_1=0$ and $A_2=0$.

2.4.1. Classification Table

Table 22 presents all solvable equations of the form $''=A_1\;^1\;^1+A_2\;^2\;^2$ whose solutions are outlined in Subsection 2.4.2. Two-parameter families (in the space of the parameters $_1$, $_2$, $_1$, and $_2$), one-parameter families, and isolated points are presented in a consecutive fashion. Equations are arranged in accordance with the growth of $_1$, the growth of $_2$ (for identical $_1$), the growth of $_1$ (for identical $_1$ and $_2$), and the growth of $_2$ (for identical $_1$, $_2$, and $_1$). The number of the equation sought is indicated in the last column in this table.

TABLE 22

Solvable equations of the form $'' = A_1 \;{}^{1}\;{}^{1} + A_2 \;{}^{2}\;{}^{2}$

No	$_1$	$_2$	$_1$	$_2$	A_1	A_2	Equation
1	arbitrary	arbitrary	0	0	arbitrary	arbitrary	2.4.2.1
2	arbitrary	arbitrary	$-\ _1-3$	$-\ _2-3$	arbitrary	arbitrary	2.4.2.2
3	arbitrary	arbitrary	$-\frac{1}{2}(\ _1+3)$	$-\frac{1}{2}(\ _2+3)$	arbitrary	arbitrary	2.4.2.3
4	arbitrary	0	0	0	arbitrary	arbitrary	2.4.2.19
5	arbitrary	0	$-\ _1-3$	-3	arbitrary	arbitrary	2.4.2.20
6	1	arbitrary	-2	-2	$-\frac{2(\ _2+1)}{(\ _2+3)^2}$	arbitrary	2.4.2.4
7	1	arbitrary	-2	$-\ _2-1$	$-\frac{2(\ _2+1)}{(\ _2+3)^2}$	arbitrary	2.4.2.5
8	1	-3	arbitrary ($\ _1\neq-2$)	0	arbitrary	arbitrary	2.4.2.83
9	-7	-7	4	3	arbitrary	arbitrary	2.4.2.39
10	-5	-5	2	0	arbitrary	arbitrary	2.4.2.16
11	-3	-7	0	1	arbitrary	arbitrary	2.4.2.42
12	-3	-7	0	3	arbitrary	arbitrary	2.4.2.43
13	-3	-4	0	0	arbitrary	arbitrary	2.4.2.17
14	-3	-4	0	1	arbitrary	arbitrary	2.4.2.18
15	-2	-3	-2	0	arbitrary	arbitrary	2.4.2.88
16	-2	-3	1	0	arbitrary	arbitrary	2.4.2.87
17	-2	-2	-1	-2	arbitrary	arbitrary	2.4.2.28
18	$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{7}{3}$	$-\frac{10}{3}$	arbitrary	arbitrary	2.4.2.48
19	$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{4}{3}$	$-\frac{10}{3}$	arbitrary	arbitrary	2.4.2.49
20	$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{4}{3}$	$-\frac{7}{3}$	arbitrary	arbitrary	2.4.2.24
21	$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{2}{3}$	$-\frac{4}{3}$	arbitrary	arbitrary	2.4.2.90
22	$-\frac{5}{3}$	$-\frac{5}{3}$	0	$-\frac{2}{3}$	arbitrary	arbitrary	2.4.2.89
23	$-\frac{5}{3}$	$-\frac{5}{3}$	2	0	arbitrary	arbitrary	2.4.2.47
24	$-\frac{5}{3}$	$-\frac{5}{3}$	2	1	arbitrary	arbitrary	2.4.2.46
25	$-\frac{3}{2}$	-2	$-\frac{3}{2}$	-2	arbitrary	arbitrary	2.4.2.81
26	$-\frac{3}{2}$	-2	0	1	arbitrary	arbitrary	2.4.2.80
27	$-\frac{7}{5}$	$-\frac{7}{5}$	$-\frac{8}{5}$	$-\frac{13}{5}$	arbitrary	arbitrary	2.4.2.25
28	$-\frac{4}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{7}{3}$	arbitrary	arbitrary	2.4.2.102
29	$-\frac{4}{3}$	$-\frac{5}{3}$	0	1	arbitrary	arbitrary	2.4.2.101
30	$-\frac{3}{5}$	$-\frac{7}{5}$	$-\frac{12}{5}$	$-\frac{13}{5}$	arbitrary	arbitrary	2.4.2.53
31	$-\frac{3}{5}$	$-\frac{7}{5}$	0	1	arbitrary	arbitrary	2.4.2.52
32	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{5}{2}$	$-\frac{7}{2}$	arbitrary	arbitrary	2.4.2.23

TABLE 22 (*Continued*)

Solvable equations of the form $'' = A_1 \quad {}^1 \quad {}^1 + A_2 \quad {}^2 \quad {}^2$

No	$_1$	$_2$	$_1$	$_2$	A_1	A_2	Equation
33	$-\frac{1}{3}$	$-\frac{5}{3}$	$-\frac{8}{3}$	$-\frac{10}{3}$	arbitrary	arbitrary	2.4.2.55
34	$-\frac{1}{3}$	$-\frac{5}{3}$	$-\frac{8}{3}$	$-\frac{7}{3}$	arbitrary	arbitrary	2.4.2.59
35	$-\frac{1}{3}$	$-\frac{5}{3}$	$-\frac{8}{3}$	$-\frac{4}{3}$	arbitrary	arbitrary	2.4.2.57
36	$-\frac{1}{3}$	$-\frac{5}{3}$	0	0	arbitrary	arbitrary	2.4.2.56
37	$-\frac{1}{3}$	$-\frac{5}{3}$	0	1	arbitrary	arbitrary	2.4.2.58
38	$-\frac{1}{3}$	$-\frac{5}{3}$	0	2	arbitrary	arbitrary	2.4.2.54
39	0	–2	–3	–2	arbitrary	arbitrary	2.4.2.108
40	0	–2	0	1	arbitrary	arbitrary	2.4.2.107
41	0	–1	–3	–2	arbitrary	arbitrary	2.4.2.22
42	0	–1	0	0	arbitrary	arbitrary	2.4.2.21
43	0	$-\frac{2}{3}$	–3	$-\frac{7}{3}$	arbitrary	arbitrary	2.4.2.73
44	0	$-\frac{2}{3}$	0	0	arbitrary	arbitrary	2.4.2.72
45	0	$-\frac{1}{2}$	–4	$-\frac{5}{2}$	arbitrary	arbitrary	2.4.2.96
46	0	$-\frac{1}{2}$	–3	$-\frac{7}{2}$	arbitrary	arbitrary	2.4.2.51
47	0	$-\frac{1}{2}$	–3	$-\frac{5}{2}$	arbitrary	arbitrary	2.4.2.45
48	0	$-\frac{1}{2}$	–3	–2	arbitrary	arbitrary	2.4.2.106
49	0	$-\frac{1}{2}$	–3	$-\frac{1}{2}$	arbitrary	arbitrary	2.4.2.85
50	0	$-\frac{1}{2}$	$-\frac{5}{3}$	$-\frac{7}{6}$	arbitrary	arbitrary	2.4.2.41
51	0	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{2}$	arbitrary	arbitrary	2.4.2.100
52	0	$-\frac{1}{2}$	$-\frac{3}{2}$	–2	arbitrary	arbitrary	2.4.2.79
53	0	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	arbitrary	arbitrary	2.4.2.78
54	0	$-\frac{1}{2}$	$-\frac{3}{2}$	0	arbitrary	arbitrary	2.4.2.99
55	0	$-\frac{1}{2}$	$-\frac{4}{3}$	$-\frac{4}{3}$	arbitrary	arbitrary	2.4.2.40
56	0	$-\frac{1}{2}$	0	–2	arbitrary	arbitrary	2.4.2.86
57	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	arbitrary	arbitrary	2.4.2.105
58	0	$-\frac{1}{2}$	0	0	arbitrary	arbitrary	2.4.2.44
59	0	$-\frac{1}{2}$	0	1	arbitrary	arbitrary	2.4.2.50
60	0	$-\frac{1}{2}$	1	0	arbitrary	arbitrary	2.4.2.95
61	$\frac{1}{3}$	$-\frac{5}{3}$	$-\frac{10}{3}$	$-\frac{7}{3}$	arbitrary	arbitrary	2.4.2.98
62	$\frac{1}{3}$	$-\frac{5}{3}$	0	1	arbitrary	arbitrary	2.4.2.97
63	1	–7	–2	–2	$\frac{15}{4}$	arbitrary	2.4.2.35
64	1	–7	–2	6	$\frac{15}{4}$	arbitrary	2.4.2.36
65	1	–4	–2	–2	6	arbitrary	2.4.2.31
66	1	–4	–2	3	6	arbitrary	2.4.2.32

TABLE 22 (*Continued*)
Solvable equations of the form $'' = A_1 \quad {}_1 \quad {}^1 + A_2 \quad {}_2 \quad {}^2$

No	1	2	1	2	A_1	A_2	Equation
67	1	–3	–5	0	arbitrary	arbitrary	2.4.2.84
68	1	–3	1	0	arbitrary	arbitrary	2.4.2.82
69	1	$-\frac{5}{2}$	–2	–2	12	arbitrary	2.4.2.64
70	1	$-\frac{5}{2}$	–2	$\frac{3}{2}$	12	arbitrary	2.4.2.65
71	1	–2	–2	–2	2	arbitrary	2.4.2.6
72	1	–2	–2	1	2	arbitrary	2.4.2.7
73	1	$-\frac{5}{3}$	–2	–2	$-\frac{3}{16}$	arbitrary	2.4.2.26
74	1	$-\frac{5}{3}$	–2	–2	$-\frac{9}{100}$	arbitrary	2.4.2.10
75	1	$-\frac{5}{3}$	–2	–2	$\frac{3}{4}$	arbitrary	2.4.2.12
76	1	$-\frac{5}{3}$	–2	–2	$\frac{63}{4}$	arbitrary	2.4.2.66
77	1	$-\frac{5}{3}$	–2	$\frac{2}{3}$	$-\frac{3}{16}$	arbitrary	2.4.2.27
78	1	$-\frac{5}{3}$	–2	$\frac{2}{3}$	$-\frac{9}{100}$	arbitrary	2.4.2.11
79	1	$-\frac{5}{3}$	–2	$\frac{2}{3}$	$\frac{3}{4}$	arbitrary	2.4.2.13
80	1	$-\frac{5}{3}$	–2	$\frac{2}{3}$	$\frac{63}{4}$	arbitrary	2.4.2.67
81	1	$-\frac{7}{5}$	–2	–2	$-\frac{5}{36}$	arbitrary	2.4.2.29
82	1	$-\frac{7}{5}$	–2	$\frac{2}{5}$	$-\frac{5}{36}$	arbitrary	2.4.2.30
83	1	$-\frac{1}{2}$	–2	–2	$-\frac{2}{9}$	arbitrary	2.4.2.14
84	1	$-\frac{1}{2}$	–2	–2	$-\frac{4}{25}$	arbitrary	2.4.2.8
85	1	$-\frac{1}{2}$	–2	–2	20	arbitrary	2.4.2.33
86	1	$-\frac{1}{2}$	–2	$-\frac{1}{2}$	$-\frac{2}{9}$	arbitrary	2.4.2.15
87	1	$-\frac{1}{2}$	–2	$-\frac{1}{2}$	$-\frac{4}{25}$	arbitrary	2.4.2.9
88	1	$-\frac{1}{2}$	–2	$-\frac{1}{2}$	20	arbitrary	2.4.2.34
89	1	0	–5	–3	arbitrary	arbitrary	2.4.2.77
90	1	0	1	0	arbitrary	arbitrary	2.4.2.76
91	1	$\frac{1}{2}$	–2	–2	$-\frac{12}{49}$	arbitrary	2.4.2.37
92	1	$\frac{1}{2}$	–2	$-\frac{3}{2}$	$-\frac{12}{49}$	arbitrary	2.4.2.38
93	2	0	–5	–4	arbitrary	arbitrary	2.4.2.92
94	2	0	–5	–3	arbitrary	arbitrary	2.4.2.69
95	2	0	$-\frac{20}{7}$	$-\frac{13}{7}$	arbitrary	arbitrary	2.4.2.94
96	2	0	$-\frac{20}{7}$	$-\frac{12}{7}$	arbitrary	arbitrary	2.4.2.71
97	2	0	$-\frac{15}{7}$	$-\frac{9}{7}$	arbitrary	arbitrary	2.4.2.70
98	2	0	$-\frac{15}{7}$	$-\frac{8}{7}$	arbitrary	arbitrary	2.4.2.93
99	2	0	0	0	arbitrary	arbitrary	2.4.2.68
100	2	0	0	1	arbitrary	arbitrary	2.4.2.91

TABLE 22 (*Continued*)
Solvable equations of the form $'' = A_1 \;{}^{1}\;{}^{1} + A_2\;{}^{2}\;{}^{2}$

No	$_1$	$_2$	$_1$	$_2$	A_1	A_2	Equation
101	2	1	−3	−2	arbitrary	$-\frac{6}{25}$	2.4.2.61
102	2	1	−3	−2	arbitrary	$\frac{6}{25}$	2.4.2.63
103	2	1	−2	−2	arbitrary	$-\frac{6}{25}$	2.4.2.60
104	2	1	−2	−2	arbitrary	$\frac{6}{25}$	2.4.2.62
105	3	1	−6	−5	arbitrary	arbitrary	2.4.2.104
106	3	1	0	1	arbitrary	arbitrary	2.4.2.103
107	3	2	$-\frac{18}{5}$	$-\frac{14}{5}$	arbitrary	arbitrary	2.4.2.74
108	3	2	$-\frac{12}{5}$	$-\frac{11}{5}$	arbitrary	arbitrary	2.4.2.75

2.4.2. Exact Solutions

1. $= {}_1\;{}^{1} + {}_2\;{}^{2}, \quad {}_1 \neq -1, \quad {}_2 \neq -1.$

1 . Solution in parametric form:

$$= a \;\;({}_1 + \tau \;{}^{1+1}\;\; \tau\;{}^{2+1})^{-1\;2}\;\tau + {}_2, \qquad = \tau,$$

where $A_1 = \frac{1}{2}a^{-2\;1-\;1}(\;{}_1+1)$, $A_2 = \;\frac{1}{2}a^{-2\;1-\;2}(\;{}_2+1)$.

2 . Solution in parametric form:

$$= a \;\;({}_1 - \tau \;{}^{1+1}\;\; \tau\;{}^{2+1})^{-1\;2}\;\tau + {}_2, \qquad = \tau,$$

where $A_1 = -\frac{1}{2}a^{-2\;1-\;1}(\;{}_1+1)$, $A_2 = \;\frac{1}{2}a^{-2\;1-\;2}(\;{}_2+1)$.

2. $= {}_1\;{}^{-\;\;1-3}\;\;{}^{1} + {}_2\;{}^{-\;\;2-3}\;\;{}^{2}, \quad {}_1 \neq -1, \quad {}_2 \neq -1.$

1 . Solution in parametric form:

$$= a \;\left(\int \frac{\tau}{\sqrt{{}_1 + \tau\;{}^{1+1}\;\;\tau\;{}^{2+1}}} + {}_2\right)^{-1}, \qquad = \tau\;\left(\int \frac{\tau}{\sqrt{{}_1 + \tau\;{}^{1+1}\;\;\tau\;{}^{2+1}}} + {}_2\right)^{-1},$$

where $A_1 = \frac{1}{2}a^{1+\;1\;1-\;1}(\;{}_1+1)$, $A_2 = \;\frac{1}{2}a^{1+\;2\;1-\;2}(\;{}_2+1)$.

2 . Solution in parametric form:

$$= a \;\left(\int \frac{\tau}{\sqrt{{}_1 - \tau\;{}^{1+1}\;\;\tau\;{}^{2+1}}} + {}_2\right)^{-1}, \qquad = \tau\;\left(\int \frac{\tau}{\sqrt{{}_1 - \tau\;{}^{1+1}\;\;\tau\;{}^{2+1}}} + {}_2\right)^{-1},$$

where $A_1 = -\frac{1}{2}a^{1+\;1\;1-\;1}(\;{}_1+1)$, $A_2 = \;\frac{1}{2}a^{1+\;2\;1-\;2}(\;{}_2+1)$.

3. $= {}_1\;{}^{-\frac{\;1+3}{2}}\;\;{}^{1} + {}_2\;{}^{-\frac{\;2+3}{2}}\;\;{}^{2}.$

1 . Solution in parametric form with ${}_1 \neq -1$ and ${}_2 \neq -1$:

$$= \;{}^{2}_{1}\exp \;\; {}_2 + \frac{1}{4}\tau^2 + \frac{2A_1}{\;{}_1+1}\tau\;{}^{1+1} + \frac{2A_2}{\;{}_2+1}\tau\;{}^{2+1}\;{}^{-1\;2}\;\;\tau\;,$$

$$= \;{}_1\tau\exp\;\frac{1}{2}\;\;\; {}_2 + \frac{1}{4}\tau^2 + \frac{2A_1}{\;{}_1+1}\tau\;{}^{1+1} + \frac{2A_2}{\;{}_2+1}\tau\;{}^{2+1}\;{}^{-1\;2}\;\;\tau\;.$$

2 . Solution in parametric form with $_1 \neq -1$ and $_2 = -1$:

$$= \ {}_1^2 \exp \quad {}_2 + \frac{1}{4}\tau^2 + \frac{2A_1}{_1+1}\tau \ {}^{_1+1} + 2A_2 \ln|\tau| \quad {}^{-1\ 2} \quad \tau \ ,$$

$$= \ {}_1\tau \exp \ \frac{1}{2} \quad {}_2 + \frac{1}{4}\tau^2 + \frac{2A_1}{_1+1}\tau \ {}^{_1+1} + 2A_2 \ln|\tau| \quad {}^{-1\ 2} \quad \tau \ .$$

4. $\quad = -\dfrac{2(\quad +1)}{(\quad +3)^2} \ {}^{-2} \ + \quad {}^{-2} \quad , \qquad \neq -3, \qquad \neq -1.$

Solution in parametric form:

$$= \ {}_1 \quad \frac{\tau}{1 \quad \tau \ {}^{+1}} + \ {}_2 \quad {}^{\frac{+3}{-1}}, \qquad = \tau \quad \frac{\tau}{1 \quad \tau \ {}^{+1}} + \ {}_2 \quad {}^{\frac{2}{-1}},$$

where $A = \dfrac{(\quad+1)(\quad-1)^2}{2(\quad+3)^2} \ {}^{1-} \quad .$

5. $\quad = -\dfrac{2(\quad +1)}{(\quad +3)^2} \ {}^{-2} \ + \quad - \ {}^{-1} \quad , \qquad \neq -3, \qquad \neq -1.$

Solution in parametric form:

$$= \ {}_1 \quad \frac{\tau}{1 \quad \tau \ {}^{+1}} + \ {}_2 \quad {}^{-\frac{+3}{-1}}, \qquad = \ {}_1\tau \quad \frac{\tau}{1 \quad \tau \ {}^{+1}} + \ {}_2 \quad {}^{-\frac{+1}{-1}},$$

where $A = \dfrac{(\quad+1)(\quad-1)^2}{2(\quad+3)^2} \ {}^{1-} \quad .$

6. $\quad = 2 \ {}^{-2} \ + \quad {}^{-2} \ {}^{-2}.$

Solution in parametric form:

$$= \ {}_1\big[\ \overline{\tau(\tau+1)} - \ln\big(\ \overline{\tau} + \ \overline{\tau+1} \ \ + \ {}_2\big]^{-1\ 3},$$

$$= \tau\big[\ \overline{\tau(\tau+1)} - \ln\big(\ \overline{\tau} + \ \overline{\tau+1} \ \ + \ {}_2\big]^{-2\ 3},$$

where $A = -\frac{9}{2} \ {}^{3}.$

7. $\quad = 2 \ {}^{-2} \ + \quad {}^{-2}.$

Solution in parametric form:

$$= \ {}_1\big[\ \overline{\tau(\tau+1)} - \ln\big(\ \overline{\tau} + \ \overline{\tau+1} \ \ + \ {}_2\big]^{1\ 3},$$

$$= \ {}_1\tau\big[\ \overline{\tau(\tau+1)} - \ln\big(\ \overline{\tau} + \ \overline{\tau+1} \ \ + \ {}_2\big]^{-1\ 3},$$

where $A = -\frac{9}{2} \ {}^{3}.$

8. $\quad = -\frac{4}{25} \ {}^{-2} \ + \quad {}^{-2} \ {}^{-1\ 2}.$

Solution in parametric form:

$$= \ {}_1(\tau^3 - 3\tau + \ {}_2)^{-5\ 3}, \qquad = (\tau^2-1)^2(\tau^3 - 3\tau + \ {}_2)^{-4\ 3},$$

where $A = \frac{4}{25} \ {}^{3\ 2}.$

9. $\quad = -\frac{4}{25} \ {}^{-2} \ + \quad {}^{-1\ 2} \ {}^{-1\ 2}.$

Solution in parametric form:

$$= \ {}_1(\tau^3 - 3\tau + \ {}_2)^{5\ 3}, \qquad = \ {}_1(\tau^2-1)^2(\tau^3 - 3\tau + \ {}_2)^{1\ 3},$$

where $A = \frac{4}{25} \ {}^{3\ 2}.$

10. $= -\frac{9}{100}\ {}^{-2} + \ {}^{-2}\ {}^{-5\ 3}$.

Solution in parametric form:

$$= \ {}_1\big[\ (\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)\big]^{-5\ 4}, \qquad = (\tau^3 - 3\tau + \ {}_2)^{3\ 2}\big[\ (\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)\big]^{-9\ 8},$$

where $A = \ \frac{9}{100}\ {}^{8\ 3}$.

11. $= -\frac{9}{100}\ {}^{-2} + \ {}^{2\ 3}\ {}^{-5\ 3}$.

Solution in parametric form:

$$= \ {}_1[\ (\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)]^{5\ 4}, \qquad = \ {}_1(\tau^3 - 3\tau + \ {}_2)^{3\ 2}[\ (\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)]^{1\ 8},$$

where $A = \ \frac{9}{100}\ {}^{8\ 3}$.

12. $= \frac{3}{4}\ {}^{-2} + \ {}^{-2}\ {}^{-5\ 3}$.

Solution in parametric form:

$$= \ {}_1(\tau^3 \quad 3\tau + \ {}_2)^{-1\ 2}, \qquad = (\tau^2 \quad 1)^{3\ 2}(\tau^3 \quad 3\tau + \ {}_2)^{-3\ 4}, \qquad \text{where} \quad A = \ \tfrac{4}{3}\ {}^{8\ 3}.$$

13. $= \frac{3}{4}\ {}^{-2} + \ {}^{2\ 3}\ {}^{-5\ 3}$.

Solution in parametric form:

$$= \ {}_1(\tau^3 \quad 3\tau + \ {}_2)^{1\ 2}, \qquad = \ {}_1(\tau^2 \quad 1)^{3\ 2}(\tau^3 \quad 3\tau + \ {}_2)^{-1\ 4}, \qquad \text{where} \quad A = \ \tfrac{4}{3}\ {}^{8\ 3}.$$

14. $= -\frac{2}{9}\ {}^{-2} + \ {}^{-2}\ {}^{-1\ 2}$.

Solution in parametric form:

$$\begin{aligned} &= \ {}_1(\ {}_1e^{2\ \tau} + \ {}_2e^{-\ \tau}\sin\omega)^{-3}, \quad \omega = \ \overline{3}\,k\tau, \\ &= k^2(\ {}_1e^{2\ \tau} + \ {}_2e^{-\ \tau}\sin\omega)^{-2}\big[2\ \ {}_1e^{2\ \tau} + \ {}_2e^{-\ \tau}(\ \overline{3}\cos\omega - \sin\omega\ \big]^2, \end{aligned}$$

where $A = \frac{16}{9}\ k^3$.

15. $= -\frac{2}{9}\ {}^{-2} + \ {}^{-1\ 2}\ {}^{-1\ 2}$.

Solution in parametric form:

$$\begin{aligned} &= \ {}_1(\ {}_1e^{2\ \tau} + \ {}_2e^{-\ \tau}\sin\omega)^{3}, \quad \omega = \ \overline{3}\,k\tau, \\ &= k^2\ \ {}_1(\ {}_1e^{2\ \tau} + \ {}_2e^{-\ \tau}\sin\omega)\big[2\ \ {}_1e^{2\ \tau} + \ {}_2e^{-\ \tau}(\ \overline{3}\cos\omega - \sin\omega\ \big]^2, \end{aligned}$$

where $A = \frac{16}{9}\ k^3$.

16. $= \ {}_1\ {}^{2}\ {}^{-5} + \ {}_2\ {}^{-5}$.

Solution in parametric form:

$$\begin{aligned} &= \frac{A_2}{A_1}^{1\ 2}\tan \qquad {}_1 - \frac{1}{2A_1A_2}\tau^{-4} - \tau^2 \quad {}^{-1\ 2} \quad \tau + \ {}_2\ , \\ &= A_2^{1\ 2}\tau\ \cos \qquad {}_1 - \frac{1}{2A_1A_2}\tau^{-4} - \tau^2 \quad {}^{-1\ 2} \quad \tau + \ {}_2 \qquad {}^{-1}. \end{aligned}$$

17. $= \ {}_1\ {}^{-3} + \ {}_2\ {}^{-4}$.

1 . Solution in parametric form:

$$= a \quad (\ {}_1 + \tau^{-3} \quad \tau^{-2})^{-1\ 2} \quad \tau + \ {}_2\ , \qquad = \ \tau, \qquad \text{where} \quad A_1 = \mp a^{-2\ 4}, \ \ A_2 = -\tfrac{3}{2}a^{-2\ 5}.$$

2 . Solution in parametric form:

$$= a \quad (\;_1 - \tau^{-3} \quad \tau^{-2})^{-1\;\;2} \quad \tau + \quad _2 \;, \qquad = \;\tau, \qquad \text{where} \quad A_1 = \mp a^{-2\;\;4}, \;\; A_2 = \tfrac{3}{2} a^{-2\;\;5}.$$

18. $= \;_1 \;\;^{-3} + \;\;_2 \;\;\;^{-4}.$

1 . Solution in parametric form:

$$= a \quad (\;_1 + \tau^{-3} \quad \tau^{-2})^{-1\;\;2} \quad \tau + \quad _2 \;\;^{-1}, \qquad = \;\tau \quad (\;_1 + \tau^{-3} \quad \tau^{-2})^{-1\;\;2} \quad \tau + \quad _2 \;\;^{-1},$$

where $A_1 = \mp a^{-2\;\;4}$, $A_2 = -\tfrac{3}{2} a^{-3\;\;5}$.

2 . Solution in parametric form:

$$= a \quad (\;_1 - \tau^{-3} \quad \tau^{-2})^{-1\;\;2} \quad \tau + \quad _2 \;\;^{-1}, \qquad = \;\tau \quad (\;_1 - \tau^{-3} \quad \tau^{-2})^{-1\;\;2} \quad \tau + \quad _2 \;\;^{-1},$$

where $A_1 = \mp a^{-2\;\;4}$, $A_2 = \tfrac{3}{2} a^{-3\;\;5}$.

19. $= \;_1 \quad + \;\;_2, \qquad \neq -1.$

1 . Solution in parametric form:

$$= a \quad (\;_1 + \tau \;\;^{+1} \quad \tau)^{-1\;\;2} \quad \tau + \quad _2 \;, \qquad = \;\tau,$$

where $A_1 = \tfrac{1}{2} a^{-2\;\;1-} \;\; (\quad + 1)$, $A_2 = \;\; \tfrac{1}{2} a^{-2}\;\;$.

2 . Solution in parametric form:

$$= a \quad (\;_1 - \tau \;\;^{+1} \quad \tau)^{-1\;\;2} \quad \tau + \quad _2 \;, \qquad = \;\tau,$$

where $A_1 = -\tfrac{1}{2} a^{-2\;\;1-} \;\; (\quad + 1)$, $A_2 = \;\; \tfrac{1}{2} a^{-2}\;\;$.

3 . For the case $\quad = -1$, see equation 2.4.2.21.

20. $= \;_1 \;\;^{-} \;\;^{-3} \quad + \;\;_2 \;\;^{-3}, \qquad \neq -1.$

1 . Solution in parametric form:

$$= a \quad (\;_1 + \tau \;\;^{+1} \quad \tau)^{-1\;\;2} \quad \tau + \quad _2 \;\;^{-1}, \qquad = \;\tau \quad (\;_1 + \tau \;\;^{+1} \quad \tau)^{-1\;\;2} \quad \tau + \quad _2 \;\;^{-1},$$

where $A_1 = \tfrac{1}{2} a^{1+ \quad 1-} \;\; (\quad + 1)$, $A_2 = \;\; \tfrac{1}{2} a\;\;$.

2 . Solution in parametric form:

$$= a \quad (\;_1 - \tau \;\;^{+1} \quad \tau)^{-1\;\;2} \quad \tau + \quad _2 \;\;^{-1}, \qquad = \;\tau \quad (\;_1 - \tau \;\;^{+1} \quad \tau)^{-1\;\;2} \quad \tau + \quad _2 \;\;^{-1},$$

where $A_1 = -\tfrac{1}{2} a^{1+ \quad 1-} \;\; (\quad + 1)$, $A_2 = \;\; \tfrac{1}{2} a\;\;$.

21. $= \;_1 + \;\;_2 \;\;^{-1}.$

Solution: $\quad = \;\; (\;_1 + 2A_1 \quad + 2A_2 \ln| \;\; |)^{-1\;\;2} \quad + \;\;_2.$

22. $= \;_1 \;\;^{-3} + \;\;_2 \;\;^{-2} \;\;^{-1}.$

Solution in parametric form:

$$= \quad (\;_1 + 2A_1\tau + 2A_2 \ln|\tau|)^{-1\;\;2} \quad \tau + \quad _2 \;\;^{-1}, \qquad = \tau \quad (\;_1 + 2A_1\tau + 2A_2 \ln|\tau|)^{-1\;\;2} \quad \tau + \quad _2 \;\;^{-1}.$$

23. $= \;_1 \;\;^{-5\;\;2} \;\;^{-1\;\;2} + \;\;_2 \;\;^{-7\;\;2} \;\;^{-1\;\;2}.$

Solution in parametric form:

$$= \frac{1}{\;}, \qquad = \frac{k^2}{\;} \;\; 2 \;\;_1 e^{2\;\;\tau} + \;\;_2 e^{-\;\;\tau} \big[\;\; \overline{3} \cos(\omega\tau) - \sin(\omega\tau) \big] \;\;^{2},$$

where $\quad = \;_1 e^{2\;\;\tau} + \;\;_2 e^{-\;\;\tau} \sin(\omega\tau) - A_1 \;\; A_2$, $A_2 = 16k^3$, $\omega = k \;\; \overline{3}$.

24. $= {}_1 {}^{-4/3} {}^{-5/3} + {}_2 {}^{-7/3} {}^{-5/3}.$

Solution in parametric form:

$$=(\tfrac{1}{36}A_2\tau^4 + {}_1\tau^3 + {}_2\tau + {}_3)^{-1}, \qquad =(\tfrac{1}{9}A_2\tau^3 + 3\,{}_1\tau^2 + {}_2)^{3/2}(\tfrac{1}{36}A_2\tau^4 + {}_1\tau^3 + {}_2\tau + {}_3)^{-1},$$

where the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint $9\;{}_1\;{}_2 = A_1 + A_2\;{}_3$.

25. $= {}_1 {}^{-8/5} {}^{-7/5} + {}_2 {}^{-13/5} {}^{-7/5}.$

Solution in parametric form:

$$= a\;{}^4_1 \;-\frac{A_1}{A_2}\;{}^{-1}, \qquad = {}^5_1\;{}^{5/2}\;a\;{}^4_1\;-\frac{A_1}{A_2}\;{}^{-1},$$

where $= {}_1 e^{2\,\tau} + {}_2 e^{-\,\tau}\sin(\sqrt{3}\,k\tau)$, $=({}'_\tau)^2 - 2\;{}''_{\tau\tau}$, $A_2 = -\dfrac{5}{1024}\dfrac{{}^{12/5}}{a^3k^6}$.

26. $= -\frac{3}{16}\;{}^{-2} \;+\; {}^{-2}\;{}^{-5/3}.$

1 . Solution in parametric form with $A<0$:

$$= {}_1[\cosh(\tau + {}_2)\cos\tau]^{-2}[\tanh(\tau + {}_2) + \tan\tau]^{-2}, \qquad = [\tanh(\tau + {}_2) + \tan\tau]^{-3/2},$$

where $A = -\frac{3}{64}\;{}^{8/3}$.

2 . Solution in parametric form with $A>0$:

$$= {}_1[\sinh\tau + \cos(\tau + {}_2)]^{-2}, \qquad = [\cosh\tau - \sin(\tau + {}_2)]^{3/2}[\sinh\tau + \cos(\tau + {}_2)]^{-3/2},$$

where $A = \frac{3}{16}\;{}^{8/3}$.

27. $= -\frac{3}{16}\;{}^{-2} \;+\; {}^{2/3}\;{}^{-5/3}.$

1 . Solution in parametric form with $A<0$:

$$= {}_1[\cosh(\tau + {}_2)\cos\tau]^2[\tanh(\tau + {}_2) + \tan\tau]^2,$$
$$= {}_1[\cosh(\tau + {}_2)\cos\tau]^2[\tanh(\tau + {}_2) + \tan\tau]^{1/2},$$

where $A = -\frac{3}{64}\;{}^{8/3}$.

2 . Solution in parametric form with $A>0$:

$$= {}_1[\sinh\tau + \cos(\tau + {}_2)]^2, \qquad = {}_1[\cosh\tau - \sin(\tau + {}_2)]^{3/2}[\sinh\tau + \cos(\tau + {}_2)]^{1/2},$$

where $A = \frac{3}{16}\;{}^{8/3}$.

28. $= {}_1 {}^{-1} {}^{-2} + {}_2 {}^{-2} {}^{-2}.$

Solution in parametric form:

$$= a\;{}_1\tau^{-2/3}\big[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \;\; \tau^2 Z^2\big] - \frac{A_1}{A_2}\;{}^{-1},$$
$$= {}_1\tau^{2/3}Z^2\;\; a\;{}_1\tau^{-2/3}\big[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \;\; \tau^2 Z^2\big] - \frac{A_1}{A_2}\;{}^{-1},$$

where

$$Z = \begin{cases} {}_1\;{}_{1/3}(\tau) + {}_2\;{}_{1/3}(\tau) & \text{for the upper sign,} \\ {}_1\;{}_{1/3}(\tau) + {}_2\;{}_{1/3}(\tau) & \text{for the lower sign,} \end{cases}$$

${}_{1/3}(\tau)$ and ${}_{1/3}(\tau)$ are the Bessel functions, and ${}_{1/3}(\tau)$ and ${}_{1/3}(\tau)$ are the modified Bessel functions; $A_2 = -\frac{9}{2}a^{-3/3}$.

In the solutions of equations 29 and 30, the following notation is used:

$$_1 = \quad _1 e^{2\ \tau} + \quad _2 e^{-\ \tau} \sin(\ \overline{3}\, k\tau\ ,$$
$$_2 = 2k\ \ _1 e^{2\ \tau} + k\ \ _2 e^{-\ \tau}[\ \overline{3}\cos(\ \overline{3}\, k\tau\ \ - \sin(\ \overline{3}\, k\tau\ ],$$
$$_3 = \ \ _2^2 - 2\ \ _1(\ \ _2)'_\tau.$$

29. $\quad = -\frac{5}{36}\ \ ^{-2}\ \ + \quad ^{-2}\ \ ^{-7\ 5}.$

Solution in parametric form:

$$= \ \ _1\ \ _3^{-3\ 2}, \qquad = \ \ _1^{5\ 2}\ \ _3^{-5\ 4}, \qquad \text{where} \quad A = -\tfrac{5}{2304}\ \ ^{12\ 5}k^{-6}.$$

30. $\quad = -\frac{5}{36}\ \ ^{-2}\ \ + \quad ^{2\ 5}\ \ ^{-7\ 5}.$

Solution in parametric form:

$$= \ \ _1\ \ _3^{3\ 2}, \qquad = \ \ _1\ \ _1^{5\ 2}\ \ _3^{1\ 4}, \qquad \text{where} \quad A = -\tfrac{5}{2304}\ \ ^{12\ 5}k^{-6}.$$

In the solutions of equations 31–39, the following notation is used:

$$R = \ \ \overline{(4\tau^3 - 1)}, \qquad = \ \tau R^{-1}\ \ \tau, \qquad _1 = 2\tau\ \ + \ \ _2\tau \mp R, \qquad _2 = \tau^{-1}(R\ \ _1 - 1),$$

where $= \ (\tau)$ *is the incomplete elliptic integral of the second kind in the form of Weierstrass.*

31. $\quad = 6\ \ ^{-2}\ \ + \quad ^{-2}\ \ ^{-4}.$

Solution in parametric form:

$$= \ \ _1\tau^{-1\ 5}\ \ _1^{1\ 5}, \qquad = \ \tau^{-3\ 5}\ \ _1^{-2\ 5}, \qquad \text{where} \quad A = \mp 150\ \ ^5.$$

32. $\quad = 6\ \ ^{-2}\ \ + \quad ^{3}\ \ ^{-4}.$

Solution in parametric form:

$$= \ \ _1\tau^{1\ 5}\ \ _1^{-1\ 5}, \qquad = \ \ _1\tau^{-2\ 5}\ \ _1^{-3\ 5}, \qquad \text{where} \quad A = \mp 150\ \ ^5.$$

33. $\quad = 20\ \ ^{-2}\ \ + \quad ^{-2}\ \ ^{-1\ 2}.$

Solution in parametric form:

$$= \ \ _1\ \ _1^{1\ 3}, \qquad = \ \ _1^{-4\ 3}\ \ _2^{2}, \qquad \text{where} \quad A = \ \ 108\ \ ^{3\ 2}.$$

34. $\quad = 20\ \ ^{-2}\ \ + \quad ^{-1\ 2}\ \ ^{-1\ 2}.$

Solution in parametric form:

$$= \ \ _1\ \ _1^{-1\ 3}, \qquad = \ \ _1\ \ _1^{-5\ 3}\ \ _2^{2}, \qquad \text{where} \quad A = \ \ 108\ \ ^{3\ 2}.$$

35. $\quad = \frac{15}{4}\ \ ^{-2}\ \ + \quad ^{-2}\ \ ^{-7}.$

Solution in parametric form:

$$= \ \ _1(4\tau\ \ _1^2 \mp \ \ _2^2)^{1\ 4}, \qquad = \ \ _1^{1\ 2}(4\tau\ \ _1^2 \mp \ \ _2^2)^{-3\ 8}, \qquad \text{where} \quad A = \ \tfrac{3}{4}\ \ ^8.$$

36. $\quad = \frac{15}{4}\ \ ^{-2}\ \ + \quad ^{6}\ \ ^{-7}.$

Solution in parametric form:

$$= \ \ _1(4\tau\ \ _1^2 \mp \ \ _2^2)^{-1\ 4}, \qquad = \ \ _1\ \ _1^{1\ 2}(4\tau\ \ _1^2 \mp \ \ _2^2)^{-5\ 8}, \qquad \text{where} \quad A = \ \tfrac{3}{4}\ \ ^8.$$

37. $= -\frac{12}{49} \quad {}^{-2} \quad + \quad {}^{-2} \; {}^{1} \; {}^{2}.$

Solution in parametric form:

$$= \;_1(\; + \;_2)^{-7}, \qquad = \tau^2(\; + \;_2)^{-4}, \qquad \text{where} \quad A = \tfrac{12}{49} \; {}^{1} \; {}^{2}.$$

38. $= -\frac{12}{49} \quad {}^{-2} \quad + \quad {}^{-3} \; {}^{2} \; {}^{1} \; {}^{2}.$

Solution in parametric form:

$$= \;_1(\; + \;_2)^{7}, \qquad = \;_1\tau^2(\; + \;_2)^{3}, \qquad \text{where} \quad A = \tfrac{12}{49} \; {}^{1} \; {}^{2}.$$

39. $= \;_1 \; {}^{4} \; {}^{-7} + \;_2 \; {}^{3} \; {}^{-7}.$

Solution in parametric form:

$$= a \;{}_1^8(4\tau \;{}_1^2 \mp \;{}_2^2) - \frac{A_1}{A_2} \;{}^{-1}, \qquad = \;{}_1^3 \;{}_1^{1} \;{}^{2} \; a \;{}_1^8(4\tau \;{}_1^2 \mp \;{}_2^2) - \frac{A_1}{A_2} \;{}^{-1},$$

where $A_2 = \tfrac{3}{64} a^{-3} \; {}^{8}.$

In the solutions of equations 40–43, the following notation is used:

$$R_1 = \overline{\;_1 + \tau^{-3} \quad \tau^{-2}}, \qquad _1 = \frac{\tau}{R_1} + \;_2, \qquad _1 = \tau - R_1 \;_1, \qquad _1 = 3\tau^3 \;{}_1^2 + 3(1 \quad \tau) \;{}_1^2,$$

$$R_2 = \overline{\;_1 - \tau^{-3} \quad \tau^{-2}}, \qquad _2 = \frac{\tau}{R_2} + \;_2, \qquad _2 = \tau - R_2 \;_2, \qquad _2 = 3\tau^3 \;{}_2^2 + 3(-1 \quad \tau) \;{}_2^2.$$

40. $= \;_1 \; {}^{-4} \; {}^{3} + \;_2 \; {}^{-4} \; {}^{3} \; {}^{-1} \; {}^{2}.$

Solutions in parametric form:

$$= a\tau^{-3} \; {}^{3}, \qquad = \; {}^{2},$$

where $A_1 = \mp\tfrac{2}{9} a^{-2} \; {}^{3}$, $A_2 = \tfrac{1}{3} a^{-2} \; {}^{3} \; {}^{3} \; {}^{2}(-1)$; $k = 1$ and $k = 2$.

41. $= \;_1 \; {}^{-5} \; {}^{3} + \;_2 \; {}^{-7} \; {}^{6} \; {}^{-1} \; {}^{2}.$

Solutions in parametric form:

$$= a\tau^3 \; {}^{-3}, \qquad = \tau^3 \; {}^{-3} \; {}^{2},$$

where $A_1 = \tfrac{2}{9} a^{-1} \; {}^{3}$, $A_2 = \tfrac{1}{3} a^{-5} \; {}^{6} \; {}^{3} \; {}^{2}(-1) \; {}^{+1}$; $k = 1$ and $k = 2$.

42. $= \;_1 \; {}^{-3} + \;_2 \; {}^{-7}.$

Solutions in parametric form:

$$= a\tau^{-3} \quad , \qquad = \tau^{-1} \; {}^{2} \; {}^{1} \; {}^{2},$$

where $A_1 = \mp\tfrac{1}{36} a^{-2} \; {}^{4}$, $A_2 = -\tfrac{1}{36} a^{-3} \; {}^{8}$; $k = 1$ and $k = 2$.

43. $= \;_1 \; {}^{-3} + \;_2 \; {}^{3} \; {}^{-7}.$

Solutions in parametric form:

$$= a\tau^3 \; {}^{-1}, \qquad = \tau^5 \; {}^{2} \; {}^{1} \; {}^{2} \; {}^{-1},$$

where $A_1 = \tfrac{1}{36} a^{-2} \; {}^{4}$, $A_2 = -\tfrac{1}{36} a^{-5} \; {}^{8}$; $k = 1$ and $k = 2$.

In the solutions of equations 44 and 45, the following notation is used:

$$
{}_1 = \begin{cases} {}_1 e^{\ \tau} + \ {}_2 e^{-\ \tau} - \dfrac{A_2}{A_1}\tau & \textit{if } A_1 > 0, \\ {}_1 \sin(k\tau) + \ {}_2 \cos(k\tau) - \dfrac{A_2}{A_1}\tau & \textit{if } A_1 < 0, \end{cases}
$$

$$
{}_2 = \begin{cases} k(\ {}_1 e^{\ \tau} - \ {}_2 e^{-\ \tau}) - \dfrac{A_2}{A_1} & \textit{if } A_1 > 0, \\ k[\ {}_1 \cos(k\tau) - \ {}_2 \sin(k\tau)] - \dfrac{A_2}{A_1} & \textit{if } A_1 < 0, \end{cases}
$$

where $k = \sqrt{\tfrac{1}{2}|A_1|}$.

44. $\quad = \ {}_1 + \ {}_2 \ {}^{-1\ 2}.$

Solution in parametric form:

$$= \ {}_1, \qquad = \ {}_2^2.$$

45. $\quad = \ {}_1 \ {}^{-3} + \ {}_2 \ {}^{-5\ 2\ \ -1\ 2}.$

Solution in parametric form:

$$= \ {}_1^{-1}, \qquad = \ {}_1^{-1}\ {}_2^2.$$

In the solutions of equations 46 and 47, the following notation is used:

For $A_1 > 0$,

$$
\begin{aligned}
{}_1 &= \ {}_1 e^{\ \tau} + \ {}_2 e^{-\ \tau} + \ {}_3 \sin(k\tau), \quad k = (\tfrac{4}{3}A_1)^{1\ 4}, \\
{}_2 &= k(\ {}_1 e^{\ \tau} - \ {}_2 e^{-\ \tau}) + k\ \ {}_3 \cos(k\tau).
\end{aligned}
$$

For $A_1 < 0$,

$$
\begin{aligned}
{}_1 &= e^{\ \tau}[\ {}_1 \sin(\ \tau) + \ {}_2 \cos(\ \tau)] + \ {}_3 e^{-\ \tau} \sin(\ \tau), \quad = (-\tfrac{1}{3}A_1)^{1\ 4}, \\
{}_2 &= \ e^{\ \tau}[(\ {}_1 - \ {}_2)\sin(\ \tau) + (\ {}_1 + \ {}_2)\cos(\ \tau)] - \ {}_3 e^{-\ \tau}[\sin(\ \tau) - \cos(\ \tau)].
\end{aligned}
$$

46. $\quad = \ {}_1 \ {}^{2\ \ -5\ 3} + \ {}_2 \ {}^{-5\ 3}.$

Solution in parametric form:

$$= \ {}_1 - \frac{A_2}{2A_1}, \qquad = \ {}_2^{3\ 2},$$

where the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint

$$
\begin{aligned}
4\ {}_1\ {}_2 + \ {}_3^2 &= \tfrac{1}{4}A_1^{-2}A_2^2 && \text{if } A_1 > 0, \\
{}_1\ {}_3 &= \tfrac{1}{16}A_1^{-2}A_2^2 && \text{if } A_1 < 0.
\end{aligned}
$$

47. $\quad = \ {}_1 \ {}^{2\ \ -5\ 3} + \ {}_2 \ {}^{-5\ 3}.$

Solution in parametric form:

$$= \ {}_1, \qquad = \ {}_2^{3\ 2},$$

where the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint

$$
\begin{aligned}
4\ {}_1\ {}_2 + \ {}_3^2 &= -\tfrac{1}{2}A_1^{-1}A_2 && \text{if } A_1 > 0, \\
{}_1\ {}_3 &= -\tfrac{1}{4}A_1^{-1}A_2 && \text{if } A_1 < 0.
\end{aligned}
$$

In the solutions of equations 48 and 49, the following notation is used:

For $A_2 > 0$,

$$_1 = \ _1 e^{\ \tau} + \ _2 e^{-\ \tau} + \ _3 \sin(k\tau), \quad k = (\tfrac{4}{3}A_2)^{1\ 4},$$

$$_2 = k(\ _1 e^{\ \tau} - \ _2 e^{-\ \tau}) + k\ \ _3 \cos(k\tau).$$

For $A_2 < 0$,

$$_1 = e^{\ \tau}[\ _1 \sin(\ \tau) + \ _2 \cos(\ \tau)] + \ _3 e^{-\ \tau} \sin(\ \tau), \qquad = (-\tfrac{1}{3}A_2)^{1\ 4},$$

$$_2 = \ e^{\ \tau}[(\ _1 - \ _2)\sin(\ \tau) + (\ _1 + \ _2)\cos(\ \tau)] - \ _3 e^{-\ \tau}[\sin(\ \tau) - \cos(\ \tau)].$$

48. $\ \ = \ _1 \ ^{-7\ 3\ \ -5\ 3} + \ _2 \ ^{-10\ 3\ \ -5\ 3}$.

Solution in parametric form:

$$= \ _1 - \frac{A_1}{2A_2}\ ^{-1}, \qquad = \ ^{3\ 2}_{2}\ \ _1 - \frac{A_1}{2A_2}\ ^{-1},$$

where the constants $_1$, $_2$, and $_3$ are related by the constraint

$$4\ _1\ _2 + \ ^2_3 = \tfrac{1}{4}A_1^2 A_2^{-2} \quad \text{if } A_2 > 0,$$

$$_1\ _3 = \tfrac{1}{16}A_1^2 A_2^{-2} \qquad \text{if } A_2 < 0.$$

49. $\ \ = \ _1 \ ^{-4\ 3\ \ -5\ 3} + \ _2 \ ^{-10\ 3\ \ -5\ 3}$.

Solution in parametric form:

$$= \ ^{-1}_1, \qquad = \ ^{-1}_1\ \ ^{3\ 2}_2,$$

where the constants $_1$, $_2$, and $_3$ are related by the constraint

$$4\ _1\ _2 + \ ^2_3 = -\tfrac{1}{2}A_1 A_2^{-1} \quad \text{if } A_2 > 0,$$

$$_1\ _3 = -\tfrac{1}{4}A_1 A_2^{-1} \qquad \text{if } A_2 < 0.$$

In the solutions of equations 50–53, the following notation is used:

$$R_1 = \ _1\tau^{\ 1} + \ _2\tau^{\ 2} + \ _3\tau^{\ 3},$$

$$R_2 = (\ _1 + \ _2\tau)e^{\ \tau} + \ _3 e^{\ \tau},$$

$$R_3 = \ _1 e^{\ \tau} + e^{\ \tau}(\ _2 \sin\omega\tau + \ _3 \cos\omega\tau),$$

$$Q_1 = \ _1 k_1 \tau^{\ 1} + \ _2 k_2 \tau^{\ 2} + \ _3 k_3 \tau^{\ 3},$$

$$Q_2 = (k\ \ _1 + \ _2 + k\ \ _2\tau)e^{\ \tau} + \omega\ \ _3 e^{\ \tau},$$

$$Q_3 = k\ \ _1 e^{\ \tau} + e^{\ \tau}[(\ _2 - \omega\ \ _3)\sin\omega\tau + (\ _3 + \omega\ \ _2)\cos\omega\tau],$$

$$_1 = \tau(Q_1)'_\tau, \qquad _2 = (Q_2)'_\tau, \qquad _3 = (Q_3)'_\tau,$$

where k_1, k_2, *and* k_3 *(real numbers) or* k *and* $\ \ \omega$ *(one real and two complex numbers) are roots of the cubic equation* $\lambda^3 - \frac{1}{2}B_2\lambda - \frac{1}{2}B_1 = 0$. *The subscripts of the functions* $R\ $, $Q\ $, *and* $\ $ *(* $\ = 1, 2, 3$*) are selected depending on the sign of the expression* $\Delta = 2B_2^3 - 27B_1^2$*:*

$$\Delta > 0 \quad \textit{subscript} \quad = 1,$$

$$\Delta = 0 \quad \textit{subscript} \quad = 2,$$

$$\Delta < 0 \quad \textit{subscript} \quad = 3.$$

If $2B_2^3 = 27B_1^2$ *(subscript 2), then*

$$k = (\tfrac{1}{6}B_2)^{1\ 2}, \quad \omega = -2(\tfrac{1}{6}B_2)^{1\ 2} \quad \textit{if} \quad B_1 < 0,$$

$$k = -(\tfrac{1}{6}B_2)^{1\ 2}, \quad \omega = 2(\tfrac{1}{6}B_2)^{1\ 2} \quad \textit{if} \quad B_1 > 0.$$

The expressions for $R\ $, $Q\ $ *contain three constants* $_1$, $_2$, *and* $_3$. *One of them can be arbitrarily fixed to set it equal to any nonzero number (for example, we can set* $_3 = \ 1$*), and the other constants can be arbitrary.*

50. $\quad = {}_1 + {}_2 \quad {}^{-1\ 2}$**.**

Solution in parametric form:

$$= R \quad, \qquad = Q^2 \quad, \qquad \text{where} \quad A_1 = B_2, \quad A_2 = B_1.$$

51. $\quad = {}_1 \ {}^{-3} + {}_2 \ {}^{-7\ 2\ \ -1\ 2}$**.**

Solution in parametric form:

$$= R^{-1}, \qquad = R^{-1}Q^2 \quad, \qquad \text{where} \quad A_1 = B_2, \quad A_2 = B_1.$$

52. $\quad = {}_1 \ {}^{-3\ 5} + {}_2 \quad {}^{-7\ 5}$**.**

Solution in parametric form:

$$= a(2Q^2 \ -4R \qquad + B_2R^2 \ ), \qquad = \ R^{5\ 2},$$

where $A_1 = -a^{\ -4\ 5}A_2B_2, \ A_2 = -\frac{5}{32}a^{-3\ \ 12\ 5}B_1^{-2}$.

53. $\quad = {}_1 \ {}^{-12\ 5\ \ -3\ 5} + {}_2 \ {}^{-13\ 5\ \ -7\ 5}$**.**

Solution in parametric form:

$$= a(2Q^2 \ -4R \qquad + B_2R^2 \ )^{-1}, \qquad = \ R^{5\ 2}(2Q^2 \ -4R \qquad + B_2R^2 \ )^{-1},$$

where $A_1 = \frac{5}{32}a^{2\ 5\ \ 8\ 5}B_1^{-2}B_2, \ A_2 = -\frac{5}{32}a^{3\ 5\ \ 12\ 5}B_1^{-2}$.

In the solutions of equations 54 and 55, the following notation is used:

1 . *For* $A_2 > 0, \ A_1 \neq 0$:

$$_1 = \ _1e^{\ \tau} + \ _2e^{-\ \tau} + \ _3\sin\omega\tau, \qquad _2 = k(\ _1e^{\ \tau} - \ _2e^{-\ \tau}) + \omega \ \ _3\cos\omega\tau,$$

where $k = \{\frac{2}{3}[(A_1^2 + 3A_2)^{1\ 2} + A_1]\}^{1\ 2}, \ \omega = \{\frac{2}{3}[(A_1^2 + 3A_2)^{1\ 2} - A_1]\}^{1\ 2}$; *the constants* $_1$, $_2$, *and* $_3$ *are related by the constraint* $4k^2 \ \ _1 \ \ _2 + \omega^2 \ \ _3^2 = 0$.

2 . *For* $-A_1^2 < 3A_2 < 0, \ A_1 > 0$:

$$_1 = \ _1\tau^{\ 1} + \ _2\tau^{-\ 1} + \ _3\tau^{\ 2} + \ _4\tau^{-\ 2}, \qquad _2 = k_1(\ _1\tau^{\ 1} - \ _2\tau^{-\ 1}) + k_2(\ _3\tau^{\ 2} - \ _4\tau^{-\ 2}),$$

where $k_1 = \{\frac{2}{3}[A_1 + (A_1^2 + 3A_2)^{1\ 2}]\}^{1\ 2}, \ k_2 = \{\frac{2}{3}[A_1 - (A_1^2 + 3A_2)^{1\ 2}]\}^{1\ 2}$; *the constants* $_1$, $_2$, *and* $_3$ *are related by the constraint* $(\ _1\ \ _2 + \ _3\ \ _4)(A_1^2 + 3A_2)^{1\ 2} + (\ _1\ \ _2 - \ _3\ \ _4)A_1 = 0$.

3 . *For* $-A_1^2 < 3A_2 < 0, \ A_1 < 0$:

$$_1 = \ _1\sin\omega_1\tau + \ _2\cos\omega_1\tau + \ _3\sin\omega_2\tau, \qquad _2 = \omega_1(\ _1\cos\omega_1\tau - \ _2\sin\omega_1\tau) + \omega_2 \ \ _3\cos\omega_2\tau,$$

where $\omega_1 = \{-\frac{2}{3}[A_1 + (A_1^2 + 3A_2)^{1\ 2}]\}^{1\ 2}, \ \omega_2 = \{-\frac{2}{3}[A_1 - (A_1^2 + 3A_2)^{1\ 2}]\}^{1\ 2}$; *the constants* $_1$, $_2$, *and* $_3$ *are related by the constraint* $\omega_1^2(\ _1^2 + \ _2^2) - \omega_2^2 \ \ _3^2 = 0$.

4 . *For* $A_1^2 + 3A_2 = 0, \ A_1 > 0$:

$$_1 = (\ _1 + \ _2\tau)e^{\ \tau} + (\ _3 + \ _4\tau)e^{-\ \tau}, \qquad _2 = (k \ \ _1 + \ _2 + k \ \ _2\tau)e^{\ \tau} - (k \ \ _3 - \ _4 + k \ \ _4\tau)e^{-\ \tau},$$

where $k = (\frac{2}{3}A_1)^{1\ 2}$; *the constants* $_1$, $_2$, *and* $_3$ *are related by the constraint* $_2 \ \ _4 + (\ _1\ \ _4 - \ _2\ \ _3)(\frac{1}{6}A_1)^{1\ 2} = 0$.

5 . *For* $A_1^2 + 3A_2 = 0, \ A_1 < 0$:

$$_1 = (\ _1 + \ _2\tau)\sin\omega\tau + \ _3\tau\cos\omega\tau, \qquad _2 = (\omega \ \ _1 + \ _3 + \omega \ \ _2\tau)\cos\omega\tau + (\ _2 - \omega \ \ _3\tau)\sin\omega\tau,$$

where $\omega = (-\frac{2}{3}A_1)^{1\ 2}$; *the constants* $_1$, $_2$, *and* $_3$ *are related by the constraint* $_2^2 + \ _3^2 + \ _1\ \ _3(-\frac{2}{3}A_1)^{1\ 2} = 0$.

6 . *For* $3A_2 < -A_1^2$:

$$_1 = e^{\ \tau}(\ _1\sin\omega\tau + \ _2\cos\omega\tau) + \ _3e^{-\ \tau}\sin\omega\tau,$$

$$_2 = e^{\ \tau}[(k \ \ _2 + \omega \ \ _1)\cos\omega\tau + (k \ \ _1 - \omega \ \ _2)\sin\omega\tau)] + \ _3e^{-\ \tau}(\omega\cos\omega\tau - k\sin\omega\tau),$$

where $k = \{\frac{1}{3}[A_1 + (-3A_2)^{1\ 2}]\}^{1\ 2}, \ \omega = \{\frac{1}{3}[-A_1 + (-3A_2)^{1\ 2}]\}^{1\ 2}$; *the constants* $_1$, $_2$, *and* $_3$ *are related by the constraint* $_2A_1 + \ _1(-A_1^2 - 3A_2)^{1\ 2} = 0$.

54. $\quad = \ _1 \ ^{-1\ 3} + \ _2 \ ^{2\ -5\ 3}.$

Solution in parametric form:

$$= \ _1, \qquad = \ _2^{3\ 2}.$$

55. $\quad = \ _1 \ ^{-8\ 3\ -1\ 3} + \ _2 \ ^{-10\ 3\ -5\ 3}.$

Solution in parametric form:

$$= \ _1^{-1}, \qquad = \ _1^{-1} \ _2^{3\ 2}.$$

In the solutions of equations 56–59, the following notation is used:

$$_1 = \begin{cases} _1 e^{\ \tau} + \ _2 e^{-\ \tau} + \ _3 \tau & \text{if } A_1 > 0, \\ _1 \sin\omega\tau + \ _2 \cos\omega\tau + \ _3 \tau & \text{if } A_1 < 0, \end{cases} \qquad \textit{where} \quad \omega = |\tfrac{4}{3}A_1|^{1\ 2},$$

$$_2 = \begin{cases} \omega(\ _1 e^{\ \tau} - \ _2 e^{-\ \tau}) + \ _3 & \text{if } A_1 > 0, \\ \omega(\ _1 \cos\omega\tau - \ _2 \sin\omega\tau) + \ _3 & \text{if } A_1 < 0, \end{cases} \qquad \textit{where} \quad \omega = |\tfrac{4}{3}A_1|^{1\ 2}.$$

56. $\quad = \ _1 \ ^{-1\ 3} + \ _2 \ ^{-5\ 3}.$

Solution in parametric form:

$$= \ _1, \qquad = \ _2^{3\ 2},$$

where the constants $_1$, $_2$, and $_3$ are related by the constraint

$$\begin{aligned} &3(A_1 \ _3^2 + A_2) + 16A_1^2 \ _1 \ _2 = 0 && \text{if } A_1 > 0, \\ &3(A_1 \ _3^2 + A_2) + 4A_1^2(\ _1^2 + \ _2^2) = 0 && \text{if } A_1 < 0. \end{aligned}$$

57. $\quad = \ _1 \ ^{-8\ 3\ -1\ 3} + \ _2 \ ^{-4\ 3\ -5\ 3}.$

Solution in parametric form:

$$= \ _1^{-1}, \qquad = \ _1^{-1} \ _2^{3\ 2},$$

where the constants $_1$, $_2$, and $_3$ are related by the constraint

$$\begin{aligned} &3(A_1 \ _3^2 + A_2) + 16A_1^2 \ _1 \ _2 = 0 && \text{if } A_1 > 0, \\ &3(A_1 \ _3^2 + A_2) + 4A_1^2(\ _1^2 + \ _2^2) = 0 && \text{if } A_1 < 0. \end{aligned}$$

58. $\quad = \ _1 \ ^{-1\ 3} + \ _2 \ ^{-5\ 3}.$

Solution in parametric form:

$$= \ _1 - \frac{A_2}{4A_1}\tau^2, \qquad = \ _2 - \frac{A_2}{2A_1}\tau \ ^{3\ 2},$$

where the constants $_1$, $_2$, and $_3$ are related by the constraint

$$\begin{aligned} &3A_1 \ _3^2 + 16A_1^2 \ _1 \ _2 + \tfrac{9}{16}A_1^{-2}A_2^2 = 0 && \text{if } A_1 > 0, \\ &3A_1 \ _3^2 + 4A_1^2(\ _1^2 + \ _2^2) + \tfrac{9}{16}A_1^{-2}A_2^2 = 0 && \text{if } A_1 < 0. \end{aligned}$$

59. $\quad = \ _1 \ ^{-8\ 3\ -1\ 3} + \ _2 \ ^{-7\ 3\ -5\ 3}.$

Solution in parametric form:

$$= \ _1 - \frac{A_2}{4A_1}\tau^2 \ ^{-1}, \qquad = \ _1 - \frac{A_2}{4A_1}\tau^2 \ ^{-1} \ _2 - \frac{A_2}{2A_1}\tau \ ^{3\ 2},$$

where the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint

$$3A_1 \ {}_3^2 + 16A_1^2 \ {}_1 \ {}_2 + \tfrac{9}{16}A_1^{-2}A_2^2 = 0 \qquad \text{if } A_1 > 0,$$
$$3A_1 \ {}_3^2 + 4A_1^2(\ {}_1^2 + \ {}_2^2) + \tfrac{9}{16}A_1^{-2}A_2^2 = 0 \quad \text{if } A_1 < 0.$$

In the solutions of equations 60–67, the following notation is used:

$$= \overline{(4\wp^3 - 1)}, \qquad \tau = \frac{\wp}{\overline{(4\wp^3-1)}} - \ {}_2.$$

The function $\wp = \wp(\tau)$ is defined implicitly in terms of the above elliptic integral of the first kind. For the upper sign, $\wp$ coincides with the classical elliptic Weierstrass function $\wp = \wp(\tau + \ {}_2, 0, 1)$. In the solution given below, one can take $\wp$ as the parameter instead of τ and use the explicit dependence $\tau = \tau(\wp)$.

60. $= \ ^{-2} \ ^2 - \frac{6}{25} \ ^{-2}$ **.**

Solution in parametric form:

$$= \ {}_1\tau^5, \qquad = \tau^2\wp, \qquad \text{where} \quad A = \tfrac{6}{25} \ ^{-1}.$$

61. $= \ ^{-3} \ ^2 - \frac{6}{25} \ ^{-2}$ **.**

Solution in parametric form:

$$= \ {}_1\tau^{-5}, \qquad = \ {}_1\tau^{-3}\wp, \qquad \text{where} \quad A = \tfrac{6}{25} \ ^{-1}.$$

62. $= \ ^{-2} \ ^2 + \frac{6}{25} \ ^{-2}$ **.**

Solution in parametric form:

$$= \ {}_1\tau^5, \qquad = (\tau^2\wp \mp 1), \qquad \text{where} \quad A = \tfrac{6}{25} \ ^{-1}.$$

63. $= \ ^{-3} \ ^2 + \frac{6}{25} \ ^{-2}$ **.**

Solution in parametric form:

$$= \ {}_1\tau^{-5}, \qquad = \ {}_1\tau^{-5}(\tau^2\wp \mp 1), \qquad \text{where} \quad A = \tfrac{6}{25} \ ^{-1}.$$

64. $= 12 \ ^{-2} + \ ^{-2} \ ^{-5} \ ^2$**.**

Solution in parametric form:

$$= \ {}_1\wp^{2 \ 7}(\ 2\tau\wp^2)^{-1 \ 7}, \qquad = \wp^{-6 \ 7}(\ 2\tau\wp^2)^{-4 \ 7}, \qquad \text{where} \quad A = \mp 147 \ ^{7 \ 2}.$$

65. $= 12 \ ^{-2} + \ ^{3 \ 2} \ ^{-5 \ 2}$**.**

Solution in parametric form:

$$= \ {}_1\wp^{-2 \ 7}(\ 2\tau\wp^2)^{1 \ 7}, \qquad = \ {}_1\wp^{-8 \ 7}(\ 2\tau\wp^2)^{-3 \ 7}, \qquad \text{where} \quad A = \mp 147 \ ^{7 \ 2}.$$

66. $= \frac{63}{4} \ ^{-2} + \ ^{-2} \ ^{-5 \ 3}$**.**

Solution in parametric form:

$$= \ {}_1(\tau \ + 2\wp)^{-1 \ 4}, \qquad = (\tau \ + 2\wp)^{-9 \ 8}(\ 2\tau\wp^2)^{3 \ 2}, \qquad \text{where} \quad A = -\tfrac{32}{3} \ ^{8 \ 3}.$$

67. $= \frac{63}{4} \ ^{-2} + \ ^{2 \ 3} \ ^{-5 \ 3}$**.**

Solution in parametric form:

$$= \ {}_1(\tau \ + 2\wp)^{1 \ 4}, \qquad = \ {}_1(\tau \ + 2\wp)^{-7 \ 8}(\ 2\tau\wp^2)^{3 \ 2}, \qquad \text{where} \quad A = -\tfrac{32}{3} \ ^{8 \ 3}.$$

In the solutions of equations 68–73, the following notation is used:

$$_1 = \quad \overline{4\wp_1^3 - 2\wp_1 - \quad _2}, \qquad \tau = \int \frac{\wp_1}{\overline{4\wp_1^3 - 2\wp_1 - \quad _2}} - \quad _1;$$

$$_2 = \quad \overline{4\wp_2^3 + 2\wp_2 - \quad _2}, \qquad \tau = \int \frac{\wp_2}{\overline{4\wp_2^3 + 2\wp_2 - \quad _2}} - \quad _1.$$

The functions $\wp_1 = \wp_1(\tau)$ and $\wp_2 = \wp_2(\tau)$ are the inverses of the above elliptic integrals. For the upper signs, they are the classical Weierstrass functions $\wp_1 = \wp(\tau + \quad _1, 2, \quad _2)$ and $\wp_2 = \wp(\tau + \quad _1, -2, \quad _2)$.

68. $= \quad _1 \quad ^2 + \quad _2.$

Solutions in parametric form:

$$= a\tau, \qquad = \quad \wp \quad ,$$

where $A_1 = \quad 6a^{-2} \quad ^{-1}$, $A_2 = a^{-2} \quad (-1) \quad$; $k = 1$ and $k = 2$.

69. $= \quad _1 \quad ^{-5} \quad ^2 + \quad _2 \quad ^{-3}.$

Solutions in parametric form:

$$= a\tau^{-1}, \qquad = \quad \tau^{-1}\wp \quad ,$$

where $A_1 = \quad 6a^3 \quad ^{-1}$, $A_2 = a \quad (-1) \quad$; $k = 1$ and $k = 2$.

70. $= \quad _1 \quad ^{-15 \quad 7} \quad ^2 + \quad _2 \quad ^{-9 \quad 7}.$

Solutions in parametric form:

$$= a\tau^7, \qquad = \quad \tau(\tau^2\wp \quad \mp 1),$$

where $A_1 = \quad \frac{6}{49}a^{1 \quad 7} \quad ^{-1}$, $A_2 = \frac{1}{49}a^{-5 \quad 7} \quad (-1) \quad$; $k = 1$ and $k = 2$.

71. $= \quad _1 \quad ^{-20 \quad 7} \quad ^2 + \quad _2 \quad ^{-12 \quad 7}.$

Solutions in parametric form:

$$= a\tau^{-7}, \qquad = \quad \tau^{-6}(\tau^2\wp \quad \mp 1),$$

where $A_1 = \quad \frac{6}{49}a^{6 \quad 7} \quad ^{-1}$, $A_2 = \frac{1}{49}a^{-2 \quad 7} \quad (-1) \quad$; $k = 1$ and $k = 2$.

72. $= \quad _1 + \quad _2 \quad ^{-2 \quad 3}.$

Solutions in parametric form:

$$= a[\quad - (-1) \quad \tau], \qquad = \quad \wp^3,$$

where $A_1 = \quad \frac{1}{2}a^{-2} \quad$, $A_2 = \frac{1}{12}a^{-2 \quad 5 \quad 3}(-1) \quad$; $k = 1$ and $k = 2$.

73. $= \quad _1 \quad ^{-3} + \quad _2 \quad ^{-7 \quad 3} \quad ^{-2 \quad 3}.$

Solutions in parametric form:

$$= a[\quad - (-1) \quad \tau]^{-1}, \qquad = \quad \wp^3[\quad - (-1) \quad \tau]^{-1},$$

where $A_1 = \quad \frac{1}{2}a \quad$, $A_2 = \frac{1}{12}a^{1 \quad 3 \quad 5 \quad 3}(-1) \quad$; $k = 1$ and $k = 2$.

In the solutions of equations 74 and 75, the following notation is used:

$$= \quad (1 \quad \tau^4)^{-1 \quad 2} \quad \tau + \quad _2, \qquad k^2 = \quad 1.$$

The function can be expressed in terms of elliptic integrals or lemniscate functions.

74. $= \quad _1 \quad ^{-18 \quad 5} \quad ^3 + \quad _2 \quad ^{-14 \quad 5} \quad ^2.$

Solutions in parametric form:

$$= a \quad _1^5 \quad ^{-5}, \qquad = \quad _1^4 \quad ^{-4}(\tau \quad - k), \qquad \text{where} \quad A_1 = \quad \frac{2}{25}a^{8 \quad 5} \quad ^{-2}, \; A_2 = \quad \frac{6}{25}a^{4 \quad 5} \quad ^{-1}k.$$

75. $= {}_1 {}^{-12\ 5\ \ 3} + {}_2 {}^{-11\ 5\ \ 2}.$

Solutions in parametric form:

$$= a\ {}_1^5\ {}^5, \qquad = {}_1\ (\tau \quad - k), \qquad \text{where} \quad A_1 = \tfrac{2}{25}a^{2\ 5\ -2},\ A_2 = \tfrac{6}{25}a^{1\ 5\ -1}k.$$

In the solutions of equations 76–81, the following notation is used:

$$= \begin{cases} {}_{1\ 3}(\tau) & \textit{for the upper sign (Bessel function),} \\ {}_{1\ 3}(\tau) & \textit{for the lower sign (modified Bessel function),} \end{cases}$$

$$g = \begin{cases} {}_{1\ 3}(\tau) & \textit{for the upper sign (Bessel function),} \\ {}_{1\ 3}(\tau) & \textit{for the lower sign (modified Bessel function),} \end{cases}$$

$$= {}_1 \ + \ {}_2 g + \beta\omega \ g \qquad \tau - \qquad g \ \ \tau \ , \quad \omega = \begin{cases} \frac{1}{2} & \textit{for the upper sign,} \\ -1 & \textit{for the lower sign.} \end{cases}$$

76. $= {}_1 \quad + \ {}_2.$

Solutions in parametric form:

$$= a\tau^{2\ 3}, \qquad = \tau^{1\ 3} \quad , \qquad \text{where} \quad A_1 = \mp\tfrac{9}{4}a^{-3},\ A_2 = \tfrac{9}{4}a^{-2}\beta.$$

77. $= {}_1 {}^{-5} + {}_2 {}^{-3}.$

Solutions in parametric form:

$$= a\tau^{-2\ 3}, \qquad = \tau^{-1\ 3} \quad , \qquad \text{where} \quad A_1 = \mp\tfrac{9}{4}a^3,\ A_2 = \tfrac{9}{4}a\beta.$$

78. $= {}_1 {}^{-3\ 2} + {}_2 {}^{-1\ 2\ \ -1\ 2}.$

Solutions in parametric form:

$$= a\tau^{2\ 3\ \ 2}, \qquad = \tau^{-2\ 3}(\tau \ \ '_\tau + \tfrac{1}{3} \ \ )^2, \qquad \text{where} \quad A_1 = -\tfrac{1}{2}a^{-1\ 2}\ \beta,\ A_2 = \mp\tfrac{1}{3}a^{-3\ 2\ \ 3\ 2}.$$

79. $= {}_1 {}^{-3\ 2} + {}_2 {}^{-2\ \ -1\ 2}.$

Solutions in parametric form:

$$= a\tau^{-2\ 3\ \ -2}, \qquad = \tau^{-4\ 3\ \ -2}(\tau \ \ '_\tau + \tfrac{1}{3} \ \ )^2, \qquad \text{where} \quad A_1 = -\tfrac{1}{2}a^{-1\ 2}\ \beta,\ A_2 = \mp\tfrac{1}{3}\ {}^{3\ 2}.$$

80. $= {}_1 {}^{-3\ 2} + {}_2 \ \ {}^{-2}.$

Solutions in parametric form:

$$= a\tau^{-2\ 3}\big[\mp\tau^2 \ \ {}^2 + 2\beta\tau \quad - (\tau \ \ '_\tau + \tfrac{1}{3} \ \ )^2\big], \qquad = \tau^{2\ 3\ \ 2},$$

where $A_1 = -a\ {}^{-1\ 2}\beta A_2,\ A_2 = \tfrac{9}{2}a^{-3\ \ 3}.$

81. $= {}_1 {}^{-3\ 2\ \ -3\ 2} + {}_2 {}^{-2\ \ -2}.$

Solutions in parametric form:

$$= a\tau^{2\ 3}\big[\mp\tau^2 \ \ {}^2 + 2\beta\tau \quad - (\tau \ \ '_\tau + \tfrac{1}{3} \ \ )^2\big]^{-1}, \qquad = \tau^{4\ 3\ \ 2}\big[\mp\tau^2 \ \ {}^2 + 2\beta\tau \quad - (\tau \ \ '_\tau + \tfrac{1}{3} \ \ )^2\big]^{-1},$$

where $A_1 = -\tfrac{9}{2}a^{-1\ 2\ \ 5\ 2}\beta,\ A_2 = \tfrac{9}{2}\ {}^{3}.$

In the solutions of equations 82–88, the following notation is used:

$$= \begin{cases} {}_1 \ (\tau) & \textit{for the upper sign (Bessel function)}, \\ {}_1 \ (\tau) & \textit{for the lower sign (modified Bessel function)}, \end{cases}$$

$$= \begin{cases} {}_2 \ (\tau) & \textit{for the upper sign (Bessel function)}, \\ {}_2 \ (\tau) & \textit{for the lower sign (modified Bessel function)}, \end{cases}$$

$$Z = {}_1 \ + {}_2 \ , \ X = \beta_1 \ + \beta_2 \ , \qquad = \tau Z' + \ Z \ , \qquad = \tau X' + \ X \ ,$$

$$N = \begin{cases} Z \ X & \textit{if } \Delta = -({}_1\beta_2 - {}_2\beta_1)^2, \\ {}^2 + \beta \qquad + \quad {}^2 & \textit{if } \Delta = 4 \quad - \beta^2, \end{cases}$$

$$N_1 = \begin{cases} Z \qquad + X & \textit{if } \Delta = -({}_1\beta_2 - {}_2\beta_1)^2, \\ \tau N' + 2 \ N & \textit{if } \Delta = 4 \quad - \beta^2, \end{cases}$$

$$N_2 = N_1^2 \quad 4\tau^2 N^2 + \omega^2 \Delta, \qquad \omega = \begin{cases} 2 & \textit{for the upper sign}, \\ -1 & \textit{for the lower sign}. \end{cases}$$

The prime denotes differentiation with respect to τ.

82. $= {}_1 \quad + {}_2 \ {}^{-3}.$

Solutions in parametric form:

$$= a\tau^{2 \ 3}, \qquad = \ \tau^{1 \ 3} N^{1 \ 2},$$

where $= \frac{1}{3}, \ A_1 = \mp \frac{9}{4} a^{-3}, \ A_2 = \frac{9}{16} a^{-2 \ 4} \omega^2 \Delta.$

83. $= {}_1 \quad + {}_2 \ {}^{-3}, \qquad \neq -2.$

Solutions in parametric form:

$$= a\tau^{2} \ , \qquad = \ \tau \ N^{1 \ 2},$$

where $= \dfrac{1}{+2}, \ A_1 = \mp \dfrac{1}{4 \ {}^2} a^{- \ -2}, \ A_2 = \dfrac{1}{16 \ {}^2} a^{-2 \ 4} \omega^2 \Delta.$

84. $= {}_1 \ {}^{-5} + {}_2 \ {}^{-3}.$

Solutions in parametric form:

$$= a\tau^{-2 \ 3}, \qquad = \ \tau^{-1 \ 3} N^{1 \ 2},$$

where $= \frac{1}{3}, \ A_1 = \mp \frac{9}{4} a^3, \ A_2 = \frac{9}{16} a^{-2 \ 4} \omega^2 \Delta.$

85. $= {}_1 \ {}^{-3} + {}_2 \ {}^{-1 \ 2 \ -1 \ 2}.$

Solutions in parametric form:

$$= a\tau^{2 \ 3} N, \qquad = \ \tau^{-2 \ 3} N^{-1} N_1^2,$$

where $= \frac{1}{3}, \ A_1 = -2a \ \omega^2 \Delta, \ A_2 = \mp \frac{8}{3} a^{-3 \ 2 \ 3 \ 2}.$

86. $= {}_1 + {}_2 \ {}^{-2 \ -1 \ 2}.$

Solutions in parametric form:

$$= a\tau^{-2 \ 3} N^{-1}, \qquad = \ \tau^{-4 \ 3} N^{-2} N_1^2,$$

where $= \frac{1}{3}, \ A_1 = -2a^{-2} \ \omega^2 \Delta, \ A_2 = \mp \frac{8}{3} \ {}^{3 \ 2}.$

87. $= {}_1 \quad {}^{-2} + \quad {}_2 \quad {}^{-3}.$

Solutions in parametric form:

$$= a\tau^{-2\ \ 3} N^{-1} N_2, \qquad = \tau^{2\ \ 3} N,$$

where $= \frac{1}{3},\ A_1 = -\frac{9}{128} a^{-3\ \ 3},\ A_2 = \frac{9}{64} a^{-2\ \ 4} \omega^2 \Delta.$

88. $= {}_1 \quad {}^{-2} \quad {}^{-2} + \quad {}_2 \quad {}^{-3}.$

Solutions in parametric form:

$$= a\tau^{2\ \ 3} N N_2^{-1}, \qquad = \tau^{4\ \ 3} N^2 N_2^{-1},$$

where $= \frac{1}{3},\ A_1 = -\frac{9}{128}\ \ {}^{3},\ A_2 = \frac{9}{64} a^{-2\ \ 4} \omega^2 \Delta.$

In the solutions of equations 89 and 90, the following notation is used:

$$\Delta = \ {}_2^2 - 2 \ \ {}_1, \quad R = (36\Delta + 54B\tau - 2\tau^3)^{1\ \ 2}, \quad z = 3 \quad \tau^{-1} R^{-1} \quad \tau,$$

$$(z) = \begin{cases} \dfrac{\overline{-\Delta}}{{}_1} \tan\left(\overline{-\Delta}\, z + \dfrac{{}_2}{{}_1} \right) & \textit{if } \Delta < 0, \\ \dfrac{\overline{\Delta}}{{}_1} \tanh\left(\mp \overline{\Delta}\, z + \dfrac{{}_2}{{}_1} \right) & \textit{if } \Delta > 0, \\ \mp \dfrac{1}{{}_1 z} - \dfrac{\overline{{}_2}}{\overline{|{}_1|}} & \textit{if } \Delta = 0, \quad {}_2 < 0, \\ \mp \dfrac{1}{{}_1 z} + \dfrac{\overline{{}_2}}{\overline{|{}_1|}} & \textit{if } \Delta = 0, \quad {}_2 > 0. \end{cases}$$

89. $= {}_1 \quad {}^{-5\ \ 3} + \quad {}_2 \quad {}^{-2\ \ 3} \quad {}^{-5\ \ 3}.$

Solutions in parametric form:

$$= a\tau^{-3\ \ 2}({}_1 \quad {}^2 - 2 \ {}_2 \quad + 2)^{3\ \ 2}, \qquad = \tau^{-9\ \ 4}({}_1 \quad {}^2 - 2 \ {}_2 \quad + 2)^{3\ \ 4}(6 \ {}_1 \quad - 6 \ {}_2 \mp R)^{3\ \ 2},$$

where $A_1 = 24a^{-2\ \ 8\ \ 3}\ \ {}_1,\ A_2 = -36a^{-4\ \ 3\ \ 8\ \ 3} B.$

90. $= {}_1 \quad {}^{-2\ \ 3} \quad {}^{-5\ \ 3} + \quad {}_2 \quad {}^{-4\ \ 3} \quad {}^{-5\ \ 3}.$

Solutions in parametric form:

$$= a\tau^{3\ \ 2}({}_1 \quad {}^2 - 2 \ {}_2 \quad + 2)^{-3\ \ 2}, \qquad = \tau^{-3\ \ 4}({}_1 \quad {}^2 - 2 \ {}_2 \quad + 2)^{-3\ \ 4}(6 \ {}_1 \quad - 6 \ {}_2 \mp R)^{3\ \ 2},$$

where $A_1 = -36a^{-4\ \ 3\ \ 8\ \ 3} B,\ A_2 = 24a^{-2\ \ 3\ \ 8\ \ 3}\ \ {}_1.$

In the solutions of equations 91–102, the following notation is used:

The functions ${}_1$ and ${}_2$ are the general solutions of the four modifications of the first Painlevé equation:

$$ {}_1'' = \ 6 \ {}_1^2 + \tau, \qquad {}_2'' = \ 6 \ {}_2^2 - \tau$$

(in the case of the upper sign, the equation for ${}_1$ is the canonical form of the first Painlevé equation, see Paragraph 2.8.2-2). In addition,

$$Q_1 = \ 6 \ {}_1^2 + \tau, \quad {}_1 = \tau^2 \ {}_1 \mp 1, \quad {}_1 = ({}_1')^2 - 2 \ {}_1 Q_1 \quad 8 \ {}_1^3, \quad {}_1 = \ {}_1' Q_1' + \ {}_1' - Q_1^2,$$

$$Q_2 = \ 6 \ {}_2^2 - \tau, \quad {}_2 = \tau^2 \ {}_2 \mp 1, \quad {}_2 = ({}_2')^2 - 2 \ {}_2 Q_2 \quad 8 \ {}_2^3, \quad {}_2 = \ {}_2' Q_2' - \ {}_2' - Q_2^2.$$

The prime denotes differentiation with respect to τ.

91. $= {}_1 \quad {}^2 + \quad {}_2 \quad .$

Solutions in parametric form:

$$= a\tau, \qquad = \quad ,$$

where $A_1 = \ 6a^{-2\ \ -1},\ A_2 = a^{-3}\ (-1)\ \ {}^{+1};\ k = 1$ and $k = 2$.

92. $= {}_1 {}^{-5}\ {}^{2} + {}_2 {}^{-4}$.

Solutions in parametric form:

$$= a\tau^{-1}, \qquad = \tau^{-1} \quad ,$$

where $A_1 = 6a^{3}\ {}^{-1}$, $A_2 = a^2\ (-1)\ {}^{+1}$; $k = 1$ and $k = 2$.

93. $= {}_1 {}^{-15\ 7\ 2} + {}_2 {}^{-8\ 7}$.

Solutions in parametric form:

$$= a\tau^{7}, \qquad = \tau \quad ,$$

where $A_1 = \frac{6}{49}a^{1\ 7\ -1}$, $A_2 = \frac{1}{49}a^{-6\ 7}\ (-1)\ {}^{+1}$; $k = 1$ and $k = 2$.

94. $= {}_1 {}^{-20\ 7\ 2} + {}_2 {}^{-13\ 7}$.

Solutions in parametric form:

$$= a\tau^{-7}, \qquad = \tau^{-6} \quad ,$$

where $A_1 = \frac{6}{49}a^{6\ 7\ -1}$, $A_2 = \frac{1}{49}a^{-1\ 7}\ (-1)\ {}^{+1}$; $k = 1$ and $k = 2$.

95. $= {}_1 + {}_2 {}^{-1\ 2}$.

Solutions in parametric form:

$$= a \quad , \qquad = (\ \ ')^2,$$

where $A_1 = 24a^{-3}$, $A_2 = 2a^{-2\ 3\ 2}(-1)\ {}^{+1}$; $k = 1$ and $k = 2$.

96. $= {}_1 {}^{-4} + {}_2 {}^{-5\ 2\ -1\ 2}$.

Solutions in parametric form:

$$= a\ \ {}^{-1}, \qquad = \ {}^{-1}(\ \ ')^2,$$

where $A_1 = 24a^{2}$, $A_2 = 2a^{1\ 2\ 3\ 2}(-1)\ {}^{+1}$; $k = 1$ and $k = 2$.

97. $= {}_1 {}^{1\ 3} + {}_2 {}^{-5\ 3}$.

Solutions in parametric form:

$$= a \quad , \qquad = \ {}^{3\ 2},$$

where $A_1 = \mp 8a\ {}^{-2}A_2$, $A_2 = -\frac{3}{16}a^{-3\ 8\ 3}$; $k = 1$ and $k = 2$.

98. $= {}_1 {}^{-10\ 3\ 1\ 3} + {}_2 {}^{-7\ 3\ -5\ 3}$.

Solutions in parametric form:

$$= a\ \ {}^{-1}, \qquad = \ {}^{3\ 2\ -1},$$

where $A_1 = \mp 8a\ {}^{-2}A_2$, $A_2 = -\frac{3}{16}a^{1\ 3\ 8\ 3}$; $k = 1$ and $k = 2$.

99. $= {}_1 {}^{-3\ 2} + {}_2 {}^{-1\ 2}$.

Solutions in parametric form:

$$= a(\ \ ')^2, \qquad = Q^2,$$

where $A_1 = \frac{1}{2}a^{-1\ 2}\ (-1)$, $A_2 = 6a^{-2\ 3\ 2}$; $k = 1$ and $k = 2$.

100. $= {}_1 {}^{-3\ 2} + {}_2 {}^{-5\ 2\ -1\ 2}$.

Solutions in parametric form:

$$= a(\ \ ')^{-2}, \qquad = (\ \ ')^{-2}Q^2,$$

where $A_1 = \frac{1}{2}a^{-1\ 2}\ (-1)$, $A_2 = 6a^{1\ 2\ 3\ 2}$; $k = 1$ and $k = 2$.

101. $= {}_{1}{}^{-4\ 3} + {}_{2}{}^{-5\ 3}$**.**

Solutions in parametric form:

$$= a \quad , \qquad = (\ ')^{3},$$

where $A_1 = a^{-1\ 3} A_2(-1)\ ,\ A_2 = \frac{1}{36} a^{-3\ 8\ 3};\ k = 1$ and $k = 2$.

102. $= {}_{1}{}^{-5\ 3\ -4\ 3} + {}_{2}{}^{-7\ 3\ -5\ 3}$**.**

Solutions in parametric form:

$$= a^{\ -1}, \qquad = (\ ')^{3\ -1},$$

where $A_1 = \frac{1}{36} a^{-1\ 3\ 7\ 3}(-1)\ ,\ A_2 = \frac{1}{36} a^{1\ 3\ 8\ 3};\ k = 1$ and $k = 2$.

In the solutions of equations 103–108, the following notation is used:

The functions $_1$ and $_2$ are the general solutions of the four modifications of the second Painlevé equation (with parameter $a = 0$):

$$_1'' = \tau\ _1\ \ 2\ _1^3, \qquad _2'' = -\tau\ _2\ \ 2\ _2^3,$$

where the primes denote differentiation with respect to τ. In the case of the upper sign, the equation for $_1$ is the canonical form of the second Painlevé equation (with parameter $a = 0$, see Paragraph 2.8.2-3).

103. $= {}_{1}{}^{3} + {}_{2}$ **.**

Solutions in parametric form:

$$= a\tau, \qquad = \quad ,$$

where $A_1 = \ 2a^{-2\ -2},\ A_2 = a^3(-1)^{\ +1};\ k = 1$ and $k = 2$.

104. $= {}_{1}{}^{-6\ 3} + {}_{2}{}^{-5}$ **.**

Solutions in parametric form:

$$= a\tau^{-1}, \qquad = \tau^{-1} \quad ,$$

where $A_1 = \ 2a^{4\ -2},\ A_2 = a^3(-1)^{\ +1};\ k = 1$ and $k = 2$.

105. $= {}_{1} + {}_{2}{}^{-1\ 2\ -1\ 2}$**.**

Solutions in parametric form:

$$= a^{\ 2}, \qquad = (\ ')^{2}, \qquad ' = (\quad)'_{\tau},$$

where $A_1 = \ 2a^{-2}\ ,\ A_2 = \frac{1}{2} a^{-3\ 2\ 3\ 2}(-1)^{\ +1};\ k = 1$ and $k = 2$.

106. $= {}_{1}{}^{-3} + {}_{2}{}^{-2\ -1\ 2}$**.**

Solutions in parametric form:

$$= a^{\ -2}, \qquad = {}^{-2}(\ ')^{2}, \qquad ' = (\quad)'_{\tau},$$

where $A_1 = \ 2a\ ,\ A_2 = \frac{1}{2}{}^{\ 3\ 2}(-1)^{\ +1};\ k = 1$ and $k = 2$.

107. $= {}_{1} + {}_{2}{}^{-2}$**.**

Solutions in parametric form:

$$= a[\tau^{\ 2}\quad{}^{4} - (\ ')^{2}], \qquad = {}^{2}, \qquad ' = (\quad)'_{\tau},$$

where $A_1 = \mp 2a^{-2}\ (-1)\ ,\ A_2 = 2a^{-3\ 3}(-1)^{\ +1};\ k = 1$ and $k = 2$.

108. $= {}_{1}{}^{-3} + {}_{2}{}^{-2\ -2}$**.**

Solutions in parametric form:

$$= a\left[\tau^{\ 2}\quad{}^{4} - (\ ')^{2}\right]^{-1}, \qquad = {}^{2}\left[\tau^{\ 2}\quad{}^{4} - (\ ')^{2}\right]^{-1},$$

where $A_1 = \ 2a\ ,\ A_2 = 2^{\ 3};\ k = 1$ and $k = 2$.

2.5. Generalized Emden–Fowler Equation $y \quad = \quad y \;\; (y \;\;)$

2.5.1. Classification Table

The case $l = 0$ corresponding to the classical Emden–Fowler equation is outlined in Section 2.3. In this section, the case $l \neq 0$ is considered.

Table 23 presents all solvable equations of the form $\;'' = A \qquad (\;')$ whose solutions are outlined in Subsection 2.5.2. Two-parameter families (in the space of the parameters $\;,\;\;$, and l), one-parameter families, and isolated points are presented in a consecutive fashion. Equations are arranged in accordance with the growth of l, the growth of $\;\;$ (for identical l), and the growth of $\;\;$ (for identical $\;\;$ and l). The number of the equation sought is indicated in the last column in this table.

TABLE 23
Solvable cases of the generalized Emden–Fowler equation $\;'' = A \qquad (\;')$

No	l			Equation
		Two-parameter families		
1	arbitrary	arbitrary	0	2.5.2.1
2	arbitrary	0	arbitrary	2.5.2.2
3	$\frac{2\; +\;\; +3}{\;+\;\; +2}$	arbitrary ($\neq -1$)	arbitrary ($\neq -1$)	2.5.2.3
		One-parameter families		
4	arbitrary ($l \neq 1, 2$)	-1	-1	2.5.2.6
5	arbitrary ($l \neq \frac{3}{2}$)	$-\frac{1}{2}$	$-\frac{1}{2}$	2.5.2.97
6	$\frac{3\;\; +5}{2\;\; +3}$	arbitrary ($\neq -\frac{3}{2}$)	$-\frac{1}{2}$	2.5.2.13
7	$\frac{3\;\; +5}{2\;\; +3}$	arbitrary ($\neq -\frac{3}{2}$)	1	2.5.2.10
8	$\frac{3\;\; +4}{2\;\; +3}$	$-\frac{1}{2}$	arbitrary ($\neq -\frac{3}{2}$)	2.5.2.11
9	$\frac{3\;\; +4}{2\;\; +3}$	1	arbitrary ($\neq -\frac{3}{2}$)	2.5.2.12
10	$\frac{3\;\; +4}{2\;\; +3}$	$-\;\; -3$	arbitrary ($\neq -\frac{3}{2}$)	2.5.2.107
11	1	arbitrary ($\neq -1, 0$)	-1	2.5.2.5
12	2	-1	arbitrary ($\neq -1, 0$)	2.5.2.4
13	3	arbitrary ($\neq -2$)	1	2.5.2.96
14	3	$-\;\; -3$	arbitrary	2.5.2.9

TABLE 23 (*Continued*)

Solvable cases of the generalized Emden–Fowler equation ″ = A (′)

No	l			Equation
		Isolated points		
15	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{5}{2}$	2.5.2.33
16	$\frac{1}{2}$	1	$-\frac{15}{8}$	2.5.2.83
17	$\frac{1}{2}$	1	$-\frac{20}{13}$	2.5.2.86
18	$\frac{1}{2}$	1	$-\frac{5}{4}$	2.5.2.80
19	$\frac{1}{2}$	1	0	2.5.2.78
20	$\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{7}{6}$	2.5.2.53
21	$\frac{4}{5}$	$-\frac{5}{2}$	$-\frac{1}{2}$	2.5.2.76
22	1	-2	1	2.5.2.14
23	1	-1	-1	2.5.2.8
24	$\frac{8}{7}$	1	$-\frac{3}{4}$	2.5.2.54
25	$\frac{8}{7}$	1	$-\frac{1}{2}$	2.5.2.52
26	$\frac{6}{5}$	$-\frac{1}{2}$	$-\frac{2}{3}$	2.5.2.68
27	$\frac{5}{4}$	1	$-\frac{1}{2}$	2.5.2.58
28	$\frac{5}{4}$	1	0	2.5.2.56
29	$\frac{9}{7}$	$-\frac{13}{8}$	1	2.5.2.39
30	$\frac{9}{7}$	$-\frac{1}{2}$	1	2.5.2.38
31	$\frac{13}{10}$	$-\frac{1}{2}$	$-\frac{5}{2}$	2.5.2.47
32	$\frac{27}{20}$	$-\frac{1}{2}$	$-\frac{2}{3}$	2.5.2.72
33	$\frac{18}{13}$	$-\frac{1}{2}$	$-\frac{7}{2}$	2.5.2.40
34	$\frac{7}{5}$	$-\frac{7}{4}$	1	2.5.2.18
35	$\frac{7}{5}$	$-\frac{10}{7}$	1	2.5.2.46
36	$\frac{7}{5}$	$-\frac{2}{3}$	1	2.5.2.32
37	$\frac{7}{5}$	$-\frac{1}{2}$	1	2.5.2.17
38	$\frac{7}{5}$	1	0	2.5.2.89
39	$\frac{7}{5}$	1	1	2.5.2.91
40	$\frac{7}{5}$	5	1	2.5.2.75
41	$\frac{10}{7}$	$-\frac{1}{2}$	$-\frac{5}{2}$	2.5.2.19
42	$\frac{22}{15}$	$-\frac{1}{2}$	$-\frac{2}{3}$	2.5.2.70
43	$\frac{3}{2}$	-2	$-\frac{1}{2}$	2.5.2.106
44	$\frac{3}{2}$	-2	1	2.5.2.99
45	$\frac{3}{2}$	$-\frac{1}{2}$	-2	2.5.2.100
46	$\frac{3}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	2.5.2.29

TABLE 23 (*Continued*)

Solvable cases of the generalized Emden–Fowler equation $'' = A \quad (\,'\,)$

No	l			Equation
47	$\frac{3}{2}$	$-\frac{1}{2}$	1	2.5.2.98
48	$\frac{3}{2}$	1	–2	2.5.2.105
49	$\frac{3}{2}$	1	$-\frac{1}{2}$	2.5.2.103
50	$\frac{23}{15}$	$-\frac{2}{3}$	$-\frac{1}{2}$	2.5.2.84
51	$\frac{11}{7}$	$-\frac{5}{2}$	$-\frac{1}{2}$	2.5.2.27
52	$\frac{8}{5}$	0	1	2.5.2.73
53	$\frac{8}{5}$	1	$-\frac{7}{4}$	2.5.2.26
54	$\frac{8}{5}$	1	$-\frac{10}{7}$	2.5.2.48
55	$\frac{8}{5}$	1	$-\frac{2}{3}$	2.5.2.35
56	$\frac{8}{5}$	1	$-\frac{1}{2}$	2.5.2.24
57	$\frac{8}{5}$	1	1	2.5.2.74
58	$\frac{8}{5}$	1	5	2.5.2.94
59	$\frac{21}{13}$	$-\frac{7}{2}$	$-\frac{1}{2}$	2.5.2.45
60	$\frac{33}{20}$	$-\frac{2}{3}$	$-\frac{1}{2}$	2.5.2.87
61	$\frac{17}{10}$	$-\frac{5}{2}$	$-\frac{1}{2}$	2.5.2.49
62	$\frac{12}{7}$	1	$-\frac{13}{8}$	2.5.2.44
63	$\frac{12}{7}$	1	$-\frac{1}{2}$	2.5.2.42
64	$\frac{7}{4}$	$-\frac{1}{2}$	1	2.5.2.51
65	$\frac{7}{4}$	0	1	2.5.2.50
66	$\frac{9}{5}$	$-\frac{2}{3}$	$-\frac{1}{2}$	2.5.2.81
67	$\frac{13}{7}$	$-\frac{3}{4}$	1	2.5.2.65
68	$\frac{13}{7}$	$-\frac{1}{2}$	1	2.5.2.61
69	2	–1	–1	2.5.2.7
70	2	1	–2	2.5.2.16
71	$\frac{11}{5}$	$-\frac{1}{2}$	$-\frac{5}{2}$	2.5.2.95
72	$\frac{7}{3}$	$-\frac{7}{6}$	$-\frac{1}{2}$	2.5.2.64
73	$\frac{5}{2}$	$-\frac{5}{2}$	$-\frac{1}{2}$	2.5.2.36
74	$\frac{5}{2}$	$-\frac{15}{8}$	1	2.5.2.69
75	$\frac{5}{2}$	$-\frac{20}{13}$	1	2.5.2.71
76	$\frac{5}{2}$	$-\frac{5}{4}$	1	2.5.2.67
77	$\frac{5}{2}$	0	1	2.5.2.66
78	3	–5	2	2.5.2.79
79	3	$-\frac{7}{2}$	$-\frac{1}{2}$	2.5.2.31
80	3	$-\frac{10}{3}$	$-\frac{5}{3}$	2.5.2.37

TABLE 23 (*Continued*)

Solvable cases of the generalized Emden–Fowler equation $'' = A \quad (\ ')$

No	l			Equation
81	3	$-\frac{20}{7}$	2	2.5.2.85
82	3	$-\frac{5}{2}$	$-\frac{1}{2}$	2.5.2.22
83	3	$-\frac{13}{5}$	$-\frac{7}{5}$	2.5.2.43
84	3	$-\frac{7}{3}$	$-\frac{5}{3}$	2.5.2.25
85	3	$-\frac{15}{7}$	2	2.5.2.82
86	3	–2	–2	2.5.2.104
87	3	–2	–1	2.5.2.15
88	3	–2	$-\frac{1}{2}$	2.5.2.101
89	3	–2	1	2.5.2.28
90	3	$-\frac{4}{3}$	$-\frac{1}{2}$	2.5.2.59
91	3	$-\frac{7}{6}$	$-\frac{1}{2}$	2.5.2.60
92	3	$-\frac{5}{6}$	$-\frac{5}{3}$	2.5.2.93
93	3	$-\frac{1}{2}$	$-\frac{5}{2}$	2.5.2.90
94	3	$-\frac{1}{2}$	$-\frac{5}{3}$	2.5.2.92
95	3	0	–4	2.5.2.55
96	3	0	$-\frac{5}{2}$	2.5.2.88
97	3	0	$-\frac{1}{2}$	2.5.2.20
98	3	0	2	2.5.2.77
99	3	1	–7	2.5.2.62
100	3	1	–4	2.5.2.57
101	3	1	–2	2.5.2.102
102	3	1	$-\frac{5}{3}$	2.5.2.23
103	3	1	$-\frac{7}{5}$	2.5.2.41
104	3	1	$-\frac{1}{2}$	2.5.2.30
105	3	1	0	2.5.2.21
106	3	2	$-\frac{5}{3}$	2.5.2.34
107	3	3	–7	2.5.2.63

2.5.2. Exact Solutions

1. $=\quad(\ )$.

1 . Solution in parametric form with $\neq -1,\ l \neq 2$:

$$= a \ {}_1^{1-\ -} \ (1 \ \ \tau^{+1})^{\frac{1}{-2}} \ \tau + \ {}_2, \qquad = {}_1^{2-}\,\tau, \qquad \text{where} \quad A = \frac{+1}{2-l}\,a^{-2\ 1-\ -}.$$

2 . Solution in parametric form with $= -1,\ l \neq 2$:

$$= a \ {}_1 \ \ \tau^{\overline{-2}} \exp(\mp\tau^2) \ \ \tau + \ {}_2, \qquad = {}_1 \exp(\mp\tau^2), \qquad \text{where} \quad A = \mp\frac{4\ ^2}{a^2(2-l)} \ \mp\frac{a}{2} \ .$$

3 . Solution in parametric form with $\neq -1,\ l = 2$:

$$= \ _1 \ \tau^{\frac{1-}{1+}} \exp(\mp\tau^2) \ \tau + \ _2, \qquad = \tau^{\frac{2}{+1}}, \qquad \text{where} \quad A = \ (\ +1)^{-1-} \ .$$

4 . Solution for $= -1,\ l = 2$:

$$= \begin{cases} (\ _1 \ + \ _2)^{\frac{1}{1-}} & \text{if } A \neq 1, \\ _2 \exp(\ _1 \ ) & \text{if } A = 1. \end{cases}$$

2. $=$ $(\ )$ **.**

1 . Solution in parametric form with $\neq -1,\ l \neq 1$:

$$= a \ _1^{1-} \tau, \qquad = \ _1^{2+ \ -} \ (1 \ \ \tau^{+1})^{\frac{1}{1-}} \ \tau + \ _2, \qquad \text{where} \quad A = \frac{+1}{1-l} a^{- \ -2 \ 1-} .$$

2 . Solution in parametric form with $= -1,\ l \neq 1$:

$$= a \ _1 \exp(\mp\tau^2), \qquad = \ _1 \ \tau^{\frac{3-}{1-}} \exp(\mp\tau^2) \ \tau + \ _2, \qquad \text{where} \quad A = \mp\frac{4a^2}{^2(1-l)} \ \mp\frac{}{2a}^{3-} .$$

3 . Solution in parametric form with $\neq -1,\ l = 1$:

$$= a\tau^{\frac{2}{+1}}, \qquad = \ _1 \ \tau^{\frac{1-}{1+}} \exp(\mp\tau^2) \ \tau + \ _2, \qquad \text{where} \quad A = \mp(\ +1)a^{1-} .$$

4 . Solution for $= -1,\ l = 1$:

$$= \begin{cases} _1| \ |^{+1} + \ _2 & \text{if } A \neq 1, \\ _1 \ln| \ | + \ _2 & \text{if } A = 1. \end{cases}$$

3. $=$ $(\ )^{\frac{2 \ + \ +3}{+ \ +2}}$ **.**

Solution in parametric form with $\neq -1,\ \neq -1$:

$$= \exp \ \frac{\tau}{(\tau)} + \ _2 \ , \qquad = \tau \exp -\frac{+1}{+1} \ \frac{\tau}{(\tau)} - \frac{+1}{+1} \ _2 \ ,$$

where the function $= (\tau)$ is defined implicitly by the formula

$$[\ + (\sigma - 1)\tau](\ + \sigma\tau)^{\frac{\sigma}{1-\sigma}} = \ _1 + \frac{A\tau^{+2}}{+ \ +2}, \qquad \sigma = -\frac{+1}{+1}.$$

For the case $= -1$, see equation 2.5.2.5. For $= -1$, see equation 2.5.2.4.

4. $=$ $^{-1}(\ )^2$**.**

Solution in parametric form with $\neq -1,\ \neq 0$:

$$= a\tau^{\frac{1}{}}, \qquad = \ \exp \ \tau^{\frac{1-}{}} \ \frac{}{+1}\tau^{\frac{+1}{}} + \ \tau^{\frac{1}{}} + \ _1^{-1} \ \tau + \ _2 \ ,$$

where $A = -a^{-}$.

For the case $= -1$, see equation 2.5.2.7. For $= 0$, see equation 2.5.2.1.

5. $=$ $^{-1}$ **.**

Solution in parametric form with $\neq -1,\ \neq 0$:

$$= \ \exp \ \tau^{\frac{1-}{}} \ \frac{}{+1}\tau^{\frac{+1}{}} + \ \tau^{\frac{1}{}} + \ _1^{-1} \ \tau + \ _2 \ , \qquad = (A\tau)^{\frac{1}{}}.$$

For the case $= -1$, see equation 2.5.2.8. For $= 0$, see equation 2.5.2.2.

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6. $= \ ^{-1} \ ^{-1}(\ ) .$

Solution in parametric form with $l \neq 1$, $l \neq 2$:

$$= \quad -\frac{\tau}{\lambda} \ ^{-1} \exp \frac{1}{\lambda} \quad \frac{\tau}{(\tau)} + \ _2 , \qquad = \exp \quad \frac{\tau}{(\tau)} + \lambda \ _2 , \qquad \lambda = \frac{l-1}{l-2},$$

where the function $= (\tau)$ is defined implicitly by the formula

$$\ln \ \frac{}{\tau} - \frac{1}{\lambda} \quad - \frac{\tau}{\lambda \ - \tau} = \frac{A}{\lambda}\tau^{\lambda} - \ln \tau + \ _1, \qquad \lambda = \frac{l-1}{l-2}.$$

For the case $l = 2$, see equation 2.5.2.7. For $l = 1$, see equation 2.5.2.8.

7. $= \ ^{-1} \ ^{-1}(\ )^2.$

Solution in parametric form:

$$= e^{\tau}, \qquad = \ _2(\mp A\tau + e^{\tau} + \ _1) \exp \quad A \quad (\mp A\tau + e^{\tau} + \ _1)^{-1} \ \tau \ .$$

8. $= \ ^{-1} \ ^{-1} \ .$

Solution in parametric form:

$$= \ _2(\ A\tau + e^{\tau} + \ _1) \exp \mp A \quad (\ A\tau + e^{\tau} + \ _1)^{-1} \ \tau \ , \qquad = e^{\tau}.$$

9. $= \ - \ ^{-3}(\ )^3.$

Solution in parametric form with $\neq -1$:

$$= a \ _1^{\ +1} \tau \quad \frac{\tau}{\overline{1 \quad \tau^{\ +1}}} + \ _2 \ ^{-1}, \qquad = \ _1^{\ -1} \quad \frac{\tau}{\overline{1 \quad \tau^{\ +1}}} + \ _2 \ ^{-1},$$

where $A = \mp \frac{+1}{2} a^{1- \quad +1}$.

For the case $= -1$, see equation 2.5.2.15.

10. $= \quad (\ )^{\frac{3 \ +5}{2 \ +3}}.$

Solution in parametric form with $\neq -3 \ 2$:

$$= a \ _1^{-2} \ (1 \quad \tau^{\ +1})^{1 \ 2} \quad (1 \quad \tau^{\ +1})^{-1 \ 2} \ \tau + \ _2 - \tau \ ,$$

$$= \ _1^{(\ +1)(\ -2)} \quad (1 \quad \tau^{\ +1})^{-1 \ 2} \ \tau + \ _2 \ ^{+2},$$

where $= -\frac{2 \quad +3}{+1}$, $A = -\frac{\frac{1}{+2}}{(\ +2)a} \quad \frac{(\ +1)a}{2(\ +2)} \ ^{\frac{1}{}}$.

11. $= \ ^{-\frac{1}{2}}(\ )^{\frac{3 \ +4}{2 \ +3}}.$

Solution in parametric form with $\neq -3 \ 2$:

$$= a \ _1^{(\ +1)^2} \tau^{\ +1} \quad (1 \quad \tau^{\ +1})^{-1 \ 2} \ \tau + \ _2 \ ^{- \ -1},$$

$$= \ _1^4 \ (1 \quad \tau^{\ +1})^{1 \ 2} \quad (1 \quad \tau^{\ +1})^{-1 \ 2} \ \tau + \ _2 - \tau \ ^2,$$

where $= -\frac{}{+1}$, $A = \frac{+3}{a(\ +1)} a^{\overline{\ +1}} \ \frac{1}{2} \quad \frac{a}{} \ ^{\frac{1}{+3}}$.

12. $= \quad (\quad)^{\frac{3\ \ +4}{2\ \ +3}}$**.**

Solution in parametric form with $\ \neq -3\ \ 2$:

$$= a\ _1^{(\ \ -1)(\ \ +2)}\ \ (1\ \ \tau\ ^{+1})^{-1\ \ 2}\ \tau + \ _2\ \ ^{+2},$$

$$=\ \ _1^{-2}\ (1\ \ \tau\ ^{+1})^{1\ \ 2}\ \ (1\ \ \tau\ ^{+1})^{-1\ \ 2}\ \tau + \ _2 - \tau\ \ ,$$

where $\ = -\dfrac{2\ \ +3}{\ +1},\ A = \dfrac{a^{\frac{1}{\ +2}}}{(\ \ +2)}\ \ \dfrac{(\ \ +1)}{2(\ \ +2)a}^{\ \frac{1}{\ }}.$

13. $= \ ^{-\frac{1}{2}}\ \ (\quad)^{\frac{3\ \ +5}{2\ \ +3}}$**.**

Solution in parametric form with $\ \neq -3\ \ 2$:

$$= a\ _1^{4}\ (1\ \ \tau\ ^{+1})^{1\ \ 2}\ \ (1\ \ \tau\ ^{+1})^{-1\ \ 2}\ \tau + \ _2 - \tau\ \ ^{2},$$

$$=\ \ _1^{(\ \ +1)^2}\tau\ ^{+1}\ \ (1\ \ \tau\ ^{+1})^{-1\ \ 2}\ \tau + \ _2\ \ ^{-\ \ -1},$$

where $\ = -\dfrac{\quad}{\ +1},\ A = -\dfrac{\ +3}{(\ \ +1)}a^{\frac{1}{2}}\ \ \overline{\ \ +1}\ \ \ \dfrac{\ }{a}^{\frac{1}{\ +3}}.$

14. $= \quad ^{-2}\quad$**.**

Solution in parametric form:

$$= a\ _1\ 2\tau\ \ \exp(\mp\tau^2)\ \tau + \ _2\ \ \exp(\mp\tau^2)\ \ , \qquad =\ _1\ \ \exp(\mp\tau^2)\ \tau + \ _2\ ,$$

where $A = \mp\frac{1}{2}a^{-2\ \ 2}$.

15. $= \quad ^{-1\ \ -2}(\quad)^3$**.**

Solution in parametric form:

$$= a\exp(\mp\tau^2)\ \ \exp(\mp\tau^2)\ \tau + \ _2\ \ ^{-1}, \qquad =\ _1\ \ \exp(\mp\tau^2)\ \tau + \ _2\ \ ^{-1},$$

where $A = \ \ 2a^2$.

16. $= \quad ^{-2}\ (\quad)^2$**.**

Solution in parametric form:

$$= a\ _1\ \ \exp(\mp\tau^2)\ \tau + \ _2\ , \qquad =\ _1\ 2\tau\ \ \exp(\mp\tau^2)\ \tau + \ _2\ \ \exp(\mp\tau^2)\ \ ,$$

where $A = \ \ a^2\ ^{-2}$.

17. $= \quad ^{-1\ \ 2}(\quad)^{7\ \ 5}$**.**

Solution in parametric form:

$$=\ a\ \ _1(\tau^2-1)(\tau^3-3\tau+\ _2)^{-1\ \ 2}, \qquad =\ _1^{16}(\tau^4-6\tau^2+4\ \ _2\tau-3)^2,$$

where $A = \ \ 15a^{-2\ \ 1\ \ 2}\ \ \dfrac{a}{16}^{\ \ 2\ \ 5}.$

18. $=\quad {}^{-7\ 4}(\quad)^{7\ 5}$.

Solution in parameter form:

$$= \ a \quad {}_1^{27}(\tau^3 - 3\tau + \quad {}_2)^{-1\ 2}(\tau^6 - 15\tau^4 + 20 \quad {}_2\tau^3 - 45\tau^2 + 12 \quad {}_2\tau + 27 - 8 \quad {}_2^2),$$
$$= \quad {}_1^{32}(\tau^4 - 6\tau^2 + 4 \quad {}_2\tau - 3)^{4\ 3},$$

where $A = \dfrac{5}{12}a^{-2\ 7\ 4}\ \dfrac{a}{9}^{\ 2\ 5}$.

19. $=\quad {}^{-5\ 2\ \ -1\ 2}(\quad)^{10\ 7}$.

Solution in parametric form:

$$= a \quad {}_1^{-1}(\tau^3 - 3\tau + \quad {}_2)^{-1}(\tau^4 - 6\tau^2 + 4 \quad {}_2\tau - 3)^{2\ 3},$$
$$= \quad {}_1^{27}(\tau^3 - 3\tau + \quad {}_2)^{-1}(\tau^6 - 15\tau^4 + 20 \quad {}_2\tau^3 - 45\tau^2 + 12 \quad {}_2\tau + 27 - 8 \quad {}_2^2)^2,$$

where $A = 28a(a \quad)^{1\ 2}\ \dfrac{a}{27}^{\ 3\ 7}$.

20. $=\quad {}^{-1\ 2}(\quad)^3$.

Solution in parametric form:

$$= a \quad {}_1^4(\tau^2 - 1)^2, \qquad = \quad {}_1^3(\tau^3 - 3\tau + \quad {}_2), \qquad \text{where} \quad A = \tfrac{4}{9}a^{3\ 2\ -2}.$$

21. $=\quad (\quad)^3$.

Solution in parametric form:

$$= a \quad {}_1^3(\tau^3 - 3\tau + \quad {}_2), \qquad = \quad {}_1\tau, \qquad \text{where} \quad A = -6a^{\ -3}.$$

22. $=\quad {}^{-1\ 2\ \ -5\ 2}(\quad)^3$.

Solution in parametric form:

$$= a \quad {}_1(\tau^2 - 1)^2(\tau^3 - 3\tau + \quad {}_2)^{-1}, \qquad = \quad {}_1^{-3}(\tau^3 - 3\tau + \quad {}_2)^{-1}, \qquad \text{where} \quad A = \mp\tfrac{4}{9}a^{3\ 2\ 1\ 2}.$$

23. $=\quad {}^{-5\ 3}\ (\quad)^3$.

Solution in parametric form:

$$= a \quad {}_1^9(\tau^3 - 3\tau + \quad {}_2)^{3\ 2}, \qquad = \quad {}_1^8(\tau^4 - 6\tau^2 + 4 \quad {}_2\tau - 3), \qquad \text{where} \quad A = \mp\tfrac{9}{64}a^{8\ 3\ -3}.$$

24. $=\quad {}^{-1\ 2}\ (\quad)^{8\ 5}$.

Solution in parametric form:

$$= a \quad {}_1^{16}(\tau^4 - 6\tau^2 + 4 \quad {}_2\tau - 3)^2, \qquad = \quad {}_1(\tau^2 - 1)(\tau^3 - 3\tau + \quad {}_2)^{-1\ 2},$$

where $A = \mp 15a^{1\ 2\ -2}\ \dfrac{}{16a}^{\ 2\ 5}$.

25. $=\quad {}^{-5\ 3\ \ -7\ 3}(\quad)^3$.

Solution in parametric form:

$$= \ a \quad {}_1(\tau^3 - 3\tau + \quad {}_2)^{3\ 2}(\tau^4 - 6\tau^2 + 4 \quad {}_2\tau - 3)^{-1}, \qquad = \quad {}_1^{-8}(\tau^4 - 6\tau^2 + 4 \quad {}_2\tau - 3)^{-1},$$

where $A = \mp\tfrac{9}{64}a^{8\ 3\ 1\ 3}$.

26. $= \quad {}^{-7\ 4}\ (\quad)^{8\ 5}$.

Solution in parametric form:

$$= a \quad {}_1^{32}(\tau^4 - 6\tau^2 + 4 \quad {}_2\tau - 3)^{4\ \ 3},$$
$$= \quad {}_1^{27}(\tau^3 - 3\tau + \quad {}_2)^{-1\ \ 2}(\tau^6 - 15\tau^4 + 20 \quad {}_2\tau^3 - 45\tau^2 + 12 \quad {}_2\tau + 27 - 8 \quad {}_2^2),$$

where $A = \mp\frac{5}{12}a^{7\ 4\ -2} \quad \overline{9a} \quad {}^{2\ 5}$.

27. $= \quad {}^{-1\ 2\ \ -5\ 2}(\quad)^{11\ 7}$.

Solution in parametric form:

$$= a \quad {}_1^{27}(\tau^3 - 3\tau + \quad {}_2)^{-1}(\tau^6 - 15\tau^4 + 20 \quad {}_2\tau^3 - 45\tau^2 + 12 \quad {}_2\tau + 27 - 8 \quad {}_2^2)^2,$$
$$= \quad {}_1^{-1}(\tau^3 - 3\tau + \quad {}_2)^{-1}(\tau^4 - 6\tau^2 + 4 \quad {}_2\tau - 3)^{2\ \ 3},$$

where $A = -28 \ (a \quad)^{1\ \ 2} \quad \overline{27a} \quad {}^{3\ 7}$.

28. $= \quad {}^{-2}(\quad)^3$.

1 . Solution in parametric form with $A < \frac{1}{4}$:

$$= \tau(\quad {}_1\tau \quad + \quad {}_2\tau^{-} \quad), \qquad = \tau^2, \qquad \text{where} \qquad = \quad \overline{1 - 4A}.$$

2 . Solution in parametric form with $A = \frac{1}{4}$:

$$= \tau(\quad {}_1 \ln|\tau| + \quad {}_2), \qquad = \tau^2.$$

3 . Solution in parametric form with $A > \frac{1}{4}$:

$$= \tau \quad {}_1 \sin(\quad \ln\tau + \quad {}_2), \qquad = \tau^2, \qquad \text{where} \qquad = \quad \overline{4A - 1}.$$

29. $= \quad {}^{-1\ 2\ \ -1\ 2}(\quad)^{3\ 2}$.

Solution in parametric form:

$$= \quad \tau^2(\quad {}_1\tau \quad + \quad {}_2\tau^{-} \quad)^2, \qquad = \tfrac{1}{4}\tau^{-2}[(1 + \quad) \quad {}_1\tau \quad + (1 - \quad) \quad {}_2\tau^{-} \quad]^2,$$

where $A = \mp k^2, \quad = k^{-2}(k^4 + 4)^{1\ \ 2}$.

30. $= \quad {}^{-1\ 2}\ (\quad)^3$.

Solution in parametric form:

$$= a \quad {}_1^2 \exp(-2\tau)\big[2\exp(3\tau) - \quad {}_2 \sin\big(\quad \overline{3}\,\tau \quad + \quad \overline{3} \quad {}_2 \cos\big(\quad \overline{3}\,\tau \quad\big]^2,$$
$$= \quad {}_1 \exp(-\tau)\big[\exp(3\tau) + \quad {}_2 \sin\big(\quad \overline{3}\,\tau \quad\big],$$

where $A = -16a^{3\ \ 2\ \ -3}$.

31. $= \quad {}^{-1\ 2\ \ -7\ 2}(\quad)^3$.

Solution in parametric form:

$$= \frac{a \quad {}_1 e^{-\tau}\big[2\exp(3\tau) - \quad {}_2 \sin\big(\quad \overline{3}\,\tau \quad + \quad \overline{3} \quad {}_2 \cos\big(\quad \overline{3}\,\tau \quad\big]^2}{\exp(3\tau) + \quad {}_2 \sin\big(\quad \overline{3}\,\tau}, \qquad = \frac{{}_1^{-1} e^{\tau}}{\exp(3\tau) + \quad {}_2 \sin\big(\quad \overline{3}\,\tau},$$

where $A = -16(a \quad)^{3\ \ 2}$.

32. $=\quad {}^{-2\ 3}(\quad)^{7\ 5}$.

1 . Solution in parametric form with $A<0$:

$$= a\quad {}_1[\cosh(\tau+\quad {}_2)\cos\tau]^{1\ \ 2}[\tanh(\tau+\quad {}_2)-\tan\tau],$$
$$=\quad {}_1^6\cosh^3(\tau+\quad {}_2)\cos^3\tau\,[\tanh(\tau+\quad {}_2)+\tan\tau]^3,$$

where $A=-5a^{-2\ \ 2\ \ 3}\ \dfrac{a}{12}^{\ 2\ \ 5}$.

2 . Solution in parametric form with $A>0$:

$$= a\quad {}_1[\cosh\tau-\sin(\tau+\quad {}_2)]^{-1\ \ 2}[\sinh\tau-\cos(\tau+\quad {}_2)],\qquad =\quad {}_1^6[\sinh\tau+\cos(\tau+\quad {}_2)]^3,$$

where $A=5a^{-2\ \ 2\ \ 3}\ \dfrac{a}{6}^{\ 2\ \ 5}$.

33. $=\quad {}^{-5\ 2\ \ -1\ 2}(\quad)^{1\ 2}$.

Solution in parametric form:

$$= a\quad {}_1^{-1}[\cosh(\tau+\quad {}_2)\cos\tau]^{-1},\qquad =\quad {}_1\cosh(\tau+\quad {}_2)\cos\tau\,[\tanh(\tau+\quad {}_2)-\tan\tau]^2,$$

where $A=-4a\ $.

34. $=\quad {}^{-5\ 3\ \ 2}(\quad)^3$.

1 . Solution in parametric form with $A>0$:

$$= a\quad {}_1^3[\cosh(\tau+\quad {}_2)\cos\tau]^{3\ \ 2},\qquad =\quad {}_1^2\cosh(\tau+\quad {}_2)\cos\tau\,[\tanh(\tau+\quad {}_2)+\tan\tau],$$

where $A=\frac{3}{16}a^{8\ \ 3\ \ -4}$.

2 . Solution in parametric form with $A<0$:

$$= a\quad {}_1^3[\cosh\tau-\sin(\tau+\quad {}_2)]^{3\ \ 2},\qquad =\quad {}_1^2[\sinh\tau+\cos(\tau+\quad {}_2)],$$

where $A=-\frac{3}{4}a^{8\ \ 3\ \ -4}$.

35. $=\quad {}^{-2\ 3}\ (\quad)^{8\ 5}$.

1 . Solution in parametric form with $A>0$:

$$= a\quad {}_1^6\cosh^3(\tau+\quad {}_2)\cos^3\tau[\tanh(\tau+\quad {}_2)+\tan\tau]^3,$$
$$=\quad {}_1[\cosh(\tau+\quad {}_2)\cos\tau]^{1\ \ 2}[\tanh(\tau+\quad {}_2)-\tan\tau],$$

where $A=5a^{2\ \ 3\ \ -2}\ \dfrac{\quad}{12a}^{\ 2\ \ 5}$.

2 . Solution in parametric form with $A<0$:

$$= a\quad {}_1^6[\sinh\tau+\cos(\tau+\quad {}_2)]^3,\qquad =\quad {}_1[\cosh\tau-\sin(\tau+\quad {}_2)]^{-1\ \ 2}[\sinh\tau-\cos(\tau+\quad {}_2)],$$

where $A=-5a^{2\ \ 3\ \ -2}\ \dfrac{\quad}{6a}^{\ 2\ \ 5}$.

36. $=\quad {}^{-1\ 2\ \ -5\ 2}(\quad)^{5\ 2}$.

Solution in parametric form:

$$= a\quad {}_1\cosh(\tau+\quad {}_2)\cos\tau\,[\tanh(\tau+\quad {}_2)-\tan\tau]^2,\qquad =\quad {}_1^{-1}[\cosh(\tau+\quad {}_2)\cos\tau]^{-1},$$

where $A=4a\ $.

37. $= \;{}^{-5\;3}\;{}^{-10\;3}(\quad)^{3}$.

1 . Solution in parametric form with $A > 0$:

$$= a \;{}_1[\cosh(\tau + \;{}_2)\cos\tau]^{1\;\;2}[\tanh(\tau + \;{}_2) + \tan\tau]^{-1},$$
$$= \;{}_1^{-2}[\cosh(\tau + \;{}_2)\cos\tau]^{-1}[\tanh(\tau + \;{}_2) + \tan\tau]^{-1},$$

where $A = \frac{3}{16}a^{8\;\;3\;\;4\;\;3}$.

2 . Solution in parametric form with $A < 0$:

$$= a \;{}_1[\cosh\tau - \sin(\tau + \;{}_2)]^{3\;\;2}[\sinh\tau + \cos(\tau + \;{}_2)]^{-1},$$
$$= \;{}_1^{-2}[\sinh\tau + \cos(\tau + \;{}_2)]^{-1},$$

where $A = -\frac{3}{4}a^{8\;\;3\;\;4\;\;3}$.

In the solutions of equations 38–45, the following notation is used:

$$= \exp(3\tau), \quad {}_1 = \; + \;{}_2\sin(\;\;\overline{3}\,\tau\;\;, \quad {}_2 = 2 \;\; - \;{}_2\sin(\;\;\overline{3}\,\tau \; + \;\;\overline{3}\;\;{}_2\cos(\;\;\overline{3}\,\tau\;\;,$$
$${}_3 = 2\;{}_1(\;{}_2)'_\tau - (\;{}_1)'_\tau\;\;{}_2 - \;{}_1\;{}_2, \quad {}_4 = 2\;{}_1(\;{}_3)'_\tau - 5(\;{}_1)'_\tau\;\;{}_3 + \;{}_1\;{}_3.$$

38. $= \;{}^{-1\;\;2}(\quad)^{9\;\;7}$.

Solution in parametric form:

$$= a\;{}_1\;{}^{-1\;6}\;{}_1^{-1\;2}\;{}_2, \qquad = \;{}_1^{8}\;{}^{-4\;3}\;{}_3^{2}, \qquad \text{where} \quad A = 7a^{-2\;\;1\;\;2}\;\frac{a}{64}^{\;2\;\;7}.$$

39. $= \;{}^{-13\;\;8}(\quad)^{9\;\;7}$.

Solution in parametric form:

$$= a\;{}_1^{25}\;{}^{-5\;6}\;{}_1^{-1\;2}\;{}_4, \qquad = \;{}_1^{32}\;{}^{-16\;15}\;{}_3^{8\;5}, \qquad \text{where} \quad A = 7a^{-2\;\;13\;\;8}\;\frac{25a}{256}^{\;2\;\;7}.$$

40. $= \;{}^{-7\;\;2}\;{}^{-1\;\;2}(\quad)^{18\;\;13}$.

Solution in parametric form:

$$= a\;{}_1^{-1}\;{}^{1\;15}\;{}_1^{-1}\;{}_3^{2\;5}, \qquad = \;{}_1^{25}\;{}^{-5\;3}\;{}_1^{-1}\;{}_4^{2}, \qquad \text{where} \quad A = -208a^{5\;\;2\;\;1\;\;2}\;\frac{a}{25}^{\;5\;\;13}.$$

41. $= \;{}^{-7\;\;5}\;(\quad)^{3}$.

Solution in parametric form:

$$= a\;{}_1^{5}\;{}^{-5\;6}\;{}_1^{5\;2}, \qquad = \;{}_1^{4}\;{}^{-2\;3}\;{}_3, \qquad \text{where} \quad A = -\frac{5}{1024}a^{12\;\;5\;\;-3}.$$

42. $= \;{}^{-1\;\;2}\;(\quad)^{12\;\;7}$.

Solution in parametric form:

$$= a\;{}_1^{8}\;{}^{-4\;3}\;{}_3^{2}, \qquad = \;{}_1\;{}^{-1\;6}\;{}_1^{-1\;2}\;{}_2, \qquad \text{where} \quad A = -7a^{1\;\;2\;\;-2}\;\frac{}{64a}^{\;2\;\;7}.$$

43. $= \;{}^{-7\;5\;\;-13\;5}(\quad)^{3}$.

Solution in parametric form:

$$= a \;{}_{1}\;\;{}^{-1\;6}\;{}^{5\;2}_{1}\;{}^{-1}_{3}, \qquad = \;{}^{-4}_{1}\;\;{}^{2\;3}\;{}^{-1}_{3}, \qquad \text{where} \quad A = -\tfrac{5}{1024}a^{12\;5\;\;3\;5}.$$

44. $= \;{}^{-13\;8}\;(\quad)^{12\;7}$.

Solution in parametric form:

$$= a \;{}^{32}_{1}\;\;{}^{-16\;\;15}\;{}^{8\;5}_{3}, \qquad = \;{}^{25}_{1}\;\;{}^{-5\;\;6}\;{}^{-1\;2}_{1}\;\;{}_{4}, \qquad \text{where} \quad A = -7a^{13\;8\;\;-2}\;\frac{25}{256a}\;{}^{2\;7}.$$

45. $= \;{}^{-1\;2\;\;-7\;2}(\quad)^{21\;13}$.

Solution in parametric form:

$$= a \;{}^{25}_{1}\;\;{}^{-5\;3}\;{}^{-1}_{1}\;{}^{2}_{4}, \qquad = \;{}^{-1}_{1}\;\;{}^{1\;\;15}\;{}^{-1}_{1}\;{}^{2\;5}_{3}, \qquad \text{where} \quad A = 208a^{1\;2\;\;5\;2}\;\frac{}{25a}\;{}^{5\;13}.$$

In the solutions of equations 46–49, the following notation is used:

$$_1 = \cosh(\tau + \;_2)\cos\tau, \qquad _2 = \tanh(\tau + \;_2) + \tan\tau, \qquad _3 = \tanh(\tau + \;_2) - \tan\tau, \qquad _4 = 3\;_2\;_3 - 4,$$
$$\theta_1 = \cosh\tau - \sin(\tau + \;_2), \quad \theta_2 = \sinh\tau + \cos(\tau + \;_2), \quad \theta_3 = \sinh\tau - \cos(\tau + \;_2), \quad \theta_4 = 3\theta_2\theta_3 - 2\theta_1^2.$$

46. $= \;{}^{-10\;7}(\quad)^{7\;5}$.

1 . Solution in parametric form with $A < 0$:

$$= a \;{}^{9}_{1}\;{}^{3\;2}_{1}\;\;{}_{4}, \qquad = \;{}^{14}_{1}\;{}^{7\;3}_{1}\;{}^{7\;3}_{2}, \qquad \text{where} \quad A = -\frac{5}{9}a^{-2\;\;10\;7}\;\frac{9a}{28}\;{}^{2\;5}.$$

2 . Solution in parametric form with $A > 0$:

$$= a \;{}^{9}_{1}\theta_1^{-1\;2}\theta_4, \qquad = \;{}^{14}_{1}\theta_2^{7\;3}, \qquad \text{where} \quad A = \frac{5}{9}a^{-2\;\;10\;7}\;\frac{9a}{14}\;{}^{2\;5}.$$

47. $= \;{}^{-5\;2\;\;-1\;2}(\quad)^{13\;10}$.

Solution in parametric form:

$$= a \;{}^{-1}_{1}\;{}^{-1\;3}_{1}\;{}^{2\;3}_{2}, \qquad = \;{}^{9}_{1}\;{}^{3}_{1}\;{}^{2}_{4}, \qquad \text{where} \quad A = -20a(a\;\;)^{1\;2}\;\frac{a}{9}\;{}^{3\;10}.$$

48. $= \;{}^{-10\;7}\;(\quad)^{8\;5}$.

1 . Solution in parametric form with $A > 0$:

$$= a \;{}^{14}_{1}\;{}^{7\;3}_{1}\;{}^{7\;3}_{2}, \qquad = \;{}^{9}_{1}\;{}^{3\;2}_{1}\;\;{}_{4}, \qquad \text{where} \quad A = \frac{5}{9}a^{10\;7\;\;-2}\;\frac{9}{28a}\;{}^{2\;5}.$$

2 . Solution in parametric form with $A < 0$:

$$= a \;{}^{14}_{1}\theta_2^{7\;3}, \qquad = \;{}^{9}_{1}\theta_1^{-1\;2}\theta_4, \qquad \text{where} \quad A = -\frac{5}{9}a^{10\;7\;\;-2}\;\frac{9}{14a}\;{}^{2\;5}.$$

49. $= \;{}^{-1\;2\;\;-5\;2}(\quad)^{17\;10}$.

Solution in parametric form:

$$= a \;{}^{9}_{1}\;{}^{3}_{1}\;{}^{2}_{4}, \qquad = \;{}^{-1}_{1}\;{}^{-1\;3}_{1}\;{}^{2\;3}_{2}, \qquad \text{where} \quad A = 20\;(a\;\;)^{1\;2}\;\frac{}{9a}\;{}^{3\;10}.$$

In the solutions of equations 50–65, the following notation is used:

$$R = \overline{\quad(4\tau^3 - 1)}, \qquad {}_1 = 2\tau \ \ (\tau) + \ \ {}_2\tau \mp R, \qquad {}_2 = \tau^{-1}(R \ \ {}_1 - 1), \qquad {}_3 = 4\tau \ \ {}_1^2 \mp \ \ {}_2^2,$$

where $(\tau) = \dfrac{\tau \quad \tau}{R}$ *is the incomplete elliptic integral of the second kind in the form of Weierstrass.*

50. $= \quad (\quad)^{7 \ \ 4}$**.**

Solution in parametric form:

$$= a \ \ {}_1^{-3}R, \qquad = \ \ {}_1^5\tau^{-1} \ \ {}_1, \qquad \text{where} \quad A = \mp\tfrac{2}{3}a^{-2}(\mp 6a \quad)^{3 \ \ 4}.$$

51. $= \quad {}^{-1 \ \ 2}(\quad)^{7 \ \ 4}$**.**

Solution in parametric form:

$$= a \ \ {}_1^{-1} \ \ {}_2, \qquad = \ \ {}_1^5\tau^2 \ \ {}_1^{-2}, \qquad \text{where} \quad A = \mp\tfrac{2}{3}a^{-2 \ \ 1 \ \ 2}(\ \ 3a \quad)^{3 \ \ 4}.$$

52. $= \quad {}^{-1 \ \ 2} \ (\quad)^{8 \ \ 7}$**.**

Solution in parametric form:

$$= a \ \ {}_1^{-16} \ \ {}_3^2, \qquad = \ \ {}_1^5 \ \ {}_1^{-3 \ \ 2} \ \ {}_2, \qquad \text{where} \quad A = \mp\frac{7}{16}a^{-1 \ \ 2 \ \ -1} \ \ \frac{16a}{3}^{\ 1 \ \ 7}.$$

53. $= \quad {}^{-7 \ \ 6 \ \ -1 \ \ 2}(\quad)^{2 \ \ 3}$**.**

Solution in parametric form:

$$= a \ \ {}_1^5 \ \ {}_1^{-3} \ \ {}_3^6, \qquad = \ \ {}_1 \ \ {}_1^{-3}(\ \ {}_2 \ \ {}_3 - 8 \ \ {}_1^2)^2, \qquad \text{where} \quad A = \mp 4a^{-5 \ \ 6 \ \ 3 \ \ 2}(a \quad)^{2 \ \ 3}.$$

54. $= \quad {}^{-3 \ \ 4} \ (\quad)^{8 \ \ 7}$**.**

Solution in parametric form:

$$= a \ \ {}_1^{-32} \ \ {}_3^{-4}, \qquad = \ \ {}_1^3 \ \ {}_1^{-3 \ \ 2}(\ \ {}_2 \ \ {}_3 - 8 \ \ {}_1^2), \qquad \text{where} \quad A = \mp\frac{7}{32}a^{-1 \ \ 4 \ \ -1} \ \ \frac{32a}{3}^{\ 1 \ \ 7}.$$

55. $= \quad {}^{-4}(\quad)^3$**.**

Solution in parametric form:

$$= a \ \ {}_1^2\tau^{-1}, \qquad = \ \ {}_1^5\tau^{-1} \ \ {}_1, \qquad \text{where} \quad A = \ \ 6a^{5 \ \ -2}.$$

56. $= \quad (\quad)^{5 \ \ 4}$**.**

Solution in parametric form:

$$= a \ \ {}_1^5\tau^{-1} \ \ {}_1, \qquad = \ \ {}_1^{-3}R, \qquad \text{where} \quad A = \ \ \tfrac{2}{3} \ \ {}^{-2}(\mp 6 \quad a)^{3 \ \ 4}.$$

57. $= \quad {}^{-4} \ (\quad)^3$**.**

Solution in parametric form:

$$= a \ \ {}_1^3 \ \ {}_1^{-1}, \qquad = \ \ {}_1^5\tau \ \ {}_1^{-1}, \qquad \text{where} \quad A = \ \ 6a^{5 \ \ -3}.$$

58. $= \quad {}^{-1 \ \ 2} \ (\quad)^{5 \ \ 4}$**.**

Solution in parametric form:

$$= a \ \ {}_1^5\tau^2 \ \ {}_1^{-2}, \qquad = \ \ {}_1^{-1} \ \ {}_2, \qquad \text{where} \quad A = \ \ \tfrac{2}{3}a^{1 \ \ 2 \ \ -2}(\ \ 3 \quad a)^{3 \ \ 4}.$$

59. $\quad = \quad {}^{-1\ 2}\ {}^{-4\ 3}(\quad)^{3}.$

Solution in parametric form:

$$= a\ {}^{4}_{1}\ {}^{2}_{2}, \qquad = \ {}^{9}_{1}\ {}^{3}_{1}, \qquad \text{where} \quad A = \mp\tfrac{4}{3}a^{3\ 2\ -2\ 3}.$$

60. $\quad = \quad {}^{-1\ 2}\ {}^{-7\ 6}(\quad)^{3}.$

Solution in parametric form:

$$= a\ {}^{5}_{1}\ {}^{-3}_{1}\ {}^{2}_{2}, \qquad = \ {}^{9}_{1}\ {}^{-3}_{1}, \qquad \text{where} \quad A = \mp\tfrac{4}{3}a^{3\ 2\ -5\ 6}.$$

61. $\quad = \quad {}^{-1\ 2}(\quad)^{13\ 7}.$

Solution in parametric form:

$$= a\ {}^{5}_{1}\ {}^{-3\ 2}_{1}\ \ {}_{2}, \qquad = \ {}^{-16}_{1}\ {}^{2}_{3}, \qquad \text{where} \quad A = \frac{7}{16}a^{-1\ -1\ 2}\ \frac{16}{3a}\ {}^{1\ 7}.$$

62. $\quad = \quad {}^{-7}\ (\quad)^{3}.$

Solution in parametric form:

$$= a\ {}^{3}_{1}\ {}^{1\ 2}_{1}, \qquad = \ {}^{8}_{1}\ {}_{3}, \qquad \text{where} \quad A = \mp\tfrac{3}{64}a^{8\ -3}.$$

63. $\quad = \quad {}^{-7\ 3}(\quad)^{3}.$

Solution in parametric form:

$$= a\ {}^{5}_{1}\ {}^{1\ 2}_{1}\ {}^{-1}_{3}, \qquad = \ {}^{8}_{1}\ {}^{-1}_{3}, \qquad \text{where} \quad A = \mp\tfrac{3}{64}a^{8\ -5}.$$

64. $\quad = \quad {}^{-1\ 2}\ {}^{-7\ 6}(\quad)^{7\ 3}.$

Solution in parametric form:

$$= a\ {}_{1}\ {}^{-3}_{1}(\ {}_{2}\ {}_{3} - 8\ {}^{2}_{1})^{2}, \qquad = \ {}^{5}_{1}\ {}^{-3}_{1}\ {}^{6}_{3}, \qquad \text{where} \quad A = \ 4a^{3\ 2\ -5\ 6}(\quad a)^{2\ 3}.$$

65. $\quad = \quad {}^{-3\ 4}(\quad)^{13\ 7}.$

Solution in parametric form:

$$= a\ {}^{3}_{1}\ {}^{-3\ 2}_{1}(\ {}_{2}\ {}_{3} - 8\ {}^{2}_{1}), \qquad = \ {}^{-32}_{1}\ {}^{-4}_{3}, \qquad \text{where} \quad A = \frac{7}{32}a^{-1\ -1\ 4}\ \frac{32}{3a}\ {}^{1\ 7}.$$

In the solutions of equations 66–95, the following notation is used:

$$\tau = \frac{\wp}{\overline{(4\wp^{3}-1)}} - \ {}_{2}, \qquad = \overline{(4\wp^{3}-1)}.$$

The function $\wp = \wp(\tau)$ is defined implicitly. The upper sign in the formulas corresponds to the classical elliptic Weierstrass function $\wp = \wp(\tau + \ {}_{2}, 0, 1)$. The solutions given below are written in parametric form. One can assume as the parameter either τ, hence $\wp = \wp(\tau)$, or $\wp$, hence $\tau = \tau(\wp)$.

66. $\quad = \quad (\quad)^{5\ 2}.$

Solution in parametric form:

$$= a\ {}^{-3}_{1}\ \ , \qquad = \ {}_{1}\tau, \qquad \text{where} \quad A = \mp\frac{2}{\ }\ \frac{6}{a}\ {}^{1\ 2}.$$

67. $= \quad {}^{-5\ 4}(\quad)^{5\ 2}$.

Solution in parametric form:

$$= a \quad {}_1^{-1}(\tau \quad - \wp), \qquad = \quad {}_1^{2}\tau^4, \qquad \text{where} \quad A = -\frac{1}{2}a^{-1\ \ 1\ \ 4} \quad \frac{3a}{2} \quad {}^{1\ \ 2}.$$

68. $= \quad {}^{-2\ \ 3\ \ -1\ \ 2}(\quad)^{6\ 5}$.

Solution in parametric form:

$$= a \quad {}_1^{9}\tau^{-3}\wp^3, \qquad = \quad {}_1^{4}(\tau \quad - \wp)^2, \qquad \text{where} \quad A = \frac{5}{3}a^{-1\ \ 3\ \ 1\ \ 2} \quad \frac{a}{4} \quad {}^{1\ \ 5}.$$

69. $= \quad {}^{-15\ \ 8}(\quad)^{5\ 2}$.

Solution in parametric form:

$$= a \quad {}_1^{3}\tau^{-6}(\tau^3 \quad + 3\tau^2\wp \mp 1), \qquad = \quad {}_1^{4}\tau^{-8}, \qquad \text{where} \quad A = \tfrac{1}{8}a^{-1\ \ 7\ \ 8}(\mp 3a \quad)^{1\ \ 2}.$$

70. $= \quad {}^{-2\ \ 3\ \ -1\ \ 2}(\quad)^{22\ 15}$.

Solution in parametric form:

$$= a \quad {}_1^{-1}\tau^3(\tau^2\wp \mp 1)^3, \qquad = \quad {}_1^{4}\tau^{-12}(\tau^3 \quad + 3\tau^2\wp \mp 1)^2, \qquad \text{where} \quad A = -5a^{-1\ \ 3\ \ 1\ \ 2} \quad \frac{a}{4} \quad {}^{7\ \ 15}.$$

71. $= \quad {}^{-20\ \ 13}(\quad)^{5\ 2}$.

Solution in parametric form:

$$= a \quad {}_1\tau(\tau^3 \quad - 4\tau^2\wp \quad 6), \qquad = \quad {}_1^{13}\tau^{13}, \qquad \text{where} \quad A = \mp\frac{2}{13}a^{-1\ \ 7\ \ 13} \quad \frac{6a}{13} \quad {}^{1\ \ 2}.$$

72. $= \quad {}^{-2\ \ 3\ \ -1\ \ 2}(\quad)^{27\ 20}$.

Solution in parametric form:

$$= a \quad {}_1^{-9}\tau^{-18}(\tau^2\wp \mp 1)^3, \qquad = \quad {}_1\tau^2(\tau^3 \quad - 4\tau^2\wp \quad 6)^2, \qquad \text{where} \quad A = \frac{20}{3}a^{-1\ \ 3\ \ 1\ \ 2} \quad \frac{a}{4} \quad {}^{7\ \ 20}.$$

73. $= \quad (\quad)^{8\ 5}$.

Solution in parametric form:

$$= a \quad {}_1^{-3} \quad , \qquad = \quad {}_1^{7}\wp^{-2}(\quad 2\tau\wp^2), \qquad \text{where} \quad A = \mp\tfrac{5}{6}a^{-2}(3a \quad)^{3\ \ 5}.$$

74. $= \quad (\quad)^{8\ 5}$.

Solution in parametric form:

$$= a \quad {}_1^{-8}(\tau \quad + 2\wp), \qquad = \quad {}_1^{7}\wp(\quad 2\tau\wp^2)^{-1\ \ 2}, \qquad \text{where} \quad A = \tfrac{10}{3}a^{-2\ \ -1}(3a \quad)^{3\ \ 5}.$$

75. $= \quad {}^{5}(\quad)^{7\ 5}$.

Solution in parametric form:

$$= a \quad {}_1^{-27}(\tau^2\wp \mp 1)(\quad 2\tau\wp^2)^{-1\ \ 2}, \qquad = \quad {}_1^{8}(\tau \quad + 2\wp)^{-1\ \ 3}, \qquad \text{where} \quad A = -10a^{-2\ \ -5}(a \quad)^{2\ \ 5}.$$

76. $= \quad {}^{-1\ \ 2\ \ -5\ \ 2}(\quad)^{4\ 5}$.

Solution in parametric form:

$$= a \quad {}_1^{27}(\tau^2\wp \mp 1)^2(\quad 2\tau\wp^2)^{-1}, \qquad = \quad {}_1^{7}(\quad 2\tau\wp^2)^{-1}(\tau \quad + 2\wp)^{4\ \ 3},$$

where $A = -5a^{-3\ \ 2\ \ 7\ \ 2} \quad \dfrac{a}{2} \quad {}^{4\ \ 5}$.

77. $= \; {}^{2}(\quad)^{3}$.

Solution in parametric form:

$$= a \;\; {}^{2}_{1}\wp, \qquad = \;\; {}^{-1}_{1}\tau, \qquad \text{where} \quad A = \mp 6a^{-1} \;\; {}^{-2}.$$

78. $= \; (\quad)^{1 \;\; 2}$.

Solution in parametric form:

$$= a \;\; {}_{1}\tau, \qquad = \;\; {}^{-3}_{1} \;\; , \qquad \text{where} \quad A = \; \frac{2}{a} \quad \frac{6}{a} \;\; {}^{1 \;\; 2}.$$

79. $= \; {}^{2 \;\; -5}(\quad)^{3}$.

Solution in parametric form:

$$= a \;\; {}^{3}_{1}\tau^{-1}\wp, \qquad = \;\; {}_{1}\tau^{-1}, \qquad \text{where} \quad A = \mp 6a^{-1} \;\; {}^{3}.$$

80. $= \; {}^{-5 \;\; 4} \;\; (\quad)^{1 \;\; 2}$.

Solution in parametric form:

$$= a \;\; {}^{2}_{1}\tau^{4}, \qquad = \;\; {}^{-1}_{1}(\tau \quad - \wp), \qquad \text{where} \quad A = \frac{1}{2}a^{1 \;\; 4 \;\; -1} \quad \frac{3}{2a} \;\; {}^{1 \;\; 2}.$$

81. $= \; {}^{-1 \;\; 2 \;\; -2 \;\; 3}(\quad)^{9 \;\; 5}$.

Solution in parametric form:

$$= a \;\; {}^{4}_{1}(\tau \quad - \wp)^{2}, \qquad = \;\; {}^{9}_{1}\tau^{-3}\wp^{3}, \qquad \text{where} \quad A = \mp\frac{5}{3}a^{1 \;\; 2 \;\; -1 \;\; 3} \quad \frac{}{4a} \;\; {}^{1 \;\; 5}.$$

82. $= \; {}^{2 \;\; -15 \;\; 7}(\quad)^{3}$.

Solution in parametric form:

$$= a \;\; {}_{1}\tau(\tau^{2}\wp \mp 1), \qquad = \;\; {}^{7}_{1}\tau^{7}, \qquad \text{where} \quad A = \mp\tfrac{6}{49}a^{-1 \;\; 1 \;\; 7}.$$

83. $= \; {}^{-15 \;\; 8} \;\; (\quad)^{1 \;\; 2}$.

Solution in parametric form:

$$= a \;\; {}^{4}_{1}\tau^{-8}, \qquad = \;\; {}^{3}_{1}\tau^{-6}(\tau^{3} \quad + 3\tau^{2}\wp \mp 1), \qquad \text{where} \quad A = -\tfrac{1}{8}a^{7 \;\; 8 \;\; -1}(\mp 3 \quad a)^{1 \;\; 2}.$$

84. $= \; {}^{-1 \;\; 2 \;\; -2 \;\; 3}(\quad)^{23 \;\; 15}$.

Solution in parametric form:

$$= a \;\; {}^{4}_{1}\tau^{-12}(\tau^{3} \quad + 3\tau^{2}\wp \mp 1)^{2}, \qquad = \;\; {}^{-1}_{1}\tau^{3}(\tau^{2}\wp \mp 1)^{3}, \qquad \text{where} \quad A = 5a^{1 \;\; 2 \;\; -1 \;\; 3} \quad \frac{}{4a} \;\; {}^{7 \;\; 15}.$$

85. $= \; {}^{2 \;\; -20 \;\; 7}(\quad)^{3}$.

Solution in parametric form:

$$= a \;\; {}^{6}_{1}\tau^{-6}(\tau^{2}\wp \mp 1), \qquad = \;\; {}^{7}_{1}\tau^{-7}, \qquad \text{where} \quad A = \mp\tfrac{6}{49}a^{-1 \;\; 6 \;\; 7}.$$

86. $= \; {}^{-20 \;\; 13} \;\; (\quad)^{1 \;\; 2}$.

Solution in parametric form:

$$= a \;\; {}^{13}_{1}\tau^{13}, \qquad = \;\; {}_{1}\tau(\tau^{3} \quad - 4\tau^{2}\wp \quad 6), \qquad \text{where} \quad A = \; \frac{2}{13}a^{7 \;\; 13 \;\; -1} \quad \frac{6}{13a} \;\; {}^{1 \;\; 2}.$$

87. $= \ {}^{-1\ 2\ \ -2\ 3}(\quad)^{33\ 20}.$

Solution in parametric form:

$$= a\ {}_1\tau^2(\tau^3\quad -4\tau^2\wp\quad 6)^2, \qquad = \ {}_1^{-9}\tau^{-18}(\tau^2\wp \mp 1)^3, \quad \text{where } A = -\frac{20}{3}a^{1\ 2\ -1\ 3}\quad \frac{}{4a}^{\ 7\ 20}.$$

88. $= \ {}^{-5\ 2}(\quad)^3.$

Solution in parametric form:

$$= a\ {}_1^4\wp^{-2}, \qquad = \ {}_1^7\wp^{-2}(\quad 2\tau\wp^2), \qquad \text{where} \quad A = \ 3a^{7\ 2\ -2}.$$

89. $= \ (\quad)^{7\ 5}.$

Solution in parametric form:

$$= a\ {}_1^7\wp^{-2}(\quad 2\tau\wp^2), \qquad = \ {}_1^{-3}\ , \qquad \text{where} \quad A = \ \tfrac{5}{6}\ {}^{-2}(3\quad a)^{3\ 5}.$$

90. $= \ {}^{-5\ 2\ \ -1\ 2}(\quad)^3.$

Solution in parametric form:

$$= a\ {}_1^3(\quad 2\tau\wp^2)^{-1}, \qquad = \ {}_1^7\wp^2(\quad 2\tau\wp^2)^{-1}, \qquad \text{where} \quad A = \ 3a^{7\ 2\ -3\ 2}.$$

91. $= \ (\quad)^{7\ 5}.$

Solution in parametric form:

$$= a\ {}_1^7\wp(\quad 2\tau\wp^2)^{-1\ 2}, \qquad = \ {}_1^{-8}(\tau\quad +2\wp), \qquad \text{where} \quad A = -\tfrac{10}{3}a^{-1\ -2}(3\quad a)^{3\ 5}.$$

92. $= \ {}^{-5\ 3\ \ -1\ 2}(\quad)^3.$

Solution in parametric form:

$$= a\ {}_1^9(\quad 2\tau\wp^2)^{3\ 2}, \qquad = \ {}_1^{16}(\tau\quad +2\wp)^2, \qquad \text{where} \quad A = \tfrac{1}{6}a^{8\ 3\ -3\ 2}.$$

93. $= \ {}^{-5\ 3\ \ -5\ 6}(\quad)^3.$

Solution in parametric form:

$$= a\ {}_1^7(\quad 2\tau\wp^2)^{3\ 2}(\tau\quad +2\wp)^{-2}, \qquad = \ {}_1^{16}(\tau\quad +2\wp)^{-2}, \qquad \text{where} \quad A = \tfrac{1}{6}a^{8\ 3\ -7\ 6}.$$

94. $= \ {}^{5}\ (\quad)^{8\ 5}.$

Solution in parametric form:

$$= a\ {}_1^8(\tau\quad +2\wp)^{-1\ 3}, \qquad = \ {}_1^{-27}(\tau^2\wp \mp 1)(\quad 2\tau\wp^2)^{-1\ 2}, \quad \text{where } A = 10a^{-5\ -2}(\quad a)^{2\ 5}.$$

95. $= \ {}^{-5\ 2\ \ -1\ 2}(\quad)^{11\ 5}.$

Solution in parametric form:

$$= a\ {}_1^7(\quad 2\tau\wp^2)^{-1}(\tau\quad +2\wp)^{4\ 3}, \qquad = \ {}_1^{27}(\tau^2\wp \mp 1)^2(\quad 2\tau\wp^2)^{-1},$$

where $A = 5a^{7\ 2\ -3\ 2}\ \frac{}{2a}^{\ 4\ 5}.$

In the solutions of equations 96 and 97, the following notation is used:

$$Z = \begin{cases} {}_1 \ (\tau) + {}_2 \ (\tau) & \textit{for the upper sign,} \\ {}_1 \ (\tau) + {}_2 \ (\tau) & \textit{for the lower sign,} \end{cases}$$

where (τ) *and* (τ) *are the Bessel functions, and* (τ) *and* (τ) *are the modified Bessel functions.*

96. $= \quad (\quad)^3.$

Solution in parametric form with $\neq -2$:

$$= \tau \ Z, \qquad = \tau^2 \ , \qquad \text{where} \quad = \frac{1}{+2}, \quad A = \frac{+2}{2} \ ^2 .$$

For the case $= -2$, see equation 2.5.2.28.

97. $= {}^{-1\ 2\ -1\ 2}(\quad) .$

Solution in parametric form with $l \neq 3 \ 2$:

$$= a\tau^2 \ Z^2, \qquad = \tau^{-2} \ (\tau Z'_\tau + \ Z)^2,$$

where $= \dfrac{1-l}{3-2l}$, $A = \dfrac{1}{3-2l} \mp \dfrac{}{a} \ ^{\frac{3}{2}-}$.

For the case $l = 3 \ 2$, see equation 2.5.2.29.

In the solutions of equations 98–106, the following notation is used:

$$Z = \begin{cases} {}_1 \ {}_{1\ 3}(\tau) + {}_2 \ {}_{1\ 3}(\tau) & \textit{for the upper sign,} \\ {}_1 \ {}_{1\ 3}(\tau) + {}_2 \ {}_{1\ 3}(\tau) & \textit{for the lower sign,} \end{cases}$$

$${}_1 = \tau Z'_\tau + \tfrac{1}{3} Z, \qquad {}_2 = \ {}^2_1 \ \tau^2 Z^2, \qquad {}_3 = \ \tfrac{2}{3}\tau^2 Z^3 - 2 \ {}_1 \ {}_2,$$

where ${}_{1\ 3}(\tau)$ *and* ${}_{1\ 3}(\tau)$ *are the Bessel functions, and* ${}_{1\ 3}(\tau)$ *and* ${}_{1\ 3}(\tau)$ *are the modified Bessel functions.*

98. $= {}^{-1\ 2}(\quad)^{3\ 2}.$

Solution in parametric form:

$$= a\tau^{-2\ 3} Z^{-1} \ {}_1, \qquad = \tau^{-4\ 3} \ {}^2_2, \qquad \text{where} \quad A = -\frac{2}{a} \ \mp \frac{3}{a} \ ^{1\ 2}.$$

99. $= {}^{-2}(\quad)^{3\ 2}.$

Solution in parametric form:

$$= a\tau^{-4\ 3} Z^{-1} \ {}_3, \qquad = \ \tau^{-2\ 3} \ {}_2, \qquad \text{where} \quad A = -a^{-2} \ (\ 3a \ )^{1\ 2}.$$

100. $= {}^{-2\ -1\ 2}(\quad)^{3\ 2}.$

Solution in parametric form:

$$= a\tau^{-4\ 3} Z^{-2} \ {}_2, \qquad = \tau^{-8\ 3} Z^{-2} \ {}^2_3, \qquad \text{where} \quad A = \ \tfrac{2}{3} a^{3\ 2}.$$

101. $= {}^{-1\ 2\ -2}(\quad)^3.$

Solution in parametric form:

$$= a\tau^{-4\ 3} Z^{-2} \ {}^2_1, \qquad = \ \tau^{-2\ 3} Z^{-2}, \qquad \text{where} \quad A = \ \tfrac{1}{3} a^{3\ 2}.$$

102. $= \quad ^{-2} \; (\quad)^3$**.**

Solution in parametric form:

$$= a\tau^{2 \; 3} Z^2, \qquad = \; \tau^{-2 \; 3} \quad {}_2, \qquad \text{where} \quad A = \tfrac{9}{2}(a \quad)^3.$$

103. $= \quad ^{-1 \; 2} \; (\quad)^{3 \; 2}$**.**

Solution in parametric form:

$$= \tau^{-4 \; 3} \quad {}^2_2, \qquad = \; \tau^{-2 \; 3} Z^{-1} \quad {}_1, \qquad \text{where} \quad A = \frac{2}{\;} \mp \frac{3}{\;} \quad {}^{1 \; 2}.$$

104. $= \quad ^{-2 \;\; -2}(\quad)^3$**.**

Solution in parametric form:

$$= a\tau^{4 \; 3} Z^2 \quad {}^{-1}_2, \qquad = \tau^{2 \; 3} Z^{-1} \quad {}^{-1}_2, \qquad \text{where} \quad A = \tfrac{9}{2} a^3.$$

105. $= \quad ^{-2} \; (\quad)^{3 \; 2}$**.**

Solution in parametric form:

$$= a\tau^{-2 \; 3} \quad {}_2, \qquad = \; \tau^{-4 \; 3} Z^{-1} \quad {}_3, \qquad \text{where} \quad A = a \quad ^{-2}(\;\; 3a \quad)^{1 \; 2}.$$

106. $= \quad ^{-1 \; 2 \;\; -2}(\quad)^{3 \; 2}$**.**

Solution in parametric form:

$$= \tau^{-8 \; 3} Z^{-2} \quad {}^2_3, \qquad = \; \tau^{-4 \; 3} Z^{-2} \quad {}_2, \qquad \text{where} \quad A = \mp\tfrac{2}{3} \quad {}^{3 \; 2}.$$

107. $= \qquad - \; ^{-3}(\quad)^{\frac{3 \;\; +4}{2 \;\; +3}}$**.**

In the books by Zaitsev & Polyanin (1993, 1994) it was shown that this equation is reducible to a Riccati equation whose solution is expressed in terms of associated Legendre functions.

2.5.3. Some Formulas and Transformations

For the sake of visualization, we use the symbolic notation

$$\{ \; , \quad , l \}$$

to denote the generalized Emden–Fowler equation

$$'' = A \qquad (\; ' \;) \; .$$

Hereinafter we omit the insignificant parameter A (which can be reduced to $\;\; 1$ by scaling the variables in accordance with the rule $\qquad a \;\; , \qquad\quad$, selecting appropriate constants a and $\;$).

2.5.3-1. A particular solution.

If $\quad + l \neq 1$, the generalized Emden–Fowler equation has a particular solution:

$$= B \;\; ^{\frac{+2-}{1- \;\; -}}, \qquad \text{where} \quad B = \left[\frac{+2-l}{1- \quad -l} \right]^{\frac{1-}{+\;-1}} \left[\frac{+ \quad +1}{A(1- \quad -l)} \right]^{\frac{1}{+\;-1}}.$$

$\boxed{\text{2.5.3-2. Discrete transformations of the generalized Emden–Fowler equation.}}$

1 . Taking as the independent variable and as the dependent one, we obtain a generalized Emden–Fowler equation for $=\ (\ )$ with changed parameters:

$$'' \ = -A \qquad (\ '\ )^{3-} .$$

Denote this transformation by and represent it as follows:

$$\{\ ,\ \ ,l\} \qquad\qquad \{\ \ ,\ \ ,3-l\} \qquad \textit{transformation}\ \ .$$

The twofold transformation yields the original equation.

2 . For $\neq 0$, $\neq -1$, $l \neq 1$, the transformation $=(\ '\ )^{1-}$, $=\ ^{+1}$ leads to a generalized Emden–Fowler equation for $=\ (\ )$ with changed parameters:

$$'' = B^{\ \frac{1}{1-}}\ \ ^{-\frac{}{+1}}(\ \ '\ )^{\frac{2\quad +1}{}},$$

where $B = -\dfrac{}{+1}\left(\dfrac{A(1-l)}{+1}\right)^{\frac{1}{}}$. Denote this transformation by $\mathcal{G}$ and represent it as follows:

$$\{\ ,\ \ ,l\} \qquad\qquad \left\{\frac{1}{1-l},\ -\frac{}{+1},\ \frac{2\quad +1}{}\right\} \qquad \textit{transformation}\ \mathcal{G}.$$

The threefold transformation $\mathcal{G}$ yields the original equation.

Whenever the solution of the transformed equation is obtained in the form $=\ (\ )$, the solution of the original equation can be written in parametric form as:

$$=\ \ ^{\frac{1}{+1}}, \qquad = k(\ \ '\ )^{-\frac{1}{}}, \qquad \text{where}\quad k=\left(\frac{+1}{A(1-l)}\right)^{\frac{1}{}}.$$

Different compositions of the transformations and $\mathcal{G}$ generate six different generalized Emden–Fowler equations, whose parameters are shown in Figure 1 (see Subsection 0.6.5).

3 . In the special case $l = 0$, the transformation $=\ \ $, $= 1$ leads to an Emden–Fowler equation with the independent variable raised to a different power:

$$'' = A\ ^{-\ \ -\ \ -3} \quad .$$

Denote this transformation by and represent it as follows:

$$\{\ ,\ \ ,0\} \qquad\qquad \{-\ \ -\ \ -3,\ \ ,0\} \qquad \textit{transformation}\ \ .$$

If $l = 0$, different compositions of the transformations , $\mathcal{G}$, and generate 12 different generalized Emden–Fowler equations, whose parameters are shown in Figure 2 (see Subsection 0.6.5).

If $l = 0$ and $= 1$, different compositions of the transformations , $\mathcal{G}$, and generate 24 different generalized Emden–Fowler equations, whose parameters are presented in Figure 3 (see Subsection 0.6.5).

$\boxed{\text{2.5.3-3. Reduction of the generalized Emden–Fowler equation to an Abel equation.}}$

The transformation

$$z = -\ \ ', \qquad v = A\ \ ^{-\ +2}\ \ ^{+\ -1}$$

reduces the generalized Emden–Fowler equation to the equation

$$(z\ v - z^2 + z)v' = [(\ \ + l - 1)z + \ \ - l + 2]v.$$

Furthermore, using the substitution $\xi = v - z^{2-} + z^{1-}$, we obtain an Abel equation of the second kind:

$$\xi\xi' = [(\ \ + 2l - 3)z + \ \ - 2l + 3]z^{-}\ \xi + [(\ \ + l - 1)z^2 + (\ \ -\ \ - 2l + 3)z - \ \ + l - 2]z^{1-2} .$$

2.6. Equations of the Form $y \quad = \quad {}_1 \quad {}^1 y \quad {}^1(y \quad)^{1} + \quad {}_2 \quad {}^2 y \quad {}^2(y \quad)^{2}$

2.6.1. Modified Emden–Fowler Equation $\quad = \quad {}_1 \quad ^{-1} \quad + \quad {}_2$

2.6.1-1. Preliminary remarks. Classification table.

For the sake of clarity, below in this subsection we use the conventional notation

$$'' \quad - k \quad ' = A \qquad ^{+1}$$

for the modified Emden–Fowler equation. For $k = 0$, see Section 2.3. For $k \neq -1$, the substitution $z = \quad ^{+1}$ leads to the Emden–Fowler equation:

$$'' \quad = \frac{A}{(k+1)^2} z^{\frac{-2}{+1}} \qquad ,$$

which is discussed in Section 2.3.

The classification Table 24 represents all solvable equations whose solutions are outlined in Subsection 2.6.1. Equations are arranged in accordance with the growth of parameter . The number of the equation sought is indicated in the last column in this table.

TABLE 24

Solvable cases of the modified Emden–Fowler equation $'' \quad - k \quad ' = A \qquad ^{+1}$

No			k	Equation
1	arbitrary ($\neq -1$)	arbitrary ($\neq -2$)	$\frac{1}{2}$	2.6.1.1
2	arbitrary ($\neq -1$)	arbitrary ($\neq -2$)	$-\frac{\quad + \quad + 3}{\quad + 1}$	2.6.1.2
3	arbitrary ($\neq -1$)	arbitrary ($\neq -2$)	$\frac{2 \quad + \quad + 3}{1 - \quad}$	2.6.1.3
4	arbitrary ($\neq -1$)	-2	-1	2.6.1.6
5	-7	arbitrary ($\neq -2$)	$\frac{1}{3}(\quad - 1)$	2.6.1.45
6	-7	arbitrary ($\neq -2$)	$\frac{1}{5}(\quad - 3)$	2.6.1.46
7	-4	arbitrary ($\neq -2$)	$\frac{1}{2}$	2.6.1.40
8	-4	arbitrary ($\neq -2$)	$\frac{1}{3}(\quad - 1)$	2.6.1.42
9	-4	-2	-1	2.6.1.41
10	-2	arbitrary ($\neq -2$)	$\frac{1}{3}(\quad - 1)$	2.6.1.28
11	-2	-2	arbitrary ($k \neq -1$)	2.6.1.29

TABLE 24 (*Continued*)
Solvable cases of the modified Emden–Fowler equation $'' - k\ ' = A \quad {}^{+1}$

No			k	Equation
12	$-\frac{5}{2}$	arbitrary ($\neq -2$)	$\frac{1}{2}$	2.6.1.35
13	$-\frac{5}{2}$	arbitrary ($\neq -2$)	$\frac{1}{3}(2\quad +1)$	2.6.1.37
14	$-\frac{5}{2}$	-2	-1	2.6.1.36
15	$-\frac{5}{3}$	arbitrary ($\neq -2$)	$-3\quad -7$	2.6.1.14
16	$-\frac{5}{3}$	arbitrary ($\neq -2$)	$\frac{1}{2}$	2.6.1.8
17	$-\frac{5}{3}$	arbitrary ($\neq -2$)	$\frac{1}{2}(3\quad +4)$	2.6.1.9
18	$-\frac{5}{3}$	arbitrary ($\neq -2$)	$\frac{1}{3}(\quad -1)$	2.6.1.13
19	$-\frac{5}{3}$	arbitrary ($\neq -2$)	$\frac{1}{3}(2\quad +1)$	2.6.1.38
20	$-\frac{5}{3}$	arbitrary ($\neq -2$)	$\frac{1}{4}(\quad -2)$	2.6.1.18
21	$-\frac{5}{3}$	arbitrary ($\neq -2$)	$-\frac{1}{4}(3\quad +10)$	2.6.1.19
22	$-\frac{5}{3}$	arbitrary ($\neq -2$)	$\frac{1}{7}(6\quad +5)$	2.6.1.39
23	$-\frac{5}{3}$	-2	-1	2.6.1.22
24	$-\frac{7}{5}$	arbitrary ($\neq -2$)	$\frac{1}{3}(\quad -1)$	2.6.1.24
25	$-\frac{7}{5}$	arbitrary ($\neq -2$)	$-\frac{1}{3}(5\quad +13)$	2.6.1.25
26	-1	arbitrary ($\neq -2$)	$\quad +1$	2.6.1.5
27	-1	arbitrary ($\neq -2$)	$\frac{1}{2}$	2.6.1.4
28	-1	-2	arbitrary ($k \neq -1$)	2.6.1.7
29	-1	-2	-1	2.6.1.20
30	$-\frac{1}{2}$	arbitrary ($\neq -2$)	$-2\quad -5$	2.6.1.12
31	$-\frac{1}{2}$	arbitrary ($\neq -2$)	$\frac{1}{2}$	2.6.1.11
32	$-\frac{1}{2}$	arbitrary ($\neq -2$)	$\frac{1}{2}(3\quad +4)$	2.6.1.43
33	$-\frac{1}{2}$	arbitrary ($\neq -2$)	$\frac{1}{3}(\quad -1)$	2.6.1.16
34	$-\frac{1}{2}$	arbitrary ($\neq -2$)	$\frac{1}{3}(2\quad +1)$	2.6.1.26
35	$-\frac{1}{2}$	arbitrary ($\neq -2$)	$-\frac{1}{3}(2\quad +7)$	2.6.1.17

TABLE 24 (*Continued*)

Solvable cases of the modified Emden–Fowler equation $'' - k\ ' = A \quad ^{+1}$

No			k	Equation
36	$-\frac{1}{2}$	arbitrary ($\neq -2$)	$\frac{1}{5}(6 \quad +7)$	2.6.1.44
37	$-\frac{1}{2}$	-2	arbitrary ($k \neq -1$)	2.6.1.27
38	$-\frac{1}{2}$	-2	-1	2.6.1.21
39	$\frac{1}{2}$	arbitrary ($\neq -2$)	$\frac{1}{2}$	2.6.1.15
40	$\frac{1}{2}$	arbitrary ($\neq -2$)	$-\frac{1}{3}(2 \quad +7)$	2.6.1.10
41	$\frac{1}{2}$	-2	-1	2.6.1.23
42	2	arbitrary ($\neq -2$)	$-7 \quad -15$	2.6.1.33
43	2	arbitrary ($\neq -2$)	$\frac{1}{2}$	2.6.1.30
44	2	arbitrary ($\neq -2$)	$-\frac{1}{3}(\quad +5)$	2.6.1.32
45	2	arbitrary ($\neq -2$)	$-\frac{1}{6}(7 \quad +20)$	2.6.1.34
46	2	-2	-1	2.6.1.31

2.6.1-2. Solvable equations and their solutions.

1. $\quad - \frac{1}{2} \quad = \quad ^{+1} \quad , \qquad \neq -1, \quad \neq -2.$

Solution in parametric form:

$$= a \ _1^{1-} \quad (1 \quad \tau \quad ^{+1})^{-1\ 2} \ \tau + \ _2 \quad ^{\frac{2}{+2}}, \qquad = \ _1^{+2}\tau,$$

where $A = \frac{1}{8}(\quad +1)(\quad +2)^2 a^{- \ -2 \ 1-} \quad .$

2. $\quad + \frac{+ \quad +3}{+1} \quad = \quad ^{+1} \quad , \qquad \neq -1, \quad \neq -2.$

Solution in parametric form:

$$= a \ _1^{1-} \quad (1 \quad \tau \quad ^{+1})^{-1\ 2} \quad \tau + \ _2 \quad ^{\frac{+1}{+2}}, \qquad = \ _1^{+2}\tau \quad (1 \quad \tau \quad ^{+1})^{-1\ 2} \quad \tau + \ _2 \quad ^{-1},$$

where $A = \dfrac{(\quad +2)^2}{2(\quad +1)} a^{- \ -2 \ 1-} \quad .$

3. $\quad + \frac{2 \quad + \quad +3}{-1} \quad = \quad ^{+1} \quad , \qquad \neq -1, \quad \neq -2.$

Solution in parametric form:

$$= \exp \frac{1-}{+2} \ _2 \qquad _1 + \frac{1}{4}\tau^2 + \frac{2B}{+1}\tau \quad ^{+1} \quad ^{-1\ 2} \quad \tau \ ,$$

$$= \tau \exp \ _2 \qquad _1 + \frac{1}{4}\tau^2 + \frac{2B}{+1}\tau \quad ^{+1} \quad ^{-1\ 2} \quad \tau \ ,$$

where $A = \dfrac{4(\quad +2)^2}{(\quad -1)^2} B.$

4. $\quad -\frac{1}{2} \quad = \quad {}^{+1\quad -1}, \quad \neq -2.$

Solution in parametric form:

$$= a \;{}_{1}^{2} \quad \exp(\mp\tau^2) \;\; \tau + \;\; {}_{2} \;\; {}^{\frac{2}{+2}}, \qquad = \;\; {}_{1}^{\;+2} \exp(\mp\tau^2),$$

where $A = \mp\frac{1}{2}(\quad +2)^2 a^{-\;\;-2\;\;2}.$

5. $\quad -(\quad +1) \quad = \quad {}^{+1\quad -1}, \quad \neq -2.$

Solution in parametric form:

$$= \exp \; \frac{2 \;\; {}_{2}}{+2} \qquad {}_{1} + \frac{1}{4}\tau^2 + \frac{2A}{(\quad +2)^2} \ln|\tau| \quad {}^{-1\;\;2} \quad \tau \;\;,$$

$$= \tau \exp \;\; {}_{2} \qquad {}_{1} + \frac{1}{4}\tau^2 + \frac{2A}{(\quad +2)^2} \ln|\tau| \quad {}^{-1\;\;2} \quad \tau \;\; .$$

6. $\quad + \quad = \quad {}^{-1} \quad , \quad \neq -1.$

Solution in parametric form:

$$= \;\; {}_{2} \exp \quad (\;{}_{1} \quad \tau \quad {}^{+1})^{-1\;\;2} \quad \tau\;, \qquad = \;\tau, \qquad \text{where} \quad A = \;\; \tfrac{1}{2} \;{}^{1-} \quad (\quad +1).$$

7. $\quad - \quad = \quad {}^{-1\quad -1}, \quad \neq -1.$

Solution in parametric form:

$$= \qquad \frac{2A}{(k+1)^2} \ln\tau + \;\; {}_{1} \quad {}^{-1\;\;2} \quad \tau + \;\; {}_{2} \quad {}^{-\frac{1}{+1}}, \qquad = \tau \qquad \frac{2A}{(k+1)^2} \ln\tau + \;\; {}_{1} \quad {}^{-1\;\;2} \quad \tau + \;\; {}_{2} \quad {}^{-1}.$$

8. $\quad -\frac{1}{2} \quad = \quad {}^{+1\quad -5\;\;3}, \quad \neq -2.$

Solution in parametric form:

$$= a \;{}_{1}^{8}(\tau^3 \quad 3\tau + \quad {}_{2})^{\frac{2}{+2}}, \qquad = \;\; {}_{1}^{3} \;{}^{+6}(\tau^2 \quad 1)^{3\;\;2}, \qquad \text{where} \quad A = \;\; \tfrac{1}{12} a^{-\;\;-2\;\;8\;\;3}(\quad +2)^2.$$

9. $\quad -\frac{1}{2}(3 \quad +4) \quad = \quad {}^{+1\quad -5\;\;3}, \quad \neq -2.$

Solution in parametric form:

$$= a \;{}_{1}^{8}(\tau^3 \quad 3\tau + \quad {}_{2})^{-\frac{2}{3\;\;+6}}, \qquad = \;\; {}_{1}^{3} \;{}^{+6}(\tau^2 \quad 1)^{3\;\;2}(\tau^3 \quad 3\tau + \quad {}_{2})^{-1},$$

where $A = \;\; \frac{3}{4} a^{-\;\;-2\;\;8\;\;3}(\quad +2)^2.$

10. $\quad +\frac{1}{3}(2 \quad +7) \quad = \quad {}^{+1\quad 1\;\;2}, \quad \neq -2.$

Solution in parametric form:

$$= a \;\; {}_{1} \qquad \frac{\tau \quad \tau}{(4\tau^3 - 1)} + \;\; {}_{2} \;\; {}^{\frac{3}{2\;\;+4}}, \qquad = \;\; {}_{1}^{2} \;{}^{+4}\tau^2 \qquad \frac{\tau \quad \tau}{(4\tau^3 - 1)} + \;\; {}_{2} \;\; {}^{-1},$$

where $A = \;\; \frac{16}{3} a^{-\;\;-2\;\;1\;\;2}(\quad +2)^2.$

11. $\quad -\frac{1}{2} \quad = \quad {}^{+1\ -1\ 2}, \qquad \neq -2.$

Solution in parametric form:

$$= a \ {}_1^3(\tau^3 - 3\tau + \ {}_2)^{\frac{2}{+2}}, \qquad = \ {}_1^2\ {}^{+4}(\tau^2-1)^2, \qquad \text{where} \quad A = \tfrac{1}{9}(\ +2)^2 a^{-\ -2\ 3\ 2}.$$

12. $\quad +(2 \ \ +5) \quad = \quad {}^{+1\ -1\ 2}, \qquad \neq -2.$

Solution in parametric form:

$$= a \ {}_1^3(\tau^3 - 3\tau + \ {}_2)^{\frac{1}{2\ +4}}, \qquad = \ {}_1^2\ {}^{+4}(\tau^2-1)^2(\tau^3 - 3\tau + \ {}_2)^{-1},$$

where $A = \frac{16}{9}(\ +2)^2 a^{-\ -2\ 3\ 2}$.

13. $\quad -\frac{1}{3}(\ \ -1) \quad = \quad {}^{+1\ -5\ 3}, \qquad \neq -2.$

Solution in parametric form:

$$= a \ {}_1^8\big[\ (\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)\big]^{\frac{3}{+2}}, \qquad = \ {}_1^3\ {}^{+6}(\tau^3 - 3\tau + \ {}_2)^{3\ 2},$$

where $A = \frac{1}{64}(\ +2)^2 a^{-\ -2\ 8\ 3}$.

14. $\quad +(3 \ \ +7) \quad = \quad {}^{+1\ -5\ 3}, \qquad \neq -2.$

Solution in parametric form:

$$= a \ {}_1^8\big[\ (\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)\big]^{\frac{1}{3\ +6}}, \qquad = \ {}_1^3\ {}^{+6}(\tau^3 - 3\tau + \ {}_2)^{3\ 2}(\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)^{-1},$$

where $A = \frac{81}{64}(\ +2)^2 a^{-\ -2\ 8\ 3}$.

15. $\quad -\frac{1}{2} \quad = \quad {}^{+1\ \ 1\ 2}, \qquad \neq -2.$

Solution in parametric form:

$$= a \ \ {}_1 \ \ \frac{\tau \ \ \tau}{(4\tau^3-1)} + \ \ {}_2 \ {}^{\frac{2}{+2}}, \qquad = \ {}_1^2\ {}^{+4}\tau^2, \qquad \text{where} \quad A = \ 3a^{-\ -2\ 1\ 2}(\ +2)^2.$$

16. $\quad -\frac{1}{3}(\ \ -1) \quad = \quad {}^{+1\ -1\ 2}, \qquad \neq -2.$

Solution in parametric form:

$$= \big[\ \ {}_1 e^{2\ \tau} + \ \ {}_2 e^{-\ \tau}\sin(\ \overline{3}\ \ \tau\ \big]^{\frac{3}{+2}},$$
$$= \ 2\ \ {}_1\ e^{2\ \tau} + \ \ {}_2\ e^{-\ \tau}\big[\ \overline{3}\cos(\ \overline{3}\ \ \tau\ \ -\sin(\ \overline{3}\ \ \tau)\big]\ \ {}^2,$$

where $A = \frac{16}{9}\ {}^3(\ +2)^2$.

17. $\quad +\frac{1}{3}(2 \ \ +7) \quad = \quad {}^{+1\ -1\ 2}, \qquad \neq -2.$

Solution in parametric form:

$$= \big[\ \ {}_1 e^{2\ \tau} + \ \ {}_2 e^{-\ \tau}\sin(\ \overline{3}\ \ \tau\ \big]^{\frac{3}{2\ +4}},$$
$$= \frac{2\ \ {}_1\ e^{2\ \tau} + \ \ {}_2\ e^{-\ \tau}\big[\ \overline{3}\cos(\ \overline{3}\ \ \tau\ \ -\sin(\ \overline{3}\ \ \tau\ \big]\ \ {}^2}{{}_1 e^{2\ \tau} + \ \ {}_2 e^{-\ \tau}\sin(\ \overline{3}\ \ \tau},$$

where $A = \frac{64}{9}\ {}^3(\ +2)^2$.

18. $\quad -\tfrac{1}{4}(\quad -2) \quad = \quad {}^{+1\ \ -5\ \ 3}, \qquad \neq -2.$

1 . Solution in parametric form with $A<0$:

$$= a \ \ {}^{8}_{1}\big[\cosh(\tau+\ \ {}_2)\cos\tau\big]^{-\frac{4}{\ +2}}\big[\tanh(\tau+\ \ {}_2)+\tan\tau\big]^{-\frac{4}{\ +2}},$$
$$= \quad {}^{3}_{1}\ {}^{+6}\big[\cosh(\tau+\ \ {}_2)\cos\tau\big]^{3\ \ 2},$$

where $A=-\frac{3}{256}a^{-\ \ -2\ \ 8\ \ 3}(\ \ +2)^2.$

2 . Solution in parametric form with $A>0$:

$$= a \ \ {}^{8}_{1}\big[\sinh\tau+\cos(\tau+\ \ {}_2)\big]^{-\frac{4}{\ +2}}, \qquad = \quad {}^{3}_{1}\ {}^{+6}\big[\cosh\tau-\sin(\tau+\ \ {}_2)\big]^{3\ \ 2},$$

where $A=\frac{3}{64}a^{-\ \ -2\ \ 8\ \ 3}(\ \ +2)^2.$

19. $\quad +\tfrac{1}{4}(3\quad +10) \quad = \quad {}^{+1\ \ -5\ \ 3}, \qquad \neq -2.$

1 . Solution in parametric form with $A<0$:

$$= a \ \ {}^{8}_{1}\big[\cosh(\tau+\ \ {}_2)\cos\tau\big]^{-\frac{4}{3\ \ +6}}\big[\tanh(\tau+\ \ {}_2)+\tan\tau\big]^{-\frac{4}{3\ \ +6}},$$
$$= \quad {}^{3}_{1}\ {}^{+6}\big[\cosh(\tau+\ \ {}_2)\cos\tau\big]^{1\ \ 2}\big[\tanh(\tau+\ \ {}_2)+\tan\tau\big]^{-1},$$

where $A=-\frac{27}{256}a^{-\ \ -2\ \ 8\ \ 3}(\ \ +2)^2.$

2 . Solution in parametric form with $A>0$:

$$= a \ \ {}^{8}_{1}\big[\sinh\tau+\cos(\tau+\ \ {}_2)\big]^{-\frac{4}{3\ \ +6}},$$
$$= \quad {}^{3}_{1}\ {}^{+6}\big[\cosh\tau-\sin(\tau+\ \ {}_2)\big]^{3\ \ 2}\big[\sinh\tau+\cos(\tau+\ \ {}_2)\big]^{-1},$$

where $A=\frac{27}{64}a^{-\ \ -2\ \ 8\ \ 3}(\ \ +2)^2.$

20. $\quad + \quad = \quad {}^{-1\ \ -1}.$

Solution in parametric form:

$$= \quad {}_2\exp \quad (2A\ln|\tau|+\ \ {}_1)^{-1\ \ 2}\ \ \tau \ \ , \qquad = \tau.$$

21. $\quad + \quad = \quad {}^{-1\ \ -1\ \ 2}.$

Solution in parametric form:

$$= \exp(\ \ \tau^3-3\ \ {}_1\tau+\ \ {}_2), \qquad = (\ \ \tau^2-\ \ {}_1)^2, \qquad \text{where} \quad A=\ \ \tfrac{4}{9}\ \ {}^{3\ \ 2}.$$

22. $\quad + \quad = \quad {}^{-1\ \ -5\ \ 3}.$

Solution in parametric form:

$$= \exp(\ \ {}_1\tau^3\quad 3\tau+\ \ {}_2), \qquad = (\ \ 3A\quad {}_1)^{3\ \ 8}(\ \ {}_1\tau^2\quad 1)^{3\ \ 2}.$$

23. $\quad + \quad = \quad {}^{-1\ \ 1\ \ 2}.$

Solution in parametric form:

$$= \quad {}_1\exp \quad \frac{\tau\ \ \tau}{(4\tau^3-\ \ {}_1)}\ \ , \qquad = \tau^2, \qquad \text{where} \quad A=\ \ 12\ \ {}^{1\ \ 2}.$$

In the solutions of equations 24 and 25, the following notation is used:

$${}_1 = {}_1 e^{2\tau} + {}_2 e^{-\tau} \sin(\ \overline{3}\ \tau\ ), \qquad {}_2 = 2\ {}_1\, e^{2\tau} + {}_2\ e^{-\tau}[\ \overline{3}\cos(\ \overline{3}\ \tau\ ) - \sin(\ \overline{3}\ \tau\ )],$$
$${}_3 = {}^2_2 - 2\ {}_1(\ {}_2)'_\tau.$$

24. $\quad - \frac{1}{3}(\quad - 1) \quad = \quad {}^{+1}\ {}^{-7}\ {}^{5}, \qquad \neq -2.$

Solution in parametric form:

$$= a\ {}_3^{\frac{3}{+2}}, \qquad = {}^5_1\ {}^2, \qquad \text{where} \quad A = -\tfrac{5}{9216} a^{-}\ {}^{-2}\ {}^{12}\ {}^{5}\ {}^{-6}(\quad + 2)^2.$$

25. $\quad + \frac{1}{3}(5 \quad + 13) \quad = \quad {}^{+1}\ {}^{-7}\ {}^{5}, \qquad \neq -2.$

Solution in parametric form:

$$= a\ {}_3^{\frac{3}{5\ +10}}, \qquad = {}^5_1\ {}^2\ {}_3^{-1}, \qquad \text{where} \quad A = -\tfrac{125}{9216} a^{-}\ {}^{-2}\ {}^{12}\ {}^{5}\ {}^{-6}(\quad + 2)^2.$$

In the solutions of equations 26–29, the following notation is used:

$$Z = \begin{cases} {}_1\ {}_{1\ 3}(\tau) + {}_2\ {}_{1\ 3}(\tau) & \textit{for the upper sign,} \\ {}_1\ {}_{1\ 3}(\tau) + {}_2\ {}_{1\ 3}(\tau) & \textit{for the lower sign,} \end{cases}$$

where ${}_{1\ 3}(\tau)$ *and* ${}_{1\ 3}(\tau)$ *are the Bessel functions, and* ${}_{1\ 3}(\tau)$ *and* ${}_{1\ 3}(\tau)$ *are the modified Bessel functions.*

26. $\quad - \frac{1}{3}(2 \quad + 1) \quad = \quad {}^{+1}\ {}^{-1}\ {}^{2}, \qquad \neq -2.$

Solution in parametric form:

$$= a\ {}^3_1 \tau^{\frac{1}{+2}} Z^{\frac{3}{+2}}, \qquad = {}^2_1\ {}^{+4} \tau^{-2\ 3} (\tau Z'_\tau + \tfrac{1}{3} Z)^2, \qquad \text{where} \quad A = \mp \tfrac{4}{27} a^{-}\ {}^{-2}\ {}^{3}\ {}^{2}(\quad + 2)^2.$$

27. $\quad - \quad = \quad {}^{-1}\ {}^{-1}\ {}^{2}, \qquad \neq -1.$

Solution in parametric form:

$$= {}_1(\tau^{1\ 3} Z)^{-\frac{2}{+1}}, \qquad = \tau^{-4\ 3} Z^{-2} (\tau Z'_\tau + \tfrac{1}{3} Z)^2, \qquad \text{where} \quad A = \mp \tfrac{1}{3}\ {}^{3\ 2}(k+1)^2.$$

28. $\quad - \frac{1}{3}(\quad - 1) \quad = \quad {}^{+1}\ {}^{-2}, \qquad \neq -2.$

Solution in parametric form:

$$= a\ {}^3_1 \tau^{-\frac{2}{+2}} \left[(\tau Z'_\tau + \tfrac{1}{3} Z)^2 \quad \tau^2 Z^2 \right]^{\frac{3}{+2}}, \qquad = {}_1\ {}^{+2} \tau^{2\ 3} Z^2,$$

where $A = -\frac{1}{2} a^{-}\ {}^{-2}\ {}^{3}(\quad + 2)^2.$

29. $\quad - \quad = \quad {}^{-1}\ {}^{-2}, \qquad \neq -1.$

Solution in parametric form:

$$= {}_1 \tau^{\frac{2}{3\ +3}} \left[(\tau Z'_\tau + \tfrac{1}{3} Z)^2 \quad \tau^2 Z^2 \right]^{-\frac{1}{+1}}, \qquad = \tau^{4\ 3} Z^2 \left[(\tau Z'_\tau + \tfrac{1}{3} Z)^2 \quad \tau^2 Z^2 \right]^{-1},$$

where $A = -\frac{9}{2}\ {}^{3}(k+1)^2.$

In the solutions of equations 30–39, the following notation is used:

$$\tau = \quad \frac{\wp}{\overline{(4\wp^3-1)}} - \quad {}_2, \qquad = \quad \overline{(4\wp^3-1)}.$$

The function $\wp$ *is defined implicitly by a first integral; the upper sign in the formulas corresponds to the classical Weierstrass elliptic function* $\wp = \wp(\tau + \quad {}_2,\, 0,\, 1)$.

30. $\quad -\frac{1}{2} \quad = \quad ^{+1 \quad 2}, \qquad \neq -2.$

Solution in parametric form:

$$= a \quad {}_1^{-1}\tau^{\frac{2}{\quad +2}}, \qquad = \quad {}_1^{\quad +2}\wp, \qquad \text{where} \quad A = \quad \tfrac{3}{2}a^{-\quad -2 \quad -1}(\quad +2)^2.$$

31. $\quad + \quad = \quad ^{-1 \quad 2}.$

Solution in parametric form:

$$= \quad {}_2 e^{\tau}, \qquad = \quad \wp(\tau,\, 0,\, \quad {}_1),$$

where $A = \quad 6 \quad ^{-1}$, and the elliptic Weierstrass function $\wp = \wp(\tau,\, 0,\, \quad {}_1)$ is defined implicitly by the integral $\tau = \quad (4z^3 - \quad {}_1)^{-1 \quad 2} \quad z$.

32. $\quad + \frac{1}{3}(\quad + 5) \quad = \quad ^{+1 \quad 2}, \qquad \neq -2.$

Solution in parametric form:

$$= a \quad {}_1^{-1}\tau^{\frac{3}{\quad +2}}, \qquad = \quad {}_1^{\quad +2}\tau^{-1}\wp, \qquad \text{where} \quad A = \quad \tfrac{2}{3}a^{-\quad -2 \quad -1}(\quad +2)^2.$$

33. $\quad + (7 \quad + 15) \quad = \quad ^{+1 \quad 2}, \qquad \neq -2.$

Solution in parametric form:

$$= a \quad {}_1^{-1}\tau^{-\frac{1}{\quad +2}}, \qquad = \quad {}_1^{\quad +2}\tau(\tau^2\wp \mp 1), \qquad \text{where} \quad A = \quad 6a^{-\quad -2 \quad -1}(\quad +2)^2.$$

34. $\quad + \frac{1}{6}(7 \quad + 20) \quad = \quad ^{+1 \quad 2}, \qquad \neq -2.$

Solution in parametric form:

$$= a \quad {}_1^{-1}\tau^{\frac{6}{7(\quad +2)}}, \qquad = \quad {}_1^{\quad +2}\tau^{-6}(\tau^2\wp \mp 1), \qquad \text{where} \quad A = \quad \tfrac{1}{6}a^{-\quad -2 \quad -1}(\quad +2)^2.$$

35. $\quad -\frac{1}{2} \quad = \quad ^{+1 \quad -5 \quad 2}, \qquad \neq -2.$

Solution in parametric form:

$$= a \quad {}_1^{7}\wp^{-\frac{4}{\quad +2}}(\quad 2\tau\wp^2)^{\frac{2}{\quad +2}}, \qquad = \quad {}_1^{2 \quad +4}\wp^{-2}, \qquad \text{where} \quad A = \mp\tfrac{3}{4}a^{-\quad -2 \quad 7 \quad 2}(\quad +2)^2.$$

36. $\quad + \quad = \quad ^{-1 \quad -5 \quad 2}.$

Solution in parametric form:

$$= \quad {}_2 \exp[\wp^{-2}(\quad 2\tau\wp^2)], \qquad = \quad \wp^{-2},$$

where $A = \mp 3 \quad ^{7 \quad 2}$, and the elliptic Weierstrass function $\wp = \wp(\tau,\, 0,\, \quad {}_1)$ is defined implicitly by the integral $\tau = \quad (4z^3 - \quad {}_1)^{-1 \quad 2} \quad z$.

37. $-\frac{1}{3}(2 \quad +1) \quad = \quad {}^{+1 \;\; -5 \;\; 2},\qquad \neq -2.$

Solution in parametric form:

$$= a \;\; {}^{7}_{1}\wp^{\frac{3}{\;\;+2}}(\quad 2\tau\wp^2)^{-\frac{3}{2\;\;+4}}, \qquad = \;\; {}^{2}_{1}{}^{\;+4}(\quad 2\tau\wp^2)^{-1},$$

where $A = \mp\frac{4}{3}a^{- \;\; -2 \;\; 7 \;\; 2}(\;\; +2)^2$.

38. $-\frac{1}{3}(2 \quad +1) \quad = \quad {}^{+1 \;\; -5 \;\; 3},\qquad \neq -2.$

Solution in parametric form:

$$= a \;\; {}^{8}_{1}(\tau \quad +2\wp)^{\frac{3}{\;\;+2}}, \qquad = \;\; {}^{3}_{1}{}^{\;+6}(\quad 2\tau\wp^2)^{3 \;\; 2}, \qquad \text{where} \quad A = -\frac{2}{27}a^{- \;\; -2 \;\; 8 \;\; 3}(\;\; +2)^2.$$

39. $-\frac{1}{7}(6 \quad +5) \quad = \quad {}^{+1 \;\; -5 \;\; 3},\qquad \neq -2.$

Solution in parametric form:

$$= a \;\; {}^{8}_{1}(\tau \quad +2\wp)^{-\frac{7}{3\;\;+6}}, \qquad = \;\; {}^{3}_{1}{}^{\;+6}(\quad 2\tau\wp^2)^{3 \;\; 2}(\tau \quad +2\wp)^{-2},$$

where $A = -\frac{6}{49}a^{- \;\; -2 \;\; 8 \;\; 3}(\;\; +2)^2$.

In the solutions of equations 40–46, the following notation is used:

$$R = \sqrt{\quad (4\tau^3 - 1)}, \qquad {}_1 = 2\tau \;\; (\tau) + \;\; {}_2\tau \mp R, \qquad {}_2 = \tau^{-1}(R \;\; {}_1 - 1),$$

where $(\tau) = \dfrac{\tau \quad \tau}{R}$ *is the incomplete elliptic integral of the second kind in the Weierstrass form.*

40. $-\frac{1}{2} \quad = \quad {}^{+1 \;\; -4},\qquad \neq -2.$

Solution in parametric form:

$$= a \;\; {}^{5}_{1}(\tau^{-1} \quad {}_1)^{\frac{2}{\;\;+2}}, \qquad = \quad {}_1{}^{+2}\tau^{-1}, \qquad \text{where} \quad A = \mp\frac{3}{2}a^{- \;\; -2 \;\; 5}(\;\; +2)^2.$$

41. $+ \quad = \quad {}^{-1 \;\; -4}.$

Solution in parametric form:

$$= \;\; {}_2\exp\Big[\,2 \quad \frac{\tau \quad \tau}{\overline{\quad (4\tau^3 - \quad {}_1)}} + \;\; {}_2 \mp \frac{1}{\tau}\sqrt{\quad (4\tau^3 - \quad {}_1)}\,\Big], \qquad = \mp(A \;\; {}^{2}_{1} \;\; 6)^{1 \;\; 5}\tau^{-1}.$$

42. $-\frac{1}{3}(\quad -1) \quad = \quad {}^{+1 \;\; -4},\qquad \neq -2.$

Solution in parametric form:

$$= a \;\; {}^{5}_{1}(\tau \quad {}^{-1}_{1})^{\frac{3}{\;\;+2}}, \qquad = \quad {}_1{}^{+2} \quad {}^{-1}_{1}, \qquad \text{where} \quad A = \mp\frac{2}{3}a^{- \;\; -2 \;\; 5}(\;\; +2)^2.$$

43. $-\frac{1}{2}(3 \quad +4) \quad = \quad {}^{+1 \;\; -1 \;\; 2},\qquad \neq -2.$

Solution in parametric form:

$$= a \;\; {}^{3}_{1} \quad {}_1^{\frac{2}{\;\;+2}}, \qquad = \;\; {}^{2}_{1}{}^{\;+4} \quad {}^{2}_{2}, \qquad \text{where} \quad A = \quad 3a^{- \;\; -2 \;\; 3 \;\; 2}(\;\; +2)^2.$$

44. $-\frac{1}{5}(6 \quad +7) \quad = \quad {}^{+1 \;\; -1 \;\; 2},\qquad \neq -2.$

Solution in parametric form:

$$= a \;\; {}^{3}_{1} \quad {}_1^{-\frac{5}{2\;\;+4}}, \qquad = \;\; {}^{2}_{1}{}^{\;+4} \quad {}^{-3}_{1} \quad {}^{2}_{2}, \qquad \text{where} \quad A = \quad \frac{48}{25}a^{- \;\; -2 \;\; 3 \;\; 2}(\;\; +2)^2.$$

45. $-\frac{1}{3}(\quad -1) \quad = \quad {}^{+1 \;\; -7},\qquad \neq -2.$

Solution in parametric form:

$$= a \;\; {}^{8}_{1}(4\tau \quad {}^{2}_{1} \mp \quad {}^{2}_{2})^{\frac{3}{\;\;+2}}, \qquad = \quad {}_1{}^{+2} \quad {}^{1 \;\; 2}_{1}, \qquad \text{where} \quad A = \quad \frac{1}{192}a^{- \;\; -2 \;\; 8}(\;\; +2)^2.$$

46. $-\frac{1}{5}(\quad -3) \quad = \quad {}^{+1 \;\; -7},\qquad \neq -2.$

Solution in parametric form:

$$= a \;\; {}^{8}_{1}(4\tau \quad {}^{2}_{1} \mp \quad {}^{2}_{2})^{-\frac{5}{\;\;+2}}, \qquad = \quad {}_1{}^{+2} \quad {}^{1 \;\; 2}_{1}(4\tau \quad {}^{2}_{1} \mp \quad {}^{2}_{2})^{-1}, \qquad \text{where} \quad A = \quad \frac{3}{1600}a^{- \;\; -2 \;\; 8}(\;\; +2)^2.$$

2.6.2. Equations of the Form $y''_{xx} = (A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2})(y'_x)^l$

See Section 2.4 for the case $l = 0$; see Section 2.5 for the cases $A_1 = 0$ or $A_2 = 0$.

2.6.2-1. Classification table.

Table 25 presents all solvable equations whose solutions are outlined in Subsection 2.6.2. Equations are arranged in accordance with the growth of l, the growth of n_1 (for identical l), the growth of n_2 (for identical l and n_1, $n_1 \geq n_2$), the growth of m_1 (for identical l, n_1, and n_2), and the growth of m_2 (for identical l, n_1, n_2, and m_1). The number of the equation sought is indicated in the last column in this table.

TABLE 25

Solvable cases of the equation $y''_{xx} = (A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2})(y'_x)^l$

l	n_1	n_2	m_1	m_2	A_1	A_2	Equation
Any ($l \neq 2$)	Any ($n_1 \neq -1$)	Any ($n_2 \neq -1$)	0	0	Any	Any	2.6.2.1
$\dfrac{n_1+2m_1+3}{n_1+m_1+2}$	Any	Any	Any	$\dfrac{n_2(m_1+1)-n_1+m_1}{n_1+1}$	Any	Any	2.6.2.98
Any ($l \neq 1$)	0	0	Any ($m_1 \neq -1$)	Any ($m_2 \neq -1$)	Any	Any	2.6.2.5
Any ($l \neq 2$)	Any ($n_1 \neq -1$)	-1	0	0	Any	Any	2.6.2.2
Any ($l \neq 2$)	0	0	Any ($m_1 \neq -1$)	-1	Any	Any	2.6.2.6
$\dfrac{3n_1+5}{2n_1+3}$	Any	$-n_1-2$	1	0	Any	Any	2.6.2.21
$\dfrac{n_1+5}{n_1+3}$	Any	$\dfrac{n_1-1}{2}$	1	0	Any	Any	2.6.2.94
$\dfrac{3m_1+4}{m_1+1}$	1	0	Any	$-m_1-2$	Any	Any	2.6.2.22
$\dfrac{2(m_1+2)}{m_1+3}$	1	0	Any	$\dfrac{m_1-1}{2}$	Any	Any	2.6.2.95
Any ($l \neq 1, 2$)	1	0	0	1	Any	Any	2.6.2.20
$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{7}{4}$	0	1	Any	Any	2.6.2.71
$\frac{1}{2}$	1	0	$-\frac{15}{8}$	$-\frac{7}{4}$	Any	Any	2.6.2.81
$\frac{1}{2}$	1	0	$-\frac{15}{8}$	$-\frac{13}{8}$	Any	Any	2.6.2.66
$\frac{1}{2}$	1	0	$-\frac{20}{13}$	$-\frac{15}{13}$	Any	Any	2.6.2.68
$\frac{1}{2}$	1	0	$-\frac{20}{13}$	$-\frac{14}{13}$	Any	Any	2.6.2.84
$\frac{1}{2}$	1	0	$-\frac{5}{4}$	$-\frac{3}{4}$	Any	Any	2.6.2.64
$\frac{1}{2}$	1	0	$-\frac{5}{4}$	$-\frac{1}{2}$	Any	Any	2.6.2.78

TABLE 25 (*Continued*)

Solvable cases of the equation $'' = (A_1\ {}^{1}\ {}^{1} + A_2\ {}^{2}\ {}^{2})(\ ')$

l	$_1$	$_2$	$_1$	$_2$	A_1	A_2	Equation
$\frac{1}{2}$	1	0	0	1	Any	Any	2.6.2.62
$\frac{1}{2}$	1	0	0	2	Any	Any	2.6.2.75
1	0	0	Any ($_1 \neq -1$)	Any ($_2 \neq -1$)	Any	Any	2.6.2.7
1	0	0	Any ($_1 \neq -1$)	-1	Any	Any	2.6.2.8
1	0	-2	0	1	Any	Any	2.6.2.25
1	1	0	0	1	Any	Any	2.6.2.23
$\frac{3}{2}$	Any	Any	$_1$	$_2$	Any	Any	2.6.2.97
$\frac{3}{2}$	0	-2	0	1	Any	Any	2.6.2.107
$\frac{3}{2}$	0	$-\frac{1}{2}$	0	1	Any	Any	2.6.2.105
$\frac{3}{2}$	1	0	-2	0	Any	Any	2.6.2.108
$\frac{3}{2}$	1	0	$-\frac{1}{2}$	0	Any	Any	2.6.2.106
2	Any ($_1 \neq -1$)	Any ($_2 \neq -1$)	0	0	Any	Any	2.6.2.3
2	Any ($_1 \neq -1$)	-1	0	0	Any	Any	2.6.2.4
2	1	0	-2	0	Any	Any	2.6.2.26
2	1	0	0	1	Any	Any	2.6.2.24
$\frac{5}{2}$	$-\frac{7}{4}$	$-\frac{15}{8}$	0	1	Any	Any	2.6.2.80
$\frac{5}{2}$	$-\frac{13}{8}$	$-\frac{15}{8}$	0	1	Any	Any	2.6.2.65
$\frac{5}{2}$	$-\frac{15}{13}$	$-\frac{20}{13}$	0	1	Any	Any	2.6.2.67
$\frac{5}{2}$	$-\frac{14}{13}$	$-\frac{20}{13}$	0	1	Any	Any	2.6.2.83
$\frac{5}{2}$	$-\frac{3}{4}$	$-\frac{5}{4}$	0	1	Any	Any	2.6.2.63
$\frac{5}{2}$	$-\frac{1}{2}$	$-\frac{5}{4}$	0	1	Any	Any	2.6.2.77
$\frac{5}{2}$	1	0	$-\frac{7}{4}$	$-\frac{1}{4}$	Any	Any	2.6.2.72
$\frac{5}{2}$	1	0	0	1	Any	Any	2.6.2.61
$\frac{5}{2}$	2	0	0	1	Any	Any	2.6.2.74
3	Any	Any	$-\ _1-3$	$-\ _2-3$	Any	Any	2.6.2.9
3	Any	Any	$-2\ _1-3$	$-2\ _2-3$	Any	Any	2.6.2.93
3	Any ($_1 \neq -2$)	Any	1	0	Any	Any	2.6.2.49
3	Any	-3	$-\ _1-3$	0	Any	Any	2.6.2.19

TABLE 25 (*Continued*)

Solvable cases of the equation $'' = (A_1 \;^{1} \;^{1} + A_2 \;^{2} \;^{2})(\;')$

l	$_1$	$_2$	$_1$	$_2$	A_1	A_2	Equation
3	Any ($_1 \neq -2$)	0	1	−3	Any	Any	2.6.2.51
3	−5	−6	1	3	Any	Any	2.6.2.100
3	−4	−5	0	2	Any	Any	2.6.2.76
3	−3	−5	0	1	Any	Any	2.6.2.44
3	−3	−5	0	2	Any	Any	2.6.2.58
3	−3	$-\frac{7}{2}$	0	$-\frac{1}{2}$	Any	Any	2.6.2.28
3	$-\frac{14}{5}$	$-\frac{18}{5}$	2	3	Any	Any	2.6.2.112
3	$-\frac{8}{3}$	$-\frac{10}{3}$	$-\frac{1}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.38
3	$-\frac{5}{2}$	−4	$-\frac{1}{2}$	0	Any	Any	2.6.2.86
3	$-\frac{5}{2}$	$-\frac{7}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	Any	Any	2.6.2.13
3	$-\frac{5}{2}$	−3	$-\frac{1}{2}$	0	Any	Any	2.6.2.32
3	$-\frac{12}{5}$	$-\frac{13}{5}$	$-\frac{3}{5}$	$-\frac{7}{5}$	Any	Any	2.6.2.30
3	$-\frac{7}{3}$	$-\frac{10}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.34
3	$-\frac{7}{3}$	$-\frac{10}{3}$	$-\frac{5}{3}$	$\frac{1}{3}$	Any	Any	2.6.2.88
3	$-\frac{7}{3}$	−3	$-\frac{2}{3}$	0	Any	Any	2.6.2.70
3	$-\frac{7}{3}$	$-\frac{8}{3}$	$-\frac{5}{3}$	$-\frac{1}{3}$	Any	Any	2.6.2.42
3	$-\frac{11}{5}$	$-\frac{12}{5}$	2	3	Any	Any	2.6.2.113
3	−2	$-\;_2-1$	1	Any	$\frac{2(\;_2+1)}{(\;_2+3)^2}$	Any	2.6.2.132
3	−2	−3	−2	0	Any	Any	2.6.2.104
3	−2	−3	−1	0	Any	Any	2.6.2.12
3	−2	−3	1	2	$-\frac{6}{25}$	Any	2.6.2.145
3	−2	−3	1	2	$\frac{6}{25}$	Any	2.6.2.144
3	−2	−3	$-\frac{1}{2}$	0	Any	Any	2.6.2.102
3	−2	−2	1	Any	$\frac{2(\;_2+1)}{(\;_2+3)^2}$	Any	2.6.2.116
3	−2	−2	1	−7	$-\frac{15}{4}$	Any	2.6.2.117
3	−2	−2	1	−4	−6	Any	2.6.2.118
3	−2	−2	1	$-\frac{5}{2}$	−12	Any	2.6.2.119
3	−2	−2	1	−2	−2	Any	2.6.2.120
3	−2	−2	1	$-\frac{5}{3}$	$-\frac{63}{4}$	Any	2.6.2.124

TABLE 25 (*Continued*)
Solvable cases of the equation $'' = (A_1 \quad {}^{1} \quad {}^{1} + A_2 \quad {}^{2} \quad {}^{2})(\quad ')$

l	1	2	1	2	A_1	A_2	Equation
3	–2	–2	1	$-\frac{5}{3}$	$-\frac{3}{4}$	Any	2.6.2.123
3	–2	–2	1	$-\frac{5}{3}$	$\frac{9}{100}$	Any	2.6.2.122
3	–2	–2	1	$-\frac{5}{3}$	$\frac{3}{16}$	Any	2.6.2.121
3	–2	–2	1	$-\frac{7}{5}$	$\frac{5}{36}$	Any	2.6.2.125
3	–2	–2	1	$-\frac{1}{2}$	–20	Any	2.6.2.128
3	–2	–2	1	$-\frac{1}{2}$	$\frac{4}{25}$	Any	2.6.2.127
3	–2	–2	1	$-\frac{1}{2}$	$\frac{2}{9}$	Any	2.6.2.126
3	–2	–2	1	$\frac{1}{2}$	$\frac{12}{49}$	Any	2.6.2.129
3	–2	–2	2	1	Any	$-\frac{6}{25}$	2.6.2.131
3	–2	–2	2	1	Any	$\frac{6}{25}$	2.6.2.130
3	$-\frac{13}{7}$	$-\frac{20}{7}$	0	2	Any	Any	2.6.2.82
3	$-\frac{12}{7}$	$-\frac{20}{7}$	0	2	Any	Any	2.6.2.60
3	$-\frac{5}{3}$	$-\frac{7}{3}$	$-\frac{4}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.92
3	$-\frac{8}{5}$	$-\frac{13}{5}$	$-\frac{7}{5}$	$-\frac{7}{5}$	Any	Any	2.6.2.109
3	$-\frac{3}{2}$	$-\frac{5}{2}$	0	$-\frac{1}{2}$	Any	Any	2.6.2.90
3	$-\frac{3}{2}$	–2	$-\frac{3}{2}$	–2	Any	Any	2.6.2.48
3	$-\frac{3}{2}$	–2	0	$-\frac{1}{2}$	Any	Any	2.6.2.46
3	$-\frac{3}{2}$	–2	$\frac{1}{2}$	1	Any	$\frac{12}{49}$	2.6.2.143
3	$-\frac{4}{3}$	$-\frac{10}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.36
3	$-\frac{4}{3}$	$-\frac{8}{3}$	$-\frac{5}{3}$	$-\frac{1}{3}$	Any	Any	2.6.2.40
3	$-\frac{4}{3}$	$-\frac{7}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.14
3	$-\frac{4}{3}$	$-\frac{4}{3}$	0	$-\frac{1}{2}$	Any	Any	2.6.2.15
3	$-\frac{9}{7}$	$-\frac{15}{7}$	0	2	Any	Any	2.6.2.59
3	$-\frac{7}{6}$	$-\frac{5}{3}$	$-\frac{1}{2}$	0	Any	Any	2.6.2.16
3	$-\frac{8}{7}$	$-\frac{15}{7}$	0	2	Any	Any	2.6.2.79
3	–1	–2	–2	–2	Any	Any	2.6.2.110
3	$-\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.115
3	$-\frac{1}{2}$	–3	$-\frac{1}{2}$	0	Any	Any	2.6.2.53
3	$-\frac{1}{2}$	–2	$-\frac{1}{2}$	1	Any	–20	2.6.2.141
3	$-\frac{1}{2}$	–2	$-\frac{1}{2}$	1	Any	$\frac{4}{25}$	2.6.2.134

TABLE 25 (*Continued*)
Solvable cases of the equation $'' = (A_1 \quad ^1 \quad ^1 + A_2 \quad ^2 \quad ^2)(\ ')$

l	$_1$	$_2$	$_1$	$_2$	A_1	A_2	Equation
3	$-\frac{1}{2}$	-2	$-\frac{1}{2}$	1	Any	$\frac{2}{9}$	2.6.2.137
3	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	0	Any	Any	2.6.2.45
3	0	-5	-3	1	Any	Any	2.6.2.52
3	0	-2	-3	-2	Any	Any	2.6.2.56
3	0	-2	0	$-\frac{1}{2}$	Any	Any	2.6.2.54
3	0	$-\frac{3}{2}$	$-\frac{1}{2}$	0	Any	Any	2.6.2.89
3	0	$-\frac{2}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.114
3	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	Any	Any	2.6.2.101
3	0	0	$-\frac{1}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.39
3	0	0	0	-1	Any	Any	2.6.2.11
3	0	0	0	$-\frac{2}{3}$	Any	Any	2.6.2.69
3	0	0	0	$-\frac{1}{2}$	Any	Any	2.6.2.31
3	0	0	2	0	Any	Any	2.6.2.57
3	$\frac{2}{5}$	-2	$-\frac{7}{5}$	1	Any	$\frac{5}{36}$	2.6.2.139
3	$\frac{2}{3}$	-2	$-\frac{5}{3}$	1	Any	$-\frac{63}{4}$	2.6.2.147
3	$\frac{2}{3}$	-2	$-\frac{5}{3}$	1	Any	$-\frac{3}{4}$	2.6.2.136
3	$\frac{2}{3}$	-2	$-\frac{5}{3}$	1	Any	$\frac{9}{100}$	2.6.2.135
3	$\frac{2}{3}$	-2	$-\frac{5}{3}$	1	Any	$\frac{3}{16}$	2.6.2.138
3	1	-2	-2	1	Any	-2	2.6.2.133
3	1	0	-7	-3	Any	Any	2.6.2.17
3	1	0	-4	-3	Any	Any	2.6.2.10
3	1	0	-2	-3	Any	Any	2.6.2.55
3	1	0	-2	$-\frac{3}{2}$	Any	Any	2.6.2.47
3	1	0	-2	0	Any	Any	2.6.2.103
3	1	0	$-\frac{5}{3}$	$-\frac{4}{3}$	Any	Any	2.6.2.91
3	1	0	$-\frac{5}{3}$	$-\frac{1}{3}$	Any	Any	2.6.2.41
3	1	0	$-\frac{5}{3}$	$\frac{1}{3}$	Any	Any	2.6.2.87
3	1	0	$-\frac{7}{5}$	$-\frac{3}{5}$	Any	Any	2.6.2.29
3	1	0	$-\frac{1}{2}$	0	Any	Any	2.6.2.27
3	1	0	0	$-\frac{1}{2}$	Any	Any	2.6.2.85

TABLE 25 (*Continued*)

Solvable cases of the equation $'' = (A_1 \;{}^{1}\;{}^{1} + A_2 \;{}^{2}\;{}^{2})(\;')$

l	$_1$	$_2$	$_1$	$_2$	A_1	A_2	Equation
3	1	0	0	2	Any	Any	2.6.2.73
3	1	0	1	−3	Any	Any	2.6.2.50
3	1	0	1	0	Any	Any	2.6.2.43
3	1	0	1	3	Any	Any	2.6.2.99
3	$\frac{3}{2}$	−2	$-\frac{5}{2}$	1	Any	−12	2.6.2.146
3	2	0	−5	−5	Any	Any	2.6.2.96
3	2	0	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.35
3	2	0	$-\frac{5}{3}$	$-\frac{1}{3}$	Any	Any	2.6.2.37
3	2	1	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.33
3	3	−2	−4	1	Any	−6	2.6.2.140
3	3	0	−7	−3	Any	Any	2.6.2.18
3	4	3	−7	−7	Any	Any	2.6.2.111
3	6	−2	−7	1	Any	$-\frac{15}{4}$	2.6.2.142

2.6.2-2. Solvable equations and their solutions.

1. $= (\;_1\;{}^1 + \;_2\;{}^2)(\;)$, $\neq 2$, $_1 \neq -1$, $_2 \neq -1$.

1 . Solution in parametric form:

$$= a \;(\;_1 + \tau\;{}^{_1+1}\;\tau\;{}^{_2+1})^{\frac{1}{-2}}\;\tau + \;_2, \qquad = \tau,$$

where $A_1 = \frac{_1+1}{2-l} a^{-2\;1-\;\;1-}$, $A_2 = \frac{_2+1}{2-l} a^{-2\;1-\;\;2-}$.

2 . Solution in parametric form:

$$= a \;(\;_1 - \tau\;{}^{_1+1}\;\tau\;{}^{_2+1})^{\frac{1}{-2}}\;\tau + \;_2, \qquad = \tau,$$

where $A_1 = \frac{_1+1}{l-2} a^{-2\;1-\;\;1-}$, $A_2 = \frac{_2+1}{2-l} a^{-2\;1-\;\;2-}$.

2. $= (\;_1 \;+\; _2\;{}^{-1})(\;)$, $\neq 2$, $\neq -1$.

Solution: $= \;_1 + \frac{A_1(2-l)}{+1}\;{}^{+1} + (2-l)A_2 \ln\;{}^{\frac{1}{-2}}\; + \;_2.$

3. $= (\;_1\;{}^1 + \;_2\;{}^2)(\;)^2$, $_1 \neq -1$, $_2 \neq -1$.

Solution: $= \;_1\; \exp -\frac{A_1}{_1+1}\;{}^{_1+1} - \frac{A_2}{_2+1}\;{}^{_2+1}\; + \;_2.$

4. $= (\ _1 \quad + \quad _2 \ ^{-1})(\quad)^2, \qquad \neq -1.$

Solution: $= \ _1 \quad ^{-} \ ^{2} \exp \ -\dfrac{A_1}{+1} \quad ^{+1} \quad + \ _2.$

5. $= (\ _1 \ ^1 + \ _2 \ ^2)(\quad), \qquad \neq 1, \quad _1 \neq -1, \quad _2 \neq -1.$

1 . Solution in parametric form:

$$= a\tau, \qquad = \quad (\ _1 + \tau \ ^{_1+1} \quad \tau \ ^{_2+1} \ ^{\frac{1}{1-}} \quad \tau + \ _2,$$

where $A_1 = \dfrac{_1+1}{1-l} a^{- \ _1-2 \ \ 1-}$, $A_2 = \dfrac{_2+1}{1-l} a^{- \ _2-2 \ \ 1-}$.

2 . Solution in parametric form:

$$= a\tau, \qquad = \quad (\ _1 - \tau \ ^{_1+1} \quad \tau \ ^{_2+1} \ ^{\frac{1}{1-}} \quad \tau + \ _2,$$

where $A_1 = \dfrac{_1+1}{l-1} a^{- \ _1-2 \ \ 1-}$, $A_2 = \dfrac{_2+1}{1-l} a^{- \ _2-2 \ \ 1-}$.

6. $= (\ _1 \quad + \quad _2 \ ^{-1})(\quad), \qquad \neq 1, \quad \neq -1.$

Solution: $= \quad _1 + \dfrac{A_1(1-l)}{+1} \quad ^{+1} + (1-l)A_2 \ln \quad ^{\frac{1}{1-}} \quad + \ _2.$

7. $= (\ _1 \ ^1 + \ _2 \ ^2) \quad , \qquad _1 \neq -1, \quad _2 \neq -1.$

Solution: $= \ _1 \quad \exp \ \dfrac{A_1}{_1+1} \quad ^{_1+1} + \dfrac{A_2}{_2+1} \quad ^{_2+1} \quad + \ _2.$

8. $= (\ _1 \quad + \quad _2 \ ^{-1}) \quad , \qquad \neq -1.$

Solution: $= \ _1 \quad ^{2} \exp \ \dfrac{A_1}{+1} \quad ^{+1} \quad + \ _2.$

9. $= (\ _1 \ ^{-} \ ^{_1-3} \quad ^1 + \ _2 \ ^{-} \ ^{_2-3} \quad ^2)(\quad)^3, \qquad _1 \neq -2, \quad _2 \neq -2.$

1 . Solution in parametric form:

$$= a\tau \quad (\ _1 + \tau^{- \ _1-2} \quad \tau^{- \ _2-2} \ ^{-1} \ ^{2} \quad \tau + \ _2 \ ^{-1},$$

$$= \quad (\ _1 + \tau^{- \ _1-2} \quad \tau^{- \ _2-2} \ ^{-1} \ ^{2} \quad \tau + \ _2 \ ^{-1},$$

where $A_1 = \frac{1}{2} a \ ^{_1+4 \ - \ _1-2}(\ _1 + 2)$, $A_2 = \frac{1}{2} a \ ^{_2+4 \ - \ _2-2}(\ _2 + 2)$.

2 . Solution in parametric form:

$$= a\tau \quad (\ _1 - \tau^{- \ _1-2} \quad \tau^{- \ _2-2} \ ^{-1} \ ^{2} \quad \tau + \ _2 \ ^{-1},$$

$$= \quad (\ _1 - \tau^{- \ _1-2} \quad \tau^{- \ _2-2} \ ^{-1} \ ^{2} \quad \tau + \ _2 \ ^{-1},$$

where $A_1 = -\frac{1}{2} a \ ^{_1+4 \ - \ _1-2}(\ _1 + 2)$, $A_2 = \frac{1}{2} a \ ^{_2+4 \ - \ _2-2}(\ _2 + 2)$.

10. $= (\ _1\ ^{-4} \ + \ _2\ ^{-3})(\ \ )^3.$

1 . Solution in parametric form:

$$= a\tau \left(\int \frac{\tau}{\overline{\ _1 + \tau^{-3} \quad \tau^{-2}}} + \ _2 \right)^{-1}, \qquad = \left(\int \frac{\tau}{\overline{\ _1 + \tau^{-3} \quad \tau^{-2}}} + \ _2 \right)^{-1},$$

where $A_1 = \frac{3}{2} a^5 \ ^{-3}, \ A_2 = \ \ a^4 \ ^{-2}.$

2 . Solution in parametric form:

$$= a\tau \left(\int \frac{\tau}{\overline{\ _1 - \tau^{-3} \quad \tau^{-2}}} + \ _2 \right)^{-1}, \qquad = \left(\int \frac{\tau}{\overline{\ _1 - \tau^{-3} \quad \tau^{-2}}} + \ _2 \right)^{-1},$$

where $A_1 = -\frac{3}{2} a^5 \ ^{-3}, \ A_2 = \ \ a^4 \ ^{-2}.$

11. $= (\ _1 + \ \ _2\ ^{-1})(\ \ )^3.$

Solution: $\quad = \quad (\ _1 - 2A_1 \ \ - 2A_2 \ln \ \ )^{-1\ 2} \quad + \ _2.$

12. $= (\ _1\ ^{-1}\ ^{-2} + \ _2\ ^{-3})(\ \ )^3.$

Solution in parametric form:

$$= \tau \left(\int \frac{\tau}{\overline{\ _1 - 2A_1 \ln \tau - 2A_2 \tau}} + \ _2 \right)^{-1}, \qquad = \left(\int \frac{\tau}{\overline{\ _1 - 2A_1 \ln \tau - 2A_2 \tau}} + \ _2 \right)^{-1}.$$

13. $= (\ _1\ ^{-1\ 2}\ ^{-5\ 2} + \ _2\ ^{-1\ 2}\ ^{-7\ 2})(\ \ )^3.$

Solution in parametric form:

$$= \frac{k^2}{\ } \ 2 \ _1 e^{2\ \tau} + \ _2 e^{-\ \tau} \left[\ \overline{3} \cos(\omega\tau) - \sin(\omega\tau) \right] \ ^2, \qquad = \frac{1}{\ },$$

where $\quad = \ _1 e^{2\ \tau} + \ _2 e^{-\ \tau} \sin(\omega\tau) - \dfrac{A_1}{A_2}, \ A_2 = -16k^3, \ \omega = k \ \ \overline{3}.$

14. $= (\ _1\ ^{-5\ 3}\ ^{-4\ 3} + \ _2\ ^{-5\ 3}\ ^{-7\ 3})(\ \ )^3.$

Solution in parametric form:

$$= \left(\tfrac{1}{36} A_2 \tau^4 + \ _1 \tau^3 + \ _2 \tau^2 + \ _3 \tau \ \right)^{-1} \left(\tfrac{1}{9} A_2 \tau^3 + 3 \ _1 \tau^2 + 2 \ _2 \tau + \ _3 \ \right)^{3\ 2},$$
$$= \left(\tfrac{1}{36} A_2 \tau^4 + \ _1 \tau^3 + \ _2 \tau^2 + \ _3 \tau \ \right)^{-1},$$

where $A_1 = 9 \ _1 \ _3 - 3 \ _2^2.$

In the solutions of equations 15–18, the following notation is used:

$$R_1 = \overline{\ _1 + \tau^{-3} \quad \tau^{-2}}, \qquad _1 = \frac{\tau}{R_1} + \ _2, \qquad _1 = \tau - R_1 \ _1, \qquad _1 = 3\tau^3 \ _1^2 + 3(1 \quad \tau) \ _1^2,$$
$$R_2 = \overline{\ _1 - \tau^{-3} \quad \tau^{-2}}, \qquad _2 = \frac{\tau}{R_2} + \ _2, \qquad _2 = \tau - R_2 \ _2, \qquad _2 = 3\tau^3 \ _2^2 + 3(-1 \quad \tau) \ _2^2.$$

15. $= (\ _1\ ^{-4\ 3} + \ _2\ ^{-1\ 2}\ ^{-4\ 3})(\ \ )^3.$

Solution in parametric form:

$$= a \ \ ^2, \qquad = \ \tau^{-3} \ \ ^3,$$

where $A_1 = \ \frac{2}{9} a \ ^{-2\ 3}, \ A_2 = \frac{1}{3} a^{3\ 2} \ ^{-2\ 3} (-1) \ ^{+1}; \ k = 1$ and $k = 2$.

16. $\mathbf{= (\ \ _1\ ^{-1\ 2\ \ -7\ 6} + \ \ _2\ ^{-5\ \ 3})(\ \ )^3.}$

Solution in parametric form:

$$= a\tau^{3\quad -3\quad 2}, \qquad = \ \tau^{3\quad -3},$$

where $A_1 = \frac{1}{3}a^{3\ 2\ \ -5\ \ 6}(-1)\ \ ,\ A_2 = \mp\frac{2}{9}a\ \ ^{-1\ \ 3}$; $k = 1$ and $k = 2$.

17. $\mathbf{= (\ \ _1\ ^{-7}\ \ + \ \ _2\ ^{-3})(\ \ )^3.}$

Solution in parametric form:

$$= a\tau^{-1\ 2\quad 1\ 2}, \qquad = \ \tau^{-3}\qquad ,$$

where $A_1 = \frac{1}{36}a^{8\ \ -3}$, $A_2 = \ \ \frac{1}{36}a^{4\ \ -2}$; $k = 1$ and $k = 2$.

18. $\mathbf{= (\ \ _1\ ^{-7\ \ 3} + \ \ _2\ ^{-3})(\ \ )^3.}$

Solution in parametric form:

$$= a\tau^{5\ 2\quad 1\ 2\quad -1}, \qquad = \ \tau^{3\quad -1},$$

where $A_1 = \frac{1}{36}a^{8\ \ -5}$, $A_2 = \mp\frac{1}{36}a^{4\ \ -2}$; $k = 1$ and $k = 2$.

In the solutions of equations 19–22, the following notation is used:

$$R_1 = \sqrt{\ \ _1\ \ \tau\ ^{+1} + \tau}, \qquad _1 = \frac{\tau}{R_1} + \ \ _2, \qquad _1 = 2R_1 - \ \ _1, \qquad _1 = 4(\tau - R_1\ \ _1) + \ \ _1^2,$$

$$R_2 = \sqrt{\ \ _1\ \ \tau\ ^{+1} - \tau}, \qquad _2 = \frac{\tau}{R_2} + \ \ _2, \qquad _2 = 2R_2 + \ \ _2, \qquad _2 = 4(\tau - R_2\ \ _2) - \ \ _2^2.$$

19. $\mathbf{= (\ \ _1\ ^{-\ \ -3}\ \ + \ \ _2\ ^{-3})(\ \ )^3, \qquad \neq -2.}$

Solution in parametric form:

$$= a\tau\ \ ^{-1}, \qquad = \ \ ^{-1}, \qquad = -\ \ -3,$$

where $A_1 = \ \ \frac{1}{2}a\ \ ^{+4\ -\ \ -2}(\ \ +2)$, $A_2 = \frac{1}{2}a\ \ (-1)\ \ $; $k = 1$ and $k = 2$.

20. $\mathbf{= (\ \ _1\ \ + \ \ _2\ \ )(\ \ )\ ,\qquad \neq 1, \quad \neq 2.}$

Solution in parametric form:

$$= a\ \ , \qquad = \ \ , \qquad = \frac{1}{l-2},$$

where $A_1 = a\ \ ^{-1}A_2(-1)\ \ ^{+1}$, $A_2 = -\frac{\quad}{2a}\ \ \frac{(\ \ +1)a\ \ ^{1}}{\quad}$; $k = 1$ and $k = 2$.

21. $\mathbf{= (\ \ _1\ \ \ \ + \ \ _2\ ^{-\ \ -2})(\ \ )^{\frac{3\ \ +5}{2\ \ +3}}.}$

Solution in parametric form:

$$= a\ \ , \qquad = \ \ ^{+2}, \qquad = -\frac{2\ \ +3}{\ \ +1},$$

where $A_1 = \frac{\quad}{4(\ \ +2)}a^{-1\ \ \frac{1}{+2}} \mp \frac{2(\ \ +1)a\ \ ^{\frac{1}{\ }}}{(\ \ +2)}$, $A_2 = a\ \ ^{-\frac{2}{+2}}A_1(-1)\ \ $; $k = 1$ and $k = 2$.

22. $= (\ _1 \quad + \ _2 \ ^{-} \ ^{-2})(\quad)^{\frac{3 \ +4}{+1}}$**.**

Solution in parametric form:

$$= a \quad ^{+2}, \qquad = \qquad , \qquad = -\frac{2 \quad +3}{+1},$$

where $A_1 = -\frac{}{4(\ +2)} a^{\frac{1}{+2}} \ ^{-1} \mp \frac{2(\ +1)}{(\ +2)a} \ ^{\frac{1}{}}$, $A_2 = a^{-\frac{2}{+2}} \ A_1(-1)$; $k = 1$ and $k = 2$.

In the solutions of equations 23–26, the following notation is used:

$$R_1 = \overline{\quad _1 + \tau \quad \ln\tau}, \quad _1 = \frac{\tau}{R_1} + \ _2, \quad _1 = 2R_1 - \ _1, \quad _1 = 4(\tau - R_1 \ _1) + \ _1^2,$$

$$R_2 = \overline{\quad _1 - \tau \quad \ln\tau}, \quad _2 = \frac{\tau}{R_2} + \ _2, \quad _2 = 2R_2 + \ _2, \quad _2 = 4(\tau - R_2 \ _2) - \ _2^2.$$

23. $= (\ _1 \ + \ _2 \ ) \quad$**.**

Solution in parametric form:

$$= a \quad , \qquad = \qquad ,$$

where $A_1 = a \ ^{-1} A_2(-1)$, $A_2 = \ \frac{1}{2} a^{-2}$; $k = 1$ and $k = 2$.

24. $= (\ _1 \ + \ _2 \ )(\quad)^2$**.**

Solution in parametric form:

$$= a \quad , \qquad = \qquad ,$$

where $A_1 = \mp\frac{1}{2} \ ^{-2}$, $A_2 = a^{-1} \ A_1(-1)$; $k = 1$ and $k = 2$.

25. $= (\ _1 + \ _2 \quad ^{-2}) \quad$**.**

Solution in parametric form:

$$= a \quad , \qquad = \qquad ,$$

where $A_1 = a \ ^{-2} A_2(-1)$, $A_2 = \ \frac{1}{8} a^{-2} \ ^{-2}$; $k = 1$ and $k = 2$.

26. $= (\ _1 \ ^{-2} \ + \ _2)(\quad)^2$**.**

Solution in parametric form:

$$= a \quad , \qquad = \qquad ,$$

where $A_1 = \mp\frac{1}{8} a^2 \ ^{-2}$, $A_2 = a^{-2} \ A_1(-1)$; $k = 1$ and $k = 2$.

In the solutions of equations 27–30, the following notation is used:

$$R_1 = \ _1\tau \ ^1 + \ _2\tau \ ^2 + \ _3\tau \ ^3,$$
$$R_2 = (\ _1 + \ _2\tau)e \ ^\tau + \ _3 e \ ^\tau,$$
$$R_3 = \ _1 e \ ^\tau + e \ ^\tau(\ _2 \sin\omega\tau + \ _3 \cos\omega\tau),$$
$$Q_1 = \ _1 k_1 \tau \ ^1 + \ _2 k_2 \tau \ ^2 + \ _3 k_3 \tau \ ^3,$$
$$Q_2 = (k \ _1 + \ _2 + k \ _2\tau)e \ ^\tau + \omega \ _3 e \ ^\tau,$$
$$Q_3 = k \ _1 e \ ^\tau + e \ ^\tau[(\ _2 - \omega \ _3)\sin\omega\tau + (\ _3 + \omega \ _2)\cos\omega\tau],$$
$$_1 = \tau(Q_1)'_\tau, \qquad _2 = (Q_2)'_\tau, \qquad _3 = (Q_3)'_\tau,$$

where k_1, k_2, *and* k_3 *(real numbers) or* k *and* ω *(one real and two complex numbers) are roots of the cubic equation* $\lambda^3 - \frac{1}{2}B_2\lambda - \frac{1}{2}B_1 = 0$. *The subscripts of the functions* R , Q , *and* ($= 1,\ 2,\ 3$) *are selected depending on the sign of the expression* $\Delta = 2B_2^3 - 27B_1^2$:

$$\begin{aligned} &\textit{if}\quad \Delta > 0 \quad \textit{subscript} \qquad = 1,\\ &\textit{if}\quad \Delta = 0 \quad \textit{subscript} \qquad = 2,\\ &\textit{if}\quad \Delta < 0 \quad \textit{subscript} \qquad = 3. \end{aligned}$$

If $2B_2^3 = 27B_1^2$ *(subscript 2), then*

$$\begin{aligned} k &= (\tfrac{1}{6}B_2)^{1\ 2}, \quad \omega = -2(\tfrac{1}{6}B_2)^{1\ 2} \quad \textit{if}\quad B_1 < 0,\\ k &= -(\tfrac{1}{6}B_2)^{1\ 2}, \quad \omega = 2(\tfrac{1}{6}B_2)^{1\ 2} \quad \textit{if}\quad B_1 > 0. \end{aligned}$$

The expressions for R , *and* Q *contain three constants* $_1$, $_2$, *and* $_3$. *One of them can be arbitrarily fixed to let it be any nonzero number (for instance, we can set* $_3 = \ 1$*), while the other constants remain arbitrary.*

27. $= (\ _1\ ^{-1\ 2} + \ _2)(\ )^3.$

Solution in parametric form:

$$= Q^2\ , \qquad = R\ , \qquad \text{where}\quad A_1 = -B_1, \quad A_2 = -B_2.$$

28. $= (\ _1\ ^{-3} + \ _2\ ^{-1\ 2\ -7\ 2})(\ )^3.$

Solution in parametric form:

$$= R^{-1}Q^2\ , \qquad = R^{-1}, \qquad \text{where}\quad A_1 = -B_2, \quad A_2 = -B_1.$$

29. $= (\ _1\ ^{-7\ 5} + \ _2\ ^{-3\ 5})(\ )^3.$

Solution in parametric form:

$$= aR^{5\ 2}, \qquad = \ (2Q^2\ - 4R \qquad + B_2R^2\ ),$$

where $A_1 = \frac{5}{32}a^{12\ 5\ -3}B_1^{-2}$, $A_2 = -a^{-4\ 5}\ A_1B_2$.

30. $= (\ _1\ ^{-3\ 5\ -12\ 5} + \ _2\ ^{-7\ 5\ -13\ 5})(\ )^3.$

Solution in parametric form:

$$= aR^{5\ 2}\big(2Q^2\ - 4R \qquad + B_2R^2\ \ ^{-1}, \qquad = \ \big(2Q^2\ - 4R \qquad + B_2R^2\ \ ^{-1},$$

where $A_1 = -\frac{5}{32}a^{8\ 5\ 2\ 5}B_1^{-2}B_2$, $A_2 = \frac{5}{32}a^{12\ 5\ 3\ 5}B_1^{-2}$.

In the solutions of equations 31 and 32, the following notation is used:

$$_1 = \begin{cases} \ _1e^{\ \tau} + \ _2e^{-\ \tau} - \dfrac{B_1}{B_2}\tau & \textit{if}\ B_2 > 0,\\ \ _1\sin(k\tau) + \ _2\cos(k\tau) - \dfrac{B_1}{B_2}\tau & \textit{if}\ B_2 < 0, \end{cases}$$

$$_2 = \begin{cases} k(\ _1e^{\ \tau} - \ _2e^{-\ \tau}) - \dfrac{B_1}{B_2} & \textit{if}\ B_2 > 0,\\ k[\ _1\cos(k\tau) - \ _2\sin(k\tau)] - \dfrac{B_1}{B_2} & \textit{if}\ B_2 < 0, \end{cases}$$

where $k = \sqrt{\frac{1}{2}|B_2|}$.

31. $= (\ _1 + \ _2\ ^{-1\ 2})(\ )^3.$

Solution in parametric form:

$$= \ _2^2, \qquad = \ _1, \qquad \text{where}\quad A_1 = -B_2, \quad A_2 = -B_1.$$

32. $= (\;{}_1\;{}^{-1\;2\;-5\;2} + \;{}_2\;{}^{-3})(\;)^3.$

Solution in parametric form:

$$= \;{}_1^{-1}\;{}_2^{2}, \qquad = \;{}_1^{-1}, \qquad \text{where} \quad A_1 = -B_1, \quad A_2 = -B_2.$$

In the solutions of equations 33–36, the following notation is used:

For $B_1 > 0$,

$$_1 = \;{}_1 e^{\;\tau} + \;{}_2 e^{-\;\tau} + \;{}_3 \sin(k\tau), \quad k = \left(\tfrac{4}{3}B_1\;{}^{1\;4}\right.,$$
$$_2 = k(\;{}_1 e^{\;\tau} - \;{}_2 e^{-\;\tau}) + k\;{}_3 \cos(k\tau).$$

For $B_1 < 0$,

$$_1 = e^{\;\tau}[\;{}_1 \sin(\;\tau) + \;{}_2 \cos(\;\tau)] + \;{}_3 e^{-\;\tau} \sin(\;\tau), \qquad = \left(-\tfrac{1}{3}B_1\;{}^{1\;4}\right.,$$
$$_2 = \;e^{\;\tau}[(\;{}_1 - \;{}_2)\sin(\;\tau) + (\;{}_1 + \;{}_2)\cos(\;\tau)] - \;{}_3 e^{-\;\tau}[\sin(\;\tau) - \cos(\;\tau)].$$

33. $= (\;{}_1\;{}^{-5\;3\;2} + \;{}_2\;{}^{-5\;3}\;)(\;)^3.$

Solution in parametric form:

$$= \;{}_2^{3\;2}, \qquad = \;{}_1 - \frac{A_2}{2A_1},$$

where $B_1 = -A_1$, $B_2 = -A_2$; the constants $_1$, $_2$, and $_3$ are related by the constraint

$$_1\;{}_3 = \tfrac{1}{16}A_1^{-2}A_2^2 \qquad \text{if } A_1 > 0,$$
$$4\;{}_1\;{}_2 + \;{}_3^2 = \tfrac{1}{4}A_1^{-2}A_2^2 \quad \text{if } A_1 < 0.$$

34. $= (\;{}_1\;{}^{-5\;3\;-7\;3} + \;{}_2\;{}^{-5\;3\;-10\;3})(\;)^3.$

Solution in parametric form:

$$= \;{}_1 - \frac{A_1}{2A_2}\;{}^{-1}\;{}_2^{3\;2}, \qquad = \;{}_1 - \frac{A_1}{2A_2}\;{}^{-1},$$

where $B_1 = -A_2$, $B_2 = -A_1$; the constants $_1$, $_2$, and $_3$ are related by the constraint

$$_1\;{}_3 = \tfrac{1}{16}A_1^2A_2^{-2} \qquad \text{if } A_2 > 0,$$
$$4\;{}_1\;{}_2 + \;{}_3^2 = \tfrac{1}{4}A_1^2A_2^{-2} \quad \text{if } A_2 < 0.$$

35. $= (\;{}_1\;{}^{-5\;3\;2} + \;{}_2\;{}^{-5\;3})(\;)^3.$

Solution in parametric form:

$$= \;{}_2^{3\;2}, \qquad = \;{}_1,$$

where $B_1 = -A_1$, $B_2 = -A_2$; the constants $_1$, $_2$, and $_3$ are related by the constraint

$$_1\;{}_3 = -\tfrac{1}{4}A_1^{-1}A_2 \qquad \text{if } A_1 > 0,$$
$$4\;{}_1\;{}_2 + \;{}_3^2 = -\tfrac{1}{2}A_1^{-1}A_2 \quad \text{if } A_1 < 0.$$

36. $= (\;{}_1\;{}^{-5\;3\;-4\;3} + \;{}_2\;{}^{-5\;3\;-10\;3})(\;)^3.$

Solution in parametric form:

$$= \;{}_1^{-1}\;{}_2^{3\;2}, \qquad = \;{}_1^{-1},$$

where $B_1 = -A_2$, $B_2 = -A_1$; the constants $_1$, $_2$, and $_3$ are related by the constraint

$$_1\;{}_3 = -\tfrac{1}{4}A_1A_2^{-1} \qquad \text{if } A_2 > 0,$$
$$4\;{}_1\;{}_2 + \;{}_3^2 = -\tfrac{1}{2}A_1A_2^{-1} \quad \text{if } A_2 < 0.$$

In the solutions of equations 37 and 38, the following notation is used:

1 . *For* $B_1 > 0,\ B_2 \neq 0$:

$$ {}_1 = \ {}_1 e^{\ \tau} + \ {}_2 e^{-\ \tau} + \ {}_3 \sin \omega\tau, \qquad {}_2 = k(\ {}_1 e^{\ \tau} - \ {}_2 e^{-\ \tau}) + \omega\ \ {}_3 \cos \omega\tau, $$

where

$$ k = \ \tfrac{2}{3}[(B_2^2 + 3B_1)^{1\ 2} + B_2]^{\ 1\ 2}, \quad \omega = \ \tfrac{2}{3}[(B_2^2 + 3B_1)^{1\ 2} - B_2]^{\ 1\ 2}, \quad 4k^2\ \ {}_1\ \ {}_2 + \omega^2\ \ {}_3^2 = 0. $$

2 . *For* $-B_2^2 < 3B_1 < 0,\ B_2 > 0$:

$$ {}_1 = \ {}_1\tau^{\ 1} + \ {}_2\tau^{-\ 1} + \ {}_3\tau^{\ 2} + \ {}_4\tau^{-\ 2}, \qquad {}_2 = k_1(\ {}_1\tau^{\ 1} - \ {}_2\tau^{-\ 1}) + k_2(\ {}_3\tau^{\ 2} - \ {}_4\tau^{-\ 2}), $$

where

$$ k_1 = \ \tfrac{2}{3}[B_2 + (B_2^2 + 3B_1)^{1\ 2}]^{\ 1\ 2}, \quad k_2 = \ \tfrac{2}{3}[B_2 - (B_2^2 + 3B_1)^{1\ 2}]^{\ 1\ 2}, $$
$$ (\ {}_1\ \ {}_2 + \ {}_3\ \ {}_4)(B_2^2 + 3B_1)^{1\ 2} + (\ {}_1\ \ {}_2 - \ {}_3\ \ {}_4)B_2 = 0. $$

3 . *For* $-B_2^2 < 3B_1 < 0,\ B_2 < 0$:

$$ {}_1 = \ {}_1 \sin \omega_1\tau + \ {}_2 \cos \omega_1\tau + \ {}_3 \sin \omega_2\tau, \qquad {}_2 = \omega_1(\ {}_1 \cos \omega_1\tau - \ {}_2 \sin \omega_1\tau) + \omega_2\ \ {}_3 \cos \omega_2\tau, $$

where

$$ \omega_1 = \ -\tfrac{2}{3}[B_2 + (B_2^2 + 3B_1)^{1\ 2}]^{\ 1\ 2}, \ \omega_2 = \ -\tfrac{2}{3}[B_2 - (B_2^2 + 3B_1)^{1\ 2}]^{\ 1\ 2}, \ \omega_1^2(\ {}_1^2 + \ {}_2^2) - \omega_2^2\ \ {}_3^2 = 0. $$

4 . *For* $B_2^2 + 3B_1 = 0,\ B_2 > 0$:

$$ {}_1 = (\ {}_1 + \ {}_2\tau)e^{\ \tau} + (\ {}_3 + \ {}_4\tau)e^{-\ \tau}, \qquad {}_2 = (k\ \ {}_1 + \ {}_2 + k\ \ {}_2\tau)e^{\ \tau} - (k\ \ {}_3 - \ {}_4 + k\ \ {}_4\tau)e^{-\ \tau}, $$

where

$$ k = \big(\tfrac{2}{3}B_2\ \ ^{1\ 2}, \qquad (\ {}_1\ \ {}_4 - \ {}_2\ \ {}_3)\big(\tfrac{2}{3}B_2\ \ ^{1\ 2} + 2\ \ {}_2\ \ {}_4 = 0. $$

5 . *For* $B_2^2 + 3B_1 = 0,\ B_2 < 0$:

$$ {}_1 = (\ {}_1 + \ {}_2\tau) \sin \omega\tau + \ {}_3\tau \cos \omega\tau, \qquad {}_2 = (\omega\ \ {}_1 + \ {}_3 + \omega\ \ {}_2\tau) \cos \omega\tau + (\ {}_2 - \omega\ \ {}_3\tau) \sin \omega\tau, $$

where

$$ \omega = \big(-\tfrac{2}{3}B_2\ \ ^{1\ 2}, \qquad {}_1\ \ {}_3\big(-\tfrac{2}{3}B_2\ \ ^{1\ 2} + \ {}_2^2 + \ {}_3^2 = 0. $$

6 . *For* $3B_1 < -B_2^2$:

$$ {}_1 = e^{\ \tau}(\ {}_1 \sin \omega\tau + \ {}_2 \cos \omega\tau) + \ {}_3 e^{-\ \tau} \sin \omega\tau, $$
$$ {}_2 = e^{\ \tau}[(k\ \ {}_2 + \omega\ \ {}_1) \cos \omega\tau + (k\ \ {}_1 - \omega\ \ {}_2) \sin \omega\tau)] + \ {}_3 e^{-\ \tau}(\omega \cos \omega\tau - k \sin \omega\tau), $$

where

$$ k = \ \tfrac{1}{3}[B_2 + (-3B_1)^{1\ 2}]^{\ 1\ 2}, \quad \omega = \ \tfrac{1}{3}[-B_2 + (-3B_1)^{1\ 2}]^{\ 1\ 2}, \qquad {}_2 B_2 + \ {}_1(-B_2^2 - 3B_1)^{1\ 2} = 0. $$

37. $\qquad = (\ {}_1\ \ ^{-5\ 3\ \ 2} + \ {}_2\ \ ^{-1\ 3})(\quad)^3.$

Solution in parametric form:

$$ = \ {}_2^{3\ 2}, \qquad = \ {}_1, \qquad \text{where} \quad B_1 = -A_1, \quad B_2 = -A_2. $$

38. $\qquad = (\ {}_1\ \ ^{-1\ 3\ \ -8\ 3} + \ {}_2\ \ ^{-5\ 3\ \ -10\ 3})(\quad)^3.$

Solution in parametric form:

$$ = \ {}_1^{-1}\ \ {}_2^{3\ 2}, \qquad = \ {}_1^{-1}, \qquad \text{where} \quad B_1 = -A_2, \quad B_2 = -A_1. $$

In the solutions of equations 39–42, the following notation is used:

$$
{}_1 = \begin{cases} {}_1 e^{\ \tau} + \ {}_2 e^{-\ \tau} + \ {}_3 \tau & \text{if } B > 0, \\ {}_1 \sin \omega\tau + \ {}_2 \cos \omega\tau + \ {}_3 \tau & \text{if } B < 0, \end{cases}
$$

$$
{}_2 = \begin{cases} \omega(\ {}_1 e^{\ \tau} - \ {}_2 e^{-\ \tau}) + \ {}_3 & \text{if } B > 0, \\ \omega(\ {}_1 \cos \omega\tau - \ {}_2 \sin \omega\tau) + \ {}_3 & \text{if } B < 0, \end{cases}
$$

where $\omega = |\frac{4}{3}B|^{1\ 2}$.

39. $= (\ {}_1 \ ^{-1\ 3} + \ {}_2 \ ^{-5\ 3})(\ )^3.$

Solution in parametric form:

$$
= \ {}_2^{3\ 2}, \qquad = \ {}_1,
$$

where $B = -A_1$; the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint

$$
\begin{aligned} &3(A_1 \ {}_3^2 + A_2) - 4A_1^2(\ {}_1^2 + \ {}_2^2) = 0 && \text{if } A_1 > 0, \\ &3(A_1 \ {}_3^2 + A_2) - 16A_1^2 \ {}_1 \ {}_2 = 0 && \text{if } A_1 < 0. \end{aligned}
$$

40. $= (\ {}_1 \ ^{-5\ 3} \ ^{-4\ 3} + \ {}_2 \ ^{-1\ 3} \ ^{-8\ 3})(\ )^3.$

Solution in parametric form:

$$
= \ {}_1^{-1} \ {}_2^{3\ 2}, \qquad = \ {}_1^{-1},
$$

where $B = -A_2$; the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint

$$
\begin{aligned} &3(A_1 + A_2 \ {}_3^2) - 4A_2^2(\ {}_1^2 + \ {}_2^2) = 0 && \text{if } A_2 > 0, \\ &3(A_1 + A_2 \ {}_3^2) - 16A_2^2 \ {}_1 \ {}_2 = 0 && \text{if } A_2 < 0. \end{aligned}
$$

41. $= (\ {}_1 \ ^{-5\ 3} \ + \ {}_2 \ ^{-1\ 3})(\ )^3.$

Solution in parametric form:

$$
= \ {}_2 - \frac{A_1}{2A_2}\tau \ ^{3\ 2}, \qquad = \ {}_1 - \frac{A_1}{4A_2}\tau^2,
$$

where $B = -A_2$; the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint

$$
\begin{aligned} &3A_2 \ {}_3^2 - 4A_2^2(\ {}_1^2 + \ {}_2^2) - \tfrac{9}{16}A_1^2 A_2^{-2} = 0 && \text{if } A_2 > 0, \\ &3A_2 \ {}_3^2 - 16A_2^2 \ {}_1 \ {}_2 - \tfrac{9}{16}A_1^2 A_2^{-2} = 0 && \text{if } A_2 < 0. \end{aligned}
$$

42. $= (\ {}_1 \ ^{-5\ 3} \ ^{-7\ 3} + \ {}_2 \ ^{-1\ 3} \ ^{-8\ 3})(\ )^3.$

Solution in parametric form:

$$
= \ {}_1 - \frac{A_1}{4A_2}\tau^2 \ ^{-1} \ {}_2 - \frac{A_1}{2A_2}\tau \ ^{3\ 2}, \qquad = \ {}_1 - \frac{A_1}{4A_2}\tau^2 \ ^{-1},
$$

where $B = -A_2$; the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint

$$
\begin{aligned} &3A_2 \ {}_3^2 - 4A_2^2(\ {}_1^2 + \ {}_2^2) - \tfrac{9}{16}A_1^2 A_2^{-2} = 0 && \text{if } A_2 > 0, \\ &3A_2 \ {}_3^2 - 16A_2^2 \ {}_1 \ {}_2 - \tfrac{9}{16}A_1^2 A_2^{-2} = 0 && \text{if } A_2 < 0. \end{aligned}
$$

In the solutions of equations 43–48, the following notation is used:

$$= \begin{cases} {}_{1\ \ 3}(\tau) & \textit{for the upper sign (Bessel function),} \\ {}_{1\ \ 3}(\tau) & \textit{for the lower sign (modified Bessel function),} \end{cases}$$

$$g = \begin{cases} {}_{1\ \ 3}(\tau) & \textit{for the upper sign (Bessel function),} \\ {}_{1\ \ 3}(\tau) & \textit{for the lower sign (modified Bessel function),} \end{cases}$$

$$= {}_{1} \;+\; {}_{2}g + \beta\omega \;\; g \qquad \tau - \qquad g \;\; \tau \;\;, \qquad \omega = \begin{cases} \frac{1}{2} & \textit{for the upper sign,} \\ -1 & \textit{for the lower sign.} \end{cases}$$

43. $= (\;{}_{1}\quad + \;{}_{2})(\quad)^{3}.$

Solution in parametric form:

$$= \tau^{1\ \ 3} \quad , \qquad = \;\tau^{2\ \ 3}, \qquad \text{where} \quad A_1 = \;\tfrac{9}{4}\;{}^{-3}, \quad A_2 = -\tfrac{9}{4}\;{}^{-2}\beta.$$

44. $= (\;{}_{1}\;{}^{-3} + \;{}_{2}\quad{}^{-5})(\quad)^{3}.$

Solution in parametric form:

$$= \tau^{-1\ \ 3} \quad , \qquad = \;\tau^{-2\ \ 3}, \qquad \text{where} \quad A_1 = -\tfrac{9}{4}\;\beta, \quad A_2 = \;\tfrac{9}{4}\;{}^{3}.$$

45. $= (\;{}_{1}\;{}^{-1\ \ 2}\;{}^{-1\ \ 2} + \;{}_{2}\;{}^{-3\ \ 2})(\quad)^{3}.$

Solution in parametric form:

$$= a\tau^{-2\ \ 3}(\tau \;\;{}'_{\tau} + \tfrac{1}{3}\quad)^{2}, \qquad = \;\tau^{2\ \ 3}\quad{}^{2}, \qquad \text{where} \quad A_1 = \;\tfrac{1}{3}a^{3\ \ 2}\;{}^{-3\ \ 2}, \quad A_2 = \tfrac{1}{2}a\;{}^{-1\ \ 2}\beta.$$

46. $= (\;{}_{1}\;{}^{-3\ \ 2} + \;{}_{2}\;{}^{-1\ \ 2}\;{}^{-2})(\quad)^{3}.$

Solution in parametric form:

$$= a\tau^{-4\ \ 3}\quad{}^{-2}(\tau \;\;{}'_{\tau} + \tfrac{1}{3}\quad)^{2}, \qquad = \;\tau^{-2\ \ 3}\quad{}^{-2}, \qquad \text{where} \quad A_1 = \tfrac{1}{2}a\;{}^{-1\ \ 2}\beta, \quad A_2 = \;\tfrac{1}{3}a^{3\ \ 2}.$$

47. $= (\;{}_{1}\;{}^{-2}\quad + \;{}_{2}\;{}^{-3\ \ 2})(\quad)^{3}.$

Solution in parametric form:

$$= a\tau^{2\ \ 3}\quad{}^{2}, \qquad = \;\tau^{-2\ \ 3}\big[\mp\tau^{2}\quad{}^{2} + 2\beta\tau \quad - (\tau \;\;{}'_{\tau} + \tfrac{1}{3}\quad)^{2}\big],$$

where $A_1 = -\frac{9}{2}a^{3}\;{}^{-3}$, $A_2 = -a^{-1\ \ 2}\;\beta A_1$.

48. $= (\;{}_{1}\;{}^{-3\ \ 2}\;{}^{-3\ \ 2} + \;{}_{2}\;{}^{-2}\;{}^{-2})(\quad)^{3}.$

Solution in parametric form:

$$= a\tau^{4\ \ 3}\quad{}^{2}\big[\mp\tau^{2}\quad{}^{2} + 2\beta\tau \quad - (\tau \;\;{}'_{\tau} + \tfrac{1}{3}\quad)^{2}\big]^{-1},$$
$$= \;\tau^{2\ \ 3}\big[\mp\tau^{2}\quad{}^{2} + 2\beta\tau \quad - (\tau \;\;{}'_{\tau} + \tfrac{1}{3}\quad)^{2}\big]^{-1},$$

where $A_1 = \frac{9}{2}a^{5\ \ 2}\;{}^{-1\ \ 2}\beta$, $A_2 = -\frac{9}{2}a^{3}$.

49. $= (\ _1 \quad ^1 + \ _2 \quad ^2)(\quad)^3, \qquad _1 \neq -2.$

Solution in parametric form:

$$= \tau \quad , \qquad = \ \tau^2 \ , \qquad = \frac{1}{_1 + 2},$$

where

$$= \ _1 \ + \ _2 g + \frac{4 \ ^2 \beta\omega}{(\ _1 + 2)^2} \ g \quad \tau \qquad \tau - \qquad \tau \ g \ \tau \ , \quad k = \frac{2 \ \ _2 - \ \ _1 + 1}{_1 + 2};$$

$$= \begin{cases} (\tau) & \text{for the upper sign (Bessel function),} \\ (\tau) & \text{for the lower sign (modified Bessel function),} \end{cases}$$

$$g = \begin{cases} (\tau) & \text{for the upper sign (Bessel function),} \\ (\tau) & \text{for the lower sign (modified Bessel function),} \end{cases}$$

$$A_1 = \ \tfrac{1}{4}(\ _1 + 2)^2 \ - \ \ ^{1-2}, \quad A_2 = - \ - \ \ ^2\beta, \quad \omega = \begin{cases} \frac{1}{2} & \text{for the upper sign,} \\ -1 & \text{for the lower sign.} \end{cases}$$

In the solutions of equations 50–56, the following notation is used:

$$= \begin{cases} _1 \ (\tau) & \textit{for the upper sign (Bessel function),} \\ _1 \ (\tau) & \textit{for the lower sign (modified Bessel function),} \end{cases}$$

$$= \begin{cases} _2 \ (\tau) & \textit{for the upper sign (Bessel function),} \\ _2 \quad (\tau) & \textit{for the lower sign (modified Bessel function),} \end{cases}$$

$$Z \ = \ _1 \ + \ _2 \ , \ X \ = \beta_1 \ \ + \beta_2 \ \ , \qquad = \tau Z' + \ Z \ , \qquad = \tau X' + \ X \ ,$$

$$N = \begin{cases} Z \ X & \textit{if } \Delta = -(\ _1\beta_2 - \ \ _2\beta_1)^2, \\ \ ^2 + \beta \qquad + \quad ^2 & \textit{if } \Delta = 4 \qquad - \beta^2, \end{cases}$$

$$N_1 = \begin{cases} Z \qquad + X & \textit{if } \Delta = -(\ _1\beta_2 - \ \ _2\beta_1)^2, \\ \tau N' + 2 \ \ N & \textit{if } \Delta = 4 \qquad - \beta^2, \end{cases}$$

$$N_2 = N_1^2 \quad 4\tau^2 N^2 + \omega^2 \Delta, \qquad \omega = \begin{cases} 2 & \textit{for the upper sign,} \\ -1 & \textit{for the lower sign,} \end{cases}$$

where the prime denotes differentiation with respect to τ.

50. $= (\ _1 \quad + \ \ _2 \ ^{-3})(\quad)^3.$

Solution in parametric form:

$$= a\tau^{1 \ \ 3} N^{1 \ \ 2}, \qquad = \ \tau^{2 \ \ 3},$$

where $\ = \frac{1}{3}, \ A_1 = \ \frac{9}{4} \ ^{-3}, \ A_2 = -\frac{9}{16} a^{4 \ \ -2} \omega^2 \Delta.$

51. $= (\ _1 \qquad + \ \ _2 \ ^{-3})(\quad)^3, \qquad \neq -2.$

Solution in parametric form:

$$= a\tau \ \ N^{1 \ \ 2}, \qquad = \ \tau^2 \ ,$$

where $\ = \dfrac{1}{\ + 2}, \ A_1 = \ \frac{1}{4} \ - \ \ ^{-2}(\quad + 2)^2, \ A_2 = -\frac{1}{16} a^{4 \ \ -2} \omega^2 \Delta(\quad + 2)^2.$

52. $= (\ _1 \ ^{-3} + \ \ _2 \quad ^{-5})(\quad)^3.$

Solution in parametric form:

$$= a\tau^{-1 \ \ 3} N^{1 \ \ 2}, \qquad = \ \tau^{-2 \ \ 3},$$

where $\ = \frac{1}{3}, \ A_1 = -\frac{9}{16} a^{4 \ \ -2} \omega^2 \Delta, \ A_2 = \ \frac{9}{4} \ \ ^3.$

53. $= (\ _1\ ^{-1\ 2}\ ^{-1\ 2} +\ _2\ ^{-3})(\ )^3$.

Solution in parametric form:

$$= a\tau^{-2\ 3}N^{-1}N_1^2, \qquad = \ \tau^{2\ 3}N,$$

where $= \frac{1}{3},\ A_1 = \ \frac{8}{3}a^3\ ^2\ ^{-3}\ ^2,\ A_2 = 2a\ \omega^2\Delta.$

54. $= (\ _1 +\ _2\ ^{-1\ 2}\ ^{-2})(\ )^3$.

Solution in parametric form:

$$= a\tau^{-4\ 3}N^{-2}N_1^2, \qquad = \ \tau^{-2\ 3}N^{-1},$$

where $= \frac{1}{3},\ A_1 = 2a\ ^{-2}\omega^2\Delta,\ A_2 = \ \frac{8}{3}a^3\ ^2.$

55. $= (\ _1\ ^{-2} +\ _2\ ^{-3})(\ )^3$.

Solution in parametric form:

$$= a\tau^{2\ 3}N, \qquad = \ \tau^{-2\ 3}N^{-1}N_2,$$

where $= \frac{1}{3},\ A_1 = \frac{9}{128}a^3\ ^{-3},\ A_2 = -\frac{9}{64}a^4\ ^{-2}\omega^2\Delta.$

56. $= (\ _1\ ^{-3} +\ _2\ ^{-2}\ ^{-2})(\ )^3$.

Solution in parametric form:

$$= a\tau^{4\ 3}N^2N_2^{-1}, \qquad = \ \tau^{2\ 3}NN_2^{-1},$$

where $= \frac{1}{3},\ A_1 = -\frac{9}{64}a^4\ ^{-2}\omega^2\Delta,\ A_2 = \frac{9}{128}a^3.$

In the solutions of equations 57–72, the following notation is used:

$$_1 = \ \overline{\ 4\wp_1^3 - 2\wp_1 - \ _2}, \qquad \tau = \int \frac{\wp_1}{\overline{\overline{4\wp_1^3 - 2\wp_1 - \ _2}}} - \ _1;$$

$$_2 = \ \overline{\ 4\wp_2^3 + 2\wp_2 - \ _2}, \qquad \tau = \int \frac{\wp_2}{\overline{\overline{4\wp_2^3 + 2\wp_2 - \ _2}}} - \ _1.$$

The functions $\wp_1 = \wp_1(\tau)$ and $\wp_2 = \wp_2(\tau)$ are defined implicitly by the above elliptic integrals. For the upper signs, they are the classical elliptic Weierstrass functions $\wp_1 = \wp(\tau + \ _1, 2, \ _2)$ and $\wp_2 = \wp(\tau + \ _1, -2, \ _2)$.

57. $= (\ _1\ ^2 +\ _2)(\ )^3$.

Solution in parametric form:

$$= a\wp\ , \qquad = \ \tau,$$

where $A_1 = \mp 6a^{-1}\ ^{-2},\ A_2 = a\ ^{-2}(-1)\ ^{+1};\ k = 1$ and $k = 2$.

58. $= (\ _1\ ^{-3} +\ _2\ ^2\ ^{-5})(\ )^3$.

Solution in parametric form:

$$= a\tau^{-1}\wp\ , \qquad = \ \tau^{-1},$$

where $A_1 = a\ (-1)\ ^{+1},\ A_2 = \mp 6a^{-1}\ ^3;\ k = 1$ and $k = 2$.

59. $= (\ _1\ ^{-9\ 7} + \ _2\ ^2\ ^{-15\ 7})(\ \ )^3$**.**

Solution in parametric form:

$$= a\tau(\tau^2\wp\ \mp 1), \qquad = \ \tau^7,$$

where $A_1 = \frac{1}{49}a\ ^{-5\ 7}(-1)\ ^{+1}$, $A_2 = \mp\frac{6}{49}a^{-1\ \ 1\ 7}$; $k = 1$ and $k = 2$.

60. $= (\ _1\ ^{-12\ 7} + \ _2\ ^2\ ^{-20\ 7})(\ \ )^3$**.**

Solution in parametric form:

$$= a\tau^{-6}(\tau^2\wp\ \mp 1), \qquad = \ \tau^{-7},$$

where $A_1 = \frac{1}{49}a\ ^{-2\ 7}(-1)\ ^{+1}$, $A_2 = \mp\frac{6}{49}a^{-1\ \ 6\ 7}$; $k = 1$ and $k = 2$.

61. $= (\ _1\ + \ _2\ )(\ \ )^{5\ 2}$**.**

Solution in parametric form:

$$= a[\ \ -(-1)\ \tau], \qquad = \ \tau,$$

where $A_1 = a\ ^{-1}A_2(-1)\ $, $A_2 = -2a^{-1\ \ -1}(\ 6a\ \ )^{1\ 2}$; $k = 1$ and $k = 2$.

62. $= (\ _1\ + \ _2\ )(\ \ )^{1\ 2}$**.**

Solution in parametric form:

$$= a\tau, \qquad = \ [\ \ -(-1)\ \tau],$$

where $A_1 = 2a^{-1\ \ -1}(\ 6\ \ a)^{1\ 2}$, $A_2 = a^{-1}\ \ A_1(-1)\ $; $k = 1$ and $k = 2$.

63. $= (\ _1\ ^{-3\ 4} + \ _2\ \ ^{-5\ 4})(\ \ )^{5\ 2}$**.**

Solution in parametric form:

$$= a[2\tau\ \ \ -2\wp\ \ +(-1)\ ^{+1}\tau^2], \qquad = \ \tau^4,$$

where $A_1 = a\ ^{-1\ \ 2}A_2(-1)\ $, $A_2 = -\frac{1}{4}a^{-1\ \ 1\ 4}(\ 3a\ \ \ )^{1\ 2}$; $k = 1$ and $k = 2$.

64. $= (\ _1\ ^{-5\ 4}\ + \ _2\ ^{-3\ 4})(\ \ )^{1\ 2}$**.**

Solution in parametric form:

$$= a\tau^4, \qquad = \ [2\tau\ \ \ -2\wp\ \ +(-1)\ ^{+1}\tau^2],$$

where $A_1 = \frac{1}{4}a^{1\ \ 4\ \ -1}(\ 3\ \ \ a)^{1\ 2}$, $A_2 = a^{-1\ 2}\ \ A_1(-1)\ $; $k = 1$ and $k = 2$.

65. $= (\ _1\ ^{-13\ 8} + \ _2\ \ ^{-15\ 8})(\ \ )^{5\ 2}$**.**

Solution in parametric form:

$$= a\tau^{-6}[2\tau^3\ \ \ +6\tau^2\wp\ \ \mp 2 + (-1)\ \ \tau^4], \qquad = \ \tau^{-8},$$

where $A_1 = a\ ^{-1\ \ 4}A_2(-1)\ $, $A_2 = \frac{1}{16}a^{-1\ \ 7\ 8}\big(\mp 6a\ \ \ \ ^{1\ 2}$; $k = 1$ and $k = 2$.

66. $= (\ _1\ ^{-15\ 8}\ + \ _2\ ^{-13\ 8})(\ \ )^{1\ 2}$**.**

Solution in parametric form:

$$= a\tau^{-8}, \qquad = \ \tau^{-6}[2\tau^3\ \ \ +6\tau^2\wp\ \ \mp 2 + (-1)\ \ \tau^4],$$

where $A_1 = -\frac{1}{16}a^{7\ \ 8\ \ -1}\big(\mp 6\ \ \ a\ \ \ ^{1\ 2}$, $A_2 = a^{-1\ \ 4}\ \ A_1(-1)\ $; $k = 1$ and $k = 2$.

67. $= (\quad _1 \quad ^{-15 \quad 13} + \quad _2 \quad ^{-20 \quad 13})(\quad)^{5 \quad 2}$.

Solution in parametric form:

$$= a\tau[5\tau^3 \quad - 20\tau^2\wp \quad 30 - (-1) \quad \tau^4], \qquad = \quad \tau^{13},$$

where $A_1 = a \quad ^{-5 \quad 13} A_2(-1) \quad , \; A_2 = -\frac{2}{65} a^{-1 \quad 7 \quad 13} \quad \frac{30a}{13} \quad ^{1 \quad 2}$; $k = 1$ and $k = 2$.

68. $= (\quad _1 \quad ^{-20 \quad 13} \quad + \quad _2 \quad ^{-15 \quad 13})(\quad)^{1 \quad 2}$.

Solution in parametric form:

$$= a\tau^{13}, \qquad = \quad \tau[5\tau^3 \quad - 20\tau^2\wp \quad 30 - (-1) \quad \tau^4],$$

where $A_1 = \frac{2}{65} a^{7 \quad 13 \quad -1} \quad \frac{30}{13a} \quad ^{1 \quad 2}$, $A_2 = a^{-5 \quad 13} \quad A_1(-1) \quad$; $k = 1$ and $k = 2$.

69. $= (\quad _1 + \quad _2 \quad ^{-2 \quad 3})(\quad)^3$.

Solution in parametric form:

$$= a\wp^3, \qquad = \quad [\quad - (-1) \quad \tau],$$

where $A_1 = \mp\frac{1}{2} a \quad ^{-2}$, $A_2 = \frac{1}{12} a^{5 \quad 3 \quad -2}(-1) \quad ^{+1}$; $k = 1$ and $k = 2$.

70. $= (\quad _1 \quad ^{-2 \quad 3 \quad -7 \quad 3} + \quad _2 \quad ^{-3})(\quad)^3$.

Solution in parametric form:

$$= a\wp^3 \left[\quad - (-1) \quad \tau \right]^{-1}, \qquad = \quad \left[\quad - (-1) \quad \tau \right]^{-1},$$

where $A_1 = \frac{1}{12} a^{5 \quad 3 \quad 1 \quad 3}(-1) \quad ^{+1}$, $A_2 = \mp\frac{1}{2} a \quad$; $k = 1$ and $k = 2$.

71. $= (\quad _1 \quad ^{-1 \quad 4} + \quad _2 \quad ^{-7 \quad 4})(\quad)^{1 \quad 2}$.

Solution in parametric form:

$$= a\left(\quad ^2 - \tau^2 \mp 4\wp^3 \quad \right), \qquad = \quad \left[\quad - (-1) \quad \tau \right]^{4 \quad 3},$$

where $A_1 = a \quad ^{-3 \quad 2} A_2, \quad A_2 = \begin{cases} a^{-3 \quad 11 \quad 4}(\mp a \quad)^{1 \quad 2} & \text{if } k = 1, \\ a^{-3 \quad 11 \quad 4}(\quad a \quad)^{1 \quad 2} & \text{if } k = 2. \end{cases}$

72. $= (\quad _1 \quad ^{-7 \quad 4} \quad + \quad _2 \quad ^{-1 \quad 4})(\quad)^{5 \quad 2}$.

Solution in parametric form:

$$= a[\quad - (-1) \quad \tau]^{4 \quad 3}, \qquad = \quad (\quad ^2 - \tau^2 \mp 4\wp^3),$$

where $A_1 = \begin{cases} \mp a^{11 \quad 4 \quad -3}(\mp \quad a)^{1 \quad 2} & \text{if } k = 1, \\ \mp a^{11 \quad 4 \quad -3}(\quad a)^{1 \quad 2} & \text{if } k = 2, \end{cases} \quad A_2 = a^{-3 \quad 2} \quad A_1.$

In the solutions of equations 73–92, the following notation is used:

The functions $_1$ and $_2$ are the general solutions of the four modifications of the first Painlevé equation:

$$ ''_1 = \ 6 \ ^2_1 + \tau, \qquad ''_2 = \ 6 \ ^2_2 - \tau $$

(in the case of the upper sign, the equation for $_1$ is the canonical form of the first Painlevé equation; see Paragraph 2.8.2-2). In addition,

$$
\begin{aligned}
Q_1 &= \ 6 \ ^2_1 + \tau, & Q_2 &= \ 6 \ ^2_2 - \tau,\\
R_1 &= 2 \ '_1 - \tau^2, & R_2 &= 2 \ '_2 + \tau^2,\\
_1 &= 3\tau \ '_1 - 3 \ _1 - \tau^3, & _2 &= 3\tau \ '_2 - 3 \ _2 + \tau^3,\\
_1 &= \tau^2 \ _1 \mp 1, & _2 &= \tau^2 \ _2 \mp 1,\\
_1 &= (\ '_1)^2 - 2 \ _1 Q_1 \ \ 8 \ ^3_1, & _2 &= (\ '_2)^2 - 2 \ _2 Q_2 \ \ 8 \ ^3_2,\\
_1 &= \ '_1 Q'_1 + \ '_1 - Q^2_1, & _2 &= \ '_2 Q'_2 - \ '_2 - Q^2_2,\\
_1 &= \tau^3 \ '_1 + 3\tau^2 \ _1 \mp 1 + \tau^5, & _2 &= \tau^3 \ '_2 + 3\tau^2 \ _2 \mp 1 - \tau^5,\\
Z_1 &= 6(\tau^3 \ '_1 - 4\tau^2 \ _1 \ \ 6) - \tau^5, & Z_2 &= 6(\tau^3 \ '_2 - 4\tau^2 \ _2 \ \ 6) + \tau^5,
\end{aligned}
$$

where the prime denotes differentiation with respect to τ.

73. $= (\ _1 \ + \ _2 \ ^2)(\ )^3.$

Solution in parametric form:

$$ = a \ , \qquad = \ \tau, $$

where $A_1 = a \ ^{-3}(-1) \ , \ A_2 = \mp 6a^{-1} \ ^{-2}$; $k = 1$ and $k = 2$.

74. $= (\ _1 \ ^2 + \ _2 \ )(\ )^{5 \ 2}.$

Solution in parametric form:

$$ = aR \ , \qquad = \ \tau, $$

where $A_1 = a \ ^{-2} A_2(-1) \ ^{+1}$, $A_2 = -2a^{-1} \ ^{-1}(\ 3a \ )^{1 \ 2}$; $k = 1$ and $k = 2$.

75. $= (\ _1 \ + \ _2 \ ^2)(\ )^{1 \ 2}.$

Solution in parametric form:

$$ = a\tau, \qquad = \ R \ , $$

where $A_1 = 2a^{-1} \ ^{-1}(\ 3 \ \ a)^{1 \ 2}$, $A_2 = a^{-2} \ A_1(-1) \ ^{+1}$; $k = 1$ and $k = 2$.

76. $= (\ _1 \ ^{-4} + \ _2 \ ^2 \ ^{-5})(\ )^3.$

Solution in parametric form:

$$ = a\tau^{-1} \ , \qquad = \ \tau^{-1}, $$

where $A_1 = a \ ^2(-1) \ , \ A_2 = \mp 6a^{-1} \ ^3$; $k = 1$ and $k = 2$.

77. $= (\ _1 \ ^{-1 \ 2} + \ _2 \ \ ^{-5 \ 4})(\ )^{5 \ 2}.$

Solution in parametric form:

$$ = a \ , \qquad = \ \tau^4, $$

where $A_1 = a \ ^{-3 \ 4} A_2(-1) \ ^{+1}$, $A_2 = -\frac{1}{4}a^{-1} \ ^{-1 \ 4}(\ 2a \ )^{1 \ 2}$; $k = 1$ and $k = 2$.

78. $= (\ _1 \ ^{-5 \ 4} \ + \ _2 \ ^{-1 \ 2})(\ )^{1 \ 2}.$

Solution in parametric form:

$$ = a\tau^4, \qquad = \ \ , $$

where $A_1 = \frac{1}{4}a^{-1 \ 4} \ ^{-1}(\ 2 \ \ a)^{1 \ 2}$, $A_2 = a^{-3 \ 4} \ A_1(-1) \ ^{+1}$; $k = 1$ and $k = 2$.

79. $= (\ _1\ ^{-8\ 7} + \ _2\ ^{2\ -15\ 7})(\ )^3$**.**

Solution in parametric form:

$$= a\tau\ ,\qquad = \ \tau^7,$$

where $A_1 = \frac{1}{49}a\ ^{-6\ 7}(-1)\ ,\ A_2 = \mp\frac{6}{49}a^{-1\ \ 1\ 7}$; $k = 1$ and $k = 2$.

80. $= (\ _1\ ^{-7\ 4} + \ _2\ ^{-15\ 8})(\ )^{5\ 2}$**.**

Solution in parametric form:

$$= a\tau^{-6}\ ,\qquad = \ \tau^{-8},$$

where $A_1 = a\ ^{-1\ 8}A_2(-1)\ ,\ A_2 = \frac{1}{8}a^{-1\ 7\ 8}(\mp 3a\ )^{1\ 2}$; $k = 1$ and $k = 2$.

81. $= (\ _1\ ^{-15\ 8} + \ _2\ ^{-7\ 4})(\ )^{1\ 2}$**.**

Solution in parametric form:

$$= a\tau^{-8},\qquad = \ \tau^{-6}\ ,$$

where $A_1 = -\frac{1}{8}a^{7\ 8\ -1}(\mp 3\ \ a)^{1\ 2}$, $A_2 = a^{-1\ 8}\ A_1(-1)\ $; $k = 1$ and $k = 2$.

82. $= (\ _1\ ^{-13\ 7} + \ _2\ ^{2\ -20\ 7})(\ )^3$**.**

Solution in parametric form:

$$= a\tau^{-6}\ ,\qquad = \ \tau^{-7},$$

where $A_1 = \frac{1}{49}a\ ^{-1\ 7}(-1)\ ,\ A_2 = \mp\frac{6}{49}a^{-1\ \ 6\ 7}$; $k = 1$ and $k = 2$.

83. $= (\ _1\ ^{-14\ 13} + \ _2\ ^{-20\ 13})(\ )^{5\ 2}$**.**

Solution in parametric form:

$$= a\tau Z\ ,\qquad = \ \tau^{13},$$

where $A_1 = a\ ^{-6\ 13}A_2(-1)\ ^{+1}$, $A_2 = -\dfrac{2}{13}a^{-1\ 7\ 13}\ \ \dfrac{a}{13}\ \ ^{1\ 2}$; $k = 1$ and $k = 2$.

84. $= (\ _1\ ^{-20\ 13} + \ _2\ ^{-14\ 13})(\ )^{1\ 2}$**.**

Solution in parametric form:

$$= a\tau^{13},\qquad = \ \tau Z\ ,$$

where $A_1 = \dfrac{2}{13}a^{7\ 13\ -1}\ \ \dfrac{}{13a}\ \ ^{1\ 2}$, $A_2 = a^{-6\ 13}\ A_1(-1)\ ^{+1}$; $k = 1$ and $k = 2$.

85. $= (\ _1 + \ _2\ ^{-1\ 2})(\ )^3$**.**

Solution in parametric form:

$$= a(\ ')^2,\qquad = \ \ ,$$

where $A_1 = \mp 24a\ ^{-3}$, $A_2 = 2a^{3\ 2\ -2}(-1)\ $; $k = 1$ and $k = 2$.

86. $= (\ _1\ ^{-1\ 2\ -5\ 2} + \ _2\ ^{-4})(\ )^3$**.**

Solution in parametric form:

$$= a\ ^{-1}(\ ')^2,\qquad = \ \ ^{-1},$$

where $A_1 = 2a^{3\ 2\ 1\ 2}(-1)\ ,\ A_2 = \mp 24a\ ^2$; $k = 1$ and $k = 2$.

87. $= (\ _{1}\ ^{-5\ 3} + \ _{2}\ ^{1\ 3})(\ )^{3}$.

Solution in parametric form:

$$= a\ ^{3\ 2}, \qquad = \quad ,$$

where $A_1 = \frac{3}{16}a^{8\ 3\ -3}$, $A_2 = \mp 8a^{-2}\ A_1$; $k = 1$ and $k = 2$.

88. $= (\ _{1}\ ^{-5\ 3\ -7\ 3} + \ _{2}\ ^{1\ 3\ -10\ 3})(\ )^{3}$.

Solution in parametric form:

$$= a\ ^{3\ 2\ -1}, \qquad = \ ^{-1},$$

where $A_1 = \frac{3}{16}a^{8\ 3\ 1\ 3}$, $A_2 = \mp 8a^{-2}\ A_1$; $k = 1$ and $k = 2$.

89. $= (\ _{1}\ ^{-1\ 2} + \ _{2}\ ^{-3\ 2})(\ )^{3}$.

Solution in parametric form:

$$= aQ^2, \qquad = (\ ')^2,$$

where $A_1 = \mp 6a^{3\ 2\ -2}$, $A_2 = \frac{1}{2}a\ ^{-1\ 2}(-1)\ ^{+1}$; $k = 1$ and $k = 2$.

90. $= (\ _{1}\ ^{-3\ 2} + \ _{2}\ ^{-1\ 2\ -5\ 2})(\ )^{3}$.

Solution in parametric form:

$$= a(\ ')^{-2}Q^2, \qquad = (\ ')^{-2},$$

where $A_1 = \frac{1}{2}a\ ^{-1\ 2}(-1)\ ^{+1}$, $A_2 = \mp 6a^{3\ 2\ 1\ 2}$; $k = 1$ and $k = 2$.

91. $= (\ _{1}\ ^{-5\ 3} + \ _{2}\ ^{-4\ 3})(\ )^{3}$.

Solution in parametric form:

$$= a(\ ')^3, \qquad = \quad ,$$

where $A_1 = -\frac{1}{36}a^{8\ 3\ -3}$, $A_2 = a^{-1\ 3}\ A_1(-1)\ $; $k = 1$ and $k = 2$.

92. $= (\ _{1}\ ^{-4\ 3\ -5\ 3} + \ _{2}\ ^{-5\ 3\ -7\ 3})(\ )^{3}$.

Solution in parametric form:

$$= a(\ ')^{3\ -1}, \qquad = \ ^{-1},$$

where $A_1 = \frac{1}{36}a^{7\ 3\ -1\ 3}(-1)\ ^{+1}$, $A_2 = -\frac{1}{36}a^{8\ 3\ 1\ 3}$; $k = 1$ and $k = 2$.

In the solutions of equations 93–96, the following notation is used:

$$= \ _2 \exp \quad \frac{\tau}{\overline{R}}\ , \qquad = \tau + 2\ \overline{R} + 4B_2, \qquad = \quad _1 - \frac{1}{2A_1A_2}\tau^{-4} - \tau^{2}\ ^{-1\ 2}\ \tau + \ _2,$$

$$R = \begin{cases} \ _1 + \frac{1}{4}\tau^2 + \frac{2B_1}{k_1+1}\tau\ ^{_1+1} + \frac{2B_2}{k_2+1}\tau\ ^{_2+1} & \text{if } k_1 \neq -1,\ k_2 \neq -1; \\ \ _1 + \frac{1}{4}\tau^2 + \frac{2B_1}{k+1}\tau\ ^{+1} + 2B_2 \ln|\tau| & \text{if } k = k_1 \neq -1,\ k_2 = -1. \end{cases}$$

93. $= (\ _{1}\ ^{-2\ \ _1-3\ \ _1} + \ _{2}\ ^{-2\ \ _2-3\ \ _2})(\ )^{3}$.

Solution in parametric form:

$$= \tau\ ^{1\ 2}, \qquad = \quad ,$$

where $k_1 = -2\ \ _1 - 3$, $k_2 = -2\ \ _2 - 3$, $A_1 = -B_1$, $A_2 = -B_2$.

94. $= \ _1 \quad + \ _2 \ \frac{-1}{2} \ (\ )^{\frac{+5}{+3}}$.

Solution in parametric form:

$$= a \quad ^{-1 \ 2} \quad , \qquad = \quad ^{\frac{1}{+1}},$$

where $k_1 = k = -\frac{+3}{+1}$, $k_2 = 0$, $A_1 = \frac{k \quad ^{+1}}{(k+1)a} - \frac{4aB_1}{(k+1)} \quad ^{1}$, $A_2 = -4a \ ^{-\frac{1}{+1}} A_1 B_2$.

95. $= \ _1 \quad + \ _2 \ \frac{-1}{2} \ (\ )^{\frac{2 \ +4}{+3}}$.

Solution in parametric form:

$$= a \ ^{\frac{1}{+1}}, \qquad = \quad ^{-1 \ 2} \quad ,$$

where $k_1 = k = -\frac{+3}{+1}$, $k_2 = 0$, $A_1 = -\frac{k}{k+1} a^{\frac{2}{+1}} \ ^{-1} - \frac{4 \ B_1}{(k+1)a} \quad ^{1}$, $A_2 = -4a^{-\frac{1}{+1}} \ A_1 B_2$.

96. $= (\ _1 \ ^{-5 \ 2} + \ _2 \ ^{-5})(\ )^3$.

Solution in parametric form:

$$= \frac{\tau \ \overline{-A_2}}{\cos}, \qquad = \sqrt{\frac{A_2}{A_1}} \tan \quad .$$

97. $= (\ _1 \quad ^1 \quad ^1 + \ _2 \quad ^2 \quad ^2)(\ )^{3 \ 2}$.

Solution in parametric form:

$$= \ _1 \tau^{1 \ 2} \exp \ -\frac{1}{2} \quad \frac{\tau}{\tau \quad \overline{\quad ^2 + 4}} \quad , \qquad = \ _1^{-1} \tau^{1 \ 2} \exp \ \frac{1}{2} \quad \frac{\tau}{\tau \quad \overline{\quad ^2 + 4}} \quad ,$$

where

$$= \begin{cases} \tau^{-1 \ 2} \quad _2 + \frac{A_1}{2(\ _1 + 1)} \tau \quad ^{_1+1} + \frac{A_2}{2(\ _2 + 1)} \tau \quad ^{_2+1} & \text{if} \quad _1 \neq -1, \quad _2 \neq -1, \\ \tau^{-1 \ 2} \quad _2 + \frac{A_1}{2(\ _1 + 1)} \tau \quad ^{_1+1} + \frac{1}{2} A_2 \ln \tau & \text{if} \quad _1 \neq -1, \quad _2 = -1. \end{cases}$$

98. $= \ _1 \quad ^1 + \ _2 \ \frac{_2(\ +1) - \ _1 +}{_1 + 1} \quad ^2 \ (\ )^{\frac{_1 + 2 \ +3}{_1 + \ +2}}$.

Solution in parametric form:

$$= \ _1 \exp \quad \frac{\tau}{\tau z} \quad , \qquad = \ _1^{-\frac{+1}{_1+1}} \tau \exp \ -\frac{+1}{_1 + 1} \quad \frac{\tau}{\tau z} \quad ,$$

where $z = z(\tau)$ is the solution of the algebraic equation

$$z - \frac{_1 + \quad + 2}{_1 + 1} \quad z - \frac{+1}{_1 + 1} \ ^{\frac{+1}{_1 + \ +2}} = \tau^{-\frac{_1 + 1}{_1 + \ +2}} \quad _2 + \frac{A_1}{_1 + \quad + 2} \tau \quad ^{_1+1} + \qquad ,$$

$$= \begin{cases} \frac{A_2(\ _1 + 1)}{(\ _1 + \quad + 2)(\ _2 + 1)} \tau \quad ^{_2+1} & \text{if} \quad _2 \neq -1, \\ \frac{A_2}{_1 + \quad + 2} \ln |\tau| & \text{if} \quad _2 = -1. \end{cases}$$

In the solutions of equations 99–108, the following notation is used:

The functions $\ _1$ *and* $\ _2$ *are the general solutions of the four modifications of the second Painlevé equation (with parameter* $a = 0$*):*

$$''_1 = \tau\ _1 \quad 2\ ^3_1, \qquad ''_2 = -\tau\ _2 \quad 2\ ^3_2.$$

In the case of the upper sign, the equation for $\ _1$ *is the canonical form of the second Painlevé equation (with parameter* $a = 0$*; see Paragraph 2.8.2-3);*

$$Q_1 = \tau\ ^2_1 \quad ^4_1 - (\ '_1)^2, \quad R_1 = \ '_1 \mp \ _1 Q_1, \qquad _1 = 2\ '_1 Q_1 - \ ^3_1 \mp \ _1 Q^2_1,$$
$$Q_2 = \tau\ ^2_2 \quad ^4_2 - (\ '_2)^2, \quad R_2 = \ '_2 \quad _2 Q_2, \qquad _2 = 2\ '_2 Q_2 + \ ^3_2 \quad _2 Q^2_2;$$

where the prime denotes differentiation with respect to τ*.*

99. $\boldsymbol{= (\ _1 \quad + \ _2\ ^3)(\quad)^3.}$

Solution in parametric form:

$$= a \quad , \qquad = \ \tau,$$

where $A_1 = \ ^3(-1)$, $A_2 = \mp 2a^{-2}\ ^{-2}$; $k = 1$ and $k = 2$.

100. $\boldsymbol{= (\ _1 \quad ^{-5} + \ _2\ ^3\ ^{-6})(\quad)^3.}$

Solution in parametric form:

$$= a\tau^{-1} \quad , \qquad = \ \tau^{-1},$$

where $A_1 = \ ^3(-1)$, $A_2 = \mp 2a^{-2}\ ^4$; $k = 1$ and $k = 2$.

101. $\boldsymbol{= (\ _1 + \ _2\ ^{-1}\ ^2\ ^{-1}\ ^2)(\quad)^3.}$

Solutions in parametric form:

$$= a(\ ')^2, \qquad = \ ^2, \qquad ' = (\quad)'_\tau,$$

where $A_1 = \mp 2a\ ^{-2}$, $A_2 = \frac{1}{2}a^3\ ^2\ ^{-3}\ ^2(-1)$; $k = 1$ and $k = 2$.

102. $\boldsymbol{= (\ _1\ ^{-1}\ ^2\ ^{-2} + \ _2\ ^{-3})(\quad)^3.}$

Solutions in parametric form:

$$= a \ ^{-2}(\ ')^2, \qquad = \ ^{-2}, \qquad ' = (\quad)'_\tau,$$

where $A_1 = \frac{1}{2}a^3\ ^2(-1)$, $A_2 = \mp 2a$; $k = 1$ and $k = 2$.

103. $\boldsymbol{= (\ _1\ ^{-2} \quad + \ _2)(\quad)^3.}$

Solutions in parametric form:

$$= a\ ^2, \qquad = \big[\tau\ ^2 \quad ^4 - (\ ')^2\big], \qquad ' = (\quad)'_\tau,$$

where $A_1 = 2a^3\ ^{-3}(-1)$, $A_2 = \ 2a\ ^{-2}(-1)$; $k = 1$ and $k = 2$.

104. $\boldsymbol{= (\ _1\ ^{-2}\ ^{-2} + \ _2\ ^{-3})(\quad)^3.}$

Solutions in parametric form:

$$= a\ ^2\big[\tau\ ^2 \quad ^4 - (\ ')^2\big]^{-1}, \qquad = \big[\tau\ ^2 \quad ^4 - (\ ')^2\big]^{-1},$$

where $A_1 = -2a^3$, $A_2 = \mp 2a$; $k = 1$ and $k = 2$.

105. $= (\ _1 + \ _2 \quad ^{-1\ 2})(\quad)^{3\ 2}$**.**

Solutions in parametric form:

$$= a \quad ^{-1} R \ , \qquad = \quad Q^2,$$

where $A_1 = \mp a \quad ^{-1\ 2} A_2(-1)$, $A_2 = \begin{cases} 2a^{-2\ 1\ 2}(2a \quad)^{1\ 2} & \text{if } k = 1, \\ -2a^{-2\ 1\ 2}(-2a \quad)^{1\ 2} & \text{if } k = 2. \end{cases}$

106. $= (\ _1 \quad ^{-1\ 2} + \quad _2)(\quad)^{3\ 2}$**.**

Solutions in parametric form:

$$= aQ^2, \qquad = \quad ^{-1} R \ ,$$

where $A_1 = \begin{cases} -2a^{1\ 2\ -2}(2 \quad a)^{1\ 2} & \text{if } k = 1, \\ 2a^{1\ 2\ -2}(-2 \quad a)^{1\ 2} & \text{if } k = 2, \end{cases}$ $A_2 = \mp a^{-1\ 2} \quad A_1(-1)$.

107. $= (\ _1 + \quad _2 \quad ^{-2})(\quad)^{3\ 2}$**.**

Solutions in parametric form:

$$= a \quad , \qquad = \quad Q \ ,$$

where $A_1 = \mp a \quad ^{-2} A_2(-1)$, $A_2 = \begin{cases} a^{-2\ 2}(2a \quad)^{1\ 2} & \text{if } k = 1, \\ a^{-2\ 2}(-2a \quad)^{1\ 2} & \text{if } k = 2. \end{cases}$

108. $= (\ _1 \quad ^{-2} + \quad _2)(\quad)^{3\ 2}$**.**

Solutions in parametric form:

$$= aQ \ , \qquad = \qquad ,$$

where $A_1 = \begin{cases} -a^{2\ -2}(2 \quad a)^{1\ 2} & \text{if } k = 1, \\ -a^{2\ -2}(-2 \quad a)^{1\ 2} & \text{if } k = 2, \end{cases}$ $A_2 = \mp a^{-2} \quad A_1(-1)$.

109. $= (\ _1 \quad ^{-7\ 5\ -8\ 5} + \quad _2 \quad ^{-7\ 5\ -13\ 5})(\quad)^3$**.**

Solution in parametric form:

$$= a \ _1^5 \ ^{5\ 2} \quad _1^4 \ - \frac{A_1}{A_2} \quad ^{-1}, \qquad = \quad _1^4 \ - \frac{A_1}{A_2} \quad ^{-1},$$

where $= \ _1 e^{2\ \tau} + \ _2 e^{-\ \tau} \sin\left(\ 3\, k\tau \ \right), \quad = (\ '_\tau)^2 - 2 \quad ''_{\tau\tau}, \ A_2 = \frac{5}{1024} a^{12\ 5\ -3} k^{-6}$.

110. $= (\ _1 \quad ^{-2\ -1} + \quad _2 \quad ^{-2\ -2})(\quad)^3$**.**

Solution in parametric form:

$$= a \quad _1 \tau^{2\ 3} Z^2 \quad _1 \tau^{-2\ 3} \left[(\tau Z'_\tau + \tfrac{1}{3} Z)^2 \quad \tau^2 Z^2 \right] - \frac{A_1}{A_2} \quad ^{-1},$$

$$= \quad _1 \tau^{-2\ 3} \left[(\tau Z'_\tau + \tfrac{1}{3} Z)^2 \quad \tau^2 Z^2 \right] - \frac{A_1}{A_2} \quad ^{-1},$$

where

$$A_2 = \frac{9}{2} a^{3\ -3}, \qquad Z = \begin{cases} \ _1 \ _{1\ 3}(\tau) + \ _2 \ _{1\ 3}(\tau) & \text{for the upper sign,} \\ \ _1 \ _{1\ 3}(\tau) + \ _2 \ _{1\ 3}(\tau) & \text{for the lower sign,} \end{cases}$$

$_{1\ 3}(\tau)$ and $_{1\ 3}(\tau)$ are the Bessel functions, and $_{1\ 3}(\tau)$ and $_{1\ 3}(\tau)$ are the modified Bessel functions.

111. $= (\ _1 \ ^{-7} \ ^4 + \ _2 \ ^{-7} \ ^3)(\ )^3.$

Solution in parametric form:

$$= a \ _1^3 \ ^{1 \ 2} \ _1^8 \ - \frac{A_1}{A_2} \ ^{-1}, \qquad = \ _1^8 \ - \frac{A_1}{A_2} \ ^{-1},$$

where $R = \overline{\ (4\tau^3 - 1)}$, $\quad = 2\tau \ \ \tau R^{-1} \ \tau + \ _2\tau \mp R$, $\quad = 4\tau \ ^2 \mp \tau^{-2}(R \ - 1)^2$, $A_2 = \mp\frac{3}{64}a^8 \ ^{-3}$.

In the solutions of equations 112 and 113, the following notation is used:

$$= \ (1 \ \ \tau^4)^{-1 \ 2} \ \tau + \ _2, \qquad k^2 = \ 1;$$

the function   can be expressed in terms of elliptic integrals or lemniscate functions.

112. $= (\ _1 \ ^2 \ ^{-14} \ ^5 + \ _2 \ ^3 \ ^{-18} \ ^5)(\ )^3.$

Solutions in parametric form:

$$= a \ _1^4 \ ^{-4}(\tau \ - k), \qquad = \ _1^5 \ ^{-5}, \qquad \text{where} \quad A_1 = \mp\tfrac{6}{25}a^{-1} \ ^4 \ ^5 k, \quad A_2 = \mp\tfrac{2}{25}a^{-2} \ ^8 \ ^5.$$

113. $= (\ _1 \ ^2 \ ^{-11} \ ^5 + \ _2 \ ^3 \ ^{-12} \ ^5)(\ )^3.$

Solutions in parametric form:

$$= a \ _1 \ (\tau \ - k), \qquad = \ _1^5 \ ^5, \qquad \text{where} \quad A_1 = \mp\tfrac{6}{25}a^{-1} \ ^1 \ ^5 k, \quad A_2 = \mp\tfrac{2}{25}a^{-2} \ ^2 \ ^5.$$

In the solutions of equations 114 and 115, the following notation is used:

$$\Delta = \ _2^2 - 2 \ _1, \quad R = (36\Delta + 54B\tau - 2\tau^3)^{1 \ 2}, \quad z = 3 \ \frac{\tau}{\tau R},$$

$$(z) = \begin{cases} \dfrac{\overline{-\Delta}}{_1}\tan\big(\ \overline{-\Delta}\, z \ + \dfrac{_2}{_1} & \textit{if } \Delta < 0; \\ \dfrac{\overline{\Delta}}{_1}\tanh\big(\mp \ \overline{\Delta}\, z \ + \dfrac{_2}{_1} & \textit{if } \Delta > 0; \\ \mp\dfrac{1}{_1 z} - \dfrac{\overline{2}}{\overline{| \ _1|}} & \textit{if } \Delta = 0, \ _2 < 0; \\ \mp\dfrac{1}{_1 z} + \dfrac{\overline{2}}{\overline{| \ _1|}} & \textit{if } \Delta = 0, \ _2 > 0. \end{cases}$$

114. $= (\ _1 \ ^{-5} \ ^3 + \ _2 \ ^{-5} \ ^3 \ ^{-2} \ ^3)(\ )^3.$

Solutions in parametric form:

$$= a\tau^{-9 \ 4}(\ _1 \ ^2 - 2 \ _2 \ + 2)^{3 \ 4}(6 \ _1 \ - 6 \ _2 \mp R)^{3 \ 2},$$
$$= \ \tau^{-3 \ 2}(\ _1 \ ^2 - 2 \ _2 \ + 2)^{3 \ 2},$$

where $A_1 = -24a^{8 \ 3 \ -2} \ _1$, $A_2 = 36a^{8 \ 3 \ -4 \ 3}B$.

115. $= (\ _1 \ ^{-5} \ ^3 \ ^{-2} \ ^3 + \ _2 \ ^{-5} \ ^3 \ ^{-4} \ ^3)(\ )^3.$

Solutions in parametric form:

$$= a\tau^{-3 \ 4}(\ _1 \ ^2 - 2 \ _2 \ + 2)^{-3 \ 4}(6 \ _1 \ - 6 \ _2 \mp R)^{3 \ 2},$$
$$= \ \tau^{3 \ 2}(\ _1 \ ^2 - 2 \ _2 \ + 2)^{-3 \ 2},$$

where $A_1 = 36a^{8 \ 3 \ -4 \ 3}B$, $A_2 = -24a^{8 \ 3 \ -2 \ 3} \ _1$.

116. $= \dfrac{2(\ \ +1)}{(\ \ +3)^2} + \ {}^{-2}(\ \ )^3, \quad \neq -3, \quad \neq -1.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.4 with respect to $=(\ )$: $'' = {}^{-2}\left[-\dfrac{2(\ \ +1)}{(\ \ +3)^2} - A\ \right].$

117. $= (-\frac{15}{4} + \ {}^{-7})\ {}^{-2}(\ \ )^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.35 with respect to $=(\ )$: $'' = {}^{-2}(\frac{15}{4} - A\ {}^{-7}).$

118. $= (-6 + \ {}^{-4})\ {}^{-2}(\ \ )^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.31 with respect to $=(\ )$: $'' = {}^{-2}(6 - A\ {}^{-4}).$

119. $= (-12 + \ {}^{-5}\ {}^{2})\ {}^{-2}(\ \ )^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.64 with respect to $=(\ )$: $'' = {}^{-2}(12 - A\ {}^{-5}\ {}^{2}).$

120. $= (-2 + \ {}^{-2})\ {}^{-2}(\ \ )^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.6 with respect to $=(\ )$: $'' = {}^{-2}(2 - A\ {}^{-2}).$

121. $= (\frac{3}{16} + \ {}^{-5}\ {}^{3})\ {}^{-2}(\ \ )^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.26 with respect to $=(\ )$: $'' = {}^{-2}(-\frac{3}{16} - A\ {}^{-5}\ {}^{3}).$

122. $= (\frac{9}{100} + \ {}^{-5}\ {}^{3})\ {}^{-2}(\ \ )^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.10 with respect to $=(\ )$: $'' = {}^{-2}(-\frac{9}{100} - A\ {}^{-5}\ {}^{3}).$

123. $= (-\frac{3}{4} + \ {}^{-5}\ {}^{3})\ {}^{-2}(\ \ )^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.12 with respect to $=(\ )$: $'' = {}^{-2}(\frac{3}{4} - A\ {}^{-5}\ {}^{3}).$

124. $= (-\frac{63}{4} + \ {}^{-5}\ {}^{3})\ {}^{-2}(\ \ )^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.66 with respect to $=(\ )$: $'' = {}^{-2}(\frac{63}{4} - A\ {}^{-5}\ {}^{3}).$

125. $= (\frac{5}{36} + \ {}^{-7}\ {}^{5})\ {}^{-2}(\ \ )^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.29 with respect to $=(\ )$: $'' = {}^{-2}(-\frac{5}{36} - A\ {}^{-7}\ {}^{5}).$

126. $= (\frac{2}{9} + \ {}^{-1}\ {}^{2})\ {}^{-2}(\ \ )^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.14 with respect to $=(\ )$: $'' = {}^{-2}(-\frac{2}{9} - A\ {}^{-1}\ {}^{2}).$

127. $= (\frac{4}{25} + \ {}^{-1}\ {}^{2})\ {}^{-2}(\ \ )^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.8 with respect to $=(\ )$: $'' = {}^{-2}(-\frac{4}{25} - A\ {}^{-1}\ {}^{2}).$

128. $= (-20 \quad + \quad ^{-1\ 2})\ ^{-2}(\quad)^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.33 with respect to $=\ (\ )$: $'' \ = \ ^{-2}(20 \ - A \ ^{-1\ 2})$.

129. $= (\frac{12}{49} \quad + \quad ^{1\ 2})\ ^{-2}(\quad)^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.37 with respect to $=\ (\ )$: $'' \ = \ ^{-2}(-\frac{12}{49} \ - A \ ^{1\ 2})$.

130. $= (\quad ^2 + \frac{6}{25} \quad)\ ^{-2}(\quad)^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.60 with respect to $=\ (\ )$: $'' \ = \ ^{-2}(-A \ ^2 - \frac{6}{25} \ )$.

131. $= (\quad ^2 - \frac{6}{25} \quad)\ ^{-2}(\quad)^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.62 with respect to $=\ (\ )$: $'' \ = \ ^{-2}(-A \ ^2 + \frac{6}{25} \ )$.

132. $= \dfrac{2(\ \ +1)}{(\ \ +3)^2} \quad ^{-2} + \quad ^{-\ \ -1}\ (\quad)^3, \qquad \neq -3, -1.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.5 with respect to $=\ (\ )$: $'' \ = -\dfrac{2(\ \ +1)}{(\ \ +3)^2} \ ^{-2} \ - A \ ^{-\ \ -1} \quad .$

133. $= (\quad ^{-2} \ - 2 \quad ^{-2})(\quad)^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.7 with respect to $=\ (\ )$: $'' \ = 2 \ ^{-2} \ - A \quad ^{-2}$.

134. $= (\quad ^{-1\ 2} \ ^{-1\ 2} + \frac{4}{25} \quad ^{-2})(\quad)^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.9 with respect to $=\ (\ )$: $'' \ = -\frac{4}{25} \ ^{-2} \ - A \ ^{-1\ 2} \ ^{-1\ 2}$.

135. $= (\quad ^{-5\ 3} \ ^{2\ 3} + \frac{9}{100} \quad ^{-2})(\quad)^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.11 with respect to $=\ (\ )$: $'' \ = -\frac{9}{100} \ ^{-2} \ - A \ ^{2\ 3} \ ^{-5\ 3}$.

136. $= (\quad ^{-5\ 3} \ ^{2\ 3} - \frac{3}{4} \quad ^{-2})(\quad)^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.13 with respect to $=\ (\ )$: $'' \ = \frac{3}{4} \ ^{-2} \ - A \ ^{2\ 3} \ ^{-5\ 3}$.

137. $= (\quad ^{-1\ 2} \ ^{-1\ 2} + \frac{2}{9} \quad ^{-2})(\quad)^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.15 with respect to $=\ (\ )$: $'' \ = -\frac{2}{9} \ ^{-2} \ - A \ ^{-1\ 2} \ ^{-1\ 2}$.

138. $= (\quad ^{-5\ 3} \ ^{2\ 3} + \frac{3}{16} \quad ^{-2})(\quad)^3.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.27 with respect to $=\ (\ )$: $'' \ = -\frac{3}{16} \ ^{-2} \ - A \ ^{2\ 3} \ ^{-5\ 3}$.

139. $= (\quad ^{-7} \; ^{5} \quad ^{2} \; ^{5} + \frac{5}{36} \quad ^{-2})(\quad)^{3}.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.30 with respect to $=$ ( ): $'' = -\frac{5}{36} \quad ^{-2} \quad - A \quad ^{2} \; ^{5} \quad ^{-7} \; ^{5}.$

140. $= (\quad ^{-4} \quad ^{3} - 6 \quad ^{-2})(\quad)^{3}.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.32 with respect to $=$ ( ): $'' = 6 \quad ^{-2} \quad - A \quad ^{3} \quad ^{-4}.$

141. $= (\quad ^{-1} \; ^{2} \quad ^{-1} \; ^{2} - 20 \quad ^{-2})(\quad)^{3}.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.34 with respect to $=$ ( ): $'' = 20 \quad ^{-2} \quad - A \quad ^{-1} \; ^{2} \quad ^{-1} \; ^{2}.$

142. $= (\quad ^{-7} \quad ^{6} - \frac{15}{4} \quad ^{-2})(\quad)^{3}.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.36 with respect to $=$ ( ): $'' = \frac{15}{4} \quad ^{-2} \quad - A \quad ^{6} \quad ^{-7}.$

143. $= (\quad ^{1} \; ^{2} \quad ^{-3} \; ^{2} + \frac{12}{49} \quad ^{-2})(\quad)^{3}.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.38 with respect to $=$ ( ): $'' = -\frac{12}{49} \quad ^{-2} \quad - A \quad ^{-3} \; ^{2} \quad ^{1} \; ^{2}.$

144. $= (\frac{6}{25} \quad ^{-2} + \quad ^{2} \quad ^{-3})(\quad)^{3}.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.61 with respect to $=$ ( ): $'' = -A \quad ^{-3} \; ^{2} - \frac{6}{25} \quad ^{-2} \quad .$

145. $= (-\frac{6}{25} \quad ^{-2} + \quad ^{2} \quad ^{-3})(\quad)^{3}.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.63 with respect to $=$ ( ): $'' = -A \quad ^{-3} \; ^{2} + \frac{6}{25} \quad ^{-2} \quad .$

146. $= (\quad ^{-5} \; ^{2} \quad ^{3} \; ^{2} - 12 \quad ^{-2})(\quad)^{3}.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.65 with respect to $=$ ( ): $'' = 12 \quad ^{-2} \quad - A \quad ^{3} \; ^{2} \quad ^{-5} \; ^{2}.$

147. $= (\quad ^{-5} \; ^{3} \quad ^{2} \; ^{3} - \frac{63}{4} \quad ^{-2})(\quad)^{3}.$

Taking to be the independent variable, we obtain an equation of the form 2.4.2.67 with respect to $=$ ( ): $'' = \frac{63}{4} \quad ^{-2} \quad - A \quad ^{2} \; ^{3} \quad ^{-5} \; ^{3}.$

2.6.3. Equations of the Form $= \qquad (\quad) + \quad ^{-1} \quad ^{+1}(\quad)^{-1}$

2.6.3-1. Classification table.

Table 26 presents all solvable equations whose solutions are outlined in Subsection 2.6.3. Two-parameter families (in the space of the parameters , , and l), one-parameter families, and isolated points are presented in a consecutive fashion. Equations are arranged in accordance with the growth of l. The number of the equation sought is indicated in the last column in this table.

TABLE 26

Solvable cases of the equation $'' = \sigma A \quad (\,'\,) + A \quad ^{-1} \quad ^{+1}(\,'\,)^{-1}$

l			σ	Equation
arbitrary ($l \neq 2$)	arbitrary ($\neq -1$)	$- \quad - 1$	-1	2.6.3.75
arbitrary ($l \neq 2$)	$1 - l$	$l - 2$	-1	2.6.3.76
arbitrary ($l \neq 3$)	-2	1	-1	2.6.3.79
arbitrary ($l \neq 1$)	0	-1	-1	2.6.3.80
$\frac{+3}{+2}$	arbitrary ($\neq -1, -2$)	1	$+1$	2.6.3.74
$\frac{3 \quad +2}{+1}$	0	arbitrary ($\neq 0, -1$)	$\frac{1}{}$	2.6.3.73
1	arbitrary ($\neq -1$)	$- \quad - 2$	-1	2.6.3.1
1	0	arbitrary ($\neq -1$)	$\frac{1}{}$	2.6.3.23
1	1	arbitrary ($\neq 0, -2$)	$\frac{2}{}$	2.6.3.37
$\frac{3}{2}$	0	arbitrary ($\neq 0, -1$)	$\frac{1}{}$	2.6.3.41
2	arbitrary ($\neq -1$)	arbitrary ($\neq 0$)	-1	2.6.3.85
2	arbitrary ($\neq -1$)	$+1$	-1	2.6.3.82
2	arbitrary ($\neq -1$)	$-2 \quad - 2$	-1	2.6.3.83
2	arbitrary ($\neq -1$)	$-\frac{+1}{2}$	-1	2.6.3.84
2	arbitrary ($\neq -1$)	0	arbitrary	2.6.3.87
2	-1	arbitrary ($\neq 0$)	arbitrary	2.6.3.86
$\frac{5}{2}$	arbitrary ($\neq -1, -2$)	1	$+1$	2.6.3.42
3	arbitrary ($\neq -2$)	$- \quad - 2$	-1	2.6.3.3
3	arbitrary ($\neq -2$)	1	$+1$	2.6.3.24
3	arbitrary ($\neq -1, -3$)	2	$\frac{+1}{2}$	2.6.3.38
0	-3	-1	2	2.6.3.65
0	-3	$\frac{1}{2}$	-4	2.6.3.61
0	-3	2	-1	2.6.3.35
0	0	-2	-1	2.6.3.48

TABLE 26 (*Continued*)

Solvable cases of the equation $'' = \sigma A \quad (\,'\,) + A \quad {}^{-1} \quad {}^{+1}(\,'\,)^{-1}$

l			σ	Equation
0	0	−1	−2	2.6.3.50
0	0	−1	−1	2.6.3.33
0	0	$-\frac{2}{5}$	$-\frac{5}{2}$	2.6.3.59
0	0	2	$\frac{1}{2}$	2.6.3.63
1	−3	1	−1	2.6.3.15
1	−2	−2	$\frac{1}{2}$	2.6.3.51
1	−2	−1	arbitrary	2.6.3.71
1	−2	−1	−1	2.6.3.7
1	−2	−1	1	2.6.3.5
1	−2	$-\frac{1}{2}$	2	2.6.3.55
1	−2	$\frac{1}{2}$	−1	2.6.3.13
1	−2	1	arbitrary	2.6.3.69
1	−2	1	−2	2.6.3.11
1	−2	1	−1	2.6.3.29
1	−1	−1	−1	2.6.3.2
1	$-\frac{1}{2}$	−2	$-\frac{1}{4}$	2.6.3.53
1	$-\frac{1}{2}$	−1	−1	2.6.3.45
1	$-\frac{1}{2}$	$-\frac{1}{2}$	−1	2.6.3.31
1	$-\frac{1}{2}$	1	$\frac{1}{2}$	2.6.3.57
1	0	−1	−1	2.6.3.77
1	0	1	arbitrary	2.6.3.67
1	0	1	−1	2.6.3.9
1	1	−4	$-\frac{1}{2}$	2.6.3.21
1	1	−1	−2	2.6.3.39
1	1	−2	−1	2.6.3.25
1	1	1	1	2.6.3.17
$\frac{3}{2}$	0	−1	−1	2.6.3.27
$\frac{3}{2}$	0	$-\frac{1}{2}$	−2	2.6.3.43
$\frac{3}{2}$	0	1	1	2.6.3.19
2	−1	0	arbitrary	2.6.3.81
$\frac{5}{2}$	−2	1	−1	2.6.3.28
$\frac{5}{2}$	$-\frac{3}{2}$	1	$-\frac{1}{2}$	2.6.3.44
$\frac{5}{2}$	0	1	1	2.6.3.20
3	−5	2	−2	2.6.3.22
3	−3	−1	2	2.6.3.52

TABLE 26 (*Continued*)

Solvable cases of the equation $'' = \sigma A \quad (\,')\, + A \quad ^{-1} \quad ^{+1}(\,')^{-1}$

l			σ	Equation
3	–3	$\frac{1}{2}$	–4	2.6.3.54
3	–3	2	–1	2.6.3.26
3	–2	–1	arbitrary	2.6.3.72
3	–2	–1	–1	2.6.3.8
3	–2	–1	1	2.6.3.6
3	–2	0	–1	2.6.3.4
3	–2	$\frac{1}{2}$	–1	2.6.3.46
3	–2	1	–1	2.6.3.78
3	–2	2	$-\frac{1}{2}$	2.6.3.40
3	$-\frac{3}{2}$	–1	$\frac{1}{2}$	2.6.3.56
3	$-\frac{3}{2}$	$\frac{1}{2}$	–1	2.6.3.32
3	$-\frac{1}{2}$	–1	–1	2.6.3.14
3	0	–2	–1	2.6.3.16
3	0	–1	arbitrary	2.6.3.70
3	0	–1	–1	2.6.3.30
3	0	–1	$-\frac{1}{2}$	2.6.3.12
3	0	$\frac{1}{2}$	2	2.6.3.58
3	0	1	arbitrary	2.6.3.68
3	0	1	–1	2.6.3.10
3	0	2	1	2.6.3.18
4	–3	1	–1	2.6.3.47
4	–2	–2	$\frac{1}{2}$	2.6.3.66
4	–2	1	–1	2.6.3.34
4	–2	1	$-\frac{1}{2}$	2.6.3.49
4	$-\frac{1}{2}$	–2	$-\frac{1}{4}$	2.6.3.62
4	$-\frac{2}{5}$	1	$-\frac{2}{5}$	2.6.3.60
4	1	–2	–1	2.6.3.36
4	1	1	2	2.6.3.64

2.6.3-2. Solvable equations and their solutions.

1. $= \quad - \;^{-2} \quad - \quad - \;^{-3} \quad ^{+1}, \qquad \neq -1.$

Solution in parametric form:

$$= a \;\; _1 \left(\frac{\tau}{1 \quad \tau^{\;+1}} + \;\; _2\right)^{-1}, \qquad = \;\; _1^{\;+1}\tau \left(\frac{\tau}{1 \quad \tau^{\;+1}} + \;\; _2\right)^{-1},$$

where $A = \mp(\quad + 1)a^{\;+1\; -}$.

2. $= \quad {}^{-1}\ {}^{-1} \quad - \quad {}^{-2}.$

Solution in parametric form:

$$= {}_1 \quad \tau^{-1}\exp(\mp\tau^2)\ \ \tau + \ {}_2 \quad {}^{-1}, \qquad = -\frac{A}{2}\exp(\mp\tau^2) \quad \tau^{-1}\exp(\mp\tau^2)\ \ \tau + \ {}_2 \quad {}^{-1}.$$

3. $= \quad - \ {}^{-2} \quad (\quad)^3 - \quad - \ {}^{-3} \quad {}^{+1}(\quad)^2, \qquad \neq -2.$

Solution in parametric form:

$$= a \ {}_1 \ {}^{+2}\tau \quad \frac{\tau}{1 \quad \tau^{-} \quad {}^{-2}} + \ {}_2 \quad {}^{-1}, \qquad = \ {}_1 \ {}^{+3} \quad \frac{\tau}{1 \quad \tau^{-} \quad {}^{-2}} + \ {}_2 \quad {}^{-1},$$

where $A = \ (\quad + 2)a \ {}^{+3} \ - \ {}^{-2}.$

4. $= \quad {}^{-2}(\quad)^3 - \quad {}^{-1}\ {}^{-2}(\quad)^2.$

Solution in parametric form:

$$= -\frac{A}{2}\exp(\mp\tau^2) \quad \tau^{-1}\exp(\mp\tau^2)\ \ \tau + \ {}_2 \quad {}^{-1}, \qquad = {}_1 \quad \tau^{-1}\exp(\mp\tau^2)\ \ \tau + \ {}_2 \quad {}^{-1}.$$

In the solutions of equations 5–12, the following notation is used:

$$= \quad \exp(\mp\tau^2)\ \ \tau + \ {}_2.$$

5. $= \quad {}^{-1}\ {}^{-2} \quad + \quad {}^{-2}\ {}^{-1}.$

Solution in parametric form:

$$= {}_1\exp(\mp\tau^2) \quad {}^{-1}, \qquad = [2\tau \quad \exp(\mp\tau^2) \quad {}^{-1}], \qquad \text{where} \quad A = \ 2 \ {}^2.$$

6. $= \quad {}^{-1}\ {}^{-2}(\quad)^3 + \quad {}^{-2}\ {}^{-1}(\quad)^2.$

Solution in parametric form:

$$= a[2\tau \quad \exp(\mp\tau^2) \quad {}^{-1}], \qquad = {}_1\exp(\mp\tau^2) \quad {}^{-1}, \qquad \text{where} \quad A = \mp 2a^2.$$

7. $= \quad {}^{-1}\ {}^{-2} \quad - \quad {}^{-2}\ {}^{-1}.$

Solution in parametric form:

$$= {}_1\big[2\tau \quad \exp(\mp\tau^2)\big]^{-1}, \qquad = \big[2\tau \quad \exp(\mp\tau^2)\big]^{-1}, \qquad \text{where} \quad A = \tfrac{1}{2} \ {}^2.$$

8. $= \quad {}^{-1}\ {}^{-2}(\quad)^3 - \quad {}^{-2}\ {}^{-1}(\quad)^2.$

Solution in parametric form:

$$= a \ \big[2\tau \quad \exp(\mp\tau^2)\big]^{-1}, \qquad = {}_1\big[2\tau \quad \exp(\mp\tau^2)\big]^{-1}, \qquad \text{where} \quad A = \tfrac{1}{2}a^2.$$

9. $= \quad - \quad .$

Solution in parametric form:

$$= a\tau, \qquad = {}_1[2\tau \quad \exp(\mp\tau^2)], \qquad \text{where} \quad A = \mp 2a^{-2}.$$

10. $= \quad (\quad)^3 - \quad (\quad)^2.$

Solution in parametric form:

$$= {}_1[2\tau \quad \exp(\mp\tau^2)], \qquad = \tau, \qquad \text{where} \quad A = \mp 2 \ {}^{-2}.$$

11. $= 2 \quad {}^{-2} \quad - \quad {}^{-1}.$

Solution in parametric form:

$$= a \;{}_1[2\tau^2 \quad \tau\exp(\mp\tau^2) \quad], \qquad = \;{}_1[2\tau \quad \exp(\mp\tau^2)], \qquad \text{where} \quad A = \mp\tfrac{1}{2}a^{-2}\;{}^2.$$

12. $= \quad {}^{-1}(\quad)^3 - 2 \quad {}^{-2} \; (\quad)^2.$

Solution in parametric form:

$$= a \;{}_1[2\tau \quad \exp(\mp\tau^2)], \qquad = \;{}_1[2\tau^2 \quad \tau\exp(\mp\tau^2) \quad], \qquad \text{where} \quad A = \mp\tfrac{1}{2}a^2\;{}^{-2}.$$

In the solutions of equations 13–22, the following notation is used:

$$= \quad \overline{\tau(\tau+1)} - \ln\big(\quad\overline{\tau} + \quad\overline{\tau+1} \quad + \;{}_2, \qquad = \sqrt{\frac{\tau+1}{\tau}} - \tau.$$

13. $= \quad {}^{1\;2\;-2} \quad - \quad {}^{-1\;2\;-1}.$

Solution in parametric form:

$$= a \;{}^4_1 \quad {}^{-2}, \qquad = \;{}^3_1\tau^{-1} \quad {}^{-2}, \qquad \text{where} \quad A = -a^{-3}\;{}^2\;{}^2.$$

14. $= \quad {}^{-1\;-1\;2}(\quad)^3 - \quad {}^{-2\;1\;2}(\quad)^2.$

Solution in parametric form:

$$= a \;{}^3_1\tau^{-1} \quad {}^{-2}, \qquad = \;{}^4_1 \quad {}^{-2}, \qquad \text{where} \quad A = -a^2\;{}^{-3}\;{}^2.$$

15. $= \quad {}^{-3} \quad - \quad {}^{-2}.$

Solution in parametric form:

$$= a \;{}^3_1 \quad {}^{-1}\sqrt{\frac{\tau+1}{\tau}}, \qquad = \;{}^2_1 \quad {}^{-1}, \qquad \text{where} \quad A = -2a^{-2}\;{}^3.$$

16. $= \quad {}^{-2}(\quad)^3 - \quad {}^{-3} \; (\quad)^2.$

Solution in parametric form:

$$= a \;{}^2_1 \quad {}^{-1}, \qquad = \;{}^3_1 \quad {}^{-1}\sqrt{\frac{\tau+1}{\tau}}, \qquad \text{where} \quad A = -2a^3\;{}^{-2}.$$

17. $= \quad + \quad {}^2.$

Solution in parametric form:

$$= a \;{}^{-1}_1\tau^{-1} \quad {}^{-1}(\tau \quad {}^2 + \tau^2 \quad - \quad {}^2), \qquad = \;{}^2_1 \quad {}^{-1}, \qquad \text{where} \quad A = a^{-2}\;{}^{-1}.$$

18. $= \quad {}^2(\quad)^3 + \quad (\quad)^2.$

Solution in parametric form:

$$= a \;{}^2_1 \quad {}^{-1}, \qquad = \;{}^{-1}_1\tau^{-1} \quad {}^{-1}(\tau \quad {}^2 + \tau^2 \quad - \quad {}^2), \qquad \text{where} \quad A = -a^{-1}\;{}^{-2}.$$

19. $= \quad (\quad)^{3\;2} + \quad (\quad)^{1\;2}.$

Solution in parametric form:

$$= a \;{}^{-1}_1 \quad \sqrt{\frac{\tau+1}{\tau}} - \quad \tau^{-1}\Big), \qquad = \;{}^3_1 \quad {}^{-1}\sqrt{\frac{\tau+1}{\tau}}, \qquad \text{where} \quad A = 2a^{-2} \quad -\frac{a}{} \quad {}^{1\;2}.$$

20. $= \quad (\quad)^{5\;2} + \quad (\quad)^{3\;2}.$

Solution in parametric form:

$$= a \;{}^3_1 \quad {}^{-1}\sqrt{\frac{\tau+1}{\tau}}, \qquad = \;{}^{-1}_1 \quad \sqrt{\frac{\tau+1}{\tau}} - \quad \tau^{-1}\Big), \qquad \text{where} \quad A = -2 \quad {}^{-2} \quad -\frac{}{a} \quad {}^{1\;2}.$$

21. $= \quad {}^{-4} \quad - 2 \quad {}^{-5\;2}.$

Solution in parametric form:

$$= a \;{}_1\tau \quad \big(\tau \quad {}^2 + \tau^2 \quad - \quad {}^2 \; {}^{-1}, \qquad = \;{}^3_1\tau \quad {}^{-1}\big(\tau \quad {}^2 + \tau^2 \quad - \quad {}^2 \; {}^{-1},$$

where $A = -2a^3\;{}^{-1}$.

22. $= 2 \quad ^{2} \ ^{-5}(\quad)^{3} - \quad ^{-4}(\quad)^{2}.$

Solution in parametric form:

$$= a \ {}_{1}^{3}\tau \quad ^{-1}\left(\tau \quad ^{2} + \tau^{2} \quad - \quad ^{2} \ ^{-1}\right., \qquad = \quad {}_{1}\tau \quad \left(\tau \quad ^{2} + \tau^{2} \quad - \quad ^{2} \ ^{-1}\right.,$$

where $A = -2a^{-1} \ ^{3}$.

23. $= \qquad + \qquad ^{-1} \quad , \qquad \neq -1.$

Solution in parametric form:

$$= a\tau^{\frac{1}{\ +1}}, \qquad = \quad {}_{1}e^{\ \tau} \quad \tau^{-\frac{}{\ +1}} e^{-\ \tau} \ \tau + \quad {}_{2} \ , \qquad \text{where} \quad A = (\ +1)a^{- \ -1}\beta.$$

24. $= \qquad (\quad)^{3} + \frac{}{\ +1} \quad ^{+1}(\quad)^{2}, \qquad \neq -2.$

Solution in parametric form:

$$= \quad {}_{1}e^{\ \tau} \quad \tau^{-\frac{\ +1}{\ +2}} e^{-\ \tau} \ \tau + \quad {}_{2} \ , \qquad = \tau^{\frac{1}{\ +2}}, \qquad \text{where} \quad A = -\frac{\ +2}{\ +1} \ ^{- \ -2}\beta.$$

In the solutions of equations 25–36, the following notation is used:

$$R = \begin{cases} \tau \quad + \quad {}_{2}\tau^{-} & \textit{for the upper sign,} \\ \sin(\ \ln\tau) + \quad {}_{2}\cos(\ \ln\tau) & \textit{for the lower sign,} \\ \ln\tau + \quad {}_{2} & \textit{for} \quad = 0, \end{cases}$$

$$Q = \begin{cases} (1 + \ )\tau \quad + (1 - \ ) \ {}_{2}\tau^{-} & \textit{for the upper sign,} \\ (1 - \quad {}_{2})\sin(\ \ln\tau) + (\ {}_{2} + \ )\cos(\ \ln\tau) & \textit{for the lower sign,} \\ \ln\tau + 1 + \quad {}_{2} & \textit{for} \quad = 0. \end{cases}$$

25. $= \quad ^{-2} \quad - \quad ^{-3} \ ^{2}.$

Solution in parametric form:

$$= a\tau^{-2}, \qquad = \ \tau^{-2}R^{-1}Q, \qquad \text{where} \quad = \quad {}_{1}, \ \ A = a \ ^{-1}.$$

The solution is valid for all three cases of the functions R and Q given above.

26. $= \quad ^{2} \ ^{-3}(\quad)^{3} - \quad ^{-2}(\quad)^{2}.$

Solution in parametric form:

$$= a\tau^{-2}R^{-1}Q, \qquad = \ \tau^{-2}, \qquad \text{where} \quad = \quad {}_{1}, \ \ A = a^{-1} \ .$$

The solution is valid for all three cases of the functions R and Q given above.

27. $= \quad ^{-1}(\quad)^{3 \ 2} - \quad ^{-2} \ (\quad)^{1 \ 2}.$

Solution in parametric form:

$$= a\tau^{-2}, \qquad = \tfrac{1}{2} \ \tau^{-2}\left(2QR^{-1} - 1 \quad ^{2}\right), \qquad \text{where} \quad = \quad {}_{1}, \ \ A = (2a \quad)^{1 \ 2}.$$

28. $= \quad ^{-2}(\quad)^{5 \ 2} - \quad ^{-1}(\quad)^{3 \ 2}.$

Solution in parametric form:

$$= \tfrac{1}{2}a\tau^{-2}(2QR^{-1} - 1 \quad ^{2}), \qquad = \ \tau^{-2}, \qquad \text{where} \quad = \quad {}_{1}, \ \ A = (2 \quad a)^{1 \ 2}.$$

29. $= \quad ^{-2} \quad - \quad ^{-1}$**.**

Solution in parametric form:

$$= a \quad {}_1\tau R, \qquad = \quad {}_1\tau Q, \qquad \text{where} \quad A = a^{-2} \; {}^{2}(1 \mp \; {}^{2}).$$

30. $= \quad ^{-1}(\quad)^3 - \quad ^{-2} \; (\quad)^2$**.**

Solution in parametric form:

$$= a \quad {}_1\tau Q, \qquad = \quad {}_1\tau R, \qquad \text{where} \quad A = a^{2} \; {}^{-2}(1 \mp \; {}^{2}).$$

31. $= \quad ^{-1 \;\; 2 \;\; -1 \;\; 2} \quad - \quad ^{-3 \;\; 2 \;\; 1 \;\; 2}$**.**

Solution in parametric form:

$$= a\tau^2 R^2, \qquad = \; \tau^2 Q^2, \qquad \text{where} \qquad = \quad {}_1, \;\; A = a^{-1 \;\; 2 \;\; 1 \;\; 2}.$$

32. $= \quad ^{1 \;\; 2 \;\; -3 \;\; 2}(\quad)^3 - \quad ^{-1 \;\; 2 \;\; -1 \;\; 2}(\quad)^2$**.**

Solution in parametric form:

$$= a\tau^2 Q^2, \qquad = \; \tau^2 R^2, \qquad \text{where} \qquad = \quad {}_1, \;\; A = a^{1 \;\; 2 \;\; -1 \;\; 2}.$$

33. $= \quad ^{-1} - \quad ^{-2} \; (\quad)^{-1}$**.**

Solution in parametric form:

$$= a\tau^2 R^2, \qquad = \; \tau^2[Q^2 + (1 \mp \; {}^{2})R^2], \qquad \text{where} \qquad = \quad {}_1, \;\; A = 2a^{-1} \; .$$

34. $= \quad ^{-2}(\quad)^4 - \quad ^{-1}(\quad)^3$**.**

Solution in parametric form:

$$= a\tau^2[Q^2 + (1 \mp \; {}^{2})R^2], \qquad = \; \tau^2 R^2, \qquad \text{where} \qquad = \quad {}_1, \;\; A = 2a \; {}^{-1}.$$

35. $= \quad ^{2 \;\; -3} - \quad ^{-2}(\quad)^{-1}$**.**

Solution in parametric form:

$$= a \quad {}_1\tau R, \qquad = \quad {}_1\tau\left[Q^2 + (1 \mp \; {}^{2})R^2\right]^{1 \;\; 2}, \qquad \text{where} \quad A = 4(1 \mp \; {}^{2})a^{-4 \;\; 4}.$$

36. $= \quad ^{-2} \; (\quad)^4 - \quad ^{-3 \;\; 2}(\quad)^3$**.**

Solution in parametric form:

$$= a \quad {}_1\tau\left[Q^2 + (1 \mp \; {}^{2})R^2\right]^{1 \;\; 2}, \qquad = \quad {}_1\tau R, \qquad \text{where} \quad A = 4(1 \mp \; {}^{2})a^{4 \;\; -4}.$$

In the solutions of equations 37–50, the following notation is used:

$$Z = \begin{cases} (\tau) + \quad {}_2 \quad (\tau) & \textit{for the upper sign,} \\ (\tau) + \quad {}_2 \quad (\tau) & \textit{for the lower sign,} \end{cases}$$

where $\quad (\tau)$ *and* $\quad (\tau)$ *are the Bessel functions, and* $\quad (\tau)$ *and* $\quad (\tau)$ *are the modified Bessel functions.*

37. $= 2 \qquad + \qquad ^{-1 \;\; 2}, \qquad \neq 0, \quad \neq -2$**.**

Solution in parametric form:

$$= a \quad {}_1^{-1}\tau^{2-2} \quad , \qquad = \quad {}_1^{\;+1}\tau^{-2} \;\; Z^{-1}(\tau Z'_\tau + \quad Z),$$

where $\quad = \dfrac{\quad + 1}{\quad + 2}, \;\; A = -\dfrac{\quad + 2}{2}a^{- \;\; -1 \;\; -1}.$

38. $= (\quad + 1) \quad ^{2} \quad (\quad)^{3} + 2 \quad ^{+1}(\quad)^{2}, \quad \neq -1, \quad \neq -3.$

Solution in parametric form:

$$= a \quad _{1}^{-} \quad ^{-2}\tau^{-2} \quad Z^{-1}(\tau Z'_{\tau} + \quad Z), \qquad = \quad _{1}\tau^{2-2} \quad ,$$

where $= \dfrac{+2}{+3}, \; A = \dfrac{+3}{2} a^{-1} \;\; ^{-} \;\; ^{-2}.$

39. $= 2 \quad ^{-1} \quad - \quad ^{-2} \;\; ^{2}.$

Solution in parametric form:

$$= \quad _{1}\tau^{2}, \qquad = \tau Z^{-1} Z'_{\tau}, \qquad \text{where} \quad = 0, \;\; A = -\tfrac{1}{2} \quad ^{-1}.$$

40. $= \quad ^{2} \;\; ^{-2}(\quad)^{3} - 2 \quad ^{-1}(\quad)^{2}.$

Solution in parametric form:

$$= a\tau Z^{-1} Z'_{\tau}, \qquad = \quad _{1}\tau^{2}, \qquad \text{where} \quad = 0, \;\; A = -\tfrac{1}{2} a^{-1}.$$

41. $= \quad (\quad)^{3 \;\; 2} + \quad ^{-1} \quad (\quad)^{1 \;\; 2}, \quad \neq 0, \quad \neq -1.$

Solution in parametric form:

$$= a \quad _{1}^{-1}\tau^{4-2} \quad , \qquad = \quad _{1}^{2} \;\; ^{+1}\tau^{-2} \quad Z^{-1}(\tau Z'_{\tau} + \quad Z) \quad \frac{1}{2(1- \quad)}\tau^{2} \quad ,$$

where $= \dfrac{2 \quad +1}{+1}, \; A = -(\quad + 1) a^{-} \;\; ^{-1} \; - \dfrac{2a}{(\quad + 1)} \quad ^{1 \;\; 2}.$

42. $= (\quad + 1) \quad (\quad)^{5 \;\; 2} + \quad ^{+1}(\quad)^{3 \;\; 2}, \quad \neq -1, \quad \neq -2.$

Solution in parametric form:

$$= a \quad _{1}^{2} \;\; ^{-3}\tau^{-2} \quad Z^{-1}(\tau Z'_{\tau} + \quad Z) \quad \frac{1}{2(1- \quad)}\tau^{2} \quad , \qquad = \quad _{1}\tau^{4-2} \quad ,$$

where $= \dfrac{2 \quad +3}{+2}, \; A = (\quad + 2) \;\; ^{-} \;\; ^{-2} \; - \dfrac{2}{(\quad + 2)a} \quad ^{1 \;\; 2}.$

43. $= 2 \quad ^{-1 \;\; 2}(\quad)^{3 \;\; 2} - \quad ^{-3 \;\; 2} \quad (\quad)^{1 \;\; 2}.$

Solution in parametric form:

$$= \quad _{1}\tau^{4}, \qquad = (\tau Z^{-1} Z'_{\tau} \quad \tfrac{1}{2}\tau^{2}), \qquad \text{where} \quad = 0, \;\; A = -\tfrac{1}{2}(- \quad)^{-1 \;\; 2}.$$

44. $= \quad ^{-3 \;\; 2}(\quad)^{5 \;\; 2} - 2 \quad ^{-1 \;\; 2}(\quad)^{3 \;\; 2}.$

Solution in parametric form:

$$= a(\tau Z^{-1} Z'_{\tau} \quad \tfrac{1}{2}\tau^{2}), \qquad = \quad _{1}\tau^{4}, \qquad \text{where} \quad = 0, \;\; A = -\tfrac{1}{2}(-a)^{-1 \;\; 2}.$$

45. $= \quad ^{-1} \;\; ^{-1 \;\; 2} \quad - \quad ^{-2} \;\; ^{1 \;\; 2}.$

Solution in parametric form:

$$= \quad _{1}Z^{-2}, \qquad = \tau^{2} Z^{-2}(Z'_{\tau})^{2}, \qquad \text{where} \quad = 0, \;\; A = - \quad ^{1 \;\; 2}.$$

46. $= \quad ^{1 \;\; 2} \;\; ^{-2}(\quad)^{3} - \quad ^{-1 \;\; 2} \;\; ^{-1}(\quad)^{2}.$

Solution in parametric form:

$$= a\tau^{2} Z^{-2}(Z'_{\tau})^{2}, \qquad = \quad _{1}Z^{-2}, \qquad \text{where} \quad = 0, \;\; A = -a^{1 \;\; 2}.$$

47. $\quad = \quad {}^{-3}(\quad)^4 - \quad {}^{-2}(\quad)^3.$

Solution in parametric form:

$$= aZ^{-1}(2\tau Z'_\tau \quad \tau^2 Z), \qquad = \ {}_1 Z^{-1}, \qquad \text{where} \qquad = 0, \quad A = 4a.$$

48. $\quad = \quad {}^{-2} - \quad {}^{-3}\ (\quad)^{-1}.$

Solution in parametric form:

$$= \ {}_1 Z^{-1}, \qquad = \ Z^{-1}(2\tau Z'_\tau \quad \tau^2 Z), \qquad \text{where} \qquad = 0, \quad A = 4\ .$$

49. $\quad = \quad {}^{-2}(\quad)^4 - 2 \quad {}^{-1}(\quad)^3.$

Solution in parametric form:

$$= a \quad {}_1[\tau^2(Z'_\tau)^2 + 2\tau Z Z'_\tau \quad \tau^2 Z^2], \qquad = \quad {}_1 Z^2, \qquad \text{where} \qquad = 0, \quad A = \tfrac{1}{2}a \quad {}^{-1}.$$

50. $\quad = 2 \quad {}^{-1} - \quad {}^{-2}\ (\quad)^{-1}.$

Solution in parametric form:

$$= a \quad {}_1 Z^2, \qquad = \quad {}_1[\tau^2(Z'_\tau)^2 + 2\tau Z Z'_\tau \quad \tau^2 Z^2], \qquad \text{where} \qquad = 0, \quad A = \tfrac{1}{2}a^{-1}\ .$$

In the solutions of equations 51–66, the following notation is used:

$$Z = \begin{cases} \ {}_{1\ 3}(\tau) + \ {}_2\ {}_{1\ 3}(\tau) & \textit{for the upper sign,} \\ \ {}_{1\ 3}(\tau) + \ {}_2\ {}_{1\ 3}(\tau) & \textit{for the lower sign,} \end{cases}$$

$$_1 = \tau Z'_\tau + \tfrac{1}{3}Z, \qquad {}_2 = \ {}^2_1 \quad \tau^2 Z^2, \qquad {}_3 = \tfrac{2}{3}\tau^2 Z^3 - 2 \ {}_1 \ {}_2,$$

where $_{1\ 3}(\tau)$ *and* $_{1\ 3}(\tau)$ *are the Bessel functions, and* $_{1\ 3}(\tau)$ *and* $_{1\ 3}(\tau)$ *are the modified Bessel functions.*

51. $\quad = \quad {}^{-2}\ {}^{-2} \quad + 2 \quad {}^{-3}\ {}^{-1}.$

Solution in parametric form:

$$= a \ {}^{-2}_1 \tau^{4\ 3} Z^2 \ {}^{-1}_2, \qquad = \quad {}_1 \tau^{-2\ 3} Z^{-1} \ {}^{-1}_2 \quad {}_3, \qquad \text{where} \quad A = 2a \quad {}^2.$$

52. $\quad = 2 \quad {}^{-1}\ {}^{-3}(\quad)^3 + \quad {}^{-2}\ {}^{-2}(\quad)^2.$

Solution in parametric form:

$$= a \ {}^{-1}_1 \tau^{-2\ 3} Z^{-1} \ {}^{-1}_2 \quad {}_3, \qquad = \ {}^2_1 \tau^{4\ 3} Z^2 \ {}^{-1}_2, \qquad \text{where} \quad A = -2a^2\ .$$

53. $\quad = \quad {}^{-2}\ {}^{-1\ 2} \quad - 4 \quad {}^{-3}\ {}^{1\ 2}.$

Solution in parametric form:

$$= a \ {}^{-1}_1 \tau^{-4\ 3} Z^{-2} \quad {}_2, \qquad = \ {}^2_1 \tau^{-4\ 3} Z^{-2} \ {}^{-2}_2 \ {}^2_3, \qquad \text{where} \quad A = \mp\tfrac{2}{3}a \ {}^{1\ 2}.$$

54. $\quad = 4 \quad {}^{1\ 2}\ {}^{-3}(\quad)^3 - \quad {}^{-1\ 2}\ {}^{-2}(\quad)^2.$

Solution in parametric form:

$$= a \ {}^{-2}_1 \tau^{-4\ 3} Z^{-2} \ {}^{-2}_2 \ {}^2_3, \qquad = \quad {}_1 \tau^{-4\ 3} Z^{-2} \quad {}_2, \qquad \text{where} \quad A = \mp\tfrac{2}{3}a^{1\ 2}\ .$$

55. $\quad = 2 \quad {}^{-1\ 2}\ {}^{-2} \quad + \quad {}^{-3\ 2}\ {}^{-1}.$

Solution in parametric form:

$$= a \ {}^4_1 \tau^{-4\ 3} Z^{-2} \ {}^2_1, \qquad = \quad {}_1 \tau^{-4\ 3} Z^{-2} \quad {}_2, \qquad \text{where} \quad A = \quad \tfrac{1}{6}a^{-1\ 2\ 2}.$$

56. $= \;{}^{-1}\;{}^{-3}\;{}^{2}(\quad)^{3} + 2\;{}^{-2}\;{}^{-1}\;{}^{2}(\quad)^{2}.$

Solution in parametric form:

$$= a\;{}_1\tau^{-4\;3}Z^{-2}\;{}_2, \qquad = \;{}^{4}_{1}\tau^{-4\;3}Z^{-2}\;{}^{2}_{1}, \qquad \text{where} \quad A = \mp\tfrac{1}{6}a^{2\;-1\;2}.$$

57. $= \;{}^{-1\;2} + 2\;{}^{1\;2}.$

Solution in parametric form:

$$= a\;{}_1\tau^{-2\;3}Z^{-1}\;{}_1, \qquad = \;{}^{4}_{1}\tau^{-8\;3}Z^{-4}\;{}^{2}_{2}, \qquad \text{where} \quad A = 2a^{-2\;1\;2}.$$

58. $= 2\;{}^{1\;2}(\quad)^{3} + \;{}^{-1\;2}\;(\quad)^{2}.$

Solution in parametric form:

$$= a\;{}^{4}_{1}\tau^{-8\;3}Z^{-4}\;{}^{2}_{2}, \qquad = \;{}_1\tau^{-2\;3}Z^{-1}\;{}_1, \qquad \text{where} \quad A = -2a^{1\;2\;-2}.$$

59. $= 5\;{}^{-2\;5} - 2\;{}^{-7\;5}\;(\quad)^{-1}.$

Solution in parametric form:

$$= a\;{}^{5}_{1}\tau^{-5\;3}Z^{-5\;2}\;{}^{5\;2}_{1}, \qquad = \;{}^{8}_{1}\tau^{-8\;3}Z^{-4}(\;{}^{2}_{2}\;\;\tfrac{4}{3}\tau^{2}Z^{3}\;\;{}_1), \qquad \text{where} \quad A = \tfrac{32}{125}a^{-8\;5}\;.$$

60. $= 2\;{}^{-2\;5}(\quad)^{4} - 5\;{}^{-7\;5}(\quad)^{3}.$

Solution in parametric form:

$$= a\;{}^{8}_{1}\tau^{-8\;3}Z^{-4}(\;{}^{2}_{2}\;\;\tfrac{4}{3}\tau^{2}Z^{3}\;\;{}_1), \qquad = \;{}^{5}_{1}\tau^{-5\;3}Z^{-5\;2}\;{}^{5\;2}_{1}, \qquad \text{where} \quad A = \tfrac{32}{125}a^{\;-8\;5}.$$

61. $= 4\;{}^{1\;2\;-3} - \;{}^{-1\;2\;-2}(\quad)^{-1}.$

Solution in parametric form:

$$= a\;{}^{8}_{1}\tau^{-4\;3}Z^{-2}\;{}^{2}_{1}, \qquad = \;{}^{5}_{1}\tau^{-4\;3}Z^{-2}(\;{}^{2}_{2}\;\;\tfrac{4}{3}\tau^{2}Z^{3}\;\;{}_1)^{1\;2}, \qquad \text{where} \quad A = \;\tfrac{1}{3}a^{-5\;2\;4}.$$

62. $= \;{}^{-2\;-1\;2}(\quad)^{4} - 4\;{}^{-3\;1\;2}(\quad)^{3}.$

Solution in parametric form:

$$= a\;{}^{5}_{1}\tau^{-4\;3}Z^{-2}(\;{}^{2}_{2}\;\;\tfrac{4}{3}\tau^{2}Z^{3}\;\;{}_1\;{}^{1\;2}, \qquad = \;{}^{8}_{1}\tau^{-4\;3}Z^{-2}\;{}^{2}_{1}, \qquad \text{where} \quad A = \;\tfrac{1}{3}a^{4\;-5\;2}.$$

63. $= \;{}^{2} + 2\;(\quad)^{-1}.$

Solution in parametric form:

$$= a\;{}_1\tau^{2\;3}Z\;\;{}^{-1\;2}_{2}, \qquad = \;{}^{4}_{1}\tau^{-4\;3}Z^{-2}\;{}^{-2}_{2}(\;{}^{2}_{3} - 4\;{}^{3}_{2}), \qquad \text{where} \quad A = \tfrac{32}{9}a^{-4}\;.$$

64. $= 2\;(\quad)^{4} + \;{}^{2}(\quad)^{3}.$

Solution in parametric form:

$$= a\;{}^{4}_{1}\tau^{-4\;3}Z^{-2}\;{}^{-2}_{2}(\;{}^{2}_{3} - 4\;{}^{3}_{2}), \qquad = \;{}_1\tau^{2\;3}Z\;\;{}^{-1\;2}_{2}, \qquad \text{where} \quad A = -\tfrac{32}{9}a^{\;-4}.$$

65. $= 2\;{}^{-1\;-3} + \;{}^{-2\;-2}(\quad)^{-1}.$

Solution in parametric form:

$$= a\;{}^{4}_{1}\tau^{4\;3}Z^{2}\;{}^{-1}_{2}, \qquad = \;{}_1\tau^{-2\;3}Z^{-1}\;{}^{-1}_{2}(\;{}^{2}_{3} - 4\;{}^{3}_{2}\;{}^{1\;2}, \qquad \text{where} \quad A = -\tfrac{8}{9}a^{-1\;4}.$$

66. $= \;{}^{-2\;-2}(\quad)^{4} + 2\;{}^{-3\;-1}(\quad)^{3}.$

Solution in parametric form:

$$= a\;{}_1\tau^{-2\;3}Z^{-1}\;{}^{-1}_{2}(\;{}^{2}_{3} - 4\;{}^{3}_{2}\;{}^{1\;2}, \qquad = \;{}^{4}_{1}\tau^{4\;3}Z^{2}\;{}^{-1}_{2}, \qquad \text{where} \quad A = \tfrac{8}{9}a^{4\;-1}.$$

In the solutions of equations 67–72, the following notation is used:

$$= \ _1 \ (\lambda, \tfrac{1}{2}; \ \tau) + \ _2 \ (\lambda, \tfrac{1}{2}; \ \tau),$$

where and are linearly independent solutions of the degenerate hypergeometric equation:

$$\tau \ {}''_{\tau\tau} + (\tfrac{1}{2} \ \tau) \ {}'_{\tau} - \lambda \ = 0.$$

The function $= \ (\lambda, \tfrac{1}{2}, \ \tau)$ *can be expressed in terms of a degenerate hypergeometric series (see equation 2.1.2.70).*

67. $= \ _1 \ + \ _2 \ .$

Solution in parametric form:

$$= a\tau^{1 \ 2}, \quad = \ , \quad \text{where} \quad A_1 = \ 2a^{-2}, \ A_2 = \ 4a^{-2}\lambda.$$

68. $= \ _1 \ (\ )^3 + \ _2 \ (\ )^2.$

Solution in parametric form:

$$= \ , \quad = \tau^{1 \ 2}, \quad \text{where} \quad A_1 = \mp 4 \ ^{-2}\lambda, \ A_2 = \mp 2 \ ^{-2}.$$

69. $= \ _1 \ ^{-2} \ + \ _2 \ ^{-1}.$

Solution in parametric form:

$$= \ , \quad = \ \tau^{1 \ 2} \ {}'_{\tau}, \quad \text{where} \quad A_1 = \mp \ ^2\lambda, \ A_2 = \ ^2\bigl(\lambda + \tfrac{1}{2} \ .$$

70. $= \ _1 \ ^{-1}(\ )^3 + \ _2 \ ^{-2} \ (\ )^2.$

Solution in parametric form:

$$= \ a\tau^{1 \ 2} \ {}'_{\tau}, \quad = \ , \quad \text{where} \quad A_1 = \mp a^2\bigl(\lambda + \tfrac{1}{2} \ , \ A_2 = \ a^2\lambda.$$

71. $= \ _1 \ ^{-1} \ ^{-2} \ + \ _2 \ ^{-2} \ ^{-1}.$

Solution in parametric form:

$$= \ ^{-1}, \quad = \ \tau^{1 \ 2} \ ^{-1} \ {}'_{\tau}, \quad \text{where} \quad A_1 = \ ^2\lambda, \ A_2 = \ \tfrac{1}{2} \ ^2.$$

72. $= \ _1 \ ^{-1} \ ^{-2}(\ )^3 + \ _2 \ ^{-2} \ ^{-1}(\ )^2.$

Solution in parametric form:

$$= \ a\tau^{1 \ 2} \ ^{-1} \ {}'_{\tau}, \quad = \ ^{-1}, \quad \text{where} \quad A_1 = \mp\tfrac{1}{2}a^2, \ A_2 = \mp a^2\lambda.$$

73. $= \ (\ )^{\frac{3 \ +2}{+1}} + \ ^{-1} \ (\ )^{\frac{2 \ +1}{+1}}, \quad \neq 0, \quad \neq -1.$

Solution in parametric form:

$$= a \ _1^{-2 \ -1} \ \frac{\tau}{\beta\tau \ + 1} + \ _2 \ ^{-1} \ , \quad = \ _1^{\ 2} \ \tau - \ \frac{\tau}{\beta\tau \ + 1} - \ _2 \ ,$$

where $k = -\dfrac{+1}{}, \ A = \dfrac{+1}{^2\beta} a^{1- \ \ -2} \ -\dfrac{\beta}{a}^{\frac{1}{+1}}.$

74. $= \ (\ + 1) \ (\ )^{\frac{+3}{+2}} + \ ^{+1}(\ )^{\frac{1}{+2}}, \quad \neq -1, \quad \neq -2.$

Solution in parametric form:

$$= a \ _1^{(\ +1)^2} \ \tau - \ \frac{\tau}{\beta\tau \ + 1} - \ _2 \ , \quad = \ _1^{-2 \ -3} \ \frac{\tau}{\beta\tau \ + 1} + \ _2 \ ^{-\frac{1}{+1}},$$

where $k = -\dfrac{+2}{+1}, \ A = -\dfrac{+2}{(\ +1)^2\beta} a^{-2 \ -} \ -\dfrac{a(\ +1)\beta}{}^{\frac{1}{+2}}.$

75. $= \quad ^{-\ \ -1} \quad (\quad) - \quad ^{-\ \ -2} \quad ^{+1}(\quad)^{-1}, \qquad \neq -1, \quad \neq 2, \quad + \quad -1 \neq 0.$

Solution in parametric form:

$$= a \quad _1 \exp \frac{l-2}{+l-1} \quad \frac{\tau}{\quad}, \qquad = \quad _1\tau^{\frac{-2}{+-1}} \exp \frac{l-2}{+l-1} \quad \frac{\tau}{\quad},$$

where

$$= \frac{+l-1}{l-2}\left(\beta + \quad _2\tau \quad \right)^{\frac{1}{2-}} - \tau, \quad k = \frac{(\quad +1)(l-2)}{+l-1},$$
$$A = -\frac{(\quad +1)(\quad +l-1)}{(l-2)^3} a^{\quad -1 \quad 1-} \quad \beta \quad \frac{(l-2)a}{(\quad +l-1)} \quad .$$

76. $= \quad ^{-2 \quad 1-}(\quad) - \quad ^{-3 \quad 2-}(\quad)^{-1}, \qquad \neq 2.$

Solution in parametric form:

$$= \quad _1 \exp \quad \frac{\tau}{\quad}, \qquad = \quad _2 \exp \quad \tau + \quad \frac{\tau}{\quad},$$

where $= (2-l)\left[\beta + e^{(\ -2)\tau}\right]^{\frac{1}{2-}} - 1, \ A = (2-l)^{2-}\beta.$

77. $= \quad ^{-1} \quad - \quad ^{-2} \quad .$

Solution: $= \begin{cases} \quad _1 + \quad _2|\quad| & \text{if } A \neq 1, \\ (\quad _1 + \quad _2 \ln|\quad| & \text{if } A = 1. \end{cases}$

78. $= \quad ^{-2}(\quad)^3 - \quad ^{-1}(\quad)^2.$

Solution in implicit form: $= \begin{cases} \quad _1 + \quad _2|\quad| & \text{if } A \neq 1, \\ (\quad _1 + \quad _2 \ln|\quad| & \text{if } A = 1. \end{cases}$

In the solutions of equations 79 and 80, the following notation is used:

$$= \begin{cases} \frac{1}{\beta+1}\tau^{\quad +1} + \frac{1}{\beta}\tau \quad + \quad _2 & \textit{if } \beta \neq 0, \\ \tau + \ln|\tau| + \quad _2 & \textit{if } \beta = 0. \end{cases}$$

79. $= \quad ^{-2}(\quad) - \quad ^{-1}(\quad)^{-1}, \qquad \neq 3.$

Solution in parametric form:

$$= a \quad _1[\tau^{\quad +1} - (\beta+1) \quad]\exp - \quad \tau^{\quad -1 \quad -1} \quad \tau, \qquad = \quad _1 \exp - \quad \tau^{\quad -1 \quad -1} \quad \tau,$$

where $\beta = \frac{2-l}{l-3}, \ A = -a^{-3 \quad 3-}.$

80. $= \quad ^{-1}(\quad) - \quad ^{-2} \ (\quad)^{-1}, \qquad \neq 1.$

Solution in parametric form:

$$= a \quad _1 \exp - \quad \tau^{\quad -1 \quad -1} \quad \tau, \qquad = \quad _1[\tau^{\quad +1} - (\beta+1) \quad]\exp - \quad \tau^{\quad -1 \quad -1} \quad \tau,$$

where $\beta = \frac{l-2}{1-l}, \ A = -a^{-1 \quad 1-}.$

81. $= {}_1 {}^{-1}(\quad)^2 + {}_2 {}^{-1}$.

Solution: $= \begin{cases} ({}_1| \; | {}^{2+1} + {}_2) {}^{1-1} & \text{if } A_1 \neq 1,\ A_2 \neq -1; \\ ({}_1 \ln| \; | + {}_2) {}^{1-1} & \text{if } A_1 \neq 1,\ A_2 = -1; \\ {}_2 \exp({}_1| \; | {}^{2+1}) & \text{if } A_1 = 1,\ A_2 \neq -1; \\ {}_2| \; | {}^{1} & \text{if } A_1 = 1,\ A_2 = -1. \end{cases}$

In the solutions of equations 82–84, the following notation is used:

$$= \exp \frac{\tau}{\tau \quad \overline{\overline{\quad {}^2 + 4}}} , \qquad = {}_2\tau^{-1 \; 2} \exp \frac{A}{2(k+1)}\tau^{\; +1} .$$

82. $= {}^{+1} (\quad)^2 - {}^{+1} , \qquad \neq -1.$

Solution in parametric form:

$$= {}_1\tau^{1 \; 2 \quad -1 \; 2}, \qquad = {}_1^{-1}\tau^{1 \; 2 \quad 1 \; 2}, \qquad k = \quad .$$

83. $= {}^{-2 \quad -2} (\quad)^2 - {}^{-2 \quad -3 \quad +1} , \qquad \neq -1.$

Solution in parametric form:

$$= {}_1\tau^{-1 \; 2 \quad 1 \; 2}, \qquad = {}^{2}_{1} , \qquad k = \quad .$$

84. $= {}^{-\frac{+1}{2}} (\quad)^2 - {}^{-\frac{+3}{2} \quad +1} , \qquad \neq -1.$

Solution in parametric form:

$$= {}^{2}_{1} , \qquad = {}_1\tau^{-1 \; 2 \quad 1 \; 2}, \qquad k = -\frac{+3}{2}.$$

85. $= (\quad)^2 - {}^{-1 \quad +1} , \qquad \neq -1, \quad \neq 0.$

Solution in parametric form:

$$= {}_1 \exp \frac{\tau}{\tau} , \qquad = {}_1 \tau \exp \; k \frac{\tau}{\tau} , \qquad k = -\frac{}{+1},$$

where $= (\tau)$ is the solution of the transcendental equation

$$\frac{(\; + k)}{(\; + k - 1) {}^{-1}} = {}_2\tau^{-1} \exp \frac{A}{+1}\tau^{\; +1} .$$

86. $= {}_1 {}^{-1}(\quad)^2 + {}_2 {}^{-1} , \qquad \neq 0.$

Solution:

$$= {}_1 \exp \quad \exp \frac{A_2}{} \quad (\; + {}_2)^{-1} , \qquad \text{where} \quad = (1 - A_1 \quad) \exp \frac{A_2}{} \qquad .$$

87. $= {}_1 (\quad)^2 + {}_2 {}^{-1 \quad +1} , \qquad \neq -1.$

Solution:

$$= {}_1 \exp \quad \exp -\frac{A_1}{+1}{}^{+1} \frac{}{+ {}_2} , \qquad \text{where} \quad = (1 + A_2 {}^{+1}) \exp -\frac{A_1}{+1}{}^{+1} \qquad .$$

2.6.4. Other Equations $({}_1 \neq {}_2)$

2.6.4-1. Classification table.

Table 27 presents all solvable equations whose solutions are outlined in Subsection 2.6.4. Equations are arranged in accordance with the growth of l_1 $(l_1 > l_2)$. The number of the equation sought is indicated in the last column in this table.

TABLE 27

Solvable cases of the equation $'' = A_1 \;{}^{1}\;{}^{1}(\;')^{1} + A_2 \;{}^{2}\;{}^{2}(\;')^{2}, \quad l_1 \neq l_2$

l_1	l_2	$_1$	$_2$	$_1$	$_2$	Equation
arbitrary	arbitrary	arbitrary	$_1$	0	0	2.6.4.3
arbitrary	arbitrary	0	0	arbitrary	$_1$	2.6.4.4
arbitrary	arbitrary	$1-l_1$	$1-l_2$	l_1-2	l_2-2	2.6.4.9
arbitrary	$3-l_1$	$1-l_1$	l_1-2	l_1-2	$1-l_1$	2.6.4.8
1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	2.6.4.1
1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	1	2.6.4.14
1	0	$-\frac{1}{2}$	0	0	0	2.6.4.16
2	0	-1	1	arbitrary	arbitrary	2.6.4.18
2	1	arbitrary	arbitrary	0	-1	2.6.4.13
2	1	arbitrary ($_1 \neq -1$)	0	0	arbitrary ($_2 \neq -1$)	2.6.4.5
2	1	arbitrary ($_1 \neq -1$)	0	0	-1	2.6.4.7
2	1	-1	0	arbitrary	arbitrary	2.6.4.12
2	1	-1	0	0	arbitrary ($_2 \neq -1$)	2.6.4.6
$\frac{5}{2}$	$\frac{1}{2}$	arbitrary	$_1+2$	$_1+2$	$_1$	2.6.4.11
3	0	arbitrary* ($_1 \neq -2$)	$_1+3$	$_1+3$	$_1$	2.6.4.10
3	1	arbitrary	arbitrary	1	-1	2.6.4.19
3	2	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	2.6.4.2
3	2	0	0	0	$-\frac{1}{2}$	2.6.4.17
3	2	1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	2.6.4.15

* For $_1 = -2$, see Equation 2.6.4.8 with $l = 3$.

2.6.4-2. Solvable equations and their solutions.

1. $\;= \;_1 \;{}^{-1}\;{}^{2} \;+\; _2 \;{}^{-1}\;{}^{2}.$

Solution in parametric form:

$$= \;_1 \exp(A_1\tau) - \frac{A_2}{4A_1}\tau^2 + \;_2, \qquad = A_1 \;_1 \exp(A_1\tau) - \frac{A_2}{2A_1}\tau \;{}^{2}.$$

2. $\;= \;_1 \;{}^{-1}\;{}^{2}(\;)^3 + \;_2 \;{}^{-1}\;{}^{2}(\;)^2.$

Solution in parametric form:

$$= A_2 \;_1 \exp(-A_2\tau) + \frac{A_1}{2A_2}\tau \;{}^{2}, \qquad = \;_1 \exp(-A_2\tau) - \frac{A_1}{4A_2}\tau^2 + \;_2.$$

3. $= {}_1 \quad (\quad)^1 + {}_2 \quad (\quad)^2.$

1 . Solution in parametric form with $\neq -1$:

$$= {}_2 + \quad \left(A_1\tau^{\ 1} + A_2\tau^{\ 2}\right)^{-1} \ - \frac{}{+1} \quad \tau, \qquad = \frac{1}{+1},$$

where $= {}_1 + (\quad +1) \quad \tau\left(A_1\tau^{\ 1} + A_2\tau^{\ 2}\right)^{-1} \quad \tau.$

2 . Solution in parametric form with $= -1$:

$$= {}_2 + \quad \left(A_1\tau^{\ 1} + A_2\tau^{\ 2}\right)^{-1} e \quad \tau, \qquad = e \ ,$$

where $= {}_1 \quad \tau\left(A_1\tau^{\ 1} + A_2\tau^{\ 2}\right)^{-1} \quad \tau.$

4. $= {}_1 \quad (\quad)^1 + {}_2 \quad (\quad)^2.$

1 . Solution in parametric form with $\neq -1$:

$$= \frac{1}{+1}, \qquad = {}_2 + \quad \tau\left(A_1\tau^{\ 1} + A_2\tau^{\ 2}\right)^{-1} \ - \frac{}{+1} \quad \tau,$$

where $= {}_1 + (\quad +1) \quad \left(A_1\tau^{\ 1} + A_2\tau^{\ 2}\right)^{-1} \quad \tau.$

2 . Solution in parametric form with $= -1$:

$$= e \ , \qquad = {}_2 + \quad \tau\left(A_1\tau^{\ 1} + A_2\tau^{\ 2}\right)^{-1} e \quad \tau,$$

where $= {}_1 \quad \left(A_1\tau^{\ 1} + A_2\tau^{\ 2}\right)^{-1} \quad \tau.$

5. $= {}_1 \quad (\quad)^2 + {}_2 \qquad , \qquad \neq -1, \qquad \neq -1.$

Solution: $\quad \exp\left(-\frac{A_1}{+1} \quad ^{+1}\right) \quad = {}_1 \quad \exp\left(\frac{A_2}{+1} \quad ^{+1}\right) \quad + {}_2.$

6. $= {}_1 \ ^{-1}(\quad)^2 + {}_2 \qquad , \qquad \neq -1.$

1 . Solution for $A_1 \neq 1$:

$$= {}_1 \quad \exp\left(\frac{A_2}{+1} \quad ^{+1}\right) \quad + {}_2 \quad ^{1-1}.$$

2 . Solution for $A_1 = 1$:

$$= {}_2 \exp\left[{}_1 \quad \exp\left(\frac{A_2}{+1} \quad ^{+1}\right) \quad \right].$$

7. $= {}_1 \quad (\quad)^2 + {}_2 \ ^{-1} \qquad , \qquad \neq -1.$

1 . Solution for $A_2 \neq -1$:

$$= {}_1 \quad \exp\left(-\frac{A_1}{+1} \quad ^{+1}\right) \quad + {}_2 \quad ^{-\ 2-1}.$$

2 . Solution for $A_2 = -1$:

$$= {}_2 \exp\left[{}_1 \quad \exp\left(-\frac{A_1}{+1} \quad ^{+1}\right) \quad \right].$$

8. $= {}_1 \quad {}^{-2} \; {}^{1-} (\quad) + {}_2 \; {}^{1-} \quad {}^{-2} (\quad)^{3-}$ **.**

Solution in parametric form:

$$= {}_2 \exp \frac{\tau}{A_1\tau \; + A_2\tau^{3-} \; - \tau^2 + \tau}, \qquad = {}_1 \exp \frac{\tau \quad \tau}{A_1\tau \; + A_2\tau^{3-} \; - \tau^2 + \tau}.$$

9. $= {}_1 \quad {}_1{}^{-2} \; {}^{1-}{}_1 (\quad)^{1} + {}_2 \quad {}_2{}^{-2} \; {}^{1-}{}_2 (\quad)^{2}$ **.**

Solution in parametric form:

$$= {}_2 \exp \frac{\tau}{A_1\tau^{1} + A_2\tau^{2} - \tau^2 + \tau}, \qquad = {}_1 \exp \frac{\tau \quad \tau}{A_1\tau^{1} + A_2\tau^{2} - \tau^2 + \tau}.$$

10. $= \quad {}^{+3} \quad (\quad)^3 - \quad {}^{+3}, \qquad \neq -2.$

Solution in parametric form:

$$= {}_1\tau^{1 \; 2} \exp -\frac{1}{2} \quad \frac{\tau}{\tau \quad \overline{\quad {}^2 + 4}}, \qquad = {}_1^{-1}\tau^{1 \; 2} \exp \frac{1}{2} \quad \frac{\tau}{\tau \quad \overline{\quad {}^2 + 4}},$$

where $= \tau^{-1 \; 2} \exp \frac{3A}{2 \quad +4} \tau \quad {}^{+2} \quad {}_2 - A \quad \tau \quad \exp \frac{3A}{\quad +2} \tau \quad {}^{+2} \quad \tau \quad {}^{-1 \; 2}.$

11. $= \quad {}^{+2} \quad (\quad)^{5 \; 2} + \quad {}^{+2} (\quad)^{1 \; 2}$ **.**

Solution in parametric form:

$$= {}_1\tau^{1 \; 2} \exp -\frac{1}{2} \quad \frac{\tau}{\tau \quad \overline{\quad {}^2 + 4}}, \qquad = {}_1^{-1}\tau^{1 \; 2} \exp \frac{1}{2} \quad \frac{\tau}{\tau \quad \overline{\quad {}^2 + 4}}.$$

Here, the function $= \quad (\tau)$ is defined in parametric form, $\tau = \tau(\quad)$, $= \quad (\quad)$, as follows:

1 . For $\neq -1$, $\neq -3 \; 2$:

$$\tau = a \quad {}^{\frac{2}{2 \quad +3}}, \qquad = \frac{(2 \quad +3)}{2(\quad +1)} \quad {}^{-1} Z^{-1}(\tau Z' + \quad Z),$$

where $Z = \begin{matrix} {}_2 \quad (\;) + \quad (\;) & \text{for the upper sign,} \\ {}_2 \quad (\;) + \quad (\;) & \text{for the lower sign,} \end{matrix}$ $(\;)$ and $(\;)$ are the Bessel functions, and $(\;)$ and $(\;)$ are the modified Bessel functions,

$$= \frac{\quad +1}{2 \quad +3}, \quad a = -\frac{2 \quad +2}{A} \quad {}^{-\frac{1}{\quad +1}}, \qquad \frac{(2 \quad +3)^2}{8(\quad +1)^2} \; {}^{\frac{2 \quad +3}{\quad +1}} = -\frac{2 \quad +2}{A} \quad {}^{\frac{1}{\quad +1}}.$$

2 . For $= -1$:

$$\tau = \frac{\quad {}^2}{2A^2}, \qquad = \frac{A}{2} Z^{-1} Z', \qquad \text{where} \quad Z = \quad {}_2 \quad {}_0(\;) + \quad {}_0(\;).$$

3 . For $= -3 \; 2$:

$$\tau = A^2 \quad {}^{-4}, \qquad = \begin{cases} \dfrac{1}{2A} \dfrac{(1+k) \quad {}_2 \quad + (1-k) \quad {}^{-}}{\quad {}_2 \quad + \quad {}^{-}} & \text{if } A^2 < \frac{1}{8}, \\ \dfrac{1}{2A} \dfrac{{}_2 \ln \quad + \quad {}_2 + 1}{{}_2 \ln \quad + 1} & \text{if } A^2 = \frac{1}{8}, \\ \dfrac{1}{2A} \dfrac{(\quad {}_2 - k)\sin(k \ln \quad) + (1 + k \quad {}_2)\cos(k \ln \quad)}{{}_2 \sin(k \ln \quad) + \cos(k \ln \quad)} & \text{if } A^2 > \frac{1}{8}, \end{cases}$$

where $k = \sqrt{|1 - 8A^2|}$.

12. $= {}_1 \; {}^1 \; {}^{-1}(\;\;)^2 + \; {}_2 \;\; {}^2$.

1 . Solution for ${}_2 \neq -1$:

$$= {}_1 \exp \qquad , \quad \text{where} \quad = \exp \frac{A_2 \quad {}^{2+1}}{{}_2+1} \quad {}_2 + \quad (1 - A_1 \quad {}^1) \exp \frac{A_2 \quad {}^{2+1}}{{}_2+1} \quad {}^{-1} .$$

2 . Solution for ${}_2 = -1$, $A_2 \neq -1$, $A_2 \neq - \; {}_1 - 1$:

$$= {}_1 \exp \quad {}^2 \quad {}_2 + \frac{1}{A_2 + 1} \quad {}^{2+1} - \frac{A_1}{{}_1 + A_2 + 1} \quad {}^{1+ \;\; 2+1} \quad {}^{-1} .$$

3 . Solution for ${}_2 = -1$, $A_2 = -1$:

$$= {}_1 \exp \quad {}^{-1} \quad {}_2 + \ln \quad - \frac{A_1}{{}_1} \quad {}^1 \quad {}^{-1} .$$

4 . Solution for ${}_2 = -1$, $A_2 = - \; {}_1 - 1$:

$$= {}_1 \exp \quad {}^{- \;\; 1-1} \quad {}_2 - \frac{1}{{}_1} \quad {}^{- \;\; 1} - A_1 \ln \quad {}^{-1} .$$

13. $= {}_1 \;\; {}^1(\;\;)^2 + \; {}_2 \; {}^{-1} \;\; {}^2$.

1 . Solution for ${}_1 \neq -1$:

$$= {}_1 \exp \qquad , \qquad = \exp \; - \frac{A_1 \quad {}^{1+1}}{{}_1+1} \quad {}_2 + \quad (1 + A_2 \quad {}^2) \exp \; - \frac{A_1 \quad {}^{1+1}}{{}_1+1} \quad {}^{-1} .$$

2 . Solution for ${}_1 = -1$, $A_1 \neq 1$, $A_1 \neq \; {}_2 + 1$:

$$= {}_1 \exp \quad {}^{- \;\; 1} \quad {}_2 + \frac{1}{1 - A_1} \quad {}^{1- \;\; 1} + \frac{A_2}{{}_2 - A_1 + 1} \quad {}^{2- \;\; 1+1} \quad {}^{-1} .$$

3 . Solution for ${}_1 = -1$, $A_1 = 1$:

$$= {}_1 \exp \quad {}^{-1} \quad {}_2 + \ln \quad + \frac{A_2}{{}_2} \quad {}^2 \quad {}^{-1} .$$

4 . Solution for ${}_1 = -1$, $A_1 = \; {}_2 + 1$:

$$= {}_1 \exp \quad {}^{- \;\; 2-1} \quad {}_2 - \frac{1}{{}_2} \quad {}^{- \;\; 2} + A_2 \ln \quad {}^{-1} .$$

In the solutions of equations 14 and 15, the following notations are used:

$$R = \begin{cases} {}_1\tau^{\;1} + \; {}_2\tau^{\;2} + \; {}_3\tau^{\;3} & \text{if } B_2(8B_1^3 + 27B_2) < 0, \\ {}_1\tau e^{\;\tau} + \; {}_2 e^{\sigma\tau} & \text{if } 8B_1^3 + 27B_2 = 0, \\ {}_1 e^{\;\tau} + \; {}_2 e^{\;\tau} \cos\omega\tau & \text{if } B_2(8B_1^3 + 27B_2) > 0, \end{cases}$$

$$Q = \begin{cases} {}_1 k_1 \tau^{\;1} + \; {}_2 k_2 \tau^{\;2} + \; {}_3 k_3 \tau^{\;3} & \text{if } B_2(8B_1^3 + 27B_2) < 0, \\ {}_1(1 + k\tau) e^{\;\tau} + \; {}_2 \sigma e^{\sigma\tau} & \text{if } 8B_1^3 + 27B_2 = 0, \\ {}_1 k e^{\;\tau} + \; {}_2 e^{\;\tau}(\;\; \cos\omega\tau - \omega \sin\omega\tau) & \text{if } B_2(8B_1^3 + 27B_2) > 0, \end{cases}$$

where k_1, k_2, and k_3 (real numbers) or k and ω (one real and two complex numbers) are roots of the cubic equation

$$\lambda^3 - B_1\lambda^2 - \tfrac{1}{2}B_2 = 0.$$

In the special case $8B_1^3 = -27B_2$, we have $k = \frac{2}{3}B_1$ (multiple root) and $\sigma = -\frac{1}{3}B_1$ (simple root).

In the expressions for R and Q, the constant ${}_3$ can be set to any nonzero number (for example, one can set ${}_3 = \; 1$).

14. $= {}_1 {}^{-1}{}^{2} + {}_2 {}^{-1}{}^{2}.$

Solution in parametric form:

$$= R, \qquad = Q^2, \qquad \text{where} \quad B_1 = A_1, \ \ B_2 = A_2.$$

15. $= {}_1 {}^{-1}{}^{2}(\ )^3 + {}_2 {}^{-1}{}^{2}(\ )^2.$

Solution in parametric form:

$$= Q^2, \qquad = R, \qquad \text{where} \quad B_1 = -A_2, \ \ B_2 = -A_1.$$

In the solutions of equations 16 and 17, the following notations are used:

$$R = \begin{cases} \tau^{\ 1\ 2}(\ {}_1\tau \ + \ {}_2\tau^{-}\ ) + \ {}_3 & \textit{if } B_1^2 + 2B_2 > 0, \\ {}_1\tau \exp(\tfrac{1}{2}B_1\tau) + \ {}_2 & \textit{if } B_1^2 + 2B_2 = 0, \\ {}_1 \exp(\tfrac{1}{2}B_1\tau)\cos(\omega\tau) + \ {}_2 & \textit{if } B_1^2 + 2B_2 < 0, \end{cases}$$

$$Q = \begin{cases} \tau^{\ 1\ 2}[(\ {}_1(B_1 + 2k)\tau \ + \ {}_2(B_1 - 2k)\tau^{-}\ ] & \textit{if } B_1^2 + 2B_2 > 0, \\ {}_1(B_1\tau + 2)\exp(\tfrac{1}{2}B_1\tau) & \textit{if } B_1^2 + 2B_2 = 0, \\ {}_1 \exp(\tfrac{1}{2}B_1\tau)[B_1\cos(\omega\tau) - 2\omega\sin(\omega\tau)] & \textit{if } B_1^2 + 2B_2 < 0, \end{cases}$$

where $k = \frac{1}{2}\sqrt{B_1^2 + 2B_2}$ *and* $\omega = \frac{1}{2}\sqrt{-(B_1^2 + 2B_2)}$.

16. $= {}_1 {}^{-1}{}^{2} + {}_2.$

Solution in parametric form:

$$= R, \qquad = \tfrac{1}{4}Q^2, \qquad \text{where} \quad B_1 = A_1, \ \ B_2 = A_2.$$

17. $= {}_1(\ )^3 + {}_2 {}^{-1}{}^{2}(\ )^2.$

Solution in parametric form:

$$= \tfrac{1}{4}Q^2, \qquad = R, \qquad \text{where} \quad B_1 = -A_2, \ \ B_2 = -A_1.$$

18. $= {}_1 {}^{1}{}^{-1}(\ )^2 + {}_2 {}^{2}.$

Solution: $= {}_1 \exp\left(-\dfrac{\ '}{(A_1 {}^{1} - 1)}\right)$, where $= (\ )$ is the general solution of the second-order linear equation $(A_1 {}^{1} - 1)\ '' - A_1 {}_1 {}^{1-1}\ ' + A_1 {}^{2}(A_1 {}^{1} - 1)^2 = 0.$

19. $= {}_1 {}^{1}(\ )^3 + {}_2 {}^{-1} {}^{2}.$

Solution: $= {}_1 \exp\left(\dfrac{\ '}{(A_2 {}^{2} + 1)}\right)$, where $= (\ )$ is the general solution of the second-order linear equation $(A_2 {}^{2} + 1)\ '' - A_2 {}_2 {}^{2-1}\ ' - A_1 {}^{1}(A_2 {}^{2} + 1)^2 = 0.$

2.7. Equations of the Form $y = f(\ )(y)(y\ )$

See Section 2.3 for the case $(\ ) = \text{const}$, $g(\ ) = \text{const}$, $(\ ) = \text{const}$.

See Section 2.5 for the case $(\ ) = \text{const}$, $g(\ ) = \text{const}$, $(\ ) = \text{const}$.

2.7.1. Equations of the Form $= (\ \ )g(\ \ )$

1. $= {}^{-2} - \dfrac{2(\ \ +1)}{(\ \ +3)^2} + \quad , \quad \neq -3, \quad \neq -1.$

See equation 2.4.2.4.

2. $= {}^{-2}(\frac{15}{4} \ \ + \ \ {}^{-7}).$

See equation 2.4.2.35.

3. $= {}^{-2}(6 \ \ + \ \ {}^{-4}).$

See equation 2.4.2.31.

4. $= {}^{-2}(12 \ \ + \ \ {}^{-5}\ {}^{2}).$

See equation 2.4.2.64.

5. $= {}^{-2}(2 \ \ + \ \ {}^{-2}).$

See equation 2.4.2.6.

6. $= {}^{-2}(-\frac{3}{16} \ \ + \ \ {}^{-5}\ {}^{3}).$

See equation 2.4.2.26.

7. $= {}^{-2}(-\frac{9}{100} \ \ + \ \ {}^{-5}\ {}^{3}).$

See equation 2.4.2.10.

8. $= {}^{-2}(\frac{3}{4} \ \ + \ \ {}^{-5}\ {}^{3}).$

See equation 2.4.2.12.

9. $= {}^{-2}(\frac{63}{4} \ \ + \ \ {}^{-5}\ {}^{3}).$

See equation 2.4.2.66.

10. $= {}^{-2}(-\frac{5}{36} \ \ + \ \ {}^{-7}\ {}^{5}).$

See equation 2.4.2.29.

11. $= {}^{-2}(-\frac{2}{9} \ \ + \ \ {}^{-1}\ {}^{2}).$

See equation 2.4.2.14.

12. $= {}^{-2}(-\frac{4}{25} \ \ + \ \ {}^{-1}\ {}^{2}).$

See equation 2.4.2.8.

13. $= {}^{-2}(20 \ \ + \ \ {}^{-1}\ {}^{2}).$

See equation 2.4.2.33.

14. $= {}^{-2}(-\frac{12}{49} \ \ + \ \ {}^{1}\ {}^{2}).$

See equation 2.4.2.37.

15. $= {}^{-2}(\ \ {}^{2} - \frac{6}{25}\ \ ).$

See equation 2.4.2.60.

16. $= {}^{-2}(\ \ {}^{2} + \frac{6}{25}\ \ ).$

See equation 2.4.2.62.

17. $= {}^{-4\ 3}(\quad + B \quad {}^{-1\ 2})$.

See equation 2.4.2.40.

18. $= (\quad {}^{4} + B \quad {}^{3}) \quad {}^{-7}$.

See equation 2.4.2.39.

19. $= (\quad {}^{2} + B) \quad {}^{-5}$.

See equation 2.4.2.16.

20. $= (\quad {}^{-1} + B \quad {}^{-2}) \quad {}^{-2}$.

See equation 2.4.2.28.

21. $= (\quad {}^{-7\ 3} + B \quad {}^{-10\ 3}) \quad {}^{-5\ 3}$.

See equation 2.4.2.48.

22. $= (\quad {}^{-4\ 3} + B \quad {}^{-10\ 3}) \quad {}^{-5\ 3}$.

See equation 2.4.2.49.

23. $= (\quad {}^{-4\ 3} + B \quad {}^{-7\ 3}) \quad {}^{-5\ 3}$.

See equation 2.4.2.24.

24. $= (\quad {}^{-2\ 3} + B \quad {}^{-4\ 3}) \quad {}^{-5\ 3}$.

See equation 2.4.2.90.

25. $= (\quad + B \quad {}^{-2\ 3}) \quad {}^{-5\ 3}$.

See equation 2.4.2.89.

26. $= (\quad {}^{2} + B) \quad {}^{-5\ 3}$.

See equation 2.4.2.47.

27. $= (\quad {}^{2} + B \quad) \quad {}^{-5\ 3}$.

See equation 2.4.2.46.

28. $= \quad (\quad {}^{-2\ 3} + \quad {}^{-5\ 3})^{2} \quad {}^{-5\ 3}$.

This is a special case of equation 2.7.1.37 with $\quad = 1$ and $\quad = 0$.

29. $= (\quad {}^{-8\ 5} + B \quad {}^{-13\ 5}) \quad {}^{-7\ 5}$.

See equation 2.4.2.25.

30. $= (\quad {}^{-5\ 2} + B \quad {}^{-7\ 2}) \quad {}^{-1\ 2}$.

See equation 2.4.2.23.

31. $= \quad (\quad {}^{5} + \quad {}^{4})^{-1\ 2} \quad {}^{-1\ 2}$.

This is a special case of equation 2.7.1.38 with $\quad = 1$ and $\quad = 0$.

32. $= \quad (\quad {}^{15\ 8} + \quad {}^{7\ 8})^{-4\ 3} \quad {}^{-1\ 2}$.

This is a special case of equation 2.7.1.39 with $\quad = 1$ and $\quad = 0$.

33. $= \quad (\quad {}^{7\ 3} + \quad {}^{4\ 3})^{-15\ 7} \quad {}^{2}$.

This is a special case of equation 2.7.1.40 with $\quad = 1$ and $\quad = 0$.

34. $= (\quad {}^2 + \quad + \;) \; {}^{-5\;\;3}$.

The transformation $\; = \; (\;),\; = (\;')^{3\;\;2}$ leads to a third-order equation: $2\;\;'\;\;'''\; - (\;\;'')^2 = \frac{4}{3}(a\;\;{}^2 + \;\;+\;)$. Differentiating the latter equation with respect to and dividing it by $'$, we obtain a fourth-order constant coefficient linear equation: $3\;\;'''' = 4a\;\; + 2\;$.

35. $= (\quad {}^{-10\;\;3} + \quad {}^{-7\;\;3} + \quad {}^{-4\;\;3}) \; {}^{-5\;\;3}$.

The transformation $\; = 1\;\;,\;\; = \;$ leads to an equation of the form 2.7.1.34: $\;'' = (a\;{}^2 + \;\;+\;)\;{}^{-5\;\;3}$.

36. $= \; (\quad {}^2 + \quad + \;) \quad {}^{-2\;\;-3}$.

This is a special case of equation 2.9.1.21 with $(\;) = k\;\;{}^{-2}$. Setting $(\;) = \;(a\;\;{}^2 + \;\;+\;)^{-1\;\;2}$ and integrating the equation, we obtain a first-order separable equation: $(a\;\;{}^2 + \;\;+\;)^2(\;')^2 = (\frac{1}{4}\;{}^2 - a\;)\;{}^2 - \frac{}{+1}\;{}^{-2\;\;-2} + \;{}_1$.

37. $= \; (\quad + \;)^2 (\quad + \;)^{-10\;\;3\;\;-5\;\;3}$.

The transformation $\xi = \dfrac{a\;\;+}{\;\;+\;\;}$, $\; = \dfrac{}{\;\;+\;\;}$ leads to an Emden–Fowler equation of the form 2.3.1.9: $\;'' = A\Delta^{-2}\xi^{2}\;\;{}^{-5\;\;3}$, where $\Delta = a \;\; - \;\;$.

38. $= \; (\quad + \;)^{-1\;\;2} (\quad + \;)^{-2\;\;-1\;\;2}$.

The transformation $\xi = \dfrac{a\;\;+}{\;\;+\;\;}$, $\; = \dfrac{}{\;\;+\;\;}$ leads to an Emden–Fowler equation of the form 2.3.1.25: $\;'' = A\Delta^{-2}\xi^{-1\;\;2}\;\;{}^{-1\;\;2}$, where $\Delta = a \;\; - \;\;$.

39. $= \; (\quad + \;)^{-4\;\;3} (\quad + \;)^{-7\;\;6\;\;-1\;\;2}$.

The transformation $\xi = \dfrac{a\;\;+}{\;\;+\;\;}$, $\; = \dfrac{}{\;\;+\;\;}$ leads to an Emden–Fowler equation of the form 2.3.1.17: $\;'' = A\Delta^{-2}\xi^{-4\;\;3}\;\;{}^{-1\;\;2}$, where $\Delta = a \;\; - \;\;$.

40. $= \; (\quad + \;)^{-15\;\;7} (\quad + \;)^{-20\;\;7\;\;\;2}$.

The transformation $\xi = \dfrac{a\;\;+}{\;\;+\;\;}$, $\; = \dfrac{}{\;\;+\;\;}$ leads to an Emden–Fowler equation of the form 2.3.1.20: $\;'' = A\Delta^{-2}\xi^{-15\;\;7}\;\;{}^{2}$, where $\Delta = a \;\; - \;\;$.

41. $= \quad \mathbf{exp}(\quad {}^2 + \quad) \, \mathbf{exp}(\quad)$.

The substitution $k \;\; = k \;\; + a \;\; {}^2 + \;\;$ leads to an autonomous equation of the form 2.9.1.1: $\;'' = Ae\;\;{}^{w} + 2ak^{-1}$.

2.7.2. Equations Containing Power Functions (const)

1. $= \dfrac{2(\;\;+1)}{(\;\;+3)^2} \; + \quad {}^{-2}(\quad)^3, \qquad \neq -3, \quad \neq -1.$

See equation 2.6.2.116.

2. $= (-\frac{15}{4} \quad + \quad {}^{-7}) \; {}^{-2}(\quad)^3$.

See equation 2.6.2.117.

3. $= (-6 \quad + \quad {}^{-4}) \; {}^{-2}(\quad)^3$.

See equation 2.6.2.118.

4. $= (-12 \quad + \quad {}^{-5}\,{}^{2})\,{}^{-2}(\quad)^3.$
See equation 2.6.2.119.

5. $= (-2 \quad + \quad {}^{-2})\,{}^{-2}(\quad)^3.$
See equation 2.6.2.120.

6. $= (\frac{3}{16} \quad + \quad {}^{-5}\,{}^{3})\,{}^{-2}(\quad)^3.$
See equation 2.6.2.121.

7. $= (\frac{9}{100} \quad + \quad {}^{-5}\,{}^{3})\,{}^{-2}(\quad)^3.$
See equation 2.6.2.122.

8. $= (-\frac{3}{4} \quad + \quad {}^{-5}\,{}^{3})\,{}^{-2}(\quad)^3.$
See equation 2.6.2.123.

9. $= (-\frac{63}{4} \quad + \quad {}^{-5}\,{}^{3})\,{}^{-2}(\quad)^3.$
See equation 2.6.2.124.

10. $= (\frac{5}{36} \quad + \quad {}^{-7}\,{}^{5})\,{}^{-2}(\quad)^3.$
See equation 2.6.2.125.

11. $= (\frac{2}{9} \quad + \quad {}^{-1}\,{}^{2})\,{}^{-2}(\quad)^3.$
See equation 2.6.2.126.

12. $= (\frac{4}{25} \quad + \quad {}^{-1}\,{}^{2})\,{}^{-2}(\quad)^3.$
See equation 2.6.2.127.

13. $= (-20 \quad + \quad {}^{-1}\,{}^{2})\,{}^{-2}(\quad)^3.$
See equation 2.6.2.128.

14. $= (\frac{12}{49} \quad + \quad {}^{1}\,{}^{2})\,{}^{-2}(\quad)^3.$
See equation 2.6.2.129.

15. $= (\quad {}^{2} + \frac{6}{25} \quad)\,{}^{-2}(\quad)^3.$
See equation 2.6.2.130.

16. $= (\quad {}^{2} - \frac{6}{25} \quad)\,{}^{-2}(\quad)^3.$
See equation 2.6.2.131.

17. $= (\quad + B \;{}^{-1}\,{}^{2})\,{}^{-4}\,{}^{3}(\quad)^3.$
See equation 2.6.2.15.

18. $= \;{}^{-7}(\quad {}^{4} + B \;{}^{3})(\quad)^3.$
See equation 2.6.2.111.

19. $= \;{}^{-5}(\quad {}^{2} + B)(\quad)^3.$
See equation 2.6.2.96.

20. $= \;{}^{-2}(\quad {}^{-1} + B \;{}^{-2})(\quad)^3.$
See equation 2.6.2.110.

21. $= {}^{-5\;3}(\quad{}^{-7\;3} + B\quad{}^{-10\;3})(\quad)^3$.

See equation 2.6.2.34.

22. $= {}^{-5\;3}(\quad{}^{-4\;3} + B\quad{}^{-10\;3})(\quad)^3$.

See equation 2.6.2.36.

23. $= {}^{-5\;3}(\quad{}^{-4\;3} + B\quad{}^{-7\;3})(\quad)^3$.

See equation 2.6.2.14.

24. $= {}^{-5\;3}(\quad{}^{-2\;3} + B\quad{}^{-4\;3})(\quad)^3$.

See equation 2.6.2.115.

25. $= {}^{-5\;3}(\quad + B\quad{}^{-2\;3})(\quad)^3$.

See equation 2.6.2.114.

26. $= {}^{-5\;3}(\quad{}^{2} + B)(\quad)^3$.

See equation 2.6.2.35.

27. $= {}^{-5\;3}(\quad{}^{2} + B\quad)(\quad)^3$.

See equation 2.6.2.33.

28. $= \quad{}^{-5\;3}(\quad{}^{-2\;3} + \quad{}^{-5\;3})^2(\quad)^3$.

This is a special case of equation 2.7.2.37 with $\quad = 1$ and $\quad = 0$.

29. $= {}^{-7\;5}(\quad{}^{-8\;5} + B\quad{}^{-13\;5})(\quad)^3$.

See equation 2.6.2.109.

30. $= {}^{-1\;2}(\quad{}^{-5\;2} + B\quad{}^{-7\;2})(\quad)^3$.

See equation 2.6.2.13.

31. $= \quad{}^{-1\;2}(\quad{}^{5} + \quad{}^{4})^{-1\;2}(\quad)^3$.

This is a special case of equation 2.7.2.38 with $\quad = 1$ and $\quad = 0$.

32. $= \quad{}^{-1\;2}(\quad{}^{15\;8} + \quad{}^{7\;8})^{-4\;3}(\quad)^3$.

This is a special case of equation 2.7.2.39 with $\quad = 1$ and $\quad = 0$.

33. $= \quad{}^{2}(\quad{}^{7\;3} + \quad{}^{4\;3})^{-15\;7}(\quad)^3$.

This is a special case of equation 2.7.2.40 with $\quad = 1$ and $\quad = 0$.

34. $= {}^{-5\;3}(\quad{}^{2} + \quad + \quad)(\quad)^3$.

Taking to be the independent variable, we obtain an equation of the form 2.7.1.34 with respect to $= (\;)$: $'' = -(a\quad{}^{2} + \quad + \quad)\quad{}^{-5\;3}$.

35. $= {}^{-5\;3}(\quad{}^{-10\;3} + \quad{}^{-7\;3} + \quad{}^{-4\;3})(\quad)^3$.

Taking to be the independent variable, we obtain an equation of the form 2.7.1.35 with respect to $= (\;)$: $'' = -(a\quad{}^{-10\;3} + \quad{}^{-4\;3} + \quad{}^{-4\;3})\quad{}^{-5\;3}$.

36. $= {}^{-2\;-3}(\quad{}^{2} + \quad + \quad)\;(\quad)^3$.

Taking to be the independent variable, we obtain an equation of the form 2.9.1.21 (for $(\xi) = -\xi^{-2}\;$) with respect to $= (\;)$: $'' = -(a\quad{}^{2} + \quad + \quad)\quad{}^{-2\;-3}$.

37. $= \;\; ^{-5\;\;3}(\;\;+\;\;)^2(\;\;+\;\;)^{-10\;\;3}(\;\;)^3.$

Taking to be the independent variable, we obtain an equation of the form 2.7.1.37 with respect to $=\;(\;)$: $'' = -A(a\;+\;)^2(\;\;+\;\;)^{-10\;\;3\;\;-5\;\;3}.$

38. $= \;\; ^{-1\;\;2}(\;\;+\;\;)^{-1\;\;2}(\;\;+\;\;)^{-2}(\;\;)^3.$

Taking to be the independent variable, we obtain an equation of the form 2.7.1.38 with respect to $=\;(\;)$: $'' = -A(a\;+\;)^{-1\;\;2}(\;\;+\;\;)^{-2\;\;-1\;\;2}.$

39. $= \;\; ^{-1\;\;2}(\;\;+\;\;)^{-4\;\;3}(\;\;+\;\;)^{-7\;\;6}(\;\;)^3.$

Taking to be the independent variable, we obtain an equation of the form 2.7.1.39 with respect to $=\;(\;)$: $'' = -A(a\;+\;)^{-4\;\;3}(\;\;+\;\;)^{-7\;\;6\;\;-1\;\;2}.$

40. $= \;\; ^2(\;\;+\;\;)^{-15\;\;7}(\;\;+\;\;)^{-20\;\;7}(\;\;)^3.$

Taking to be the independent variable, we obtain an equation of the form 2.7.1.40 with respect to $=\;(\;)$: $'' = -A(a\;+\;)^{-15\;\;7}(\;\;+\;\;)^{-20\;\;7\;\;2}.$

41. $= \;\; ^{-1\;\;2\;\;-2}[(\;\;)^2 + B^2]^{1\;\;2}.$

Solution in parametric form:

$$= a(\;^2 - 1)^{-1}(\tau \quad R)^2, \qquad = \tau^{-1}(\;^2 - 1)^{-1\;\;2},$$

where $R = \sqrt{\tau^2 - 2\tau^{-1} + \;_1}$, $= \mp \tanh \;\;_2 + \;\; R^{-1}\;\;\tau\;\;$, $A = -\tfrac{1}{2}a^{-1\;\;2\;\;2}$, $B = \tfrac{1}{2}a^{-1}\;\;$.

42. $= \;\; ^{-1\;\;2\;\;-2}[(\;\;)^2 - B^2]^{1\;\;2}.$

Solution in parametric form:

$$= a(\;^2 + 1)^{-1}(\tau \quad R)^2, \qquad = \tau^{-1}(\;^2 + 1)^{-1\;\;2},$$

where $R = \sqrt{\;_1 - \tau^2 - 2\tau^{-1}}$, $= \;\tan \;\;_2 + \;\; R^{-1}\;\;\tau\;\;$, $A = -\tfrac{1}{2}a^{-1\;\;2\;\;2}$, $B = \tfrac{1}{2}a^{-1}\;\;$.

43. $= \;\; ^{-1\;\;2\;\;-2}[B^2 - (\;\;)^2]^{1\;\;2}.$

Solution in parametric form:

$$= a(1 - \;^2)^{-1}(\tau \;\mp R)^2, \qquad = \tau^{-1}(1 - \;^2)^{-1\;\;2},$$

where $R = \sqrt{\tau^2 - 2\tau^{-1} + \;_1}$, $= \;\tanh \;\;_2 + \;\; R^{-1}\;\;\tau\;\;$, $A = -\tfrac{1}{2}a^{-1\;\;2\;\;2}$, $B = \tfrac{1}{2}a^{-1}\;\;$.

44. $= \;\; ^{-2\;\;-1\;\;2}(\;\;)^2[(\;\;)^2 + B^2]^{1\;\;2}.$

Solution in parametric form:

$$= a\tau^{-1}(\;^2 - 1)^{-1\;\;2}, \qquad = (\;^2 - 1)^{-1}(\tau \quad R)^2,$$

where $R = \sqrt{\tau^2 - 2\tau^{-1} + \;_1}$, $= \mp \tanh \;\;_2 + \;\; R^{-1}\;\;\tau\;\;$, $A = -\tfrac{1}{4}a^{3\;\;-3\;\;2}$, $B = 2a^{-1}\;\;$.

45. $= \;\; ^{-2\;\;-1\;\;2}(\;\;)^2[(\;\;)^2 - B^2]^{1\;\;2}.$

Solution in parametric form:

$$= a\tau^{-1}(1 - \;^2)^{-1\;\;2}, \qquad = (1 - \;^2)^{-1}(\tau \;\mp R)^2,$$

where $R = \sqrt{\tau^2 - 2\tau^{-1} + \;_1}$, $= \;\tanh \;\;_2 + \;\; R^{-1}\;\;\tau\;\;$, $A = -\tfrac{1}{4}a^{3\;\;-3\;\;2}$, $B = 2a^{-1}\;\;$.

46. $= \;\; ^{-2\;\;-1\;\;2}(\;\;)^2[B^2 - (\;\;)^2]^{1\;\;2}.$

Solution in parametric form:

$$= a\tau^{-1}(1 + \;^2)^{-1\;\;2}, \qquad = (1 + \;^2)^{-1}(\tau \quad R)^2,$$

where $R = \sqrt{-\tau^2 - 2\tau^{-1} + \;_1}$, $= \;\tan \;\;_2 + \;\; R^{-1}\;\;\tau\;\;$, $A = -\tfrac{1}{4}a^{3\;\;-3\;\;2}$, $B = 2a^{-1}\;\;$.

2.7.3. Equations Containing Exponential Functions (const)

2.7.3-1. Preliminary remarks.

1 . If $l \neq 1 -$, the equation

$$'' = Ae \qquad (\ ') \tag{1}$$

has a particular solution:

$$= Be^{\lambda}\ , \qquad \text{where} \quad \lambda = \frac{1}{1 - \quad - l}, \qquad B = (A\lambda\ ^{-2})^{\lambda}.$$

2 . If $\neq 0$ and $l \neq 1$, equation (1) can be reduced with the aid of the transformation

$$= (\ ')^{1-}\ , \qquad = e$$

to a generalized Emden–Fowler equation with respect to $= \ (\ )$:

$$'' = B \,{}^{\frac{1}{1-}}\ {}^{-1}(\ ')^{\overline{2\ \ +1}}, \tag{2}$$

where $B = -\ \left[A(1-l)\right]^{\overline{1}}$. Equations of the form (2) are outlined in Section 2.5.

Whenever the general solution $= \ (\ )$ of the Emden–Fowler equation (2) is obtained, the solution of the original equation (1) can be written out in parametric form as:

$$= \ln\ \ , \qquad = k(\ ')^{-\overline{1}}, \qquad \text{where} \quad k = \left[A(1-l)\right]^{-\overline{1}}.$$

3 . If $l \neq \ +2$, the equation

$$'' = A \quad e\ (\ ') \tag{3}$$

has a particular solution:

$$= \lambda \ln(B\ \ ), \qquad \text{where} \quad \lambda = l - \ \ - 2, \qquad B = \left(-\frac{\lambda^{1-}}{A}\right)^{\frac{1}{\lambda}}.$$

4 . Taking to be the independent variable and to be the dependent one, we obtain from equation (3) an equation of the form (1) for $= \ (\ )$:

$$'' = -Ae \qquad (\ ')^{3-}\ .$$

5 . If $\neq -1$ and $l \neq 1$, equation (3) can be reduced with the aid of the transformation

$$= (\ ')^{1-}\ , \qquad = \quad {}^{+1}$$

to a generalized Emden–Fowler equation for $= \ (\ )$:

$$'' = -\frac{1}{\ +1}\ {}^{\frac{1}{1-}}\ {}^{-\overline{\ \ +1}}(\ ')^{2}. \tag{4}$$

Equations of this form are outlined in Section 2.5.

Whenever the general solution $= \ (\ )$ of the Emden–Fowler equation (4) is obtained, the solution of the original equation (3) can be written out in parametric form as:

$$= \ {}^{\frac{1}{\ +1}}, \qquad = -\ln(\ ') + \ln\frac{\ +1}{A(1-l)}.$$

2.7.3-2. Solvable equations and their solutions.

1. = $^{\lambda}$.

Solution: $= \begin{cases} -\dfrac{1}{\lambda}\ln \dfrac{a\lambda}{2\ \ _1^2}\sin^2(\ _1\ +\ _2) & \text{if } a\lambda > 0, \\ -\dfrac{1}{\lambda}\ln \dfrac{a\lambda}{2\ \ _1^2}\sinh^2(\ _1\ +\ _2) & \text{if } a\lambda > 0, \\ -\dfrac{1}{\lambda}\ln -\dfrac{a\lambda}{2\ \ _1^2}\cosh^2(\ _1\ +\ _2) & \text{if } a\lambda < 0. \end{cases}$

2. = ( ).

1 . Solution in parametric form with $l \neq 1$:

$$= \ln \frac{1}{A(1-l)}\ _1^{1-}\ \tau\ , \qquad = \ _1 \ \frac{1}{\tau}\big(1\ \ \tau\ ^{\frac{1}{1-}}\ \ \tau + \ _2.$$

2 . Solution in parametric form with $l = 1$:

$$= \ln \frac{\tau}{A}\ , \qquad = \ _1 \ \frac{1}{\tau}\exp(\ \tau)\ \ \tau + \ _2.$$

3. = ( )^2.

1 . Solution in parametric form with $\neq -1$:

$$= \ \frac{\tau}{\ } + \ _2, \qquad = \tau \exp -\frac{1}{\ +1}\ \ \frac{\tau}{\ } + \ _2\ ,$$

where the function = (τ) is defined implicitly by the relation

$$\ln \ \frac{\ }{\tau} - \frac{1}{\ +1} \ - \frac{\tau}{(\ +1)\ \ -\tau} = \frac{A}{\ +1}\tau^{\ +1} - \ln\tau + \ _1.$$

2 . Solution for $= -1$:

$$= \ _2 \exp \ \frac{\ }{\ + Ae\ \ + \ _1}\ .$$

4. = .

1 . Solution for $A > 0$:

$$= \ _1\ _0\big(2\ \sqrt{A}\,e\ \ ^2 + \ _2\ _0\big(2\ \sqrt{A}\,e\ \ ^2,$$

where $_0(z)$ and $_0(z)$ are the modified Bessel functions.

2 . Solution for $A < 0$:

$$= \ _1\ _0\big(2\ \sqrt{-A}\,e\ \ ^2 + \ _2\ _0\big(2\ \sqrt{-A}\,e\ \ ^2,$$

where $_0(z)$ and $_0(z)$ are the Bessel functions.

5. = $^{-1\ 2}(\ )^{3\ 2}$.

Solution in parametric form:

$$= \tau^2 - \ln(A\ \ ), \qquad = \ _1\big[2\tau\ \ - \exp(\tau^2)\big]^2, \qquad \text{where} \quad = \ \exp(\tau^2)\ \ \tau + \ _2.$$

6. $\quad = \quad (\quad)^{3\ 2}$**.**

Solution in parametric form:

$$=-\ln\left[A\ {}_1^3\left(\ \overline{\tau^2+\tau}-\ \right)\right],\qquad =2\ {}_1^2\ 1-\sqrt{\frac{\tau+1}{\tau}}\ ,\qquad \text{where}\qquad =\ln\left(\ \overline{\tau}+\ \overline{\tau+1}\ \right)+\ _2.$$

7. $\quad = \quad (\quad)$ **.**

1 . Solution in parametric form with $l \neq 2$:

$$=\ _1\ \frac{1}{\tau}(1\quad \tau)^{\frac{1}{-2}}\ \tau+\ _2,\qquad =\ln\ \frac{1}{A(2-l)}\ {}_1^{-2}\tau\ .$$

2 . Solution in parametric form with $l = 2$:

$$=\ _1\ \frac{1}{\tau}\exp(\mp\tau)\ \tau+\ _2,\qquad =\ln\ \frac{\tau}{A}\ .$$

8. $\quad = \qquad$ **.**

1 . Solution in parametric form with $\neq -1$:

$$=\tau\exp\ -\frac{1}{+1}\quad \frac{\tau}{\ }+\ _2\ ,\qquad =\ \frac{\tau}{\ }+\ _2,$$

where the function $\ = \ (\tau)$ is defined implicitly by the relation

$$\ln\ \frac{\ }{\tau}-\frac{1}{+1}\ -\frac{\tau}{(\ +1)\ -\tau}=-\frac{A}{+1}\tau^{\ +1}-\ln\tau+\ _1.$$

2 . Solution for $\ = -1$:

$$=\ _2\exp\ \frac{\ }{-Ae\ +\ _1}\ .$$

9. $\quad = \quad {}^{-1\ 2}\ (\quad)^{3\ 2}$**.**

Solution in parametric form:

$$=\ _1\left[2\tau\ -\exp(\tau^2)\right]^2,\qquad =\tau^2-\ln(-A\ ),\qquad \text{where}\qquad =\ \exp(\tau^2)\ \tau+\ _2.$$

10. $\quad = \quad (\quad)^{3\ 2}$**.**

Solution in parametric form:

$$=2\ {}_1^2\ 1-\sqrt{\frac{\tau+1}{\tau}}\Bigg),\qquad =-\ln\left[A\ {}_1^3\left(\ -\ \overline{\tau^2+\tau}\ \right)\right],\qquad \text{where}\qquad =\ln\left(\ \overline{\tau}+\ \overline{\tau+1}\ \right)+\ _2.$$

11. $\quad = \quad (\quad)^3$**.**

1 . Solution in parametric form with $A > 0$:

$$=\ _1\ _0(2\tau)+\ _2\ _0(2\tau),\qquad =\ln\left(\tau\quad \overline{A}\ \right),$$

where $_0(z)$ and $_0(z)$ are the Bessel functions.

2 . Solution in parametric form with $A < 0$:

$$=\ _1\ _0(2\tau)+\ _2\ _0(2\tau),\qquad =\ln\left(\tau\quad \overline{-A}\ \right),$$

where $_0(z)$ and $_0(z)$ are the modified Bessel functions.

12. $= \quad (\quad).$

Solution in parametric form:

$$= \frac{\tau}{\tau} + \quad _2, \qquad = \ln \frac{}{A} - \frac{\tau}{\tau} - \quad _2,$$

$$\text{where} \quad = \begin{cases} \frac{1}{2-l}\tau^{2-} + \frac{1}{1-l}\tau^{1-} + \quad _1 & \text{if } l \neq 1, 2; \\ \tau + \ln|\tau| + \quad _1 & \text{if } l = 1; \\ \ln|\tau| - \frac{1}{\tau} + \quad _1 & \text{if } l = 2. \end{cases}$$

13. $= \quad \exp(\quad) \exp(\quad ^2 + \quad)(\quad)^3.$

Taking to be the independent variable, we obtain an equation of the form 2.7.1.41 with respect to $= ( ):$ $'' = -A \exp(a \quad ^2 + \quad) \exp(k \quad).$

14. $= \quad ^{-1} \, ^2(\quad)^3 \, ^2 \quad \overline{\quad - 2B}.$

Solution in parametric form:

$$= \ln[a\tau(\cosh \quad)^{-1}], \qquad = B \cosh^2 \quad (\tau \tanh \quad R)^2,$$

where $a = -A^{-1}B^{-1} \, ^2$, $R = \overline{2\ln\tau + \tau^2 + \quad _1}$, $= \quad _2 \mp \quad R^{-1} \quad \tau.$

15. $= \quad ^{-1} \, ^2(\quad)^3 \, ^2 \quad \overline{2B - \quad }.$

Solution in parametric form:

$$= \ln[a\tau(\cos \quad)^{-1}], \qquad = B \cos^2 \quad (\tau \tan \quad R)^2,$$

where $a = -A^{-1}B^{-1} \, ^2$, $R = \overline{2\ln\tau - \tau^2 + \quad _1}$, $= \quad _2 \quad R^{-1} \quad \tau.$

16. $= \quad ^{-1} \, ^2 \quad \overline{\quad - B}.$

Solution in parametric form:

$$= \frac{1}{2B} \cos^2 \quad (\tau \tan \quad R)^2, \qquad = \ln[\quad \tau(\cos \quad)^{-1}],$$

where $= A^{-1} \quad \overline{2}$, $R = \overline{2\ln\tau - \tau^2 + \quad _1}$, $= \quad _2 \quad R^{-1} \quad \tau.$

17. $= \quad ^{-1} \, ^2 \quad \overline{B - \quad }.$

Solution in parametric form:

$$= \frac{1}{2B} \cosh^2 \quad (\tau \tanh \quad R)^2, \qquad = \ln[\quad \tau(\cosh \quad)^{-1}],$$

where $= A^{-1} \quad \overline{2}$, $R = \overline{2\ln\tau + \tau^2 + \quad _1}$, $= \quad _2 \mp \quad R^{-1} \quad \tau.$

2.7.4. Equations Containing Hyperbolic Functions (const)

1. $= \quad [\cosh(\quad)]^{-2} \quad .$

Solution in parametric form:

$$= a \cosh \quad (\tau \tanh \quad R), \qquad = \quad \lambda,$$

where $A = a^{-2}$, $R = \overline{2\ln\tau + \tau^2 + \quad _1}$, $= \quad _2 \mp \quad R^{-1} \quad \tau.$

2. $=\quad[\sinh(\quad)]^{-2}\quad.$

Solution in parametric form:

$$= a\sinh\quad(\tau\coth\qquad R),\qquad =\quad\lambda,$$

where $A = a^{-2},\ R = \overline{2\ln\tau + \tau^2 + \quad_1},\quad = \ _2\mp\quad R^{-1}\quad\tau.$

3. $=\quad\cosh(\quad)(\quad)^{3\ 2}.$

Solution in parametric form:

$$= a(\ ^2+1)^{-1\ 2}(\tau\qquad R),\qquad =\lambda^{-1}\ln\big(\quad+\quad\overline{\ ^2+1}\quad,$$

where $A = 2a^{-2}\quad\overline{a\lambda},\ R = \overline{\ _1 - \tau^2 - 2\tau^{-1}},\quad =\quad\tan\quad_2 +\quad R^{-1}\quad\tau\quad.$

4. $=\quad\sinh(\quad)(\quad)^{3\ 2}.$

Solution in parametric form:

$$= a(\ ^2-1)^{-1\ 2}(\tau\qquad R),\qquad =\quad\lambda^{-1}\ln\big(\quad+\quad\overline{\ ^2-1}\quad,$$

where $A = \quad 2a^{-2}\quad\overline{a\lambda},\ R = \overline{\ _1 + \tau^2 - 2\tau^{-1}},\quad = \mp\tanh\quad_2 +\quad R^{-1}\quad\tau\quad.$

5. $=\quad\cosh(\quad)\ (\quad)^{3\ 2}.$

Solution in parametric form:

$$= \lambda^{-1}\ln(\quad+\quad\overline{\ ^2+1}),\qquad =\ (\ ^2+1)^{-1\ 2}(\tau\qquad R),$$

where $A = -2\ ^{-2}\quad\overline{\lambda},\ R = \overline{\ _1 - \tau^2 - 2\tau^{-1}},\quad =\quad\tan\quad_2 +\quad R^{-1}\quad\tau\quad.$

6. $=\quad\sinh(\quad)\ (\quad)^{3\ 2}.$

Solution in parametric form:

$$=\quad\lambda^{-1}\ln(\quad+\quad\overline{\ ^2-1}),\qquad =\ (\ ^2-1)^{-1\ 2}(\tau\qquad R),$$

where $A = \mp 2\ ^{-2}\quad\overline{\lambda},\ R = \overline{\ _1 + \tau^2 - 2\tau^{-1}},\quad = \mp\tanh\quad_2 +\quad R^{-1}\quad\tau\quad.$

7. $=\quad[\cosh(\quad)]^{-2}\ (\quad)^2.$

Solution in parametric form:

$$=\quad\lambda,\qquad =\quad\cosh\quad(\tau\tanh\qquad R),$$

where $A = -\ ^{-2},\ R = \overline{2\ln\tau + \tau^2 + \quad_1},\quad = \ _2\mp\quad R^{-1}\quad\tau.$

8. $=\quad[\sinh(\quad)]^{-2}\ (\quad)^2.$

Solution in parametric form:

$$=\quad\lambda,\qquad =\quad\sinh\quad(\tau\coth\qquad R),$$

where $A = -\ ^{-2},\ R = \overline{2\ln\tau + \tau^2 + \quad_1},\quad = \ _2\mp\quad R^{-1}\quad\tau.$

2.7.5. Equations Containing Trigonometric Functions (const)

In the solutions of equations 1–4, the following notation is used:

$$R = \sqrt{2 \ln \tau - \tau^2 + \quad {}_1}, \qquad = \quad {}_2 \qquad R^{-1} \quad \tau.$$

1. = [cos( )]$^{-2}$.

Solution in parametric form:

$$= a \cos \quad (\tau \tan \qquad R), \qquad = \quad \lambda, \qquad \text{where} \quad A = a^{-2}.$$

2. = [sin( )]$^{-2}$.

Solution in parametric form:

$$= a \sin \quad (\tau \cot \quad \mp R), \qquad = \quad \lambda, \qquad \text{where} \quad A = a^{-2}.$$

3. = [cos( )]$^{-2}$ ( )^2.

Solution in parametric form:

$$= \lambda^{-1} \quad , \qquad = \quad \cos \quad (\tau \tan \qquad R), \qquad \text{where} \quad A = - \quad ^2.$$

4. = [sin( )]$^{-2}$ ( )^2.

Solution in parametric form:

$$= \lambda^{-1} \quad , \qquad = \quad \sin \quad (\tau \cot \quad \mp R), \qquad \text{where} \quad A = - \quad ^2.$$

In the solutions of equations 5–8, the following notation is used:

$$R = \sqrt{\tau^2 - 2\tau^{-1} + \quad {}_1}, \qquad = \tanh \quad {}_2 + \quad R^{-1} \quad \tau \quad .$$

5. = cos( ) ( )$^{3\ 2}$.

Solution in parametric form:

$$= a\big(1 - \quad ^{2 \quad -1 \quad 2}(\tau \quad \mp R), \qquad = \lambda^{-1} \arccos \quad , \qquad \text{where} \quad A = 2a^{-2}(-a\lambda)^{1 \quad 2}.$$

6. = sin( ) ( )$^{3\ 2}$.

Solution in parametric form:

$$= a\big(1 - \quad ^{2 \quad -1 \quad 2}(\tau \quad \mp R), \qquad = \lambda^{-1} \arccos \quad , \qquad \text{where} \quad A = 2a^{-2}(a\lambda)^{1 \quad 2}.$$

7. = cos( ) ( )$^{3\ 2}$.

Solution in parametric form:

$$= \lambda^{-1} \arccos \quad , \qquad = \quad \big(1 - \quad ^{2 \quad -1 \quad 2}(\tau \quad \mp R), \qquad \text{where} \quad A = -2 \quad ^{-2}(- \quad \lambda)^{1 \quad 2}.$$

8. = sin( ) ( )$^{3\ 2}$.

Solution in parametric form:

$$= \lambda^{-1} \arccos \quad , \qquad = \quad \big(1 - \quad ^{2 \quad -1 \quad 2}(\tau \quad \mp R), \qquad \text{where} \quad A = -2 \quad ^{-2}(\quad \lambda)^{1 \quad 2}.$$

2.7.6. Some Transformations

For the sake of visualization, we also use the symbolic notation $\{\ ,\ g,\ \ \}$ to denote the equation

$$'' \;=\; {}_1(\ )g_1(\ )\ {}_1(\ ').\tag{1}$$

1 . Taking to be the independent variable and to be the dependent one, we obtain an equation of similar form for $\ = \ (\ )$:

$$'' = g_1(\ )\ {}_1(\ )\ {}^*_1(\ '),\qquad \text{where}\qquad {}^*_1(\ ) = -\ {}^3\ {}_1(1\ \ ).$$

Denote this transformation by .

2 . The Bäcklund transformation

$$\bar{\ } = \frac{\ }{{}_1(\ )},\qquad \bar{\ } = \ {}_1(\ )\ \ ,\qquad \text{where}\qquad = \ ',\tag{2}$$

leads to an equation of similar form for the function $\bar{\ } = \bar{\ }(\bar{\ })$:

$$\bar{\ }''_{\bar{\ }\bar{\ }} = {}_2(\bar{\ })g_2(\bar{\ })\ {}_2(\bar{\ }'_{\bar{\ }}),$$

where the functions ${}_2$, g_2, and ${}_2$ are defined in terms of the original functions ${}_1$, g_1, and ${}_1$ parametrically by the relations

$$\begin{aligned} {}_2(\bar{\ }) &= \ , & \bar{\ } &= \frac{\ }{{}_1(\ )};\\ g_2(\bar{\ }) &= \frac{1}{{}_1(\ )}, & \bar{\ } &= \ {}_1(\ )\ \ ;\\ {}_2(\ \bar{\ }) &= -\frac{1}{[g_1(\ )]^3}\,\frac{g_1}{\ }, & \bar{\ } &= \frac{1}{g_1(\ )}. \end{aligned}$$

Denote transformation (2) by $\mathcal{G}$.

For equations of the form (1) in which ${}_1$, g_1, and ${}_1$ are power functions of their arguments, the transformation $\mathcal{G}$ (up to a constant factor) is considered in Subsection 2.5.3. For equations (1) with exponential functions ${}_1$ and g_1, the transformation $\mathcal{G}$ is discussed in Subsection 2.7.3.

Whenever the solution $\bar{\ } = \bar{\ }(\bar{\ })$ of the transformed equation is found, the formulas

$$\bar{\ } = \ {}_1(\ )\ \ ,\qquad \bar{\ }'_{\bar{\ }} = \frac{1}{g_1(\ )},$$

can be used to obtain the solution of the original equation (1) in parametric form, $\ = \ (\bar{\ })$, $\ = \ (\bar{\ })$.

3 . The twofold application of the transformation $\mathcal{G}$ to the original equation yields an equation of similar form:

$$\bar{\bar{\ }}''_{\bar{\bar{\ }}\bar{\bar{\ }}} = {}_3(\bar{\bar{\ }})g_3(\bar{\bar{\ }})\ {}_3(\bar{\bar{\ }}'_{\bar{\bar{\ }}}),$$

where the functions ${}_3$, g_3, and ${}_3$ are defined in terms of the original functions ${}_1$, g_1, and ${}_1$ parametrically by

$$\begin{aligned} {}_3(\bar{\bar{\ }}) &= \frac{1}{g_1(\ )}, & \bar{\bar{\ }} &= \ g_1(\ )\ \ ;\\ g_3(\bar{\bar{\ }}) &= \frac{1}{\ }, & \bar{\bar{\ }} &= \frac{\ }{{}_1(\ )};\\ {}_3(\ \bar{\bar{\ }}) &= \frac{1}{\ }, & \bar{\bar{\ }} &= \ {}_1(\ ). \end{aligned}$$

The threefold transformation $\mathcal{G}$ yields the original equation.

Different compositions of the transformations and $\mathcal{G}$ generate six different equations of the analogous form, which are shown in Figure 4 (see Subsection 0.6.5).

4 . In the special case $g(\ ) =\ \ ,\ \ = 1$, the transformation $\ = \frac{1}{\tau},\ \ = \frac{}{\tau}$ leads to an equation of similar form:

$$''_{\tau\tau} = \tau^{-\ \ -3}\ \ \frac{1}{\tau}\ \ .$$

Denote this transformation by .

For $g(\ ) =$ and $\ = 1$, different compositions of the transformations , $\mathcal{G}$, and generate twelve different equations of the form (1).

2.8. Some Nonlinear Equations with Arbitrary Parameters

2.8.1. Equations Containing Power Functions

2.8.1-1. Equations of the form $(\ ,\ )\ '' + g(\ ,\ ) = 0$.

1. $\ \ = \ \ ^2 + \ \ .$

The transformation $\ = k^3\ ,\ \ = kz$, where $k = 6^{1\ 5}(a\ )^{-1\ 5}$, leads to the first Painlevé transcendent: $\ '' = 6\ \ ^2 + z$ (see Paragraph 2.8.2-2).

2. $\ \ = \ \ ^3 + \ \ + \ .$

The transformation $\ = (2\ a)^{1\ 2\ 1\ 3}\ ,\ \ = \ ^{-1\ 3}z$ leads to the second Painlevé transcendent: $\ '' = 2\ \ ^3 + z\ \ + (a\ 2)^{1\ 2\ -1}$ (see Paragraph 2.8.2-3).

3. $\ \ = \ \ \ + \ \ ^{-3}.$

This is a special case of equation 2.9.1.2 with $(\ ) = -\ \ .$

4. $\ \ = \ \ \ + \ \ ^{k-1\ -1} - \ ^{2\ \ 2k\ \ -3}.$

This is a special case of equation 2.9.1.3 with $(\ ) = -a$ and $g(\ ) = \ \ .$

5. $\ \ = (\ \ ^2 + \ \ + \ )\ ^{-5}.$

This is a special case of equation 2.7.1.36 with $\ = 1$.

6. $\ \ = \ \ ^{-1\ \ -1\ 2} + \ \ ^{-2} + 2\ \ ^2.$

The solution is determined by the first-order equation $a\ (\ ' + 2\ \ ^{1\ 2}) = \ (\ ,\ ,\ )$, where the function is defined implicitly by $\ - \ln|\ \ + a^{1\ 2}| = a^{-1\ 2}(\ ^{1\ 2} + \ )^2 + \ .$

7. $\ \ = (\ \ ^2 + \ \ + \ ^2 + \ \ + \ \ + \gamma)^{-3\ 2},\ \ \ \neq 0.$

The substitution $2a\ \ = 2a\ \ + \ \ + $ leads to an equation of the form 2.9.1.21:

$$''\ = \ ^{-3}\ \ \frac{}{\overline{A\ ^2 + B\ \ +\ }}\Big),$$

where $(\xi) = \xi^3(a\xi^2 + 1)^{-3\ 2},\ A = \frac{4a\ \ -\ \ ^2}{4a},\ B = \frac{2a\beta -}{2a},\ \ = \frac{4a\ \ -\ \ ^2}{4a}.$

8. $\ \ = \ \ ^{-1\ 3} + (\ \ ^2 + \ \ + \ )\ ^{-5\ 3}.$

The transformation $\ = (\ ),\ \ = (\ ')^{3\ 2}$ leads to a third-order equation: $2\ \ '\ \ ''' - (\ '')^2 = \frac{4}{3}\lambda(\ ')^2 + \frac{4}{3}(a\ \ ^2 + \ \ + \ )$. Differentiating the latter equation with respect to and dividing it by $\ '$, we arrive at a fourth-order constant coefficient linear equation: $3\ \ '''' = 2\lambda\ \ '' + 4a\ \ + 2\ .$

9. $= \;^{-8\;3}\;^{-1\;3} + (\;^{-10\;3} + \;^{-7\;3} + \;^{-4\;3})\;^{-5\;3}.$

The transformation $= 1$, $=$ leads to an equation of the form 2.8.1.8: $'' = \lambda\;^{-1\;3} + (a\;^{2} + \; + \;)\;^{-5\;3}$.

10. $= (\; + \; + \;)\;.$

This is a special case of equation 2.9.1.4 with $(\xi) = \xi$.

11. $= (\; + \;^{2})\; + \;.$

The substitution $a \; = a \; + \;^{2}$ leads to an autonomous equation of the form 2.9.1.1: $'' = a \; + \; + 2a^{-1}$.

12. $= \;^{-2}\;^{-3}(\; + \;)\;.$

This is a special case of equation 2.9.1.15 with $(\xi) = \lambda\xi$ and $= \; = 0$.

13. $= \;(\; + \;)\;(\; + \;)^{-}\;^{-}\;^{-3}\;.$

The transformation $\xi = \dfrac{a\; +}{+}$, $= \dfrac{}{+}$ leads to the Emden–Fowler equation $'' = A(a\; - \;)^{-2}\xi$, whose solvable cases are outlined in Section 2.3.

14. $= \;^{-}\;^{k-}\;^{-3}(\; + \;)^{k}.$

This is a special case of equation 2.9.1.8 with $(\xi) = \;\xi^{-}\;^{-}\;^{-3}(a\xi\; + \;)$.

15. $= \;^{-2}\;^{k-2}\;^{-3}(\;^{2} + \;)^{k}.$

This is a special case of equation 2.9.1.9 with $(\xi) = \;\xi^{-2}\;^{-2}\;^{-3}(a\xi^{2} + \;)$.

16. $^{2}\; = \;^{+1} + \;.$

This is a special case of equation 2.9.1.11 with $(z) = az +$.

17. $^{2}\; = \;(\; + 1)\; + \;^{3}\;^{+2} + \;^{+3}\;^{+2}\;.$

This is a special case of equation 2.9.1.12 with $(\xi) = a + \;\xi$.

18. $^{2}\; = \;(\; + 1)\; + \;^{k}\;^{+3k+2}(\;^{2k+1} + \;)\;.$

The transformation $\xi = \;^{2\;+1} + \;$, $=$ leads to the Emden–Fowler equation $'' = a\;^{-2}(2k+1)^{-2}\xi$, whose solvable cases are outlined in Section 2.3.

19. $(\; + \;^{2})\; = 1.$

This is a special case of equation 2.8.1.11 with $= -1$ and $= 0$.

20. $(\; + \;)^{2}\;^{2}\; = \;.$

The transformation $\xi = \ln \dfrac{+a}{}$, $= -$ leads to an autonomous equation of the form 2.2.1.7: $'' - \;' = a^{-2}\;^{-2}$.

21. $(\;^{2} + \;^{2} + 2\; + \;)^{2}\; + s\; = 0.$

Dividing by the coefficient of $''$ and multiplying by $a\;(\;' - \;) + \;(2\;' - \;) + \;'$, we arrive at an exact differential equation. Integrating the latter, we obtain a first-order equation:

$$(a\;^{2} + 2\; + \;)(\;')^{2} - 2(a\; + \;)\;' + a\;^{2} + \frac{^{2}}{^{2} + a\;^{2} + 2\; + } = \;.$$

22. $(\ \ +\ \ )^2(\ \ +\ \ )^2 \quad = s\ \ +\ \ (\ \ +\ \ )^k(\ \ +\ \ )^{1-\ \ -k}$ **.**

The transformation $\xi = \ln \dfrac{a\ \ +}{+}$, $= \dfrac{a\ \ +}{+}$ $\overline{\ \ }^{-1}$ $\overline{\ \ \ \ \ }$ / $+$ leads to an autonomous equation: $'' - (2\ \ +1)\ \ ' + (\ \ ^2 + \ \ - \ \ \Delta^{-2}) \ \ = A\Delta^{-2}$, where $= \frac{}{-1}$, $\Delta = a\ \ -\ \ $.

23. $(\ \ +\ \ +\ \ ) \qquad = (\ \ +\ \ +\gamma)\ ^{-1}$**.**

This is a special case of equation 2.9.1.16 with $(\ \ ) = k\ \ ^{1-}$.

24. $(\ \ +\ \ +\ \ ) \qquad = (\ \ +\ \ +\gamma)\ ^{-3}$**.**

This is a special case of equation 2.9.1.17 with $(\ \ ) = k\ \ ^{3-}$.

25. $(\ \ \ +\ \ \ )\ \ \ +\ \ \ ^{-3} = 0.$

This is a special case of equation 2.9.1.8 with $(\xi) = -\ (a\xi\ \ +\ \ )^{-1}$.

26. $(\ \ \ +\ \ \ )\ \ \ +\ \ \ ^{-3} = 0.$

This is a special case of equation 2.9.1.8 with $(\xi) = -\ \xi\ \ ^{-3}(a\xi\ \ +\ \ )^{-1}$.

27. $(\ \ ^2 +\ \ \ )\ \ \ +\ \ ^{2\ \ -3} = 0.$

This is a special case of equation 2.9.1.9 with $(\xi) = -\ \xi^{2\ \ -3}(a\xi^2\ \ +\ \ )^{-1}$.

28. $(\ \ \ +\ \ \ )\ \ \ +\ \ \ \ ^{-\ \ -3} = 0.$

This is a special case of equation 2.9.1.8 with $(\xi) = -\ \xi^{\ -\ \ -3}(a\xi\ \ +\ \ )^{-1}$.

29. $(\ \ ^2 +\ \ \ )\ \ \ +\ \ \ ^{2\ \ -2\ \ -3} = 0.$

This is a special case of equation 2.9.1.9 with $(\xi) = -\ \xi^{2\ \ -2\ \ -3}(a\xi^2\ \ +\ \ )^{-1}$.

See also equations 2.7.1.1–2.7.1.40.

$\boxed{\text{2.8.1-2. Equations of the form } (\ ,\ )\ '' + g(\ ,\ )\ ' + (\ ,\ ) = 0.}$

30. $\ \ +3\ \ \ +\ \ ^3 +\ \ \ \ = 0.$

This is a special case of equation 2.9.2.1 with $(\ \ ) = a\ \ $.

31. $\ \ +(\ \ +\ \ )\ \ +\ \ \ ^{-1}\ \ = 0.$

This is a special case of equation 2.9.2.4 with $(\ \ ) = \ \ $.

32. $\ \ +(2\ \ +\ \ )\ \ +\ \ \ ^2 = \ \ $ **.**

This is a special case of equation 2.9.2.5 with $(\ \ ) = \ \ $ and $g(\ \ ) = \ \ $.

33. $\ \ = \ \ +\ \ \ +\ \ ^{2\ \ +1\ \ -3}$**.**

This is a special case of equation 2.9.2.9 with $(\ \ ) = -\ \ $.

34. $\ \ = \ \ +\ \ ^{2\ \ +1} +\ \ ^{2\ \ +1}\ \ $ **.**

This is a special case of equation 2.9.2.20 with $(\ \ ) = a + \ \ $.

35. $\ \ = -(\ \ +1)\ \ +\ \ ^{-1} +\ \ \ ^{+\ \ -1}\ \ $ **.**

This is a special case of equation 2.9.2.30 with $(\xi) = a + \ \ \xi\ \ $.

36. $= (\quad ^{k} \quad + \quad -1) \quad .$

Solution:

$$\int \frac{}{(\ )+\ _1} = \ _2 + \frac{1}{k} \ , \qquad \text{where} \qquad (\ ) = a\frac{1}{+1} \ ^{+1}.$$

37. $^{2} \quad + \quad = \quad + \ .$

This is a special case of equation 2.9.2.23 with $(\ ) = a \quad + \ .$

38. $^{2} \quad = -(\ + \quad +1) \quad - \quad + \quad ^{k+ \ -2} \quad ^{k}.$

This is a special case of equation 2.9.2.31 with $(\xi) = a\xi \ .$

39. $^{2} \quad + \quad + \quad = \quad .$

The transformation $=\xi \ , \ =\xi$, where $= \frac{1}{}$, $\beta = \frac{1-a}{2} - \frac{1}{2}$, $=(1-a)^2-4$, leads to the Emden–Fowler equation $'' = \ ^2\xi \ ^{+ \ - \ -2}$, whose solvable cases are outlined in Section 2.3.

40. $(\ ^{2} + \ ) \quad + \quad + \quad = 0.$

This is a special case of equation 2.9.2.24 with $(\ ) = \quad .$

2.8.1-3. Equations of the form $(\ ,\ )\ '' + g(\ ,\ )(\ ')^2 + (\ ,\ )\ ' + (\ ,\ ) = 0.$

41. $= (\ )^2 - 2 \quad + 2 \quad + \ .$

The substitution $= \ + \frac{1}{2}a \ ^2$ leads to an autonomous equation of the form 2.9.3.25: $'' = (\ ')^2 + 2a \ - a + \ .$

42. $= (\ + \ + \ )^2 + \ ^2 \ + \ + s.$

The substitution $= \ ' + \ +$ leads to a Riccati equation: $' = a \ ^2 + \ + (k - \ ) \ + \ + \ .$

43. $= \quad (\quad - \ )^2 + \quad .$

This is a special case of equation 2.9.3.2 with $(\ ) = \quad$, $g(\ ) = 0$, and $(\ ) = a \quad .$

44. $= \quad (\ + \ )^2 + \ ^2 \ + \quad .$

The substitution $= \ ' +$ leads to a Riccati equation: $' = a \quad ^2 + \quad + \quad .$

45. $= (\ ^2 \ ^2 + \ ) \ + \quad (\quad - \quad)^2 + \quad .$

The substitution $= \ ' - a$ leads to a Riccati equation: $' = \quad ^2 - a \quad + \quad .$

46. $= (\ + \ + \ ) \ [\ (\ )^2 + \ ]^k.$

This is a special case of equation 2.9.4.37 with $(\ ) = \quad$ and $g(v) = (\ v^2 + \beta) \ .$

47. $+ \ (\ )^2 + \frac{1}{2} \quad + \quad ^2 + \quad + \quad = 0.$

The substitution $(\ ) = \ (\ ')^2$ leads to a first-order linear equation: $' + 2a \ + 2 \ ^2 + 2 \ + 2k = 0.$

48. $+ \ (\ )^2 - \quad ^{k} \quad = 0.$

Solution:

$$\int \frac{e}{(\ )+\ _1} = \ _2 + \ln|\ |, \qquad \text{where} \qquad (\ ) = \int e \qquad + \frac{1}{a}e \quad .$$

49. $+ \ (\ )^2 = (\quad ^{k} \quad + \quad -1) \quad .$

Solution:

$$\int \frac{e}{(\ )+\ _1} = \ _2 + \frac{1}{k} \ , \qquad \text{where} \qquad (\ ) = \int e \qquad .$$

50. $\;^{2}\quad = 2\;+\quad(\quad+\;)^2 + \quad.$

The substitution $\;=\;'+\;$ leads to a Riccati equation: $\;'=a\quad^2+2\;+\quad.$

51. $\;^{2}\quad=\;(\;+1)\;+\quad(\quad+\quad)^2+\quad.$

The substitution $\;=\;'+a\;$ leads to a Riccati equation: $\;'=\quad^2+(a+1)\;+\quad.$

52. $\quad=(\quad)^2-\;.$

1 . Solution (a is any):

$$=\;_1\exp(\;_2\;)-\frac{a}{4\;_1\;_2^2}\exp(-\;_2\;).$$

2 . Solution for $a<0$:

$$=\;_1\sin\left(\frac{\overline{a}}{\overline{\;_1^2+\;_2^2}}\right)+\;_2\cos\left(\frac{\overline{a}}{\overline{\;_1^2+\;_2^2}}\right).$$

There are also singular solutions: $\;=\quad\overline{a}+\;.$

53. $\quad-\tfrac{1}{4}(\quad)^2=\quad^2+\quad+\;.$

The substitution $\;=\;^{4\;3}$ leads to a special case of the equation 2.8.1.8 with $\lambda=0$: $4\;''\;=3(a\;^2+\;+\;)\;^{-5\;3}.$

54. $3\quad-2(\quad)^2=\quad^2+\quad+\;.$

The substitution $\;=\;^3$ leads to an equation of the form 2.8.1.5: $9\;''\;=(a\;^2+\;+\;)\;^{-5}.$

55. $2\quad=(\quad)^2+\quad^2-\;.$

This is a special case of equation 2.9.3.5 with $\;(\;)=-\quad.$

56. $\quad=\;(\quad)^2-\quad^{4\;-2}+\quad^2.$

This is a special case of equation 2.9.3.8 with $\;(\;)=-\quad.$

57. $\quad=\;(\quad)^2+\quad^{k\;2}+\quad^{+1}.$

This is a special case of equation 2.9.3.9 with $\;(\;)=-a\;$ and $g(\;)=-\quad.$

58. $(\;+2)\quad-(\;+1)(\quad)^2=(\quad^2+\quad+\;)\;.$

The substitution $\;=\;^{+2}$ leads to an equation of the form 2.7.1.36:

$$''\;=\frac{1}{(\;+2)^2}(a\;^2+\;+\;)\quad^{-2\;-3}.$$

59. $\quad=(\quad)^2+\quad+\quad^2.$

This is a special case of equation 2.9.3.7 with $\;(\;)=-a\;$ and $g(\;)=-\quad.$

60. $\quad+\;(\quad)^2+(\quad+\;)\quad=0.$

Solution: $\frac{+b}{}\;=\;_1\;\exp\left(-\frac{1}{a}\;(\;+\lambda)\right)\quad+\;_2.$

61. $\quad-(\quad)^2=\quad+\quad+\;.$

1 . Solution:

$$=\;_1\sinh(\;_3\;)+\;_2\cosh(\;_3\;)+a+\quad_3^{-2},$$

where the constants $\;_1$, $\;_2$, and $\;_3$ are related by the constraint $(\;_1^2-\;_2^2)\;_3^2+a\;+\;+\;^2\;_3^{-2}=0.$

2 . Solution:

$$=\;_1\sin(\;_3\;)+\;_2\cos(\;_3\;)+a-\quad_3^{-2},$$

where the constants $\;_1$, $\;_2$, and $\;_3$ are related by the constraint $(\;_1^2+\;_2^2)\;_3^2+a\;+\;-\;^2\;_3^{-2}=0.$

There is also a singular solution: $\;=-\quad.$

62. $-(\quad)^2 = {}_2 \quad + {}_1 \quad + {}_0 \quad + \;.$

Particular solutions: $= e^{\lambda} - a_0^{-1}$, where is an arbitrary constant and $\lambda = \lambda_{1,2}$ are roots of the quadratic equation $(a_2 a_0 + \;)\lambda^2 + a_1 a_0 \lambda + a_0^2 = 0$.

63. $(\; + \;) = (\; - \;)^2.$

The substitution $= -a \; + \; z$ leads to the equation $zz'' + 2zz' - \; {}^{+3}(z')^2 = 0$. Having set $= z' \; z$, we obtain a Bernoulli equation: $' + 2 \; + \; (1 - \; {}^{+2}) \; {}^2 = 0$.

64. $(2 \; + \; + \;) \; - (\;)^2 - \; + \; = 0.$

Solution:

$$= {}_1 \; {}^2 + {}_2 \; + {}_3,$$

where the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint $4 \; {}_1 \; {}_3 - {}_2^2 + 2 \; {}_1 - a \; {}_2 + \; = 0$.

65. $= (\;)^2 - \; + \; {}^{k} \; {}^{s}.$

This is a special case of equation 2.9.4.64 with $(\xi) = a\xi$, $g(\xi) = 1$, $k = \; - 1$, and $= \; + 2$.

66. ${}^2 \; + (\;)^2 = \; + \;.$

Having set $1 \; = \; '(\;)$, we obtain a third-order equation: $- \; ' \; ''' \; + 3(\; '')^2 = (a \; + \;)(\; ')^5$. Taking to be the independent variable, we obtain a constant coefficient linear equation for $= (\;)$: $''' = a \; + \;$.

67. $(\;{}^2 - \;{}^2)(\;{}^2 - \;{}^2) \; + (\;{}^2 - \;{}^2)(\;)^2 = (\;{}^2 - \;{}^2) \;.$

Solution: $\arcsin \text{—} = {}_1 + {}_2 \arcsin \text{—}$.

68. $(\; - \; + \;) = {}_2(\;)^2 + {}_1 \; + {}_0.$

The contact transformation

$$X = \; ', \quad = \; ' - \; + a, \quad ' = \;, \quad '' = 1 \; '',$$

where $= (X)$, leads to a linear equation: $({}_2 X^2 + {}_1 X + {}_0) \; '' - \; = 0$.

Inverse transformation:

$$= \; ', \quad = X \; ' - \; + a, \quad ' = X, \quad '' = 1 \; ''.$$

2.8.1-4. Other equations.

69. $= (\; - \;) \;.$

This is a special case of equation 2.9.4.58 with $(\;) = a$ and $g(\xi) = \xi$.

70. $= {}^2 \; + (\; + \;) \;.$

The substitution $= \; ' + a$ leads to a Bernoulli equation: $' = a \; + \;$.

71. $= (\;{}^2 \;{}^2 + \;) + (\; - \;) \;.$

The substitution $= \; ' - a$ leads to a Bernoulli equation: $' = -a \; + \;$.

72. $= \; {}^{-} \; {}^{-3} (\; - \;) \;.$

This is a special case of equation 2.9.4.59 with $(\xi) = a\xi$.

73. $= \; {}^{-1} (\; - \;) \;.$

This is a special case of equation 2.9.4.60 with $(\;) = a$ and $g(\xi) = \xi$.

74. $= \quad {}^{k-1} \quad {}^{k-1}(\quad - \quad)^{\frac{2\ +}{}}$ **.**

This is a special case of equation 2.9.4.24 with $(\xi) = a\xi$.

75. $= \quad (\quad) \ (\quad - \quad) \ .$

The Legendre transformation $= \ '$, $= \ ' -$, where $= \ (\ )$, leads to the generalized Emden–Fowler equation: $'' = \frac{1}{k} \ ^{-} \ ^{-} (\ ')^{-}$. Solvable equations of this type are outlined in Section 2.3 and Section 2.5.

76. $= \quad ^{-1} \quad ^{-1}(\quad)^{\frac{2\ +\ -\ k}{+}} (\quad - \quad)^{k}$**.**

This is a special case of equation 2.9.4.25 with $(\xi) = a\xi$.

77. $= \quad (\quad - \quad) + \quad (\quad - \quad)^{k}$**.**

This is a special case of equation 2.9.4.4 with $(\ ) = a$ and $g(\ ) =$.

78. $^{2} \quad = 2 \quad + \quad (\quad + \quad) \ .$

The substitution $= \ ' +$ leads to a Bernoulli equation: $' = 2 \ + a$.

79. $^{2} \quad = (\ - 1) \ + \quad (\quad - \quad)^{k}$**.**

This is a special case of equation 2.9.4.3 with $(\ ) = a \ ^{-2}$.

80. $^{2} \quad = (\ + 1) \ + \quad (\quad + \quad) \ .$

The substitution $= \ ' + a$ leads to a Bernoulli equation: $' = (a + 1) \ +$.

81. $(\quad)^{2} = \ (\quad - \ ) + \quad + \gamma$**.**

Differentiating the equation with respect to yields:

$$'' \ (2 \ ''' \ - \quad - \beta) = 0. \qquad (1)$$

Equating the second factor to zero and integrating, one obtains:

$$= \tfrac{1}{48} \ ^{4} + \tfrac{1}{12}\beta \ ^{3} + \ _{2} \ ^{2} + \ _{1} \ + \ _{0}. \qquad (2)$$

The integration constants and the parameters , β, and are related by the constraint $4 \ _{2}^{2} = \beta \ _{1} - \ _{0} +$, which is obtained by substituting the above solution (2) into the original equation.

In addition, there is a singular solution, which corresponds to setting the first factor in (1) equal to zero:

$$= \ _{1} \ + \ _{0}, \qquad \text{where} \quad \beta \ _{1} - \ _{0} + \ = 0.$$

82. $(\quad)^{2} + 2 \quad + \quad + \quad - (\quad)^{2} - \quad + \ = 0$**.**

Solution:

$$= \ _{1} \ ^{2} + \ _{2} \ + \ _{3},$$

where the constants $_{1}$, $_{2}$, and $_{3}$ are related by the constraint $4 \ _{1}^{2} + a(4 \ _{1} \ _{3} - \ _{2}^{2}) - \ _{2} + 2 \ _{1} + k = 0$.

2.8.2. Painlevé Transcendents

2.8.2-1. Preliminary remarks. Singular points of solutions.

1 . Singular points of solutions to ordinary differential equations can be *fixed* or *movable*. The coordinates of fixed singular points remain the same for different solutions of an equation.* The coordinates of movable singular points vary depending on the particular solution selected (i.e., they depend on the initial conditions).

Listed below are simple examples of first-order ordinary differential equations and their solutions having movable singularities:

Equation	*Solution*	*Solution's singularity type*
$' = -\ ^2$	$= 1\ \ (z - z_0)$	movable pole
$' = 1$	$= 2\ \ \overline{z - z_0}$	algebraic branch point
$' = e^{-}$	$= \ln(z - z_0)$	logarithmic branch point
$' = -\ \ \ln^2$	$= \exp[1\ \ (z - z_0)]$	essential singularity

Algebraic branch points, logarithmic branch points, and essential singularities are called *movable critical points*.

2 . The Painlevé equations arise from the classification of the following second-order differential equations over the complex plane:

$$'' = R(z,\ \ ,\ \ '),$$

where $R = R(z,\ \ ,\ \ )$ is a function rational in and and analytic in z. It was shown by P. Painlevé (1897–1902) and B. Gambier (1910) that all equations of this type whose solutions do not have movable critical points (but are allowed to have fixed singular points and movable poles) can be reduced to 50 classes of equations. Moreover, 44 classes out of them are integrable by quadrature or admit reduction of order. The remaining 6 equations are irreducible; these are known as the Painlevé equations or Painlevé transcendents, and their solutions are known as the Painlevé transcendental functions.

The canonical forms of the Painlevé transcendents are given below in Paragraphs 2.8.2-2 through 2.8.2-7. Solutions of the first, second, and fourth Painlevé transcendents have movable poles (no fixed singular points). Solutions of the third and fifth Painlevé transcendents have two fixed logarithmic branch points, $z = 0$ and $z =$. Solutions of the sixth Painlevé transcendent have three fixed logarithmic branch points, $z = 0$, $z = 1$, and $z =$.

It is significant that the Painlevé equations often arise in mathematical physics.

2.8.2-2. First Painlevé transcendent.

1 . The first Painlevé transcendent has the form

$$'' = 6\ \ ^2 + z. \tag{1}$$

The solutions of the first Painlevé transcendent are single-valued functions of z.

The solutions of equation (1) can be presented, in the vicinity of movable pole $z_{\rm p}$, in terms of the series:

$$= \frac{1}{(z - z_{\rm p})^2} + \sum_{=2} a\ \ (z - z_{\rm p})\ \ ,$$

$$a_2 = -\tfrac{1}{10} z_{\rm p},\quad a_3 = -\tfrac{1}{6},\quad a_4 = \ \ ,\quad a_5 = 0,\quad a_6 = \tfrac{1}{300} z_{\rm p}^2,$$

where $z_{\rm p}$ and are arbitrary constants; the coefficients a (≥ 7) are uniquely defined in terms of $z_{\rm p}$ and .

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* Solutions of linear ordinary differential equations can only have fixed singular points, and their positions are determined by the singularities of the equation coefficients.

2 . In a neighborhood of a fixed point $z = z_0$, the solution of the Cauchy problem for the first Painlevé transcendent (1) can be represented by the Taylor series (see Paragraph 0.3.3-1):

$$= A + B(z - z_0) + \tfrac{1}{2}(6A^2 + z_0)(z - z_0)^2 + \tfrac{1}{6}(12AB + 1)(z - z_0)^3 + \tfrac{1}{2}(6A^3 + B^2 + Az_0)(z - z_0)^4 + \quad ,$$

where A and B are initial data of the Cauchy problem, so that $|_{=0} = A$ and $'|_{=0} = B$.

The solutions of the Cauchy problems for the second and fourth Painlevé transcendents can be expressed likewise (fixed singular points should be excluded from consideration for the remaining Painlevé transcendents).

3 . For large values of $|z|$, the following asymptotic formula holds:

$$z^{1\ 2}\wp\left(\tfrac{4}{5}z^{5\ 4} - a;\ 12, \quad \right),$$

where the elliptic Weierstrass function $\wp(\ ; 12, \ )$ is defined implicitly by the integral

$$= \quad \frac{\wp}{\overline{4\wp^3 - 12\wp -}};$$

a and are some constants.

The first Painlevé transcendent (1) is invariant under scaling of variables, $z = \lambda\bar{z}$, $= \lambda^{3\ -}$, where $\lambda^5 = 1$.

2.8.2-3. Second Painlevé transcendent.

1 . The second Painlevé transcendent has the form

$$'' = 2\ ^3 + z \ + \ . \tag{2}$$

The solutions of the second Painlevé transcendent are single-valued functions of z.

The solutions of equation (2) can be represented, in the vicinity of a movable pole $z_{\rm p}$, in terms of the series:

$$= \frac{}{z - z_{\rm p}} + \sum_{=1} \quad (z - z_{\rm p}) \ ,$$

$$_1 = -\tfrac{1}{6} \quad z_{\rm p}, \quad _2 = -\tfrac{1}{4}(\quad + \quad), \quad _3 = \quad , \quad _4 = \tfrac{1}{72}z_{\rm p}(\quad + 3\ ),$$

$$_5 = \tfrac{1}{3024}\left[(27 + 81\ ^2 - 2z_{\rm p}^3) \quad + 108 \quad - 216 \quad z_{\rm p}\right],$$

where $= 1$; $z_{\rm p}$ and are arbitrary constants; the coefficients $(\ \geq 6)$ are uniquely defined in terms of $z_{\rm p}$ and .

2 . For fixed , denote the solution by $(z, \ )$. Then the following relation holds:

$$(z, -\ ) = -\ (z, \ ), \tag{3}$$

while the solutions $(z, \ )$ and $(z, \ - 1)$ are related by the Bäcklund transformations:

$$\begin{aligned} (z, \ - 1) &= -\ (z, \ ) + \frac{2\ - 1}{2\ '(z, \ ) - 2\ ^2(z, \ ) - z}, \\ (z, \ ) &= -\ (z, \ - 1) - \frac{2\ - 1}{2\ '(z, \ - 1) + 2\ ^2(z, \ - 1) + z}. \end{aligned} \tag{4}$$

Therefore, in order to study the general solution of equation (2) with arbitrary , it is sufficient to construct the solution for all out of the band $0 \leq \mathrm{Re} \ < \frac{1}{2}$.

Three solutions corresponding to and 1 are related by the rational formulas:

$$_{+1} = -\frac{(\ _{-1} + \ )(4\ ^3 + 2z \quad + 2\ + 1) + (2\ - 1)}{2(\ _{-1} + \ )(2\ ^2 + z) + 2\ - 1},$$

where stands for $(z, \ )$.

The solutions $(z,\ \ )$ and $(z,-\ \ -1)$ are related by the Bäcklund transformations:

$$(z,-\ \ -1)=\ (z,\ \ )+\frac{2\ \ +1}{2\ '(z,\ \ )+2\ ^2(z,\ \ )+z},$$
$$(z,\ \ )=\ (z,-\ \ -1)-\frac{2\ \ +1}{2\ '(z,-\ \ -1)+2\ ^2(z,-\ \ -1)+z}.$$

3 . For $\ =0,$ equation (2) has the trivial solution $\ =0.$ Taking into account this fact and relations (3) and (4), we find that the second Painlevé transcendent with $\ =\ \ 1,\ \ 2,$ has the rational particular solutions:

$$(z,\ \ 1)=\mp\frac{1}{z},\qquad (z,\ \ 2)=\ \ \frac{1}{z}-\frac{3z^2}{z^3+4}\ ,$$

For $\ =\frac{1}{2},$ equation (2) admits the one-parameter family of solutions:

$$(z,\tfrac{1}{2})=-\frac{\ '}{\ },\qquad \text{where}\qquad =\ \overline{z}\ \ \ _{1\ \ 1\ 3}\big(\tfrac{\overline{2}}{3}z^{3\ \ 2}\ \ +\ \ _{2\ \ 1\ 3}\big(\tfrac{\overline{2}}{3}z^{3\ \ 2}\ \ . \tag{5}$$

(Here, the function is a solution of the second-order linear equation $\ ''+\frac{1}{2}z\ \ =0;$ see 2.1.2.2 and 2.1.2.7 with $\ =1.$) It follows from (3)–(5) that the second Painlevé transcendent for all $\ =\ \ +\frac{1}{2}$ with $\ =0,\ \ 1,\ \ 2,$ has a one-parameter family of solutions that can be expressed in terms of Bessel functions.

2.8.2-4. Third Painlevé transcendent.

1 . The third Painlevé transcendent has the form

$$''\ =\frac{(\ ')^2}{\ }-\frac{\ '}{\ }+\frac{1}{z}(\ \ ^2+\beta)+\ \ ^3+\frac{\ }{\ }. \tag{6}$$

In terms of the new independent variable defined by $z=e$, the solutions of the transformed equation will be single-valued functions of .

Any solution of the Riccati equation

$$'=k\ ^2+\frac{\ -k}{kz}\ \ +\ \ , \tag{7}$$

where $k^2=\ \ ,\ \ ^2=-\ \ ,\ k\beta+\ \ (\ \ -2k)=0,$ is a solution of equation (6). Substituting $z=\lambda\tau,$ $\ =-\frac{\ '}{k\ },$ where $\lambda^2=\frac{1}{k},$ into (7), we obtain a linear equation:

$$''_{\tau\tau}+\frac{k-\ }{k\tau}\ \ '_{\tau}+\ \ =0,$$

whose general solution is expressed in terms of Bessel functions:

$$=\tau^{\overline{2}}\ \ \ _1\ _{\overline{2}}(\tau)+\ \ _2\ _{\overline{2}}(\tau)\ \ .$$

2 . In some special cases, equation (6) can be integrated by quadrature. Rewrite equation (6) in the form of integro-differential relations in two ways:

$$\frac{\ '}{\ }\ ^2+\ \frac{\ }{\ ^2}-\ \ ^2\ e^2\ +2\ \frac{\beta}{\ }-\ \ \ e\ \ =2\ \ \ \ \frac{\ }{\ ^2}-\ \ ^2\ e^2\ +\ \frac{\beta}{\ }-\ \ \ \ e\ \ \ ; \tag{8}$$
$$\frac{\ '}{\ }=\ \ \ \frac{\ }{\ ^2}+\ \ ^2\ e^2\ +\ \frac{\beta}{\ }+\ \ \ e\ \ \ ,\qquad z=e\ . \tag{9}$$

It is obvious from (8) that for $\ =\beta=\ =\ =0$, the general solution has the form: $\ =\ _1 z\ ^2$.
Adding (9) multiplied by 2 to (8), we obtain

$$\frac{\ '}{\ }^{2}+2\frac{\ '}{\ }+\ \frac{\ }{\ ^2}-\ ^2\ e^{2\ }\ +2\ \frac{\beta}{\ }-\ \ e\ =4\ \frac{e^{2\ }\ +\beta e\ }{\ ^2}\ . \tag{10}$$

Subtracting (9) times 2 from (8) yields

$$\frac{\ '}{\ }^{2}-2\frac{\ '}{\ }+\ \frac{\ }{\ ^2}-\ ^2\ e^{2\ }\ +2\ \frac{\beta}{\ }-\ \ e\ =-4\ (\ e^{2\ }\ ^2+\ e\ )\ . \tag{11}$$

Substituting $\ =\beta=0$ into equation (10) and $\ =\ =0$ into equation (11), we arrive at

$$\frac{\ '}{\ }^{2}+2\frac{\ '}{\ }-2\ \ e\ -\ ^2 e^{2\ }=\ _1, \tag{12}$$

$$\frac{\ '}{\ }^{2}-2\frac{\ '}{\ }+\frac{\ }{\ ^2}e^{2\ }\ +\frac{2\beta}{\ }e\ =\ _2. \tag{13}$$

Equations (12) and (13) are integrable by elementary functions. Substituting $\ =e^{-\ }\ v$ into (12), we obtain an autonomous equation:

$$(v')^2=2\ v+\ +(1+\ _1)v^2. \tag{14}$$

As a result, we find:

$$\ =\begin{cases}\dfrac{2\ }{z(\ ^2\ln^2 z+2\ \ \ln z+\ ^2-\ )} & \text{if } \ _1=-1,\ \beta=\ =0;\\[2mm] \dfrac{1}{z(\ ^{-}\ln z+\ )} & \text{if } \ _1=-1,\ \ =\beta=\ =0;\\[2mm] \dfrac{z\ ^{-1}}{\ _2 z^{2\ }\ +\ _1 z\ \ +\ _2} & \text{if } \ _1\neq-1,\ \beta=\ =0,\end{cases}$$

where $\ _2\neq 0$, $\ _1=-\dfrac{\ }{\ _1+1}$, $\ _2=\dfrac{\ ^2-\ (1+\ _1)}{4\ _2(1+\ _1)^2}$, $\ ^2=1+\ _1$.

Accordingly, equation (13) is reduced to equation (14) with the substitution $\ =ve\ $.

If $\beta=-\ $ and $\ =-\ $, the substitution $\ =e^{-\ w}$ brings equation (6) to the following form: $\ ''+\dfrac{1}{z}\ '=\dfrac{2\ }{z}\sin\ \ +2\ \sin 2\ $.

2.8.2-5. Fourth Painlevé transcendent.

1 . The fourth Painlevé transcendent has the form

$$\ ''=\frac{(\ ')^2}{2\ }+\frac{3}{2}\ ^3+4z\ ^2+2(z^2-\ )\ +\frac{\beta}{\ }. \tag{15}$$

The solutions of the fourth Painlevé transcendent are single-valued functions of z.

The Laurent-series expansion of the solution of equation (15) in the vicinity of a movable pole $z_{\rm p}$ is given by:

$$\ =\frac{\ }{z-z_{\rm p}}-z_{\rm p}-\frac{\ }{3}(z_{\rm p}^2+2\ \ -4\ )(z-z_{\rm p})+\ (z-z_{\rm p})^2+\sum_{\ =3}a\ (z-z_{\rm p})\ ,$$

where $\ =\ 1$; $z_{\rm p}$ and $\ $ are arbitrary constants; and the $a\ $ ($\ \ge 3$) are uniquely defined in terms of $\ $, β, $z_{\rm p}$, and $\ $.

2 . If the condition $\beta + 2(1 + \quad)^2 = 0$, where $\quad = \quad 1$, is satisfied, then every solution of the Riccati equation

$$' = \quad ^2 + 2 \quad z \quad - 2(1 + \quad)$$

is simultaneously a solution of the fourth Painlevé equation (12).

Equation (15) is invariant under the transformation $= \lambda^{-}$, $z = \lambda \bar{z}$, $= {}^{-}\lambda^2$, $\beta = \bar{\beta}$, where $\lambda^4 = 1$.

3 . Two solutions of equation (15) corresponding to different values of the parameters and β are related to each other by the Bäcklund transformations:

$$= \frac{1}{2}(\; ' - \quad - 2 \; z \quad - \quad ^2), \qquad ^2 = -2\beta,$$

$$= -\frac{1}{2}(\; ' - \quad + 2 \; z \quad + \quad ^2), \qquad ^2 = -2\beta,$$

$$2\beta = -(\quad - 1 - \tfrac{1}{2} \quad)^2, \qquad 4 \quad = -2 \quad - 2 \quad - 3 \quad ,$$

where $= \;(z, \; , \beta)$, $= \;(z, \; , \beta)$, and is an arbitrary parameter.

2.8.2-6. Fifth Painlevé transcendent.

1 . The fifth Painlevé transcendent has the form

$$'' = \frac{3 \quad - 1}{2 \;(\quad - 1)}(\; ')^2 - \frac{'}{z} + \frac{(\quad - 1)^2}{z^2} \qquad + \frac{\beta}{} \quad + \quad \frac{}{z} + \frac{(\quad + 1)}{\quad - 1}. \qquad (16)$$

If we pass on to the new independent variable $z = e$, the solutions are single-valued functions of .

Solutions of the fifth Painlevé transcendent (16) corresponding to different values of parameters are related by:

$$(z, \; , \beta, \; , \;) = \;(-z, \; , \beta, - \; , \;),$$

$$(z, \; , \beta, \; , \;) = \frac{1}{(z, -\beta, - \; , - \; , \;)}.$$

2 . On setting $z = e$ in (16), we obtain

$$'' = \frac{3 \quad - 1}{2 \;(\quad - 1)}(\; ')^2 + (\quad - 1)^2 \qquad + \frac{\beta}{} \quad + \quad e \quad + \frac{(\quad + 1)}{\quad - 1} e^2 \;. \qquad (17)$$

If $= \; = 0$, equation (17) is reduced, by integration, to a first-order autonomous equation:

$$' = (\quad - 1) \quad \overline{2 \quad ^2 + \quad - 2\beta},$$

which is readily integrable by quadrature.

If the condition

$$= \quad \overline{-2} \;\big(1 + \quad \overline{-2\beta} - \quad \overline{2}$$

is satisfied, any solution of the Riccati equation

$$z \; ' = \quad \overline{2} \quad ^2 + \big(\quad \overline{-2} \; z - \quad \overline{2} \; - \quad \overline{-2\beta} \quad + \quad \overline{-2\beta} \qquad (18)$$

is simultaneously a solution of the fifth Painlevé transcendent (16). Equation (18) can be reduced to the degenerate hypergeometric equation 2.1.2.70.

2.8.2-7. Sixth Painlevé transcendent.

1 . The sixth Painlevé transcendent has the form

$$'' = \frac{1}{2}\left(\frac{1}{\ } + \frac{1}{\ -1} + \frac{1}{\ -z}\right)(\ ')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{\ -z}\right)\ '$$

$$+ \frac{(\ -1)(\ -z)}{z^2(z-1)^2}\left(\ + \beta\frac{z}{\ ^2} + \ \frac{z-1}{(\ -1)^2} + \ \frac{z(z-1)}{(\ -z)^2}\right). \qquad (19)$$

In equation (19), the points $z = 0$, $z = 1$, and $z =$ are fixed logarithmic branch points.

Painlevé found two integrable cases of the equation. First, if $\ = \beta = \ = \ = 0$, the general solution of equation (19) has the form:

$$= \ (\ _1\omega_1 + \ _2\omega_2,\ z),$$

where $\ (\ , z)$ is the elliptic function, defined by the integral

$$= \int_0 \frac{}{\sqrt{\ (\ -1)(\ -z)}}, \qquad (20)$$

with periods $2\omega_1$ and $2\omega_2$, which are functions of z. Secondly, if $\ = \beta = \ = 0$, $\ = \frac{1}{2}$, the general solution of equation (19) has the form:

$$= \ (\ + \ _1\omega_1 + \ _2\omega_2,\ z),$$

where $\ \ 0$ is any particular solution of the linear equation

$$'' - \frac{2z-1}{z(z-1)}\ ' + \frac{1}{4z(z-1)}\ = 0$$

and $\ (\ , z)$ is the elliptic function defined by formula (20).

2 . Solutions of the sixth Painlevé transcendent (19) corresponding to different values of parameters are related by:

$$(z, -\beta, - \ , \ , \ ) = \frac{1}{\ \left(\frac{1}{z}, \ , \beta, \ , \ \right)},$$

$$(z, -\beta, - \ , \ , \ ) = 1 - \frac{1}{\ \left(\frac{1}{1-z}, \ , \beta, \ , \ \right)},$$

$$\left(z, -\beta, - \ , - \ + \frac{1}{2}, - \ + \frac{1}{2}\right) = \frac{z}{\ (z, \ , \beta, \ , \ )}.$$

The successive application of these relations yields 24 equations of the form (19) with different values of parameters related by known transformations.

3 . Every solution of the Riccati equation

$$' = \frac{\sqrt{2\ }}{z(z-1)}\ ^2 + \frac{\lambda z + \ }{z(z-1)}\ + \frac{\sqrt{-2\beta}}{z-1},$$

where

$$\lambda = \frac{\sqrt{2\ } - (\ + \beta + \ + \ )}{\sqrt{2\ } - \sqrt{-2\beta} - 1}, \qquad \ = \frac{\sqrt{-2\beta} - (\ + \beta - \ - \ )}{\sqrt{2\ } - \sqrt{-2\beta} - 1},$$

is simultaneously a solution of equation (19) if $\sqrt{2\ } - \sqrt{-2\beta} \neq 1$ and the condition

$$2\sqrt{2\ }\,(3\beta - \ + \ - \ ) + 2\sqrt{-2\beta}\,(3\ - \beta - \ + \ ) + 4\sqrt{-\ \beta}\,(\beta - \ + \ - \ - 1)$$

$$+ (\ + \beta + \ + \ )^2 + 2(\ - \beta - \ - 4\ \beta - 2\ \ - 2\beta\ ) = 0$$

is satisfied (one should take the value of $\sqrt{-\ \beta}$ that coincides with $\sqrt{\ }\sqrt{-\beta}$).

References for Subsection 2.8.2: P. Painlevé (1900), B. Gambier (1910), G. M. Murphy (1960), E. L. Ince (1964), A. S. Fokas and M. J. Ablowitz (1982), V. I. Gromak and N. A. Lukashevich (1990), R. Conte (1999), A. R. Chowdhury (2000).

2.8.3. Equations Containing Exponential Functions

2.8.3-1. Equations of the form $(\ , \ ) '' + g(\ , \ ) = 0$.

1. $= \quad ^{\lambda \ +} \quad + \ .$

The substitution $= \ + (\lambda \ \beta)$ leads to an autonomous equation of the form 2.9.1.1: $'' = ae^{\ w} + \ .$

2. $= \quad ^{\lambda} \quad .$

The transformation $z = e^{\lambda} \quad ^{-1}, \ = \ '$ leads to a first-order equation: $z[(\ -1) \ + \lambda] \ ' = az - \ ^{2}$.

3. $= \quad ^{\lambda} \ .$

The transformation $z = \ ^{+2} e^{\lambda} \ , \ = \ '$ leads to a first-order equation: $z(\lambda \ + \ + 2) \ ' = az + \ .$

4. $= \quad ^{\lambda} \quad + \quad ^{-3}.$

This is a special case of equation 2.9.1.2 with $(\ ) = - \ e^{\lambda} \ .$

5. $= \ ^{2} \ + \ \mathrm{exp}[\ (\ + 3) \ ] \quad .$

This is a special case of equation 2.9.1.29 with $(\xi) = a\xi \ .$

6. $= \ ^{2} \ + \quad , \quad \neq 0.$

The transformation $\xi = e^{2\lambda} \ , \ = \ e^{\lambda}$ leads to the Emden–Fowler equation $'' = \dfrac{a}{4\lambda^2}\xi \quad ,$ where $= \dfrac{- 3\lambda - \ \lambda}{2\lambda}$, whose special cases are given in Section 2.3.

7. $= \ ^{2} \ + \quad ^{\lambda(\ +3)} \ (\ ^{2\lambda} \ + \ ) \quad , \quad \neq 0.$

The transformation $\xi = \ e^{2\lambda} \ + \ , \ = \ e^{\lambda}$ leads to the Emden–Fowler equation $'' = \dfrac{a}{4 \ ^{2}\lambda^2}\xi \quad$, whose special cases are given in Section 2.3.

8. $= \ ^{2} \ + \quad ^{\lambda(\ +3)} \ (\ ^{2\lambda} \ + \ ) \ (\ ^{2\lambda} \ + \ )^{- \ - \ -3} \quad .$

The transformation $\xi = \dfrac{ae^{2\lambda} \ +}{e^{2\lambda} \ +}, \quad = \dfrac{e^{\lambda}}{e^{2\lambda} \ +}$ leads to the Emden–Fowler equation $'' = A(2\Delta\lambda)^{-2}\xi \quad$, where $\Delta = a \ - \quad$ (see Section 2.3).

9. $= \ \mathrm{exp}(\ ^{2} + \ ) \, \mathrm{exp}(\gamma \ ) + \ .$

The substitution $= \ + (\ ^{2} + \beta \ )$ leads to an autonomous equation of the form 2.9.1.1: $'' = ae^{\ w} + \ + 2 \quad ^{-1}$.

10. $^{2} \quad = \quad ^{+2} \quad + \ .$

This is a special case of equation 2.9.1.31 with $(\xi) = a\xi$.

2.8.3-2. Equations of the form $(\ , \ ) '' + g(\ , \ ) ' + (\ , \ ) = 0$.

11. $= \quad + \ ^{2} \quad .$

This is a special case of equation 2.9.2.17 with $(\ ) = \quad .$

12. $= - \quad + \quad ^{-1}$.

This is a special case of equation 2.9.2.36 with $(\xi) = \ \xi^{\ -1}$.

13. $+ \quad + \quad = \ ^{\lambda}$.

The substitution $\xi = e$ leads to an equation of the form 2.8.1.39: $\xi^2 \ '' + (a+1)\xi \ ' + \ = \xi^{\lambda}$.

14. $= -(\ + \ ) \ - \ + \ ^{(\ -2\nu)} \ ^{-1}$.

This is a special case of equation 2.9.2.37 with $(\xi) = a\xi^{\ -1}$.

15. $= \quad + \quad + \ ^{2\lambda} \ ^{-3}$.

This is a special case of equation 2.9.2.14 with $(\ ) = -$.

16. $= \quad + \quad + \ ^{2\lambda} \ ^{-3}$.

This is a special case of equation 2.9.2.14 with $(\ ) = - \ e$.

17. $= \quad + \ \exp(2 \quad + \quad)$.

This is a special case of equation 2.9.2.17 with $(\ ) = \ \exp(\quad)$.

18. $+ 3 \quad + \ ^3 + \ ^{\lambda} \ = 0$.

This is a special case of equation 2.9.2.1 with $(\ ) = ae^{\lambda}$.

19. $= \quad + \quad$.

Solution: $= \ _1 - \ln \ -a \quad e^{\ 1} \quad + \ _2$.

20. $= 2 \quad + \ ^2$.

Solution in parametric form:

$$= \ln \frac{\tau^2}{2 \ _1}, \qquad = -a^{-1} \ _1\tau^{-2}Z^{-1}(\tau Z'_\tau + Z).$$

Here, $Z = \ _1 \ _1(\tau) + \ _2 \ _1(\tau)$ or $Z = \ _1 \ _1(\tau) + \ _2 \ _1(\tau)$, where $_1(\tau)$ and $_1(\tau)$ are the Bessel functions, and $_1(\tau)$ and $_1(\tau)$ are the modified Bessel functions.

21. $= \quad + \ ^{-1}$.

Solution: $= \ _1 - \ln \ _2 - a \quad \exp(\ _1 \ )$.

22. $= \ ^{-1} \ ^2 + 2 \ ^1 \ ^2$.

Solution in parametric form:

$$= \ln \frac{1}{a} \mp \tau^2, \quad = \ _1^2\left[2\tau \ \exp(\mp\tau^2) \ \right]^2, \quad \text{where} \quad = \ \exp(\mp\tau^2) \ \tau + \ _2 \ ^{-1}.$$

23. $+ (2 \quad + \ ^{\lambda} \ ) \quad + \ ^{\lambda} \ = 0$.

Integrating yields a Riccati equation: $' + a \ ^2 + \ e^{\lambda} \ = \ $.

24. $= (\quad + \quad$.

Solution: $2 \ \dfrac{}{a \ ^2 + \ _1} = \ _2 + \dfrac{1}{\beta}e$.

25. $= \quad ^{+} (\quad +1).$

Solution: $= -\ln \quad _1 e^{- \quad 2} - \dfrac{a}{1+ \quad 2} e$. To the limiting case $_2 \quad -1$ there corresponds $= - \quad - \ln(\quad _1 - a \quad)$.

26. $+ \quad = \quad \lambda$ **.**

This equation is encountered in combustion theory and hydrodynamics. The transformation $\xi = \ln \quad , \quad = \lambda \quad + (\quad +1) \ln$ leads to an autonomous equation of the form 2.7.3.1: $'' = a\lambda e^{w}$.

Solution in parametric form:

$$= \exp[\quad _1 \quad (\,)], \qquad = \frac{}{\lambda} - \frac{+1}{\lambda} [\quad _1 \quad (\,)],$$

where

$$(\,) = \begin{cases} \dfrac{1}{\quad _2} \ln \dfrac{\overline{\quad _2 + 2a\lambda e} - \overline{\quad _2}}{\overline{\quad _2 + 2a\lambda e} + \overline{\quad _2}} & \text{if} \quad _2 > 0, \\ -\dfrac{2}{\overline{2a\lambda e}} & \text{if} \quad _2 = 0, \\ \dfrac{2}{\overline{- \quad _2}} \arctan \dfrac{\overline{\quad _2 + 2a\lambda e}}{\overline{- \quad _2}} & \text{if} \quad _2 < 0. \end{cases}$$

27. $= \quad + \quad ^{2 \quad +1 \quad \lambda}$ **.**

This is a special case of equation 2.9.2.20 with $(\,) = ae^{\lambda}$.

28. $= \quad + \quad ^{2 \quad +1}$ **exp( ).**

This is a special case of equation 2.9.2.20 with $(\,) = a \exp(\lambda \quad)$.

29. $^{2} \quad + \quad = \quad ^{\lambda}$ **.**

The substitution $= \ln| \quad |$ leads to an equation of the form 2.7.3.1: $'' = ae^{\lambda}$.

Solution:

$$= -\frac{1}{\lambda} \ln \frac{a\lambda}{2 \quad _1^2} \sin^2(\quad _1 \ln| \quad | + \quad _2) \qquad \text{if} \quad a\lambda > 0,$$

$$= -\frac{1}{\lambda} \ln \frac{a\lambda}{2 \quad _1^2} \sinh^2(\quad _1 \ln| \quad | + \quad _2) \qquad \text{if} \quad a\lambda > 0,$$

$$= -\frac{1}{\lambda} \ln \left(-\frac{a\lambda}{2 \quad _1^2} \right) \cosh^2(\quad _1 \ln| \quad | + \quad _2) \qquad \text{if} \quad a\lambda < 0.$$

30. $^{2} \quad + \quad = \quad ^{\lambda} + \quad$ **.**

This is a special case of equation 2.9.2.23 with $(\,) = ae^{\lambda} \; + \;$.

31. $^{2} \quad + \quad = \quad + \quad$ **.**

This is a special case of equation 2.9.2.40 with $(\xi) = k\xi + \;$.

32. $(\quad ^{2} + \;) \quad + \quad + \quad ^{\lambda}$ **= 0.**

This is a special case of equation 2.9.2.24 with $(\,) = \quad e^{\lambda}$.

33. $(\quad ^{2} + \;) \quad + \quad ^{2} \quad + \quad$ **= 0.**

This is a special case of equation 2.9.2.34 with $g(\,) = ae^{2} \; + \;$ and $(\,) = - \;$.

34. $(\quad ^{2} + \;) \quad + \quad ^{2} \quad + \quad ^{\lambda}$ **= 0.**

This is a special case of equation 2.9.2.34 with $g(\,) = ae^{2} \; + \;$ and $(\,) = - \; e^{\lambda}$.

2.8.3-3. Equations of the form $(\ ,\ )\ ''+g(\ ,\ )(\ ')^2+(\ ,\ )\ '+(\ ,\ )=0$.

35. $=(\quad)^2-\ ^4\ +\ .$

This is a special case of equation 2.9.3.18 with $(\ )=-\ .$

36. $=(\quad)^2-\ ^4\ +\ ^{\lambda}\ .$

This is a special case of equation 2.9.3.18 with $(\ )=-\ e^{\lambda}\ .$

37. $=(\quad)^2+\ \ +\ .$

This is a special case of equation 2.9.3.17 with $(\ )=-\ $ and $g(\ )=-\ .$

38. $=(\quad)^2+\ ^{+}\ +\ .$

This is a special case of equation 2.9.3.17 with $(\ )=-\ e^{c}$ and $g(\ )=-k\ .$

39. $=(\quad)^2+\ ^{+\lambda}\ +\ .$

This is a special case of equation 2.9.3.17 with $(\ )=-\ e^{\lambda}$ and $g(\ )=-\ e\ .$

40. $+\ (\quad)^2+\ ^{\lambda}\ +\ =0.$

This is a special case of equation 2.9.3.25 with $(\ )=a$ and $g(\ )=\ e^{\lambda}\ +\ .$

41. $+\ ^{\lambda}(\quad)^2+\ \ +\ =0.$

This is a special case of equation 2.9.3.25 with $(\ )=ae^{\lambda}$ and $g(\ )=\ \ +\ .$

42. $+\ ^{\lambda}(\quad)^2+\ \ +\ =0.$

This is a special case of equation 2.9.3.25 with $(\ )=ae^{\lambda}$ and $g(\ )=\ e\ \ +\ .$

43. $=\ (\quad)^2+\ ^{\lambda}\ .$

This is a special case of equation 2.9.3.38 with $(\ )=a$ and $g(\ )=\ e^{\lambda}\ .$

44. $=\ ^{\lambda}(\quad)^2+\ .$

This is a special case of equation 2.9.3.38 with $(\ )=ae^{\lambda}$ and $g(\ )=\ .$

45. $=\ ^{\lambda}(\quad)^2+\ .$

This is a special case of equation 2.9.3.38 with $(\ )=ae^{\lambda}$ and $g(\ )=\ e\ .$

46. $=\ ^{\lambda}(\quad-\quad)^2+\ .$

This is a special case of equation 2.9.3.2 with $(\ )=\ e\ $, $g(\ )=0$, and $(\ )=ae^{\lambda}\ .$

47. $-(\quad)^2=\ ^{\lambda}\ .$

The substitutions $=\exp\bigl(\frac{1}{2}\ +\frac{1}{2}\lambda\ $ leads to an autonomous equation of the form 2.7.3.1: $''=2ae^{-w}$.

48. $2\ =(\quad)^2+\ ^{\lambda}\ ^2-\ .$

This is a special case of equation 2.9.3.5 with $(\ )=-\ e^{\lambda}\ .$

49. $=(\quad)^2-\ ^4\ ^{-2}+\ ^{\lambda}\ ^2.$

This is a special case of equation 2.9.3.8 with $(\ )=-\ e^{\lambda}\ .$

50. $-(\quad)^2 = \quad \lambda \quad k.$

1 . For $k \neq 2$, the substitution $\quad = \exp \quad + \dfrac{\lambda}{2-k}$ leads to an autonomous equation of the form 2.7.3.1: $\quad'' \quad = ae^{(\quad -2)w}$.

2 . Solution for $k = 2$: $\ln|\quad| = \quad_1 \quad + \quad_2 + a\lambda^{-2}e^{\lambda}$.

51. $= \quad (\quad)^2 + \quad\quad ^2 + \quad \lambda \quad\quad ^{+1}.$

This is a special case of equation 2.9.3.9 with $(\quad) = -a$ and $g(\quad) = -\ e^{\lambda}$.

52. $= \quad (\quad)^2 + \quad \lambda \quad ^2 + \quad\quad ^{+1}.$

This is a special case of equation 2.9.3.9 with $(\quad) = -ae^{\lambda}$ and $g(\quad) = -$.

53. $= \quad (\quad)^2 + \quad \lambda \quad ^2 + \quad\quad ^{+1}.$

This is a special case of equation 2.9.3.9 with $(\quad) = -ae^{\lambda}$ and $g(\quad) = -\ e$.

54. $-(\quad)^2 = \quad \exp(\quad ^2 + \quad .$

The substitutions $\quad = \quad \exp\left(\frac{1}{2} \quad + \frac{1}{2}\beta \quad ^2 + \frac{1}{2}\lambda\right.$ leads to an autonomous equation of the form 2.9.1.1: $\quad'' \quad = 2ae^{-w} - 2\beta$.

55. $-(\quad)^2 + \quad ^2 = \quad \exp(\quad ^2 + \quad .$

The substitutions $\quad = \quad \exp\left(\frac{1}{2} \quad + \frac{1}{2}\beta \quad ^2 + \frac{1}{2}\lambda\right.$ leads to an autonomous equation of the form 2.9.1.1: $\quad'' \quad = 2\ e^{-w} - 2(a+\beta)$.

56. $-(\quad)^2 = \quad \exp(\quad ^2 + \quad\quad ^k.$

1 . For $k \neq 2$, the substitution $\quad = \exp \quad + \dfrac{1}{2-k}(\beta \quad ^2 + \lambda$ leads to an autonomous equation of the form 2.9.1.1: $\quad'' \quad = ae^{(\quad -2)w} - \dfrac{2\beta}{2-k}$.

2 . Solution for $k = 2$:

$$\ln|\quad| = \quad_1 \quad + \quad_2 + a \quad_0 (\quad - \quad)\exp(\beta \quad ^2 + \lambda \quad\quad .$$

57. $-(\quad)^2 + \quad ^2 = \quad \exp(\quad ^2 + \quad\quad ^k.$

1 . For $k \neq 2$, the substitution $\quad = \exp \quad + \dfrac{1}{2-k}(\beta \quad ^2 + \lambda$ leads to an autonomous equation of the form 2.9.1.1: $\quad'' \quad = \quad e^{(\quad -2)w} - a - \dfrac{2\beta}{2-k}$.

2 . For $k=2$, the substitutions $\quad = \quad e^{w}$ leads to a linear equation: $\quad'' \quad = \quad \exp(\beta \quad ^2 + \lambda \quad -a$.
Solution:

$$\ln|\quad| = -\frac{a}{2} \quad ^2 + \quad_1 \quad + \quad_2 + \quad\quad_0 (\quad - \quad)\exp(\beta \quad ^2 + \lambda \quad\quad .$$

58. $+ \quad (\quad)^2 - \frac{1}{2} \quad = \quad (\ _2 \quad ^2 + \quad_1 \quad + \quad_0).$

The substitution $(\quad) = e^{-} \quad (\quad ')^2$ leads to a first-order linear equation: $\quad ' + 2a \quad = 2 \quad_2 \quad ^2 + 2 \quad_1 \quad + 2 \quad_0$.

59. $= (\quad)^2 + \quad\quad\quad + \quad \lambda \quad ^2.$

This is a special case of equation 2.9.3.7 with $(\quad) = -a$ and $g(\quad) = -\ e^{\lambda}$.

60. $= (\quad)^2 + \quad ^{\lambda} \quad + \quad ^2.$

This is a special case of equation 2.9.3.7 with $(\ ) = -ae^{\lambda}$ and $g(\ ) = -\quad .$

61. $= (\quad)^2 + \quad ^{\lambda} \quad + \quad ^2.$

This is a special case of equation 2.9.3.7 with $(\ ) = -ae^{\lambda}$ and $g(\ ) = -\ e\quad .$

62. $+ \ (\quad)^2 = (\quad ^{+} \quad + \quad .$

1 . Solution with $\ne -\ $:

$$\int \frac{e}{(\ ) + \ _1} = \ _2 + \frac{1}{\beta} e \quad , \qquad \text{where} \qquad (\ ) = \frac{}{+} e^{(\ +\ )} .$$

2 . Solution with $= -\ $:

$$\int \frac{e}{+ \ _1} = \ _2 + \frac{1}{\beta} e \quad .$$

63. $= \quad (\quad + \quad)^2 + \ ^2 \ + \quad ^{\lambda} .$

The substitution $= \ ' +$ leads to a Riccati equation: $' = ae \quad ^2 + \quad + \ e^{\lambda} .$

2.8.3-4. Other equations.

64. $+ \quad (\quad)^3 + \quad = 0.$

This is a special case of equation 2.9.3.35 with $(\ ) = \quad .$

65. $+ \quad ^{+\lambda} (\quad)^3 + \quad = 0.$

This is a special case of equation 2.9.3.35 with $(\ ) = \ e^{\lambda} .$

66. $= \quad (\quad)^3 + \quad (\quad)^2.$

Solution: $= \ _1 - \ln \ a \quad e^{\ _1} \quad + \ _2 .$

67. $= \quad (\quad)^3 + \quad (\quad)^2.$

Solution in parametric form:

$$= \ _1 e^{-\ \tau} \quad \tau^{-1} e^{\ \tau} \ \tau + \ _2 , \qquad = \ln \tau.$$

68. $= \quad ^2 \ (\quad)^3 + 2 \quad (\quad)^2.$

Solution in parametric form:

$$= a^{-1} \ _1 \tau^{-2} Z^{-1} (\tau Z'_{\tau} + Z), \qquad = \ln \frac{\tau^2}{2 \ _1} \quad .$$

Here, $Z = \ _1 \ _1(\tau) + \ _2 \ _1(\tau)$ or $Z = \ _1 \ _1(\tau) + \ _2 \ _1(\tau)$, where $_1(\tau)$ and $_1(\tau)$ are the Bessel functions, and $_1(\tau)$ and $_1(\tau)$ are the modified Bessel functions.

69. $= 2 \quad ^{1\ 2} \ (\quad)^3 + \quad ^{-1\ 2} \ (\quad)^2.$

Solution in parametric form:

$$= \ _1^2 \big[2\tau \quad \exp(\mp\tau^2) \quad \big]^2, \qquad = \ln \mp \frac{1}{a} \quad \mp \tau^2, \qquad \text{where} \qquad = \quad \exp(\mp\tau^2) \ \tau + \ _2 \quad ^{-1} .$$

70. $= \quad ^{-1} (\quad)^3 + \quad (\quad)^2.$

Solution: $= \ _1 - \ln \ a \quad e^{\ _1} \quad + \ _2 .$

71. $= \quad ^{+} [(\quad)^3 + (\quad)^2]$.

Solution: $= -\ln \quad {}_1 e^{\ 2} + \dfrac{a}{1 - \quad {}_2} e$. To the limiting case $\quad {}_2 \quad 1$ there corresponds $= - \quad - \ln(\quad {}_1 + a \quad)$.

72. $= \quad (\quad)^{3\ 2} + \quad (\quad)^{1\ 2}$.

Solution in parametric form:

$$= \ln \tau^2, \qquad = -2a^{-2}\tau^{-4}\left[Z^{-1}(\tau Z'_\tau + 2Z) \mp \tfrac{1}{2}\tau^2\right],$$

where

$$Z = \begin{cases} {}_1 \ {}_2(\tau) + \ {}_2 \ {}_2(\tau) & \text{for the upper sign,} \\ {}_1 \ {}_2(\tau) + \ {}_2 \ {}_2(\tau) & \text{for the lower sign,} \end{cases}$$

${}_2(\tau)$ and ${}_2(\tau)$ are the Bessel functions, and ${}_2(\tau)$ and ${}_2(\tau)$ are the modified Bessel functions.

73. $= \quad (\quad)^{5\ 2} + \quad (\quad)^{3\ 2}$.

Solution in parametric form:

$$= -2a^{-2}\tau^{-4}\left[Z^{-1}(\tau Z'_\tau + 2Z) \mp \tfrac{1}{2}\tau^2\right], \qquad = \ln \tau^2,$$

where

$$Z = \begin{cases} {}_1 \ {}_2(\tau) + \ {}_2 \ {}_2(\tau) & \text{for the upper sign,} \\ {}_1 \ {}_2(\tau) + \ {}_2 \ {}_2(\tau) & \text{for the lower sign,} \end{cases}$$

${}_2(\tau)$ and ${}_2(\tau)$ are the Bessel functions, and ${}_2(\tau)$ and ${}_2(\tau)$ are the modified Bessel functions.

74. $= - \quad + \quad ^{k}(\quad) \ ^{+2}$.

This is a special case of equation 2.9.4.17 with $(\ ) = -$ and $= \quad + 2$.

75. $= -\dfrac{2 -}{1 -} \quad + \quad ^{-k+1}(\quad)^{k}$.

This is a special case of equation 2.9.4.31 with $(\xi) = \ \xi$.

76. $= \quad + \quad \exp[(\quad + 2 - \ ) \ ] \quad (\ )$.

The substitution $\xi = e$ leads to the generalized Emden–Fowler equation $'' = A\xi \quad (\ ')$, which is discussed in Section 2.5.

77. $= -(\quad)^2 + \quad \exp[(\quad + \quad - 1) \ ](\quad)$.

The substitution $= e$ leads to the generalized Emden–Fowler equation $'' = A \quad (\ ')$, which is discussed in Section 2.5.

78. $= -\dfrac{1 -}{2 -}(\quad)^2 + \quad ^{+k-2} \quad (\quad)^{k}$.

This is a special case of equation 2.9.4.30 with $(\xi) = \ \xi$.

79. $= \quad (\quad)^2 + \ ^{\lambda} \ (\quad)^{k}$.

This is a special case of equation 2.9.4.13 with $(\ ) = a$ and $g(\ ) = \ e^{\lambda}$.

80. $= \ ^{\lambda} \ (\quad)^2 + \quad (\quad)^{k}$.

This is a special case of equation 2.9.4.13 with $(\ ) = ae^{\lambda}$ and $g(\ ) = \quad$.

81. $= \ ^{\lambda} (\ )^2 + \ (\ )^k.$

This is a special case of equation 2.9.4.13 with $(\ ) = ae^{\lambda}$ and $g(\ ) = \ e$.

82. $= \ ^2 + \ (\ + \ )^k.$

The substitution $= \ ' + a$ leads to a Bernoulli equation: $' = a \ + \ e$.

83. $= \ ^{+} [(\ )^k + (\ )^{k-1}], \quad \neq 2.$

Solution in parametric form:

$$= \tau - \frac{\tau}{} - {}_2, \qquad = \frac{\tau}{} + {}_2, \qquad \text{where} \qquad = \left[a(2-k)e^{\tau} + {}_1\right]^{\frac{1}{-2}} + 1.$$

84. $= \ (\ - \ ) + \ ^{\lambda} (\ - \ )^k.$

This is a special case of equation 2.9.4.4 with $(\ ) = a$ and $g(\ ) = \ e^{\lambda}$.

85. $= \ ^{\lambda} (\ - \ ) + \ (\ - \ )^k.$

This is a special case of equation 2.9.4.4 with $(\ ) = ae^{\lambda}$ and $g(\ ) =$.

86. $= \ ^{\lambda} (\ - \ ) + \ (\ - \ )^k.$

This is a special case of equation 2.9.4.4 with $(\ ) = ae^{\lambda}$ and $g(\ ) = \ e$.

87. $+ \ = \ ^{\lambda} (\ ) \ .$

The transformation $= \ ', \ = \ ^{- \ +1}e^{\lambda}$ leads to a first-order linear equation: $a \ ' = \lambda \ + \ - \ + 1.$

88. $+ \ + \ ^{-2 \ +1 \ \lambda} (\ ) \ = 0.$

This is a special case of equation 2.9.4.14 with $(\ ) = ae^{\lambda}$.

89. $+ \ = (\ ^{\lambda} + \ ^{-1})(\ ) \ .$

The transformation $= \ ', \ = \ ^{- \ +1}e^{\lambda}$ leads to a first-order separable equation: $(a \ + \ ) \ ' = (\lambda \ + \ - \ + 1) \ .$

90. $= (\ )^2 + \ (\ )^k.$

This is a special case of equation 2.9.4.68 with $(\xi) = \ \xi, g(\ ) =$, and $= \ - k + 2.$

91. $= (\ )^2 + (\ ^{\lambda} \ + \ ^{2-} \ )(\ ) \ .$

The transformation $\xi = \ ' \ , \ = e^{\lambda} \ + \ ^{-2}$ leads to a first-order separable equation: $\xi \ (a \ + \ ) \ ' = [(\ + \ - 2)\xi + \lambda] \ .$

2.8.4. Equations Containing Hyperbolic Functions

2.8.4-1. Equations with hyperbolic sine.

1. $= \ ^2 + \ (\sinh \ )^{- \ -3} \ .$

This is a special case of equation 2.9.1.34 with $(\xi) = a\xi$.

2. $= \ \sinh(\ ) \ + \ ^{-3}.$

This is a special case of equation 2.9.1.2 with $(\ ) = - \ \sinh(\lambda \ ).$

3. = **sinh** (+) + .

This is a special case of equation 2.9.1.4 with ( ) = sinh $+\beta$ and = 0.

4. = (+ **sinh**) − **sinh** + .

The substitution = + sinh leads to an autonomous equation of the form 2.9.1.1: $''$ $= a$ + .

5. + 3 + ^{3} + **sinh**( ) = 0.

This is a special case of equation 2.9.2.1 with ( ) = a sinh(λ).

6. + ($)^2$ + **sinh** + = 0.

This is a special case of equation 2.9.3.25 with ( ) = a and g( ) = sinh + .

7. + **sinh** ($)^2$ + + = 0.

This is a special case of equation 2.9.3.25 with ( ) = a sinh and g( ) = + .

8. + **sinh** ($)^2$ + **sinh** ( ) + = 0.

This is a special case of equation 2.9.3.25 with ( ) = a sinh and g( ) = sinh (λ) + .

9. = ($)^2$ + **sinh** ($)^k$.

This is a special case of equation 2.9.4.13 with ( ) = a and g( ) = sinh .

10. = **sinh** ($)^2$ + ($)^k$.

This is a special case of equation 2.9.4.13 with ( ) = a sinh and g( ) = .

11. = + $^{2\ +1}$ **sinh** ( ).

This is a special case of equation 2.9.2.20 with ( ) = a sinh (λ).

12. 2 = ($)^2$ + **sinh** ( ) ^{2} − .

This is a special case of equation 2.9.3.5 with ( ) = − sinh (λ).

13. = ($)^2$ − $^{4\ -2}$ + **sinh** ( ) ^{2}.

This is a special case of equation 2.9.3.8 with ( ) = − sinh (λ).

14. = ($)^2$ + ^{2} + **sinh**^{k}( ) $^{+1}$.

This is a special case of equation 2.9.3.9 with ( ) = $-a$ and g( ) = − sinh (λ).

15. = ($)^2$ + + **sinh** ( ) ^{2}.

This is a special case of equation 2.9.3.7 with ( ) = $-a$ and g( ) = − sinh (λ).

16. = ($)^2$ + **sinh** ( ) + ^{2}.

This is a special case of equation 2.9.3.7 with ( ) = $-a$ sinh (λ) and g( ) = − .

17. ^{2} + = **sinh** ( ) + .

This is a special case of equation 2.9.2.23 with ( ) = a sinh (λ) + .

18. (^{2} +) + + **sinh** ( ) + = 0.

This is a special case of equation 2.9.2.24 with ( ) = sinh (λ) + .

2.8.4-2. Equations with hyperbolic cosine.

19. $= {}^{2} + (\cosh\ )^{-\ -3}\ .$

This is a special case of equation 2.9.1.35 with $(\xi) = a\xi\ $.

20. $= \cosh(\ )\ +\ {}^{-3}.$

This is a special case of equation 2.9.1.2 with $(\ ) = -\ \cosh(\lambda\ )$.

21. $= \cosh(\ +\ ) + \ .$

This is a special case of equation 2.9.1.4 with $(\ ) = \ \cosh\ + \beta$ and $\ = 0$.

22. $= (\ + \cosh\ ) - \cosh\ + \ .$

The substitution $\ = \ + \cosh\ $ leads to an autonomous equation of the form 2.9.1.1: $'' = a\ + \ $.

23. $+3\ +\ {}^{3} + \cosh(\ )\ = 0.$

This is a special case of equation 2.9.2.1 with $(\ ) = a\cosh(\lambda\ )$.

24. $+\ (\ )^{2} + \cosh\ +\ = 0.$

This is a special case of equation 2.9.3.25 with $(\ ) = a\ $ and $g(\ ) = \ \cosh\ +\ $.

25. $+ \cosh\ (\ )^{2} +\ +\ = 0.$

This is a special case of equation 2.9.3.25 with $(\ ) = a\cosh\ $ and $g(\ ) = \ +\ $.

26. $+ \cosh\ (\ )^{2} + \cosh(\ ) + \ = 0.$

This is a special case of equation 2.9.3.25 with $(\ ) = a\cosh\ $ and $g(\ ) = \cosh\ (\lambda\ ) + \ $.

27. $= \ (\ )^{2} + \cosh\ (\ )^{k}.$

This is a special case of equation 2.9.4.13 with $(\ ) = a\ $ and $g(\ ) = \ \cosh\ $.

28. $= \cosh\ (\ )^{2} +\ (\ )^{k}.$

This is a special case of equation 2.9.4.13 with $(\ ) = a\cosh\ $ and $g(\ ) = \ $.

29. $= \ +\ {}^{2\ +1}\cosh(\ ).$

This is a special case of equation 2.9.2.20 with $(\ ) = a\cosh\ (\lambda\ )$.

30. $2\ = (\ )^{2} + \cosh\ (\ )\ {}^{2} - \ .$

This is a special case of equation 2.9.3.5 with $(\ ) = -\ \cosh\ (\lambda\ )$.

31. $= (\ )^{2} - \ {}^{4\ -2} + \cosh\ (\ )\ {}^{2}.$

This is a special case of equation 2.9.3.8 with $(\ ) = -\ \cosh\ (\lambda\ )$.

32. $= (\ )^{2} + \ {}^{2} + \cosh^{k}(\ )\ {}^{+1}.$

This is a special case of equation 2.9.3.9 with $(\ ) = -a\ $ and $g(\ ) = -\ \cosh\ (\lambda\ )$.

33. $= (\ )^{2} + \ + \cosh\ (\ )\ {}^{2}.$

This is a special case of equation 2.9.3.7 with $(\ ) = -a\ $ and $g(\ ) = -\ \cosh\ (\lambda\ )$.

34. $= (\ )^{2} + \cosh\ (\ )\ +\ {}^{2}.$

This is a special case of equation 2.9.3.7 with $(\ ) = -a\cosh\ (\lambda\ )$ and $g(\ ) = -\ $.

35. ^{2} $+$ $=$ **cosh** ( ) + .

This is a special case of equation 2.9.2.23 with ( ) = a cosh (λ) + .

36. (^{2} +) + + **cosh** ( ) + = **0.**

This is a special case of equation 2.9.2.24 with ( ) = cosh (λ) + .

2.8.4-3. Equations with hyperbolic tangent.

37. = **tanh**( ) + $^{-3}$.

This is a special case of equation 2.9.1.2 with ( ) = − tanh(λ).

38. = **tanh** (+) + .

This is a special case of equation 2.9.1.4 with ( ) = tanh + β and = 0.

39. + 3 + ^{3} + **tanh**( ) = **0.**

This is a special case of equation 2.9.2.1 with ( ) = a tanh(λ).

40. + ( )^{2} + **tanh** + = **0.**

This is a special case of equation 2.9.3.25 with ( ) = a and g( ) = tanh + .

41. + **tanh** ( )^{2} + + = **0.**

This is a special case of equation 2.9.3.25 with ( ) = a tanh and g( ) = + .

42. + **tanh** ( )^{2} + **tanh** ( ) + = **0.**

This is a special case of equation 2.9.3.25 with ( ) = a tanh and g( ) = tanh (λ) + .

43. = ( )^{2} + **tanh** ( )^{k}.

This is a special case of equation 2.9.4.13 with ( ) = a and g( ) = tanh .

44. = **tanh** ( )^{2} + ( )^{k}.

This is a special case of equation 2.9.4.13 with ( ) = a tanh and g( ) = .

45. = + ^{2} $^{+1}$ **tanh** ( ).

This is a special case of equation 2.9.2.20 with ( ) = a tanh (λ).

46. 2 = ( )^{2} + **tanh** ( ) ^{2} − .

This is a special case of equation 2.9.3.5 with ( ) = − tanh (λ).

47. = ( )^{2} − ^{4} $^{-2}$ + **tanh** ( ) ^{2}.

This is a special case of equation 2.9.3.8 with ( ) = − tanh (λ).

48. = ( )^{2} + ^{2} + **tanh**^{k}( ) $^{+1}$.

This is a special case of equation 2.9.3.9 with ( ) = −a and g( ) = − tanh (λ).

49. = ( )^{2} + + **tanh** ( ) ^{2}.

This is a special case of equation 2.9.3.7 with ( ) = −a and g( ) = − tanh (λ).

50. = ( )^{2} + **tanh** ( ) + ^{2}.

This is a special case of equation 2.9.3.7 with ( ) = −a tanh (λ) and g( ) = − .

51. $\;{}^{2}\quad + \quad = \;\mathbf{tanh}\;(\quad) + \;.$

This is a special case of equation 2.9.2.23 with ( ) = a tanh (λ) + .

52. $(\quad{}^{2} + \;)\quad + \quad + \mathbf{tanh}\;(\quad) + \quad = 0.$

This is a special case of equation 2.9.2.24 with ( ) = tanh (λ) + .

2.8.4-4. Equations with hyperbolic cotangent.

53. $= \;\mathbf{coth}(\quad)\; + \quad{}^{-3}.$

This is a special case of equation 2.9.1.2 with ( ) = − coth(λ).

54. $= \;\mathbf{coth}\;(\quad + \quad) + \;.$

This is a special case of equation 2.9.1.4 with ( ) = coth + β and = 0.

55. $+\,3\quad + \quad{}^{3} + \quad\mathbf{coth}(\quad)\quad = 0.$

This is a special case of equation 2.9.2.1 with ( ) = a coth(λ).

56. $+ \quad (\quad)^{2} + \quad\mathbf{coth}\quad + \quad = 0.$

This is a special case of equation 2.9.3.25 with ( ) = a and g( ) = coth + .

57. $+ \quad\mathbf{coth}\quad(\quad)^{2} + \quad + \quad = 0.$

This is a special case of equation 2.9.3.25 with ( ) = a coth and g( ) = + .

58. $+ \quad\mathbf{coth}\quad(\quad)^{2} + \quad\mathbf{coth}\;(\quad) + \quad = 0.$

This is a special case of equation 2.9.3.25 with ( ) = a coth and g( ) = coth (λ) + .

59. $= \quad(\quad)^{2} + \quad\mathbf{coth}\quad(\quad)^{k}.$

This is a special case of equation 2.9.4.13 with ( ) = a and g( ) = coth .

60. $= \quad\mathbf{coth}\quad(\quad)^{2} + \quad(\quad)^{k}.$

This is a special case of equation 2.9.4.13 with ( ) = a coth and g( ) = .

61. $= \quad + \quad{}^{2}\;{}^{+1}\,\mathbf{coth}\;(\quad).$

This is a special case of equation 2.9.2.20 with ( ) = a coth (λ).

62. $2 \quad = (\quad)^{2} + \quad\mathbf{coth}\;(\quad)\;{}^{2} - \;.$

This is a special case of equation 2.9.3.5 with ( ) = − coth (λ).

63. $= \;(\quad)^{2} - \quad{}^{4}\;{}^{-2} + \quad\mathbf{coth}\;(\quad)\;{}^{2}.$

This is a special case of equation 2.9.3.8 with ( ) = − coth (λ).

64. $= \;(\quad)^{2} + \quad{}^{2} + \quad\mathbf{coth}^{k}(\quad)\quad{}^{+1}.$

This is a special case of equation 2.9.3.9 with ( ) = −a and g( ) = − coth (λ).

65. $= (\quad)^{2} + \quad + \quad\mathbf{coth}\;(\quad)\;{}^{2}.$

This is a special case of equation 2.9.3.7 with ( ) = −a and g( ) = − coth (λ).

66. $= (\quad)^{2} + \quad\mathbf{coth}\;(\quad)\quad + \quad{}^{2}.$

This is a special case of equation 2.9.3.7 with ( ) = −a coth (λ) and g( ) = − .

67. $\quad ^{2} \quad + \quad = \quad$ **coth** $(\quad) + \quad .$

This is a special case of equation 2.9.2.23 with $(\ ) = a \coth (\lambda \ ) + \ .$

68. $(\quad ^{2} + \ ) \quad + \quad + \coth (\quad) + \quad = 0.$

This is a special case of equation 2.9.2.24 with $(\ ) = \coth (\lambda \ ) + \ .$

2.8.4-5. Equations containing combinations of hyperbolic functions.

69. $\quad = \ ^{2} \ + \ $ **sinh** $(\quad)$ **cosh**$^{- \ - \ -3}(\quad) \quad .$

The transformation $\xi = \tanh(\lambda \ )$, $\quad = \dfrac{}{\cosh(\lambda \ )}$ leads to the Emden–Fowler equation $'' = a\lambda^{-2}\xi$, which is discussed in Section 2.3.

70. $\quad = \ ^{2} \ + \ $ **cosh** $(\quad)$ **sinh**$^{- \ - \ -3}(\quad) \quad .$

The transformation $\xi = \coth(\lambda \ )$, $\quad = \dfrac{}{\sinh(\lambda \ )}$ leads to the Emden–Fowler equation $'' = a\lambda^{-2}\xi$, which is discussed in Section 2.3.

71. $\quad = (\quad$ **sinh** $\quad -$ **cosh** $\quad)^{k} + \quad .$

The substitution $\quad = \ ' \sinh \quad - \quad \cosh$ leads to a first-order separable equation: $' = a \sinh \quad .$

72. $\quad = (\quad$ **cosh** $\quad -$ **sinh** $\quad)^{k} + \quad .$

The substitution $\quad = \ ' \cosh \quad - \quad \sinh$ leads to a first-order separable equation: $' = a \cosh \quad .$

73. **sinh** $\quad + \frac{1}{2}$ **cosh** $\quad =$ **sinh**$(\quad) + \ .$

This is a special case of equation 2.9.2.34 with $g(\ ) = \sinh$ and $(\ ) = a \sinh(\lambda \ ) + \ .$

74. **cosh** $\quad + \frac{1}{2}$ **sinh** $\quad =$ **cosh**$(\quad) + \ .$

This is a special case of equation 2.9.2.34 with $g(\ ) = \cosh$ and $(\ ) = a \cosh(\lambda \ ) + \ .$

75. $\quad + (\quad)^{2} +$ **sinh**$(\quad) \quad +$ **cosh**$(\quad) + \quad = 0.$

This is a special case of equation 2.9.3.6 with $(\ ) = a \sinh(\beta \ )$ and $g(\ ) = \ \cosh(\lambda \ ) + \ .$

76. $\quad - (\quad)^{2} +$ **sinh**$(\quad) \quad +$ **cosh**$(\quad) \quad ^{2} = 0.$

This is a special case of equation 2.9.3.7 with $(\ ) = a \sinh(\beta \ )$ and $g(\ ) = \ \cosh(\lambda \ ).$

2.8.5. Equations Containing Logarithmic Functions

2.8.5-1. Equations of the form $(\ , \ ) '' + g(\ , \ ) ' + (\ , \ ) = 0.$

1. $\quad = (\quad + \quad$ **ln** $\ ).$

This is a special case of equation 2.9.1.27 with $(z) = \ \ln z.$

2. $\quad = \ ^{-2}(\quad + \quad$ **ln** $\ ).$

This is a special case of equation 2.9.1.28 with $(z) = \ \ln z.$

3. $\quad = \ ^{-3}($**ln** $\quad -$ **ln** $\ ).$

This is a special case of equation 2.9.1.8 with $(\xi) = a \ln \xi.$

4. $= \ ^{-3}\ ^{2}(2\ln \ -\ln \ ).$

This is a special case of equation 2.9.1.9 with $(\xi) = 2a\ln\xi$.

5. $= \ \ln\ (\ + \ ) + s.$

This is a special case of equation 2.9.1.4 with $(\ ) = k\ln \ + \$ and $= 0$.

6. $= \ ^{-3}\big[2\ln \ -\ln(\ ^{2} + \ )\big].$

This is a special case of equation 2.9.1.21 with $(\ ) = 2\ln$ and $= 0$.

7. $^{2} \ = \ ^{2}(\ + \ \ln \ + \ ) \ + \ .$

This is a special case of equation 2.9.1.36 with $(\xi) = \xi$.

8. $^{2} \ = \ (\ +1) \ + \ ^{3} \ ^{+2}(\ln \ + \ \ln \ ).$

This is a special case of equation 2.9.1.12 with $(\xi) = a\ln\xi$.

9. $^{2} \ + \tfrac{1}{4} \ + \ \frac{1-}{2}(\ \ln \ + \ ) \ = 0.$

The transformation $\xi = a\ln \ + \ ,\ \ = \ ^{-1} \ ^{2}$ leads to the Emden–Fowler equation: $'' + Aa^{-2}\xi \ = 0$ (see Section 2.3).

10. $= \ + \ ^{2} \ ^{+1}\ln\ (\ ).$

This is a special case of equation 2.9.2.20 with $(\ ) = a\ln\ (\lambda \ )$.

11. $= -(\ +1) \ + \ ^{-1}(\ln \ + \ \ln \ ).$

This is a special case of equation 2.9.2.30 with $(\xi) = a\ln\xi$.

12. $= (\ + \ \ln \ ) \ .$

This is a special case of equation 2.9.2.39 with $(\xi) = \ln\xi$.

13. $+ \ (2 \ + \ln \ + \ ) \ + \ = 0.$

Integrating yields a Riccati equation: $' + a \ ^{2} + (\ln \ + \ ) \ = \ .$

14. $= \ \ln^{k}(\ ) \ .$

This is a special case of equation 2.9.2.21 with $(\ ) = a\ln\ (\ )$.

15. $^{2} \ + \ = \ \ln\ (\ ) + \ .$

This is a special case of equation 2.9.2.23 with $(\ ) = a\ln\ (\lambda \ ) + \ .$

16. $(\ ^{2} + \ ) \ + \ + \ \ln\ (\ ) = 0.$

This is a special case of equation 2.9.2.24 with $(\ ) = \ \ln\ (\lambda \ )$.

2.8.5-2. Other equations.

17. $+ \ (\ )^{2} + \ \ln \ + \ = 0.$

This is a special case of equation 2.9.3.25 with $(\ ) = a$ and $g(\ ) = \ \ln \ + \ .$

18. $= \ (\ )^{2} + \ \ln\ (\ ) \ .$

This is a special case of equation 2.9.3.38 with $(\ ) = a$ and $g(\ ) = \ \ln\ (\lambda \ )$.

19. $+ \;\; \ln \;\; (\;\;)^2 + \;\;\; + \; = 0.$

This is a special case of equation 2.9.3.25 with $(\;) = a \ln$ and $g(\;) = \;\; + \; .$

20. $= \;\; \ln \;\; (\;\;)^2 + \;\;\; .$

This is a special case of equation 2.9.3.38 with $(\;) = a \ln$ and $g(\;) = \;\; .$

21. $= \;\; \ln \;\; (\;\;)^2 + \;\; \ln \;\; (\;\;) \;\; .$

This is a special case of equation 2.9.3.38 with $(\;) = a \ln$ and $g(\;) = \;\; \ln \;\; (\lambda \;\;).$

22. $= \;\; {}^{-2} \;\; {}^{-1}(\;\;)^4 - 2 \;\; {}^{-3} \ln \;\; (\;\;)^3.$

Solution in parametric form:

$$= \lambda\bigl[(\;\;\; 2\tau)^2 \;\;\; 4\ln(\;\; {}_1 \;\;)\bigr]^{1 \;\; 2}, \qquad = \;\; {}_1 \;\; ,$$

where $= \exp(\mp\tau^2) \;\;\; \exp(\mp\tau^2) \;\; \tau + \;\; {}_2 \;\; {}^{-1}, \; \lambda = \bigl(\; \tfrac{1}{2}a \;\; {}^2_1 \;\; {}^{1 \;\; 4}.$

23. $= 2 \;\; \ln \;\; {}^{-3} - \;\; {}^{-1} \;\; {}^{-2}(\;\;)^{-1}.$

Solution in parametric form:

$$= \;\; {}_1 \;\; , \qquad = \lambda\bigl[(\;\;\; 2\tau)^2 \;\;\; 4\ln(\;\; {}_1 \;\;)\bigr]^{1 \;\; 2},$$

where $= \exp(\mp\tau^2) \;\;\; \exp(\mp\tau^2) \;\; \tau + \;\; {}_2 \;\; {}^{-1}, \; \lambda = \bigl(\; \tfrac{1}{2}a \;\; {}^2_1 \;\; {}^{1 \;\; 4}.$

24. $= \;\;\; (\;\;)^2 + \;\; \ln \;\;\; (\;\;)^k.$

This is a special case of equation 2.9.4.13 with $(\;) = a$ and $g(\;) = \;\; \ln \;\; .$

25. $= \;\; \ln \;\; (\;\;)^2 + \;\;\; (\;\;)^k.$

This is a special case of equation 2.9.4.13 with $(\;) = a \ln$ and $g(\;) = \;\; .$

26. $= \; (\;\;)^2 + \;\;\; {}^2 + \;\; \ln^k(\;\;) \;\;\; {}^{+1}.$

This is a special case of equation 2.9.3.9 with $(\;) = -a$ and $g(\;) = - \;\; \ln \;\; (\lambda \;\;).$

27. $= (\;\;)^2 + \;\; \ln \;\; (\;\;) \;\;\; + \;\;\; {}^2.$

This is a special case of equation 2.9.3.7 with $(\;) = -a \ln \;\; (\lambda \;\;)$ and $g(\;) = - \;\; .$

28. $= (\;\;)^2 + \;\;\; + \;\; \ln \;\; (\;\;) \;\; {}^2.$

This is a special case of equation 2.9.3.7 with $(\;) = -a$ and $g(\;) = - \;\; \ln \;\; (\lambda \;\;).$

29. $= (\;\; + \;\; \ln \;\;)(\;\;)^2.$

This is a special case of equation 2.9.3.40 with $(\xi) = \ln\xi.$

2.8.6. Equations Containing Trigonometric Functions

2.8.6-1. Equations with sine.

1. $= - \;\; {}^2 \;\; + \;\; (\sin \;\;) \;\; {}^{- \;\; -3}.$

This is a special case of equation 2.9.1.40 with $(\xi) = a\xi^{- \;\; -3}.$

2. $= -\ {}^{2}\ +\ \ \sin\ (\ \ +\ )\sin\ (\ \ +\ )\ {}^{-\ -\ -3}.$

The transformation $\xi = \dfrac{\sin(\lambda\ \ + a)}{\sin(\lambda\ \ +\ )}$, $\ = \dfrac{}{\sin(\lambda\ \ +\ )}$ leads to the Emden–Fowler equation: $'' = A[\lambda \sin(\ \ - a)]^{-2}\xi^{\ -\ -\ -3}$ (see Section 2.3).

3. $=\ \sin(\ \ )\ +\ \ {}^{-3}.$

This is a special case of equation 2.9.1.2 with $(\ ) = -\ \sin(\lambda\ )$.

4. $=\ \ \sin\ (\ \ +\ \ ) +\ .$

This is a special case of equation 2.9.1.4 with $(\ ) =\ \ \sin\ \ \ + \beta$ and $\ = 0$.

5. $=\ (\ +\ \sin\ )\ +\ \sin\ \ +\ .$

The substitution $\ =\ \ +\ \sin\ $ leads to an autonomous equation of the form 2.9.1.1: $'' = a\ \ +\ .$

6. $+ 3\ \ \ +\ {}^{3} +\ \sin(\ \ )\ = 0.$

This is a special case of equation 2.9.2.1 with $(\ ) = a\sin(\lambda\ )$.

7. $+\ \ \ (\ \ )^2 +\ \sin\ \ \ +\ = 0.$

This is a special case of equation 2.9.3.25 with $(\ ) = a$ and $g(\ ) =\ \sin\ \ \ +\ .$

8. $=\ \ \ (\ \ )^2 +\ \sin\ (\ \ )\ \ .$

This is a special case of equation 2.9.3.38 with $(\ ) = a$ and $g(\ ) =\ \sin\ (\lambda\ )$.

9. $=\ \ \ (\ \ )^2 +\ \sin\ \ \ (\ \ )^k.$

This is a special case of equation 2.9.4.13 with $(\ ) = a$ and $g(\ ) =\ \sin\ \ .$

10. $+\ \sin\ \ (\ \ )^2 +\ \ \ \ +\ = 0.$

This is a special case of equation 2.9.3.25 with $(\ ) = a\sin$ and $g(\ ) =\ \ \ +\ .$

11. $+\ \sin\ \ (\ \ )^2 +\ \sin\ (\ \ ) +\ = 0.$

This is a special case of equation 2.9.3.25 with $(\ ) = a\sin$ and $g(\ ) =\ \sin\ (\lambda\ ) +\ .$

12. $=\ \sin\ \ (\ \ )^2 +\ \ \ \ \ .$

This is a special case of equation 2.9.3.38 with $(\ ) = a\sin$ and $g(\ ) =\ \ .$

13. $=\ \sin\ \ (\ \ )^2 +\ \sin\ (\ \ )\ \ .$

This is a special case of equation 2.9.3.38 with $(\ ) = a\sin$ and $g(\ ) =\ \sin\ (\lambda\ )$.

14. $=\ \sin\ \ (\ \ )^2 +\ \ \ (\ \ )^k.$

This is a special case of equation 2.9.4.13 with $(\ ) = a\sin$ and $g(\ ) =\ \ .$

15. $=\ \ \ +\ \ {}^{2\ \ +1}\sin\ (\ \ ).$

This is a special case of equation 2.9.2.20 with $(\ ) = a\sin\ (\lambda\ )$.

16. $2\ \ \ = (\ \ )^2 +\ \sin(\ \ )\ {}^{2} -\ .$

This is a special case of equation 2.9.3.5 with $(\ ) = -\ \sin(\lambda\ )$.

17. $=\ (\ \ )^2 -\ \ {}^{4\ \ -2} +\ \sin(\ \ )\ {}^{2}.$

This is a special case of equation 2.9.3.8 with $(\ ) = -\ \sin(\lambda\ )$.

18. $= \ (\ )^2 + \quad {}^2 + \ \sin^k(\quad) \quad {}^{+1}$.

This is a special case of equation 2.9.3.9 with $(\ ) = -a \quad$ and $g(\ ) = - \ \sin \ (\lambda \ )$.

19. $= \ (\ )^2 + \ \sin(\quad) \ {}^2 + \ \sin(\quad) \quad {}^{+1}$.

This is a special case of equation 2.9.3.9 with $(\ ) = -a \sin(\lambda \ )$ and $g(\ ) = - \ \sin(\quad)$.

20. $= (\ )^2 + \qquad + \ \sin \ (\quad) \ {}^2$.

This is a special case of equation 2.9.3.7 with $(\ ) = -a \quad$ and $g(\ ) = - \ \sin \ (\lambda \ )$.

21. $= (\ )^2 + \ \sin \ (\quad) \qquad + \qquad {}^2$.

This is a special case of equation 2.9.3.7 with $(\ ) = -a \sin \ (\lambda \ )$ and $g(\ ) = - \quad$.

22. ${}^2 \quad + \quad = \ \sin \ (\quad) + \ $.

This is a special case of equation 2.9.2.23 with $(\ ) = a \sin \ (\lambda \ ) + \ $.

23. $(\quad {}^2 + \ ) \quad + \qquad + \sin \ (\quad) + \quad = 0$.

This is a special case of equation 2.9.2.24 with $(\ ) = \sin \ (\lambda \ ) + \ $.

24. $\sin^2 \qquad = \ (\ + 1 - \ \sin^2 \ ) \ + \ (\sin \ ) \quad {}^{+3 \ +2} \quad$.

This is a special case of equation 2.9.1.44 with $(\xi) = a\xi \quad$.

2.8.6-2. Equations with cosine.

25. $= - \ {}^2 \ + \ (\cos \quad) \quad {}^{- \ -3}$.

This is a special case of equation 2.9.1.41 with $(\xi) = a\xi^{- \ -3}$.

26. $= - \ {}^2 \ + \ \cos \ (\quad + \ ) \cos \ (\quad + \ ) \quad {}^{- \ - \ -3}$.

The transformation $\xi = \dfrac{\cos(\lambda \ + a)}{\cos(\lambda \ + \ )}, \quad = \dfrac{}{\cos(\lambda \ + \ )}$ leads to the Emden–Fowler equation: $'' = A[\lambda \sin(\ - a)]^{-2}\xi \quad {}^{- \ - \ -3}$ (see Section 2.3).

27. $= \ \cos(\quad) \ + \quad {}^{-3}$.

This is a special case of equation 2.9.1.2 with $(\ ) = - \ \cos(\lambda \ )$.

28. $= \ \cos \ (\quad + \quad) + \ $.

This is a special case of equation 2.9.1.4 with $(\ ) = \ \cos \quad + \beta$ and $= 0$.

29. $= \ (\ + \ \cos \ ) \ + \ \cos \ + \ $.

The substitution $= \ + \ \cos \ $ leads to an autonomous equation of the form 2.9.1.1: $'' = a \quad + \ $.

30. $+ 3 \quad + \ {}^3 + \ \cos(\quad) \ = 0$.

This is a special case of equation 2.9.2.1 with $(\ ) = a \cos(\lambda \ )$.

31. $+ \quad (\ )^2 + \ \cos \quad + \ = 0$.

This is a special case of equation 2.9.3.25 with $(\ ) = a \quad$ and $g(\ ) = \ \cos \quad + \ $.

32. $= \quad (\ )^2 + \ \cos \ (\quad) \quad$.

This is a special case of equation 2.9.3.38 with $(\ ) = a \quad$ and $g(\ ) = \ \cos \ (\lambda \ )$.

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33. $= \quad (\quad)^2 + \cos \quad (\quad)^k.$

This is a special case of equation 2.9.4.13 with $(\ ) = a$ and $g(\ ) = \ \cos$.

34. $+ \cos \quad (\quad)^2 + \quad + \quad = 0.$

This is a special case of equation 2.9.3.25 with $(\ ) = a\cos$ and $g(\ ) = \quad + \ .$

35. $+ \cos \quad (\quad)^2 + \cos \ (\quad) + \quad = 0.$

This is a special case of equation 2.9.3.25 with $(\ ) = a\cos$ and $g(\ ) = \ \cos \ (\lambda\ ) + \ .$

36. $= \cos \quad (\quad)^2 + \quad .$

This is a special case of equation 2.9.3.38 with $(\ ) = a\cos$ and $g(\ ) = \quad .$

37. $= \cos \quad (\quad)^2 + \cos \ (\quad) \quad .$

This is a special case of equation 2.9.3.38 with $(\ ) = a\cos$ and $g(\ ) = \ \cos \ (\lambda\ ).$

38. $= \cos \quad (\quad)^2 + \quad (\quad)^k.$

This is a special case of equation 2.9.4.13 with $(\ ) = a\cos$ and $g(\ ) = \quad .$

39. $= \quad + \quad {}^{2\ \ +1} \cos \ (\quad).$

This is a special case of equation 2.9.2.20 with $(\ ) = a\cos \ (\lambda\ ).$

40. $2 \quad = (\quad)^2 + \cos(\quad) \ {}^2 - \ .$

This is a special case of equation 2.9.3.5 with $(\ ) = - \ \cos(\lambda\ ).$

41. $= \ (\quad)^2 - \quad {}^{4\ \ -2} + \cos(\quad) \ {}^2.$

This is a special case of equation 2.9.3.8 with $(\ ) = - \ \cos(\lambda\ ).$

42. $= \ (\quad)^2 + \quad {}^2 + \cos^k(\quad) \quad {}^{+1}.$

This is a special case of equation 2.9.3.9 with $(\ ) = -a$ and $g(\ ) = - \ \cos \ (\lambda\ ).$

43. $= \ (\quad)^2 + \cos(\quad) \ {}^2 + \cos(\quad) \quad {}^{+1}.$

This is a special case of equation 2.9.3.9 with $(\ ) = -a\cos(\lambda\ )$ and $g(\ ) = - \ \cos(\quad).$

44. $= (\quad)^2 + \quad + \cos \ (\quad) \ {}^2.$

This is a special case of equation 2.9.3.7 with $(\ ) = -a$ and $g(\ ) = - \ \cos \ (\lambda\ ).$

45. $= (\quad)^2 + \cos \ (\quad) \quad + \quad {}^2.$

This is a special case of equation 2.9.3.7 with $(\ ) = -a\cos \ (\lambda\ )$ and $g(\ ) = - \quad .$

46. ${}^2 \quad + \quad = \cos \ (\quad) + \ .$

This is a special case of equation 2.9.2.23 with $(\ ) = a\cos \ (\lambda\ ) + \ .$

47. $(\ \ {}^2 + \ ) \quad + \quad + \cos \ (\quad) + \quad = 0.$

This is a special case of equation 2.9.2.24 with $(\ ) = \cos \ (\lambda\ ) + \ .$

48. $\cos^2 \quad = \ (\ + 1 - \cos^2 \ ) + (\cos \ ) \quad {}^{+3\ \ +2} \quad .$

This is a special case of equation 2.9.1.45 with $(\xi) = a\xi \ .$

2.8.6-3. Equations with tangent.

49. $= \tan(\quad) + \quad {}^{-3}.$

This is a special case of equation 2.9.1.2 with $(\quad) = - \tan(\lambda\quad)$.

50. $= \tan(\quad + \quad) + \quad .$

This is a special case of equation 2.9.1.4 with $(\quad) = \tan \quad + \beta$ and $= 0$.

51. $+ \quad (\quad)^2 + \tan \quad + \quad = 0.$

This is a special case of equation 2.9.3.25 with $(\quad) = a$ and $g(\quad) = \tan \quad + \quad$.

52. $= \quad (\quad)^2 + \tan(\quad) \quad .$

This is a special case of equation 2.9.3.38 with $(\quad) = a$ and $g(\quad) = \tan(\lambda\quad)$.

53. $= \quad (\quad)^2 + \tan \quad (\quad)^k.$

This is a special case of equation 2.9.4.13 with $(\quad) = a$ and $g(\quad) = \tan \quad$.

54. $+ \tan \quad (\quad)^2 + \quad + \quad = 0.$

This is a special case of equation 2.9.3.25 with $(\quad) = a \tan$ and $g(\quad) = \quad + \quad$.

55. $+ \tan \quad (\quad)^2 + \tan(\quad) + \quad = 0.$

This is a special case of equation 2.9.3.25 with $(\quad) = a \tan$ and $g(\quad) = \tan(\lambda\quad) + \quad$.

56. $= \tan \quad (\quad)^2 + \quad .$

This is a special case of equation 2.9.3.38 with $(\quad) = a \tan$ and $g(\quad) = \quad$.

57. $= \tan \quad (\quad)^2 + \tan(\quad) \quad .$

This is a special case of equation 2.9.3.38 with $(\quad) = a \tan$ and $g(\quad) = \tan(\lambda\quad)$.

58. $= \tan \quad (\quad)^2 + \quad (\quad)^k.$

This is a special case of equation 2.9.4.13 with $(\quad) = a \tan$ and $g(\quad) = \quad$.

59. $= \quad + \quad {}^{2 \quad +1} \tan(\quad).$

This is a special case of equation 2.9.2.20 with $(\quad) = a \tan(\lambda\quad)$.

60. $= (\quad)^2 + \quad {}^2 + \tan^k(\quad) \quad {}^{+1}.$

This is a special case of equation 2.9.3.9 with $(\quad) = -a$ and $g(\quad) = - \tan(\lambda\quad)$.

61. $= (\quad)^2 + \quad + \tan(\quad) \quad {}^2.$

This is a special case of equation 2.9.3.7 with $(\quad) = -a$ and $g(\quad) = - \tan(\lambda\quad)$.

62. ${}^2 \quad + \quad = \tan(\quad) + \quad .$

This is a special case of equation 2.9.2.23 with $(\quad) = a \tan(\lambda\quad) + \quad$.

63. $(\quad {}^2 + \quad) \quad + \quad + \tan(\quad) + \quad = 0.$

This is a special case of equation 2.9.2.24 with $(\quad) = \tan(\lambda\quad) + \quad$.

2.8.6-4. Equations with cotangent.

64. **= cot( ) + $^{-3}$.**

This is a special case of equation 2.9.1.2 with ( ) = − cot(λ).

65. **= cot (+) + .**

This is a special case of equation 2.9.1.4 with ( ) = cot + β and = 0.

66. **+ ( )^2 + cot + = 0.**

This is a special case of equation 2.9.3.25 with ( ) = a and g( ) = cot + .

67. **= ( )^2 + cot ( ) .**

This is a special case of equation 2.9.3.38 with ( ) = a and g( ) = cot (λ).

68. **= ( )^2 + cot ( )^k.**

This is a special case of equation 2.9.4.13 with ( ) = a and g( ) = cot .

69. **+ cot ( )^2 + + = 0.**

This is a special case of equation 2.9.3.25 with ( ) = a cot and g( ) = + .

70. **+ cot ( )^2 + cot ( ) + = 0.**

This is a special case of equation 2.9.3.25 with ( ) = a cot and g( ) = cot (λ) + .

71. **= cot ( )^2 + .**

This is a special case of equation 2.9.3.38 with ( ) = a cot and g( ) = .

72. **= cot ( )^2 + cot ( ) .**

This is a special case of equation 2.9.3.38 with ( ) = a cot and g( ) = cot (λ).

73. **= cot ( )^2 + ( )^k.**

This is a special case of equation 2.9.4.13 with ( ) = a cot and g( ) = .

74. **= + $^{2\ \ +1}$ cot ( ).**

This is a special case of equation 2.9.2.20 with ( ) = a cot (λ).

75. **= ( )^2 + ^{2} + cot^k( ) $^{+1}$.**

This is a special case of equation 2.9.3.9 with ( ) = $-a$ and g( ) = − cot (λ).

76. **= ( )^2 + + cot ( ) ^{2}.**

This is a special case of equation 2.9.3.7 with ( ) = $-a$ and g( ) = − cot (λ).

77. **^{2} + = cot ( ) + .**

This is a special case of equation 2.9.2.23 with ( ) = a cot (λ) + .

78. **(^{2} +) + + cot ( ) + = 0.**

This is a special case of equation 2.9.2.24 with ( ) = cot (λ) + .

2.8.6-5. Equations containing combinations of trigonometric functions.

79. $= - {}^{2} + \cos(\ )\sin(\ )\ {}^{- - -3}.$

The transformation $\xi = \cot(\lambda\ )$, $= \dfrac{}{\sin(\lambda\ )}$ leads to the Emden–Fowler equation: $'' = a\lambda^{-2}\xi^{\ - - -3}$ (see Section 2.3).

80. $= - {}^{2} + \sin(\ )[\sin(\ ) + \cos(\ )]\ {}^{- - -3}.$

The transformation $\xi = 1 + \cot(\lambda\ )$, $= \dfrac{}{\sin(\lambda\ )}$ leads to the Emden–Fowler equation: $'' = a(\ \lambda)^{-2}\xi^{\ - - -3}$ (see Section 2.3).

81. $= (\ \sin - \cos\ )^{k} - \ .$

The substitution $= {}' \sin - \cos$ leads to a first-order separable equation: $' = a \sin$.

82. $= (\ \cos + \sin\ )^{k} - \ .$

The substitution $= {}' \cos + \sin$ leads to a first-order separable equation: $' = a \cos$.

83. $= 2(\cos\ )^{-2} + (\cot\ )^{\ +3}\ .$

This is a special case of equation 2.9.1.43 with $(\xi) = a\xi$.

84. $= 2(\sin\ )^{-2} + (\tan\ )^{\ +3}\ .$

This is a special case of equation 2.9.1.42 with $(\xi) = a\xi$.

85. $= (\ + 1)(\tan\ ) + + (\cos\ )^{\ -2\ -1}.$

This is a special case of equation 2.9.2.44 with $(\xi) = a\xi^{\ -1}$.

86. $+ (2 + \sin\ ) + (\cos\ ) = 0.$

Integrating yields a Riccati equation: $' + a\ {}^{2} + (\sin\ ) = \ .$

87. ${}^{2} + {}^{2}\tan + (\ \tan - - 1) = {}^{+2}(\cos\ )^{2}\ {}^{-3}.$

This is a special case of equation 2.9.2.47 with $(\xi) = \ \xi^{\ -3}$.

88. $\sin + \tfrac{1}{2}\cos = + \ .$

This is a special case of equation 2.9.2.34 with $g(\ ) = \sin$ and $(\ ) = a + \ .$

89. $\sin + \tfrac{1}{2}\cos = \sin(\ ) + \ .$

This is a special case of equation 2.9.2.34 with $g(\ ) = \sin$ and $(\ ) = a\sin(\lambda\ ) + \ .$

90. $\sin + \tfrac{1}{2}\cos = \cos(\ ) + \ .$

This is a special case of equation 2.9.2.34 with $g(\ ) = \sin$ and $(\ ) = a\cos(\lambda\ ) + \ .$

91. $\sin + \tfrac{1}{2}\cos = \tan(\ ) + \ .$

This is a special case of equation 2.9.2.34 with $g(\ ) = \sin$ and $(\ ) = a\tan(\lambda\ ) + \ .$

92. $+ (\ )^{2} + \sin(\ ) + \cos(\ ) + = 0.$

This is a special case of equation 2.9.3.6 with $(\ ) = a\sin(\beta\ )$ and $g(\ ) = \cos(\lambda\ ) + \ .$

93. $- (\ )^{2} + \sin(\ ) + \cos(\ )\ {}^{2} = 0.$

This is a special case of equation 2.9.3.7 with $(\ ) = a\sin(\beta\ )$ and $g(\ ) = \cos(\lambda\ ).$

2.8.7. Equations Containing the Combinations of Exponential, Hyperbolic, Logarithmic, and Trigonometric Functions

1. $= {}^2 + {}^{3\lambda} (\ln +).$

This is a special case of equation 2.9.1.29 with $(\xi) = a \ln \xi.$

2. $= - + (\ln +).$

This is a special case of equation 2.9.2.36 with $(\xi) = \ln \xi.$

3. $= + {}^2 \ln ( ).$

This is a special case of equation 2.9.2.17 with $( ) = \ln (\lambda).$

4. $= + {}^2 \sin ( ).$

This is a special case of equation 2.9.2.17 with $( ) = \sin (\lambda).$

5. $= + {}^2 \tan ( ).$

This is a special case of equation 2.9.2.17 with $( ) = \tan (\lambda).$

6. $+ \tan + (\tan -) = (\cos)^2 {}^{-3}.$

This is a special case of equation 2.9.2.46 with $(\xi) = \xi^{-3}.$

7. $= ( )^2 - {}^4 + \sinh( ).$

This is a special case of equation 2.9.3.18 with $( ) = - \sinh(\lambda).$

8. $= ( )^2 + \cosh ( ) + .$

This is a special case of equation 2.9.3.17 with $( ) = - \cosh (\lambda)$ and $g( ) = - .$

9. $= ( )^2 + \ln ( ) + .$

This is a special case of equation 2.9.3.17 with $( ) = - \ln (\lambda)$ and $g( ) = - .$

10. $= ( )^2 + \ln ( ) + {}^{\nu}.$

This is a special case of equation 2.9.3.17 with $( ) = - \ln (\lambda)$ and $g( ) = - e .$

11. $= ( )^2 - {}^4 + \sin( ).$

This is a special case of equation 2.9.3.18 with $( ) = - \sin(\lambda).$

12. $= ( )^2 + \sin ( ) + .$

This is a special case of equation 2.9.3.17 with $( ) = - \sin (\lambda)$ and $g( ) = - .$

13. $= ( )^2 + \sin ( ) + {}^{\nu}.$

This is a special case of equation 2.9.3.17 with $( ) = - \sin (\lambda)$ and $g( ) = - e .$

14. $+ {}^{\lambda} ( )^2 + \ln + = 0.$

This is a special case of equation 2.9.3.25 with $( ) = ae^{\lambda}$ and $g( ) = \ln + .$

15. $+ {}^{\lambda} ( )^2 + \sin + = 0.$

This is a special case of equation 2.9.3.25 with $( ) = ae^{\lambda}$ and $g( ) = \sin + .$

16. $= {}^{\lambda} ( )^2 + \ln ( ) .$

This is a special case of equation 2.9.3.38 with $( ) = ae^{\lambda}$ and $g( ) = \ln ( ).$

17. $= \lambda ( )^2 +$ **sin** $( )$ **.**

This is a special case of equation 2.9.3.38 with $( ) = ae^{\lambda}$ and $g( ) =$ sin $( )$.

18. $= \lambda ( )^2 +$ **tan** $( )$ **.**

This is a special case of equation 2.9.3.38 with $( ) = ae^{\lambda}$ and $g( ) =$ tan $( )$.

19. $+$ **ln** $( )^2 + \lambda + = 0.$

This is a special case of equation 2.9.3.25 with $( ) = a \ln$ and $g( ) = e^{\lambda} + .$

20. $=$ **ln** $( )^2 + \lambda$ **.**

This is a special case of equation 2.9.3.38 with $( ) = a \ln$ and $g( ) = e^{\lambda}$.

21. $=$ **ln** $( )^2 +$ **sin** $( )$ **.**

This is a special case of equation 2.9.3.38 with $( ) = a \ln$ and $g( ) =$ sin (λ).

22. $+$ **sin** $( )^2 + \lambda + = 0.$

This is a special case of equation 2.9.3.25 with $( ) = a \sin$ and $g( ) = e^{\lambda} + .$

23. $=$ **sin** $( )^2 + \lambda$ **.**

This is a special case of equation 2.9.3.38 with $( ) = a \sin$ and $g( ) = e^{\lambda}$.

24. $=$ **sin** $( )^2 +$ **ln** $( )$ **.**

This is a special case of equation 2.9.3.38 with $( ) = a \sin$ and $g( ) =$ ln (λ).

25. $=$ **tan** $( )^2 + \lambda$ **.**

This is a special case of equation 2.9.3.38 with $( ) = a \tan$ and $g( ) = e^{\lambda}$.

26. $+$ **cosh**$( )( )^3 + = 0.$

This is a special case of equation 2.9.3.35 with $( ) =$ cosh(λ).

27. $+$ **sin**$( )( )^3 + = 0.$

This is a special case of equation 2.9.3.35 with $( ) =$ sin(λ).

28. $=$ **ln** $+$ **.**

Solution: $= -\ln e^{1} \quad {}_2 - \frac{a}{1} \quad {}^{-1}e^{-1} \quad + \frac{a}{1} \ln \quad .$

29. $= ( )^3 +$ **ln** $( )^2.$

Solution: $= -\ln e^{1} \quad {}_2 + \frac{a}{1} \quad {}^{-1}e^{-1} \quad - \frac{a}{1} \ln \quad .$

30. $= ( )^2 + \lambda +$ **sin** $( )\ ^2.$

This is a special case of equation 2.9.3.7 with $( ) = -ae^{\lambda}$ and $g( ) = -$ sin $( )$.

31. $(\ ^2 +) \quad + \ ^2 \quad =$ **cosh** $( ) + .$

This is a special case of equation 2.9.2.34 with $g( ) = ae^2 +$ and $( ) = \cosh (\lambda) + .$

32. $(\ ^2 +) \quad + \ ^2 \quad =$ **tanh** $( ) + .$

This is a special case of equation 2.9.2.34 with $g( ) = ae^2 +$ and $( ) = \tanh (\lambda) + .$

33. **(^{2} +) + ^{2} = ln ( ) + .**

This is a special case of equation 2.9.2.34 with $g( ) = ae^2$ + and ( ) = ln (λ) + .

34. **(^{2} +) + ^{2} = sin ( ) + .**

This is a special case of equation 2.9.2.34 with $g( ) = ae^2$ + and ( ) = sin (λ) + .

35. **(^{2} +) + ^{2} = tan ( ) + .**

This is a special case of equation 2.9.2.34 with $g( ) = ae^2$ + and ( ) = tan (λ) + .

36. **sin + $\frac{1}{2}$ cos = $^\lambda$ + .**

This is a special case of equation 2.9.2.34 with $g( )$ = sin and ( ) = ae^λ + .

37. **sin + $\frac{1}{2}$ cos = cosh ( ) + .**

This is a special case of equation 2.9.2.34 with $g( )$ = sin and ( ) = a cosh (λ) + .

38. **sin + $\frac{1}{2}$ cos = sinh ( ) + .**

This is a special case of equation 2.9.2.34 with $g( )$ = sin and ( ) = a sinh (λ) + .

39. **sin + $\frac{1}{2}$ cos = tanh ( ) + .**

This is a special case of equation 2.9.2.34 with $g( )$ = sin and ( ) = a tanh (λ) + .

40. **sin + $\frac{1}{2}$ cos = ln + .**

This is a special case of equation 2.9.2.34 with $g( )$ = sin and ( ) = a ln + .

2.9. Equations Containing Arbitrary Functions

Notation: , g, , , and are arbitrary composite functions of their arguments indicated in parentheses just after the function name (the arguments can depend on , , ′).

2.9.1. Equations of the Form $F(,) + G(,) = 0$

2.9.1-1. Arguments of arbitrary functions are algebraic and power functions of and .

1. **= f( ).**

This autonomous equation arises in mechanics, combustion theory, and the theory of mass transfer with chemical reactions. The substitution ( ) = ′ leads to a first-order separated equation: ′ = ( ).

Solution: $_1 + 2$ ( ) $^{-1\ 2}$ = $_2$.

Particular solutions: = A , where A are roots of the algebraic (transcendental) equation (A) = 0.

2. **+ f( ) = $^{-3}$.**

Yermakov's equation. Let = ( ) be a nontrivial solution of the second-order linear equation ″ + ( ) = 0. The transformation $\xi = \frac{}{2}$, $z = \frac{}{}$ leads to an autonomous equation of the form 2.9.1.1: $z'' = az^{-3}$.

Solution: $_1\ ^2 = a\ ^2 + \ ^2\ _2 + \ _1\ \frac{}{2}\ ^2$.

Reference: V. P. Yermakov (1880).

3. $+ f(\;)\;=\;(\;)^{-1} - {}^{2}(\;)^{-3}.$

Generalized Yermakov's equation.

Solution: $= \;+2\;\frac{g(\;)}{2}\;{}^{1\;2}$, where $=(\;)$ is the general solution of the linear equation $'' - (\;) = 0.$

4. $= f(\;+\;+\;).$

The substitution $= a\;+\;+$ leads to an equation of the form 2.9.1.1: $'' = a\;(\;).$

5. $= f(\;+\;{}^{2}+\;+\;).$

The substitution $= +a\;{}^{2}+\;+$ leads to an equation of the form 2.9.1.1: $'' = (\;)+2a.$

6. $= f(\;+\;+\;{}^{2}+\;) - (\;-1)\;{}^{-2}.$

The substitution $= +a\;+\;{}^{2}+$ leads to an equation of the form 2.9.1.1: $'' = (\;)+2\;.$

7. $= {}^{-1} f(\;{}^{-1}).$

Homogeneous equation. The transformation $= -\ln|\;|,\; z =$ leads to an autonomous equation: $z'' - z' = (z)$. Reducing its order with the substitution $(z) = \;'$, we arrive at the Abel equation $' - \;= (\;)$, which is discussed in Subsection 1.3.1.

8. $= {}^{-3} f(\;{}^{-1}).$

The transformation $\xi = 1\;,\; =$ leads to an equation of the form 2.9.1.1: $'' = (\;).$

9. $= {}^{-3\;2} f(\;{}^{-1\;2}).$

Having set $= \;{}^{-1\;2}$, we obtain $—(\;')^2 = \frac{1}{2}\;' + 2\;(\;)\;'$. Integrating the latter equation, we arrive at a separable equation.

Solution: $\;_1 + \frac{1}{4}\;{}^{2} + 2\;(\;)\;{}^{-1\;2} = \;_2\;\ln\;.$

10. $= {}^{k-2} f(\;{}^{-k}\;).$

Generalized homogeneous equation.

1 . The transformation $= \ln\;,\; z = \;{}^{-}$ leads to an autonomous equation of the form 2.9.6.2: $z'' + (2k-1)z' + k(k-1)z = (z).$

2 . The transformation $z = \;{}^{-}\;,\; = \;'$ leads to a first-order equation: $z(\;-k)\;' = z^{-1}\;(z) + \;-\;{}^{2}.$

11. $= {}^{-2} f(\;).$

Generalized homogeneous equation. The transformation $z = \;,\; = \;'$ leads to a first-order equation: $z(\;+\;)\;' = (z) + \;-\;{}^{2}.$

12. $= (\;+1)\;{}^{-2} + {}^{3}\; f(\;).$

This is a special case of equation 2.9.1.46 with $= \;{}^{-}\;.$

13. $= {}^{-3\;2} f(\;{}^{-1\;2} + \;{}^{1\;2}).$

The substitution $= a\;+$ leads to an equation of the form 2.9.1.9: $'' = a\;{}^{-3\;2}\;(\;{}^{-1\;2}).$

14. $(\;+\;)^2\; = f(\;).$

The transformation $\xi = \ln\frac{+a}{}\;,\; z = —$ leads to an autonomous equation of the form 2.9.6.2: $z'' - z' = a^{-2}\;(z)$. Reducing its order with the substitution $(z) = z'$, we arrive at an Abel equation of the second kind: $' - \;= a^{-2}\;(z)$. See Subsection 1.3.1 for information about an Abel equation of the second kind.

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15. $= \frac{1}{\ \ ^3} f \left(\frac{}{} + \frac{}{\ \ ^2} + \frac{}{} + \right).$

The transformation $\xi = 1\ ,\ =$ leads to an equation of the form 2.9.1.5: $'' = (\ + a\xi^2 + \ \xi + \ ).$

16. $= \frac{1}{\ + \ +} f \left(\frac{\ + \ +}{\ + \ + \gamma} \right).$

This is a special case of equation 2.9.1.18.

1 . For $a\beta - \ = 0,$ we have an equation of the form 2.9.1.4.

2 . For $a\beta - \ \neq 0,$ the transformation

$$z = \ - \ _0, \qquad = \ - \ _0,$$

where $_0$ and $_0$ are the constants determined by the linear algebraic system of equations

$$a\ _0 + \ _0 + \ = 0, \qquad _0 + \beta\ _0 + \ = 0,$$

leads to a homogeneous equation of the form 2.9.1.7:

$$'' = \frac{1}{z} \left(\frac{}{z} \right), \quad \text{where} \quad (\xi) = \frac{1}{a + \ \xi} \left(\frac{a + \ \xi}{\ + \beta\xi} \right).$$

17. $= \frac{1}{(\ + \ + \ )^3} f \left(\frac{\ + \ +}{\ + \ + \gamma} \right).$

This is a special case of equation 2.9.1.20.

1 . For $a\beta - \ = 0,$ we have an equation of the form 2.9.1.4.

2 . For $a\beta - \ \neq 0,$ the transformation

$$z = \ - \ _0, \qquad = \ - \ _0,$$

where $_0$ and $_0$ are the constants determined by the linear algebraic system of equations

$$a\ _0 + \ _0 + \ = 0, \qquad _0 + \beta\ _0 + \ = 0,$$

leads to a solvable equation of the form 2.9.1.8:

$$'' = \frac{1}{z^3} \left(\frac{}{z} \right), \quad \text{where} \quad (\xi) = \frac{1}{(a + \ \xi)^3} \left(\frac{a + \ \xi}{\ + \beta\xi} \right).$$

18. $= \frac{1}{\ _1 + \ _1 + \ _1} f \left(\frac{\ _2 + \ _2 + \ _2}{\ _3 + \ _3 + \ _3} \right).$

Let the following condition be satisfied: $\begin{vmatrix} a_1 & _1 & _1 \\ a_2 & _2 & _2 \\ a_3 & _3 & _3 \end{vmatrix} = 0.$

For $a_2\ _3 - a_3\ _2 \neq 0,$ the transformation

$$z = \ - \ _0, \qquad = \ - \ _0,$$

where $_0$ and $_0$ are the constants determined by the linear algebraic system of equations

$$a_2\ _0 + \ _2\ _0 + \ _2 = 0, \quad a_3\ _0 + \ _3\ _0 + \ _3 = 0,$$

leads to a homogeneous equation of the form 2.9.1.7:

$$'' = \frac{1}{z} \left(\frac{}{z} \right), \quad \text{where} \quad (\xi) = \frac{1}{a_1 + \ _1\xi} \left(\frac{a_2 + \ _2\xi}{a_3 + \ _3\xi} \right).$$

19. $= \dfrac{1}{{}^2(\;+\; +\;)} f \left(\dfrac{\;+\;+\;}{\;+\;+\gamma} \right).$

For $a\beta - \;\neq 0,$ the transformation

$$z = \;-\;_0, \qquad = \;-\;_0,$$

where $_0$ and $_0$ are the constants determined by the linear algebraic system of equations

$$a\;_0 + \;_0 + \; = 0, \qquad _0 + \beta\;_0 + \; = 0,$$

leads to an equation of the form 2.9.1.14:

$$z(z + \;_0)^2 \;'' = \; \frac{}{z}, \quad \text{where} \quad (\xi) = \frac{1}{a + \;\xi} \; \frac{a + \;\xi}{\; + \beta\xi}.$$

20. $= \dfrac{1}{(\;_1 + \;_1 + \;_1)^3} f \left(\dfrac{\;_2 + \;_2 + \;_2}{\;_3 + \;_3 + \;_3} \right).$

Let the following condition be satisfied: $\begin{vmatrix} a_1 & _1 & _1 \\ a_2 & _2 & _2 \\ a_3 & _3 & _3 \end{vmatrix} = 0.$

For $a_2\;_3 - a_3\;_2 \neq 0,$ the transformation

$$z = \;-\;_0, \qquad = \;-\;_0,$$

where $_0$ and $_0$ are the constants determined by the linear algebraic system of equations

$$a_2\;_0 + \;_2\;_0 + \;_2 = 0, \qquad a_3\;_0 + \;_3\;_0 + \;_3 = 0,$$

leads to a solvable equation of the form 2.9.1.8:

$$'' = \frac{1}{z^3} \; \frac{}{z}, \quad \text{where} \quad (\xi) = \frac{1}{(a_1 + \;_1\xi)^3} \; \frac{a_2 + \;_2\xi}{a_3 + \;_3\xi}.$$

21. $= \;^{-3} f \left(\dfrac{}{\sqrt{\;^2 + \;+\;}} \right).$

Setting $(\;) = \;(a\;^2 + \;+\;)^{-1/2}$ and integrating the equation, we obtain a first-order separable equation:

$$(a\;^2 + \;+\;)^2 (\;')^2 = (\tfrac{1}{4}\;^2 - a\;)\;^2 + 2\;\;^{-3}\;(\;)\; + \;_1.$$

22. $(\;^2 + \;+\;)^{3/2}\; = f \left(\dfrac{\;+\;+\gamma}{\sqrt{\;^2 + \;+\;}} \right).$

Setting $= \;+\beta\;+\;$ and denoting $(z) = \dfrac{1}{z^3}\;(z),$ we obtain an equation of the form 2.9.1.21: $'' = \;^{-3}\;\left(\dfrac{}{\sqrt{a\;^2 + \;+\;}}\right).$

23. $(\;+\;)\; = \;(\;-1)\;^{-2} + \;^{-2} f \left(\dfrac{}{\;+\;} \right).$

The transformation $\xi = \dfrac{}{(a\;+\;)^2}, \quad = \dfrac{}{a\;+\;}$ leads to an autonomous equation of the form 2.9.1.1: $'' = \;^{-2}\;(\;).$

24. $= (\quad + \quad)^{-2 \quad -1 \quad 2} f(\quad ^{1 \; 2} + \quad) + 2 \quad ^2$.

The solution is determined by the first-order equation

$$(a \quad + \quad)(\quad ' + 2 \quad ^{-}) = \quad (\quad + \quad ^{-}, \quad), \tag{1}$$

where the function $\quad = (\;,\;)$ is the general solution of the Abel equation of the second kind:

$$' - 2(a \quad + \quad) \quad = 2 \quad (\quad).$$

Abel equations of this type are discussed in Subsection 1.3.3. By the change of variable $^{1 \; 2} = \quad - \quad$, equation (1) is reduced to the form $2(a \quad + \quad)(\quad - \quad) \quad ' = \quad (\;,\;)$.

25. $= \dfrac{(\quad + \quad) \quad ^{-1}}{(\quad + \quad) \quad ^{+2}} f \left(\dfrac{(\quad + \quad)}{(\quad + \quad) \quad ^{+1}} \right)$.

The transformation $\xi = \ln \dfrac{a \quad +}{+}$, $\quad = \dfrac{(a \quad + \quad)}{(\quad + \quad) \quad ^{+1}}$ leads to an autonomous equation of the form 2.9.6.2:

$$'' - (2 \quad + 1) \quad ' + \quad (\quad + 1) \quad = \Delta^{-2} \quad (\quad), \qquad \text{where} \quad \Delta = a \quad - \quad .$$

2.9.1-2. Arguments of the arbitrary functions are other functions.

26. $= \quad ^{-} \quad f(\qquad)$.

1 . The substitution $z = e \qquad$ leads to an autonomous equation: $z'' - 2az' + a^2 z = \quad (z)$.

2 . The transformation $z = e \quad$, $\quad = \quad '$ leads to a first-order equation: $z(\quad + a) \quad ' = z^{-1} \quad (z) - \quad ^2$.

27. $= \quad f(\qquad)$.

Equation invariant under "translation–dilatation" transformation. The transformation $z = e \qquad$, $\quad = \quad '$ leads to a first-order equation: $z(\qquad + a) \quad ' = \quad (z) - \quad ^2$.

28. $= \quad ^{-2} f(\qquad)$.

Equation invariant under "dilatation–translation" transformation. The transformation $z = \quad e \quad$, $\quad = \quad '$ leads to a first-order equation: $z(a \quad + \quad) \quad ' = \quad (z) + \quad$.

29. $= \quad ^2 + \quad ^{3\lambda} f(\quad ^{\lambda} \quad)$.

This is a special case of equation 2.9.1.46 with $\quad = e^{-\lambda} \quad$.

30. $= f(\quad + \quad ^{\lambda} + \quad) - \quad ^{2 \; \lambda}$.

The substitution $\quad = \quad + ae^{\lambda} + \quad$ leads to an equation of the form 2.9.1.1: $'' = \quad (\quad)$.

31. $^2 \quad = \quad ^2 f(\qquad) + \quad$.

The substitution $\quad = \quad - \quad \ln$ leads to an equation of the form 2.9.1.1: $'' = \quad (e^w)$.

32. $= f(\quad + \quad \sinh \quad + \quad) - \quad \sinh \quad$.

The substitution $\quad = \quad + a \sinh \quad + \quad$ leads to an equation of the form 2.9.1.1: $'' = \quad (\quad)$.

33. $= f(\quad + \quad \cosh \quad + \quad) - \quad \cosh \quad$.

The substitution $\quad = \quad + a \cosh \quad + \quad$ leads to an equation of the form 2.9.1.1: $'' = \quad (\quad)$.

34. $= {}^{2} + (\sinh\ )^{-3} f\,\dfrac{}{\sinh}\ .$

This is a special case of equation 2.9.1.46 with $= \sinh\lambda$.

35. $= {}^{2} + (\cosh\ )^{-3} f\,\dfrac{}{\cosh}\ .$

This is a special case of equation 2.9.1.46 with $= \cosh\lambda$.

36. ${}^{2} \quad = {}^{2} f(\ +\ \ln\ +\ ) +\ .$

The substitution $= +a\ln\ +$ leads to an equation of the form 2.9.1.1: $'' = (\ ).$

37. $= -\dfrac{}{{}^{2}\ln} + \dfrac{1}{(\ln\ )^{3}} f\,\dfrac{}{\ln}\ .$

This is a special case of equation 2.9.1.46 with $= \ln$.

38. $= f(\ +\ \sin\ +\ ) +\ \sin\ .$

The substitution $= +a\sin\ +$ leads to an equation of the form 2.9.1.1: $'' = (\ ).$

39. $= f(\ +\ \cos\ +\ ) +\ \cos\ .$

The substitution $= +a\cos\ +$ leads to an equation of the form 2.9.1.1: $'' = (\ ).$

40. $= -\ {}^{2} + (\sin\ )^{-3} f\,\dfrac{}{\sin}\ .$

This is a special case of equation 2.9.1.46 with $= \sin\lambda$.

41. $= -\ {}^{2} + (\cos\ )^{-3} f\,\dfrac{}{\cos}\ .$

This is a special case of equation 2.9.1.46 with $= \cos\lambda$.

42. $= 2(\sin\ )^{-2} + (\tan\ )^{3} f(\ \tan\ ).$

This is a special case of equation 2.9.1.46 with $= \cot$.

43. $= 2(\cos\ )^{-2} + (\cot\ )^{3} f(\ \cot\ ).$

This is a special case of equation 2.9.1.46 with $= \tan$.

44. $\sin^{2} \quad = (\ +1-\ \sin^{2}\ ) + \sin^{3\ +2} f(\ \sin\ ).$

The substitution $= \xi + \frac{\pi}{2}$ leads to an equation of the form 2.9.1.45:

$$\cos^{2}\xi\ '' = (\ +1-\ \cos^{2}\xi) + \cos^{3\ +2}\xi\ (\ \cos\ \xi).$$

45. $\cos^{2} \quad = (\ +1-\ \cos^{2}\ ) + \cos^{3\ +2} f(\ \cos\ ).$

The transformation $\xi = \ \cos^{2}\ ,\ = \ \cos$ leads to an autonomous equation of the form 2.9.1.1: $'' = (\ ).$

46. $= \dfrac{}{} + {}^{-3} f\,\dfrac{}{}\ , \qquad = (\ ).$

The transformation $\xi = \dfrac{}{{}^{2}},\ = \dfrac{}{}$ leads to an equation of the form 2.9.1.1: $'' = (\ ).$

Solution: $\quad {}_{1} + 2 \quad (\ ) \quad {}^{-1\ 2} \quad = {}_{2} \quad \dfrac{}{{}^{2}(\ )}.$

47. $= \ {}^{-3}f \ — + \quad + —— \quad - \quad -2 \quad , \quad = (\ ), \quad = (\ ).$

The transformation $= \frac{}{2}, \quad = — +$ leads to an autonomous equation of the form 2.9.1.1: $'' = (\ ).$

Solution: $\frac{}{\sqrt{2 \ (\ ) + \ _1}} = \ \frac{}{2} + \ _2,$ where $(\ ) = \ (\ ) \ .$

2.9.2. Equations of the Form $F(\ , \ ) \quad + G(\ , \ ) \quad + \quad (\ , \ ) = 0$

2.9.2-1. Argument of the arbitrary functions is .

1. $+ 3 \quad + \ ^3 + f(\ ) \ = 0.$

The substitution $= \ ^{-1}$ leads to a second-order linear equation: $'' + (\ ) \ = 0.$

2. $+ 6 \quad + 4 \ ^3 + 4f(\ ) \ + f \ (\ ) = 0.$

The substitution $= \ '$ leads to a third-order equation of the form 3.5.3.11: $''' + 3 \ ' \ '' + 4 \ (\ ) \ ' + \ '(\ ) \ ^2 = 0.$

3. $+ 3f \quad + f^2 \ ^3 + f \ ^2 + (\ ) \ = 0, \quad f = f(\ ).$

The substitution $= \ ^{-1}$ leads to a second-order linear equation: $'' + g(\ ) \ = 0.$

4. $+ [\quad + f(\ )] \quad + f \ (\ ) \ = 0.$

Integrating yields a Riccati equation: $' + (\ ) \ + \frac{1}{2}a \ ^2 = \ .$

5. $+ [2 \quad + f(\ )] \quad + \ f(\ ) \ ^2 = (\ ).$

On setting $= \ ' + a \ ^2$, we obtain $' + (\ ) \ = g(\ )$. Thus, the original equation is reduced to a first-order linear equation and a Riccati equation.

6. $+ [3 \ + f(\ )] \quad + \ ^3 + f(\ ) \ ^2 + (\ ) \ + (\ ) = 0.$

The substitution $= \ '$ leads to a third-order linear equation: $''' + (\ ) \ '' + g(\ ) \ ' + (\ ) \ = 0.$

7. $+ [\ + 3f(\ )] \quad - \ ^3 + f(\ ) \ ^2 + [f \ (\ ) + 2f^2(\ )] \ = 0.$

The transformation

$$= (\ ) \ (z), \quad z = \int \ (\ ) \ , \quad \text{where} \quad (\ ) = \exp - \int \ (\ ) \ ,$$

leads to an autonomous equation of the form 2.2.3.2: $'' + \ ' - \ ^3 = 0.$

8. $- (\ + 1) \ (\ ) \ ^{-1} \ = f(\ ) \ + \ (\ ) \ - \ ^2(\ ) \ ^{2 \ -1}.$

Solution: $= \ + (1 - \ ) \ g(\ ) \ ^{-1} \ ^{\frac{1}{1-}}$, where $= (\ )$ is the general solution of the second-order linear equation $'' = (\ ) \ .$

9. $\quad - \quad + f(\ ) \ = \ ^{2\ +1\ -3}.$

The substitution $\ = \ ^{-}\ ^{2}$ leads to Yermakov's equation 2.9.1.2:

$$'' + \ ^{-2}[\ \ (\ ) - \tfrac{1}{4}\ (\ +2)]\ \ = a\ ^{-3}.$$

10. $\quad + (2f\ + \ )\ \ + f\ \ ^{2} + \ \ \ = 0, \qquad f = f(\ ), \qquad = (\ ).$

Integrating yields a Riccati equation: $\ ' + \ \ ^{2} + g\ \ = \ .$

11. $\quad + \ 3f\ + \frac{}{}\ \ + f^{2}\ ^{3} + f\ \ ^{2} + (2f\ + \ )\ + \ \ + \frac{\ ^{2}}{} = 0.$

Here, $\ = (\ ),\ g = g(\ ),\ \ = (\ ).$

Solution: $\ = \frac{\ '}{(\ )}$, where $\ = (\ )$ is the general solution of the linear equation:

$$'' - \ \frac{\ '}{} + \frac{\ '}{}\ \ ' + \ g\ \ = 0,$$

and $\ = (\ )$ is the general solution of another linear equation: $\ '' + (\ )\ \ = 0.$

12. $\quad + [3f(\ )\ + 2\ (\ ) + \ (\ )\ ^{-1}]\ \ + f^{2}(\ )\ ^{3} + [f\ (\ ) + 2f(\ )\ (\ )]\ ^{2}$
$\quad + [\ \ (\ ) + \ ^{2}(\ ) + 2f(\ )\ (\ ) - p(\ )]\ \ + \ \ (\ ) + 2\ (\ )\ (\ ) + \ ^{2}(\ )\ ^{-1} = 0.$

The solution satisfies the Riccati equation $\ ' + \ (\ )\ ^{2} + [g(\ ) - z(\ ,\ \ )]\ \ + \ (\ ) = 0$, where $z = z(\ ,\ \ )$ is the general solution of another Riccati equation: $z' + z^{2} = \ (\ ).$

13. $\quad + \ 2f\ \ + 2\ -\frac{f}{} - \frac{\ 2}{\ ^{2}}\ \ + \ \ + f\ \ + \frac{f^{2} + }{} + \frac{2f}{\ ^{2}} + \frac{\ ^{2}}{\ ^{3}} = 0.$

Here, $\ = (\ ),\ g = g(\ )$, and $\ = (\ ).$

The solution is determined by the Abel equation of the second kind $\ ' = (\ln\ )'\ \ ^{2} - (\ )\ - g(\ )$, where $\ = (\ )$ is the general solution of the linear equation: $\ '' + (\ )\ \ = 0.$ Abel equations of the second kind are discussed in Section 1.3.

14. $\quad - \quad + f(\ ) \ = \ ^{2\lambda}\ \ ^{-3}.$

The substitution $\ = e^{-\lambda}\ \ ^{2}$ leads to Yermakov's equation 2.9.1.2:

$$'' + [\ (\ ) - \tfrac{1}{4}\lambda^{2}]\ \ = a\ ^{-3}.$$

15. $\quad + \quad + f\ \ = \ ^{-2}\ \ ^{-3}, \qquad f = f(\ ), \qquad = (\ ).$

The substitution $\ = e\ \ ^{2}$ leads to Yermakov's equation 2.9.1.2:

$$'' + [\ \ - \tfrac{1}{4}(g')^{2} - \tfrac{1}{2}g''\ ]\ \ = a\ ^{-3}.$$

16. $\quad + \ \ \tan(\ \ )\ \ + f(\ ) \ = \ [\cos(\ \ )]^{2}\ \ ^{-3}.$

This is a special case of equation 2.9.2.15 with $g = -\ \ \ln\cos(\lambda\ \ ).$

2.9.2-2. Argument of the arbitrary functions is .

17. $\quad = \quad + \ ^{2}\ \ f(\ ).$

Multiplying both sides by e^{-2} , we obtain an equation of the form 2.9.2.34.

Solution: $\ _{1} + 2\ \ (\ )\ \ ^{-1\ 2} \ = \ _{2}\ \frac{1}{a}e\ \ .$

18. $= f(\ )$.

Solution: $\dfrac{}{(\ ) + \ _1} = \ _2 + \ $, where $(\ ) = \ (\ )$.

19. $= [\ f(\ ) + \ ]$.

The substitution $(\ ) = e^{-}\ '$ leads to a first-order separable equation: $' = (\ )$.

Solution: $\dfrac{}{(\ ) + \ _1} = \ _2 + \dfrac{1}{}e$, where $(\ ) = \ (\ )$.

20. $= \ + \ ^{2\ +1} f(\ )$.

Multiplying both sides by $^{-2\ -1}$, we obtain an equation of the form 2.9.2.34.

1 . Solution for $\neq -1$:

$$_1 + 2\ (\ )\ ^{-1\ 2} = \frac{^{+1}}{+1} + \ _2.$$

2 . Solution for $= -1$:

$$_1 + 2\ (\ )\ ^{-1\ 2} = \ln|\ | + \ _2.$$

21. $= f(\ )$.

The substitution $(\ ) = \ '$ leads to a first-order linear equation: $' = -\ + 1 + \ (\ )$.

22. $= [\ ^k f(\ ) + \ - 1]$.

Solution: $\dfrac{}{(\ ) + \ _1} = \ _2 + \dfrac{1}{k}$, where $(\ ) = \ (\ )$.

23. $^2\ + \ = f(\ )$.

The substitution $= e$ leads to an equation of the form 2.9.1.1: $'' = (\ )$.

24. $(\ ^2 + \ )\ + \ + f(\ ) = 0$.

The substitution $\xi = \dfrac{}{\overline{a\ ^2 + }}$ leads to an autonomous equation of the form 2.9.1.1: $'' + (\ ) = 0$.

25. $(\ ^{2\lambda} + \ )\ + \ ^{2\lambda}\ + f(\ ) = 0$.

This is a special case of equation 2.9.2.34 with $g(\ ) = ae^{2\lambda} + \ $.

26. $\sin\ + \frac{1}{2}\cos\ = f(\ )$.

This is a special case of equation 2.9.2.34 with $g = \sin\ $.

27. $\cos\ - \frac{1}{2}\sin\ = f(\ )$.

This is a special case of equation 2.9.2.34 with $g = \cos\ $.

2.9.2-3. Other arguments of the arbitrary functions.

28. $= f(\ )\ + (\ )$.

Integrating yields a first-order equation: $' = \ (\ )\ + \ g(\ )\ + \ $.

29. $\quad + [f(\ ) + \ (\ )] \quad + f\ (\ ) \quad = 0.$

Integrating yields a first-order equation: $\ ' + \ (\ ) + \ g(\ ) \ = \ .$

30. $\quad + (\ + 1) \quad = \ ^{-1} f(\quad).$

The transformation $\xi = \ ,\ = \ $ leads to an autonomous equation of the form 2.9.1.1: $^{2}\ '' = \ (\ ).$

31. $^{2} \quad + (\ + \ + 1) \quad + \quad = \ ^{-2}\ f(\quad).$

1 . For $\ \neq \ $, the transformation $\xi = \ ^{-}\ ,\ = \ $ leads to an autonomous equation of the form 2.9.1.1: $(\ - \ )^{2}\ '' = \ (\ ).$

2 . For $\ = \ $, the transformation $\xi = \ln\ ,\ = \ $ leads to an autonomous equation of the form 2.9.1.1: $\ '' = \ (\ ).$

32. $^{2} \quad = f(\quad)(\quad - \ ).$

This is a special case of equation 2.9.3.42 with $g(\ ) = \ (\ ) = 0.$

33. $^{3} \quad = f(\quad)(\quad - \ ).$

This is a special case of equation 2.9.4.61 with $g(z) = z.$

34. $\quad + \tfrac{1}{2} \quad = f(\ ), \quad = \ (\ ).$

Integrating yields a first-order separable equation: $g(\ )(\ ')^{2} = 2 \quad (\ ) \quad + \ _{1}.$

Solution for $g(\ ) \geq 0$:

$$_{1} + 2 \quad (\ ) \quad ^{-1\ 2} \quad = \ _{2} \quad \frac{}{g(\ )}.$$

35. $\quad - \frac{}{} \quad + \frac{}{}\frac{}{} - \frac{}{} \quad = \frac{2}{3} f \frac{}{}, \quad = \ (\ ), \quad = \ (\ ).$

The transformation $\xi = \frac{}{2}\ ,\ = \frac{}{}$ leads to an autonomous equation of the form 2.9.1.1: $\ '' = \ (\ ).$

36. $\quad = - \quad + \quad f(\quad).$

The transformation $\xi = e\ ,\ = \ e\ $ leads to an equation of the form 2.9.1.1: $\ '' = a^{-2}\ (\ ).$

37. $\quad + (\ + \ ) \quad + \quad = \ ^{(\ -2\nu)}\ f(\quad).$

1 . For $\ \neq \ $, the transformation $\xi = e^{(\ -\ )}\ ,\ = \ e\ $ leads to an autonomous equation of the form 2.9.1.1: $(\ - \ )^{2}\ '' = \ (\ ).$

2 . For $\ = \ $, the substitution $\ = \ e\ $ leads to an autonomous equation of the form 2.9.1.1: $\ '' = \ (\ ).$

38. $\quad - \quad - (\ + \ ) = \ ^{2\ +1\ \ 3}\ f(\quad).$

This is a special case of equation 2.9.2.35 with $\ = \ $ and $\ = e^{-}\ .$

39. $\quad = f(\quad)\ .$

The transformation $z = \ e\ ,\ = \ '$ leads to a first-order separable equation: $z(a\ + \ )\ ' = [\ (z) + 1]\ .$

40. $\;^2 \quad + \quad = f(\quad)$.

The transformation $z = \quad e\;, \quad = \quad '$ leads to a first-order separable equation: $z(a \quad + \quad)\;' = \quad (z)$.

41. $\;^2 \quad - \quad ^2 \quad - (\quad + \quad + 1) \;=\; ^{3\;+2\;\;2} \quad f(\quad)$.

This is a special case of equation 2.9.2.35 with $\;= e\;$ and $\;= \;^-\;$.

42. $\quad - \text{—} \quad - \quad \text{—} + \quad = \;^3 \quad ^2 f(\quad , \quad = (\;)$.

The transformation $\xi = \quad e^2 \quad , \quad = \quad e$ leads to an equation of the form 2.9.1.1: $\;'' = (\;)$.

43. $\;^2 \quad + \quad = f(\;+\;\ln\;+\;\ln^2\;)$.

The substitution $\;= e\;$ leads to an equation of the form 2.9.1.5: $\;'' = (\;+ a \;+\;^2)$.

44. $\quad - (\;+1)\tan \quad - \quad = \cos\;^{-2} \quad f(\;\cos\;)$.

This is a special case of equation 2.9.2.35 with $\;= \cos^{-\;-1}$ and $\;= \cos^-\;$.

45. $\quad + (\;-\;)\tan \quad - \;[(\;+1)\tan^2 \;+1] \;= \cos^2\;^+ \quad f(\;\cos\;)$.

This is a special case of equation 2.9.2.35 with $\;= \cos\;^-$ and $\;= \cos^-\;$.

46. $\quad + \;\tan \quad + (\;\tan\;-\;) \;= \cos^2 \quad ^3 \;f(\quad)$.

This is a special case of equation 2.9.2.35 with $\;= \cos\;$ and $\;= e^{-b}\;$.

47. $\;^2 \quad + \;^2\tan \quad + (\quad\tan\;-\;-1) \;= \;^{3\;+2}\cos^2 \quad f(\quad)$.

This is a special case of equation 2.9.2.35 with $\;= \cos\;$ and $\;= \;^-\;$.

48. $\;^2 \quad - \;^2\cot \quad - (\quad\cot\;+\;+1) \;= \;^{3\;+2}\sin^2 \quad f(\quad)$.

This is a special case of equation 2.9.2.35 with $\;= \sin\;$ and $\;= \;^-\;$.

2.9.3. Equations of the Form

$$F(\;,\;) \quad + \sum_{=0} G\;(\;,\;)(\;) \quad = 0 \quad (M = 2, 3, 4)$$

2.9.3-1. Argument of the arbitrary functions is .

1. $\quad = f(\;)(\;+\;)^2 + \;^2\;$.

The substitution $\;= \;' + a\;$ leads to a Bernoulli equation: $\;' = a \;+ (\;)\;^2$.

2. $\quad = f(\;) + (\;)(\;-\;) + (\;)(\;-\;)^2$.

The substitution $(\;) = \;' -\;$ leads to a Riccati equation: $\;' = \;(\;) + g(\;) \;+ \;(\;)\;^2$.

3. $\quad + (\;+1) \;= f(\;)(\;+\;)^2$.

The substitution $\;= \;' + a\;$ leads to a first-order separable equation: $\;' = (\;)\;^2$.

4. $\quad - (\;)^2 = f(\;)$.

1 . The substitution $\;= ae^{\lambda}\;$ leads to a similar equation: $\;'' - (\;')^2 = a^{-2}e^{-2\lambda} \;(\;)$.

2 . The substitutions $\;= \;e\;^2$ leads to the equation $\;'' = 2\;(\;)e^-\;$. For $(\;) = ke\;^{2+}\;$, see 2.7.1.41.

5. $\mathbf{2 \quad - (\quad)^2 + f(\ ) \ ^2 + \quad = 0.}$

1 . Differentiating with respect to , we obtain a third-order linear equation:

$$2 \ ''' \ + 2 \ (\ ) \ ' + \ '(\ ) \ = 0.$$

2 . The substitution $= z^2$ leads to Yermakov's equation 2.9.1.2: $z'' + \frac{1}{4} \ (\ )z = -\frac{1}{4}az^{-3}$.

3 . If and v are two solutions of the second-order linear equation $4 \ '' + \ (\ ) \ = 0$, which satisfy the condition $(\ v' - \ ' v)^2 = a$, then $= \ v$ is a solution of the original equation.

6. $\mathbf{+ (\quad)^2 + f(\ ) \quad + \ (\ ) = 0.}$

The substitution $= \ ^2$ leads to a linear equation: $'' + \ (\ ) \ ' + 2g(\ ) = 0$.

7. $\mathbf{- (\quad)^2 + f(\ ) \quad + \ (\ ) \ ^2 = 0.}$

The substitution $= \ '$ leads to a first-order linear equation: $' + \ (\ ) \ + g(\ ) = 0$.

8. $\mathbf{- \ (\quad)^2 + f(\ ) \ ^2 + \quad ^4 \ ^{-2} = 0.}$

1 . For $= 1$, this is an equation of the form 2.9.3.7.

2 . For $\neq 1$, the substitution $= \ ^{1-}$ leads to Yermakov's equation 2.9.1.2: $'' + (1- \ ) \ (\ ) \ + a(1- \ ) \ ^{-3} = 0$.

9. $\mathbf{- \ (\quad)^2 + f(\ ) \ ^2 + \ (\ ) \quad ^{+1} = 0.}$

The substitution $= \ ^{1-}$ leads to a nonhomogeneous linear equation: $'' + (1- \ ) \ (\ ) \ + (1- \ )g(\ ) = 0$.

10. $\mathbf{+ \ (\quad)^2 + f(\ ) \quad + \ (\ ) \ ^2 = 0.}$

The substitution $= \ ^{+1}$ leads to a linear equation: $'' + \ (\ ) \ ' + (a+1)g(\ ) \ = 0$.

11. $\mathbf{- 2(\quad)^2 - (f \quad + 2 \ ) \quad + f \quad ^2 + \quad = 0, \qquad f = f(\ ), \quad = \ (\ ).}$

Integrating yields a Riccati equation: $' + \quad ^2 + \quad + g = 0$.

12. $\mathbf{- (\quad)^2 + (f \ ^2 + \ ) \quad + f \quad ^3 - \quad = 0, \qquad f = f(\ ), \quad = \ (\ ).}$

Integrating yields a Riccati equation: $' + \quad ^2 + \quad - g = 0$.

13. $\mathbf{+ (\quad)^2 + [2f \ ^2 + 2(f + \ ) \ + \ + 2 \ ] \quad + f^2 \ ^4 + 2f \quad ^3}$

$$\mathbf{+ (2f \ + \ ^2 + 2f \ ) \ ^2 + (\quad + 2 \quad) \ + \quad + \ ^2 - p = 0.}$$

Here, $= \ (\ ), g = g(\ ), \ = \ (\ ), \ = \ (\ )$.

The solution satisfies the Riccati equation $' + \ (\ ) \ ^2 + g(\ ) \ + \ (\ ) - z(\ , \ ) = 0$, where $z = z(\ , \ )$ is the general solution of another Riccati equation: $z' + z^2 = \ (\ )$.

14. $\mathbf{= f(\ )(\quad)^2.}$

The substitution $(\ ) = \ '$ leads to a Bernoulli equation 1.1.5: $' = \ + [\ (\ ) - 1] \ ^2$.

15. $\mathbf{+ f(\ )(\quad)^2 + \ (\ ) \quad + \ (\ ) \ ^2 = 0.}$

The substitution $= \ '$ leads to a Riccati equation: $' + (1 + \ ) \ ^2 + g \ + \ = 0$.

16. $\mathbf{(\ + \quad) \quad = f(\ )(\quad - \ )^2.}$

The substitution $= -a \ + \ z$ leads to the equation $zz'' + 2zz' - \ ^3 \ (\ )(z')^2 = 0$. On setting $= z' \ z$, we obtain a Bernoulli equation: $' + 2 \ + [\ - \ ^3 \ (\ )] \ ^2 = 0$.

17. $\mathbf{- \ (\quad)^2 + f(\ ) \quad + \ (\ ) = 0.}$

The substitution $= e^{-}$ leads to a nonhomogeneous linear equation: $'' - ag(\ ) \ = a \ (\ )$.

18. $-\ (\quad)^2 + \quad{}^4 \quad + f(\ ) = 0.$

The substitution $\quad = e^{-}\quad$ leads to Yermakov's equation 2.9.1.2: $\quad'' - a\ (\ )\quad = a\quad{}^{-3}$.

19. $= f(\ )(\quad \sinh \quad - \quad \cosh \quad)^2 + \quad .$

The substitution $\quad = \quad' \sinh \quad - \quad \cosh \quad$ leads to a first-order separable equation: $\quad' =$ $\sinh \quad (\ )\quad{}^2$.

20. $= f(\ )(\quad \cosh \quad - \quad \sinh \quad)^2 + \quad .$

The substitution $\quad = \quad' \cosh \quad - \quad \sinh \quad$ leads to a first-order separable equation: $\quad' =$ $\cosh \quad (\ )\quad{}^2$.

21. $= f(\ )(\quad \sin \quad - \quad \cos \quad)^2 - \quad .$

The substitution $\quad = \quad' \sin \quad - \quad \cos \quad$ leads to a first-order separable equation: $\quad' =$ $\sin \quad (\ )\quad{}^2$.

22. $= f(\ )(\quad \cos \quad + \quad \sin \quad)^2 - \quad .$

The substitution $\quad = \quad' \cos \quad + \quad \sin \quad$ leads to a first-order separable equation: $\quad' =$ $\cos \quad (\ )\quad{}^2$.

$\boxed{\text{2.9.3-2. Argument of the arbitrary functions is } \quad .}$

23. $+ \ (\quad)^2 - \tfrac{1}{2} \quad = \quad f(\ ).$

The substitution $\ (\ ) = e^{-}\ (\ ')^2$ leads to a first-order linear equation: $\quad' + 2a \quad = 2\ (\ )$.

24. $+ \ (\quad)^2 = [\quad f(\ ) + \quad]\quad .$

Solution:

$$\int \frac{e}{(\ ) + \quad{}_1} = \quad{}_2 + \frac{1}{\beta} e \quad, \qquad \text{where} \quad (\ ) = \int e \quad (\ ) \quad .$$

25. $+ f(\ )(\quad)^2 + \ (\ ) = 0.$

The substitution $\ (\ ) = (\ ')^2$ leads to a first-order linear equation: $\quad' + 2\ (\ )\quad + 2g(\ ) = 0$.

26. $+ f(\ )(\quad)^2 - \tfrac{1}{2} \quad = \quad (\ ).$

The substitution $\ (\ ) = e^{-}\ (\ ')^2$ leads to a first-order linear equation: $\quad' + 2\ (\ )\quad = 2g(\ )$.

27. $= \quad f(\ )(\quad)^3.$

Taking $\quad$ to be the independent variable, we obtain a linear equation with respect to $\quad = \ (\ )$: $\quad'' = -\ (\ )\ .$

28. $+ [\ f(\ ) + \ (\ )](\quad)^3 + \ (\ )(\quad)^2 = 0.$

Taking $\quad$ to be the independent variable, we obtain a linear equation with respect to $\quad = \ (\ )$: $\quad'' - \ (\ )\ ' - \ (\ )\ - g(\ ) = 0$.

29. $+ f(\ )(\quad)^4 + \ (\ )(\quad)^2 + \ (\ ) = 0.$

The substitution $\ (\ ) = (\ ')^2$ leads to a Riccati equation: $\quad' + 2\ (\ )\ {}^2 + 2g(\ )\quad + 2\ (\ ) = 0$ (see Section 1.2).

30. $+ \quad (\quad)^2 - f(\quad) \quad = 0.$

Solution:

$$\int \frac{e}{(\;) + \quad _1} = \quad _2 + \ln| \quad |, \qquad \text{where} \qquad (\;) = \int e \quad (\;) \quad + \frac{1}{a} e \quad .$$

31. $+ \frac{1}{2} \quad = \quad f(\;)(\quad)^2 + \; (\;).$

The substitution $(\;) = \; (\; ')^2$ leads to a first-order linear equation: $\; ' = 2 \; (\;) \quad + 2g(\;).$

32. $+ \quad (\quad)^2 = \big[\; ^k f(\;) + \quad - 1 \big] \quad .$

Solution:

$$\int \frac{e}{(\;) + \quad _1} = \quad _2 + \frac{1}{k} \quad , \qquad \text{where} \qquad (\;) = \int e \quad (\;) \quad .$$

33. $^3 \quad + [\; ^4 f(\;) + \quad](\quad)^3 = 0.$

Taking to be the independent variable, we obtain an equation of the form 2.9.1.2 for $= \; (\;)$: $'' - \; (\;) \quad - a \quad ^{-3} = 0.$

34. $= \quad ^{-1} [f(\;) + \; (\;)(\quad - \;) + \; (\;)(\quad - \;)^2] \quad .$

The substitution $(\;) = \quad ' -$ leads to a Riccati equation: $\; ' = \; (\;) + g(\;) \quad + \; (\;) \quad ^2.$

35. $+ \quad f(\;)(\quad)^3 + \quad = 0.$

The substitution $\xi = e^{-}$ leads to an equation of the form 2.9.4.36 with $g(z) = az^3$: $'' - a \; (\;)(\; ')^3 = 0.$

36. $+ f(\;)(\quad)^4 + \; (\;)(\quad)^2 + \; (\;) = 0.$

The substitution $(\;) = (\; ')^2$ leads to a Riccati equation: $\; ' + 2 \; (\;) \quad ^2 + 2g(\;) \quad + 2 \; (\;) = 0.$

37. $+ \quad ^2 \quad ^{+1} f(\;)(\quad)^4 + \quad = 0.$

This is a special case of equation 2.9.4.18 with $= 4$ and $= \; ^{-} \quad .$

2.9.3-3. Other arguments of arbitrary functions.

38. $= f(\;)(\quad)^2 + \; (\;) \quad .$

Dividing by $\; '$, we obtain an exact differential equation. Its solution follows from the equation:

$$\ln| \; ' | = \quad (\;) \quad + \quad g(\;) \quad + \quad .$$

Solving the latter for $\; '$, we arrive at a separable equation. In addition, $= \quad _1$ is a singular solution, with $_1$ being an arbitrary constant.

39. $= \frac{f(\qquad)}{\qquad} (\quad - \;) \quad .$

The transformation $z = \qquad , \quad = \frac{\quad '}{\qquad}$ leads to a first-order separable equation: $z(\quad + \;) \; ' = [\; (z) - 1](\; ^2 - \;).$

40. $= f(\qquad)(\quad)^2.$

The transformation $z = e \qquad , \quad = \; '$ leads to a first-order separable equation: $z(\quad + a) \; ' = [\; (z) - 1] \quad ^2.$

41. $^{2} \quad = f(\quad)(\quad - \quad)(\quad)^2.$

This is a special case of equation 2.9.4.20 with $k = 2$.

42. $= \;^{-2}(\quad - \quad)\, f \;—\; + \;—\; + \;—\; (\quad)^2\;.$

The transformation $z = \quad, \quad = \quad '$ leads to a Riccati equation: $z\ ' = z^2\ (z)\ ^2 + [zg(z) - 1]\ + \ (z)$.

43. $= \;^{-5\;2}\;(\quad ^{-3\;2})(\quad)^4.$

This is a special case of equation 2.9.4.21 with $= -3\;\;2, \quad = 1$, and $(z) = z\ (z)$.

2.9.4. Equations of the Form $F(\ ,\ ,\ ) \quad + G(\ ,\ ,\ ) = 0$

2.9.4-1. Arguments of the arbitrary functions depend on or .

1. $= f(\ )(\quad + \quad)^k + \;^2\;.$

The substitution $= \ ' + a$ leads to a Bernoulli equation: $' = a \quad + (\ )\quad .$

2. $+ f(\ ) \quad + (\ )(\quad)^k = 0.$

The substitution $(\ ) = \ '$ leads to a Bernoulli equation: $' + (\ ) + g(\ ) \quad = 0.$

3. $= (\ -1)\ ^{-2} + f(\ )(\quad - \quad)^k.$

The substitution $= \quad ' - \quad ^{-1}$ leads to a first-order separable equation: $' = \;^{+\,-}\ (\ )\quad .$

4. $= f(\ )(\quad - \ ) + (\ )(\quad - \ )^k.$

The substitution $(\ ) = \quad ' -$ leads to a Bernoulli equation: $' = \quad (\ ) \ + \ g(\ )\quad .$

5. $+ (\ + 1) \quad = f(\ )(\quad + \quad)^k.$

The substitution $= \quad ' + a$ leads to a first-order separable equation: $' = (\ )\quad .$

6. $= \ + f(\ )\ \overline{(\quad)^2 - 2\quad}.$

Setting $= \ '$, we rewrite the equation as follows: $(\ ' - a)^2[\ (\ )]^{-2} = \;^2 - 2a\ .$ Differentiating both sides with respect to and dividing by $(\ ' - a)$, we obtain a second-order linear equation: $'' = \ '\ ' + \;^3\ - a\ '.$

There is also the solution: $= \frac{1}{2}a(\quad + \quad)^2.$

7. $= f(\ )\ \overline{\quad(\quad - \ )}.$

The Legendre transformation $= \ ',\quad = \ ' -$ leads to an equation of the form 2.9.4.40: $'' = \dfrac{1}{(\ ')\ \overline{\quad}}.$

8. $f_1(\ ) \quad + f_2(\ ) \quad + f_3(\ )(\quad)^2 + f_4(\ ) \quad + f_5(\ )\ ^2 = 0.$

The substitution $(\ ) = \ '$ leads to the Abel equation $(\ _1 \ + \ _2)\ ' + \ _1\ ^3 + (\ _2 + \ _3)\ ^2 + \ _4 \ + \ _5 = 0$, where $= \ (\ ).$

9. $= f(\ )(\ \sinh \ - \ \cosh \ )^k + \ .$

The substitution $= \ '\sinh \ - \ \cosh$ leads to a first-order separable equation: $' = \sinh \ (\ )\quad .$

10. $= f(\ )(\ \ \cosh\ \ -\ \ \sinh\ \ )^k +\ \ .$

The substitution $\ = {}'\cosh\ -\ \sinh\ $ leads to a first-order separable equation: $\ ' = \cosh\ (\ )\ .$

11. $= f(\ )(\ \ \sin\ \ -\ \ \cos\ \ )^k -\ \ .$

The substitution $\ = {}'\sin\ -\ \cos\ $ leads to a first-order separable equation: $\ ' = \sin\ (\ )\ .$

12. $= f(\ )(\ \ \cos\ \ +\ \ \sin\ \ )^k -\ \ .$

The substitution $\ = {}'\cos\ +\ \sin\ $ leads to a first-order separable equation: $\ ' = \cos\ (\ )\ .$

13. $= f(\ )(\ \ )^2 + (\ )(\ \ )^k.$

The substitution $(\ ) = {}'$ leads to a Bernoulli equation: $\ ' = (\ )\ + g(\ )\ ^{-1}.$

14. $+\ \ ^{-2}\ ^{+1} f(\ )(\ \ )\ +\ \ = 0.$

This is a special case of equation 2.9.4.18 with $\ = {}^{-}\ .$

15. $= {}^{-1}\left[f(\ )(\ \ -\ ) + (\ )(\ \ -\ )^k\right]\ .$

The substitution $(\ ) = {}' -$ leads to a Bernoulli equation: $\ ' = (\ )\ + g(\ )\ .$

16. $= f(\ )(\ \ -\ )^{1\ 2}(\ \ )^2.$

The transformation $= {}' -\ ,\ \ = -\ '$, where $\ = (\ )$, leads to an equation of the form 2.9.4.40: $\ '' = \dfrac{1}{(-\ ')\ }\ .$

17. $+\ \ ^{(\ -2)} f(\ )(\ \ )\ +\ \ = 0.$

This is a special case of equation 2.9.4.18 with $\ = e^{-}\ .$

18. $+\ \ ^{2-} f(\ )(\ \ ) - \frac{}{}\ = 0, \qquad = (\ ).$

The substitution $\xi = \ (\ )$ leads to an equation of the form 2.9.4.36: $\ '' + (\ )(\ ')\ = 0.$

19. $f\ \ + \tfrac{1}{2} f\ \ = f\ (\ )(\ \ )^2 + f\ \ (\ )(\ \ )^2\ , \qquad f = f(\ ).$

The substitution $\xi = \dfrac{}{(\ )}$ leads to an autonomous equation of the form 2.9.4.13: $\ '' = g(\ )(\ ')^2 + (\ )(\ ')^2\ .$

2.9.4-2. Arguments of the arbitrary functions depend on and .

20. $= {}^{-2} f(\ \ )(\ \ -\ )(\ \ )^k.$

The transformation $z = \ ,\ \ = \ '$ leads to a Bernoulli equation: $z\ ' = -\ + z\ (z)\ .$

There are particular solutions: $\ =\ $ and $\ = \ _1$ (for $k > 0$).

21. $= \frac{f(\quad)}{\quad}(\quad)^{\frac{2\ +}{+}}.$

The transformation $z =\quad ,\ = \ '$ yields: $z(\ +\ )\ ' = z^{-\frac{1}{+1}}\ (z)\ ^{\frac{2\ +}{+}} + \ - \ ^2.$
We divide both sides of this equation by $\ ^{\frac{2\ +}{+}}$ and introduce the new dependent variable $= \ \overline{\ +}\ - \ ^{-}\overline{\ +}$. As a result, we obtain a first-order linear equation:

$$(\ +\ )z\ ' = -\ + z^{-\frac{1}{+}}\ (z).$$

22. $= \frac{-\quad -}{\quad}\ ^{-1} + \ ^{k-1}\ ^{-k} f(\quad)(\quad)^{k+1}.$

Passing on to the new variables $z =$ and $= \ '$, we arrive at a first-order equation:

$$z(\ +\ )\ ' = \frac{(1-k)}{k}\ - \ ^2 + \ (z)\ ^{+1}.$$

The substitution $= \frac{}{1-k}\ ^{1-} - \frac{}{k}\ ^{-}$ leads to a linear equation: $z\ ' = \frac{k-1}{}\ + \ (z).$

23. $= \frac{+\quad +}{\quad}\ ^{-1}(\quad)^2 + \ ^{k}\ ^{-k-1} f(\quad)(\quad)^{k+2}.$

Passing on to the new variables $z =\quad ,\ = \ '$, we arrive at a first-order equation:

$$z(\ +\ )\ ' = \ + \frac{(1+k)}{k}\ ^2 + \ (z)\ ^{+2}.$$

The substitution $= \frac{}{k}\ ^{-} + \frac{}{k+1}\ ^{-\ -1}$ leads to a linear equation: $z\ ' = -\frac{k+1}{}\ - \ (z).$

24. $= \frac{f(\quad)}{\quad}(\quad - \quad)^{\frac{2\ +}{}}.$

This is a special case of equation 2.9.4.25 with $k = \frac{2\ +}{}$.

25. $= \frac{f(\quad)}{\quad}(\quad)^{\frac{2\ +\ -\ k}{+}}(\quad - \quad)^k.$

The transformation $z =\quad ,\ = \ '$ leads to a first-order equation:

$$z(\ +\ )\ ' = z^{\frac{-1}{+}}\ (z)\ ^{\frac{2\ +\ -}{+}}(\ - 1)\ + \ - \ ^2.$$

Multiplying both sides by $\ ^{-\frac{2\ +}{+}}$ and passing on to the new variable $= \ \overline{\ +}\ - \ ^{-}\overline{\ +}$, we arrive at a Bernoulli equation: $(\ +\ )z\ ' = -\ + z^{\frac{-1}{+}}\ (z)\ .$

26. $= \ ^{-2}(\quad - \quad)\ f \ — \ + \ — \ (\quad)^k\ .$

The transformation $z =\quad ,\ = \ '$ leads to a Bernoulli equation: $z\ ' = [z\ (z) - 1]\ + z\ g(z)\ .$

27. $= \ ^{-3}(\quad - \quad)^2 f \ — \ + \ ^{-3}(\quad - \quad)^k\ — \ .$

The transformation $= -1\ ,\ = -$ leads to an autonomous equation of the form 2.9.4.13: $\ '' = \ (\ )(\ ')^2 + g(\ )(\ ')\ .$

28. $= \dfrac{f(\quad)}{}(\quad)^{\frac{2+\,-\,k}{+}}(\quad - \;)^k + \dfrac{(\quad)}{}\;(\quad - \;).$

The transformation $z = \quad , \quad = \; '$, followed by the substitution $\quad = \overline{\quad +} - {}^{-}\overline{\quad +}$, leads to a Bernoulli equation: $(\;+\;)z\;' = [g(z)-1] \; + z^{\frac{-1}{+}}\;(z)\;$.

29. $= \dfrac{f(\quad)}{}(\quad)^{\frac{2\,+}{+}} + \dfrac{(\quad)}{}\;(\quad - \;) + \dfrac{(\quad)}{}(\quad)^{\overline{\;+\;}}(\quad - \;)^2.$

The transformation $z = \quad , \quad = \; '$, followed by the substitution $(z) = \overline{\;+\;} - {}^{-}\overline{\;+\;}$, leads to a Riccati equation: $(\;+\;)z\;' = z^{\frac{1}{+}}\;(z)\;^2 + [g(z)-1]\; + z^{-\frac{1}{+}}\;(z).$

30. $= -\dfrac{1-}{2-}(\quad)^2 + {}^{k-2}f(\quad)(\quad)^k.$

Passing on to the new variables $z = \quad e$ and $\quad = \; '$, we have

$$z(a \;+\;)\;' = \frac{a}{}\,\frac{1-k}{2-k}\;{}^2 + \;+\;(z)\;.$$

Multiplying both sides by ${}^{-}$ and introducing the new variable $v = \dfrac{a}{2-k}\;{}^{2-} + \dfrac{}{1-k}\;{}^{1-}$, we obtain a first-order linear equation: $zv' = \dfrac{1-k}{}v + \;(z).$

31. $= -\dfrac{2-}{1-} \quad + {}^{1-k}f(\quad)(\quad)^k.$

Passing on to the new variables $z = e$ and $\quad = \; '$, we have

$$z(\quad + a)\;' = -\;{}^2 - \frac{a}{}\,\frac{2-k}{1-k}\; + \;(z)\;.$$

Multiplying both sides by ${}^{-}$ and introducing the new variable $v = \dfrac{}{2-k}\;{}^{2-} + \dfrac{a}{1-k}\;{}^{1-}$, we obtain a first-order linear equation: $zv' = (k-2)v + \;(z).$

32. $= -\quad \ln \quad + f(\quad)\;.$

The transformation $z = e \quad , \quad = \; '$ leads to a first-order equation:

$$z(\quad + a)\;' = -\frac{a}{}\;\ln\; - \;{}^2 + \;(z)\;.$$

Dividing both sides by and passing on to the new variable $v = \quad + a\ln\;$, we obtain a first-order linear equation: $zv' = -v + \;(z).$

33. $= {}^{-2}f(\quad)\exp\;-\quad .$

The transformation $z = \; e\;, \quad = \; '$ leads to the first-order equation $z(a \;+\;)\;' = \;+\;(z)\exp\;-\dfrac{a}{}$, which can be reduced, with the aid of the substitution $\; = \exp\;\dfrac{a}{}$, to a linear equation: $z\;' = \;+\;(z).$

34. $= -(\quad)^2\ln(\quad) + f(\quad)(\quad)^2.$

The transformation $z = \; e\;, \quad = \; '$ leads to the equation

$$z(a \;+\;)\;' = \; -\frac{a}{}\;{}^2\ln\; + \;{}^2\;(z).$$

Dividing both sides by ^{2} and passing on to the new variable $v = a\ln\; - \;{}^{-1}$, we obtain a first-order linear equation: $zv' = -v + \;(z).$

2.9.4-3. Arguments of the arbitrary functions depend on , , and $'$.

35. $= f(\ )\ (\ ).$

The substitution $(\ ) = \ '$ leads to a first-order separable equation: $' = (\ )g(\ )$.

36. $= f(\ )\ (\ ).$

The substitution $(\ ) = \ '$ leads to a first-order separable equation: $' = (\ )g(\ )$.

In addition, there may exist solutions of the form $= A \ +$, where A are roots of the equation $g(A) = 0$, is an arbitrary number, or $= B$, where B are roots of the equation $(B) = 0$.

37. $= f(\ \ +\ \ +\ )\ (\ ).$

For $= 0$, we have an equation of the form 2.9.4.35. For $\neq 0$, the substitution $(\ ) = \ + (a\ +\ )$ leads to an equation of the form 2.9.4.36: $'' = (\ )g\ \ ' - \frac{a}{}$.

38. $= \ ^{-}\ ^{-1} f(\ \ ).$

The substitution $(\ ) = \ '$ leads to a first-order separable equation: $' = (\ ) +$.

39. $= \ ^{-2}\ ^{-1} f(\ \ ).$

The substitution $(\ ) = \ '$ leads to a first-order separable equation: $' = (\ ) + \ ^2$.

40. $= \dfrac{f(\ )}{\overline{\ \ }}.$

Setting $= \ '$ and passing on to the new variables $= \dfrac{}{(\ )}$ and $= 2\ \overline{\ }$, we have $= (\ ')^2$. Differentiating the latter with respect to , we obtain a second-order linear equation: $'' = g(\ )$. Here, the function $g(\ )$ is defined parametrically: $g = \frac{1}{4}$, $= \dfrac{}{(\ )}$.

41. $= \dfrac{f(\ )}{\overline{\ \ +\ \ }}.$

Setting $= \ '$, we rewrite the equation as follows: $\left[\ \overline{\ }\ '\ (\ )\right]^{-2} = a\ +\ ^{-1}$. Differentiating both sides with respect to and passing on to the new variables $= \dfrac{1}{2}\ \dfrac{}{(\ )}$, $z = \overline{\ }$, we obtain an equation of the form 2.9.1.2: $z'' = a\ (\ )z - \ z^{-3}$.

42. $= \dfrac{f(\ )}{\overline{\ \ +\ \ ^2}}.$

Setting $= \ '$, we rewrite the equation as follows: $\left[\ '\ (\ )\right]^{-2} = a\ +\ ^2$. Differentiating both sides with respect to and passing on to the new variable $= \dfrac{}{(\ )}$, we obtain a second-order linear equation for $= (\ )$ integrable by quadrature: $2\ '' = 2\ + a\ (\ )$. Here, the function $= (\ )$ is defined implicitly: $= \dfrac{}{(\ )}$.

43. $= \dfrac{f(\ )}{\overline{\ \ +\ \ ^2}}.$

Taking to be the independent variable, we obtain an equation of the form 2.9.4.42 for $= (\ )$: $'' = -(a\ +\ ^2)^{-1\ 2}\ (1\ \ ')(\ ')^3$.

44. $= (\quad ^2 + \quad + \quad ^2 + \quad + \quad + \gamma)^{-1\ 2} f(\quad).$

The transformation $= A + B +$, $= \quad + \quad + Q$, where $= (\ )$, reduces this equation by selecting appropriate constants A, B, , , , and Q, to an equation of the form 2.9.4.41, 2.9.4.42, or 2.9.4.43.

45. $= \dfrac{f(\quad)}{\overline{\overline{\quad + \quad ^{3\ 2} +}}}.$

Setting $= \ '$, we rewrite the equation as follows: $[\ ^{-} \ ' \quad (\ )]^{-2} = a \quad + \quad ^{-} + \ .$ Differentiating both sides with respect to and passing to the new variables $= \frac{1}{2} \quad \frac{}{(\ )}$ and $z = \ ^{-}$, we obtain a second-order linear equation: $2z'' = 2a \ (\ ) z +$. Here, the function $= \ (\ )$ is defined implicitly: $= \frac{1}{2} \quad \frac{}{(\ )}$.

46. $= \dfrac{f(\quad)}{\overline{\overline{\quad + \quad ^{3\ 2} +}}}.$

Taking to be the independent variable, we obtain an equation of the form 2.9.4.45 for $= \ (\ )$: $'' = -(a \quad + \quad ^{3\ 2} + \quad)^{-1\ 2} \ (1 \quad ')(\ ')^3$.

47. $= \dfrac{f(\quad)}{\overline{\overline{\quad + \quad ^2 + \quad ^{3\ 2} +}}}.$

The substitution $a \quad = a \quad +$ leads to an equation of the form 2.9.4.45 for $= \ (\ )$: $'' \ = \dfrac{(\ ' - \quad a)}{\overline{\overline{a \quad + \quad ^{3\ 2} +}}}$.

48. $= \dfrac{f(\quad)}{\overline{\overline{\quad + \quad ^2 + \quad ^{3\ 2} +}}}.$

Taking to be the independent variable, we obtain an equation of the form 2.9.4.47 for $= \ (\ )$: $'' = -(a \quad + \quad ^2 + \quad ^{3\ 2} + \quad)^{-1\ 2} \ (1 \quad ')(\ ')^3$.

49. $= \ ^{-2}(\quad - \quad) f(\quad).$

The Legendre transformation $= \ '$, $= \ ' - \quad (\ ' = \ , \ '' = 1 \quad '')$ leads to an equation of the form 2.9.3.14: $'' = [\ (\ )]^{-1}(\ ')^2$.

50. $= [\ f(\quad) + \quad (\quad) + \quad (\quad)]^{-1}.$

The Legendre transformation $= \ '$, $= \ ' - \quad (\ ' = \ , \ '' = 1 \quad '')$ leads to a second-order linear equation: $'' = [\ (\ ) + \ g(\ )] \ ' - g(\ ) \quad + \ (\ )$.

51. $(\quad + \quad - \ + \ ) \quad = f(\quad).$

The contact transformation

$$X = \ ', \quad = \ ' + a \ ' - \ + \ , \quad ' = \ + a, \quad '' = 1 \quad '',$$

where $= \ (X)$, leads to a linear equation: $(X) \ '' \ - \quad = 0$.

Inverse transformation:

$$= \ ' - a, \quad = X \ ' - \ + \ , \quad ' = X, \quad '' = 1 \quad ''.$$

52. $\boldsymbol{f(\quad)\quad+\ (\quad)\quad+\ (\quad)=0.}$

Integrating yields a first-order equation:

$$(\ )\quad+\quad g(\ )\quad+\quad(\ )\quad=\ ,\qquad\text{where}\quad=\ '.$$

53. $\boldsymbol{=\ {}^{2}+f(\ )\ (\quad+\quad).}$

The substitution $=\ '+a$ leads to a first-order equation: $'=a\quad+\ (\ )g(\ )$.

54. $\boldsymbol{f(\quad+\quad)\quad+(\ {}^{2}+2\quad)(\quad+\ )=0.}$

The contact transformation

$$X=\ '+a\ ,\qquad=\tfrac{1}{2}(\ ')^2+a\ ,\qquad'=\ ',\qquad''\ =\frac{''}{''\ +a},$$

where $=\ (X)$, leads to a linear equation: $(X)\ ''\ +2\ =0$.

Inverse transformation:

$$=\frac{1}{a}\big(X-\ '\ ,\qquad=\frac{1}{2a}\big[2\ -(\ ')^2\big],\qquad'=\ '.$$

55. $\boldsymbol{=\ {}^{-1}f\quad-\!-\ .}$

The substitution $=\ '-\!-$ leads to a first-order separable equation: $'=-\ +\ (\ )$.

56. $\boldsymbol{(\quad-\ )(\ {}^{2}\quad+\quad-\ )=\ {}^{2}f\quad+\ -\ .}$

The contact transformation ($a\neq-1$)

$$X=\ '+a-,\qquad=\ {}^{+1}\ '-\ ,\qquad'=\ {}^{+1},\qquad''\ =\frac{(a+1)\ {}^{+2}}{{}^{2}\ ''\ +a\ '-a}$$

leads to a linear equation: $(X)\ ''\ =(a+1)\ $.

Inverse transformation:

$$=(\ ')^{\frac{1}{+1}},\qquad=\frac{1}{a+1}(X\ '-\ )(\ ')^{-\frac{}{+1}},\qquad'=\frac{X\ '+a}{(a+1)\ '}.$$

57. $\boldsymbol{=\ {}^{-3}f(\quad-\ )(\ )^{-1}.}$

The Legendre transformation $=\ ',\ =\ '-$ leads to an equation of the form 2.9.3.27: $''=\ [\ (\ )]^{-1}(\ ')^3$.

58. $\boldsymbol{=f(\ )\ (\quad-\ ).}$

The substitution $=\ '-$ leads to a first-order separable equation: $'=\ (\ )g(\ )$.

59. $\boldsymbol{=\ {}^{-\ -3}\ f(\quad-\ ).}$

The transformation $=1\ ,\ =$ leads to an autonomous equation of the form 2.9.4.36: $''=\ (-\ ')$.

60. $\boldsymbol{=\ {}^{-1}f(\ )\ (\quad-\ )\ .}$

The substitution $(\ )=\ '-$ leads to a first-order separable equation: $'=\ (\ )g(\ )$.

61. $\boldsymbol{=\ {}^{-3}f(\quad)\ (\quad-\ ).}$

The transformation $=1\ ,\ =$ leads to an autonomous equation of the form 2.9.4.36: $''=\ (\ )g(-\ ')$.

62. $^{3} \quad + 2 \ ^{2}(\quad)^{2} = f(\quad + \quad).$

The contact transformation

$$X = \ ' + \ , \qquad = \ ^{2} \ ', \qquad ' = \ , \qquad '' \ = \frac{1}{'' \ + 2 \ '}$$

where $= \ (X)$, leads to a linear equation: $(X) \ '' \ = \ .$

Inverse transformation:

$$= \ ', \qquad = X - \frac{}{'}, \qquad ' = \frac{}{(\ ')^{2}}, \qquad '' \ = \frac{1}{' \ ''} - \frac{2}{(\ ')^{3}}.$$

63. $= \frac{}{2} f \ \frac{}{} \ .$

The substitution $(\ ) = \ '$ leads to a first-order separable equation: $' = (\ ) + \ - \ ^{2}.$

64. $= \ ^{-1}(\quad)^{2} - \ ^{-1} \quad + \ ^{-2} \ f(\quad) \ \frac{}{} \ .$

The transformation $z = \quad , \quad = \ '$ leads to a first-order separable equation: $z(\quad + \ ) \ ' = \ (z)g(\ ).$

65. $= \ (\ - 1) \ ^{-2} \ + f(\ ) \ (\quad - \quad ^{-1} \ ).$

The substitution $= \ ' - \quad ^{-1}$ leads to a first-order separable equation: $' = (\ )g(\ ).$

66. $^{2} \quad + \quad - \quad = \ ^{+2} f(\ ^{+1} \quad - \quad .$

The contact transformation $(a \neq -1)$:

$$X = \ ' + a -, \qquad = \ ^{+1} \ ' - \quad , \qquad ' = \ ^{+1}, \qquad '' \ = \frac{(a+1) \quad ^{+2}}{^{2} \ '' \ + a \quad ' - a}$$

leads to an autonomous equation of the form 2.9.1.1: $(\ ) \ '' \ = a + 1.$

Inverse transformation:

$$= (\ ')^{\frac{1}{+1}}, \qquad = \frac{1}{a+1}(X \ ' - \ )(\ ')^{-\frac{}{+1}}, \qquad ' = \frac{X \ ' + a}{(a+1) \ '}.$$

67. $= - \ ^{-1} \quad + \ ^{-2} f(\quad) \ (\quad).$

The transformation $z = \ e \ , \quad = \ '$ leads to a first-order separable equation: $z(a \quad + \ ) \ ' = \ (z)g(\ ).$

68. $= \ ^{-1}(\quad)^{2} + \ f(\quad) \ (\quad).$

The transformation $z = e \quad , \quad = \ '$ leads to a first-order separable equation: $z(\quad + a) \ ' = \ (z)g(\ ).$

69. $(\quad + \ )(\quad + \ ) = \ ^{-2} \ f(\quad).$

The contact transformation

$$X = e \quad ', \qquad = \ ' + \ , \qquad ' = e^{-} \ , \qquad '' \ = -\frac{e^{-2}}{'' \ + \ '},$$

where $= \ (X)$, leads to a linear equation: $(X) \ '' \ = - \ .$

Inverse transformation:

$$= -\ln \ ', \qquad = \quad - X \ ', \qquad ' = X \ '.$$

70. $- \quad = \ ^{2} \ f(\quad - \quad (\quad - \ .$

1 . The substitution $= \ ' - \ $ leads to a first-order equation: $' = e^{2} \ (e \quad)(\ ' + \ ).$

2 . The contact transformation

$$X = e\ (\ ' - \ ), \qquad = (\ ')^2 - \ ^2, \qquad ' = 2e^{-}\ ', \qquad '' = 2e^{-2}\ \frac{''\ -\ '}{''\ -}$$

leads to a linear equation: $'' \ = 2\ (X)$.

71. $\ = f(\ )\ (\ \sinh \ - \ \cosh \ ) + \ .$

The substitution $\ = \ ' \sinh \ - \ \cosh \ $ leads to a first-order separable equation: $\ ' = \sinh \ (\ )g(\ )$.

72. $\ = f(\ )\ (\ \cosh \ - \ \sinh \ ) + \ .$

The substitution $\ = \ ' \cosh \ - \ \sinh \ $ leads to a first-order separable equation: $\ ' = \cosh \ (\ )g(\ )$.

73. $\ = f(\ )\ (\ \sin \ - \ \cos \ ) - \ .$

The substitution $\ = \ ' \sin \ - \ \cos \ $ leads to a first-order separable equation: $\ ' = \sin \ (\ )g(\ )$.

74. $\ = f(\ )\ (\ \cos \ + \ \sin \ ) - \ .$

The substitution $\ = \ ' \cos \ + \ \sin \ $ leads to a first-order separable equation: $\ ' = \cos \ (\ )g(\ )$.

75. $\ + \frac{1}{2}\ \ = f(\ )\ (\ \ -\ ,\ \ = \ (\ ).$

The substitution $(\ ) = \ '\ \ \overline{g}$ leads to a first-order separable equation: $\ ' = \ (\ )\ (\ )$.

76. $f(\ \ ^2 - \ \ + (2\ \ ^3 - 3\ \ (2\ \ - \ = 0, \ \ \neq 0.$

The contact transformation

$$X = a(\ ')^2 - \ , \qquad = 2a(\ ')^3 - 3\ , \qquad ' = 3\ ', \qquad '' = \frac{3\ ''}{2a\ '\ ''\ -}$$

leads to a linear equation: $(X)\ ''\ + 3\ \ = 0$.

Inverse transformation:

$$= \frac{a}{9}(\ ')^2 - \frac{1}{}X, \qquad = \frac{2a}{81}(\ ')^3 - \frac{1}{3}\ , \qquad ' = \frac{1}{3}\ ', \qquad '' = \frac{3\ ''}{2a\ '\ ''\ - 9}.$$

77. $\ = f(\ \ ^2 + \ \ .$

The substitution $(\ ) = (\ ')^2 + a$ leads to a first-order separable equation: $\ ' = 2\ (\ ) + a$.

78. $\ = f(\ )\ (\ \ ^2 + 2\ \ - \ .$

The substitution $(\ ) = (\ ')^2 + 2a$ leads to a first-order separable equation: $\ ' = 2\ (\ )g(\ )$.

79. $\ = f(\ \ ^2 - 2\ \ + 2\ \ .$

The substitution $\ = \ + \frac{1}{2}a\ ^2$ leads to an autonomous equation of the form 2.9.4.78: $'' = (\ '^2 + 2a\ ) - a$.

80. $\ - \ = \ ^2\ f(\ ^2 - \ ^2\ (\ \ - \ .$

The contact transformation

$$X = e\ (\ ' - \ ), \qquad = (\ ')^2 - \ ^2, \qquad ' = 2e^{-}\ ', \qquad '' = 2e^{-2}\ \frac{''\ -\ '}{''\ -}$$

leads to an autonomous equation of the form 2.9.1.1: $'' \ = 2\ (\ )$.

81. $+\frac{1}{2} \quad = f(\quad ^2 + \quad .$

The substitution $(\ ) = (\ ')^2 + a$ leads to a first-order separable equation: $' = 2\ (\ ) + a$.

82. $+\frac{1}{2} \quad = \ ^- f(\quad ^2 + \quad .$

The substitution $(\ ) = e\ (\ ')^2 + a$ leads to a first-order separable equation: $' = 2\ (\ ) + a$.

83. $f(\ (\ )^k - \qquad = [\ (\ )^{k+1} - (\ +1)\ ][\ (\ )^{k-1} \quad - \ ].$

The contact transformation ($a \ \neq 0$, $k \neq -1$)

$$X = a(\ ')\ - \quad , \qquad = ak(\ ')\ ^{+1} - (k+1)\ , \qquad ' = (k+1)\ ', \qquad '' = \frac{(k+1)\ ''}{ak(\ ')\ ^{-1}\ ''\ -}$$

leads to a linear equation: $(X)\ '' \ = (k+1)\ .$

Inverse transformation:

$$= \frac{a(\ ')}{(k+1)} - \frac{X}{\ }, \qquad = \frac{ak(\ ')\ ^{+1}}{(k+1)\ ^{+2}} - \frac{\ }{(k+1)}, \qquad ' = \frac{'}{k+1}.$$

84. $+\frac{1}{2} \quad = f(\quad ^2 + \quad , \qquad = (\ ).$

The substitution $(\ ) = (\ )(\ ')^2 + a$ leads to a first-order separated equation: $' = 2\ (\ ) + a$.

85. $+\frac{1}{2} \quad = f(\ )\ (\quad ^2 + 2 \quad - \ , \qquad = (\ ).$

The substitution $(\ ) = (\ )(\ ')^2 + 2a$ leads to a first-order separable equation: $' = 2\ (\ )g(\ )$.

86. $= \quad (\ )^{\frac{2\ +\ +3}{\ +\ +2}}\ (\ ), \qquad = (\ \ -\ )(\ \ )^{-\frac{+1}{\ +\ +2}}.$

The Legendre transformation $= \ ', \ = \ ' -$ leads to an equation of the form 2.9.4.25: $'' = \frac{1}{\ } \frac{\ }{(\ \ )}\ (\ ')^{\frac{2\ +1-\ b}{+1}} (\ \ ' - \ )^b$, where $a = -\frac{+1}{+\ \ +2}$, $= -\ \ .$

2.9.5. Equations Not Solved for Second Derivative

1. $f(\ )(\ \ -\ )^2 = (\ \ )^2 - 2 \quad + \ .$

Differentiating with respect to , we obtain

$$(\ '' - a)(2\ \ ''' \ + \ '\ '' - 2\ ' - a\ ') = 0. \tag{1}$$

Equating the second factor to zero and making the transformation $\xi = \frac{\ }{\ }$, $= \ '$, we arrive at a second-order constant coefficient linear equation of the form 2.1.9.1:

$$'' - \quad = \tfrac{1}{2}a\ ', \tag{2}$$

whose right-hand side is to be expressed in terms of ξ. Substituting the solution of equation (2) into the original one, we obtain a relation connecting integration constants.

Equating the first factor in (1) to zero, we find the singular solution: $= \frac{1}{2}a(\ +\ )^2 + \frac{\ }{2a}$.

---

2. $f(\ )(\quad -\quad)^2 = (\quad)^2 - \quad ^2 + \ .$

Differentiating with respect to , we obtain

$$(\ '' \ - a\ )[2\ (\ ''' \ - a\ ')+\ '(\ ''\ - a\ ) - 2\ '] = 0.$$

Equating the second factor to zero, we arrive at a third-order linear equation:

$$2\ (\ '''\ - a\ ') + \ '(\ ''\ - a\ ) - 2\ ' = 0.$$

Equating the first factor to zero, one can find the singular solution.

3. $= f(\quad).$

The substitution $(\ ) = \ '$ leads to an equation of the form 1.8.1.7: $= (\ ').$

4. $= f(\quad).$

The substitution $(\ ) = \frac{1}{2}(\ ')^2$ leads to an equation of the form 1.8.1.8: $= (\ ').$

5. $= \quad ^2 + \quad + \ + f(\quad).$

The substitution $= -a\ ^2 - \quad -$ leads to an equation of the form 2.9.5.4: $= (\ '' + 2a).$

6. $= \ + (\quad)^2 + \quad + \ + f(\quad).$

The Legendre transformation $= \ ',\ = \ ' -\ (\ ' = ,\ '' = 1\quad '')$ leads to an equation of the form 2.9.5.5: $= a\ ^2 + \quad + \ + (1\quad '').$

7. $f(\quad) + \quad = \ .$

Solution: $= \frac{1}{2}\ _1\ ^2 + \ (\ _1) + \ _2.$

8. $f(\quad + \ ) = (\quad)^2 + (\quad)^2.$

Differentiating with respect to , we obtain

$$[\ '(\ ''\ + \ ) - 2\ ''\ ](\ '''\ + \ ') = 0.$$

From the equation $'''\ + \ ' = 0$, it follows that:

$$= A\sin(\ + \ _1) + \ _2, \qquad \text{where} \quad A^2 = \ (\ _2).$$

Equating the expression in square brackets to zero, we arrive at the singular solution in parametric form:

$$= \int \frac{[2 - \ ''(\ )]}{\overline{4\ (\ ) - [\ '(\ )]^2}}, \qquad = \ -\tfrac{1}{2}\ '(\ ).$$

9. $= \ f(\qquad).$

The transformation $= \ ^2,\ = (\ ')^2$ leads to an equation of the form 1.8.1.11: $= \ ^2(\ ').$

10. $= (\quad)^2 + f(\qquad .$

1 . Solution:

$$= \ _1 \exp(\ _2\ ) + \frac{(\ _2^2)}{4\ _1\ _2^2} \exp(-\ _2\ ).$$

2 . Solution:

$$= \ _1 \sin(\ _3\ ) + \ _2 \cos(\ _3\ ),$$

where the constants $_1$, $_2$, and $_3$ are related by the constraint $(\ _1^2 + \ _2^2)\ _3^2 + (-\ _3^2) = 0.$

3 . Solutions: $= \quad \overline{-\ (0)} + \ .$

11. $= f(\ ^3\ )$.

The transformation $= 1\ ,\ =$ leads to an equation of the form 2.9.5.4: $= (\ '')$.

12. $- = f(\ )$.

This is a special case of equation 2.9.5.16 with $=$.

13. $- = f(\ ^\lambda\ )$.

This is a special case of equation 2.9.5.16 with $= e^\lambda$.

14. $- = f(\ln\ )$.

This is a special case of equation 2.9.5.16 with $= \ln$.

15. $- = f(\sin\ )$.

This is a special case of equation 2.9.5.16 with $= \sin$.

16. $- = f(\ ),\quad = (\ )$.

The transformation $\xi =\ -\ ,\ =\ ' -$ leads to an equation of the form 1.8.1.8: $= (\ ')$.

2.9.6. Equations of General Form

2.9.6-1. Equations containing arbitrary functions of two variables.

1. $= (\ ,\ )$.

The substitution $(\ ) =\ '$ leads to a first-order equation: $' = (\ ,\ )$.

2. $= (\ ,\ )$.

Autonomous equation. The substitution $(\ ) =\ '$ leads to a first-order equation: $' = (\ ,\ )$.

3. $= (\ +\ ,\ )$.

The substitution $= a\ +$ leads to an equation of the form 2.9.6.2: $'' =\ ,\ ' - \frac{a}{}$.

4. $= \frac{1}{}\ \frac{}{},\ $.

Homogeneous equation. This is a special case of equation 2.9.6.6 with $k = 1$.

5. $= \frac{1}{+\ +}\ \frac{+\ +}{+\ +\gamma},\ $.

1 . For $a\beta - \ = 0$, the substitution $= a\ +\ +$ leads to an autonomous equation of the form 2.9.6.2.

2 . For $a\beta - \ \neq 0$, the transformation

$$z = \ - \ _0, \qquad = \ - \ _0,$$

where $_0$ and $_0$ are the constants determined by the linear algebraic system of equations

$$a\ _0 + \ _0 + \ = 0, \qquad _0 + \beta\ _0 + \ = 0,$$

leads to a homogeneous equation of the form 2.9.6.4:

$$'' = \frac{1}{z} \quad \frac{}{z}, \quad ' \quad , \qquad \text{where} \qquad (\xi, \ ) = \frac{1}{a + \ \xi} \quad \frac{a + \ \xi}{\ + \beta\xi}, \qquad .$$

6. $= \ ^{k-2} \ (\ ^{-k} \ , \ ^{1-k} \ ).$

Generalized homogeneous equation. The transformation $= \ln \ ,\ \ = \ ^{-}$ leads to an equation of the form 2.9.6.2: $'' + (2k-1) \ ' + k(k-1) \ = \ (\ , \ ' + k \ ).$

7. $= \frac{}{\ ^2} \qquad , \frac{}{} \qquad .$

Generalized homogeneous equation. The transformation $z = \qquad , \quad = \quad '$ leads to a first-order equation: $z(\quad + \ ) \ ' = \ (z, \ ) + \ - \ ^2.$

8. $= \ ^2 \ + \ (\ , \ + \ ).$

The substitution $= \ ' + a$ leads to a first-order equation: $' = a \ + \ (\ , \ ).$

9. $= (\ ^2 \ ^2 + \ ) \ + \ (\ , \ - \ ).$

The substitution $= \ ' - a$ leads to a first-order equation: $' = -a \ + \ (\ , \ ).$

10. $= \qquad , \quad - \frac{}{} \ .$

The substitution $(\ ) = \ ' - \frac{}{}$ leads to a first-order equation: $' = - \ + \ (\ , \ ).$

11. $= \ (\ , \ - \ ).$

The substitution $(\ ) = \ ' - \$ leads to a first-order equation: $' = \ (\ , \ ).$

12. $= \ ^{-2} \ (\ , \ - \ ).$

The substitution $(\ ) = \ ' - \$ leads to a first-order equation: $(\ + \ ) \ ' = \ (\ , \ ).$

13. $+ (\ + 1) \ = \ (\ , \ + \ ).$

The substitution $= \ ' + a$ leads to a first-order equation: $' = \ (\ , \ ).$

14. $^2 \ = 2 \ + \ (\ , \ + \ ).$

The substitution $= \ ' + \$ leads to a first-order equation: $' = 2 \ + \ (\ , \ ).$

15. $^2 \ = \ (\ + 1) \ + \ (\ , \ + \ ).$

The substitution $= \ ' + a$ leads to a first-order equation: $' = (a+1) \ + \ (\ , \ ).$

16. $= 2 \ + \ (\ , \ - \ ^2).$

The substitution $= \ ' - a \ ^2$ leads to a first-order equation: $' = \ (\ , \ ).$

17. $= \ ^{-} \ (\ , \ ).$

The substitution $= e$ leads to a second-order autonomous equation of the form 2.9.6.2: $'' - 2a \ ' + a^2 \ = \ (\ , \ ' - a \ ).$

18. $= \ (\ , \ ).$

Equation invariant under "translation–dilatation" transformation. The transformation $z = e \quad , \quad = \ '$ leads to a first-order equation: $z(\ + a) \ ' = \ (z, \ ) - \ ^2.$ See also Paragraph 0.3.2-7.

19. $= {}^{-2} (\quad , \quad).$

Equation invariant under "dilatation–translation" transformation. The transformation $z = \ e \ , \ = \ '$ leads to a first-order equation: $z(a \ + \ ) \ ' = \ (z, \ ) + \ .$ See also Paragraph 0.3.2-7.

20. $= {}^{2} (\quad , {}^{-} \quad).$

The transformation $z = \ e \ , \ = e^{-} \ '$ leads to a first-order equation: $(az \ + 1) \ ' = \ (z, \ ) - a \ ^{2}.$

21. $= \quad + (\ , \quad - \quad).$

The substitution $\ = \ ' - ae$ leads to a first-order equation: $\ ' = \ (\ , \ ).$

22. $= ({}^{2} + \quad) + \big(\ , \quad - \quad .$

The substitution $\ = \ ' - e$ leads to a first-order equation: $\ ' = -e \quad + \ (\ , \ ).$

23. $= \ (\ , \ \sinh \ - \ \cosh \ ) + \ .$

The substitution $\ = \ ' \sinh \ - \ \cosh$ leads to a first-order equation: $\ ' = \ (\ , \ ) \sinh \ .$

24. $= \ (\ , \ \cosh \ - \ \sinh \ ) + \ .$

The substitution $\ = \ ' \cosh \ - \ \sinh$ leads to a first-order equation: $\ ' = \ (\ , \ ) \cosh \ .$

25. $= {}^{-2} (\quad + \ \ln \ , \quad).$

The transformation $z = a \ + \ \ln \ , \ = \ '$ leads to a first-order equation: $(a \ + \ ) \ ' = \ (z, \ ) + \ .$

26. $= \ (\quad + \ \ln \ , \quad).$

The transformation $z = a \ + \ \ln \ , \ = \ '$ leads to a first-order equation: $(\quad + a) \ ' = \ (z, \ ) - \ ^{2}.$

27. $= \ (\ , \ \sin \ - \ \cos \ ) - \ .$

The substitution $\ = \ ' \sin \ - \ \cos$ leads to a first-order equation: $\ ' = \ (\ , \ ) \sin \ .$

28. $= \ (\ , \ \cos \ + \ \sin \ ) - \ .$

The substitution $\ = \ ' \cos \ + \ \sin$ leads to a first-order equation: $\ ' = \ (\ , \ ) \cos \ .$

29. $= ({}^{2} + \quad) + \ (\ , \quad - \quad), \qquad = \ (\ ).$

The substitution $\ = \ ' -$ leads to a first-order equation: $\ ' = - \quad + \ (\ , \ ).$

30. $= \frac{\quad}{\quad} + \quad , \ - \frac{\quad}{\quad} \ , \qquad = \ (\ ).$

The substitution $\ = \ ' - \frac{\ '}{\quad}$ leads to a first-order equation: $\ ' = -\frac{\ '}{\quad} + \ (\ , \ ).$

31. $= \quad + \quad + \ (\ , \quad - \ ), \qquad = \ (\ , \ ).$

The substitution $\ = \ ' - \ (\ , \ )$ leads to a first-order equation: $\ ' = \ (\ , \ ).$

32. $f^{2} \quad + f f \quad = \ (\ , f \quad), \qquad f = f(\ ).$

The substitution $\ (\ ) = \ '$ leads to a first-order equation: $\ ' = \ (\ , \ ).$

33. $= \ (\ , \ ).$

Let $\neq \ (\ ) \ + \ (\ )$, i.e., the equation is nonlinear. Then its order can be reduced by one if the right-hand side of the equation has the following form:

$$(\ , \ ) = \ ^{-3 \ 2} \quad (\ ) + \ \left[\tfrac{1}{2} \ \ ''' \ (\ + \ ) + \ ^{1 \ 2} g'' \ \ ^{-1}\right] \quad , \tag{1}$$

where

$$= \exp \ k \quad ^{-1} \quad , \qquad = \ ^{-3 \ 2} g \ \ ^{-1} \quad , \qquad = \ ^{-1 \ 2} \ \ ^{-1} \ - \ ;$$

$= \ (\ )$, $= \ (\ )$, and $g = g(\ )$ are arbitrary functions, and k is an arbitrary constant.

The integral in (1) can always be expressed in terms of and . The following cases are possible:

1 . For $''' \ \neq 0$,

$$(\ , \ ) = \ ^{-3 \ 2} \quad (\ ) + \tfrac{1}{4} \ ^{-2}\left[2 \ \ '' \ - (\ ')^2\right] \ + \tfrac{1}{2} \ ^{-2}\left(2 \ g' - \ ' g + 2kg \ + k^2 \ ^{-3 \ 2} \quad .$$

2 . For $= a \ ^2 + \ + \ ,$ $' \neq -2k$, $' \neq \tfrac{2}{3}k$,

$$(\ , \ ) = \ ^{-3 \ 2} \quad (\ ) + \tfrac{1}{2} \ ^{-2}\left(2 \ g' - \ ' g + 2kg \ + \left(k^2 + \tfrac{1}{4}\Delta \quad ^{-3 \ 2} \quad , \qquad \text{where} \quad \Delta = 4a \ - \ ^2.$$

3 . For $= \beta - 2k \ ,$

$$(\ , \ ) = \ ^{-2}[\ (\ + \ \ ) + \ g' + 2kg], \qquad \text{where} \qquad = - \quad ^{-1} g \quad .$$

4 . For $= \tfrac{2}{3}k \ + \beta,$

$$(\ , \ ) = \ \left(\ ^{-2} \ - \ \ + \ ^{-2}\left(\ g' + \tfrac{2}{3}kg \ + \tfrac{8}{9}k^2 \right.\right. \ , \qquad \text{where} \qquad = \quad ^{-3} g \quad .$$

In all these cases, the transformation

$$= \quad ^{-1} \quad , \qquad = \ ^{-1 \ 2} \ \ ^{-1} \ -$$

leads to the autonomous equation $'' + 2k \ ' + k^2 \ = \ (\ )$, which is reducible, with the aid of the substitution $z(\ ) = \ '$, to an Abel equation: $zz' + 2kz + k^2 \ = \ (\ )$ (see Subsection 1.3.1).

If $k = 0$, the solution of the original equation for case 1 is as follows:

$$\frac{}{\overline{2 \ (\ ) + \ _1}} = \ \frac{}{} + \ _2, \qquad \text{where} \qquad (\ ) = \quad (\ ) \quad .$$

If $k = 0$, the solution of the original equation for case 2 is given by:

$$2 \quad \frac{}{\overline{8 \ (\ ) - \Delta \ ^2 + \ _1}} = \quad \frac{}{a \ ^2 + \ + \ } + \ _2, \qquad \text{where} \qquad (\ ) = \quad (\ ) \quad .$$

34. $+ \quad + \quad = \ ^{\lambda} \quad (\quad , \quad .$

1 . The substitution $= e^{\lambda}$ leads to an autonomous equation:

$$'' \ + (2\lambda + a) \ ' + (\lambda^2 + a\lambda + \ ) \ = \ \left(\ ' + \lambda \ , \ '' \ + 2\lambda \ ' + \lambda^2 \right. \quad .$$

2 . Particular solution: $= ke^{\lambda}$, where k is a root of the algebraic (transcendental) equation $k(\lambda^2 + a\lambda + \ ) = \ (k\lambda, k\lambda^2)$.

35. $_1(\quad)^2 + {}_2 \quad + {}_3 \quad + {}_4(\quad)^2 + {}_5 \quad + {}_6{}^2 = {}^\lambda \quad (\quad, \quad .$

The substitution $= e^{\lambda} \;\; {}^2$ leads to an autonomous equation.

36. $= {}_1(\quad, \quad + {}_2(\quad, \quad + {}^\lambda \; {}_3(\quad, \quad .$

The substitution $= e^{\lambda}$ leads to an autonomous equation.

37. $(\quad, {}^2 \quad - \quad + \quad = 0.$

Solution:

$$= {}_1 \ln \; + {}_2 + {}_3,$$

where ${}_2$ is an arbitrary constant and the constants ${}_1$ and ${}_3$ are related by the constraint $({}_1, {}_3) = 0$.

38. $(\quad, \quad - {}^2) = 0.$

1 . Solution:

$$= {}_1 \exp({}_3 \;) + {}_2 \exp(- {}_3 \;),$$

where the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint $({}_3^2, 4\;{}_1\;{}_2\;{}_3^2) = 0.$

2 . Solution:

$$= {}_1 \cos({}_3 \;) + {}_2 \sin({}_3 \;),$$

where the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint $(-{}_3^2, -({}_1^2 + {}_2^2)\;{}_3^2 \;= 0.$

39. $({}^3 \quad, \quad + {}^2) = 0.$

Solutions:

$$= \sqrt{{}_1{}^2 + 2\;{}_2 \; + {}_3},$$

where the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint $({}_1\;{}_3 - {}_2^2, {}_1) = 0.$

40. $\quad + \frac{{}^2}{\quad}, \quad + \frac{- \;{}^3}{2}\Big) = 0.$

Solutions can be found from the relation $(\; - {}_1)^2 = 2\;{}_2(\; - A) + {}_2^2$, where $({}_1, A) = 0$. The question of whether there are other solutions calls for further investigation.

41. $\frac{}{} + \quad , \;{}^{+1}\frac{}{} \quad = 0.$

A solution of this equation is any function that solves the first-order separable equation:

$$' = {}_1{}^{-} + {}_2,$$

where the constants ${}_1$ and ${}_2$ are related by the constraint $(a\;{}_2, -a\;{}_1) = 0.$

42. $\frac{}{} + \quad , \quad \frac{}{} \quad = 0.$

A solution of this equation is any function that solves the first-order separable equation:

$$' = {}_1 e^{-} + {}_2,$$

where the constants ${}_1$ and ${}_2$ are related by the constraint $({}_2, -{}_1) = 0.$

2.9.6-2. Equations containing arbitrary functions of three variables.

43. **(, − , ^{2} − 2 + 2) = 0.**

Solution:

$$= {}_1{}^2 + {}_2 + {}_3,$$

where the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint $(2\ {}_1, -\ {}_2, 2\ {}_3) = 0$.

Reference: E. L. Ince (1964).

44. **(, − , 2 − ( )^2 = 0.**

Solution:

$$= {}_1{}^2 + {}_2 + {}_3,$$

where the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint $(2\ {}_1, -\ {}_2, 4\ {}_1\ {}_3 - {}_2^2) = 0$.

45. **(^{3} , + 2 , ^{2} + − = 0.**

Solution:

$$= {}_1 + {}_2 + \frac{}{}{}_3,$$

where the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint $(2\ {}_3, 2\ {}_1, -\ {}_2) = 0$.

46. **($^{+2}$, + (+ 1) , ^{2} + − = 0.**

Solution:

$$= {}_1{}^- + {}_2 + {}_3.$$

The constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint $\big(a(a+1)\ {}_1, (a+1)\ {}_2, -a\ {}_3\ \big) = 0$.

47. **(^{3} , + ^{2}, + ^{2} − = 0.**

Solution:

$${}^2 = {}_1{}^2 + 2\ {}_2 + {}_3,$$

where the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint $({}_1\ {}_3 - {}_2^2, {}_1, -\ {}_2) = 0$.

48. **(, − , − , − = 0.**

The substitution $= {}' -$ leads to a first-order equation: $(, , {}' + , {}') = 0$.

49. **(, + , − ^{2} , + = 0.**

The substitution $= {}' + a$ leads to a first-order equation: $(, , {}' - a , {}') = 0$.

50. **$-\dfrac{{}^2+1}{}\ ,\ +\dfrac{{}^2+1}{},\ \dfrac{({}^2+1)^3\ {}^2}{}\Big) = 0.$**

It is known that all functions of the form $(- {}_1)^2 + (- {}_2)^2 = A^2$, where $A = A({}_1, {}_2)$ is determined from the algebraic (transcendental) equation $({}_1, {}_2, A) = 0$, are solutions of the original equation.

51. **(, + , − − (+ 1) = 0.**

Solution:

$$= {}_1 e^{-} + {}_2 + {}_3,$$

where the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint $({}_1, {}_2, {}_3) = 0$.

52. $\quad , \quad -(\quad)^2, \ (\quad + \quad \overline{\qquad} \quad \exp(- \quad \overline{\qquad} \quad = 0.$

Solution:

$$= \ _1 \exp(\ _3\ ) + \ _2 \exp(-\ _3\ ),$$

where the constants $_1$, $_2$, and $_3$ are related by the constraint $(\ ^2_3, 4\ _1\ _2\ ^2_3, 2\ _1) = 0.$

53. $(\quad - \ , \quad \sinh \quad - \quad \cosh \ , (\quad)^2 - (\quad)^2 \ = 0.$

Solution:

$$= \ _1 \sinh \ + \ _2 \cosh \ + \ _3,$$

where the constants $_1$, $_2$, and $_3$ are related by the constraint $(-\ _3, -\ _1,\ ^2_1 - \ ^2_2) = 0.$

54. $(\quad , \ ^2 \quad - \quad + \ , \quad - \quad \ln \quad = 0.$

Solution:

$$= \ _1 \ \ln \ + \ _2 \ + \ _3,$$

where the constants $_1$, $_2$, and $_3$ are related by the constraint $(\ _1,\ _3,\ _1 + \ _2) = 0.$

55. $\quad \frac{}{}, \quad - \ \frac{}{}, \quad \frac{}{} - \ln \frac{(\quad)^2}{}\Big) = 0.$

Solution:

$$= \ _1 \exp(\ _2\ ) + \ _3,$$

where the constants $_1$, $_2$, and $_3$ are related by the constraint $(\ _2, -\ _2\ _3, -\ln\ _1) = 0.$

56. $(\quad + \ , \quad \sin \quad - \quad \cos \ , (\quad)^2 + (\quad)^2 \ = 0.$

Solution:

$$= \ _1 \sin \ + \ _2 \cos \ + \ _3,$$

where the constants $_1$, $_2$, and $_3$ are related by the constraint $(\ _3, -\ _1,\ ^2_1 + \ ^2_2) = 0.$

57. $(\ , \quad ^2, \quad + \tfrac{1}{2} \quad = 0.$

The substitution $(\ ) = (\ ')^2$ leads to a first-order equation: $(\ ,\ , \tfrac{1}{2}\ '\ = 0.$

58. $(\ , \quad ^2, \quad + \tfrac{1}{2} \quad = 0, \quad = (\ ).$

The substitution $(\ ) = (\ )(\ ')^2$ leads to a first-order equation: $(\ ,\ , \tfrac{1}{2}\ '\ = 0.$

2.9.7. Some Transformations

1. $\quad + \ ^{-3} \quad \frac{1}{}, \ \frac{}{} \ = 0.$

The transformation $\xi = 1 \quad , \quad = \quad$ leads to the equation $\ '' + (\xi, \ ) = 0.$

2. $\quad = (\quad + 1)\ ^{-2} + \ ^3 \quad (\ ^{2\ +1}, \quad).$

The transformation $\xi = \ ^{2\ +1}, \quad = \quad$ leads to the equation $(2\ + 1)^2\ '' = (\xi, \ ).$

3. $\quad + (\quad + \ )^{-3} \quad \frac{+}{+}, \ \frac{}{+} \ = 0.$

The transformation $\xi = \dfrac{+}{a \ +}, \quad = \dfrac{}{a\ +}$ leads to the equation $\ '' + \Delta^{-2}\ (\xi, \ ) = 0,$ where $\Delta = a\ - \ .$

4. $\quad {}^{2} \quad + \quad + \quad + \quad (\;,\;) = 0.$

The transformation $= \xi \;,\; = \xi$, where the parameters and are found from the simultaneous algebraic equations

$$2 \;\; + 1 + (a-1) \;\; = 0, \qquad {}^{2} + (a-1) \quad + \quad {}^{2} = 0,$$

leads to an equation of the form

$$'' \; + \; {}^{2}\xi^{-} \; {}^{-2} \;\; (\xi \;, \xi \quad) = 0.$$

5. $\quad = \;(\;\; + 1) \;\; {}^{-2} \;+\; {}^{3} \;\; (\;\; {}^{2} \;\; {}^{+1} + \;, \qquad).$

The transformation $\xi = a \;\; {}^{2} \;\; {}^{+1} + \;, \quad = \quad$ leads to the equation $a^{2}(2 \;\; + 1)^{2} \;\; '' = \;\; (\xi, \;\;).$

6. $\quad = \;\; {}^{2} \;+\; {}^{3\lambda} \;\; (\;\; {}^{2\lambda} \;+\; , \;\; {}^{\lambda} \;\;).$

The transformation $\xi = ae^{2\lambda} \;\; + \;, \quad = e^{\lambda}$ leads to the equation $'' = (2a\lambda)^{-2} \;\; (\xi, \;\;).$

7. $$= \;\; {}^{2} \;+\; \frac{{}^{3\lambda}}{(\;\; {}^{2\lambda} \;+\;)^{3}} \qquad \frac{{}^{2\lambda} \;+\;}{{}^{2\lambda} \;+\;}, \; \frac{{}^{\lambda}}{{}^{2\lambda} \;+\;}\Bigg).$$

The transformation $\xi = \dfrac{ae^{2\lambda} \;\; +}{e^{2\lambda} \;\; +}, \quad = \dfrac{e^{\lambda}}{e^{2\lambda} \;\; +}$ leads to the equation

$$'' = (2\Delta\lambda)^{-2} \;\; (\xi, \;\;), \qquad \text{where} \quad \Delta = a \;\; - \quad .$$

8. $$= \;\; {}^{2} \;\; + \sinh^{-3}(\quad) \qquad \coth(\quad), \; \frac{}{\sinh(\quad)}\Bigg).$$

The transformation $\xi = \coth(\lambda \;\;), \quad = \dfrac{}{\sinh(\lambda \;\;)}$ leads to the equation $'' = \lambda^{-2} \;\; (\xi, \;\;).$

9. $$= \;\; {}^{2} \;\; + \cosh^{-3}(\quad) \qquad \tanh(\quad), \; \frac{}{\cosh(\quad)}\Bigg).$$

The transformation $\xi = \tanh(\lambda \;\;), \quad = \dfrac{}{\cosh(\lambda \;\;)}$ leads to the equation $'' = \lambda^{-2} \;\; (\xi, \;\;).$

10. $$ {}^{2} \quad + \tfrac{1}{4} \;\; + \quad \overline{} \qquad \ln \quad + \;\; , \; \frac{}{}\Bigg) = 0.$$

The transformation $\xi = a \ln \;\; + \;, \quad = \dfrac{}{}$ leads to the equation $'' + a^{-2} \;\; (\xi, \;\;) = 0.$

11. $$| \;\; {}^{2} - 1|^{3} \;\; {}^{2} \qquad = \qquad \ln \frac{-}{+1}, \; \frac{}{\overline{| \;\; {}^{2} - 1|}}\Bigg).$$

The transformation $\xi = \ln \dfrac{a \;\; - a}{+1}, \quad = \dfrac{}{\overline{| \;\; {}^{2} - 1|}}$ leads to the equation $4 \;\; '' = \;\; (\xi, \;\;) + \quad .$

12. $$+ \;\; {}^{2} \;\; + \sin^{-3}(\quad) \qquad \cot(\quad), \; \frac{}{\sin(\quad)}\Bigg) = 0.$$

The transformation $\xi = \cot(\lambda \;\;), \quad = \dfrac{}{\sin(\lambda \;\;)}$ leads to the equation $'' + \lambda^{-2} \;\; (\xi, \;\;) = 0.$

13. $$+ \;\; {}^{2} \;\; + \cos^{-3}(\quad) \qquad \tan(\quad), \; \frac{}{\cos(\quad)}\Bigg) = 0.$$

The transformation $\xi = \tan(\lambda \;\;), \quad = \dfrac{}{\cos(\lambda \;\;)}$ leads to the equation $'' + \lambda^{-2} \;\; (\xi, \;\;) = 0.$

14. $\;\; + \;\;^{2} \;\; + \sin^{-3}(\;\; + \;\;) \;\; \left(\frac{\sin(\;\; + \;\;)}{\sin(\;\; + \;\;)}, \frac{\;}{\sin(\;\; + \;\;)} \right) = 0.$

The transformation $\xi = \dfrac{\sin(\lambda \;\; + a)}{\sin(\lambda \;\; + \;\;)}, \quad = \dfrac{\;}{\sin(\lambda \;\; + \;\;)}$ leads to the equation

$$'' \;\; + [\lambda \sin(\;\; - a)]^{-2} \;\; (\xi, \;\;) = 0.$$

15. $(\;^{2} + 1)^{3 \;\; 2} \;\; + \;\; \arctan \;\; + \;\;, \frac{\;}{\overline{\;^{2} + 1}} \Big) = 0.$

The transformation $\xi = \arctan \;\; + \;\;, \quad = \dfrac{\;}{\overline{\;^{2} + 1}}$ leads to the equation $\;'' + \;\; + \;\; (\xi, \;\;) = 0.$

16. $(\;^{2} + 1)^{3 \;\; 2} \;\; + \;\; \operatorname{arccot} \;\; + \;\;, \frac{\;}{\overline{\;^{2} + 1}} \Big) = 0.$

The transformation $\xi = \operatorname{arccot} \;\; + \;\;, \quad = \dfrac{\;}{\overline{\;^{2} + 1}}$ leads to the equation $\;'' + \;\; + \;\; (\xi, \;\;) = 0.$

17. $\;\; + \;\; (\;,\; \;) = 0.$

The transformation $\;\; = \;\; (z), \quad = \;\; \overline{a \;\;'}$ leads to the equation

$$'' + \frac{1}{2} \frac{'''}{'} - \frac{3}{4} \frac{''}{'} \;^{2} \;\; + a^{-2}(a \;\;')^{3 \;\; 2} \;\; (\;,\; \;\; \overline{a \;\;'} \;\; = 0.$$

The sign of the parameter a must coincide with that of the derivative $\;\;'$.

18. $\;\; + f(\;,\; \;)(\;\;)^{3} + \;\; (\;,\; \;)(\;\;)^{2} = 0.$

Taking $\;$ to be the independent variable, we obtain the following equation with respect to $\;\; = \;\; (\;):\;\; '' - g(\;,\; \;) \;\;' - \;\; (\;,\; \;) = 0.$

19. $\;\; (\;,\; \;,\; \;,\; \;) = 0.$

Applying the Legendre transformation $\;\; = \;\;', \;\; = \;\;' - \;\;,$ where $\;\; = \;\; (\;),$ and using the relations $\;' = \;\;$ and $\;'' = 1 \;\; ''$, we arrive at the equation

$$', \;\; ' - \;\; , \;, \frac{1}{''} \;\; = 0.$$

Given a solution of the original equation, the corresponding solution of the transformed equation is written in parametric form as:

$$= \;\;', \qquad = \;\;' - \;\;, \qquad \text{where} \qquad = \;\; (\;).$$

Chapter 3

Third-Order Differential Equations

3.1. Linear Equations

3.1.1. Preliminary Remarks

1 . A homogeneous linear equation of the third order has the general form

$$ {}_3(\ )\ ''' \ +\ {}_2(\ )\ '' \ +\ {}_1(\ )\ ' +\ {}_0(\ )\ = 0. \qquad (1)$$

Let ${}_0 = {}_0(\ )$ be a nontrivial particular solution of this equation. The substitution

$$ = {}_0(\ )\quad z(\ )$$

leads to a second-order linear equation:

$$ {}_3\ {}_0 z'' + (3\ {}_3\ {}_0' + \ {}_2\ {}_0)z' + (3\ {}_3\ {}_0'' + 2\ {}_2\ {}_0' + \ {}_1\ {}_0)z = 0, \qquad (2)$$

where the prime denotes differentiation with respect to .

2 . Let ${}_1 = {}_1(\ )$ and ${}_2 = {}_2(\ )$ be two nontrivial linearly independent particular solutions of equation (1). Then the general solution of this equation can be written in the form:

$$ = \ {}_1\ {}_1 + \ {}_2\ {}_2 + \ {}_3\ \ {}_2\ \ {}_1\ \ -\ {}_1\ \ {}_2\ \ , \qquad (3)$$

where

$$ = \exp\ -\ \frac{{}_2}{{}_3}\ \ (\ {}_1\ {}_2' - \ {}_1'\ {}_2\ ^{-2}.$$

For specific equations described below in 3.1.2–3.1.9, often only particular solutions will be given, and the general solution can be obtained by formula (3).

3 . A nonhomogeneous linear equation of the third-order has the form

$$ {}_3(\ )\ ''' \ +\ {}_2(\ )\ '' \ +\ {}_1(\ )\ ' +\ {}_0(\ )\ = g(\ ). \qquad (4)$$

Let ${}_1 = {}_1(\ )$ and ${}_2 = {}_2(\ )$ be two linearly independent particular solutions of the corresponding homogeneous equation (1). Then the general solution of equation (4) is defined by formula (3) with:

$$ = \Delta^{-2} e^{-F}\ \ 1 + \frac{1}{{}_3}\ \ \frac{g}{{}_3}\Delta e^{F}\ \ \ ,\quad \text{where}\quad = \ \frac{{}_2}{{}_3}\ ,\ \ \Delta = \ {}_1\ {}_2' - \ {}_1'\ {}_2.$$

4 . The substitution $= z\exp\ -\frac{1}{3}\ \ \frac{{}_2}{{}_3}$ reduces equation (1) to a form from which the second derivative is absent:

$$ z''' + \big(-\ {}_2' - \tfrac{1}{3}\ {}_2^2 + \ {}_1\ z' + \big(-\tfrac{1}{3}\ {}_2'' - \tfrac{1}{3}\ {}_1\ {}_2 + \tfrac{2}{27}\ {}_2^3 + \ {}_0\ z = 0,$$

where $= \ \ {}_3\ (k = 0,\ 1,\ 2).$

3.1.2. Equations Containing Power Functions

3.1.2-1. Equations of the form $_3(\;)\;''' + \;_0(\;)\; = g(\;)$.

1. $+ \quad = 0.$

Solution:

$$= \begin{cases} _1 + \;_2 \; + \;_3 \;^2 & \text{if } \lambda = 0, \\ _1\exp(-k\;) + \;_2\exp\left(\tfrac{1}{2}k\right)\cos\left(\tfrac{3}{2}k\right) + \;_3\exp\left(\tfrac{1}{2}k\right)\sin\left(\tfrac{3}{2}k\right) & \text{if } \lambda \neq 0, \end{cases}$$

where $k = \lambda^{1\;3}$.

2. $+ \quad = \;^2 + \quad + \;.$

Solution: $= \; + \dfrac{1}{\lambda}(a\;^2 + \quad + \;)$, where is the general solution of the equation 3.1.2.1: $''' + \lambda \; = 0$.

3. $= \quad + \;.$

This is a special case of equation 5.1.2.3 with $= 3$.

4. $+ (\quad + \;) \; = 0.$

For $a = 0$, this is an equation of the form 3.1.2.1. For $a \neq 0$, the substitution $a\xi = a \; +$ leads to an equation of the form 3.1.2.3: $''' + a\xi \; = 0$.

5. $+ \;^3 \; = \;.$

The substitution $\xi = \;^2$ leads to an equation of the form 3.1.2.126: $2\xi \; ''' + 3 \; '' + \frac{1}{4}a\xi \; = \frac{1}{4}\;$.

6. $+ (3\;^2 - \;^3\;^3) \; = 0.$

Integrating, we obtain a second-order nonhomogeneous linear equation: $'' + a \; ' + (a^2\;^2 - a) \; = \; \exp\left(\tfrac{1}{2}a\;^2\right)$ (see 2.1.2.31 for the solution of the corresponding homogeneous equation).

7. $= \quad .$

This is a special case of equation 5.1.2.4 with $= 3$.

1 . For $\beta = -9, -7, -6, -9\;2, -3, -3\;2, 1$, and 3, see equations 3.1.2.17, 3.1.2.14, 3.1.2.11, 3.1.2.19, 3.1.2.10, 3.1.2.18, 3.1.2.3, and 3.1.2.5, respectively.

2 . The transformation $= \;^{-1}$, $= \;^{-2}$ leads to an equation of similar form: $''' = -a \;^{-\;-6}\;$.

3 . For $\beta \neq -3$, the transformation $\xi = \;^{(\;+3)\;3}$, $= \;^3$ leads to an equation of the form 3.1.2.69: $\xi^3 \; ''' + (1 - \;^2)\xi \; ' + (\;^2 - 1 - a\;^3\xi^3) \; = 0$, where $= \frac{3}{+3}$.

8. $+ [\;^3\;^3 - 3\;^2\;^2\;^{-1} + \;(\; - 1)\;^{-2}] \; = 0.$

Particular solution: $_0 = \exp\left(-\dfrac{a\;^{+1}}{+1}\right)$. The substitution $= \exp\left(-\dfrac{a\;^{+1}}{+1}\right) z(\;)$ leads to a second-order linear equation of the form 2.1.2.47: $z'' - 3a \; z' + (3a^2\;^2 - 3a\;^{-1})z = 0$.

9. $- \;^2(\; + 3) \; = 0.$

Particular solution: $_0 = \; e\;$. The substitution $= \; e \; z(\;)$ leads to a second-order equation of the form 2.1.2.108: $z'' + 3(a\; + 1)z' + 3a(a\; + 2)z = 0$.

10. $^{3} \quad = (\ ^{2} - 1)\ .$

This is a special case of equation 3.1.2.175. Solution: $= (\ _1\ ^{1} + \ _2\ ^{2} + \ _3\ )$, where $_1$ and $_2$ are roots of the quadratic equation $^{2} + a \ + a^2 - 1 = 0$.

11. $^{6} \quad = \quad + \ ^{2}.$

The transformation $= \ ^{-1}, \ = \ ^{-2}$ leads to a constant coefficient linear equation of the form 3.1.2.2: $''' + a \ + \ = 0$.

12. $(\ -\ )^3(\ -\ )^3 \quad - \quad = 0, \qquad \neq \ .$

The transformation $= \ln \dfrac{-a}{-}$, $= \dfrac{}{(\ -\ )^2}$ leads to a constant coefficient linear equation: $(a -\ )^3(\ ''' - 3\ '' + 2\ ') - \quad = 0$.

13. $(\ ^{2} + \ + \ )^3 \quad = \ .$

The transformation $\xi = \dfrac{}{a\ ^{2} + \ + }$, $= \dfrac{}{a\ ^{2} + \ + }$ leads to a constant coefficient linear equation: $''' + (4a \ - \ ^{2})\ ' = k\ .$

14. $^{7} \quad = \quad + \ ^{3}.$

The transformation $= \ ^{-1}, \ = \ ^{-2}$ leads to a linear equation of the form 3.1.2.3: $''' + a \ + \ = 0$.

15. $^{7} \quad + (\ + \ ) \ = 0.$

The transformation $= \ ^{-1}, \ = \ ^{-2}$ leads to a linear equation of the form 3.1.2.4: $''' - (\ + a) \ = 0$.

16. $^{9} \quad + (\ ^{3} - 3\ ^{2}\ ^{2}) \ = 0.$

The transformation $= \ ^{-1}, \ = \ ^{-2}$ leads to a linear equation of the form 3.1.2.6: $''' + (3a^2 \ - a^3\ ^{3}) \ = 0$.

17. $^{9} \quad = \ .$

The transformation $= \ ^{-1}, \ = \ ^{-2}$ leads to an equation of the form 3.1.2.5: $''' + a\ ^{3} \ = 0$.

18. $^{3\ 2} \quad = \ .$

This is a special case of equation 5.1.2.8 with $\ = 1$.

19. $^{9\ 2} \quad = \ .$

This is a special case of equation 5.1.2.9 with $\ = 1$.

$\boxed{\text{3.1.2-2. Equations of the form } \ _3(\ )\ ''' + \ _1(\ )\ ' + \ _0(\ )\ = g(\ ).}$

20. $\quad + \quad + \ ^{2}\ (3 - \ - \ ^{2}) \ = 0.$

Integrating, we obtain a second-order nonhomogeneous linear equation: $'' + a\ ' + (a^2\ ^{2} + a\ - a) \ = \ \exp\left(\tfrac{1}{2}a\ ^{2}\right.$ (see 2.1.2.31 for the solution of the corresponding homogeneous equation).

21. $\quad + \quad + \quad = 0, \qquad = 1, 2, 3, \ldots$

Solution: $= \ ^{(\ -1)}$, where is the solution of the second-order linear equation $'' + a \ = \ .$

22. $\quad + \quad - 2 \quad = 0.$

The substitution $= \;' - 2$ leads to a second-order linear equation of the form 2.1.2.2: $'' + a \quad = 0.$

23. $\quad + \quad + (\quad + {}^2) \quad = 0.$

Particular solution: $_0 = e^{-b}$. The substitution $= \;' +$ leads to a second-order linear equation of the form 2.1.2.12: $'' - \;' + (a \; + {}^2) \quad = 0.$

24. $\quad + \quad + (\quad + \quad + {}^3) \quad = 0.$

Integrating yields a second-order linear equation: $'' - \;' + (a \; + {}^2) \; = \; e^{-b}$ (see 2.1.2.108 for the solution of the corresponding homogeneous equation with $= 0$).

25. $\quad + (\quad + \;) \quad + \quad = 0.$

Integrating yields a second-order nonhomogeneous linear equation: $'' + (a \; + \;) \; =$ (see 2.1.2.2 for the solution of the corresponding homogeneous equation).

26. $\quad + (\quad + \;) \quad - \quad = 0.$

Particular solution: $_0 = a \; + \;$. The transformation $\xi = a \; + \;, \; z = \dfrac{'}{a \; +} - \dfrac{a}{(a \; + \;)^2}$ leads to a second-order linear equation of the form 2.1.2.67: $\xi z'' + 3z' + a^{-2}\xi^2 z = 0.$

27. $\quad + (\quad + \;) \quad + 3 \quad = 0.$

The substitution $a\xi = a \; +$ leads to a linear equation of the form 3.1.2.21 with $= 3$: $''' + a\xi \;' + 3a \; = 0.$

28. $\quad + (2 \quad + \;) \quad + \quad = 0.$

The substitution $a\xi = a \; + \frac{1}{2}$ leads to a linear equation of the form 3.1.2.48 with $= 1$: $''' + 2a\xi \;' + a \; = 0.$

29. $\quad + (\quad - {}^2) \quad + \quad = 0.$

The substitution $= \;' +$ leads to a second-order linear equation of the form 2.1.2.108: $'' - \;' + a \quad = 0.$

30. $\quad + (\quad - {}^2) \quad + (\quad + 1) \quad = 0.$

Integrating yields a second-order linear equation: $'' - \;' + a \quad = \; e^{-b}$ (see 2.1.2.108 for the solution of the corresponding homogeneous equation with $= 0$).

31. $\quad + (\quad + \;) \quad + (\quad + \quad + {}^2) \quad = 0.$

The substitution $= \;' +$ leads to a second-order linear equation of the form 2.1.2.12: $'' - \;' + (a \; + \; + {}^2) \quad = 0.$

32. $\quad + (\quad + \;) \quad + (\; {}^2 \; {}^2 + \quad + \; - 3 \;) \quad = 0.$

Particular solution: $_0 = \exp\bigl(-\frac{1}{2} \; {}^2$. The substitution $= \exp\bigl(-\frac{1}{2} \; {}^2 \quad z(\;)$ leads to a second-order linear equation of the form 2.1.2.31: $z'' - 3 \quad z' + (3 \; {}^2 \; {}^2 + a \; + \; - 3 \;)z = 0.$

33. $\quad + \; {}^2 \quad + \quad = 0.$

This is a special case of equation 3.1.2.47 with $= 1$.

34. $\quad + \quad ^2 \quad -2 \quad = 0.$

The substitution $\quad = \quad ' - 2 \quad$ leads to a second-order linear equation of the form 2.1.2.7: $\quad '' \; + a \quad ^2 \quad = 0.$

35. $\quad - \quad ^2 \; ^2 \quad + \; ^2 \quad = 0.$

Integrating yields a second-order linear equation: $\quad '' \; + a \quad ' - a \quad = \quad \exp\bigl(\tfrac{1}{2}a \quad ^2$ (see 2.1.2.108 for the solution of the corresponding homogeneous equation with $\quad = 0$).

36. $\quad + \quad ^2 \quad + \; (\quad ^2 + \; ^2) \quad = 0.$

The substitution $\quad = \; ' + \quad$ leads to a second-order linear equation of the form 2.1.2.31: $\quad '' \; - \quad ' + (a \; ^2 + \; ^2) \quad = 0.$

37. $\quad + (\quad -1) \; ^2 \; ^2 \quad + \; ^2 \; (\quad ^2 + 2 \quad + 1) \quad = 0.$

Integrating, we obtain a second-order nonhomogeneous linear equation: $\quad '' \; - \quad ' + (a \; ^2 \; ^2 + \;) \quad = \quad \exp\bigl(-\tfrac{1}{2} \quad ^2$ (see 2.1.2.31 for the solution of the corresponding homogeneous equation).

38. $\quad + (\quad ^2 + \;) \quad + 2 \quad = 0.$

Integrating yields a second-order nonhomogeneous linear equation: $\quad '' \; + (a \; ^2 + \;) \quad =$ (see 2.1.2.4 for the solution of the corresponding homogeneous equation).

39. $\quad + (\quad ^2 - \; ^2) \quad + \quad (2 - \quad) \quad = 0.$

Integrating yields a second-order linear equation: $\quad '' \; + \quad ' + a \quad ^2 \quad = \quad e^b$ (see 2.1.2.13 for the solution of the corresponding homogeneous equation with $\quad = 0$).

40. $\quad + (\quad ^2 + \;) \quad + \; (\quad ^2 + \quad + \; ^2) \quad = 0.$

The substitution $\quad = \; ' + \quad$ leads to a second-order linear equation of the form 2.1.2.13: $\quad '' \; - \quad ' + (a \; ^2 + \quad + \; ^2) \quad = 0.$

41. $\quad - (3 \; ^2 \; ^2 + \quad + 3 \;) \quad + 2 \quad (\; ^2 \; ^2 - \;) \quad = 0.$

1 . Particular solutions with $a > 0$: $\quad _1 = \exp\bigl(\tfrac{1}{2} \quad ^2 + \quad \overline{a} \quad , \quad _2 = \exp\bigl(\tfrac{1}{2} \quad ^2 - \quad \overline{a} \quad .$

2 . Particular solutions with $a < 0$: $\quad _1 = \exp\bigl(\tfrac{1}{2} \quad ^2 \; \cos\bigl(\quad \overline{|a|} \quad , \quad _2 = \exp\bigl(\tfrac{1}{2} \quad ^2 \; \sin\bigl(\quad \overline{|a|} \quad .$

3 . Particular solutions with $a = 0$: $\quad _1 = \exp\bigl(\tfrac{1}{2} \quad ^2 \quad , \quad _2 = \quad \exp\bigl(\tfrac{1}{2} \quad ^2 \quad .$

42. $\quad + (\quad ^2 + \quad + \;) \quad + \quad [(\quad + \; ^2) \; ^2 + \quad + \quad -3 \quad] \quad = 0.$

Particular solution: $\quad _0 = \exp\bigl(-\tfrac{1}{2}k \quad ^2$. The substitution $\quad = \exp\bigl(-\tfrac{1}{2}k \quad ^2 \quad z(\;)$ leads to a second-order equation of the form 2.1.2.31: $z'' \; -3k \quad z' + [(a+3k^2) \; ^2 + \quad + \quad -3k]z = 0.$

43. $\quad + (\quad ^4 + \quad) \quad -2(\quad ^3 + \;) \quad = 0.$

This is a special case of equation 3.1.2.49 with $\quad = 2.$

44. $\quad + \quad -2 \quad ^{-1} \quad = 0.$

The substitution $\quad = \quad ' - 2 \quad$ leads to a second-order linear equation of the form 2.1.2.7: $\quad '' \; + a \quad = 0.$

45. $\quad + \quad + \quad ^{-1} \quad = 0.$

Integrating yields a second-order nonhomogeneous linear equation: $\quad '' \; + a \quad =$ (see 2.1.2.7 for the solution of the corresponding homogeneous equation).

46. $+ \quad {}^{+1} \quad + \; (\quad + 3) \qquad = 0.$

The substitution $= {}^{-1}, \quad = \quad {}^{-2}$ leads to an equation of the form 3.1.2.45 with $= - \quad - 5$: $''' + a \; {}^{-} \; {}^{-5} \; {}' - a(\quad + 5) \; {}^{-} \; {}^{-6} \quad = 0.$

47. $+ \quad {}^{2} \quad + \quad {}^{2} \; {}^{-1} \quad = 0.$

Solution: $= {}_{1} \quad {}^{2}(\;) + {}_{2} \quad (\;) \quad (\;) + {}_{3} \quad {}^{2}(\;)$, where $= \dfrac{1}{2(\; + 1)}, \quad = \dfrac{\overline{a} \quad {}^{+1}}{2(\; + 1)}$; $(\;)$ and $(\;)$ are the Bessel functions.

48. $+ 2 \qquad + \qquad {}^{-1} \quad = 0.$

Solution: $= {}_{1} \; {}^{2}_{1} + {}_{2} \; {}_{1} \; {}_{2} + {}_{3} \; {}^{2}_{2}$. Here, ${}_{1}$ and ${}_{2}$ are a fundamental set of solutions of a second-order linear equation of the form 2.1.2.7: $2 \quad '' + a \qquad = 0.$

49. $+ (\quad {}^{2} + \qquad {}^{-1}) \quad - 2(\quad {}^{2} \; {}^{-1} + \qquad {}^{-2}) \quad = 0.$

The substitution $= \quad ' - 2$ leads to a second-order linear equation of the form 2.1.2.10: $'' + (a \; {}^{2} + \qquad {}^{-1}) \quad = 0.$

50. $+ (\quad {}^{2} + \qquad {}^{-1}) \quad - \; [(\quad + \; {}^{2}) \; {}^{3} \; + (\; + 3 \quad) \; {}^{2} \; {}^{-1} + \; (\quad - 1) \quad {}^{-2}] \quad = 0.$

Particular solution: ${}_{0} = \exp \dfrac{+1}{+1}$. The substitution $= \exp \dfrac{+1}{+1} \qquad z(\;)$ leads to a second-order equation of the form 2.1.2.47: $z'' + 3 \qquad z' + [(a + 3 \; {}^{2}) \; {}^{2} + (\; + 3 \quad) \quad {}^{-1}] z = 0.$

51. $+ \qquad + \; (\; {}^{2} \quad + \quad) \quad = 0.$

The substitution $= \; ' + $ leads to a second-order linear equation of the form 2.1.2.108: $'' - \qquad ' + (\; {}^{2} \; + a) \quad = 0.$

52. $+ \qquad - [\; (\quad + \; {}^{2}) \quad + \quad + 3 \; {}^{2}] \quad = 0.$

Particular solution: ${}_{0} = \; e^{b}$. The substitution $= \; e^{b} \qquad z(\;)$ leads to a second-order linear equation of the form 2.1.2.108: $z'' + 3(\quad + 1) z' + [(a + 3 \; {}^{2}) \; + 6 \;] z = 0.$

53. $+ (\; - \; {}^{2} \;) \quad + \qquad = 0.$

The substitution $= \; ' + a$ leads to a second-order linear equation of the form 2.1.2.108: $'' - a \quad ' + \qquad = 0.$

54. $+ (\quad {}^{2} + \quad) \quad - 2(\quad + \;) \quad = 0.$

The substitution $= \quad ' - 2$ leads to a second-order linear equation of the form 2.1.2.2: $'' + (a \; + \;) \quad = 0.$

55. $+ (\quad {}^{3} + \quad) \quad - 2(\quad {}^{2} + \;) \quad = 0.$

The substitution $= \quad ' - 2$ leads to a second-order linear equation of the form 2.1.2.4: $'' + (a \; {}^{2} + \;) \quad = 0.$

56. $(\quad + \;) \qquad + \qquad + \; (\quad {}^{2} \; + \quad {}^{2} + \;) \quad = 0.$

The substitution $= \; ' + k$ leads to a second-order linear equation of the form 2.1.2.108: $(a \; + \;) \; '' - k(a \; + \;) \; ' + (ak^{2} \; + \; k^{2} + \;) \quad = 0.$

57. $(\quad + 2) \qquad - \; {}^{3} \qquad + 2 \; {}^{3} \quad = 0.$

Particular solutions: ${}_{1} = \; {}^{2}, \quad {}_{2} = e^{-}$.

58. $(\quad + \quad - \quad) \qquad - {}^3(\quad + \quad) \quad + \quad {}^3 \quad = 0.$

Particular solutions: $_1 = a \quad + \quad, \quad _2 = e^c \quad.$

59. $(\quad + \quad) \qquad + (\quad + \quad) \quad + s[(\quad s^2 + \quad) \quad + \quad s^2 + \quad] \quad = 0.$

The substitution $\quad = \quad ' + \quad$ leads to a second-order linear equation of the form 2.1.2.108: $(a \quad + \quad) \quad '' - (a \quad + \quad) \quad ' + [(a \ ^2 + \quad) \quad + \quad ^2 + \quad] \quad = 0.$

60. $(\quad + \quad) \qquad + [(\quad - \quad ^2) \quad + \quad - \quad ^2] \quad + \quad (\quad + \quad) \quad = 0.$

The substitution $\quad = \quad ' + k$ leads to a second-order linear equation of the form 2.1.2.108: $(a \quad + \quad) \quad '' - k(a \quad + \quad) \quad ' + (\quad + \quad) \quad = 0.$

61. $(\quad + \quad) \qquad - (\ ^3 \ ^3 - 3 \ ^2 \quad + \ ^3) \quad + \quad (\ ^2 \ ^2 - 3 \quad - \ ^2) \quad = 0.$

Particular solutions: $_1 = e^b \quad, \quad _2 = \exp\left(-\tfrac{1}{2}a \quad ^2\right.$.

62. $^2 \qquad - 6 \quad + \quad ^2 \quad + 2 \quad = 0.$

The substitution $\quad = \quad ^2$ leads to an equation of the form 3.1.2.173: $^3 \quad ''' \quad + 6 \ ^2 \quad '' \ +$ $(a \ ^3 - 12) \quad + 2 \quad = 0.$

63. $^2 \qquad + (\quad ^2 + \quad - \quad ^2 - \quad) \quad + (\quad - 1)(\quad + \quad) \quad = 0.$

The substitution $\quad = \quad ' + (\quad - 1)$ leads to a second-order linear equation of the form 2.1.2.108: $'' \ - (\quad + 1) \quad ' + (a \quad + \quad) \quad = 0.$

64. $^2 \qquad + (\quad ^2 + \quad + \quad) \quad - \quad [(\quad + \quad ^2) \ ^2 + \quad + \quad] \quad = 0.$

The substitution $\quad = \quad ' - k$ leads to a second-order linear equation of the form 2.1.2.135: $^2 \quad '' \ + k \ ^2 \quad ' + [(a + k^2) \ ^2 + \quad + \quad] \quad = 0.$

65. $^2 \qquad + (\qquad - \ ^2 - \quad) \quad + \quad (\quad - 1) \quad ^{-1} \quad = 0.$

The substitution $\quad = \quad ' + (\quad - 1)$ leads to a second-order equation of the form 2.1.2.67: $'' \ - (\quad + 1) \quad ' + a \quad ^{-1} \quad = 0.$

66. $^2 \qquad + (\quad ^{+1} - \ ^2 - \quad) \quad + \quad (\quad - 1) \qquad = 0.$

The substitution $\quad = \quad ' + (\quad - 1)$ leads to a second-order linear equation of the form 2.1.2.67: $'' \ - (\quad + 1) \quad ' + a \qquad = 0.$

67. $^2 \qquad - 3 \quad ^{+1}(\quad + \quad ^{+1}) \quad + \quad (\quad - \quad ^2 + 2 \ ^2 \ ^2 \ ^{+2}) \quad = 0.$

1 . Particular solutions with $\quad \neq -1$: $\quad _1 = \exp \dfrac{a \quad ^{+1}}{\quad + 1}, \quad _2 = \quad \exp \dfrac{a \quad ^{+1}}{\quad + 1}.$

2 . Particular solutions with $\quad = -1$: $\quad _1 = \quad, \quad _2 = \quad ^{+1}.$

68. $(\quad + \quad) \qquad + \quad (\quad + \quad) \quad - 2(\quad + \quad) \quad = 0.$

The substitution $\quad = \quad ' - 2$ leads to a second-order linear equation of the form 2.1.2.108: $(a \quad + \quad) \quad '' + (\quad + \quad) \quad = 0.$

69. $^3 \qquad + (1 - \quad ^2) \qquad + (\quad ^3 + \quad ^2 - 1) \quad = 0.$

For $a = \quad 1$, we have a constant coefficient equation of the form 3.1.2.1. For $\quad = 0$, we obtain the Euler equation 3.1.2.175.

1 . If $\neq 0$ and a is a positive integer greater than 1, then the solution is:

$$= \quad {}^{1-} \sum_{=1}^{3} \exp(-\lambda \quad)\quad(\ ),$$

where λ_1, λ_2, and λ_3 are roots of the cubic equation $\lambda^3 =$ and $(\ )$ are polynomials of degree $\leq 3(a-1)$.

2 . Denote the solution of the original equation for arbitrary (including complex) a by . Then the following recurrence relation holds:

$$_{+3} = \quad + (2a+3) \quad ^{-1} \quad '' - (a+1)(2a+3)(\quad ^{-2} \quad ' - \quad ^{-3} \quad), \tag{1}$$

where the prime denotes differentiation with respect to .

Since the functions $_1 = e^{-\lambda}$, corresponding to three values of λ determined by the equation $\lambda^3 =$, form a fundamental set of solutions, formula (1) makes it possible to find all for any integer values of not divisible by 3. In particular, $_2 = (\ ^{-1} + \lambda)e^{-\lambda}$, where $\lambda^3 =$.

70. $^3 \quad + (4 \ ^3 + \quad) \quad - \quad = 0.$

Solution: $= \ _1 \ ^2(\ ) + \ _2 \quad (\ ) \quad (\ ) + \ _3 \ ^2(\ )$, where $(\ )$ and $(\ )$ are the Bessel functions; $4 \ ^2 = 1 - a$.

71. $^3 \quad + [\ ^2 + 3\ (1 - \ )] \quad + 2\ (\ ^2 + \ ^2 - 1) \quad = 0.$

1 . Particular solutions with $a > 0$: $_1 = \ ^b \sin(\ \overline{a}\ $, $_2 = \ ^b \cos(\ \overline{a}\ $.

2 . Particular solutions with $a < 0$: $_1 = \ ^b \exp(- \ \overline{-a}\ $, $_2 = \ ^b \exp(\ \overline{-a}\ $.

3 . Particular solutions with $a = 0$: $_1 = \ ^b$, $_2 = \ ^{b+1}$.

72. $^3 \quad + (\ ^2 + \quad + \ ) \quad + (\ - 1)(\ ^2 + \quad + \ + \ ^2 + \ ) \quad = 0.$

The substitution $= \ ' + (k-1)$ leads to a second-order linear equation of the form 2.1.2.131: $^2 \ '' - (k+1) \quad ' + (a \ ^2 + \quad + \ + k^2 + k) \quad = 0$.

73. $^3 \quad + \quad + (\ - 1)(\ ^{-1} + \ ^2 + \ ) \quad = 0.$

The substitution $= \ ' + (\ - 1)$ leads to a second-order linear equation of the form 2.1.2.132: $^2 \ '' - (\ + 1) \quad ' + (a \ ^{-1} + \ ^2 + \ ) \quad = 0$.

74. $^3 \quad + (\quad + \ ) \quad - 2(\quad + \ ) \quad = 0.$

The substitution $= \ ' - 2$ leads to a second-order linear equation of the form 2.1.2.118: $^2 \ '' + (a \quad + \ ) \quad = 0$.

75. $^3 \quad + (\quad + \ - \ ) \quad + (\ - 1)(\quad + \ + \ ^2) \quad = 0.$

The substitution $= \ ' + (\ - 1)$ leads to a second-order linear equation of the form 2.1.2.132: $^2 \ '' - (\ + 1) \quad ' + (a \quad + \ + \ ^2) \quad = 0$.

76. $^3 \quad + (\ ^2 \ + 1 - \ ^2) \quad + [\ ^3 + \ (\ - 1) \ ^2 + \ ^2 - 1] \quad = 0.$

The transformation $\xi =$, $z = \ ^{-1}$ leads to a constant coefficient linear equation: $z''' + az' + \ z = 0$.

77. $^2(\quad + \ ) \quad + (\ - \ ^2 - \quad) \quad + (\ - 1)(\ + \ ^2 + \quad) \quad = 0.$

The substitution $= \ ' + (\ - 1)$ leads to a second-order linear equation of the form 2.1.2.172: $(a \ + \ ) \ '' - (\ + 1)(a \ + \ ) \ ' + (\ + a \ ^2 + a \ ) \quad = 0$.

78. $(\ {}^2 + \ \ + \ )\ \ \ + \ \ \ - 2\ \ = 0.$

The substitution $= {}' - 2$ leads to a second-order linear equation of the form 2.1.2.179: $(a\ {}^2 + \ \ + \ )\ {}'' + \ \ = 0.$

79. ${}^5\ \ \ = (\ \ \ - 2\ ).$

Solution: $= \begin{cases} {}^2[\ {}_1 + \ {}_2 \exp(\ \overline{a}\ \ ) + \ {}_3 \exp(-\ \overline{a}\ \ )] & \text{if } a > 0, \\ {}^2[\ {}_1 + \ {}_2 \cos(\ \overline{-a}\ \ ) + \ {}_3 \sin(\ \overline{-a}\ \ )] & \text{if } a < 0. \end{cases}$

80. ${}^6\ \ \ + \ {}^2\ \ + (\ \ - 2\ \ \ )\ = 0.$

The transformation $= {}^{-1}$, $= {}^{-2}$ leads to a constant coefficient linear equation of the form 3.1.2.82 with $a_2 = 0$: ${}''' + a\ {}' - \ \ = 0.$

3.1.2-3. Equations of the form ${}_3(\ )\ {}''' + {}_2(\ )\ {}'' + {}_1(\ )\ {}' + {}_0(\ )\ = g(\ ).$

81. $+ 3\ \ \ + 3\ {}^2\ \ + \ {}^3\ \ = 0.$

Solution: $= e^{-}\ \ (\ {}_1 + \ {}_2\ + \ {}_3\ {}^2).$

82. $+ \ {}_2\ \ + \ {}_1\ \ + \ {}_0\ \ = 0.$

A third-order constant coefficient linear equation.

Denote $(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$.

1 . Let the characteristic polynomial (λ) be factorizable:

$(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$, where λ_1, λ_2, and λ_3 are real numbers.

Solution: $= \begin{cases} {}_1 e^{\lambda_1} + {}_2 e^{\lambda_2} + {}_3 e^{\lambda_3} & \text{if all the roots } \lambda \text{ are different,} \\ (\ {}_1 + \ {}_2\ )e^{\lambda_1} + {}_3 e^{\lambda_3} & \text{if } \lambda_1 = \lambda_2 \neq \lambda_3, \\ (\ {}_1 + \ {}_2\ + \ {}_3\ {}^2)e^{\lambda_1} & \text{if } \lambda_1 = \lambda_2 = \lambda_3. \end{cases}$

2 . Let $(\lambda) = (\lambda - \lambda_1)(\lambda^2 + 2\ {}_1\lambda + \ {}_0)$, where ${}^2_1 < \ {}_0$.

Solution: $= {}_1 e^{\lambda_1} + e^{-b_1}\ (\ {}_2 \cos\ \ + \ {}_3 \sin\ \ )$, where $= \sqrt{\ {}_0 - \ {}^2_1}$.

83. $+ \ \ \ + (\ \ + \ )\ \ + (\ \ \ + \ \ + \ )\ \ = 0.$

Integrating yields a second-order linear equation: ${}'' + (\ \ + \ )\ \ = \ \ e^{-}$ (see 2.1.2.2 for the solution of the corresponding homogeneous equation with $= 0$).

84. $+ 3\ \ \ + 2(\ \ + \ {}^2)\ \ + (2\ \ \ + 1)\ \ = 0.$

This is a special case of equation 3.1.2.113 with $= 0$ and $= 1$.

85. $+ \ \ \ + (\ {}^2 + \ \ + \ )\ \ + (\ {}^2 + \ \ + \ )\ \ = 0.$

The substitution $= {}' + a$ leads to a second-order linear equation of the form 2.1.2.6: ${}'' + (\ {}^2 + \ \ + \ )\ \ = 0.$

86. $+ \ \ \ + \ \ \ \ + \ \ \ \ \ = 0.$

The substitution $= {}' + a$ leads to a second-order linear equation of the form 2.1.2.7: ${}'' + \ \ \ \ = 0.$

87. $+ 3\ \ \ \ + 3\ {}^2\ {}^2\ \ + (\ {}^3\ {}^3 + \ )\ \ = 0.$

The substitution $= \exp\bigl(-\tfrac{1}{2}a\ {}^2$ leads to a constant coefficient linear equation of the form 3.1.2.82 with $a_2 = 0$: ${}''' - 3a\ {}' + \ \ = 0.$

88. $+ \quad + (\quad + \quad - \;^{2}) \quad + \quad = 0.$

Particular solutions: $_1 = e^{-b}$, $_2 = e^{-b}$ $\exp\bigl(2 \quad - \tfrac{1}{2}a \;^{2}$.

89. $+ 3 \quad + (2 \;^{2} \;^{2} + \quad + \;) \quad + \quad = 0.$

Solution: $= {}_1 \;^2_1 + {}_2 \;_1 \;_2 + {}_3 \;^2_2$. Here, $_1$ and $_2$ are linearly independent solutions of a second-order equation of the form 2.1.2.28: $'' + a \quad ' + \tfrac{1}{4} \quad = 0.$

90. $+ 3 \quad + (2 \;^{2} \;^{2} + 2 \quad + \;) \quad + (2 \;^{2} + 1) \quad = 0.$

This is a special case of equation 3.1.2.113 with $= 1$ and $= 1$.

91. $+ 3 \quad + 3(\;^{2} \;^{2} + \;) \quad + (\;^{3} \;^{3} + \quad + \;) \quad = 0.$

The substitution $= \exp\bigl(-\tfrac{1}{2}a \;^{2}$ leads to a linear equation of the form 3.1.2.4: $''' + [(\; - 3a^2) \; + \;] \quad = 0.$

92. $+ 3 \quad + [2(\;^{2} + \;) \;^{2} + \;] \quad + 2 \quad (\;^{2} + 1) \quad = 0.$

This is a special case of equation 3.1.2.113 with $= 1$ and $= 2$.

93. $+ (\quad + \;) \quad + (\quad + \quad + \;) \quad + \quad = 0.$

Integrating yields a second-order linear equation: $'' + a \quad ' + \quad = \quad e^{-b}$ (see 2.1.2.28 for the solution of the corresponding homogeneous equation with $= 0$).

94. $+ (\quad + \quad + \;) \quad + \;^{2} \quad - \;^{2} \quad = 0.$

Particular solutions: $_1 = \;$, $_2 = e^{-b}$.

95. $+ (\quad + \;) \quad + [(\quad + \;) \quad + \;] \quad + (\quad + 1) \quad = 0.$

Integrating yields a second-order linear equation: $'' + a \quad ' + \quad = \quad e^{-b}$ (see 2.1.2.28 for the solution of the corresponding homogeneous equation with $= 0$).

96. $+ (\quad + \quad + \;) \quad + (\quad + \quad + s) \quad + s(\quad + \;) \quad = 0.$

Particular solutions: $_1 = e^{\lambda_1}$, $_2 = e^{\lambda_2}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + \; \lambda + \; = 0$.

97. $+ (\quad + \;) \quad + (\quad + 2 \;) \quad + \quad [(\quad - \quad) \;^{2} + \;] \quad = 0.$

Particular solution: $_0 = \exp\bigl(-\tfrac{1}{2}a \;^{2}$. The substitution $= \exp\bigl(-\tfrac{1}{2}a \;^{2} \quad z(\;)$ leads to a second-order linear equation of the form 2.1.2.31: $z'' + (\; - 2a \;)z' + [a^2 \;^{2} + (\; - 2a \;) \; - a]z = 0.$

98. $+ (\quad + \;) \quad + (\quad + \;) \quad + [\quad ^{2} + (\quad + \quad) \quad + \quad + \quad] \quad = 0.$

Integrating yields a second-order linear equation: $'' + (\quad + \;) \quad = \quad \exp\bigl(-\tfrac{1}{2}a \;^{2} -$ (see 2.1.2.2 for the solution of the corresponding homogeneous equation with $= 0$).

99. $+ (\quad + \;) \quad + (\;^{2} + \quad + \gamma) \quad - \quad [\quad ^{2} + (\quad + \;) \quad + \;^{2} + \quad + \gamma] \quad = 0.$

The substitution $= \; ' - k$ leads to a second-order linear equation of the form 2.1.2.31: $'' + (a \quad + \quad + k) \; ' + [\quad ^{2} + (ak + \beta) \quad + k^2 + \quad k + \quad] \quad = 0.$

100. $\mathbf{- \quad ^2 \quad + (\quad + \quad - 1) \quad - \quad = 0.}$

The following three series, converging for any , make up a fundamental set of solutions:

$$_1 = 1 + \sum_{=1} \frac{a \ (a-3)(\ -3) \quad (a-3 \ +3)(\ -3 \ +3)}{(3 \ )!} \ ^3,$$

$$_2 = \ + \sum_{=1} \frac{(a-1)(\ -1)(a-4)(\ -4) \quad (a-3 \ +2)(\ -3 \ +2)}{(3 \ +1)!} \ ^{3 \ +1},$$

$$_3 = \frac{^2}{2} + \sum_{=1} \frac{(a-2)(\ -2)(a-5)(\ -5) \quad (a-3 \ +1)(\ -3 \ +1)}{(3 \ +2)!} \ ^{3 \ +2}.$$

101. $\mathbf{+ \quad - 2 \quad ^{-2} = 0.}$

The substitution $= \ ' - 2$ leads to a second-order linear equation of the form 2.1.2.45: $'' + a \quad ' + a \ ^{-1} = 0.$

102. $\mathbf{+ \quad - \quad - \quad = 0.}$

1 . Particular solutions with > 0: $_1 = \exp(- \ \overline{\ } \ )$, $_2 = \exp(\ \overline{\ } \ )$.

2 . Particular solutions with < 0: $_1 = \cos\bigl(\ \overline{\ -\ } \ $, $_2 = \sin\bigl(\ \overline{\ -\ } \ $.

103. $\mathbf{+ \quad - 2 \quad ^{-1} \quad + 2 \quad ^{-2} \quad = 0.}$

Particular solutions: $_1 = \ $, $_2 = \ ^2$.

104. $\mathbf{+ \quad + \quad ^{-1} \quad - 2(\ + \ ) \quad ^{-2} \quad = 0.}$

The substitution $= \ ' - 2$ leads to a second-order linear equation of the form 2.1.2.45: $'' + a \quad ' + (a + \ ) \ ^{-1} = 0.$

105. $\mathbf{+ \quad + \quad + \ ^2(\quad - \ ) \quad = 0.}$

Particular solutions: $_1 = \exp\bigl(-\frac{1}{2} \quad \cos\bigl(\frac{\ 3}{2}$, $_2 = \exp\bigl(-\frac{1}{2} \quad \sin\bigl(\frac{\ 3}{2}$.

106. $\mathbf{+ \quad - (\quad ^{-1} - \quad ^2) \quad + \quad (\quad ^{+1} + 3) \quad = 0.}$

Particular solutions: $_1 = \cos\bigl(\frac{1}{2} \ ^2 \ \overline{\ }$, $_2 = \sin\bigl(\frac{1}{2} \ ^2 \ \overline{\ }$.

107. $\mathbf{+ \quad + (\quad + \quad ^{-1} - \quad ^2) \quad + \quad ^{-1} \quad = 0.}$

Particular solutions: $_1 = e^{-b}$, $_2 = e^{-b} \quad \exp \ 2 \quad - \frac{a}{+1} \ ^{+1}$.

108. $\mathbf{+ \quad + \quad - \quad ^{-1} \quad = 0.}$

Particular solution: $_0 = \ $.

109. $\mathbf{+ \quad + \quad + \quad ^{-1}(\quad ^{+1} + \quad) \quad = 0.}$

The substitution $= \ '' + \ $ leads to a first-order linear equation: $' + a \quad = 0.$

110. $\mathbf{+ \quad - \ (2 \quad + 3 \ ) \quad + \ ^2(\quad + 2 \ ) \quad = 0.}$

Particular solutions: $_1 = e^{b}$, $_2 = \ e^{b}$.

111. $\mathbf{+ \quad + (\quad - \ ^2 + \ ) \quad + \ (\quad - \ ) \quad = 0.}$

Particular solutions: $_1 = e^{\lambda_1}$, $_2 = e^{\lambda_2}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + \ \lambda + \ = 0.$

112. + + (− ^{2}) − (+) = 0.

Particular solution: $_0 = e^c$.

113. + 3 + (2 ^{2} ^{2} + 2 + $^{-1}$) + (2 $^+$ + $^{-1}$) = 0.

Solution: = $_1$ $^2_1$+ $_2$ $_1$ $_2$+ $_3$ $^2_2$. Here, $_1$ and $_2$ form a fundamental set of solutions of the second-order linear equation: $''$ $+a$ $'$ $+\frac{1}{2}$ $= 0$.

114. = (−) + (−) + .

Particular solutions: $_1 = e^{\lambda_1}$, $_2 = e^{\lambda_2}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda +$ $= 0$.

115. + (+) + (+ +) + (+ ^{2})(+ −) = 0.

Particular solutions: $_1 = e^{\lambda_1}$, $_2 = e^{\lambda_2}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 +$ $\lambda +$ $+$ $^2 = 0$.

116. + (−) + − (+) = 0.

Particular solution: $_0 = e^b$.

117. + (+) + (+) + = 0.

Particular solution: $_0 = e^-$.

118. + (+) + [+ (+) $^{-1}$] + [$^{-1}$ + (− 1) $^{-2}$] = 0.

Integrating yields a second-order linear equation: $''$ $+a$ $'$ $+$ $^{-1}$ $=$ e^{-c} (see 2.1.2.45 for the solution of the corresponding homogeneous equation with $= 0$).

119. + (+) + ($^{+1}$ + 2) + = 0.

Particular solutions: $_1 = \exp\bigl(-\frac{1}{2}$ ^{2} , $_2 = \exp\bigl(-\frac{1}{2}$ ^{2} $\exp\bigl(\frac{1}{2}$ ^{2} .

120. + (+) + ($^{+1}$ + + − ^{2}) + ($^{+1}$ − +) = 0.

Particular solutions: $_1 = e^{-c}$, $_2 = e^{-c}$ $\exp\bigl(2$ $-\frac{1}{2}$ ^{2} .

121. + (+ $^{-1}$ +) + ^{2} − ^{2} $^{-1}$ = 0.

Particular solutions: $_1 =$, $_2 = e^{-b}$.

122. + (+) + + (+) = 0.

1 . Particular solutions with > 0: $_1 = \cos\bigl($ $-$, $_2 = \sin\bigl($ $-$.

2 . Particular solutions with < 0: $_1 = \exp\bigl(-$ $\overline{-}$, $_2 = \exp\bigl($ $\overline{-}$.

123. + (+) + ($^+$ + + $^{-1}$ − ^{2})
+ ($^+$ − + $^{-1}$) = 0.

Particular solutions: $_1 = e^{-c}$, $_2 = e^{-c}$ $\exp$ 2 $-\frac{^{+1}}{+1}$.

124. + 3 + = 0.

The substitution = leads to a constant coefficient linear equation of the form 3.1.2.1: $'''$ $+a$ $= 0$.

125. **− 3 + = 0, = 0, 1, 2, . . .**

Solution: $= {}^{3\ +2}\ \frac{1}{2}$ —— $\frac{}{2}$, where is the general solution of equation 3.1.2.1: $''' + a = 0$.

126. **2 + 3 + = , ≠ 0.**

Solution: $= \sum_{=1}^{4} \int_0^{\lambda} \frac{e \quad z}{\overline{2z^3 + a}}$, where λ_1, λ_2, and λ_3 are roots of the cubic equation $2\lambda^3 + a = 0$; $\lambda_4 = -$ for > 0 and $\lambda_4 = +$ for < 0. In addition, the constants are related by the constraint $\overline{a}(\ _1 + \ _2 + \ _3 + \ _4) + = 0$, and the integrals are taken along straight lines.

127. **+ 3 + ^{2} = .**

The substitution = leads to an equation of the form 3.1.2.3: $''' + a =$.

128. **+ 3 + ^{4} = .**

The substitution = leads to an equation of the form 3.1.2.5: $''' + a\ ^3 =$.

129. **+ + + ^{3} = 0.**

The substitution $= ' +$ leads to a second-order linear equation of the form 2.1.2.108: $'' + (a - \ )\ ' + \ ^2 = 0$.

130. **+ (+) − − = 0, > 0, > 0.**

Solution:

$$= \sum_{=1}^{3} \quad |\ |\ ^{-1}|\ ^2 - 1|^{(b-2)}\ ^2 e^{-} \quad ,$$

where $_1 = -1$, $\beta_1 = \ _2 = 0$, $\beta_2 = 1$; for > 0, $_3 = 1$ and $\beta_3 = +$; for < 0, $_3 = -$ and $\beta_3 = -1$.

131. **+ + (− ^{2}) − (+) = 0.**

The substitution $= ' -$ leads to a second-order linear equation of the form 2.1.2.108: $'' + (\ + a)\ ' + (\ + a\ ) = 0$.

132. **+ + [(− ^{2}) +] + (−) = 0.**

Particular solutions: $_1 = e^{\lambda_1}$, $_2 = e^{\lambda_2}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + \ \lambda + \ = 0$.

133. **+ + + (+ − 1) $^{-1}$ = 0.**

The substitution $= '' + \ ^{-1}$ leads to a first-order linear equation: $' + a = 0$.

134. **+ (+) − ^{2} = 0.**

The substitution $= ' + a$ leads to a second-order linear equation of the form 2.1.2.108: $'' + \ ' - a = 0$.

135. **+ (+) + − = 0.**

The substitution $= ' -$ leads to a second-order linear equation of the form 2.1.2.108: $'' + (a + \ - 1)\ ' + \ = 0$.

136. $+(\quad +3) \quad +(\quad +2\ ) \quad +(\quad + \ ) \quad = 0.$

The substitution $\quad = \quad$ leads to a constant coefficient linear equation of the form 3.1.2.82: $''' \ + a \ '' + \ \ ' + \quad = 0.$

137. $+(\quad +3) \quad + \ (\quad +2) \quad + \ [\ (\ - \ ) \ + \ ] \ = 0.$

Solution: $\ = \ ^{-1}\big[\ _1 e^{(b- \ )} \ + \ _2 e^{-b \quad 2}\cos\big(\frac{\ 3}{2} \quad + \ _3 e^{-b \quad 2}\sin\big(\frac{\ 3}{2} \quad \big].$

138. $+[\ (\ +1) \ + \ ] \quad + \ ^2 \quad - \ ^2 \quad = 0.$

Particular solutions: $_1 = \ , \quad _2 = e^{-} \ .$

139. $-(\ +2\ ) \quad -(\ -2 \ -1) \quad +(\ -1) \quad = 0.$

Solution: $\ = \ _1 e \ + \ ^{+1}[\ _2 \ _{+1}(\ ) + \ _3 \ _{+1}(\ )]$, where $(\ )$ and $(\ )$ are the modified Bessel functions.

140. $2 \quad -4(\ + \ -1) \quad +(2 \ +6 \ -5) \quad +(1-2\ ) \quad = 0.$

Solution: $\ = \ _1 e \ + \ e \ ^2[\ _2 \ (\ 2) + \ _3 \ (\ 2)]$, where (z) and (z) are the modified Bessel functions.

141. $2 \quad +3(2 \ + \ ) \quad +6(\ + \ ) \quad +(2 \ +3 \ ) \quad = 0, \qquad > 0.$

Solution:

$$= \sum_{=1}^{4} \int_0^{\lambda} e \ [\ (z)]^{(\ -2)\ 2} \ z, \qquad _4 = - \ _1 - \ _2 - \ _3,$$

where $(z) = z^3 + 3az^2 + 3 \ z + \ $; λ_1, λ_2, and λ_3 are roots of this polynomial, which are assumed to be different; $\lambda_4 = -\ $ for $\ > 0$ and $\lambda_4 = +\ $ for $\ < 0$.

142. $+(\ + \ ) \quad +[(\ +s- \ ^2) \ + \ ] \quad +s[(\ - \ ) \ + \ ] \quad = 0.$

Particular solutions: $_1 = e^{\lambda_1}\ , \quad _2 = e^{\lambda_2}\ ,$ where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + \ \lambda + \ = 0$.

143. $+(\ ^2 + \ +2) \quad - \ (\ +1) \quad = 0.$

This is a special case of equation 3.1.2.145 with $\ = 2$.

144. $+(\ ^2 + \ ) \quad +4 \quad +2 \quad = 0.$

Integrating the equation twice, we arrive at a first-order linear equation: $\ ' + (a \ ^2 + \ -2) \ = \ _1 + \ _2 \ .$

145. $+(\quad + \ +2) \quad - \ (\ +1) \ ^{-2} \quad = 0.$

The substitution $\ = \ ' + \ $ leads to a second-order linear equation of the form 2.1.2.45: $'' \ + a \ ^{-1} \ ' - a(\ +1) \ ^{-2} \ = 0.$

146. $+(\quad +3) \quad +(2 \quad ^{-1} + \quad) \quad + \ (\quad +1) \quad = 0.$

Particular solutions: $_1 = \ ^{-1}\cos\big(\ \overline{\ } \ , \quad _2 = \ ^{-1}\sin\big(\ \overline{\ } \ .$

147. $+(\quad ^{+1} + 3) \quad + \ (\quad +2) \quad + \ (\quad ^{+1} + \quad - \ ^2 \ ) \quad = 0.$

Particular solutions: $_1 = \ ^{-1}\exp\big(-\frac{1}{2} \quad \cos\big(\frac{\ 3}{2} \quad , \quad _2 = \ ^{-1}\exp\big(-\frac{1}{2} \quad \sin\big(\frac{\ 3}{2} \quad .$

148. $+(\quad +3) \quad +(\quad +2 \quad ^{-1} - \ ^2 \ ) \quad + \ (\quad ^{-1} - \ ) \quad = 0.$

Particular solutions: $_1 = \ ^{-1}, \quad _2 = \ ^{-1}e^{-b} \ .$

149. $(\quad + \quad) \qquad + [\; (\quad + 1) \quad + \quad {}^2 + 1] \qquad + \quad {}^2 \qquad - \quad {}^2 \quad = 0.$

Particular solutions: $\quad _1 = \quad , \quad _2 = e^{-b} \quad .$

150. $(\quad + \quad) \qquad + \; (\quad + \quad) \qquad + (\quad + \quad) \quad + \; (\quad + \quad) \; = 0.$

The substitution $\quad = \quad ' + k \quad$ leads to a second-order linear equation of the form 2.1.2.108: $(a \quad + \quad) \quad '' \; + (\quad + \quad) \quad = 0.$

151. $(\quad + \quad) \qquad + (\quad + \quad) \qquad + [(\quad + \quad) \; + \quad + \quad] \quad + (\quad + \quad {}^2)[(\; - \quad) \; + \; - \quad] \; = 0.$

Particular solutions: $\quad _1 = \exp(\; _1 \;), \quad _2 = \exp(\; _2 \;)$, where $\; _1$ and $\; _2$ are roots of the quadratic equation $\quad {}^2 + \quad + \lambda + \quad {}^2 = 0.$

152. $(\quad + \quad) \qquad + (\quad + \quad) \qquad - \; [(3 \quad + 2 \;) \; + 3 \quad + 2 \;] \quad + \; {}^2[(2 \quad + \quad) \; + 2 \quad + \quad] \; = 0.$

Particular solutions: $\quad _1 = e \quad , \quad _2 = \quad e \quad .$

153. $(\quad + \quad) \qquad + (\quad + \quad) \qquad + s \; (\quad + \quad) \qquad + s[\quad {}^2 + (\quad + \quad) \; + \quad] \; = 0.$

The substitution $\quad = \; '' + \qquad$ leads to a first-order linear equation: $(a \; + \;) \quad ' + (\quad + \quad) \; = 0.$

154. $(1 - \quad) \qquad + \; (\quad - 2 \quad + 1) \qquad + (- \quad {}^2 + 2 \quad - 1) \qquad + 2 \; (\quad - 1) \quad = 0.$

Particular solutions: $\quad _1 = \quad {}^2, \quad _2 = e \quad .$

155. $(\quad + \quad) \qquad + (\quad + \quad) \qquad + s \quad (\quad + \quad) \qquad + s \quad {}^{-1}[\quad {}^2 + (\quad + \quad) \; + \quad] \; = 0.$

The substitution $\quad = \; '' + \qquad$ leads to a first-order linear equation: $(a \; + \;) \quad ' + (\quad + \quad) \; = 0.$

156. $(\quad - 1) \qquad + \; (\qquad {}^{+1} - 2 \quad - \quad {}^2) \qquad + (2 \quad - \quad {}^2 \qquad {}^{+2} + \quad {}^2) \quad + 2 \quad (\quad - 1) \qquad = 0.$

Particular solutions: $\quad _1 = \quad {}^2, \quad _2 = e \quad .$

157. $\quad {}^2 \qquad + 3 \qquad - 3 \quad + \quad {}^2 \quad + \quad = 0.$

Solution: $\quad = (\qquad)'$, where the function $\quad = \quad (\;)$ satisfies a constant coefficient linear equation of the form 3.1.2.2: $\quad ''' \quad + a \quad = \quad .$

158. $\quad {}^2 \qquad + 6 \qquad + 6 \quad + \quad {}^2 \quad = \; .$

The substitution $\quad = \quad {}^2 \quad$ leads to a constant coefficient linear equation of the form 3.1.2.2: $\quad ''' \quad + a \quad = \; .$

159. $\quad {}^2 \qquad - 3(\quad + \quad) \qquad + 3 \; (3 \quad + 1) \quad - \quad {}^2 \quad = 0, \qquad , \quad = 1, 2, 3, \ldots$

Solution:

$$= \sum_{=0}^{-1} (\quad - 3 \quad - 1) \sum_{=0}^{-1} (\quad - 3 \quad - 2) \sum_{=1}^{3} \qquad e \qquad , \qquad = \frac{}{},$$

where the ω are three roots of the cubic equation $\omega^3 = 1$.

160. $\quad {}^2 \qquad + 6 \qquad + 6 \quad + \quad {}^3 \quad = \; .$

The substitution $\quad = \quad {}^2 \quad$ leads to a linear equation of the form 3.1.2.3: $\quad ''' \quad + a \qquad = \; .$

161. $\quad {}^2 \qquad - 2(\quad + 1) \qquad - (\quad {}^2 - 6 \quad) \quad + 2 \qquad = 0, \qquad = 1, 2, 3, \ldots$

Solution:

$$= \begin{cases} _1 + \quad _2 \quad {}^4 + \quad _3 \quad {}^{2 \quad +1} & \text{if } a = 0, \\ _1(a \quad {}^2 - 4 \quad + 2) + \quad _2 e^{\quad -} \quad (\;) + \quad _3 e^{-} \quad {}^{-} Q(\;) & \text{if } a \neq 0, \end{cases}$$

where $\quad (\;)$ and $Q(\;)$ are some polynomials of degree $\leq 2 \quad + 2$.

162. $^{2} \quad + 3 \quad + (4 \;^{2}\;^{2} + 1 - 4 \;^{2}\;^{2}) \quad + 4 \;^{3}\;^{2}\;^{-1} \quad = 0.$

Solution: $= {}_1\,{}_b^2(\ ) + {}_2\,{}_b(\ )\,{}_b(\ ) + {}_3\,{}_b^2(\ )$, where ${}_b(z)$ and ${}_b(z)$ are the Bessel functions.

163. $^{2} \quad + \;^{2} \quad + (\ \ + \ ) \quad + (\ \ + \ ) \ = 0.$

The substitution $= \ ' + a$ leads to a second-order linear equation of the form 2.1.2.111: $^{2}\ '' + (\ \ + \ ) \ = 0.$

164. $^{2} \quad + \;^{2} \quad + (\ \ + \ ) \quad + (\ \ + \ ) \ = 0.$

The substitution $= \ ' + a$ leads to a second-order linear equation of the form 2.1.2.118: $^{2}\ '' + (\ \ + \ ) \ = 0.$

165. $^{2} \quad - (\ + \ ) \quad + (2 \ + 1) \quad - (\ + 1) \ = 0.$

Solution: $= {}_1 e \ + \ ^{(\ +1)\ 2}\big[\ {}_2\ {}_{+1}\big(2\ \overline{a}\ \big) + {}_3\ {}_{+1}\big(2\ \overline{a}\ \big)\big]$, where (z) and (z) are the Bessel functions.

166. $^{2} \quad - (\;^{2} - 2\ ) \quad - (\;^{2} + \;^{2} - \tfrac{1}{4}) \quad + (\;^{2} - 2 \ + \;^{2} - \tfrac{1}{4}) \ = 0.$

Solution: $= {}_1 e \ + \ ^{-}\big[\ {}_2\ (\ ) + {}_3\ (\ )\big]$, where $(\ )$ and $(\ )$ are the modified Bessel functions.

167. $^{2} \quad - 2\ (\ - 1) \quad + (\;^{2} - 2 \ + \tfrac{1}{4} - \;^{2}) \quad + (\;^{2} - \tfrac{1}{4}) \ = 0.$

Solution: $= {}_1 e \ + \ ^{-} e \ ^{2}\big[\ {}_2\ (\ \ 2) + {}_3\ (\ \ 2)\big]$, where (z) and (z) are the modified Bessel functions.

168. $^{2} \quad - 3(\ - \ ) \quad + [2 \;^{2} + 4(\ - \ ) \ + (2 \ - 1)] \quad - 2\ (2 \ - 2 \ + 1) \ = 0.$

Solution: $= {}_1\ {}_1^2 + {}_2\ {}_1\ {}_2 + {}_3\ {}_2^2$. Here, ${}_1$ and ${}_2$ are a fundamental set of solutions of a second-order linear equation of the form 2.1.2.108: $'' + (a - \ )\ ' + \ = 0.$

169. $^{2} \quad + [(\ + \ ) \ + \ ] \quad + [(\ \ + \ )\;^{2} + (\ \ + \ ) \ + \gamma] \quad + (\ \;^{2} + \ \ + \gamma) \ = 0.$

The substitution $= \ ' + \ $ leads to a second-order linear equation of the form 2.1.2.146 with $= 1$: $^{2}\ '' + (a \ + \ )\ ' + (\ ^{2} + \beta \ + \ ) \ = 0.$

170. $^{2} \quad + (\ \ ^{+1} + \ ) \quad + [\ (\ - 2) \ + \ ] \quad + (\ - \ + 2) \ ^{-1} \ = 0.$

Particular solutions: ${}_1 = \ ^{1}$, ${}_2 = \ ^{2}$, where ${}_1$ and ${}_2$ are roots of the quadratic equation $^{2} + (\ - 3) \ + \ - \ + 2 = 0.$

171. $2\ (\ - 1) \quad + 3(2 \ - 1) \quad + (2 \ \ + \ ) \quad + \ \ = 0.$

Solution: $= {}_1\ {}_1^2 + {}_2\ {}_1\ {}_2 + {}_3\ {}_2^2$. Here, ${}_1$ and ${}_2$ form a fundamental set of solutions of the equation $2\ (\ -1)\ '' + (2 \ -1)\ ' + \big(\tfrac{1}{2}a \ + \tfrac{1}{4} \ - \tfrac{1}{2} \ \ \big) = 0$, which is reduced, by means of the substitution $= \cos^2\xi$, to the Mathieu equation 2.1.6.29: $2\ '' = (a + \ - 2 + a\cos 2\xi)\ .$

172. $(\ {}_2 \;^{2} + \ {}_1 \ + \ {}_0) \quad + (\ {}_1 \ + \ {}_0) \quad + (\ {}_1 \ + \ {}_0) \quad - \ \ {}_1 \ = 0.$

Here, ${}_1 \neq 0$ and is a positive integer. A solution of this equation is a polynomial of degree that can represented as follows:

$$= \sum_{=0} \ -\frac{1}{\ {}_1} \quad \ ^{-\ -1}[(a\ ^{2} + a_1 \ + a_0)\ ^{3} + (\ {}_1 \ + \ {}_0)\ ^{2} + \ {}_0\ ] \quad ,$$

where $= \ \overline{\ \ }$, $= \dfrac{\ ^{+1}}{\ + 1}$ with $\neq -1.$

173. $^3 \quad + 6 \; ^2 \quad + (\quad ^3 - 12) \quad + 2 \quad = 0.$

Solution: $\quad = (\quad ^2)'$, where $\quad = \; (\;)$ satisfies a constant coefficient linear equation of the form 3.1.2.2: $\quad ''' \quad + a \quad = \; .$

174. $^3 \quad + \quad ^2 \quad + \quad + (\quad - 2) \quad = 0.$

This is a special case of equation 3.1.2.175. Solution: $\quad = \; _1 \; ^{2-} + \; _2 \; ^1 + \; _3 \; ^2$, where $_1$ and $_2$ are roots of the quadratic equation $^2 - \quad + \quad = 0.$

175. $^3 \quad + \quad ^2 \quad + \quad + \quad = 0.$

The Euler equation. The substitution $\quad = \ln| \; |$ leads to a constant coefficient linear equation of the form 3.1.2.82: $\quad ''' + (a - 3) \quad '' + (2 - a + \;) \; ' + \quad = 0.$

176. $^3 \quad + 3 \quad ^2 \quad + 3 \; (\quad - 1) \quad + [\quad ^3 + \; (\quad - 1)(\quad - 2)] \quad = 0.$

The substitution $\quad = \quad$ leads to a constant coefficient linear equation of the form 3.1.2.1: $\quad ''' \quad + \quad = 0.$

177. $^3 \quad + 3 \quad ^2 \quad + 3 \; (\quad - 1) \quad + [\quad + \; (\quad - 1)(\quad - 2)] \quad = 0.$

The substitution $\quad = \quad$ leads to an equation of the form 3.1.2.7: $\quad ''' \quad + \quad ^{-3} \quad = 0.$

178. $^3 \quad + 3(1 - \;) \; ^2 \quad + \; [4 \; ^2 \; ^2 \; ^2 \; + 1 - 4 \; ^2 \; ^2 + 3 \; (\quad - 1)]$

$+ [4 \; ^2 \; ^2(\; - \;) \; ^2 \; + \; (4 \; ^2 \; ^2 - \; ^2)] \quad = 0.$

Solution: $\quad = \; _1 \quad ^2(\quad ^c) + \; _2 \quad (\quad ^c) \; (\quad ^c) + \; _3 \quad ^2(\quad ^c)$, where $\; (\;)$ and $\; (\;)$ are the Bessel functions.

179. $^3 \quad + (\quad ^2 + \;) \quad + 2(2 \quad - 9) \quad + 2(\quad - 6) \quad = 0.$

Integrating the equation twice, we arrive at a first-order linear equation: $\; ^3 \; ' + [(a-6) \; ^2 + \;] \; = \; _1 + \; _2 \; .$

180. $^3 \quad + \; ^2(\quad + \;) \quad + \quad + (\quad + \quad - 2) \quad = 0.$

Particular solutions: $\; _1 = \quad ^1, \quad _2 = \quad ^2$, where $_1$ and $_2$ are roots of the quadratic equation $^2 - \quad + \quad = 0.$

181. $^3 \quad + \; ^2(2 \quad + \;) \quad + \; (\; ^2 \; ^2 + 2 \quad + \;) \quad + (\; ^2 \quad ^2 + \quad - 2 \;) \quad = 0.$

Particular solutions: $\; _1 = e^{-} \quad ^1, \quad _2 = e^{-} \quad ^2$, where $_1$ and $_2$ are roots of the quadratic equation $^2 - \quad + \quad = 0.$

182. $^3 \quad + \quad + \quad + \; (\quad ^{-2} - 2) \quad = 0.$

Particular solutions: $\; _1 = \quad ^1, \quad _2 = \quad ^2$, where $_1$ and $_2$ are roots of the quadratic equation $^2 - \quad + \quad = 0.$

183. $^3 \quad + \; ^2(\quad + \;) \quad + \; (\quad + \quad - 1) \quad + (\quad + \quad - 3) \quad = 0.$

Particular solutions: $\; _1 = \cos(\ln \;), \quad _2 = \sin(\ln \;).$

184. $^3 \quad + \; ^2(\quad + \quad + \quad + 1)$

$+ \; [\quad ^2 \; + (\quad + \;) \quad + \gamma + \quad] \quad + (\quad - 1)(\quad ^2 \; + \quad + \gamma) \quad = 0.$

The substitution $\quad = \quad ' + (\quad - 1)$ leads to a second-order linear equation of the form 2.1.2.146: $\; ^2 \quad '' \; + \; (a \quad + \;) \; ' + (\quad ^2 + \beta \quad + \;) \quad = 0.$

185. $(\quad + \;) \; ^3 \quad + (\quad + \;) \; ^2 \quad + s(\quad + \;) \quad + s[(\quad - 2 \;) \quad + \quad - 2 \;] \quad = 0.$

Particular solutions: $\; _1 = \quad ^1, \quad _2 = \quad ^2$, where $_1$ and $_2$ are roots of the quadratic equation $^2 - \quad + \quad = 0.$

186. $\quad {}^{6} \quad + 6 \; {}^{5} \quad - \quad + 2 \quad = 0.$

The substitution $\; = {}^{-1}$ leads to an equation of the form 3.1.2.62: $\; {}^{2} \; ''' - 6 \; ' + a \; {}^{2} \; - 2 \; = 0.$

187. $\quad {}^{2}(\quad + \;) \quad + \; (\quad {}^{k+1} + 2 \quad + \;) \quad + [2 \quad {}^{k+1} + \; (\; - 1) \quad] \quad + \quad (\; - 1) \; {}^{k} \; = 0.$

Integrating the equation twice, we arrive at a first-order linear equation: $(\quad + a) \; ' + (\quad + \;) \; = \; {}_{1} + \; {}_{2} \; .$

3.1.3. Equations Containing Exponential Functions

3.1.3-1. Equations with exponential functions.

1. $\quad - \quad {}^{\lambda} \; (\; {}^{2} \; {}^{2\lambda} \; + 3 \quad {}^{\lambda} \; + \; {}^{2}) \; = 0.$

Particular solution: $\; {}_{0} = \exp \; \frac{a}{\lambda} e^{\lambda} \quad .$ The substitution $\; = \exp \; \frac{a}{\lambda} e^{\lambda} \quad z(\;)$ leads to a second-order linear equation of the form 2.1.3.27: $z'' \; + 3ae^{\lambda} \; z' + (3a^{2}e^{2\lambda} \; + 3a\lambda e^{\lambda} \;)z = 0.$

2. $\quad + \quad {}^{\lambda} \quad + \quad {}^{\lambda} \quad = \quad .$

Integrating yields a second-order linear equation: $\; '' \; + ae^{\lambda} \quad = \quad {}^{-1}e \quad + \quad$ (see 2.1.3.1 for the solution of the corresponding homogeneous equation with $\; = \; = 0$).

3. $\quad + \quad {}^{\lambda} \quad + \; (\quad {}^{\lambda} \; + \; {}^{2}) \quad = 0.$

The substitution $\; = \; ' + \quad$ leads to a second-order linear equation of the form 2.1.3.10: $\; '' \; - \quad ' + (ae^{\lambda} \; + \; {}^{2}) \quad = 0.$

4. $\quad + \quad {}^{\lambda} \quad + [\; (\; - \;) \; {}^{\lambda} \; - \; {}^{3}] \quad = 0.$

Integrating yields a second-order linear equation: $\; '' \; + \quad ' + (ae^{\lambda} \; + \; {}^{2}) \quad = \quad e^{b}$ (see 2.1.3.10 for the solution of the corresponding homogeneous equation with $\; = 0$).

5. $\quad + (\quad {}^{\lambda} \; - \; {}^{2}) \quad + \quad {}^{\lambda} \quad = 0.$

The substitution $\; = \; ' + \quad$ leads to a second-order linear equation of the form 2.1.3.10: $\; '' \; - \quad ' + ae^{\lambda} \quad = 0.$

6. $\quad + (\quad {}^{\lambda} \; - \; {}^{2}) \quad + \; (\; - \;) \; {}^{\lambda} \quad = 0.$

Integrating yields a second-order linear equation: $\; '' \; + \quad ' + ae^{\lambda} \quad = \quad e^{b}$ (see 2.1.3.10 for the solution of the corresponding homogeneous equation with $\; = 0$).

7. $\quad + (\quad {}^{\lambda} \; + \;) \quad + \; (\quad {}^{\lambda} \; + \; + \; {}^{2}) \quad = 0.$

The substitution $\; = \; ' + \quad$ leads to a second-order linear equation of the form 2.1.3.10: $\; '' \; - \quad ' + (ae^{\lambda} \; + \; + \; {}^{2}) \quad = 0.$

8. $\quad + (\quad {}^{2\lambda} \; + \quad {}^{\lambda} \;) \quad - \; (\quad {}^{2\lambda} \; + \quad {}^{\lambda} \; + \; {}^{2}) \quad = 0.$

The substitution $\; = \; ' - \quad$ leads to a second-order linear equation of the form 2.1.3.27: $\; '' \; + \quad ' + (ae^{2\lambda} \; + \; e^{\lambda} \; + \; {}^{2}) \quad = 0.$

9. $\quad - 3 \quad {}^{\lambda} \; (\quad {}^{\lambda} \; + \;) \quad + \quad {}^{\lambda} \; (2 \; {}^{2} \; {}^{2\lambda} \; - \; {}^{2}) \quad = 0.$

Particular solutions: $\; {}_{1} = \exp \; \frac{a}{\lambda} e^{\lambda} \quad , \quad {}_{2} = \quad \exp \; \frac{a}{\lambda} e^{\lambda} \quad .$

10. $\quad - (3 \; {}^{2} \; {}^{2\lambda} \; + 3 \quad {}^{\lambda} \; + \;) \quad + \quad {}^{\lambda} \; (2 \; {}^{2} \; {}^{2\lambda} \; - 2 \; - \; {}^{2}) \quad = 0.$

1 . Particular solutions with $\; > 0$: $\; {}_{1} = \exp \; \frac{a}{\lambda} e^{\lambda} \; - \quad \overline{} \quad , \quad {}_{2} = \exp \; \frac{a}{\lambda} e^{\lambda} \; + \quad \overline{} \quad .$

2 . Particular solutions with $\; < 0$: $\; {}_{1} = \exp \; \frac{a}{\lambda} e^{\lambda} \quad \cos(\quad \overline{} \quad , \quad {}_{2} = \exp \; \frac{a}{\lambda} e^{\lambda} \quad \sin(\quad \overline{} \quad .$

11. $+ \quad + \quad {}^{\lambda} \quad + \quad {}^{\lambda} \quad = 0.$

The substitution $\ = \ ' + a$ leads to a second-order linear equation of the form 2.1.3.1: $'' + \ e^{\lambda} \ = 0.$

12. $+ \quad + (\ {}^{\lambda} + \ ) \quad + [\ (\ + \ ) \ {}^{\lambda} \ + \ ] \ = 0.$

Integrating yields a second-order linear equation: $'' + (\ e^{\lambda} \ + \ ) \ = \ e^{-}$ (see 2.1.3.2 for the solution of the corresponding homogeneous equation with $\ = 0$).

13. $+ \quad + (\ {}^{2\lambda} \ + \ {}^{\lambda} \ ) \quad + \ (\ {}^{2\lambda} \ + \ {}^{\lambda} \ ) \ = 0.$

The substitution $\ = \ ' + a$ leads to a second-order linear equation of the form 2.1.3.27: $'' + (\ e^{2\lambda} \ + \ e^{\lambda} \ ) \ = 0.$

14. $+ \ {}^{\lambda} \quad - \ {}^{2}(\ {}^{\lambda} \ + \ ) \ = 0.$

The substitution $\ = \ ' - \ $ leads to a second-order linear equation of the form 2.1.3.27: $'' + (ae^{\lambda} \ + \ ) \ ' + \ (ae^{\lambda} \ + \ ) \ = 0.$

15. $+ \ {}^{\lambda} \quad - \quad - \ {}^{\lambda} \quad = 0.$

1 . Particular solutions with $\ > 0$: $\ _1 = \exp\bigl(- \ \bar{} \ $, $\ _2 = \exp\bigl(\ \bar{} \ $.

2 . Particular solutions with $\ < 0$: $\ _1 = \cos\bigl(\ \overline{\ -\ } \ $, $\ _2 = \sin\bigl(\ \overline{\ -\ } \ $.

16. $+ \ {}^{\lambda} \quad + \ {}^{\lambda} \quad + \ {}^{3} \ = 0.$

The substitution $\ = \ ' + \ $ leads to a second-order linear equation of the form 2.1.3.27: $'' + (ae^{\lambda} \ - \ ) \ ' + \ {}^{2} \ = 0.$

17. $+ \ {}^{\lambda} \quad + \ {}^{\lambda} \quad + \ {}^{2}(\ {}^{\lambda} \ - \ ) \ = 0.$

Particular solutions: $\ _1 = \exp\bigl(-\frac{1}{2} \ \ \cos\bigl(\frac{\ 3}{2} \ $, $\ _2 = \exp\bigl(-\frac{1}{2} \ \ \sin\bigl(\frac{\ 3}{2} \ $.

18. $+ \ {}^{\lambda} \quad - \ (2 \ {}^{\lambda} \ + 3 \ ) \quad + \ {}^{2}(\ {}^{\lambda} \ + 2 \ ) \ = 0.$

Particular solutions: $\ _1 = e^{b} \ $, $\ _2 = \ e^{b} \ $.

19. $+ \ {}^{\lambda} \quad + (\quad - \ {}^{2}) \quad - \ (\ {}^{\lambda} \ + \quad) \ = 0.$

Particular solution: $\ _0 = e^{c} \ $.

20. $+ \ {}^{\lambda} \quad + (\ {}^{\lambda} \ - \ {}^{2} + \ ) \quad + \ (\ {}^{\lambda} \ - \ ) \ = 0.$

Particular solutions: $\ _1 = e^{\ _1}$, $\ _2 = e^{\ _2}$, where β_1 and β_2 are roots of the quadratic equation $\beta^2 + \ \beta + \ = 0$.

21. $+ \ {}^{\lambda} \quad + [\ (\ + \ ) \ {}^{\lambda} \ - \ {}^{2}] \quad + \quad {}^{\lambda} \quad = 0.$

Particular solutions: $\ _1 = e^{-b} \ $, $\ _2 = e^{-b} \ \ \exp \ 2 \ \ - \frac{a}{\lambda} e^{\lambda} \ $.

22. $+ (\ {}^{\lambda} \ + \ ) \quad - \ {}^{2} \ {}^{\lambda} \quad = 0.$

The substitution $\ = \ ' + \ $ leads to a second-order linear equation of the form 2.1.3.27: $'' + ae^{\lambda} \ ' - a \ e^{\lambda} \ = 0.$

23. $+ (\ {}^{\lambda} \ + \ ) \quad - \ {}^{2}(\ {}^{\lambda} \ + \ + \ ) \ = 0.$

The substitution $\ = \ ' - \ $ leads to a second-order linear equation of the form 2.1.3.27: $'' + (ae^{\lambda} \ + \ + \ ) \ ' + \ (ae^{\lambda} \ + \ + \ ) \ = 0.$

24. $+(\quad^{\lambda}+\ )\quad+\ (\quad^{\lambda}+\ )\quad+\quad^{3}\ =0.$

The substitution $=\ '+$ leads to a second-order linear equation of the form 2.1.3.27: $''\ +(ae^{\lambda}\ +\ -\ )\ '+\ ^{2}\ =0.$

25. $+(\qquad+2\ )\quad-\ (\qquad+\ )\quad-2\ ^{3}\ =0.$

Particular solutions: $_{1}=e\ ,\quad _{2}=e^{-}\quad+\quad a.$

26. $=(\ ^{\lambda}\ -\ )\quad+(\quad^{\lambda}-\ )\quad+\quad^{\lambda}\ .$

Particular solutions: $_{1}=e^{\ 1}\ ,\quad _{2}=e^{\ 2}\ ,$ where β_1 and β_2 are roots of the quadratic equation $\beta^2+a\beta+\ =0.$

27. $+(\quad^{\lambda}+\ )\quad+(\quad^{\lambda}+\ )\quad-s[(\ s+\ )\ ^{\lambda}\ +\ s+\ \ +s^{2}]\ =0.$

The substitution $=\ '-$ leads to a second-order linear equation of the form 2.1.3.27: $''\ +(ae^{\lambda}\ +\ +\ )\ '+[(a\ +\ )e^{\lambda}\ +\ \ +\ +\ ^{2}]\ =0.$

28. $+(\quad^{\lambda}+\ )\quad+(\quad^{2\lambda}+\ )\quad-s(\quad^{2\lambda}+\ s\ ^{\lambda}+\ s+\ \ +s^{2})\ =0.$

The substitution $=\ '-$ leads to a second-order linear equation of the form 2.1.3.27: $''\ +(ae^{\lambda}\ +\ +\ )\ '+(\ e^{2\lambda}\ +a\ e^{\lambda}\ +\ \ +\ +\ ^{2})\ =0.$

29. $+(\quad^{\lambda}+\ )\quad+(\quad^{\lambda}+\ )\quad-\quad^{\lambda}[\ (\ +\ )\ ^{2\lambda}$

$+(\quad+3\quad+\quad+\ )\ ^{\lambda}+\ ^{2}+\quad+\ ]\ =0.$

Particular solution: $_{0}=\exp\ \frac{k}{\lambda}e^{\lambda}$. The substitution $=\exp\ \frac{k}{\lambda}e^{\lambda}\quad z(\ )$ leads to a second-order linear equation of the form 2.1.3.27.

30. $+(2\quad^{\lambda}+\ )\quad+\quad^{\lambda}(\quad^{\lambda}+2\ +3\ )\quad+\quad^{\lambda}[\ (\ +2\ )\ ^{\lambda}+\quad+\ ^{2}]\ =0.$

Particular solutions: $_{1}=\exp\ -\frac{a}{\lambda}e^{\lambda}\ ,\quad _{2}=\quad\exp\ -\frac{a}{\lambda}e^{\lambda}\ .$

31. $+(\quad^{\lambda}-\ )\quad+\qquad-\ (\quad^{\lambda}+\qquad)\ =0.$

Particular solution: $_{0}=e^{b}\ .$

32. $+(\ ^{\lambda}+\ )\quad+(\quad^{\lambda}+\qquad)\quad+\qquad=0.$

Particular solution: $_{0}=e^{-}\ .$

33. $+(\quad^{\lambda}+\qquad)\quad+\quad+\ (\quad^{\lambda}+\qquad)\ =0.$

The substitution $=\ ''+$ leads to a first-order linear equation: $'+(ae^{\lambda}\ +\ e\ \ )\ =0.$

34. $+(\quad^{\lambda}+\quad^{\nu}\ )\quad+[\quad^{(\lambda+\nu)}+\ (\ +\ )\ ^{\nu}\ -\ ^{2}]$

$+\ [\quad^{(\lambda+\nu)}-\quad^{\lambda}+\quad^{\nu}\ ]\ =0.$

Particular solutions: $_{1}=e^{-c}\ ,\quad _{2}=e^{-c}\quad\exp\ 2\quad-\ -e\qquad.$

35. $+\quad^{\lambda}(\qquad+2\ )\quad-\ [\quad^{(\lambda+\ )}+\ ]\quad-2\quad^{3\ \lambda}\ =0.$

Particular solutions: $_{1}=e\ ,\quad _{2}=e^{-}\quad+\quad.$

36. $(\quad+\ )\quad-\qquad=0.$

Particular solution: $_{0}=ae\ \ +\ .$

37. $(\qquad+\ +\ )\quad-(\ ^{3}\quad+\ ^{3}+\ ^{3})\quad+\ (\ ^{2}-\ ^{2})\ =0.$

Particular solutions: $_{1}=e^{c}\ ,\quad _{2}=e^{-}\quad+\ .$

38. $(\quad ^{\lambda} + \quad) \quad + (\quad ^{\lambda} + \quad) \quad + \ (\quad ^{\lambda} + \quad) \quad + \ (\quad ^{\lambda} + \quad) \quad = 0.$

1 . Particular solutions with $k > 0$: $\quad _1 = \cos\bigl(\quad \overline{k}$, $\quad _2 = \sin\bigl(\quad \overline{k}$.

2 . Particular solutions with $k < 0$: $\quad _1 = \exp\bigl(- \quad \overline{-k}$, $\quad _2 = \exp\bigl(\quad \overline{-k}$.

3.1.3-2. Equations with power and exponential functions.

39. $\quad + \quad ^{\lambda} \quad + \quad (\quad ^{\lambda} + \ ^{2} \ ^{2} - 3 \ ) \quad = 0.$

Particular solution: $\quad _0 = \exp\bigl(-\tfrac{1}{2} \quad ^{2}$.

40. $\quad + (\quad + \ ) \ ^{\lambda} \quad - \quad ^{\lambda} \quad = 0.$

Particular solution: $\quad _0 = a \quad + \quad .$

41. $\quad + (\quad + \ ) \ ^{\lambda} \quad - 2 \quad ^{\lambda} \quad = 0.$

Particular solution: $\quad _0 = (a \quad + \quad)^2.$

42. $\quad + \quad ^{\lambda} \quad + \quad + \quad ^{-1}(\quad ^{\lambda} + \quad) \quad = 0.$

The substitution $\quad = \quad '' \ + \quad$ leads to a first-order linear equation: $\quad ' + ae^{\lambda} \quad = 0.$

43. $\quad + \quad ^{\lambda} \quad + (\ ^{2} - \quad ^{\lambda} \ ) \quad + \quad (\ ^{2} \ ^{\lambda} \ + 3) \quad = 0.$

1 . Particular solutions with $\quad > 0$: $\quad _1 = \cos\bigl(\tfrac{1}{2} \ ^{2} \ \overline{\ }$, $\quad _2 = \sin\bigl(\tfrac{1}{2} \ ^{2} \ \overline{\ }$.

2 . Particular solutions with $\quad < 0$: $\quad _1 = \exp\bigl(-\tfrac{1}{2} \ ^{2} \ \overline{-}$, $\quad _2 = \exp\bigl(-\tfrac{1}{2} \ ^{2} \ \overline{-}$.

44. $\quad + \ ^{2} \ ^{\lambda} \quad - 2 \quad ^{\lambda} \quad + 2 \quad ^{\lambda} \quad = 0.$

Particular solutions: $\quad _1 = \quad , \quad _2 = \quad ^2.$

45. $\quad + (\quad + \quad ^{\lambda} \ ) \quad + \ (\quad ^{\lambda} + 2) \quad + \quad ^{\lambda} \quad = 0.$

Particular solutions: $\quad _1 = \exp\bigl(-\tfrac{1}{2}a \ ^{2}$, $\quad _2 = \exp\bigl(-\tfrac{1}{2}a \ ^{2} \quad \exp\bigl(\tfrac{1}{2}a \ ^{2}$.

46. $\quad + (\quad ^{\lambda} + \quad ^{\lambda} + \ ) \quad + \ ^{2} \quad ^{\lambda} \quad - \ ^{2} \quad ^{\lambda} \quad = 0.$

Particular solutions: $\quad _1 = \quad , \quad _2 = e^{-} \quad .$

47. $\quad + \quad (\ ^{\lambda} + 2 \ ) \quad - \ (\quad ^{\lambda} + \ ) \quad - 2 \quad ^{3} \quad = 0.$

Particular solutions: $\quad _1 = e^{\lambda} \ , \quad _2 = e^{-\lambda} \ + \quad \lambda.$

48. $\quad + (\quad - 2 \quad ^{\lambda} \ ) \quad - \quad ^{\lambda} \ (2 \quad - \quad ^{\lambda} + 3 \ )$

$$+ \quad ^{\lambda} \ [\quad (\quad ^{\lambda} - \ ) + 2 \quad ^{\lambda} - \ ^{2}] \quad = 0.$$

Particular solutions: $\quad _1 = \exp \ \frac{}{\lambda} e^{\lambda} \quad , \quad _2 = \quad \exp \ \frac{}{\lambda} e^{\lambda} \quad .$

49. $\quad + \quad + \ (\ ^{\lambda} + \ ) \quad + [\ (\quad + \ ) \ ^{\lambda} + \quad] \quad = 0.$

The substitution $\quad = \quad '' \ + (\ e^{\lambda} \ + \ )$ leads to a first-order linear equation: $\quad ' + a \quad = 0.$

50. $\quad + \quad ^{\lambda} \quad - 2 \quad ^{\lambda} \quad = 0.$

The substitution $\quad = \quad ' - 2 \quad$ leads to a second-order linear equation of the form 2.1.3.1: $\quad '' \ + ae^{\lambda} \quad = 0.$

51. $= (\ {}^{\lambda} -\ )\quad + (\ {}^{\lambda} -\ )\quad +\ {}^{\lambda}\ .$

Particular solutions: $_1 = e^{\ 1}$, $_2 = e^{\ 2}$, where β_1 and β_2 are roots of the quadratic equation $\beta^2 + a\beta + \ = 0$.

52. $+ (\ {}^{\lambda} + 3)\quad + (\ + 2)\ {}^{\lambda}\quad + (\ {}^{\lambda} +\ {}^{\lambda} -\ {}^{2}\ ) = 0.$

Particular solutions: $_1 = \ ^{-1}\exp\bigl(-\tfrac{1}{2}\ \ \cos\bigl(\tfrac{\ 3}{2}$, $_2 = \ ^{-1}\exp\bigl(-\tfrac{1}{2}\ \ \sin\bigl(\tfrac{\ 3}{2}$.

53. ${}^{2}\quad + (\ {}^{\lambda} +\ )\quad + [\ (\ - 2)\ {}^{\lambda} +\ ]\quad + (\ -\ + 2)\ {}^{\lambda}\quad = 0.$

Particular solutions: $_1 = \ ^{1}$, $_2 = \ ^{2}$, where $_1$ and $_2$ are roots of the quadratic equation $^{2} + (\ - 3)\ +\ -\ + 2 = 0$.

54. ${}^{3}\quad +\ {}^{2}\ {}^{\lambda}\quad +\quad + (\ {}^{\lambda} - 2)\ = 0.$

Particular solutions: $_1 = \ ^{1}$, $_2 = \ ^{2}$, where $_1$ and $_2$ are roots of the quadratic equation $^{2} -\ + a = 0$.

55. ${}^{3}\quad +\ {}^{2}(\ {}^{\lambda} +\ )\quad + (\ {}^{\lambda} +\ -\ )\quad + (\ {}^{\lambda} - 2)\ = 0.$

Particular solutions: $_1 = \ ^{1}$, $_2 = \ ^{2}$, where $_1$ and $_2$ are roots of the quadratic equation $^{2} + (\ - 1)\ +\ = 0$.

56. $(\ +\ )\quad -\quad = 0.$

Particular solution: $_0 = ae\ +\ .$

57. $(\ +\ {}^{2})\quad -\quad = 0.$

Particular solution: $_0 = ae\ +\ ^{2}.$

58. $(\ +\ )\quad +\ = 0.$

Particular solution: $_0 = a\ +\ e^{-}\ .$

59. $(\ {}^{2} +\ )\quad +\ = 0.$

Particular solution: $_0 = a\ ^{2} +\ e^{-}\ .$

3.1.4. Equations Containing Hyperbolic Functions

3.1.4-1. Equations with hyperbolic sine.

1. $+\quad + \sinh^{2}\quad + \sinh^{2}\quad = 0.$

The substitution $= \ ' + a$ leads to a second-order linear equation of the form 2.1.4.1: $''\ +\ \sinh^{2}\ \ = 0$.

2. $+ \sinh (\ )\quad +\quad + \sinh (\ )\ = 0.$

1 . Particular solutions with > 0: $_1 = \cos\bigl(\ \overline{\ }$, $_2 = \sin\bigl(\ \overline{\ }$.

2 . Particular solutions with < 0: $_1 = \exp\bigl(-\ \overline{-}$, $_2 = \exp\bigl(\ \overline{-}$.

3. $+ \sinh (\ )\quad +\quad +\ {}^{-1}[\ \sinh (\ ) +\ ]\ = 0.$

The substitution $= \ '' +$ leads to a first-order linear equation: $' + a \sinh\ (\lambda\ )\ = 0$.

4. $+ \sinh (\ )\quad + \sinh (\ )\quad +\ {}^{2}[\ \sinh (\ ) -\ ]\ = 0.$

Particular solutions: $_1 = \exp\bigl(-\tfrac{1}{2}\ \ \cos\bigl(\tfrac{\ 3}{2}$, $_2 = \exp\bigl(-\tfrac{1}{2}\ \ \sin\bigl(\tfrac{\ 3}{2}$.

5. $+ \ \mathbf{sinh}\ (\quad) \quad - \ [2\ \ \mathbf{sinh}\ (\quad) + 3\ ] \quad + \ {}^2[\ \ \mathbf{sinh}\ (\quad) + 2\ ] \ = 0.$

Particular solutions: $\ _1 = e^{b}\ ,\quad _2 = \ \ e^{b}\ .$

6. $+ \ \mathbf{sinh} \qquad + (\quad \mathbf{sinh} \quad + \ - \ {}^2) \quad + \ (\ \mathbf{sinh} \quad - \ ) \ = 0.$

Particular solutions: $\ _1 = \exp(\lambda_1\ ),\quad _2 = \exp(\lambda_2\ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + \ \lambda + \ = 0.$

7. $+ \quad \mathbf{sinh} \qquad + (\quad {}^2 - \quad \mathbf{sinh} \quad) \quad + \quad (\quad {}^2\mathbf{sinh} \quad + 3) \ = 0.$

Particular solutions: $\ _1 = \cos\big(\tfrac{1}{2}\ \ {}^2\ \ -\ ,\quad _2 = \sin\big(\tfrac{1}{2}\ \ {}^2\ \ -\ .$

8. $+ \quad {}^2\mathbf{sinh}\ (\quad) \quad - 2 \quad \mathbf{sinh}\ (\quad) \quad + 2\ \ \mathbf{sinh}\ (\quad) \ = 0.$

Particular solutions: $\ _1 = \ ,\quad _2 = \ {}^2.$

9. $= (\mathbf{sinh} \quad - \ ) \quad + (\ \mathbf{sinh} \quad - \ ) \quad + \ \mathbf{sinh} \quad .$

Particular solutions: $\ _1 = \exp(\lambda_1\ ),\quad _2 = \exp(\lambda_2\ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0.$

10. $+ (\ \mathbf{sinh} \quad + \ ) \quad + \ (\quad \mathbf{sinh} \quad + 2) \quad + \quad \mathbf{sinh} \qquad = 0.$

Particular solutions: $\ _1 = \exp\big(-\tfrac{1}{2}\ \ {}^2\ ,\quad _2 = \exp\big(-\tfrac{1}{2}\ \ {}^2 \quad \exp\big(\tfrac{1}{2}\ \ {}^2 \quad .$

11. $+ \ (\ \mathbf{sinh}^2 \quad + \ ) \quad - 2(\ \mathbf{sinh}^2 \quad + \ ) \ = 0.$

The substitution $\ = \ \ ' - 2\ $ leads to a second-order linear equation of the form 2.1.4.1: $\ '' \ + (a \sinh^2 \ + \ ) \ = 0.$

12. ${}^2 \qquad + (\quad {}^2\mathbf{sinh} \quad + \quad) \quad + [\ (\ - 2)\ \ \mathbf{sinh} \quad + \ ] \quad + \ (\ - \ + 2)\,\mathbf{sinh} \qquad = 0.$

Particular solutions: $\ _1 = \ \ {}^1,\quad _2 = \ \ {}^2$, where $\ _1$ and $\ _2$ are roots of the quadratic equation $\ {}^2 + (\ - 3)\ \ + \ - \ + 2 = 0.$

13. ${}^3 \qquad + \ {}^2(\ \mathbf{sinh} \quad + \ ) \quad + \ (\quad \mathbf{sinh} \quad + \ - \ ) \quad + \ (\ \mathbf{sinh} \quad - 2) \ = 0.$

Particular solutions: $\ _1 = \ \ {}^1,\quad _2 = \ \ {}^2$, where $\ _1$ and $\ _2$ are roots of the quadratic equation $\ {}^2 + (\ - 1)\ \ + \ = 0.$

14. $\mathbf{sinh} \qquad + \qquad + \qquad + \ {}^2(\ - \ \ \mathbf{sinh} \quad) \ = 0.$

Particular solutions: $\ _1 = \exp\big(-\tfrac{1}{2} \quad \cos\big(\tfrac{\ \overline{3}}{2} \quad ,\quad _2 = \exp\big(-\tfrac{1}{2} \quad \sin\big(\tfrac{\ \overline{3}}{2} \quad .$

15. $\mathbf{sinh} \qquad + \qquad + \ \mathbf{sinh} \qquad + \qquad = 0.$

1 . Particular solutions with $\ > 0$: $\ _1 = \cos\big(\quad -\ ,\quad _2 = \sin\big(\quad -\ .$

2 . Particular solutions with $\ < 0$: $\ _1 = \exp\big(-\ \ \overline{-}\ ,\quad _2 = \exp\big(\ \ \overline{-}\ .$

16. $\mathbf{sinh} \qquad + \qquad - \ (2\ \ + 3\ \ \mathbf{sinh} \quad) \quad + \ {}^2(\ + 2\ \ \mathbf{sinh} \quad) \ = 0.$

Particular solutions: $\ _1 = e^{b}\ ,\quad _2 = \ \ e^{b}\ .$

17. $\mathbf{sinh} \qquad + \qquad + [(\ - \ {}^2)\,\mathbf{sinh} \quad + \ ] \quad + \ (1 - \ \ \mathbf{sinh} \quad) \ = 0.$

Particular solutions: $\ _1 = e^{\lambda_1}\ ,\quad _2 = e^{\lambda_2}\ $, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0.$

18. $\mathbf{sinh}\ (\quad) \qquad + \quad {}^2 \qquad - 2 \qquad + 2 \qquad = 0.$

Particular solutions: $\ _1 = \ ,\quad _2 = \ {}^2.$

19. **sinh + (sinh + + 1) + ^{2} − ^{2} = 0.**

Particular solutions: $_1 =\ ,\quad _2 = e^{-}$.

20. **sinh + (sinh + 1) + (+ 2 sinh) + = 0.**

Particular solutions: $_1 = \exp\bigl(-\tfrac{1}{2}a\ ^2$, $_2 = \exp\bigl(-\tfrac{1}{2}a\ ^2$ $\exp\bigl(\tfrac{1}{2}a\ ^2$.

21. **sinh + (3 sinh +) + (sinh + 2) + (sinh +) = 0.**

Particular solutions: $_1 = \ ^{-1}\cos\bigl(\ \overline{a}$, $_2 = \ ^{-1}\sin\bigl(\ \overline{a}$.

22. **^{3} sinh + ^{2} − 2 sinh + 2(2 sinh −) = 0.**

Particular solutions: $_1 = \ ^{-1}$, $_2 = \ ^2$.

23. **^{3} sinh + ^{2} − 6 sinh + 6(2 sinh −) = 0.**

Particular solutions: $_1 = \ ^{-2}$, $_2 = \ ^3$.

24. **^{3} sinh + ^{2} + (− sinh) + (− 3 sinh) = 0.**

Particular solutions: $_1 = \cos(\ln\ )$, $_2 = \sin(\ln\ )$.

25. **^{3} sinh + ^{2}(sinh +) + [− (+ 1) sinh] + (2 sinh −) = 0.**

Particular solutions: $_1 = \ ^{-\ \overline{b}}$, $_2 = \ ^{\overline{b}}$.

3.1.4-2. Equations with hyperbolic cosine.

26. **+ + cosh(2) + cosh(2) = 0.**

The substitution $=\ ' + a$ leads to a second-order linear equation of the form 2.1.4.9: $'' + \cosh(2\ ) = 0$.

27. **+ + cosh^2 + cosh^2 = 0.**

The substitution $=\ ' + a$ leads to a second-order linear equation of the form 2.1.4.10: $'' + \cosh^2\ = 0$.

28. **+ cosh ( ) + + cosh ( ) = 0.**

1 . Particular solutions with > 0: $_1 = \cos\bigl(\ \overline{\ }$, $_2 = \sin\bigl(\ \overline{\ }$.

2 . Particular solutions with < 0: $_1 = \exp\bigl(-\ \overline{-}$, $_2 = \exp\bigl(\ \overline{-}$.

29. **+ cosh ( ) + + $^{-1}$[cosh ( ) +] = 0.**

The substitution $=\ '' +$ leads to a first-order linear equation: $' + a\cosh\ (\lambda\ ) = 0$.

30. **+ cosh ( ) + cosh ( ) + ^{2}[cosh ( ) −] = 0.**

Particular solutions: $_1 = \exp\bigl(-\tfrac{1}{2}$ $\cos\bigl(\tfrac{\ 3}{2}$, $_2 = \exp\bigl(-\tfrac{1}{2}$ $\sin\bigl(\tfrac{\ 3}{2}$.

31. **+ cosh ( ) − [2 cosh ( ) + 3] + ^{2}[cosh ( ) + 2] = 0.**

Particular solutions: $_1 = e^{b}$, $_2 = \ e^{b}$.

32. **+ cosh + (cosh + − ^{2}) + (cosh −) = 0.**

Particular solutions: $_1 = \exp(\lambda_1\ )$, $_2 = \exp(\lambda_2\ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + \ \lambda + \ = 0$.

33. $+ \quad \cosh \quad + (\quad ^2 - \quad \cosh \quad) \quad + \quad (\quad ^2 \cosh \quad + 3) \quad = 0.$

Particular solutions: $\ _1 = \cos\big(\tfrac{1}{2}\ ^2 \ -\ \ , \quad _2 = \sin\big(\tfrac{1}{2}\ ^2\ -\ \ .$

34. $+ \quad ^2 \cosh \ (\quad) \quad - 2 \quad \cosh \ (\quad) \quad + 2 \quad \cosh \ (\quad) \quad = 0.$

Particular solutions: $\ _1 = \ , \quad _2 = \ ^2.$

35. $= (\cosh \quad - \ ) \quad + (\ \cosh \quad - \ ) \quad + \ \cosh \quad .$

Particular solutions: $\ _1 = \exp(\lambda_1 \ ), \quad _2 = \exp(\lambda_2 \ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0.$

36. $+ (\ \cosh \quad + \quad) \quad + \ (\quad \cosh \quad + 2) \quad + \quad \cosh \quad = 0.$

Particular solutions: $\ _1 = \exp\big(-\tfrac{1}{2}\ ^2 \ , \quad _2 = \exp\big(-\tfrac{1}{2}\ ^2 \quad \exp\big(\tfrac{1}{2}\ ^2 \quad .$

37. $+ \ [\ \cosh(2 \ ) + \ ] \quad - 2[\ \cosh(2 \ ) + \ ] \quad = 0.$

The substitution $\ = \ ' - 2 \ $ leads to a second-order linear equation of the form 2.1.4.9: $'' \ + [a \cosh(2 \ ) + \ ] \ = 0.$

38. $+ \ (\ \cosh^2 \quad + \ ) \quad - 2(\ \cosh^2 \quad + \ ) \quad = 0.$

The substitution $\ = \ ' - 2 \ $ leads to a second-order linear equation of the form 2.1.4.10: $'' \ + (a \cosh^2 \quad + \ ) \ = 0.$

39. $= (\cosh \quad - \quad) \quad + (\ \cosh \quad - \quad) \quad + \ \cosh \quad .$

Particular solutions: $\ _1 = \exp(\lambda_1 \ ), \quad _2 = \exp(\lambda_2 \ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0.$

40. $^2 \quad + (\ ^2 \cosh \quad + \quad) \quad + [\ (\ - 2) \ \cosh \quad + \ ] \quad + \ (\ - \ + 2) \cosh \quad = 0.$

Particular solutions: $\ _1 = \ ^1, \quad _2 = \ ^2$, where $\ _1$ and $\ _2$ are roots of the quadratic equation $\ ^2 + (\ - 3) \ + \ - \ + 2 = 0.$

41. $^3 \quad + \ ^2(\ \cosh \quad + \ ) \quad + \ (\ \cosh \quad + \ - \ ) \quad + \ (\ \cosh \quad - 2) \quad = 0.$

Particular solutions: $\ _1 = \ ^1, \quad _2 = \ ^2$, where $\ _1$ and $\ _2$ are roots of the quadratic equation $\ ^2 + (\ - 1) \ + \ = 0.$

42. $\cosh \quad + \quad + \quad + \ ^2(\ - \ \cosh \quad) \quad = 0.$

Particular solutions: $\ _1 = \exp\big(-\tfrac{1}{2} \quad \cos\big(\tfrac{\ 3}{2} \quad , \quad _2 = \exp\big(-\tfrac{1}{2} \quad \sin\big(\tfrac{\ 3}{2} \quad .$

43. $\cosh \quad + \quad + \ \cosh \quad + \quad = 0.$

1 . Particular solutions with $\ > 0$: $\ _1 = \cos\big(\quad , \quad _2 = \sin\big(\quad .$

2 . Particular solutions with $\ < 0$: $\ _1 = \exp\big(- \quad , \quad _2 = \exp\big(\quad .$

44. $\cosh \quad + \quad - \ (2 \ + 3 \ \cosh \quad) \quad + \ ^2(\ + 2 \ \cosh \quad) \quad = 0.$

Particular solutions: $\ _1 = e^b \ , \quad _2 = \ e^b \ .$

45. $\cosh \quad + \quad + [(\ - \ ^2) \cosh \quad + \ ] \quad + \ (1 - \ \cosh \quad) \quad = 0.$

Particular solutions: $\ _1 = e^{\lambda_1} \ , \quad _2 = e^{\lambda_2} \ $, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0.$

46. $\cosh \ (\quad) \quad + \quad ^2 \quad - 2 \quad + 2 \quad = 0.$

Particular solutions: $\ _1 = \ , \quad _2 = \ ^2.$

47. **cosh + (cosh + + 1) + ^{2} − ^{2} = 0.**

Particular solutions: $_1 =$, $_2 = e^{-}$.

48. **cosh + (cosh + 1) + (+ 2 cosh) + = 0.**

Particular solutions: $_1 = \exp\bigl(-\tfrac{1}{2}a\ ^2$, $_2 = \exp\bigl(-\tfrac{1}{2}a\ ^2$ $\exp\bigl(\tfrac{1}{2}a\ ^2$.

49. **cosh + (3 cosh +) + (cosh + 2) + (cosh +) = 0.**

Particular solutions: $_1 =\ ^{-1}\cos\bigl(\ \overline{a}$, $_2 =\ ^{-1}\sin\bigl(\ \overline{a}$.

50. **^{3} cosh + ^{2} − 2 cosh + 2(2 cosh −) = 0.**

Particular solutions: $_1 =\ ^{-1}$, $_2 =\ ^2$.

51. **^{3} cosh + ^{2} − 6 cosh + 6(2 cosh −) = 0.**

Particular solutions: $_1 =\ ^{-2}$, $_2 =\ ^3$.

52. **^{3} cosh + ^{2} + (− cosh) + (− 3 cosh) = 0.**

Particular solutions: $_1 = \cos\bigl(\ln|\ |$, $_2 = \sin\bigl(\ln|\ |$.

53. **^{3} cosh + ^{2}(cosh +) + [− (+ 1) cosh] + (2 cosh −) = 0.**

Particular solutions: $_1 =\ ^{-\ \overline{b}}$, $_2 =\ ^{\overline{b}}$.

3.1.4-3. Equations with hyperbolic sine and cosine.

54. **+ [sinh(2) +] + cosh(2) = 0.**

This is a special case of equation 3.1.9.26 with ( ) = $\tfrac{1}{2}$[a sinh(2) +].

55. **+ [sinh(2) +] + 2 cosh(2) = 0.**

Integrating yields a second-order linear equation: $''$ + [a sinh(2) +] = .

56. **+ [cosh(2) +] + sinh(2) = 0.**

Solution: = $_1\ ^2_1 +\ _2\ _1\ _2 +\ _3\ ^2_2$. Here, $_1$ and $_2$ form a fundamental set of solutions of the modified Mathieu equation 2.1.4.9: 4 $''$ + [a cosh(2) +] = 0.

57. **+ [cosh(2) +] + 2 sinh(2) = 0.**

Integrating yields a second-order linear equation: $''$ + [a cosh(2) +] = (see 2.1.4.9 for the solution of the corresponding modified homogeneous Mathieu equation with = 0).

58. **+ (cosh − ^{2}) + (sinh − cosh) = 0.**

This is a special case of equation 3.1.9.29 with ( ) = cosh .

3.1.4-4. Equations with hyperbolic tangent.

59. **− ^{3} tanh( ) = 0.**

Particular solution: $_0$ = cosh(a). The substitution = cosh(a) $z($) leads to a second-order linear equation of the form 2.1.4.43: z'' + $3a$ tanh(a)z' + $3a^2z = 0$.

60. **= + (1 –) tanh .**

This is a special case of equation 3.1.9.30 with ( ) = a and g( ) = cosh .

61. **– 3 ^{2} + 2 ^{3} tanh( ) = 0.**

Particular solutions: $_1$ = cosh(a), $_2$ = cosh(a).

62. **= tanh + tanh (1 – tanh) .**

This is a special case of equation 3.1.9.30 with ( ) = a tanh and g( ) = cosh .

63. **+ + [tanh( ) +] + [tanh( ) +] = 0.**

The substitution = $'$ + a leads to a second-order linear equation of the form 2.1.4.22: $''$ + [tanh(λ) +] = 0.

64. **+ – [2 tanh( ) + 3] + ^{2}[2 tanh^{2}( ) + 2 tanh( ) –] = 0.**

Particular solutions: $_1$ = cosh(λ), $_2$ = cosh(λ).

65. **– tanh – + tanh = 0.**

1 . Solution for $a > 0$: = $_1$ exp(– $\overline{a}$) + $_2$ exp($\overline{a}$) + $_3$ cosh .

2 . Solution for $a < 0$: = $_1$ cos($\overline{-a}$) + $_2$ sin($\overline{-a}$) + $_3$ cosh .

66. **+ tanh ( ) + + $^{-1}$[tanh ( ) +] = 0.**

The substitution = $''$ + leads to a first-order linear equation: $'$ + a tanh (λ) = 0.

67. **+ tanh ( ) + tanh ( ) + ^{2}[tanh ( ) –] = 0.**

Particular solutions: $_1$ = exp$\bigl(-\frac{1}{2}$ cos$\bigl(\frac{\ \ 3}{2}$, $_2$ = exp$\bigl(-\frac{1}{2}$ sin$\bigl(\frac{\ \ 3}{2}$.

68. **+ tanh ( ) – [2 tanh ( ) + 3] + ^{2}[tanh ( ) + 2] = 0.**

Particular solutions: $_1$ = e^{b} , $_2$ = e^{b} .

69. **+ tanh ( ) + [tanh ( ) + – ^{2}] + [tanh ( ) –] = 0.**

Particular solutions: $_1$ = exp(β_1), $_2$ = exp(β_2), where β_1 and β_2 are roots of the quadratic equation β^2 + β + = 0.

70. **+ – (2 tanh + 3) + [(2 tanh^{2} – 1) + 2 tanh] = 0.**

Particular solutions: $_1$ = cosh , $_2$ = cosh .

71. **+ tanh – (2 tanh $^{+1}$ + 3) + (2 tanh $^{+2}$ – tanh + 2 tanh) = 0.**

Particular solutions: $_1$ = cosh , $_2$ = cosh .

72. **+ tanh + (^{2} – tanh) + (^{2} tanh + 3) = 0.**

Particular solutions: $_1$ = cos$\bigl(\frac{1}{2}$ ^{2} – , $_2$ = sin$\bigl(\frac{1}{2}$ ^{2} – .

73. **+ ^{2} tanh ( ) – 2 tanh ( ) + 2 tanh ( ) = 0.**

Particular solutions: $_1$ = , $_2$ = ^{2}.

74. **= (tanh –) + (tanh –) + tanh .**

Particular solutions: $_1$ = exp(λ_1), $_2$ = exp(λ_2), where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda +$ = 0.

75. $+ (\ \tanh\ +\ )\ +\ (\ \tanh\ + 2)\ +\ \tanh\ = 0.$

Particular solutions: $_1 = \exp\bigl(-\tfrac{1}{2}\ ^2\ ,\ _2 = \exp\bigl(-\tfrac{1}{2}\ ^2\ \exp\bigl(\tfrac{1}{2}\ ^2\ .$

76. $+ [\ (\tanh\ -\ ) -\ ]\ + [\ (\ ^2 - 1)\ - 1]\ +\ [\ (1 -\ \tanh\ ) + 1]\ = 0.$

Particular solutions: $_1 = e^b\ ,\ _2 = \cosh\ .$

77. $+ [\ \tanh(\ )(\ - 1) -\ ^{-1}]\ -\ ^2\ +\ ^2\ ^{-1}\ = 0.$

Particular solutions: $_1 =\ ,\ _2 = \cosh(\lambda\ ).$

78. $+ (\ \tanh\ ^{+1}\ -\ \tanh\ -\ )$

$+ [\ (\ ^2 - 1) \tanh\ - 1]\ +\ (-\ \tanh\ ^{+1}\ +\ \tanh\ + 1)\ = 0.$

Particular solutions: $_1 = e^b\ ,\ _2 = \cosh\ .$

79. $^2\ + (\ ^2 \tanh\ +\ )\ + [\ (\ - 2)\ \tanh\ +\ ]\ +\ (\ -\ + 2) \tanh\ = 0.$

Particular solutions: $_1 =\ ^1,\ _2 =\ ^2$, where $_1$ and $_2$ are roots of the quadratic equation $^2 + (\ - 3)\ +\ -\ + 2 = 0.$

80. $^3\ +\ ^2(\ \tanh\ +\ )\ +\ (\ \tanh\ +\ -\ )\ +\ (\ \tanh\ - 2)\ = 0.$

Particular solutions: $_1 =\ ^1,\ _2 =\ ^2$, where $_1$ and $_2$ are roots of the quadratic equation $^2 + (\ - 1)\ +\ = 0.$

81. $\tanh\ +\ +\ +\ ^2(\ -\ \tanh\ )\ = 0.$

Particular solutions: $_1 = \exp\bigl(-\tfrac{1}{2}\ \cos\bigl(\tfrac{\ 3}{2}\ ,\ _2 = \exp\bigl(-\tfrac{1}{2}\ \sin\bigl(\tfrac{\ 3}{2}\ .$

82. $\tanh\ +\ +\ \tanh\ +\ = 0.$

1 . Particular solutions with > 0: $_1 = \cos\bigl(\ \ ,\ _2 = \sin\bigl(\ \ .$

2 . Particular solutions with < 0: $_1 = \exp\bigl(-\ \ ,\ _2 = \exp\bigl(\ \ .$

83. $\tanh\ +\ -\ (2\ + 3\ \tanh\ )\ +\ ^2(\ + 2\ \tanh\ )\ = 0.$

Particular solutions: $_1 = e^b\ ,\ _2 =\ e^b\ .$

84. $\tanh\ +\ + [(\ -\ ^2) \tanh\ +\ ]\ +\ (1 -\ \tanh\ )\ = 0.$

Particular solutions: $_1 = e^{\lambda_1}\ ,\ _2 = e^{\lambda_2}\ $, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda +\ = 0.$

85. $\tanh\ (\ )\ +\ ^2\ - 2\ + 2\ = 0.$

Particular solutions: $_1 =\ ,\ _2 =\ ^2.$

86. $\tanh\ + (\ \tanh\ +\ + 1)\ +\ ^2\ -\ ^2\ = 0.$

Particular solutions: $_1 =\ ,\ _2 = e^{-}\ .$

87. $\tanh\ + (\ \tanh\ + 1)\ +\ (\ + 2 \tanh\ )\ +\ = 0.$

Particular solutions: $_1 = \exp\bigl(-\tfrac{1}{2}a\ ^2\ ,\ _2 = \exp\bigl(-\tfrac{1}{2}a\ ^2\ \exp\bigl(\tfrac{1}{2}a\ ^2\ .$

88. $\tanh\ + (3 \tanh\ +\ )\ + (\ \tanh\ + 2)\ +\ (\tanh\ +\ )\ = 0.$

Particular solutions: $_1 =\ ^{-1} \cos\bigl(\ \overline{a}\ ,\ _2 =\ ^{-1} \sin\bigl(\ \overline{a}\ .$

89. $\;{}^{3}\,\mathbf{tanh}\qquad + \quad {}^{2} \quad - 2 \quad \mathbf{tanh} \qquad + 2(2\,\mathbf{tanh} \quad - \;)\; = 0.$

Particular solutions: $\;_{1} = \;{}^{-1}, \quad _{2} = \;{}^{2}.$

90. $\;{}^{3}\,\mathbf{tanh}\qquad + \quad {}^{2} \quad - 6 \quad \mathbf{tanh} \qquad + 6(2\,\mathbf{tanh} \quad - \;)\; = 0.$

Particular solutions: $\;_{1} = \;{}^{-2}, \quad _{2} = \;{}^{3}.$

91. $\;{}^{3}\,\mathbf{tanh}\qquad + \quad {}^{2} \quad + \;(\quad - \mathbf{tanh} \quad) \quad + (\quad - 3\,\mathbf{tanh} \quad) \; = 0.$

Particular solutions: $\;_{1} = \cos\bigl(\ln|\;|\;, \quad _{2} = \sin\bigl(\ln|\;|\;.$

92. $\;{}^{3}\,\mathbf{tanh}\qquad + \;{}^{2}(\mathbf{tanh} \quad + \;) \quad + \;[\;\; - (\;\; + 1)\,\mathbf{tanh} \quad] \quad + \;(2\,\mathbf{tanh} \quad - \;) \; = 0.$

Particular solutions: $\;_{1} = \; - \;\overline{b}, \quad _{2} = \quad \overline{b}.$

3.1.4-5. Equations with hyperbolic cotangent.

93. $\quad - \;{}^{3}\,\mathbf{coth}(\quad) \; = 0.$

Particular solution: $\;_{0} = \sinh(a\;)$. The substitution $\; = \sinh(a\;)\;\; z(\;)$ leads to a second-order linear equation of the form 2.1.4.52: $z'' \; + 3a\coth(a\;)z' + 3a^2 z = 0.$

94. $\quad = \quad + (1 - \;)\,\mathbf{coth} \quad .$

This is a special case of equation 3.1.9.30 with $\;(\;) = a$ and $g(\;) = \sinh\;$.

95. $\quad - 3 \;{}^{2} \quad + 2 \;{}^{3}\,\mathbf{coth}(\quad) \; = 0.$

Particular solutions: $\;_{1} = \sinh(a\;), \quad _{2} = \quad \sinh(a\;).$

96. $\quad = \;\mathbf{coth} \qquad + \mathbf{coth} \;\; (1 - \quad \mathbf{coth} \quad)\;.$

This is a special case of equation 3.1.9.30 with $\;(\;) = a\coth\;$ and $g(\;) = \sinh\;$.

97. $\quad + \qquad + [\;\;\mathbf{coth}(\quad) + \;] \quad + \;[\;\;\mathbf{coth}(\quad) + \;] \; = 0.$

The substitution $\; = \;' + a\;$ leads to a second-order linear equation of the form 2.1.4.44: $\;'' \; + [\;\;\coth(\lambda\;) + \;] \quad = 0.$

98. $\quad + \qquad - \;[2\;\;\mathbf{coth}(\quad) + 3\;] \quad + \;{}^{2}[2\;\;\mathbf{coth}^{2}(\quad) + 2\;\;\mathbf{coth}(\quad) - \;] \; = 0.$

Particular solutions: $\;_{1} = \sinh(\lambda\;), \quad _{2} = \quad \sinh(\lambda\;).$

99. $\quad - \mathbf{coth} \qquad - \qquad + \quad \mathbf{coth} \qquad = 0.$

1 . Solution for $a > 0$: $\; = \;_{1}\exp(- \quad \overline{a}\,) + \;_{2}\exp(\quad \overline{a}\,) + \;_{3}\sinh\;$.

2 . Solution for $a < 0$: $\; = \;_{1}\cos(\quad \overline{-a}) + \;_{2}\sin(\quad \overline{-a}) + \;_{3}\sinh\;$.

100. $\quad + (\;\;\mathbf{coth} \quad - \quad - \;) \quad + (\;\;{}^{2} - \quad - 1) \quad + \;(- \quad \mathbf{coth} \quad + \quad + 1) \; = 0.$

Particular solutions: $\;_{1} = e^{b}\;, \quad _{2} = \sinh\;$.

101. $\quad + \;\;\mathbf{coth}\;\;(\quad) \quad + \qquad + \quad {}^{-1}[\quad \mathbf{coth}\;\;(\quad) + \quad] \; = 0.$

The substitution $\; = \;'' + \;$ leads to a first-order linear equation: $\;' + a\coth\;\;(\lambda\;) \; = 0.$

102. $\quad + \;\;\mathbf{coth}\;\;(\quad) \quad + \quad \mathbf{coth}\;\;(\quad) \quad + \;{}^{2}[\;\;\mathbf{coth}\;\;(\quad) - \;] \; = 0.$

Particular solutions: $\;_{1} = \exp\bigl(-\tfrac{1}{2} \quad \cos\bigl(\tfrac{\overline{3}}{2} \quad , \quad _{2} = \exp\bigl(-\tfrac{1}{2} \quad \sin\bigl(\tfrac{\overline{3}}{2} \quad .$

103. $+ \coth(\quad) \quad - [2 \coth(\quad) + 3\ ] \quad + {}^2[\ \coth(\quad) + 2\ ] \ = 0.$

Particular solutions: $_1 = e^{b}\ ,\quad _2 = \ e^{b}\ .$

104. $+ \quad \coth \quad + (\ ^2 - \ \coth \ ) \quad + \quad (\ ^2 \coth \quad + 3) \ = 0.$

Particular solutions: $_1 = \cos\big(\tfrac{1}{2}\ ^2\ \ ^{-}\ ,\quad _2 = \sin\big(\tfrac{1}{2}\ ^2\ \ ^{-}\ .$

105. $+ \ ^2\coth(\quad) \quad - 2 \quad \coth(\quad) \quad + 2 \ \coth(\quad) \ = 0.$

Particular solutions: $_1 = \ ,\quad _2 = \ ^2.$

106. $+ \quad\quad - (2 \quad \coth \quad + 3) \quad + [\quad (2\coth^2 \quad - 1) + 2\coth \ ] \ = 0.$

Particular solutions: $_1 = \sinh\ ,\quad _2 = \ \sinh\ .$

107. $+ \ \coth \quad\quad - (2 \ \coth \ ^{+1} \ + 3) \quad + (2 \ \coth \ ^{+2} \ - \ \coth \quad + 2\coth \ ) \ = 0.$

Particular solutions: $_1 = \sinh\ ,\quad _2 = \ \sinh\ .$

108. $+ (\ \coth \quad + \quad) \quad + \ (\quad \coth \quad + 2) \quad + \quad \coth \quad = 0.$

Particular solutions: $_1 = \exp\big(-\tfrac{1}{2}\ ^2\ ,\quad _2 = \exp\big(-\tfrac{1}{2}\ ^2\quad \exp\big(\tfrac{1}{2}\ ^2\quad .$

109. $+ [\ \coth(\quad)(\quad - 1) - \quad ^{-1}] \quad - \quad ^2 \quad + \quad ^2 \ ^{-1} \ = 0.$

Particular solutions: $_1 = \ ,\quad _2 = \sinh(\lambda\ ).$

110. $^2 \quad + (\ ^2\coth \quad + \quad) \quad + [\ (\ - 2) \ \coth \quad + \ ] \quad + \ (\ - \ + 2)\coth \quad = 0.$

Particular solutions: $_1 = \ ^{1},\ _2 = \ ^{2}$, where $_1$ and $_2$ are roots of the quadratic equation $^2 + (\ - 3) \ + \ - \ + 2 = 0.$

111. $^3 \quad + \ ^2(\ \coth \quad + \ ) \quad + \ (\quad \coth \quad + \ - \ ) \quad + \ (\ \coth \quad - 2) \ = 0.$

Particular solutions: $_1 = \ ^{1},\ _2 = \ ^{2}$, where $_1$ and $_2$ are roots of the quadratic equation $^2 + (\ - 1) \ + \ = 0.$

112. $\coth \quad + \quad + \quad + \ ^2(\ - \ \coth \ ) \ = 0.$

Particular solutions: $_1 = \exp\big(-\tfrac{1}{2}\quad \cos\big(\tfrac{\sqrt{3}}{2}\quad ,\quad _2 = \exp\big(-\tfrac{1}{2}\quad \sin\big(\tfrac{\sqrt{3}}{2}\quad .$

113. $\coth \quad + \quad + \ \coth \quad + \quad = 0.$

1 . Particular solutions with $\ > 0$: $_1 = \cos\big(\quad \overline{\ }\ ,\quad _2 = \sin\big(\quad \overline{\ }\ .$

2 . Particular solutions with $\ < 0$: $_1 = \exp\big(-\quad \overline{-}\ ,\quad _2 = \exp\big(\quad \overline{-}\ .$

114. $\coth \quad + \quad - \ (2 \ + 3 \ \coth \ ) \quad + \ ^2(\ + 2 \ \coth \ ) \ = 0.$

Particular solutions: $_1 = e^{b}\ ,\quad _2 = \ e^{b}\ .$

115. $\coth \quad + \quad + [(\ - \ ^2)\coth \quad + \ ] \quad + \ (1 - \ \coth \ ) \ = 0.$

Particular solutions: $_1 = e^{\lambda_1}\ ,\quad _2 = e^{\lambda_2}\ $, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0.$

116. $\coth(\quad) \quad + \quad ^2 \quad - 2 \quad + 2 \quad = 0.$

Particular solutions: $_1 = \ ,\quad _2 = \ ^2.$

117. $\coth \quad + (\ \coth \quad + \quad + 1) \quad + \ ^2 \quad - \ ^2 \ = 0.$

Particular solutions: $_1 = \ ,\quad _2 = e^{-}\ .$

118. **coth** $+($ **coth** $+1)$ $+$ $($ $+2$ **coth** $)$ $+$ $=0.$

Particular solutions: $_1 = \exp\bigl(-\tfrac{1}{2}a\ {}^2$, $_2 = \exp\bigl(-\tfrac{1}{2}a\ {}^2$ $\exp\bigl(\tfrac{1}{2}a\ {}^2$.

119. **coth** $+(3$ **coth** $+\ )$ $+($ **coth** $+2)$ $+$ $($**coth** $+\ )$ $=0.$

Particular solutions: $_1 = {}^{-1}\cos\bigl(\ \overline{a}$, $_2 = {}^{-1}\sin\bigl(\ \overline{a}$.

120. ^{3} **coth** $+$ ^{2} -2 **coth** $+2(2$ **coth** $-\ )$ $=0.$

Particular solutions: $_1 = {}^{-1}$, $_2 = {}^2$.

121. ^{3} **coth** $+$ ^{2} -6 **coth** $+6(2$ **coth** $-\ )$ $=0.$

Particular solutions: $_1 = {}^{-2}$, $_2 = {}^3$.

122. ^{3} **coth** $+$ ^{2} $+$ $($ $-$ **coth** $)$ $+($ -3 **coth** $)$ $=0.$

Particular solutions: $_1 = \cos\bigl(\ln$, $_2 = \sin\bigl(\ln$.

123. ^{3} **coth** $+$ ^{2}(**coth** $+\ )$ $+$ $[\ -(\ +1)$ **coth** $]$ $+$ $(2$ **coth** $-\ )$ $=0.$

Particular solutions: $_1 = {}^{-\ \overline{b}}$, $_2 = {}^{\overline{b}}$.

3.1.5. Equations Containing Logarithmic Functions

3.1.5-1. Equations with logarithmic functions.

1. $+$ **ln** $(\ )$ $+$ $+$ **ln** $(\ )$ $=0.$

1 . Particular solutions with >0: $_1 = \cos\bigl(\ \overline{\ }$, $_2 = \sin\bigl(\ \overline{\ }$.

2 . Particular solutions with <0: $_1 = \exp\bigl(-\ \overline{-}$, $_2 = \exp\bigl(\ \overline{-}$.

2. $+$ **ln** $+$ **ln** $+$ $^2($ **ln** $-\ )$ $=0.$

Particular solutions: $_1 = \exp\bigl(-\tfrac{1}{2}$ $\cos\bigl(\tfrac{\sqrt{3}}{2}$, $_2 = \exp\bigl(-\tfrac{1}{2}$ $\sin\bigl(\tfrac{\sqrt{3}}{2}$.

3. $+$ **ln** $(\ )$ $-$ $[2$ **ln** $(\ )+3\ ]$ $+$ $^2[$ **ln** $(\ )+2\ ]$ $=0.$

Particular solutions: $_1 = e^b$, $_2 = e^b$.

4. $+$ **ln** $+($ **ln** $+\ -\ {}^2)$ $+$ $($ **ln** $-\ )$ $=0.$

Particular solutions: $_1 = \exp(\lambda_1\ )$, $_2 = \exp(\lambda_2\ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + \ \lambda + \ = 0$.

5. $+($ **ln** $+\ )$ $+$ $+$ $($ **ln** $+\ )$ $=0.$

1 . Particular solutions with >0: $_1 = \cos\bigl(\ \overline{\ }$, $_2 = \sin\bigl(\ \overline{\ }$.

2 . Particular solutions with <0: $_1 = \exp\bigl(-\ \overline{-}$, $_2 = \exp\bigl(\ \overline{-}$.

6. $=($**ln** $-\ )$ $+($ **ln** $-\ )$ $+$ **ln** .

Particular solutions: $_1 = \exp(\lambda_1\ )$, $_2 = \exp(\lambda_2\ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0$.

7. **ln** $+$ $+$ $+$ $^2($ $-$ **ln** $)$ $=0.$

Particular solutions: $_1 = \exp\bigl(-\tfrac{1}{2}$ $\cos\bigl(\tfrac{\sqrt{3}}{2}$, $_2 = \exp\bigl(-\tfrac{1}{2}$ $\sin\bigl(\tfrac{\sqrt{3}}{2}$.

8. $\ln \quad + \quad + [(\; - \; ^2) \ln \quad + \;] \quad + \; (1 - \; \ln \quad) \; = 0.$

Particular solutions: $\;_1 = e^{\lambda_1}\;, \quad _2 = e^{\lambda_2}\;$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + \; \lambda + a = 0$.

9. $\ln \quad + \quad - \; (2 \; + 3 \; \ln \quad) \quad + \; ^2(\; + 2 \; \ln \quad) \; = 0.$

Particular solutions: $\;_1 = e^{b} \;, \quad _2 = \; e^{b} \;.$

10. $\ln \; (\quad) \quad + \quad + \; \ln \; (\quad) \quad + \quad = 0.$

1 . Particular solutions with $\; > 0$: $\;_1 = \cos\big(\quad - \;, \quad _2 = \sin\big(\quad - \;.$

2 . Particular solutions with $\; < 0$: $\;_1 = \exp\big(- \quad \overline{-} \;, \quad _2 = \exp\big(\quad \overline{-} \;.$

3.1.5-2. Equations with power and logarithmic functions.

11. $\quad + (\quad + \;) \ln \; (\quad) \quad - \quad \ln \; (\quad) \; = 0.$

Particular solution: $\;_0 = a \; + \;.$

12. $\quad + (\quad + \;) \ln \; (\quad) \quad - 2 \quad \ln \; (\quad) \; = 0.$

Particular solution: $\;_0 = (a \; + \;)^2.$

13. $\quad + \; \ln \quad + (\quad + \;) \quad + (\quad \ln \quad + \quad \ln \quad + \;) \; = 0.$

Integrating yields a second-order linear equation: $\;'' + (\quad + \;) \; = \quad \exp \; -a \quad \ln$ (see 2.1.2.2 for the solution of the corresponding homogeneous equation with $\; = 0$).

14. $\quad + \; \ln \; (\quad) \quad + \quad + \quad ^{-1}[\quad \ln \; (\quad) + \;] \; = 0.$

The substitution $\; = \; '' + \;$ leads to a first-order linear equation: $\;' + a \ln \; (\lambda \;) \; = 0.$

15. $\quad + \quad \ln \quad + (\; ^2 - \quad \ln \quad) \quad + \quad (\quad ^2 \ln \quad + 3) \; = 0.$

Particular solutions: $\;_1 = \cos\big(\tfrac{1}{2} \; ^2 \; - \;, \quad _2 = \sin\big(\tfrac{1}{2} \; ^2 \; - \;.$

16. $\quad + (\; \ln \quad + \;) \quad + \; (\quad \ln \quad + 2) \quad + \quad \ln \quad = 0.$

Particular solutions: $\;_1 = \exp\big(-\tfrac{1}{2} \; ^2 \;, \quad _2 = \exp\big(-\tfrac{1}{2} \; ^2 \quad \exp\big(\tfrac{1}{2} \; ^2 \quad .$

17. $\quad + (\quad \ln \quad + \;) \quad + \; (\quad - 1) \ln \quad - \quad \ln \quad = 0.$

The substitution $\; = \; ' + \;$ leads to a second-order linear equation of the form 2.1.5.13: $\;'' + a \; \ln \quad ' - a \ln \quad = 0.$

18. $\quad + (\quad \ln \quad + \; \ln \quad + \;) \quad + \quad ^2 \; \ln \quad - \quad ^2 \ln \quad = 0.$

Particular solutions: $\;_1 = \;, \quad _2 = e^{-b} \;.$

19. $\quad + \quad ^2 \ln \; (\quad) \quad - 2 \quad \ln \; (\quad) \quad + 2 \; \ln \; (\quad) \; = 0.$

Particular solutions: $\;_1 = \;, \quad _2 = \; ^2.$

20. $\quad + \quad - \; (\quad \ln^2 \quad + 1) \quad - \quad (\quad \ln^2 \quad + 1) \; = 0.$

The substitution $\; = \; ' + a \;$ leads to a second-order linear equation of the form 2.1.5.3: $\;'' - (\; ^2 \; \ln^2 \; + \;) \; = 0.$

21. $+ \ln (\ ) \ + \ + \ \ln (\ ) = 0.$

1 . Particular solutions with >0: $_1 = \cos(\ \bar{} \ ,$ $_2 = \sin(\ \bar{} \ .$

2 . Particular solutions with <0: $_1 = \exp(- \ \overline{-} \ ,$ $_2 = \exp(\ \overline{-} \ .$

22. $+ \ \ln \ + (\ \ln \ - \ ^2 + \ ) \ + \ = 0.$

Particular solutions: $_1 = e^{-b} \ ,$ $_2 = e^{-b} \ \ - \ e^{(\ +2b)} \ .$

23. $= (\ln \ - \ ) \ + (\ \ln \ - \ ) \ + \ \ln \ .$

Particular solutions: $_1 = \exp(\lambda_1 \ ),$ $_2 = \exp(\lambda_2 \ ),$ where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0.$

24. $+ (\ \ln \ + 3) \ + (2 \ \ln \ + \ ) \ + (\ \ln \ + 1) \ = 0.$

Particular solutions: $_1 = \ ^{-1} \cos(\ \bar{} \ ,$ $_2 = \ ^{-1} \sin(\ \bar{} \ .$

25. $+ (\ \ln \ + 3) \ + (\ \ln \ + 2 \ \ln \ - \ ^2 \ ) \ + (\ \ln \ - \ ) \ = 0.$

Particular solutions: $_1 = \ ^{-1},$ $_2 = \ ^{-1} e^{-b} \ .$

26. $+ [\ (\ - \ln \ ) \ + 2] \ + \ ^{-1} \ - \ ^{-2} \ = 0.$

Particular solutions: $_1 = \ ,$ $_2 = \ln \ - \ + 1.$

27. $^2 \ + \ln (\ ) \ + \ ^2 \ + \ \ln (\ ) = 0.$

1 . Particular solutions with >0: $_1 = \cos(\ \bar{} \ ,$ $_2 = \sin(\ \bar{} \ .$

2 . Particular solutions with <0: $_1 = \exp(- \ \overline{-} \ ,$ $_2 = \exp(\ \overline{-} \ .$

28. $^2 \ + \ ^2(\ \ln \ + \ ) \ + 2 \ - \ = 0.$

Integrating the equation twice, we arrive at a first-order linear equation: $' + (a \ln \ + \ ) \ = \ _1 + \ _2 \ .$

29. $^2 \ - 3 \ [\ \ln^2(\ ) + 1] \ + [2 \ ^2 \ ^2 \ln^3(\ ) + 1] \ = 0.$

Particular solutions: $_1 = \exp \ a \ \ln(\lambda \ ) \ ,$ $_2 = \ \exp \ a \ \ln(\lambda \ ) \ .$

30. $^2 \ + \ ^2(\ \ln \ + \ ) \ + 2 \ (\ + \ ) \ - \ = 0.$

Integrating the equation twice, we arrive at a first-order linear equation: $' + (a \ln \ + \ ) \ = \ _1 + \ _2 \ .$

31. $^2 \ + (\ ^2 \ln \ + \ ) \ + [\ (\ - 2) \ \ln \ + \ ] \ + (\ - \ + 2) \ln \ = 0.$

Particular solutions: $_1 = \ ^1,$ $_2 = \ ^2,$ where $_1$ and $_2$ are roots of the quadratic equation $^2 + (\ - 3) \ + \ - \ + 2 = 0.$

32. $^3 \ + \ ^2(\ \ln \ + \ ) \ + 2 \ - \ = 0.$

Integrating the equation twice, we obtain a first-order linear equation: $' + (a \ln \ + \ - 2) \ = \ _1 + \ _2 \ .$

33. $^3 \ + \ln (\ ) \ + \ ^3 \ + \ \ln (\ ) = 0.$

1 . Particular solutions with >0: $_1 = \cos(\ \bar{} \ ,$ $_2 = \sin(\ \bar{} \ .$

2 . Particular solutions with <0: $_1 = \exp(- \ \overline{-} \ ,$ $_2 = \exp(\ \overline{-} \ .$

34. $^{3} \quad + \quad ^{2}\ln \quad -2 \quad +2(2- \quad \ln \quad) \quad =0.$

Particular solutions: $_{1}= \ ^{-1}, \quad _{2}= \ ^{2}.$

35. $^{3} \quad + \quad ^{2}\ln \quad -6 \quad +6(2- \quad \ln \quad) \quad =0.$

Particular solutions: $_{1}= \ ^{-2}, \quad _{2}= \ ^{3}.$

36. $^{3} \quad + \ ^{2}(\ \ln \ + \ ) \quad +2 \ (\ + \ ) \ - \quad =0.$

Integrating the equation twice, we obtain a first-order linear equation: $' + (a\ln \ + \ -2) \ = \ _{1} + \ _{2} \ .$

37. $^{3} \quad + \ ^{2}(\ \ln \ + \ ) \quad + \ (\ \ln \ + \ - \ ) \quad + \ (\ \ln \quad -2) \quad =0.$

Particular solutions: $_{1}= \ ^{1}, \quad _{2}= \ ^{2}$, where $_{1}$ and $_{2}$ are roots of the quadratic equation $^{2}+(\ -1) \ + \ =0.$

38. $\ln \quad +(\ \ln \ +1) \quad + \ (\ +2\ln \ ) \ + \quad =0.$

Particular solutions: $_{1}=\exp\bigl(-\tfrac{1}{2}a \ ^{2} \ , \quad _{2}=\exp\bigl(-\tfrac{1}{2}a \ ^{2} \quad \exp\bigl(\tfrac{1}{2}a \ ^{2} \quad .$

39. $\ln \quad +(\ \ln \ + \ +1) \quad + \ ^{2} \quad - \ ^{2} \ =0.$

Particular solutions: $_{1}= \ , \quad _{2}=e^{-} \ .$

40. $\ln \ (\ ) \quad + \ ^{2} \quad -2 \quad +2 \quad =0.$

Particular solutions: $_{1}= \ , \quad _{2}= \ ^{2}.$

3.1.6. Equations Containing Trigonometric Functions

$\boxed{\text{3.1.6-1. Equations with sine.}}$

1. $+ \ \sin \quad - \ (\ \sin \ + \ ^{2}) \quad =0.$

The substitution $=e^{b} \ ^{2}(\ ' - \ )$ leads to a second-order linear equation of the form 2.1.6.2: $'' \ + \bigl(a\sin \ + \tfrac{3}{4} \ ^{2} \quad =0.$

2. $+ \ \sin^{2} \quad - \ (\ \sin^{2} \ + \ ^{2}) \quad =0.$

The substitution $=e^{b} \ ^{2}(\ ' - \ )$ leads to a second-order linear equation of the form 2.1.6.3: $'' \ + \bigl(a\sin^{2} \ + \tfrac{3}{4} \ ^{2} \quad =0.$

3. $+[\ \sin(\ )+ \ ] \quad - \ [\ \sin(\ )+ \ + \ ^{2}] \quad =0.$

The substitution $=e^{c} \ ^{2}(\ ' - \ )$ leads to a second-order linear equation of the form 2.1.6.2: $'' \ + \bigl[a\sin(\lambda \ )+ \ + \tfrac{3}{4} \ ^{2}\bigr] \quad =0.$

4. $+ \quad +[\ \sin(\ )+ \ ] \quad + \ [\ \sin(\ )+ \ ] \quad =0.$

The substitution $= \ ' + a$ leads to a second-order linear equation of the form 2.1.6.2: $'' \ +[\ \sin(\lambda \ )+ \ ] \ =0.$

5. $+ \quad + \ \sin^{2}(\ ) \quad + \ \sin^{2}(\ ) \quad =0.$

The substitution $= \ ' + a$ leads to a second-order linear equation of the form 2.1.6.3: $'' \ + \ \sin^{2}(\lambda \ ) \quad =0.$

6. $+ \ \sin(\ ) \ - \ - \ \sin(\ ) = 0.$

1 . Particular solutions with >0: $_1 = \exp(-$ $\overline{\ }$, $_2 = \exp($ $\overline{\ }$.

2 . Particular solutions with <0: $_1 = \cos($ $\overline{\ }$, $_2 = \sin($ $\overline{\ }$.

7. $+ \ \sin(\ ) \ + \ \sin(\ ) \ + \ ^2[\ \sin(\ ) - \ ] = 0.$

Particular solutions: $_1 = \exp(-\frac{1}{2}$ $\cos(\frac{3}{2}$, $_2 = \exp(-\frac{1}{2}$ $\sin(\frac{3}{2}$.

8. $+ \ \sin(\ ) \ - [2 \ \sin(\ ) + 3\ ] \ + \ ^2[\ \sin(\ ) + 2\ ] = 0.$

Particular solutions: $_1 = e^b$, $_2 = \ e^b$.

9. $+ \ \sin \ + (\ \sin \ + \ - \ ^2) \ + (\ \sin \ - \ ) = 0.$

Particular solutions: $_1 = \exp(\lambda_1\ )$, $_2 = \exp(\lambda_2\ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + \ \lambda + \ = 0$.

10. $+ \ \sin(\ ) \ + \ + \ ^{-1}[\ \sin(\ ) + \ ] = 0.$

The substitution $= \ '' +$ leads to a first-order linear equation: $' + a \sin(\lambda\ ) \ = 0.$

11. $+ (\ \sin \ + \ ) \ + \ + (\ \sin \ + \ ) = 0.$

1 . Particular solutions with >0: $_1 = \cos($ $\overline{\ }$, $_2 = \sin($ $\overline{\ }$.

2 . Particular solutions with <0: $_1 = \exp(-$ $\overline{\ }$, $_2 = \exp($ $\overline{\ }$.

12. $= (\sin \ - \ ) \ + (\ \sin \ - \ ) \ + \ \sin \ .$

Particular solutions: $_1 = \exp(\lambda_1\ )$, $_2 = \exp(\lambda_2\ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0$.

13. $+ (\ \sin \ + \ ) \ + (\ \sin \ + 2) \ + \ \sin \ = 0.$

Particular solutions: $_1 = \exp(-\frac{1}{2}\ ^2$, $_2 = \exp(-\frac{1}{2}\ ^2$ $\exp(\frac{1}{2}\ ^2$.

14. $+ \ \sin \ + (\ ^2 - \ \sin \ ) \ + (\ ^2 \sin \ + 3) = 0.$

Particular solutions: $_1 = \cos(\frac{1}{2}\ ^2$ $\overline{\ }$, $_2 = \sin(\frac{1}{2}\ ^2$ $\overline{\ }$.

15. $+ (\ \sin \ + \ \sin \ + \ ) \ + \ ^2 \ \sin \ - \ ^2 \sin \ = 0.$

Particular solutions: $_1 =$, $_2 = e^{-b}$.

16. $+ \ ^2 \sin(\ ) \ - 2 \ \sin(\ ) \ + 2 \ \sin(\ ) = 0.$

Particular solutions: $_1 =$, $_2 = \ ^2.$

17. $+ [\ \sin(\ ) + \ ] \ - 2[\ \sin(\ ) + \ ] = 0.$

The substitution $= \ ' - 2$ leads to a second-order linear equation of the form 2.1.6.2: $'' + [a \sin(\lambda\ ) + \ ] \ = 0.$

18. $= (\sin \ - \ ) \ + (\ \sin \ - \ ) \ + \ \sin \ .$

Particular solutions: $_1 = \exp(\lambda_1\ )$, $_2 = \exp(\lambda_2\ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0$.

19. $+ (\ \sin \ + 3) \ + (2 \ \sin \ + \ ) \ + (\ \sin \ + 1) = 0.$

Particular solutions: $_1 = \ ^{-1} \cos($ $\overline{\ }$, $_2 = \ ^{-1} \sin($ $\overline{\ }$.

20. ^{2} = (sin − ^{2}) + (sin − ^{2}) + sin .

Particular solutions: $_1 = \exp(\lambda_1\ )$, $_2 = \exp(\lambda_2\ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0$.

21. ^{3} + ^{2} sin ( ) + + [sin ( ) − 2] = 0.

Particular solutions: $_1 = \ ^{1}$, $_2 = \ ^{2}$, where $_1$ and $_2$ are roots of the quadratic equation $^2 - \ + \ = 0$.

22. sin^{2} + sin^{2} + + = 0.

The substitution $= \ ' + a$ leads to a second-order linear equation of the form 2.1.6.21: $\sin^2\ \ '' + \ = 0$.

23. sin + + + ^{2}(− sin) = 0.

Particular solutions: $_1 = \exp\bigl(-\tfrac{1}{2}\ \bigr)\cos\bigl(\tfrac{\sqrt{3}}{2}\ \bigr)$, $_2 = \exp\bigl(-\tfrac{1}{2}\ \bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}\ \bigr)$.

24. sin + + sin + = 0.

1 . Particular solutions with > 0: $_1 = \cos\bigl(\ \sqrt{\ }\ \bigr)$, $_2 = \sin\bigl(\ \sqrt{\ }\ \bigr)$.

2 . Particular solutions with < 0: $_1 = \exp\bigl(-\ \sqrt{-\ }\ \bigr)$, $_2 = \exp\bigl(\ \sqrt{-\ }\ \bigr)$.

25. sin + − (2 + 3 sin) + ^{2}(+ 2 sin) = 0.

Particular solutions: $_1 = e^{b}$, $_2 = \ e^{b}$.

26. sin + + [(− ^{2}) sin +] + (1 − sin) = 0.

Particular solutions: $_1 = e^{\lambda_1}$, $_2 = e^{\lambda_2}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0$.

27. sin ( ) + ^{2} − 2 + 2 = 0.

Particular solutions: $_1 = \ $, $_2 = \ ^{2}$.

28. sin + (sin + + 1) + ^{2} − ^{2} = 0.

Particular solutions: $_1 = \ $, $_2 = e^{-\ }$.

29. sin + (sin + 1) + (+ 2 sin) + = 0.

Particular solutions: $_1 = \exp\bigl(-\tfrac{1}{2}a\ ^2\bigr)$, $_2 = \exp\bigl(-\tfrac{1}{2}a\ ^2\bigr)\int \exp\bigl(\tfrac{1}{2}a\ ^2\bigr)\ $.

30. sin + (3 sin +) + (sin + 2) + (sin +) = 0.

Particular solutions: $_1 = \ ^{-1}\cos\bigl(\ \sqrt{a}\ \bigr)$, $_2 = \ ^{-1}\sin\bigl(\ \sqrt{a}\ \bigr)$.

31. ^{3} sin + ^{2} − 2 sin + 2(2 sin −) = 0.

Particular solutions: $_1 = \ ^{-1}$, $_2 = \ ^{2}$.

32. ^{3} sin + ^{2} − 6 sin + 6(2 sin −) = 0.

Particular solutions: $_1 = \ ^{-2}$, $_2 = \ ^{3}$.

33. ^{3} sin + ^{2} + (− sin) + (− 3 sin) = 0.

Particular solutions: $_1 = \cos(\ln\ )$, $_2 = \sin(\ln\ )$.

34. ^{3} sin + ^{2}(sin +) + [− (+ 1) sin] + (2 sin −) = 0.

Particular solutions: $_1 = \ ^{-\sqrt{b}}$, $_2 = \ ^{\sqrt{b}}$.

3.1.6-2. Equations with cosine.

35. $+\ \cos(2\ )\quad -\ [\ \cos(2\ )+\ ^2]\ \ = 0.$

The substitution $= e^{b}\ ^2(\ '-\ \ )$ leads to a Mathieu equation of the form 2.1.6.29: $''\ +\big[a\cos(2\ )+\frac{3}{4}\ ^2\big]\ \ = 0.$

36. $+[\ \cos(\ \ )+\ ]\quad -\ [\ \cos(\ \ )+\ \ +\ ^2]\ \ = 0.$

The transformation $\xi = \frac{1}{2}\lambda\ ,\quad = e^{c}\ ^2(\ '-\ \ )$ leads to the Mathieu equation 2.1.6.29: $''\ +4\lambda^{-2}\big[a\cos(2\xi)+\ \ +\frac{3}{4}\ ^2\big]\ \ = 0.$

37. $+\qquad +(\ \cos 2\ \ +\ )\quad +\ (\ \cos 2\ \ +\ )\ \ = 0.$

The substitution $= \ ' + a$ leads to the Mathieu equation 2.1.6.29: $''\ +(\ \cos 2\ \ +\ )\ \ = 0.$

38. $+\ \cos\ (\ \ )\quad +\quad +\quad \cos\ (\ \ )\ \ = 0.$

1 . Particular solutions with > 0: $_1 = \cos\big(\ \bar{}\ ,\quad _2 = \sin\big(\ \bar{}\ .$

2 . Particular solutions with < 0: $_1 = \exp\big(-\ \overline{\ -\ }\ ,\quad _2 = \exp\big(\ \overline{\ -\ }\ .$

39. $+\ \cos\ (\ \ )\quad +\ \cos\ (\ \ )\quad +\ ^2[\ \cos\ (\ \ )-\ ]\ \ = 0.$

Particular solutions: $_1 = \exp\big(-\frac{1}{2}\qquad \cos\big(\frac{\sqrt{3}}{2}\qquad ,\quad _2 = \exp\big(-\frac{1}{2}\qquad \sin\big(\frac{\sqrt{3}}{2}\qquad .$

40. $+\ \cos\ (\ \ )\quad -\ [2\ \cos\ (\ \ )+3\ ]\quad +\ ^2[\ \cos\ (\ \ )+2\ ]\ \ = 0.$

Particular solutions: $_1 = e^{b}\ ,\quad _2 = \ \ e^{b}\ .$

41. $+\ \cos\qquad +(\ \cos\ \ +\ -\ ^2)\quad +\ (\ \cos\ \ -\ )\ \ = 0.$

Particular solutions: $_1 = \exp(\lambda_1\ )$, $_2 = \exp(\lambda_2\ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2+\ \lambda+\ \ =0.$

42. $+\ \cos\ (\ \ )\quad +\qquad +\qquad ^{-1}[\ \cos\ (\ \ )+\ \ ]\ \ = 0.$

The substitution $= \ ''+$ leads to a first-order linear equation: $'+a\cos\ (\lambda\ )\ \ =0.$

43. $+(\ \cos\ \ +\ )\quad +\quad +\ (\ \cos\ \ +\ )\ \ = 0.$

1 . Particular solutions with > 0: $_1 = \cos\big(\ \bar{}\ ,\quad _2 = \sin\big(\ \bar{}\ .$

2 . Particular solutions with < 0: $_1 = \exp\big(-\ \overline{\ -\ }\ ,\quad _2 = \exp\big(\ \overline{\ -\ }\ .$

44. $= (\cos\ \ -\ )\quad +(\ \cos\ \ -\ )\quad +\ \cos\qquad .$

Particular solutions: $_1 = \exp(\lambda_1\ )$, $_2 = \exp(\lambda_2\ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2+a\lambda+\ \ =0.$

45. $+(\ \cos\ \ +\ \ )\quad +\ (\ \ \cos\ \ +2)\quad +\ \ \cos\qquad = 0.$

Particular solutions: $_1 = \exp\big(-\frac{1}{2}\ \ ^2\ ,\quad _2 = \exp\big(-\frac{1}{2}\ \ ^2\qquad \exp\big(\frac{1}{2}\ \ ^2\qquad .$

46. $+\ \ \cos\qquad +(\ \ ^2-\ \cos\ \ )\quad +\ \ (\ \ ^2\cos\ \ +3)\ \ = 0.$

Particular solutions: $_1 = \cos\big(\frac{1}{2}\ ^2\ \ \bar{}\ ,\quad _2 = \sin\big(\frac{1}{2}\ ^2\ \ \bar{}\ .$

47. $+(\ \ \cos\ \ +\ \cos\ \ +\ )\quad +\ \ ^2\ \cos\qquad -\ \ ^2\cos\qquad = 0.$

Particular solutions: $_1 = \ \ ,\quad _2 = e^{-b}\ .$

48. $+\ {}^{2}\cos(\ )\quad -2\ \cos(\ )\quad +2\ \cos(\ )\quad = 0.$

Particular solutions: ${}_1 = \ ,\quad {}_2 = \ {}^2.$

49. $+\ (\ \cos 2\ +\ )\quad -2(\ \cos 2\ +\ )\quad = 0.$

The substitution $=\ {}'-2$ leads to the Mathieu equation 2.1.6.29: ${}''+(a\cos 2\ +\ )\ =0.$

50. $+\ (\ \cos^2\ +\ )\quad -2(\ \cos^2\ +\ )\quad = 0.$

The substitution $=\ {}' - 2$ leads to a second-order linear equation of the form 2.1.6.30: ${}''\ +(a\cos^2\ +\ )\ =0.$

51. $= (\cos\ -\ )\quad +(\ \cos\ -\ )\quad +\ \cos\ .$

Particular solutions: ${}_1 = \exp(\lambda_1\ ),\quad {}_2 = \exp(\lambda_2\ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0.$

52. $+(\ \cos\ +3)\quad +(2\ \cos\ +\ )\quad +\ (\ \cos\ +1)\ = 0.$

Particular solutions: ${}_1 = {}^{-1}\cos(\ \ \bar{}\ ,\quad {}_2 = {}^{-1}\sin(\ \ \bar{}\ .$

53. ${}^2\quad = (\cos\ -\ {}^2)\quad +(\ \cos\ -\ {}^2)\quad +\ \cos\ .$

Particular solutions: ${}_1 = \exp(\lambda_1\ ),\quad {}_2 = \exp(\lambda_2\ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0.$

54. ${}^3\quad +\ {}^2\cos(\ )\quad +\quad +[\ \cos(\ )-2]\ = 0.$

Particular solutions: ${}_1 = \ {}^{1},\quad {}_2 = \ {}^{2}$, where ${}_1$ and ${}_2$ are roots of the quadratic equation ${}^2 - \ + \ = 0.$

55. $\cos^2\quad +\ \cos^2\quad +\quad +\quad = 0.$

The substitution $=\xi+\frac{\pi}{2}$ leads to an equation of the form 3.1.6.22: $\sin^2\xi\ {}''' + a\sin^2\xi\ {}'' + {}' + a\ = 0.$

56. $\cos\quad +\quad +\quad +\ {}^2(\ -\ \cos\ )\ = 0.$

Particular solutions: ${}_1 = \exp\bigl(-\tfrac{1}{2}\ \bigr)\cos\bigl(\tfrac{\sqrt{3}}{2}\ \bigr),\quad {}_2 = \exp\bigl(-\tfrac{1}{2}\ \bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}\ \bigr).$

57. $\cos\quad +\quad +\ \cos\quad +\quad = 0.$

1 . Particular solutions with > 0: ${}_1 = \cos(\ \bar{}\ ,\quad {}_2 = \sin(\ \bar{}\ .$

2 . Particular solutions with < 0: ${}_1 = \exp(-\ \bar{-}\ ,\quad {}_2 = \exp(\ \bar{-}\ .$

58. $\cos\quad +\quad -\ (2\ +3\ \cos\ )\quad +\ {}^2(\ +2\ \cos\ )\ = 0.$

Particular solutions: ${}_1 = e^{b}\ ,\quad {}_2 = \ e^{b}\ .$

59. $\cos\quad +\quad +[(\ -\ {}^2)\cos\ +\ ]\quad +\ (1-\ \cos\ )\ = 0.$

Particular solutions: ${}_1 = e^{\lambda_1}\ ,\quad {}_2 = e^{\lambda_2}\ $, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0.$

60. $\cos(\ )\quad +\ {}^2\quad -2\quad +2\quad = 0.$

Particular solutions: ${}_1 = \ ,\quad {}_2 = \ {}^2.$

61. $\cos\quad +(\ \cos\ +\ +1)\quad +\ {}^2\quad -\ {}^2\ = 0.$

Particular solutions: ${}_1 = \ ,\quad {}_2 = e^{-}\ .$

62. **cos + (cos + 1) + (+ 2 cos) + = 0.**

Particular solutions: $_1 = \exp\bigl(-\tfrac{1}{2}a\ ^2$, $_2 = \exp\bigl(-\tfrac{1}{2}a\ ^2 \quad \exp\bigl(\tfrac{1}{2}a\ ^2$.

63. **cos + (3 cos +) + (cos + 2) + (cos +) = 0.**

Particular solutions: $_1 = \ ^{-1}\cos\bigl(\ \ \overline{a}$, $_2 = \ ^{-1}\sin\bigl(\ \ \overline{a}$.

64. **^{3} cos + ^{2} − 2 cos + 2(2 cos −) = 0.**

Particular solutions: $_1 = \ ^{-1}$, $_2 = \ ^2$.

65. **^{3} cos + ^{2} − 6 cos + 6(2 cos −) = 0.**

Particular solutions: $_1 = \ ^{-2}$, $_2 = \ ^3$.

66. **^{3} cos + ^{2} + (− cos) + (− 3 cos) = 0.**

Particular solutions: $_1 = \cos(\ln|\ |)$, $_2 = \sin(\ln|\ |)$.

67. **^{3} cos + ^{2}(cos +) + [− (+ 1) cos] + (2 cos −) = 0.**

Particular solutions: $_1 = |\ |^{-\ \ \overline{b}}$, $_2 = |\ |\ \ \overline{b}$.

3.1.6-3. Equations with sine and cosine.

68. **+ [sin( ) +] + cos( ) = 0.**

Integrating yields a second-order linear equation: $'' + [a\sin(\lambda\ ) + \ ]\ =$ (see 2.1.6.2 for the solution of the corresponding homogeneous equation with $= 0$).

69. **+ [sin( ) − ^{2}] + [cos( ) − sin( )] = 0.**

By integrating and substituting $= e^{b\ \ ^2}$, we obtain a second-order nonhomogeneous linear equation: $'' + \bigl[a\sin(\lambda\ ) - \tfrac{1}{4}\ ^2\bigr]\ = \ e^{3b\ \ ^2}$ (see 2.1.6.2 for the solution of the corresponding homogeneous equation).

70. **+ [cos(2) +] − sin(2) = 0.**

Solution: $= \ _1\ _1^2 + \ _2\ _1\ _2 + \ _3\ _2^2$. Here, $_1$, $_2$ is a fundamental set of solutions of the Mathieu equation 2.1.6.29: $4\ '' + [a\cos(2\ ) + \ ]\ = 0$.

71. **+ [cos(2) +] − 2 sin(2) = 0.**

Integrating yields a nonhomogeneous Mathieu equation: $'' + [a\cos(2\ ) + \ ]\ = \ $.

72. **+ [cos(2) − ^{2}] − [cos(2) + 2 sin(2)] = 0.**

By integrating and substituting $= e^{b\ \ ^2}$, we obtain a nonhomogeneous Mathieu equation: $'' + [a\cos(2\ ) - \tfrac{1}{4}\ ^2]\ = \ e^{3b\ \ ^2}$.

73. **− 3 [sin^2( ) + cos( )] + sin( )[^{2} + 2 ^{2} sin^2( )] = 0.**

Particular solutions: $_1 = \exp\ -\frac{a}{\ }\cos(\ )$, $_2 = \ \exp\ -\frac{a}{\ }\cos(\ )$.

74. **sin( ) + + 3 ^{2} sin( ) + 2 ^{3} cos( ) = 0.**

This is a special case of equation 3.1.9.105 with $(\ ) = 0$.

75. **sin( ) + [+ (2 + 1) cos( )] − (^{2} + 2) sin( ) − ^{2} cos( ) = 0.**

This is a special case of equation 3.1.9.106 with $(\ ) = 0$.

76. $\sin^2 \qquad + 3\sin \quad \cos \qquad + [\cos 2 \quad + 4\;(\quad + 1)\sin^2 \;] \quad + 2\;(\quad + 1)\sin 2 \qquad = 0.$

Solution: $\quad = {}_1\,{}^2_1 + {}_2\,{}_1\,{}_2 + {}_3\,{}^2_2$. Here, ${}_1$, ${}_2$ form a fundamental set of solutions of the Legendre equation 2.1.2.154, with the argument of the functions ${}_1$ and ${}_2$ substituted by cos .

3.1.6-4. Equations with tangent.

77. $\quad + \;{}^3\tan(\quad) \quad = 0.$

Integrating yields a second-order nonhomogeneous linear equation: $\;'' + \tan\xi \;\; ' - \quad = \quad \cos\xi$, where $\xi = a$ (see 2.1.6.53 for the solution of the corresponding homogeneous equation).

78. $\quad - \;{}^3\tan(\quad) \quad = 0.$

Particular solution: ${}_0 = \cos(a \;)$. The substitution $\; = \cos(a \;) \;\; z(\;)$ leads to a second-order linear equation of the form 2.1.6.53: $z'' - 3\tan\xi\, z' - 3z = 0$, where $\xi = a$.

79. $\quad + 3 \;{}^2 \quad + 2 \;{}^3\tan(\quad) \quad = 0.$

Particular solutions: ${}_1 = \cos(a \;)$, ${}_2 = \quad \cos(a \;)$.

80. $\quad + \quad + (\quad - 1)\tan \quad = 0.$

Particular solution: ${}_0 = \cos$. The substitution $\; = \cos \quad z(\;)$ leads to a second-order linear equation: $z'' - 3\tan \quad z' + (a - 3)z = 0$.

81. $\quad + \quad + \;(\quad - \;{}^2)\tan(\quad) \quad = 0.$

Particular solution: ${}_0 = \cos(\lambda \;)$. The substitution $\; = \cos(\lambda \;) \;\; z(\;)$ leads to a second-order linear equation of the form 2.1.6.53: $z'' - 3\tan\xi\, z' + (a\lambda^{-2} - 3)z = 0$, where $\xi = \lambda$.

82. $\quad + \;\tan^2 \quad - \;(\;\tan^2 \quad + \;{}^2) \quad = 0.$

The substitution $\; = e^{b} \;\; {}^2(\; ' - \quad)$ leads to a second-order linear equation of the form 2.1.6.51: $\;'' + \left(a\tan^2 \quad + \tfrac{3}{4} \;{}^2 \right. \quad = 0.$

83. $\quad + [\;\tan^2(\quad) + \;] \quad - \;[\;\tan^2(\quad) + \quad + \;{}^2] \quad = 0.$

The transformation $\xi = \lambda \;$, $\; = e^{c} \;\; {}^2(\; ' - \quad)$ leads to an equation of the form 2.1.6.51: $\;'' + \lambda^{-2}(a\tan^2\xi + \quad + \tfrac{3}{4} \;{}^2) \quad = 0.$

84. $\quad + \;\tan \quad + \tan \;(\;\tan \quad - 1) \quad = 0.$

Particular solution: ${}_0 = \cos$. The substitution $\; = \cos \quad z(\;)$ leads to a second-order linear equation: $z'' - 3\tan \quad z' + (a\tan \quad - 3)z = 0$.

85. $\quad + \quad + (\;\tan^2 \quad + \;) \quad + \;(\;\tan^2 \quad + \;) \quad = 0.$

The substitution $\; = \;' + a$ leads to a second-order linear equation of the form 2.1.6.51: $\;'' + (\;\tan^2 \quad + \;) \quad = 0.$

86. $\quad + \quad + \;[3 \quad + 2 \;\tan(\quad)] \quad + \;{}^2\{ \;[1 + 2\tan^2(\quad)] + 2 \;\tan(\quad)\} \quad = 0.$

Particular solutions: ${}_1 = \cos(\lambda \;)$, ${}_2 = \quad \cos(\lambda \;)$.

87. **+ tan( ) − − tan( ) = 0.**

1 . Solution for $a>0$: $= {}_1\exp(-\ \overline{a}\ + {}_2\exp(\ \overline{a}\ + {}_3\cos(\lambda\ )$.

2 . Solution for $a<0$: $= {}_1\cos(\ \overline{-a}\ + {}_2\sin(\ \overline{-a}\ + {}_3\cos(\lambda\ )$.

88. **+ tan( ) + + (+ − ^{2}) tan( ) = 0.**

Particular solution: ${}_0=\cos(\lambda\ )$. The transformation $=\frac{z}{\lambda}$, $=\cos(\lambda\ )$ leads to a second-order linear equation of the form 2.1.6.131: $'' + \big(a\lambda^{-1} - 3\ \tan z\ \ ' + \big(\ \lambda^{-2} - 3 - 2a\lambda^{-1}\tan^2 z\ = 0$.

89. **+ tan ( ) + + tan ( ) = 0.**

1 . Particular solutions with >0: ${}_1=\cos(\ \ -\ $, ${}_2=\sin(\ \ -\ $.

2 . Particular solutions with <0: ${}_1=\exp(-\ \overline{-}\ $, ${}_2=\exp(\ \overline{-}\ $.

90. **+ tan ( ) + tan ( ) + ^{2}[tan ( ) −] = 0.**

Particular solutions: ${}_1=\exp(-\frac{1}{2}\ \ \cos(\frac{\ 3}{2}\ $, ${}_2=\exp(-\frac{1}{2}\ \ \sin(\frac{\ 3}{2}\ $.

91. **+ tan ( ) − [2 tan ( ) + 3] + ^{2}[tan ( ) + 2] = 0.**

Particular solutions: ${}_1=e^b$, ${}_2=\ e^b$.

92. **+ tan + (tan + − ^{2}) + (tan −) = 0.**

Particular solutions: ${}_1=\exp(\lambda_1\ )$, ${}_2=\exp(\lambda_2\ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2+\ \lambda+\ =0$.

93. **+ tan ( ) + + $^{-1}$[tan ( ) +] = 0.**

The substitution $=\ ''+\ $ leads to a first-order linear equation: $'+a\tan\ (\lambda\ )\ =0$.

94. **+ (tan +) + + (tan +) = 0.**

1 . Particular solutions with >0: ${}_1=\cos(\ \ -\ $, ${}_2=\sin(\ \ -\ $.

2 . Particular solutions with <0: ${}_1=\exp(-\ \overline{-}\ $, ${}_2=\exp(\ \overline{-}\ $.

95. **= (tan −) + (tan −) + tan .**

Particular solutions: ${}_1=\exp(\lambda_1\ )$, ${}_2=\exp(\lambda_2\ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2+a\lambda+\ =0$.

96. **+ (tan +) + (tan + 2) + tan = 0.**

Particular solutions: ${}_1=\exp(-\frac{1}{2}\ ^2$, ${}_2=\exp(-\frac{1}{2}\ ^2\ \ \exp(\frac{1}{2}\ ^2\ $.

97. **+ tan + (^{2} − tan) + (^{2} tan + 3) = 0.**

Particular solutions: ${}_1=\cos(\frac{1}{2}\ ^2\ -\ $, ${}_2=\sin(\frac{1}{2}\ ^2\ -\ $.

98. **+ (tan + tan +) + ^{2} tan − ^{2} tan = 0.**

Particular solutions: ${}_1=\ $, ${}_2=e^{-b}$.

99. **+ ^{2} tan ( ) − 2 tan ( ) + 2 tan ( ) = 0.**

Particular solutions: ${}_1=\ $, ${}_2=\ ^2$.

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100. $-[\ (\ \ +\tan\ \ )\ \ +\ \ ]\ \ +[\ (\ ^2+1)\ \ +1]\ \ +\ [\ (\ \ \tan\ \ -1)\ \ -1]\ \ =0.$

Particular solutions: $_1=e\ \ ,\ \ _2=\cos\ \ .$

101. $-(\ \ \tan\ \ +\ \tan\ ^{+1}\ +\ )\ \ +[\ (\ ^2+1)\tan\ \ +1]$
$+\ (\ \ \tan\ ^{+1}\ -\ \tan\ \ -1)\ \ =0.$

Particular solutions: $_1=e\ \ ,\ \ _2=\cos\ \ .$

102. $+[\ \tan(\ \ )(\ \ +1)+\ \ ^{-1}]\ \ -\ ^2\ \ +\ ^2\ ^{-1}\ =0.$

Particular solutions: $_1=\ \ ,\ \ _2=\cos(\lambda\ \ ).$

103. $+\ (\ \tan^2\ \ +\ )\ \ -2(\ \tan^2\ \ +\ )\ \ =0.$

The substitution $\ \ =\ \ '-2\ \ $ leads to a second-order linear equation of the form 2.1.6.51: $''\ +(a\tan^2\ \ +\ )\ \ =0.$

104. $=(\tan\ \ -\ \ )\ \ +(\ \tan\ \ -\ \ )\ \ +\ \tan\ \ .$

Particular solutions: $_1=\exp(\lambda_1\ \ )$, $_2=\exp(\lambda_2\ \ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2+a\lambda+\ \ =0.$

105. $+(\ \ \tan\ \ +3)\ \ +(2\ \ \tan\ \ +\ \ )\ \ +\ (\ \ \tan\ \ +1)\ \ =0.$

Particular solutions: $_1=\ ^{-1}\cos(\ \ \ \overline{\ }\ \ ,\ \ _2=\ ^{-1}\sin(\ \ \ \overline{\ }\ \ .$

106. $^2\ \ =(\tan\ \ -\ ^2)\ \ +(\ \tan\ \ -\ ^2)\ \ +\ \tan\ \ .$

Particular solutions: $_1=\exp(\lambda_1\ \ )$, $_2=\exp(\lambda_2\ \ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2+a\lambda+\ \ =0.$

107. $^3\ \ +\ ^2\tan(\ \ )\ \ +\ \ +[\ \tan(\ \ )-2]\ \ =0.$

Particular solutions: $_1=\ \ ^1$, $_2=\ \ ^2$, where $_1$ and $_2$ are roots of the quadratic equation $^2-\ \ +\ \ =0.$

108. $\tan\ \ \ +\ \ +\ \ +\ ^2(\ -\ \tan\ \ )\ \ =0.$

Particular solutions: $_1=\exp(-\tfrac{1}{2}\ \ \ \cos(\tfrac{\ 3}{2}\ \ \ ,\ \ _2=\exp(-\tfrac{1}{2}\ \ \ \sin(\tfrac{\ 3}{2}\ \ \ .$

109. $\tan\ \ \ +\ \ +\ \tan\ \ \ +\ \ =0.$

1 . Particular solutions with $\ >0$: $_1=\cos(\ \ \overline{\ }\ \ ,\ \ _2=\sin(\ \ \overline{\ }\ \ .$

2 . Particular solutions with $\ <0$: $_1=\exp(-\ \ \overline{\ }\ \ ,\ \ _2=\exp(\ \ \overline{\ }\ \ .$

110. $\tan\ \ \ +\ \ -\ (2\ +3\ \tan\ \ )\ \ +\ ^2(\ +2\ \tan\ \ )\ \ =0.$

Particular solutions: $_1=e^b\ \ ,\ \ _2=\ \ e^b\ \ .$

111. $\tan\ \ \ +\ \ +[(\ -\ ^2)\tan\ \ +\ \ ]\ \ +\ (1-\ \tan\ \ )\ \ =0.$

Particular solutions: $_1=e^{\lambda_1}\ \ ,\ \ _2=e^{\lambda_2}\ \ $, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2+a\lambda+\ \ =0.$

112. $\tan(\ \ )\ \ +\ ^2\ \ -2\ \ +2\ \ =0.$

Particular solutions: $_1=\ \ ,\ \ _2=\ ^2.$

113. $\tan\ \ \ +(\ \tan\ \ +\ \ +1)\ \ +\ ^2\ \ -\ ^2\ \ =0.$

Particular solutions: $_1=\ \ ,\ \ _2=e^{-}\ \ .$

114. $\tan \quad + (\quad \tan \quad + 1) \quad + \ (\quad + 2\tan \quad) \quad + \quad = 0.$

Particular solutions: $_1 = \exp\bigl(-\tfrac{1}{2}a \quad ^2 \quad , \quad _2 = \exp\bigl(-\tfrac{1}{2}a \quad ^2 \quad \exp\bigl(\tfrac{1}{2}a \quad ^2 \quad .$

115. $\tan \quad + (3\tan \quad + \ ) \quad + (\quad \tan \quad + 2) \quad + \ (\tan \quad + \ ) \ = 0.$

Particular solutions: $_1 = \ ^{-1}\cos(\quad \overline{a} \ , \quad _2 = \ ^{-1}\sin(\quad \overline{a} \ .$

116. $^3\tan \quad + \quad ^2 \quad -2 \quad \tan \quad + 2(2\tan \quad - \ ) \ = 0.$

Particular solutions: $_1 = \ ^{-1}, \quad _2 = \ ^2.$

117. $^3\tan \quad + \quad ^2 \quad -6 \quad \tan \quad + 6(2\tan \quad - \ ) \ = 0.$

Particular solutions: $_1 = \ ^{-2}, \quad _2 = \ ^3.$

118. $^3\tan \quad + \quad ^2 \quad + \ (\quad -\tan \quad) \quad + (\quad -3\tan \quad) \ = 0.$

Particular solutions: $_1 = \cos(\ln \ ), \quad _2 = \sin(\ln \ ).$

119. $^3\tan \quad + \quad ^2(\tan \quad + \ ) \quad + \ [\quad -(\quad +1)\tan \quad] \quad + \ (2\tan \quad - \ ) \ = 0.$

Particular solutions: $_1 = \ ^{- \ \overline{b}}, \quad _2 = \ ^{\overline{b}}.$

3.1.6-5. Equations with cotangent.

120. $+ \quad ^3\cot(\quad) \quad = 0.$

The substitution $= +\frac{\quad}{2a}$ leads to a linear equation of the form 3.1.6.78: $''' - a^3\tan(a \ ) \ = 0.$

121. $- \quad ^3\cot(\quad) \quad = 0.$

The substitution $= +\frac{\quad}{2a}$ leads to a linear equation of the form 3.1.6.77: $''' + a^3\tan(a \ ) \ = 0.$

122. $+ 3 \quad ^2 \quad - 2 \quad ^3\cot(\quad) \quad = 0.$

Particular solutions: $_1 = \sin(a \ ), \quad _2 = \quad \sin(a \ ).$

123. $+ \quad + (1 - \quad)\cot \quad = 0.$

Particular solution: $_0 = \sin \quad .$

124. $+ \quad \cot^2 \quad - \ (\quad \cot^2 \quad + \quad ^2) \quad = 0.$

The substitution $= e^{b} \quad ^2(\ ' - \quad)$ leads to a second-order linear equation of the form 2.1.6.81: $'' \ + \bigl(a\cot^2 \quad + \tfrac{3}{4} \quad ^2 \quad = 0.$

125. $+ \quad + (\quad \cot^2 \quad + \ ) \quad + \ (\quad \cot^2 \quad + \ ) \quad = 0.$

The substitution $= \ ' + a$ leads to a second-order linear equation of the form 2.1.6.81: $'' \ + (\quad \cot^2 \quad + \ ) \ = 0.$

126. $+ \quad \cot \quad + \cot \quad (1 - \quad \cot \quad) \quad = 0.$

Particular solution: $_0 = \sin \quad .$

127. $+ \quad + \ [3 \quad -2 \quad \cot(\quad)] \quad + \quad ^2\{ \ [1 + 2\cot^2(\quad)] - 2 \quad \cot(\quad)\} \quad = 0.$

Particular solutions: $_1 = \sin(\lambda \ ), \quad _2 = \quad \sin(\lambda \ ).$

128. $- \cot(\quad) \quad - \quad + \quad \cot(\quad) \quad = 0.$

1 . Solution for $a > 0$: $= {}_1 \exp\big(- \quad \overline{a} \quad + \quad {}_2 \exp\big(\quad \overline{a} \quad + \quad {}_3 \sin(\lambda \quad).$

2 . Solution for $a < 0$: $= {}_1 \cos\big(\quad \overline{-a} \quad + \quad {}_2 \sin\big(\quad \overline{-a} \quad + \quad {}_3 \sin(\lambda \quad).$

129. $+ \quad \cot \;(\quad) \quad + \quad \cot \;(\quad) \quad + \;^{2}[\quad \cot \;(\quad) - \quad] \quad = 0.$

Particular solutions: ${}_1 = \exp\big(-\tfrac{1}{2} \quad \cos\big(\tfrac{\overline{3}}{2} \quad , \quad {}_2 = \exp\big(-\tfrac{1}{2} \quad \sin\big(\tfrac{\overline{3}}{2} \quad .$

130. $+ \quad \cot \;(\quad) \quad - \;[2 \quad \cot \;(\quad) + 3 \;] \quad + \;^{2}[\quad \cot \;(\quad) + 2 \;] \quad = 0.$

Particular solutions: ${}_1 = e^{b} \quad , \quad {}_2 = \quad e^{b} \quad .$

131. $+ \quad \cot \quad + (\quad \cot \quad + \quad - \quad ^{2}) \quad + \;(\quad \cot \quad - \quad) \quad = 0.$

Particular solutions: ${}_1 = \exp(\lambda_1 \quad)$, ${}_2 = \exp(\lambda_2 \quad)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + \quad \lambda + \quad = 0.$

132. $+ \quad \cot \;(\quad) \quad + \quad + \quad ^{-1}[\quad \cot \;(\quad) + \quad] \quad = 0.$

The substitution $= \quad '' \; + \quad$ leads to a first-order linear equation: $' + a \cot \;(\lambda \quad) \quad = 0.$

133. $+ (\quad \cot \quad + \quad) \quad + \quad + \;(\quad \cot \quad + \quad) \quad = 0.$

1 . Particular solutions with > 0: ${}_1 = \cos\big(\quad \overline{} \quad , \quad {}_2 = \sin\big(\quad \overline{} \quad .$

2 . Particular solutions with < 0: ${}_1 = \exp\big(- \quad \overline{-} \quad , \quad {}_2 = \exp\big(\quad \overline{-} \quad .$

134. $= (\cot \quad - \quad) \quad + (\quad \cot \quad - \quad) \quad + \quad \cot \quad .$

Particular solutions: ${}_1 = \exp(\lambda_1 \quad)$, ${}_2 = \exp(\lambda_2 \quad)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \quad = 0.$

135. $+ \quad \cot \quad + (\quad ^{2} - \quad \cot \quad) \quad + \quad (\quad ^{2}\cot \quad + 3) \quad = 0.$

Particular solutions: ${}_1 = \cos\big(\tfrac{1}{2} \quad ^{2} \quad \overline{} \quad , \quad {}_2 = \sin\big(\tfrac{1}{2} \quad ^{2} \quad \overline{} \quad .$

136. $+ (\quad \cot \quad + \quad \cot \quad + \quad) \quad + \quad ^{2} \quad \cot \quad - \quad ^{2}\cot \quad = 0.$

Particular solutions: ${}_1 = \quad , \quad {}_2 = e^{-b} \quad .$

137. $+ \quad ^{2}\cot \;(\quad) \quad - 2 \quad \cot \;(\quad) \quad + 2 \quad \cot \;(\quad) \quad = 0.$

Particular solutions: ${}_1 = \quad , \quad {}_2 = \quad ^{2}.$

138. $+ [1 - \quad (\quad + 1) \quad \cot(\quad)] \quad - \quad ^{2} \quad + \quad ^{2} \quad = 0.$

Particular solutions: ${}_1 = \quad , \quad {}_2 = \sin(\lambda \quad).$

139. $+ \quad (\quad \cot^2 \quad + \quad) \quad - 2(\quad \cot^2 \quad + \quad) \quad = 0.$

The substitution $= \quad ' - 2$ leads to a second-order linear equation of the form 2.1.6.81: $'' \; + (a \cot^2 \quad + \quad) \quad = 0.$

140. $+ (\quad \cot \quad + 3) \quad + (2 \quad \cot \quad + \quad) \quad + \;(\quad \cot \quad + 1) \quad = 0.$

Particular solutions: ${}_1 = \quad ^{-1} \cos\big(\quad \overline{} \quad , \quad {}_2 = \quad ^{-1} \sin\big(\quad \overline{} \quad .$

141. $^{2} \quad = (\cot \quad - \quad ^{2}) \quad + (\quad \cot \quad - \quad ^{2}) \quad + \quad \cot \quad .$

Particular solutions: ${}_1 = \exp(\lambda_1 \quad)$, ${}_2 = \exp(\lambda_2 \quad)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \quad = 0.$

142. $^3 \quad + \quad ^2\cot(\quad) \quad + \quad + [\cot(\quad) - 2] = 0.$

Particular solutions: $_1 = \quad ^1$, $_2 = \quad ^2$, where $_1$ and $_2$ are roots of the quadratic equation $^2 - \quad + \quad = 0$.

143. $\cot \quad + \quad + \quad + \,^2(\quad - \cot \quad) = 0.$

Particular solutions: $_1 = \exp\bigl(-\frac{1}{2} \quad \cos\bigl(\frac{\ 3}{2} \quad$, $_2 = \exp\bigl(-\frac{1}{2} \quad \sin\bigl(\frac{\ 3}{2} \quad$.

144. $\cot \quad + \quad + \cot \quad + \quad = 0.$

1 . Particular solutions with $\quad > 0$: $_1 = \cos\bigl(\quad -$, $_2 = \sin\bigl(\quad -$.

2 . Particular solutions with $\quad < 0$: $_1 = \exp\bigl(- \quad \overline{-}$, $_2 = \exp\bigl(\quad \overline{-}$.

145. $\cot \quad + \quad - (2 \quad + 3 \cot \quad) \quad + \,^2(\quad + 2 \cot \quad) = 0.$

Particular solutions: $_1 = e^b$, $_2 = \quad e^b$.

146. $\cot \quad + \quad + [(\quad - \,^2)\cot \quad + \quad] \quad + (1 - \cot \quad) = 0.$

Particular solutions: $_1 = e^{\lambda_1}$, $_2 = e^{\lambda_2}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \quad = 0$.

147. $\cot(\quad) \quad + \quad ^2 \quad - 2 \quad + 2 \quad = 0.$

Particular solutions: $_1 = \quad$, $_2 = \quad ^2$.

148. $\cot \quad + (\cot \quad + \quad + 1) \quad + \quad ^2 \quad - \quad ^2 = 0.$

Particular solutions: $_1 = \quad$, $_2 = e^{-}$.

149. $\cot \quad + (\quad \cot \quad + 1) \quad + (\quad + 2\cot \quad) \quad + \quad = 0.$

Particular solutions: $_1 = \exp\bigl(-\frac{1}{2}a \quad ^2$, $_2 = \exp\bigl(-\frac{1}{2}a \quad ^2 \quad \exp\bigl(\frac{1}{2}a \quad ^2$.

150. $\cot \quad + (3\cot \quad + \quad) \quad + (\quad \cot \quad + 2) \quad + (\cot \quad + \quad) = 0.$

Particular solutions: $_1 = \quad ^{-1}\cos\bigl(\quad \overline{a}$, $_2 = \quad ^{-1}\sin\bigl(\quad \overline{a}$.

151. $^3\cot \quad + \quad ^2 \quad - 2 \cot \quad + 2(2\cot \quad - \quad) = 0.$

Particular solutions: $_1 = \quad ^{-1}$, $_2 = \quad ^2$.

152. $^3\cot \quad + \quad ^2 \quad - 6 \cot \quad + 6(2\cot \quad - \quad) = 0.$

Particular solutions: $_1 = \quad ^{-2}$, $_2 = \quad ^3$.

153. $^3\cot \quad + \quad ^2 \quad + (\quad - \cot \quad) \quad + (\quad - 3\cot \quad) = 0.$

Particular solutions: $_1 = \cos(\ln \quad)$, $_2 = \sin(\ln \quad)$.

154. $^3\cot \quad + \,^2(\cot \quad + \quad) \quad + [\quad - (\quad + 1)\cot \quad] \quad + (2\cot \quad - \quad) = 0.$

Particular solutions: $_1 = \quad ^{-\ \overline{b}}$, $_2 = \quad ^{\overline{b}}$.

3.1.7. Equations Containing Inverse Trigonometric Functions

1. $\quad + \quad + \quad + \quad = \arcsin^k \quad .$

This is a special case of equation 5.1.6.26.

2. $+ \arcsin^k \quad + \quad + \quad \arcsin^k \quad = 0.$

1 . Particular solutions with $a > 0$: $\ _1 = \cos(\quad \overline{a}$, $\ _2 = \sin(\quad \overline{a}$.

2 . Particular solutions with $a < 0$: $\ _1 = \exp(- \quad \overline{-a}$, $\ _2 = \exp(\quad \overline{-a}$.

The substitution $= '' + a$ leads to a first-order linear equation: $' + \arcsin \quad = 0.$

3. $+ \arcsin^k \quad + \quad + \quad ^{-1}(\ \arcsin^k \ + \ ) \ = 0.$

The substitution $= '' + a$ leads to a first-order linear equation: $' + \arcsin \quad = 0.$

4. $+ \arcsin^k \quad + \ \arcsin^k \quad + \ ^2(\arcsin^k \ - \ ) \ = 0.$

Particular solutions: $\ _1 = \exp\bigl(-\frac{1}{2}a \quad \cos\bigl(\frac{\sqrt{3}}{2}a$, $\ _2 = \exp\bigl(-\frac{1}{2}a \quad \sin\bigl(\frac{\sqrt{3}}{2}a$.

5. $+ \arcsin^k \quad - \ (2\arcsin^k \ + 3 \ ) \quad + \ ^2(\arcsin^k \ + 2 \ ) \ = 0.$

Particular solutions: $\ _1 = e$, $\ _2 = \ e$.

6. $+ \arcsin^k \quad + (\ \arcsin^k \ + \ - \ ^2) \quad + \ (\arcsin^k \ - \ ) \ = 0.$

Particular solutions: $\ _1 = e^{\lambda_1}$, $\ _2 = e^{\lambda_2}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0.$

7. $= (\arcsin^k \ - \ ) \quad + (\ \arcsin^k \ - \ ) \quad + \ \arcsin^k \quad .$

The substitution $= '' + a\ ' +$ leads to a first-order linear equation: $' = \arcsin \quad .$

8. $+ \ \arcsin^k \quad + (\ ^2 - \arcsin^k \ ) \quad + \ (\ ^2\arcsin^k \ + 3) \ = 0.$

Particular solutions: $\ _1 = \cos\bigl(\frac{1}{2} \ ^2 \ \overline{a}$, $\ _2 = \sin\bigl(\frac{1}{2} \ ^2 \ \overline{a}$.

9. $+ (\arcsin^k \ + \ ) \quad + \ (\ \arcsin^k \ + 2) \quad + \ \arcsin^k \quad = 0.$

Particular solutions: $\ _1 = \exp\bigl(-\frac{1}{2}a \ ^2$, $\ _2 = \exp\bigl(-\frac{1}{2}a \ ^2 \quad \exp\bigl(\frac{1}{2}a \ ^2$.

10. $+ \ ^2\arcsin^k \quad - 2 \ \arcsin^k \quad + 2\arcsin^k \quad = 0.$

Solution:

$$= \ _1 + \ _2 \ ^2 + \ _3 \ ^2 \quad ^{-3} \quad - \quad ^{-2} \quad , \quad \text{where} \quad = \exp \ - \quad ^2 \arcsin \quad .$$

11. $+ (\ \arcsin^k \ + \arcsin^k \ + \ ) \quad + \ ^2 \ \arcsin^k \quad - \ ^2\arcsin^k \quad = 0.$

Particular solutions: $\ _1 = \ $, $\ _2 = e^{-}$.

12. $+ \arccos^k \quad + \quad + \quad \arccos^k \quad = 0.$

1 . Particular solutions with $a > 0$: $\ _1 = \cos(\quad \overline{a}$, $\ _2 = \sin(\quad \overline{a}$.

2 . Particular solutions with $a < 0$: $\ _1 = \exp(- \quad \overline{-a}$, $\ _2 = \exp(\quad \overline{-a}$.

The substitution $= '' + a$ leads to a first-order linear equation: $' + \arccos \quad = 0.$

13. $+ \arccos^k \quad + \quad + \quad ^{-1}(\ \arccos^k \ + \ ) \ = 0.$

The substitution $= '' + a$ leads to a first-order linear equation: $' + \arccos \quad = 0.$

14. $+ \arccos^k \quad + \ \arccos^k \quad + \ ^2(\arccos^k \ - \ ) \ = 0.$

Particular solutions: $\ _1 = \exp\bigl(-\frac{1}{2}a \quad \cos\bigl(\frac{\sqrt{3}}{2}a$, $\ _2 = \exp\bigl(-\frac{1}{2}a \quad \sin\bigl(\frac{\sqrt{3}}{2}a$.

15. $+ \arccos^k \quad - \ (2\arccos^k \ + 3 \ ) \quad + \ ^2(\arccos^k \ + 2 \ ) \ = 0.$

Particular solutions: $\ _1 = e$, $\ _2 = \ e$.

16. $+ \arccos^k \quad + (\quad \arccos^k \quad + \quad - \quad {}^2) \quad + \quad (\arccos^k \quad - \quad) \quad = 0.$

Particular solutions: $_1 = e^{\lambda_1}$, $_2 = e^{\lambda_2}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \quad = 0$.

17. $= (\arccos^k \quad - \quad) \quad + (\quad \arccos^k \quad - \quad) \quad + \quad \arccos^k \quad .$

The substitution $= \;'' + a \;' +$ leads to a first-order linear equation: $' = \arccos$.

18. $+ \quad \arccos^k \quad + (\quad {}^2 - \arccos^k \quad) \quad + \quad (\quad {}^2 \arccos^k \quad + 3) \quad = 0.$

Particular solutions: $_1 = \cos\bigl(\tfrac{1}{2} \quad {}^2 \quad \overline{a}$, $_2 = \sin\bigl(\tfrac{1}{2} \quad {}^2 \quad \overline{a}$.

19. $+ (\arccos^k \quad + \quad) \quad + \quad (\quad \arccos^k \quad + 2) \quad + \quad \arccos^k \quad = 0.$

Particular solutions: $_1 = \exp\bigl(-\tfrac{1}{2}a \quad {}^2$, $_2 = \exp\bigl(-\tfrac{1}{2}a \quad {}^2 \quad \exp\bigl(\tfrac{1}{2}a \quad {}^2$.

20. $+ \quad {}^2 \arccos^k \quad - 2 \quad \arccos^k \quad + 2 \arccos^k \quad = 0.$

Solution:

$$= \;_1 \; + \;_2 \;{}^2 + \;_3 \quad {}^2 \quad {}^{-3} \quad - \quad {}^{-2} \quad , \quad \text{where} \quad = \exp \; - \quad {}^2 \arccos \quad .$$

21. $+ (\quad \arccos^k \quad + \arccos^k \quad + \quad) \quad + \quad {}^2 \quad \arccos^k \quad - \quad {}^2 \arccos^k \quad = 0.$

Particular solutions: $_1 =$, $_2 = e^{-}$.

22. $+ \arctan^k \quad + \quad + \quad \arctan^k \quad = 0.$

1 . Particular solutions with $a > 0$: $_1 = \cos\bigl(\quad \overline{a}$, $_2 = \sin\bigl(\quad \overline{a}$.

2 . Particular solutions with $a < 0$: $_1 = \exp\bigl(- \quad \overline{-a}$, $_2 = \exp\bigl(\quad \overline{-a}$.

The substitution $= \;'' + a$ leads to a first-order linear equation: $' + \arctan \quad = 0$.

23. $+ \arctan^k \quad + \quad + \quad {}^{-1}(\quad \arctan^k \quad + \quad) \quad = 0.$

The substitution $= \;'' + a$ leads to a first-order linear equation: $' + \arctan \quad = 0$.

24. $+ \arctan^k \quad + \quad \arctan^k \quad + \quad {}^2(\arctan^k \quad - \quad) \quad = 0.$

Particular solutions: $_1 = \exp\bigl(-\tfrac{1}{2}a \quad \cos\bigl(\tfrac{\overline{3}}{2}a$, $_2 = \exp\bigl(-\tfrac{1}{2}a \quad \sin\bigl(\tfrac{\overline{3}}{2}a$.

25. $+ \arctan^k \quad - \quad (2 \arctan^k \quad + 3 \quad) \quad + \quad {}^2(\arctan^k \quad + 2 \quad) \quad = 0.$

Particular solutions: $_1 = e$, $_2 = \quad e$.

26. $+ \arctan^k \quad + (\quad \arctan^k \quad + \quad - \quad {}^2) \quad + \quad (\arctan^k \quad - \quad) \quad = 0.$

Particular solutions: $_1 = e^{\lambda_1}$, $_2 = e^{\lambda_2}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \quad = 0$.

27. $= (\arctan^k \quad - \quad) \quad + (\quad \arctan^k \quad - \quad) \quad + \quad \arctan^k \quad .$

The substitution $= \;'' + a \;' +$ leads to a first-order linear equation: $' = \arctan$.

28. $+ \quad \arctan^k \quad + (\quad {}^2 - \arctan^k \quad) \quad + \quad (\quad {}^2 \arctan^k \quad + 3) \quad = 0.$

Particular solutions: $_1 = \cos\bigl(\tfrac{1}{2} \quad {}^2 \quad \overline{a}$, $_2 = \sin\bigl(\tfrac{1}{2} \quad {}^2 \quad \overline{a}$.

29. $+ (\arctan^k \quad + \quad) \quad + \quad (\quad \arctan^k \quad + 2) \quad + \quad \arctan^k \quad = 0.$

Particular solutions: $_1 = \exp\bigl(-\tfrac{1}{2}a \quad {}^2$, $_2 = \exp\bigl(-\tfrac{1}{2}a \quad {}^2 \quad \exp\bigl(\tfrac{1}{2}a \quad {}^2$.

30. $+ \quad ^{2}\,\mathrm{arctan}^{k} \quad -2 \quad \mathrm{arctan}^{k} \quad +2\,\mathrm{arctan}^{k} \quad = 0.$

Solution:

$$= {}_1 + {}_2 {}^2 + {}_3 \quad {}^2 \quad {}^{-3} \quad - \quad {}^{-2} \quad , \quad \text{where} \quad = \exp \; - \quad {}^2 \,\mathrm{arctan} \quad .$$

31. $+(\quad \mathrm{arctan}^{k} \quad + \mathrm{arctan}^{k} \quad + \quad) \quad + \quad ^{2} \quad \mathrm{arctan}^{k} \quad - \quad ^{2}\,\mathrm{arctan}^{k} \quad = 0.$

Particular solutions: ${}_1 = \quad , \quad {}_2 = e^{-} \quad .$

32. $+\mathrm{arccot}^{k} \quad + \quad + \quad \mathrm{arccot}^{k} \quad = 0.$

1 . Particular solutions with $a > 0$: ${}_1 = \cos\big(\quad \overline{a} \;,\quad {}_2 = \sin\big(\quad \overline{a} \;.$

2 . Particular solutions with $a < 0$: ${}_1 = \exp\big(- \quad \overline{-a} \;,\quad {}_2 = \exp\big(\quad \overline{-a} \;.$

The substitution $= \;'' + a$ leads to a first-order linear equation: $' + \mathrm{arccot} \quad = 0.$

33. $+\mathrm{arccot}^{k} \quad + \quad + \quad ^{-1}(\quad \mathrm{arccot}^{k} \quad + \quad) \quad = 0.$

The substitution $= \;'' + a$ leads to a first-order linear equation: $' + \mathrm{arccot} \quad = 0.$

34. $+\mathrm{arccot}^{k} \quad + \quad \mathrm{arccot}^{k} \quad + \quad ^{2}(\mathrm{arccot}^{k} \quad - \quad) \quad = 0.$

Particular solutions: ${}_1 = \exp\big(-\tfrac{1}{2}a \quad \cos\big(\tfrac{\overline{3}}{2}a \quad ,\quad {}_2 = \exp\big(-\tfrac{1}{2}a \quad \sin\big(\tfrac{\overline{3}}{2}a \quad .$

35. $+\mathrm{arccot}^{k} \quad - \quad (2\,\mathrm{arccot}^{k} \quad + 3 \quad) \quad + \quad ^{2}(\mathrm{arccot}^{k} \quad + 2 \quad) \quad = 0.$

Particular solutions: ${}_1 = e \quad ,\quad {}_2 = \quad e \quad .$

36. $+\mathrm{arccot}^{k} \quad + (\quad \mathrm{arccot}^{k} \quad + \quad - \quad ^{2}) \quad + \quad (\mathrm{arccot}^{k} \quad - \quad) \quad = 0.$

Particular solutions: ${}_1 = e^{\lambda_1} \;,\quad {}_2 = e^{\lambda_2} \;,$ where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \quad = 0.$

37. $= (\mathrm{arccot}^{k} \quad - \quad) \quad + (\quad \mathrm{arccot}^{k} \quad - \quad) \quad + \quad \mathrm{arccot}^{k} \quad .$

The substitution $= \;'' + a \;' + \quad$ leads to a first-order linear equation: $' = \mathrm{arccot} \quad .$

38. $+ \quad \mathrm{arccot}^{k} \quad + (\quad ^{2} - \mathrm{arccot}^{k} \quad) \quad + \quad (\quad ^{2}\,\mathrm{arccot}^{k} \quad + 3) \quad = 0.$

Particular solutions: ${}_1 = \cos\big(\tfrac{1}{2} \quad ^2 \quad \overline{a} \;,\quad {}_2 = \sin\big(\tfrac{1}{2} \quad ^2 \quad \overline{a} \;.$

39. $+(\mathrm{arccot}^{k} \quad + \quad) \quad + \quad (\quad \mathrm{arccot}^{k} \quad + 2) \quad + \quad \mathrm{arccot}^{k} \quad = 0.$

Particular solutions: ${}_1 = \exp\big(-\tfrac{1}{2}a \quad ^2 \;,\quad {}_2 = \exp\big(-\tfrac{1}{2}a \quad ^2 \quad \exp\big(\tfrac{1}{2}a \quad ^2 \quad .$

40. $+ \quad ^{2}\,\mathrm{arccot}^{k} \quad -2 \quad \mathrm{arccot}^{k} \quad +2\,\mathrm{arccot}^{k} \quad = 0.$

Solution:

$$= {}_1 + {}_2 {}^2 + {}_3 \quad {}^2 \quad {}^{-3} \quad - \quad {}^{-2} \quad , \quad \text{where} \quad = \exp \; - \quad {}^2 \,\mathrm{arccot} \quad .$$

41. $+(\quad \mathrm{arccot}^{k} \quad + \mathrm{arccot}^{k} \quad + \quad) \quad + \quad ^{2} \quad \mathrm{arccot}^{k} \quad - \quad ^{2}\,\mathrm{arccot}^{k} \quad = 0.$

Particular solutions: ${}_1 = \quad , \quad {}_2 = e^{-} \quad .$

42. $+(\quad ^{2} + \quad) \quad + 4 \quad + 2 \quad = \mathrm{arcsin}^{k} \quad .$

Integrating the equation twice, we arrive at a first-order linear equation:

$$' + (a \quad ^2 + \quad - 2) \quad = \quad {}_1 + \quad {}_2 \quad + \int_0 (\quad - \quad)\,\mathrm{arcsin} \quad , \qquad {}_0 \text{ is any number.}$$

43. $+(\quad \mathrm{arcsin}^{k} \quad + 3) \quad + (2\,\mathrm{arcsin}^{k} \quad + \quad) \quad + \quad (\quad \mathrm{arcsin}^{k} \quad + 1) \quad = 0.$

Particular solutions: ${}_1 = \quad ^{-1}\cos\big(\quad \overline{a} \;,\quad {}_2 = \quad ^{-1}\sin\big(\quad \overline{a} \;.$

44. $+(\ \mathbf{arccos}^k\ +3)\quad +(2\,\mathbf{arccos}^k\ +\ \ )\ +(\ \mathbf{arccos}^k\ +1)\ =0.$

Particular solutions: $_1 = \ ^{-1}\cos(\ \ \overline{a}\ ,\quad _2 = \ ^{-1}\sin(\ \ \overline{a}\ .$

45. $+(\ \mathbf{arctan}^k\ +3)\quad +(2\,\mathbf{arctan}^k\ +\ \ )\ +(\ \mathbf{arctan}^k\ +1)\ =0.$

Particular solutions: $_1 = \ ^{-1}\cos(\ \ \overline{a}\ ,\quad _2 = \ ^{-1}\sin(\ \ \overline{a}\ .$

46. $+(\ \mathbf{arccot}^k\ +3)\quad +(2\,\mathbf{arccot}^k\ +\ \ )\ +(\ \mathbf{arccot}^k\ +1)\ =0.$

Particular solutions: $_1 = \ ^{-1}\cos(\ \ \overline{a}\ ,\quad _2 = \ ^{-1}\sin(\ \ \overline{a}\ .$

47. $^3\quad +[(\ \ +6)\ ^2 +\ \ ]\quad +2(2\ \ +3)\quad +2\quad =\mathbf{arcsin}^k\ .$

Integrating the equation twice, we arrive at a first-order linear equation:

$$^3\ '\ +(a\ ^2+\ \ )\ =\ _1+\ \ _2+\ \ \int_0 (\ \ -\ \ )\arcsin\ \ \ ,\qquad _0 \text{ is any number.}$$

48. $^3\quad +\ ^2\,\mathbf{arcsin}^k\quad -2\quad +2(2-\mathbf{arcsin}^k\ )\ =0.$

Particular solutions: $_1 = \ ^{-1},\quad _2 = \ ^2.$

49. $^3\quad +\ ^2\,\mathbf{arcsin}^k\quad -6\quad +6(2-\mathbf{arcsin}^k\ )\ =0.$

Particular solutions: $_1 = \ ^{-2},\quad _2 = \ ^3.$

50. $^3\quad +\ ^2\,\mathbf{arcsin}^k\quad +\ (\mathbf{arcsin}^k\ -1)\quad +(\mathbf{arcsin}^k\ -3)\ =0.$

Particular solutions: $_1 = \cos(\ln\ \ ),\quad _2 = \sin(\ln\ \ ).$

51. $^3\quad +\ ^2(\mathbf{arcsin}^k\ +1)\quad +\ (\mathbf{arcsin}^k\ -\ \ -1)\ -\ (\mathbf{arcsin}^k\ -2)\ =0.$

Particular solutions: $_1 = \ ^{-}\ \overline{\ \ }\ ,\quad _2 = \ \overline{\ \ }\ .$

52. $^3\quad +\ ^2(\mathbf{arcsin}^k\ +\ \ )\quad +(\ \mathbf{arcsin}^k\ +\ \ -\ \ )\ +\ (\mathbf{arcsin}^k\ -2)\ =0.$

Particular solutions: $_1 = \ ^{1},\ _2 = \ ^{2}$, where $_1$ and $_2$ are roots of the quadratic equation $^2+(a-1)\ \ +\ \ =0.$

53. $^3\quad +\ ^2\,\mathbf{arccos}^k\quad -2\quad +2(2-\mathbf{arccos}^k\ )\ =0.$

Particular solutions: $_1 = \ ^{-1},\quad _2 = \ ^2.$

54. $^3\quad +\ ^2\,\mathbf{arccos}^k\quad -6\quad +6(2-\mathbf{arccos}^k\ )\ =0.$

Particular solutions: $_1 = \ ^{-2},\quad _2 = \ ^3.$

55. $^3\quad +\ ^2\,\mathbf{arccos}^k\quad +\ (\mathbf{arccos}^k\ -1)\quad +(\mathbf{arccos}^k\ -3)\ =0.$

Particular solutions: $_1 = \cos(\ln\ \ ),\quad _2 = \sin(\ln\ \ ).$

56. $^3\quad +\ ^2(\mathbf{arccos}^k\ +1)\quad +\ (\mathbf{arccos}^k\ -\ \ -1)\ -\ (\mathbf{arccos}^k\ -2)\ =0.$

Particular solutions: $_1 = \ ^{-}\ \overline{\ \ }\ ,\quad _2 = \ \overline{\ \ }\ .$

57. $^3\quad +\ ^2(\mathbf{arccos}^k\ +\ \ )\quad +(\ \mathbf{arccos}^k\ +\ \ -\ \ )\ +\ (\mathbf{arccos}^k\ -2)\ =0.$

Particular solutions: $_1 = \ ^{1},\ _2 = \ ^{2}$, where $_1$ and $_2$ are roots of the quadratic equation $^2+(a-1)\ \ +\ \ =0.$

58. $^3\quad +\ ^2\,\mathbf{arctan}^k\quad -2\quad +2(2-\mathbf{arctan}^k\ )\ =0.$

Particular solutions: $_1 = \ ^{-1},\quad _2 = \ ^2.$

59. $^3 \quad + \ ^2\arctan^k \quad - 6 \quad + 6(2 - \arctan^k\ ) \ = 0.$

Particular solutions: $_1 = {}^{-2}$, $_2 = {}^{3}$.

60. $^3 \quad + \ ^2\arctan^k \quad + \ (\arctan^k \ - 1) \quad + (\arctan^k \ - 3) \ = 0.$

Particular solutions: $_1 = \cos(\ln\ )$, $_2 = \sin(\ln\ )$.

61. $^3 \quad + \ ^2(\arctan^k \ + 1) \quad + \ (\arctan^k \ - \ - 1) \quad - \ (\arctan^k \ - 2) \ = 0.$

Particular solutions: $_1 = \ ^{-} \ \bar{\ }$, $_2 = \ \bar{\ }$.

62. $^3 \quad + \ ^2(\arctan^k \ + \ ) \quad + \ (\ \arctan^k \ + \ - \ ) \quad + \ (\arctan^k \ - 2) \ = 0.$

Particular solutions: $_1 = {}^{1}$, $_2 = {}^{2}$, where $_1$ and $_2$ are roots of the quadratic equation $^2 + (a-1) \ + \ = 0$.

63. $^3 \quad + \ ^2\operatorname{arccot}^k \quad - 2 \quad + 2(2 - \operatorname{arccot}^k\ ) \ = 0.$

Particular solutions: $_1 = {}^{-1}$, $_2 = {}^{2}$.

64. $^3 \quad + \ ^2\operatorname{arccot}^k \quad - 6 \quad + 6(2 - \operatorname{arccot}^k\ ) \ = 0.$

Particular solutions: $_1 = {}^{-2}$, $_2 = {}^{3}$.

65. $^3 \quad + \ ^2\operatorname{arccot}^k \quad + \ (\operatorname{arccot}^k \ - 1) \quad + (\operatorname{arccot}^k \ - 3) \ = 0.$

Particular solutions: $_1 = \cos(\ln\ )$, $_2 = \sin(\ln\ )$.

66. $^3 \quad + \ ^2(\operatorname{arccot}^k \ + 1) \quad + \ (\operatorname{arccot}^k \ - \ - 1) \quad - \ (\operatorname{arccot}^k \ - 2) \ = 0.$

Particular solutions: $_1 = \ ^{-} \ \bar{\ }$, $_2 = \ \bar{\ }$.

67. $^3 \quad + \ ^2(\operatorname{arccot}^k \ + \ ) \quad + \ (\ \operatorname{arccot}^k \ + \ - \ ) \quad + \ (\operatorname{arccot}^k \ - 2) \ = 0.$

Particular solutions: $_1 = {}^{1}$, $_2 = {}^{2}$, where $_1$ and $_2$ are roots of the quadratic equation $^2 + (a-1) \ + \ = 0$.

3.1.8. Equations Containing Combinations of Exponential, Logarithmic, Trigonometric, and Other Functions

1. $\quad = \tan \quad + \ ^{\lambda} (\quad + \tan \quad).$

Particular solution: $_0 = \cos\ $.

2. $\quad + \ ^{\lambda} \quad + (2 \ ^{\lambda} \tan \ + 3) \quad + [\ ^{\lambda} (2\tan^2 \ + 1) + 2\tan \ ] \ = 0.$

Particular solutions: $_1 = \cos\ $, $_2 = \ \cos\ $.

3. $\quad + \ ^{\lambda} \quad + (3 - 2 \ ^{\lambda} \cot \ ) \quad + [\ ^{\lambda} (2\cot^2 \ + 1) - 2\cot \ ] \ = 0.$

Particular solutions: $_1 = \sin\ $, $_2 = \ \sin\ $.

4. $\quad + \cosh \quad + (2 \cosh \ \tan \ + 3) \quad + [\cosh \ (2\tan^2 \ + 1) + 2\tan \ ] \ = 0.$

Particular solutions: $_1 = \cos\ $, $_2 = \ \cos\ $.

5. $\quad + \cosh \quad + (3 - 2 \cosh \ \cot \ ) \quad + [\cosh \ (2\cot^2 \ + 1) - 2\cot \ ] \ = 0.$

Particular solutions: $_1 = \sin\ $, $_2 = \ \sin\ $.

6. $\quad + \sinh \quad + (2 \sinh \ \tan \ + 3) \quad + [\sinh \ (2\tan^2 \ + 1) + 2\tan \ ] \ = 0.$

Particular solutions: $_1 = \cos\ $, $_2 = \ \cos\ $.

7. $+ \sinh \quad + (3 - 2 \sinh \quad \cot \ ) \quad + [\ \sinh \quad (2\cot^2 \quad + 1) - 2\cot \quad] = 0.$

Particular solutions: $_1 = \sin \ , \quad _2 = \quad \sin \ .$

8. $+ \tanh \quad + (2 \tanh \quad \tan \quad + 3) \quad + [\ \tanh \quad (2\tan^2 \quad + 1) + 2\tan \quad] = 0.$

Particular solutions: $_1 = \cos \ , \quad _2 = \quad \cos \ .$

9. $+ \tanh \quad + (3 - 2 \tanh \quad \cot \ ) \quad + [\ \tanh \quad (2\cot^2 \quad + 1) - 2\cot \quad] = 0.$

Particular solutions: $_1 = \sin \ , \quad _2 = \quad \sin \ .$

10. $+ \coth \quad + (2 \coth \quad \tan \quad + 3) \quad + [\ \coth \quad (2\tan^2 \quad + 1) + 2\tan \quad] = 0.$

Particular solutions: $_1 = \cos \ , \quad _2 = \quad \cos \ .$

11. $+ \coth \quad + (3 - 2 \coth \quad \cot \ ) \quad + [\ \coth \quad (2\cot^2 \quad + 1) - 2\cot \quad] = 0.$

Particular solutions: $_1 = \sin \ , \quad _2 = \quad \sin \ .$

12. $+ \ln \quad - (2 \ln \quad \tanh \quad + 3) \quad + [\ \ln \quad (2\tanh^2 \quad - 1) + 2\tanh \quad] = 0.$

Particular solutions: $_1 = \cosh \ , \quad _2 = \quad \cosh \ .$

13. $+ \ln \quad - (2 \ln \quad \coth \quad + 3) \quad + [\ \ln \quad (2\coth^2 \quad - 1) + 2\coth \quad] = 0.$

Particular solutions: $_1 = \sinh \ , \quad _2 = \quad \sinh \ .$

14. $+ \ln \quad + (2 \ln \quad \tan \quad + 3) \quad + [\ \ln \quad (2\tan^2 \quad + 1) + 2\tan \quad] = 0.$

Particular solutions: $_1 = \cos \ , \quad _2 = \quad \cos \ .$

15. $+ \ln \quad + (3 - 2 \ln \quad \cot \ ) \quad + [\ \ln \quad (2\cot^2 \quad + 1) - 2\cot \quad] = 0.$

Particular solutions: $_1 = \sin \ , \quad _2 = \quad \sin \ .$

16. $+ \cos \quad - (2 \cos \quad \tanh \quad + 3) \quad + [\ \cos \quad (2\tanh^2 \quad - 1) + 2\tanh \quad] = 0.$

Particular solutions: $_1 = \cosh \ , \quad _2 = \quad \cosh \ .$

17. $+ \cos \quad - (2 \cos \quad \coth \quad + 3) \quad + [\ \cos \quad (2\coth^2 \quad - 1) + 2\coth \quad] = 0.$

Particular solutions: $_1 = \sinh \ , \quad _2 = \quad \sinh \ .$

18. $+ \sin \quad - (2 \sin \quad \tanh \quad + 3) \quad + [\ \sin \quad (2\tanh^2 \quad - 1) + 2\tanh \quad] = 0.$

Particular solutions: $_1 = \cosh \ , \quad _2 = \quad \cosh \ .$

19. $+ \sin \quad - (2 \sin \quad \coth \quad + 3) \quad + [\ \sin \quad (2\coth^2 \quad - 1) + 2\coth \quad] = 0.$

Particular solutions: $_1 = \sinh \ , \quad _2 = \quad \sinh \ .$

20. $+ \tan \quad - (2 \tan \quad \tanh \quad + 3) \quad + [\ \tan \quad (2\tanh^2 \quad - 1) + 2\tanh \quad] = 0.$

Particular solutions: $_1 = \cosh \ , \quad _2 = \quad \cosh \ .$

21. $+ \tan \quad - (2 \tan \quad \coth \quad + 3) \quad + [\ \tan \quad (2\coth^2 \quad - 1) + 2\coth \quad] = 0.$

Particular solutions: $_1 = \sinh \ , \quad _2 = \quad \sinh \ .$

22. $+ \cot \quad - (2 \cot \quad \tanh \quad + 3) \quad + [\ \cot \quad (2\tanh^2 \quad - 1) + 2\tanh \quad] = 0.$

Particular solutions: $_1 = \cosh \ , \quad _2 = \quad \cosh \ .$

23. $+ \cot \quad - (2 \cot \quad \coth \quad + 3) \quad + [\ \cot \quad (2\coth^2 \quad - 1) + 2\coth \quad] = 0.$

Particular solutions: $_1 = \sinh \ , \quad _2 = \quad \sinh \ .$

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24. + (+ 2) cosh − (cosh +) − 2 ^{3} cosh = 0.

Particular solutions: $_1 = e$, $_2 = e^{-}$ + a.

25. + (+ 2) sinh − (sinh +) − 2 ^{3} sinh = 0.

Particular solutions: $_1 = e$, $_2 = e^{-}$ + a.

26. + (+ 2) tanh − (tanh +) − 2 ^{3} tanh = 0.

Particular solutions: $_1 = e$, $_2 = e^{-}$ + a.

27. + (+ 2) coth − (coth +) − 2 ^{3} coth = 0.

Particular solutions: $_1 = e$, $_2 = e^{-}$ + a.

28. + (+ 2) ln − (ln +) − 2 ^{3} ln = 0.

Particular solutions: $_1 = e$, $_2 = e^{-}$ + a.

29. +(ln −2) − (2 ln − +3) + [ln (−1)+2 −1] =0.

Particular solutions: $_1 = \exp(e)$, $_2 = \exp(e)$.

30. + (cos − 2) − (2 cos − + 3)

+ [cos (− 1) + 2 − 1] = 0.

Particular solutions: $_1 = \exp(e)$, $_2 = \exp(e)$.

31. + (+ 2) cos − (cos +) − 2 ^{3} cos = 0.

Particular solutions: $_1 = e$, $_2 = e^{-}$ + a.

32. + (sin − 2) − (2 sin − + 3)

+ [sin (− 1) + 2 − 1] = 0.

Particular solutions: $_1 = \exp(e)$, $_2 = \exp(e)$.

33. + (+ 2) sin − (sin +) − 2 ^{3} sin = 0.

Particular solutions: $_1 = e$, $_2 = e^{-}$ + a.

34. − [$^\lambda$ (tan +) +] + [(^{2} + 1) $^\lambda$ + 1] + [$^\lambda$ (tan − 1) − 1] = 0.

Particular solutions: $_1 = e$, $_2 = \cos$.

35. + [tan ($^\lambda$ + 1) + $^\lambda$] − $^\lambda$ + $^\lambda$ = 0.

Particular solutions: $_1 =$, $_2 = \cos$.

36. + (tan − 2) − (2 tan − + 3)

+ [tan (− 1) + 2 − 1] = 0.

Particular solutions: $_1 = \exp(e)$, $_2 = \exp(e)$.

37. + (+ 2) tan − (tan +) − 2 ^{3} tan = 0.

Particular solutions: $_1 = e$, $_2 = e^{-}$ + a.

38. + [$^\lambda$ (cot +) +] + [(^{2} + 1) $^\lambda$ + 1] + [$^\lambda$ (1 − cot) + 1] = 0.

Particular solutions: $_1 = e^{-}$, $_2 = \sin$.

39. + [$^\lambda$ − cot ($^\lambda$ + 1)] − $^\lambda$ + $^\lambda$ = 0.

Particular solutions: $_1 =$, $_2 = \sin$.

40. **+ (cot − 2) − (2 cot − + 3)**
+ [cot (− 1) + 2 − 1] = 0.

Particular solutions: $_1 = \exp(\ e\ )$, $_2 = \ \exp(\ e\ )$.

41. **+ (+ 2) cot − (cot +) − 2 ^{3}cot = 0.**

Particular solutions: $_1 = e$, $_2 = e^{-}\ +\ a$.

42. **−[cosh (tan +)+] +[(^{2}+1) cosh +1] + [cosh (tan −1)−1] = 0.**

Particular solutions: $_1 = e$, $_2 = \cos$.

43. **+[cosh (cot +)+] +[(^{2}+1) cosh +1] + [cosh (1− cot)+1] = 0.**

Particular solutions: $_1 = e^{-}$, $_2 = \sin$.

44. **−[sinh (tan +)+] +[(^{2}+1) sinh +1] + [sinh (tan −1)−1] = 0.**

Particular solutions: $_1 = e$, $_2 = \cos$.

45. **+[sinh (cot +)+] +[(^{2}+1) sinh +1] + [sinh (1− cot)+1] = 0.**

Particular solutions: $_1 = e^{-}$, $_2 = \sin$.

46. **−[tanh (tan +)+] +[(^{2}+1) tanh +1] + [tanh (tan −1)−1] = 0.**

Particular solutions: $_1 = e$, $_2 = \cos$.

47. **+[tanh (cot +)+] +[(^{2}+1) tanh +1] + [tanh (1− cot)+1] = 0.**

Particular solutions: $_1 = e^{-}$, $_2 = \sin$.

48. **−[coth (tan +)+] +[(^{2}+1) coth +1] + [coth (tan −1)−1] = 0.**

Particular solutions: $_1 = e$, $_2 = \cos$.

49. **+[coth (cot +)+] +[(^{2}+1) coth +1] + [coth (1− cot)+1] = 0.**

Particular solutions: $_1 = e^{-}$, $_2 = \sin$.

50. **+ [tan (tanh −) −] + [(^{2} − 1) tan − 1]**
+ [tan (1 − tanh) + 1] = 0.

Particular solutions: $_1 = e^{b}$, $_2 = \cosh$.

51. **+ (tan + tanh) + + (tan + tanh) = 0.**

1 . Particular solutions with > 0: $_1 = \cos(\ \bar{}\ $, $_2 = \sin(\ \bar{}\ $.

2 . Particular solutions with < 0: $_1 = \exp(-\ \overline{-}\ $, $_2 = \exp(\ \overline{-}\ $.

52. **+[tan (coth −)−] +[(^{2}−1) tan −1] + [tan (1− coth)+1] = 0.**

Particular solutions: $_1 = e^{b}$, $_2 = \sinh$.

53. **+ (tan + coth) + + (tan + coth) = 0.**

1 . Particular solutions with > 0: $_1 = \cos(\ \bar{}\ $, $_2 = \sin(\ \bar{}\ $.

2 . Particular solutions with < 0: $_1 = \exp(-\ \overline{-}\ $, $_2 = \exp(\ \overline{-}\ $.

54. **+[cot (tanh −)−] +[(^{2}−1) cot −1] + [cot (1− tanh)+1] = 0.**

Particular solutions: $_1 = e^{b}$, $_2 = \cosh$.

55. $+ \cot \tanh - - \cot \tanh = 0.$

1 . Particular solutions with > 0: $_1 = \exp(- \,\bar{}\,$, $_2 = \exp(\,\bar{}\,$.

2 . Particular solutions with < 0: $_1 = \cos(\,\overline{-}\,$, $_2 = \sin(\,\overline{-}\,$.

56. $+[\cot (\coth -) -] +[(^2 - 1)\cot - 1] + [\cot (1 - \coth) + 1] = 0.$

Particular solutions: $_1 = e^b$, $_2 = \sinh$.

57. $+(\cot + \coth) + + (\cot + \coth) = 0.$

1 . Particular solutions with > 0: $_1 = \cos(\,\bar{}\,$, $_2 = \sin(\,\bar{}\,$.

2 . Particular solutions with < 0: $_1 = \exp(-\,\overline{-}\,$, $_2 = \exp(\,\overline{-}\,$.

58. $+[\ln (\tanh -) -] +[(^2 - 1)\ln - 1] + [\ln (1 - \tanh) + 1] = 0.$

Particular solutions: $_1 = e^b$, $_2 = \cosh$.

59. $+ \ln \tanh - - \ln \tanh = 0.$

1 . Particular solutions with > 0: $_1 = \exp(- \,\bar{}\,$, $_2 = \exp(\,\bar{}\,$.

2 . Particular solutions with < 0: $_1 = \cos(\,\overline{-}\,$, $_2 = \sin(\,\overline{-}\,$.

60. $+[\ln (\coth -) -] +[(^2 - 1)\ln - 1] + [\ln (1 - \coth) + 1] = 0.$

Particular solutions: $_1 = e^b$, $_2 = \sinh$.

61. $+(\ln + \coth) + + (\ln + \coth) = 0.$

1 . Particular solutions with > 0: $_1 = \cos(\,\bar{}\,$, $_2 = \sin(\,\bar{}\,$.

2 . Particular solutions with < 0: $_1 = \exp(-\,\overline{-}\,$, $_2 = \exp(\,\overline{-}\,$.

62. $-[\ln (\tan +) +] + [(^2 + 1)\ln + 1] + [\ln (\tan - 1) - 1] = 0.$

Particular solutions: $_1 = e$, $_2 = \cos$.

63. $+[\ln (\cot +) +] + [(^2 + 1)\ln + 1] + [\ln (1 - \cot) + 1] = 0.$

Particular solutions: $_1 = e^-$, $_2 = \sin$.

64. $+[\cos (\tanh -) -] +[(^2 - 1)\cos - 1] + [\cos (1 - \tanh) + 1] = 0.$

Particular solutions: $_1 = e^b$, $_2 = \cosh$.

65. $+ \cos \tanh + + \cos \tanh = 0.$

1 . Particular solutions with > 0: $_1 = \cos(\,\bar{}\,$, $_2 = \sin(\,\bar{}\,$.

2 . Particular solutions with < 0: $_1 = \exp(-\,\overline{-}\,$, $_2 = \exp(\,\overline{-}\,$.

66. $+[\cos (\coth -) -] +[(^2 - 1)\cos - 1] + [\cos (1 - \coth) + 1] = 0.$

Particular solutions: $_1 = e^b$, $_2 = \sinh$.

67. $+(\cos + \coth) + + (\cos + \coth) = 0.$

1 . Particular solutions with > 0: $_1 = \cos(\,\bar{}\,$, $_2 = \sin(\,\bar{}\,$.

2 . Particular solutions with < 0: $_1 = \exp(-\,\overline{-}\,$, $_2 = \exp(\,\overline{-}\,$.

68. $+[\sin (\tanh -) -] +[(^2 - 1)\sin - 1] + [\sin (1 - \tanh) + 1] = 0.$

Particular solutions: $_1 = e^b$, $_2 = \cosh$.

69. $+\ \sin\ \ \tanh\ \ +\ \ +\ \sin\ \ \tanh\ \ = 0.$

1 . Particular solutions with >0: $_1 = \cos($ $,\ _2 = \sin($.

2 . Particular solutions with <0: $_1 = \exp(-$ $,\ _2 = \exp($.

70. $+[\ \sin\ (\coth\ -\ )-\ ]\ \ +[\ (^2-1)\sin\ -1]\ +[\ \sin\ (1-\ \coth\ )+1]\ =0.$

Particular solutions: $_1 = e^b$, $_2 = \sinh$.

71. $+(\ \sin\ +\ \coth\ )\ +\ +(\ \sin\ +\ \coth\ )\ = 0.$

1 . Particular solutions with >0: $_1 = \cos($ $,\ _2 = \sin($.

2 . Particular solutions with <0: $_1 = \exp(-$ $,\ _2 = \exp($.

72. $+[\ ^2\ ^\lambda(\ -\ln\ )+2]\ +\ ^\lambda\ -\ ^\lambda\ = 0.$

Particular solutions: $_1 =$, $_2 = \ln\ -\ +1.$

73. $(\ ^\lambda-1)\ -(\ ^\lambda+\tan\ )\ +(\ ^\lambda+\ ^2)\ +(\ \tan\ -\ ^\lambda)\ = 0.$

Particular solutions: $_1 = e$, $_2 = \cos$.

74. $\cosh\ +[\tan\ (\ \cosh\ +\ )+1]\ -\ +\ = 0.$

Particular solutions: $_1 =$, $_2 = \cos$.

75. $\cosh\ +[1-\cot\ (\ \cosh\ +\ )]\ -\ +\ = 0.$

Particular solutions: $_1 =$, $_2 = \sin$.

76. $\sinh\ +[\tan\ (\ \sinh\ +\ )+1]\ -\ +\ = 0.$

Particular solutions: $_1 =$, $_2 = \cos$.

77. $\sinh\ +[1-\cot\ (\ \sinh\ +\ )]\ -\ +\ = 0.$

Particular solutions: $_1 =$, $_2 = \sin$.

78. $\tanh\ +[\tan\ (\ \tanh\ +\ )+1]\ -\ +\ = 0.$

Particular solutions: $_1 =$, $_2 = \cos$.

79. $\tanh\ +[1-\cot\ (\ \tanh\ +\ )]\ -\ +\ = 0.$

Particular solutions: $_1 =$, $_2 = \sin$.

80. $\coth\ +[\tan\ (\ \coth\ +\ )+1]\ -\ +\ = 0.$

Particular solutions: $_1 =$, $_2 = \cos$.

81. $\coth\ +[1-\cot\ (\ \coth\ +\ )]\ -\ +\ = 0.$

Particular solutions: $_1 =$, $_2 = \sin$.

82. $\ln\ +[\tanh\ (\ -\ \ln\ )-1]\ -\ +\ = 0.$

Particular solutions: $_1 =$, $_2 = \cosh$.

83. $\ln\ +[\coth\ (\ -\ \ln\ )-1]\ -\ +\ = 0.$

Particular solutions: $_1 =$, $_2 = \sinh$.

84. $\ln\ +[\tan\ (\ \ln\ +\ )+1]\ -\ +\ = 0.$

Particular solutions: $_1 =$, $_2 = \cos$.

85. $\ln \quad + [1 - \cot \ (\ \ln \ \ + \ \ )] \quad - \quad + \ = 0.$

Particular solutions: $_1 = \ ,\ \ _2 = \sin \ .$

86. $\cos \quad + [\tanh \ (\ - \ \cos \ ) - 1] \quad - \quad + \ = 0.$

Particular solutions: $_1 = \ ,\ \ _2 = \cosh \ .$

87. $\cos \quad + [\coth \ (\ - \ \cos \ ) - 1] \quad - \quad + \ = 0.$

Particular solutions: $_1 = \ ,\ \ _2 = \sinh \ .$

88. $\cos \quad + (2 \ \cos \ - \ ^2 \ln \ + \ ^2) \quad + \quad - \ = 0.$

Particular solutions: $_1 = \ ,\ \ _2 = \ln \ - \ + 1.$

89. $\sin \quad + [\tanh \ (\ - \ \sin \ ) - 1] \quad - \quad + \ = 0.$

Particular solutions: $_1 = \ ,\ \ _2 = \cosh \ .$

90. $\sin \quad + [\coth \ (\ - \ \sin \ ) - 1] \quad - \quad + \ = 0.$

Particular solutions: $_1 = \ ,\ \ _2 = \sinh \ .$

91. $\sin \quad + (2 \ \sin \ - \ ^2 \ln \ + \ ^2) \quad + \quad - \ = 0.$

Particular solutions: $_1 = \ ,\ \ _2 = \ln \ - \ + 1.$

92. $\tan \quad + [\tanh \ (\ - \ \tan \ ) - 1] \quad - \quad + \ = 0.$

Particular solutions: $_1 = \ ,\ \ _2 = \cosh \ .$

93. $\tan \quad + [\coth \ (\ - \ \tan \ ) - 1] \quad - \quad + \ = 0.$

Particular solutions: $_1 = \ ,\ \ _2 = \sinh \ .$

94. $\tan \quad + (2 \ \tan \ - \ ^2 \ln \ + \ ^2) \quad + \quad - \ = 0.$

Particular solutions: $_1 = \ ,\ \ _2 = \ln \ - \ + 1.$

95. $\cot \quad + [\tanh \ (\ - \ \cot \ ) - 1] \quad - \quad + \ = 0.$

Particular solutions: $_1 = \ ,\ \ _2 = \cosh \ .$

96. $\cot \quad + [\coth \ (\ - \ \cot \ ) - 1] \quad - \quad + \ = 0.$

Particular solutions: $_1 = \ ,\ \ _2 = \sinh \ .$

97. $\cot \quad + (2 \ \cot \ - \ ^2 \ln \ + \ ^2) \quad + \quad - \ = 0.$

Particular solutions: $_1 = \ ,\ \ _2 = \ln \ - \ + 1.$

3.1.9. Equations Containing Arbitrary Functions

Notation: $= \ (\ )$, $g = g(\ )$, *and* $= \ (\ )$ *are arbitrary functions of* ; *a*, , , , *and* λ *are arbitrary parameters.*

3.1.9-1. Equations of the form $_3(\ ) \ ''' \ + \ _1(\ ) \ ' + \ _0(\ ) \ = g(\ ).$

1. $= f(\ ) \ .$

The transformation $= \ ^{-1}, \ = \ ^{-2}$ leads to an equation of the same form: $''' = - \ ^{-6} \ (1 \ ) \ .$

2. $= f\,\dfrac{+}{+}\;\dfrac{}{(\quad+\quad)^6}.$

The transformation $\xi = \dfrac{a\quad+}{+}$, $\quad = \dfrac{}{(\quad+\quad)^2}$ leads to a simpler equation: $''' \;= \Delta^{-3}\;(\xi)\;$, where $\Delta = a\quad-\quad$.

3. $f\qquad - f\qquad = 0.$

Particular solution: $\;_0 = \quad$. The substitution $\quad = \quad z\quad$ leads to a second-order linear equation: $\quad z'' \;+ 3\;\,' z' + 3\;\,'' z = 0.$

4. $f\qquad + f\qquad = \;.$

Integrating yields a second-order linear equation: $\quad'' - \;'\;' + \;'' \quad = \quad g \quad + \;.$

5. $\quad + f\quad - (\;f + \;^3)\; = 0.$

Particular solution: $\;_0 = e\quad$. The substitution $\quad = \;' - a\quad$ leads to a second-order linear equation: $\quad'' \;+ a\;\;' + (\quad + a^2)\quad = 0.$

6. $\quad + f\quad + \quad(f + \;^2\;^2 - 3\;)\; = 0.$

Particular solution: $\;_0 = \exp\bigl(-\tfrac{1}{2}a\;\;^2\;$. The substitution $\quad = \exp\bigl(-\tfrac{1}{2}a\;\;^2\quad z(\;)\;$ leads to a second-order linear equation: $z'' \;- 3a\;\; z' + (\quad + 3a^2\;\;^2 - 3a)z = 0.$

7. $\quad + (f - \;^2)\quad + \;f\quad = 0.$

Particular solution: $\;_0 = e^{-}\quad$. The substitution $\quad = \;' + a\quad$ leads to a second-order linear equation: $\quad'' \;- a\;\;' + \qquad = 0.$

8. $\quad + \;f\quad - 2f\quad = 0.$

Particular solution: $\;_0 = \;^2$. The substitution $\quad = \;' - 2\quad$ leads to a second-order linear equation: $\quad'' \;+ \qquad = 0.$

9. $\quad + (\quad + \;)f\quad - \;f\quad = 0.$

Particular solution: $\;_0 = a\quad + \;.$

10. $\quad + (\quad + \;)f\quad - 2\;\;f\quad = 0.$

Particular solution: $\;_0 = (a\quad + \;)^2$. The substitution $\quad = (a\quad + \;)\;' - 2a\quad$ leads to a second-order linear equation: $\quad'' \;+ (a\quad + \;)\qquad = 0.$

11. $\quad + (f - \;^2\;^2)\quad + \quad(f - 3\;)\; = 0.$

Particular solution: $\;_0 = \exp\bigl(-\tfrac{1}{2}a\;\;^2\;$. The substitution $\quad = \exp\bigl(-\tfrac{1}{2}a\;\;^2\quad z(\;)\;$ leads to a second-order linear equation: $z'' \;- 3a\;\; z' + (2a^2\;\;^2 - 3a + \quad)z = 0.$

12. $\quad + (f - \;^2\;^2\;)\quad - \;[\quad f + 3\qquad^2\;^{-1} + \;(\quad - 1)\quad^{-2}]\; = 0.$

Particular solution: $\;_0 = \exp\;\dfrac{a}{+1}\quad^{+1}\;$. The substitution $\quad = \exp\;\dfrac{a}{+1}\quad^{+1}\quad z(\;)$ leads to a second-order linear equation: $z'' \;+ 3a\quad z' + (2a^2\;\;^2 \;+ 3a\qquad^{-1} + \quad)z = 0.$

13. $\quad + \;f\quad - [(\quad + 1)f + \;^3\quad + 3\;^2]\; = 0.$

Particular solution: $\;_0 = \quad e\quad.$

14. $^{2}\quad + (\ f - \ ^{2} - \ )\quad + (\ - 1)f\quad = 0.$

Particular solution: $_0 = \ ^{1-}$. The substitution $= \ ' + (a-1)$ leads to a second-order linear equation: $'' - (a+1)\ ' + \ = 0.$

15. $^{2}\quad + [\ (\quad + 1)f - 6]\quad + f\quad = 0.$

Particular solution: $_0 = a + \ ^{-1}.$

16. $(\ + 1)\quad + \ (f - \ - 3)\quad - (\ + 1)f\quad = 0.$

Particular solution: $_0 = \ e\ .$

17. $^{3}\quad + \ f\quad + (\ - 1)(f + \ ^{2} + \ )\quad = 0.$

Particular solution: $_0 = \ ^{1-}$. The substitution $= \ ' + (a-1)$ leads to a second-order linear equation: $^{2}\ '' - (a+1)\ \ ' + (\ + a^2 + a)\ = 0.$

18. $^{6}\quad + \ ^{2}f\quad + (\ ^{3} + \ f - 2\ f)\quad = 0.$

Particular solution: $_0 = \ ^{2}e\ .$

19. $+ (f - \ ^{2}\ ^{2\lambda}\ )\quad - \ ^{\lambda}\ (f + 3\quad ^{\lambda} + \ ^{2})\quad = 0.$

Particular solution: $_0 = \exp\ \frac{a}{\lambda}e^{\lambda}$. The substitution $= \exp\ \frac{a}{\lambda}e^{\lambda}\quad z(\ )$ leads to a second-order linear equation: $z'' + 3ae^{\lambda}\ z' + (\ + 2a^2e^{2\lambda}\ + 3a\lambda e^{\lambda}\ )z = 0.$

20. $+ [(1 + \quad)f - \ ^{2}]\quad + \ f\quad = 0.$

Particular solution: $_0 = e^{-}\ + \ .$

21. $= f\quad + \tanh\ (1 - f)\ .$

This is a special case of equation 3.1.9.30 with $g(\ ) = \cosh\ .$

22. $= f\quad + \coth\ (1 - f)\ .$

This is a special case of equation 3.1.9.30 with $g(\ ) = \sinh\ .$

23. $+ f\quad + \tan\ (f - 1)\quad = 0.$

Particular solution: $_0 = \cos\ $. The substitution $= \cos\quad z(\ )$ leads to a second-order linear equation: $z'' - 3\tan\ z' + (\ - 3)z = 0.$

24. $+ f\quad + \cot\ (1 - f)\quad = 0.$

Particular solution: $_0 = \sin\ .$

25. $+ f\quad + f\quad = \ .$

Integrating yields a second-order linear equation: $'' + \ = \ g\ + \ .$

26. $+ 2f\quad + f\quad = 0.$

Solution: $= \ _1\ _1^2 + \ _2\ _1\ _2 + \ _3\ _2^2$. Here, $_1$ and $_2$ are linearly independent solutions of the second-order linear equation: $2\ '' + \ = 0.$

27. $+ f\quad + f(2f\ - f^2)\quad = 0.$

Integrating yields a second-order linear equation: $'' + \ ' + \ ^{2} = \ \exp\ .$

28. $+ (\quad - 1) f^2 \quad - [f \quad - (2 \quad + 1) f f \quad + \quad f^3] \quad = 0.$

Integrating yields a second-order equation: $'' + \quad ' + (a \quad ^2 - \quad ') \quad = \quad \exp \quad .$

29. $+ (f - \quad ^2) \quad + (f \quad - \quad f) \quad = 0.$

The substitution $\quad = \quad '' \quad + a \quad ' + \quad$ leads to a first-order linear equation: $' - a \quad = 0.$

30. $= \text{———} \quad + f(\;) \quad - \text{——} \quad .$

The substitution $\quad = \quad ' - \dfrac{g'}{g}$ leads to a second-order linear equation.

3.1.9-2. Equations of the form $_3(\;) \; ''' \; + \; _2(\;) \; '' \; + \; _1(\;) \; ' + \; _0(\;) \; = g(\;).$

31. $+ \quad + \quad + \quad = f(\;).$

This is a special case of equation 5.1.6.26 with $\quad = 3.$

32. $+ \quad + f \quad + \quad f \quad = 0.$

The substitution $\quad = \quad ' + a \quad$ leads to a second-order linear equation: $'' + \quad = 0.$

33. $+ f \quad - \quad ^2(f + \quad) \quad = 0.$

Particular solution: $_0 = e \quad$. The substitution $\quad = \quad ' - a \quad$ leads to a second-order linear equation: $'' + (\quad + a) \quad ' + a(\quad + a) \quad = 0.$

34. $+ f \quad + \quad + \quad f \quad = 0.$

1 . Particular solutions with $a > 0$: $_1 = \cos\big(\quad \overline{a} \quad , \quad _2 = \sin\big(\quad \overline{a} \quad .$

2 . Particular solutions with $a < 0$: $_1 = \exp\big(- \quad \overline{-a} \quad , \quad _2 = \exp\big(\quad \overline{-a} \quad .$

The substitution $\quad = \quad '' \quad + a \quad$ leads to a first-order linear equation: $' + \quad = 0.$

35. $+ f \quad + \quad + \quad ^{-1}(\; f + \quad) \quad = 0.$

The substitution $\quad = \quad '' \quad + a \quad$ leads to a first-order linear equation: $' + \quad = 0.$

36. $+ f \quad + \quad f \quad + \quad ^3 \quad = 0.$

The substitution $\quad = \quad ' + a \quad$ leads to a second-order linear equation: $'' + (\quad - a) \quad ' + a^2 \quad = 0.$

37. $+ f \quad + \quad f \quad + \quad ^2(f - \quad) \quad = 0.$

Particular solutions: $_1 = \exp\big(-\tfrac{1}{2}a \quad \cos\big(\tfrac{\overline{3}}{2}a \quad , \quad _2 = \exp\big(-\tfrac{1}{2}a \quad \sin\big(\tfrac{\overline{3}}{2}a \quad .$

38. $+ f \quad + \quad + \quad = 0.$

The substitution $\quad = \quad '$ leads to a second-order linear equation: $'' + \quad ' + g \quad + \quad = 0.$

39. $+ f \quad - \quad (2f + 3 \quad) \quad + \quad ^2(f + 2 \quad) \quad = 0.$

Particular solutions: $_1 = e \quad , \quad _2 = \quad e \quad .$

40. $+ f \quad + \quad - \quad = 0.$

The substitution $\quad = \quad ' - \quad$ leads to a second-order equation: $'' + (\quad - 1) \quad ' + \quad ^2 g \quad = 0.$

41. $+ f \quad + (\quad - \quad ^2) \quad - \quad (\; f + \quad) \quad = 0.$

Particular solution: $_0 = e \quad$. The substitution $\quad = \quad ' - a \quad$ leads to a second-order linear equation: $'' + (\quad + a) \quad ' + (a \quad + g) \quad = 0.$

42. $+ f \quad + (\ f + \ - \ ^2) \quad + \ (f - \ ) \quad = 0.$

Particular solutions: $\ _1 = e^{\lambda_1}$, $\ _2 = e^{\lambda_2}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0$.

43. $+ (f - \ ) \quad - \ ^2 f \quad = 0.$

Particular solution: $\ _0 = e$. The substitution $= \ ' - a$ leads to a second-order linear equation: $'' + \ ' + a \quad = 0$.

44. $= (f - \ ) \quad + (\ f - \ ) \quad + \ f \ .$

Particular solutions: $\ _1 = \exp(\lambda_1 \ )$, $\ _2 = \exp(\lambda_2 \ )$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + \ = 0$. The substitution $= \ '' + a \ ' +$ leads to a first-order linear equation: $' = \ $.

45. $+ (f - \ ) \quad + \quad - \ (\ f + \ ) \quad = 0.$

Particular solution: $\ _0 = e$.

46. $+ (f + \ ) \quad + (\ f + \ ) \quad + \quad = 0.$

Particular solution: $\ _0 = e^{-}$.

47. $+ \ f \quad + (\ ^2 - f) \quad + \ (\ ^2 f + 3) \quad = 0.$

Particular solutions: $\ _1 = \cos\bigl(\frac{1}{2} \ ^2 \ \overline{a}$, $\ _2 = \sin\bigl(\frac{1}{2} \ ^2 \ \overline{a}$.

48. $+ (\ + \ ) f \quad + \ f \quad - 2f \quad = 0.$

Particular solution: $\ _0 = \ ^2 + 2a \ + \ $.

49. $+ (f + \ ) \quad + \ (\ f + 2) \quad + \ f \quad = 0.$

Particular solutions: $\ _1 = \exp\bigl(-\frac{1}{2}a \ ^2$, $\ _2 = \exp\bigl(-\frac{1}{2}a \ ^2 \quad \exp\bigl(\frac{1}{2}a \ ^2$.

50. $+ \ ^2 f \quad - 2 \ f \quad + 2f \quad = 0.$

Particular solutions: $\ _1 = \ $, $\ _2 = \ ^2$.

Solution: $= \ _1 + \ _2 \ ^2 + \ _3 \ ^2 \ ^{-3} \quad - \quad ^{-2}$, where $= \exp \ - \ ^2$.

51. $+ (f + \ ) \quad + (\ + 2 \ ) \quad + \ [\quad + (1 - \ ^2) f] \quad = 0.$

Particular solution: $\ _0 = \exp\bigl(-\frac{1}{2}a \ ^2$. The substitution $= \ ' + a$ leads to a second-order linear equation: $'' + \ ' + (g - a \ ) \ = 0$.

52. $+ (\ f + f + \ ) \quad + \ ^2 \ f \quad - \ ^2 f \quad = 0.$

Particular solutions: $\ _1 = \ $, $\ _2 = e^{-}$.

53. $+ (\ ^2 + \ + \ ) f \quad - 2 \ f \quad = 0.$

Particular solution: $\ _0 = a \ ^2 + \ + \ $.

54. $+ \ (\ f + \ ) \quad - \quad - 2f \quad = 0.$

Particular solution: $\ _0 = \ ^2$. The substitution $= \ ' - 2$ leads to a second-order linear equation: $'' + \ (\ + g) \ ' + \quad = 0$.

55. $- \ (\ + \ ) f \quad + (\ - \ ^2) f \quad + 2 \ f \quad = 0.$

Particular solution: $\ _0 = \ ^2 + a \ + \frac{1}{2}(a^2 - \ )$.

56. $-[(2 \quad + \quad)f + (\ ^2 + \quad + \ ) \ ] \quad + 2f \quad + 2 \quad = 0.$

Particular solution: $\ _0 = \ ^2 + a \ + \ .$

57. $+3 \quad + \ (\ ^2 + 1)f \quad - (\ ^2 - 1)f \quad = 0.$

Particular solution: $\ _0 = a \ + \ ^{-1}.$

58. $+(\ ^2 + \ ) \quad + 4 \quad + 2 \quad = f(\ ).$

Integrating the equation twice, we arrive at a first-order linear equation:

$$' + (a \ ^2 + \ - 2) \ = \ _1 + \ _2 + \ \int_0 (\ - \ ) \ (\ ) \ , \qquad _0 \text{ is any number.}$$

59. $+ \ (f - 2 \ ) \quad + \ (\ + \ ^2) \quad - [\ (\quad + 2)f + (\quad + 1) \ ] \ = 0.$

Particular solution: $\ _0 = \ e \quad .$

60. $+(\ f + 3) \quad + (2f + \quad) \quad + \ (\ f + 1) \ = 0.$

Particular solutions: $\ _1 = \ ^{-1}\cos(\ \overline{a}), \quad _2 = \ ^{-1}\sin(\ \overline{a}).$

61. $+(\ f + 3) \quad + (\quad + 2)f \quad + \ (\ f + f - \ ^2 \ ) \ = 0.$

Particular solutions: $\ _1 = \ ^{-1}\exp\bigl(-\tfrac{1}{2}a \quad \cos\bigl(\tfrac{\ 3}{2}a \quad , \quad _2 = \ ^{-1}\exp\bigl(-\tfrac{1}{2}a \quad \sin\bigl(\tfrac{\ 3}{2}a \quad .$

62. $+(\ f + 3) \quad + (\ f + 2f - \ ^2 \ ) \quad + \ (f - \ ) \ = 0.$

Particular solutions: $\ _1 = \ ^{-1}, \quad _2 = \ ^{-1}e^{-} \quad .$

63. $+(\ f + 3) \quad + (2f + \quad ^{+1}) \quad + \quad (\ f + \ + 1) \ = 0.$

The substitution $\ = \quad$ leads to an equation of the form 3.1.9.35: $\ ''' + \ '' + a \ ' + a \ ^{-1}(\ + \ ) \ = 0.$

64. $+(\ ^2 f + \ + 2) \quad - \ (\ + 1)f \ = 0.$

Particular solution: $\ _0 = \ ^{-} \ .$ The substitution $\ = \ ' + a \ $ leads to a second-order linear equation: $\ '' + \ ' - (a + 1) \ = 0.$

65. $+[\ ^2(\ ^2 + 1)f + 3] \quad - 2f \ = 0.$

Particular solution: $\ _0 = a \ + \ ^{-1}.$

66. $+[\ (\ ^2 - 1)f + \ ^2(\ ^2 + 1) \ + 3] \quad - 2f \quad - 2 \quad = 0.$

Particular solution: $\ _0 = a \ + \ ^{-1}.$

67. $(\quad - 1) \quad + \ [(\quad - 2)f - \ ^2] \quad + [(2 - \ ^2 \ ^2)f + \ ^2] \quad + 2 \ (\quad - 1)f \ = 0.$

Particular solutions: $\ _1 = \ ^2, \quad _2 = e \quad .$

68. $\ ^2 \quad + \ f \quad + [\ (\quad + 1) \ + 2f - 6] \quad + \quad = 0.$

Particular solution: $\ _0 = a + \ ^{-1}.$

69. $\ ^2 \quad + \ [\ (\quad + 1)f + 3] \quad - 2f \ = 0.$

Particular solution: $\ _0 = a + \ ^{-1}.$

70. $\ ^2 \quad + \ (\ f + \ ) \quad + [(\ - 2) \ f + \ ] \quad + (\ - \quad + 2)f \ = 0.$

By integrating, we obtain the second-order nonhomogeneous Euler equation 2.1.9.15: $\ ^2 \ '' + (a - 2) \ ' + (\ - a + 2) \ = \ \exp \ - \qquad .$

71. $(\quad + \quad) \qquad + (\quad + \quad) \qquad + \qquad + \quad = f.$

Integrating yields a second-order linear equation: $(a \quad + \quad) \quad '' + [(\quad - 2a) \quad + \beta - \quad] \; ' +$ $(\quad + 2a - \quad) \quad = \qquad + \quad .$

72. $^3 \qquad + \quad ^2 \qquad + \qquad + \qquad = f(\quad).$

Nonhomogeneous Euler equation. The substitution $\quad = \ln| \quad |$ leads to an equation of the form 3.1.9.31: $''' + (a-3) \; '' + (\quad - a + 2) \; ' + \quad = \quad (\quad e \quad).$

73. $^3 \qquad + (\quad + 2) \; ^2 \qquad + \quad f \qquad + \quad f \qquad = 0.$

Particular solution: $_0 = \quad ^{-} \quad .$

74. $^3 \qquad + [(\quad + 6) \; ^2 + \quad] \qquad + 2(2 \quad + 3) \qquad + 2 \qquad = f(\quad).$

Integrating the equation twice, we arrive at a first-order linear equation:

$$^3 \; ' + (a \; ^2 + \quad) \quad = \quad _1 + \quad _2 + \quad \int_0 (\quad - \quad) \; (\;) \quad , \qquad _0 \text{ is any number.}$$

75. $^3 \qquad + \; ^2(\quad ^{2 \quad +1} - 3 \quad) \qquad + 2 \quad (\quad + 1)(2 \quad + 1) \quad = f(\quad).$

Integrating the equation twice, we arrive at a first-order linear equation:

$$^{-2} \quad ' + (a \; ^{-2 \quad -1} + \quad) \quad = \quad _1 + \quad _2 + \quad \int_0 (\quad - \quad) \; ^{-2 \quad -3} \; (\;) \quad , \qquad _0 \text{ is any number.}$$

76. $^3 \qquad + \; ^2 f \qquad - 2 \qquad + 2(2 - f) \quad = 0.$

Particular solutions: $_1 = \quad ^{-1}, \quad _2 = \quad ^2.$

77. $^3 \qquad + \; ^2 f \qquad - 6 \qquad + 6(2 - f) \quad = 0.$

Particular solutions: $_1 = \quad ^{-2}, \quad _2 = \quad ^3.$

78. $^3 \qquad + \; ^2 f \qquad + \quad (f - 1) \qquad + (f - 3) \quad = 0.$

Particular solutions: $_1 = \cos(\ln| \quad |), \quad _2 = \sin(\ln| \quad |).$

79. $^3 \qquad + \; ^2(f + 1) \qquad + \quad (f - \quad - 1) \qquad - \quad (f - 2) \quad = 0.$

Particular solutions: $_1 = \quad ^{-} \; \bar{} \; , \quad _2 = \quad \bar{} \quad .$

80. $^3 \qquad + \; ^2(f + \quad) \qquad + \quad (\quad f + \quad - \quad) \qquad + \quad (f - 2) \quad = 0.$

Particular solutions: $_1 = \quad ^1, \quad _2 = \quad ^2$, where $_1$ and $_2$ are roots of the quadratic equation $^2 + (a - 1) \quad + \quad = 0.$

81. $^3 \qquad + \; ^2(f + \quad) \qquad + \quad [\quad + (\quad - 1)f] \qquad + (\quad - 2) \qquad = 0.$

Particular solution: $_0 = \quad ^{2-} \quad$. The substitution $\quad = \quad ' + (a - 2) \quad$ leads to a second-order linear equation: $^2 \; '' + \qquad ' + g \quad = 0.$

82. $^3 \qquad + \; ^2(f + 2 \quad) \qquad + \quad (2 \quad f + \; ^2 \; ^2 + \quad) \qquad + (\; ^2 \; ^2 f + \quad f - 2 \quad) \quad = 0.$

Particular solutions: $_1 = e^{-} \quad ^1, \quad _2 = e^{-} \quad ^2$, where $_1$ and $_2$ are roots of the quadratic equation $^2 - \quad + \quad = 0.$

83. $\qquad + \quad ^{\lambda} \qquad - 3 \; ^2 \qquad + 2 \; ^3 \quad = f(\quad).$

Integrating the equation twice, we arrive at a first-order linear equation:

$$e^{-\lambda} \quad ' + (a + 2\lambda e^{-\lambda} \quad) \quad = \quad _1 + \quad _2 + \quad \int_0 (\quad - \quad) e^{-\lambda} \quad (\;) \quad , \qquad _0 \text{ is any number.}$$

84. $\qquad + (f + \quad) \qquad + [\quad f + (1 + \qquad) \;] \qquad + \qquad = 0.$

Particular solution: $_0 = e^{-} \quad + \quad .$

85. $+(\quad +2\ )f \quad -\ (\quad f+\ )\quad -2\ ^3f\quad = 0.$

Particular solutions: $_1 = e\ \ ,\quad _2 = e^{-}\quad +\quad a.$

86. $+(f-2\quad ^{\lambda}\ )\quad -\quad ^{\lambda}\ (2f-\quad ^{\lambda}\ +3\ )\quad +\quad ^{\lambda}\ [(\quad ^{\lambda}\ -\ )f+2\quad ^{\lambda}\ -\ ^2]\ =0.$

Particular solutions: $_1 = \exp\ \frac{a}{\lambda}e^{\lambda}\quad ,\quad _2 = \quad \exp\ \frac{a}{\lambda}e^{\lambda}\quad .$

87. $+(f-\quad ^{\lambda}\ )\quad +(\ -2\quad ^{\lambda}\ )\quad -\quad ^{\lambda}\ [(\quad ^{\lambda}\ +\ )f+\ +\ ^2]\ = 0.$

Particular solution: $_0 = \exp\ \frac{a}{\lambda}e^{\lambda}$. The substitution $= \exp\ \frac{a}{\lambda}e^{\lambda}\quad z(\ )$ leads to a second-order linear equation: $z''\ +(\ +2ae^{\lambda}\ )z' + (2ae^{\lambda}\quad + g + a^2e^{2\lambda}\ + a\lambda e^{\lambda}\ )z = 0.$

88. $-[(\ +\ +\quad)f-\ +\ ]\quad +[(\ ^2-\ ^2+\quad)f-\ ]\quad +\quad (\ +\ )f\quad = 0.$

Particular solutions: $_1 = e^{c}\ ,\quad _2 = e^{-}\quad +\quad .$

89. $^{\lambda}\quad +(2\quad ^{\lambda}\ +\quad +\gamma)\quad +(\ ^2\ ^{\lambda}\ +2\quad)\quad +\quad ^2\quad = f(\ ).$

Integrating the equation twice, we arrive at a first-order linear equation:

$$e^{\lambda}\quad ' + (\beta e\quad +\ )\ =\ _1 +\ _2\ +\ _0\ (\ -\ )\ (\ )\quad ,\qquad _0 \text{ is any number.}$$

90. $+f\quad +\quad -\ [\ f+\tanh(\quad)(\ +\ ^2)]\ = 0.$

Particular solution: $_0 = \cosh(\lambda\ )$. The substitution $= \cosh(\lambda\ )\quad z(\ )$ leads to a second-order linear equation: $z''\ +[\ +3\lambda\tanh(\lambda\ )]z' + [g+3\lambda^2+2\lambda\quad \tanh(\lambda\ )]z = 0.$

91. $+f\quad -\ [2f\tanh(\quad)+3\ ]\quad +\ ^2\{[2\tanh^2(\quad)-1]f+2\ \ \tanh(\quad)\}\ = 0.$

Particular solutions: $_1 = \cosh(\lambda\ ),\quad _2 = \quad \cosh(\lambda\ ).$

92. $+f\quad -\ [2f\coth(\quad)+3\ ]\quad +\ ^2\{[2\coth^2(\quad)-1]f+2\ \ \coth(\quad)\}\ = 0.$

Particular solutions: $_1 = \sinh(\lambda\ ),\quad _2 = \quad \sinh(\lambda\ ).$

93. $+[(\tanh\ \ -\ )f-\ ]\quad +[(\ ^2-1)f-1]\quad +\ \ [(1-\quad \tanh\ \ )f+1]\ = 0.$

Particular solutions: $_1 = e\ \ ,\quad _2 = \cosh\ \ .$

94. $+[(\coth\ \ -\ )f-\ ]\quad +[(\ ^2-1)f-1]\quad +\ \ [(1-\quad \coth\ \ )f+1]\ = 0.$

Particular solutions: $_1 = e\ \ ,\quad _2 = \sinh\ \ .$

95. $+[\ \tanh(\quad)(\ f-1)-f]\quad -\ ^2\ f\quad +\ ^2f\quad = 0.$

Particular solutions: $_1 = \quad ,\quad _2 = \cosh(\lambda\ ).$

96. $+[\ \coth(\quad)(\ f-1)-f]\quad -\ ^2\ f\quad +\ ^2f\quad = 0.$

Particular solutions: $_1 = \quad ,\quad _2 = \sinh(\lambda\ ).$

97. $+[\ ^2(\ -\ln\ \ )f+2]\quad +\ f\quad -f\quad = 0.$

Particular solutions: $_1 = \quad ,\quad _2 = \ln\ \ -a+1.$

98. $+f\quad +\quad +\ [\ f+\tan(\quad)(\ -\ ^2)]\ = 0.$

Particular solution: $_0 = \cos(\lambda\ )$. The substitution $= \cos(\lambda\ )\quad z(\ )$ leads to a second-order linear equation: $z''\ +[\ -3\lambda\tan(\lambda\ )]z' + [g-3\lambda^2-2\lambda\quad \tan(\lambda\ )]z = 0.$

99. $+ f \quad + \quad [2f\tan(\quad) + 3\quad] \quad + \quad ^2\{[1 + 2\tan^2(\quad)]f + 2\quad \tan(\quad)\} \quad = 0.$

Particular solutions: $_1 = \cos(\lambda\quad)$, $_2 = \quad \cos(\lambda\quad)$.

100. $+ f \quad + \quad [3\quad - 2f\cot(\quad)] \quad + \quad ^2\{[1 + 2\cot^2(\quad)]f - 2\quad \cot(\quad)\} \quad = 0.$

Particular solutions: $_1 = \sin(\lambda\quad)$, $_2 = \quad \sin(\lambda\quad)$.

101. $-[(\quad + \tan\quad)f + \quad] \quad + [(\,^2 + 1)f + 1] \quad + \quad [(\quad \tan\quad - 1)f - 1] \quad = 0.$

Particular solutions: $_1 = e$, $_2 = \cos$.

102. $+[(\cot\quad + \quad)f + \quad] \quad + [(\,^2 + 1)f + 1] \quad + \quad [(1 - \quad \cot\quad)f + 1] \quad = 0.$

Particular solutions: $_1 = e^-$, $_2 = \sin$.

103. $+[f + \quad \tan(\quad)(\quad f + 1)] \quad - \,^2\, f \quad + \,^2 f \quad = 0.$

Particular solutions: $_1 =$, $_2 = \cos(\lambda\quad)$.

104. $+[f - \quad \cot(\quad)(\quad f + 1)] \quad - \,^2\, f \quad + \,^2 f \quad = 0.$

Particular solutions: $_1 =$, $_2 = \sin(\lambda\quad)$.

105. $\sin(\quad) \quad + \quad + 3\quad ^2\sin(\quad) \quad + 2\quad ^3\cos(\quad) \quad = f(\quad).$

Integrating the equation twice, we arrive at a first-order linear equation:

$$a\sin(\lambda\quad)\,' + [\quad - 2a\lambda\cos(\lambda\quad)] \;=\; _1 + \quad _2 + \quad \int_0 (\quad - \quad)\,(\,)\quad, \qquad _0 \text{ is any number.}$$

106. $\sin(\quad) \quad + [\quad + (2\quad + 1)\cos(\quad)] \quad - (\,^2 + 2\quad)\sin(\quad) \quad - \,^2\cos(\quad) \quad = f(\quad).$

Integrating the equation twice, we arrive at a first-order linear equation:

$$\sin(\lambda\quad)\,' + [a + \cos(\lambda\quad)] \;=\; _1 + \quad _2 + \quad \int_0 (\quad - \quad)\,(\,)\quad, \qquad _0 \text{ is any number.}$$

107. $(f - 1) \quad - [\quad f + \quad \tan(\quad)] \quad + (\,^2 f + \,^2) \quad + \quad [\quad \tan(\quad) - \quad f] \quad = 0.$

Particular solutions: $_1 = e$, $_2 = \cos(\lambda\quad)$.

108. $+ f \quad + \quad + (f \quad + \quad) \quad = 0.$

Integrating yields a second-order linear equation: $'' + g \quad = \quad \exp\,-$.

109. $+ 3f \quad + (f \quad + 2f^2 + 2\quad) \quad + (2f \quad + \quad) \quad = 0.$

Solution: $= \,_1\,^2_1 + \,_2\,_1\,_2 + \,_3\,^2_2$. Here, $_1$, $_2$ is a fundamental set of solutions of the second-order linear equation: $'' + \quad ' + \frac{1}{2}g \quad = 0$.

110. $+ (f + \quad) \quad + (f \quad + f \quad + \quad) \quad + (\quad + \quad) \quad = 0.$

Integrating yields a second-order linear equation: $'' + \quad ' + \quad = \quad \exp\,- \quad g$.

111. $+ (f + \quad) \quad + (2\quad + f \quad + \quad) \quad + (\quad + f \quad + \quad) \quad = 0.$

The substitution $= \,' + g$ leads to a second-order linear equation: $'' + \quad ' + \quad = 0$.

3.2. Equations of the Form $y \quad = \quad y\;(y\;)^{\gamma}(y\quad)$

3.2.1. Classification Table

Table 28 presents below all solvable equations whose solutions are outlined in Subsections 3.2.2–3.2.4. Two-parameter families (in the space of the parameters , β, , and), one-parameter families, and isolated points are represented in a consecutive fashion. Equations are arranged in accordance with the growth of , the growth of (for identical), the growth of β (for identical and), and the growth of (for identical , , and β). The number of the equation sought is indicated in the last column in this table.

In Subsections 3.2.2–3.2.4, the value of the insignificant parameter A is in many cases defined in the form of a function of two (one) auxiliary coefficients a and ,

$$A = \quad (a, \quad) \tag{1}$$

and the corresponding solutions are represented in parametric form,

$$= \quad {}_1(\tau, \quad {}_1, \quad {}_2, \quad {}_3, a), \qquad = \quad {}_2(\tau, \quad {}_1, \quad {}_2, \quad {}_3, \quad), \tag{2}$$

where τ is the parameter, ${}_1$, ${}_2$, and ${}_3$ are arbitrary constants, and ${}_1$ and ${}_2$ are some functions.

Having fixed the auxiliary coefficient sign $a > 0$ (or $\;> 0$), one should express the coefficient in terms of both A and a with the help of

$$= \quad (A, a).$$

Substituting this formula into (2) yields a solution of the equation under consideration (where the specific numerical value of the coefficient a can be chosen arbitrarily). The case $a < 0$ (or $\;< 0$), which may lead to a branch of the solution or to a different domain of definition of the variables and in (2), should be considered in a similar manner.

TABLE 28
Solvable equations of the form $\;''' = A \quad (\;')\;(\;'')$

		β		Equation
arbitrary	arbitrary	0	arbitrary	3.2.4.15*
arbitrary ($\neq 2$)	arbitrary ($\neq -1$)	0	0	3.2.4.1
$\dfrac{\;+4\beta+5}{\;+2\beta+3}$	arbitrary ($\neq -1$)	arbitrary ($\beta \neq -1$)	0	3.2.4.174
$\dfrac{3\;+7}{2(\;+2)}$	arbitrary ($\neq -2$)	$-\frac{1}{2}$	0	3.2.4.10
$\dfrac{3\;+7}{2(\;+2)}$	arbitrary ($\neq -2$)	1	0	3.2.4.7
arbitrary ($\neq 1, 2$)	-1	-1	0	3.2.4.175
arbitrary ($\neq 2$)	-1	0	0	3.2.4.11
$\dfrac{3\beta+4}{2\beta+3}$	0	arbitrary ($\beta \neq -\frac{3}{2}$)	0	3.2.4.8
arbitrary ($\neq \frac{3}{2}$)	0	$-\frac{1}{2}$	0	3.2.4.87

* Given are formulas of reduction to the generalized Emden–Fowler equation.

TABLE 28 (*Continued*)

Solvable equations of the form $''' = A \quad (\,') \; (\,'')$

		β		Equation
arbitrary ($\neq 1$)	1	arbitrary ($\beta \neq -1$)	0	3.2.4.2
arbitrary ($\neq 1$)	1	-1	0	3.2.4.13
arbitrary ($\neq 2$)	1	1	0	3.2.4.4
$\dfrac{3\beta+4}{2\beta+3}$	3	arbitrary ($\beta \neq -\frac{3}{2}$)	0	3.2.4.9
-1	3	$-\frac{7}{5}$	0	3.2.4.168
-1	3	0	0	3.2.4.164
0	arbitrary ($\neq -1$)	0	0	3.2.4.3
0	arbitrary	$-\frac{1}{4}(\quad + 5)$	0	3.2.4.171
0	$-2\beta - 5$	arbitrary ($\beta \neq -2$)	0	3.2.4.5
0	-13	1	0	3.2.4.153
0	-13	3	0	3.2.4.155
0	-7	0	0	3.2.4.141
0	-7	1	0	3.2.4.145
0	-4	$-\frac{1}{2}$	0	3.2.4.127
0	-4	0	0	3.2.4.123
0	-3	-2	0	3.2.4.95
0	-3	-1	0	3.2.4.30
0	-3	0	0	3.2.4.26
0	-3	1	0	3.2.4.91
0	$-\frac{7}{3}$	$-\frac{10}{3}$	0	3.2.4.76
0	$-\frac{7}{3}$	$-\frac{7}{3}$	0	3.2.4.42
0	$-\frac{7}{3}$	$-\frac{4}{3}$	0	3.2.4.52
0	$-\frac{7}{3}$	$-\frac{5}{6}$	0	3.2.4.133
0	$-\frac{7}{3}$	$-\frac{1}{2}$	0	3.2.4.131
0	$-\frac{7}{3}$	0	0	3.2.4.48
0	$-\frac{7}{3}$	1	0	3.2.4.38
0	$-\frac{7}{3}$	2	0	3.2.4.70
0	$-\frac{9}{5}$	$-\frac{13}{5}$	0	3.2.4.64
0	$-\frac{9}{5}$	1	0	3.2.4.60

TABLE 28 (*Continued*)
Solvable equations of the form $\;''' = A \quad (\;') \;(\;'')$

		β		Equation
0	−1	−2	0	3.2.4.22
0	−1	0	0	3.2.4.18
0	0	$-\frac{7}{2}$	0	3.2.2.2
0	0	$-\frac{7}{2}$	3	3.2.3.3
0	0	$-\frac{5}{2}$	0	3.2.2.3
0	0	$-\frac{5}{2}$	1	3.2.3.4
0	0	−2	0	3.2.2.6
0	0	$-\frac{4}{3}$	$-\frac{4}{3}$	3.2.3.5
0	0	$-\frac{4}{3}$	0	3.2.2.4
0	0	$-\frac{5}{4}$	$-\frac{3}{2}$	3.2.3.7
0	0	$-\frac{5}{4}$	0	3.2.2.8
0	0	$-\frac{7}{6}$	$-\frac{5}{3}$	3.2.3.6
0	0	$-\frac{7}{6}$	0	3.2.2.5
0	0	$-\frac{1}{2}$	−3	3.2.3.8
0	0	$-\frac{1}{2}$	$-\frac{3}{2}$	3.2.3.9
0	0	$-\frac{1}{2}$	0	3.2.2.7
0	0	0	arbitrary	3.2.3.1
0	0	0	0	3.2.2.1
0	0	1	arbitrary	3.2.3.2
0	1	1	0	3.2.4.35
0	2	$-\frac{7}{2}$	0	3.2.4.165
0	2	0	0	3.2.4.161
0	3	arbitrary ($\beta \neq -2$)	0	3.2.4.85
0	3	−2	0	3.2.4.82
0	5	−5	0	3.2.4.105
0	5	$-\frac{20}{7}$	0	3.2.4.117
0	5	$-\frac{15}{7}$	0	3.2.4.111
0	5	0	0	3.2.4.101
$\frac{1}{2}$	0	$-\frac{5}{2}$	0	3.2.4.74
$\frac{1}{2}$	3	$-\frac{15}{8}$	0	3.2.4.114
$\frac{1}{2}$	3	$-\frac{20}{13}$	0	3.2.4.120

TABLE 28 (*Continued*)

Solvable equations of the form $'''$ $= A$ $(\ ')$ $(\ '')$

		β		Equation
$\frac{1}{2}$	3	$-\frac{5}{4}$	0	3.2.4.108
$\frac{1}{2}$	3	0	0	3.2.4.104
$\frac{2}{3}$	0	$-\frac{7}{6}$	0	3.2.4.157
$\frac{4}{5}$	-4	$-\frac{1}{2}$	0	3.2.4.137
1	arbitrary ($\neq 1$)	-1	0	3.2.4.140
1	-3	$-\frac{1}{2}$	0	3.2.4.32
1	-3	1	0	3.2.4.24
1	-1	-1	0	3.2.4.177
1	1	arbitrary ($\beta \neq -1$)	0	3.2.4.14
1	1	-1	0	3.2.4.17
1	1	1	0	3.2.4.21
$\frac{8}{7}$	3	$-\frac{3}{4}$	0	3.2.4.159
$\frac{8}{7}$	3	$-\frac{1}{2}$	0	3.2.4.151
$\frac{6}{5}$	0	$-\frac{2}{3}$	0	3.2.4.109
$\frac{5}{4}$	-4	$-\frac{1}{2}$	0	3.2.4.57
$\frac{5}{4}$	3	$-\frac{1}{2}$	0	3.2.4.148
$\frac{5}{4}$	3	0	0	3.2.4.144
$\frac{9}{7}$	$-\frac{9}{4}$	1	0	3.2.4.66
$\frac{9}{7}$	0	$-\frac{1}{3}$	0	3.2.4.169
$\frac{9}{7}$	0	1	0	3.2.4.62
$\frac{13}{10}$	0	$-\frac{5}{2}$	0	3.2.4.80
$\frac{27}{20}$	0	$-\frac{2}{3}$	0	3.2.4.121
$\frac{18}{13}$	0	$-\frac{7}{2}$	0	3.2.4.68
$\frac{7}{5}$	-7	1	0	3.2.4.54
$\frac{7}{5}$	$-\frac{5}{2}$	1	0	3.2.4.45
$\frac{7}{5}$	$-\frac{13}{7}$	1	0	3.2.4.78
$\frac{7}{5}$	$-\frac{1}{3}$	1	0	3.2.4.72
$\frac{7}{5}$	0	1	0	3.2.4.40
$\frac{7}{5}$	1	1	0	3.2.4.50
$\frac{7}{5}$	3	0	0	3.2.4.126
$\frac{7}{5}$	3	1	0	3.2.4.130

TABLE 28 (*Continued*)
Solvable equations of the form $''' = A \quad (\ ')\ (\ '')$

		β		Equation
$\frac{7}{5}$	11	1	0	3.2.4.135
$\frac{10}{7}$	0	$-\frac{5}{2}$	0	3.2.4.43
$\frac{22}{15}$	0	$-\frac{2}{3}$	0	3.2.4.115
$\frac{3}{2}$	arbitrary	$\frac{1}{2}(\ -1)$	0	3.2.4.173
$\frac{3}{2}$	−3	$-\frac{1}{2}$	0	3.2.4.100
$\frac{3}{2}$	−3	1	0	3.2.4.97
$\frac{3}{2}$	0	−2	0	3.2.4.99
$\frac{3}{2}$	0	$-\frac{1}{2}$	0	3.2.4.84
$\frac{3}{2}$	0	1	0	3.2.4.93
$\frac{3}{2}$	1	1	0	3.2.4.28
$\frac{3}{2}$	3	−2	0	3.2.4.98
$\frac{3}{2}$	3	$-\frac{1}{2}$	0	3.2.4.94
$\frac{3}{2}$	3	0	0	3.2.4.29
$\frac{23}{15}$	$-\frac{1}{3}$	$-\frac{1}{2}$	0	3.2.4.116
$\frac{11}{7}$	−4	$-\frac{1}{2}$	0	3.2.4.47
$\frac{8}{5}$	1	1	0	3.2.4.125
$\frac{8}{5}$	3	−4	0	3.2.4.55
$\frac{8}{5}$	3	$-\frac{7}{4}$	0	3.2.4.46
$\frac{8}{5}$	3	$-\frac{10}{7}$	0	3.2.4.79
$\frac{8}{5}$	3	$-\frac{2}{3}$	0	3.2.4.73
$\frac{8}{5}$	3	$-\frac{1}{2}$	0	3.2.4.41
$\frac{8}{5}$	3	0	0	3.2.4.51
$\frac{8}{5}$	3	1	0	3.2.4.129
$\frac{8}{5}$	3	5	0	3.2.4.136
$\frac{21}{13}$	−6	$-\frac{1}{2}$	0	3.2.4.69
$\frac{33}{20}$	$-\frac{1}{3}$	$-\frac{1}{2}$	0	3.2.4.122
$\frac{17}{10}$	−4	$-\frac{1}{2}$	0	3.2.4.81
$\frac{12}{7}$	$\frac{1}{3}$	$-\frac{1}{2}$	0	3.2.4.170
$\frac{12}{7}$	3	$-\frac{13}{8}$	0	3.2.4.67
$\frac{12}{7}$	3	$-\frac{1}{2}$	0	3.2.4.63
$\frac{7}{4}$	0	$-\frac{5}{2}$	0	3.2.4.56
$\frac{7}{4}$	0	1	0	3.2.4.147

TABLE 28 (*Continued*)

Solvable equations of the form $'''\ = A\ \ (\ ')\ (\ '')$

		β		Equation
$\frac{7}{4}$	1	1	0	3.2.4.143
$\frac{9}{5}$	$-\frac{1}{3}$	$-\frac{1}{2}$	0	3.2.4.110
$\frac{13}{7}$	$-\frac{1}{2}$	1	0	3.2.4.160
$\frac{13}{7}$	0	1	0	3.2.4.152
2	arbitrary ($\neq -1$)	0	0	3.2.4.12
2	-1	arbitrary ($\beta \neq 0$)	0	3.2.4.139
2	-1	-1	0	3.2.4.176
2	-1	0	0	3.2.4.16
2	0	-2	0	3.2.4.33
2	3	-2	0	3.2.4.25
2	3	0	0	3.2.4.19
$\frac{11}{5}$	0	$-\frac{5}{2}$	0	3.2.4.138
$\frac{7}{3}$	$-\frac{4}{3}$	$-\frac{1}{2}$	0	3.2.4.158
$\frac{5}{2}$	-4	$-\frac{1}{2}$	0	3.2.4.75
$\frac{5}{2}$	$-\frac{11}{4}$	1	0	3.2.4.113
$\frac{5}{2}$	$-\frac{27}{13}$	1	0	3.2.4.119
$\frac{5}{2}$	$-\frac{3}{2}$	1	0	3.2.4.107
$\frac{5}{2}$	1	1	0	3.2.4.103
3	arbitrary ($\neq -3$)	1	0	3.2.4.86
3	$-2\beta - 5$	arbitrary ($\beta \neq -2$)	0	3.2.4.6
3	arbitrary	$-\ \ -2$	0	3.2.4.172
3	-9	2	0	3.2.4.106
3	-6	$-\frac{1}{2}$	0	3.2.4.59
3	-6	$\frac{1}{2}$	0	3.2.4.166
3	$-\frac{17}{3}$	$-\frac{5}{3}$	0	3.2.4.77
3	$-\frac{33}{7}$	2	0	3.2.4.118
3	$-\frac{21}{5}$	$-\frac{7}{5}$	0	3.2.4.65
3	-4	$-\frac{1}{2}$	0	3.2.4.37
3	$-\frac{11}{3}$	$-\frac{5}{3}$	0	3.2.4.44
3	$-\frac{23}{7}$	2	0	3.2.4.112

TABLE 28 (*Continued*)
Solvable equations of the form ‴ = A (′) (″)

		β		Equation
3	–3	–2	0	3.2.4.96
3	–3	–1	0	3.2.4.23
3	–3	$-\frac{1}{2}$	0	3.2.4.90
3	–3	1	0	3.2.4.83
3	$-\frac{5}{3}$	$-\frac{5}{3}$	0	3.2.4.53
3	$-\frac{5}{3}$	$-\frac{1}{2}$	0	3.2.4.149
3	$-\frac{4}{3}$	$-\frac{1}{2}$	0	3.2.4.150
3	–1	–2	0	3.2.4.31
3	$-\frac{2}{3}$	$-\frac{5}{3}$	0	3.2.4.134
3	0	$-\frac{5}{2}$	0	3.2.4.128
3	0	$-\frac{5}{3}$	0	3.2.4.132
3	0	$-\frac{1}{2}$	0	3.2.4.89
3	0	1	–3	3.2.4.88
3	1	–4	0	3.2.4.142
3	1	$-\frac{5}{2}$	0	3.2.4.124
3	1	–2	0	3.2.4.27
3	1	$-\frac{5}{3}$	0	3.2.4.49
3	1	–1	0	3.2.4.20
3	1	$-\frac{1}{2}$	0	3.2.4.34
3	1	$\frac{1}{2}$	0	3.2.4.162
3	1	2	0	3.2.4.102
3	3	–7	0	3.2.4.154
3	3	–4	0	3.2.4.146
3	3	–2	0	3.2.4.92
3	3	$-\frac{5}{3}$	0	3.2.4.39
3	3	$-\frac{7}{5}$	0	3.2.4.61
3	3	$-\frac{1}{2}$	0	3.2.4.58
3	3	0	0	3.2.4.36
3	5	$-\frac{5}{3}$	0	3.2.4.71
3	7	–7	0	3.2.4.156
4	$-\frac{9}{5}$	1	0	3.2.4.167
4	1	1	0	3.2.4.163

3.2.2. Equations of the Form =

1. = .

Solution: $= \frac{1}{6}A \;\; ^{3} + \;\; _{2} \;\; ^{2} + \;\; _{1} \;\; + \;\; _{0}$.

2. = $^{-7} \;\; ^{2}$.

Solution in parametric form:

$$= a \;\; ^{3}_{1} \;\; \left[\;\; _{1}e^{2\sigma\tau} + \;\; _{2}e^{-\sigma\tau}\sin\left(\;\; \overline{3}\,\sigma\tau \;\; \right) \right]^{-3 \;\; 2} \;\; \tau + \;\; _{3},$$

$$= \;\; ^{2}_{1}\left[\;\; _{1}e^{2\sigma\tau} + \;\; _{2}e^{-\sigma\tau}\sin\left(\;\; \overline{3}\,\sigma\tau \;\; \right) \right]^{-1},$$

where $A = -8a^{-3} \;\; ^{9} \;\; ^{2}\sigma^{3}$.

3. = $^{-5} \;\; ^{2}$.

Solution in parametric form:

$$= a \;\; ^{7}_{1} \;\; (\tau^{3} - 3\tau + \;\; _{2})^{-3 \;\; 2} \;\; \tau + \;\; _{3}, \qquad = \;\; ^{6}_{1}(\tau^{3} - 3\tau + \;\; _{2})^{-1}, \qquad \text{where} \quad A = -6a^{-3} \;\; ^{7} \;\; ^{2}.$$

4. = $^{-4} \;\; ^{3}$.

Solution in parametric form:

$$= a \;\; ^{7}_{1} \;\; R^{-1}(2\tau \;\; \mp R)^{2} \;\; \tau + \;\; _{3}, \qquad = \;\; ^{9}_{1}(2\tau \;\; \mp R)^{3},$$

where $R = \;\; \overline{\;\; (4\tau^{3} - 1)}, \quad = \;\; \tau R^{-1} \;\; \tau + \;\; _{2}, \; A = \;\; 18a^{-3} \;\; ^{7} \;\; ^{3}$.

5. = $^{-7} \;\; ^{6}$.

Solution in parametric form:

$$= a \;\; ^{13}_{1} \;\; R^{-1}(2\tau \;\; \mp R)^{-5 \;\; 2} \;\; \tau + \;\; _{3}, \qquad = \;\; ^{18}_{1}(2\tau \;\; \mp R)^{-3},$$

where $R = \;\; \overline{\;\; (4\tau^{3} - 1)}, \quad = \;\; \tau R^{-1} \;\; \tau + \;\; _{2}, \; A = \mp 18a^{-3} \;\; ^{13} \;\; ^{6}$.

In the solutions of equations 6 and 7, the following notation is used:

$$Z = \begin{cases} \;\; _{1} \;\; _{1 \;\; 3}(\tau) + \;\; _{2} \;\; _{1 \;\; 3}(\tau) & \textit{for the upper sign,} \\ \;\; _{1} \;\; _{1 \;\; 3}(\tau) + \;\; _{2} \;\; _{1 \;\; 3}(\tau) & \textit{for the lower sign,} \end{cases}$$

where $_{1 \;\; 3}(\tau)$ *and* $_{1 \;\; 3}(\tau)$ *are the Bessel functions, and* $_{1 \;\; 3}(\tau)$ *and* $_{1 \;\; 3}(\tau)$ *are the modified Bessel functions.*

6. = $^{-2}$.

Solution in parametric form:

$$= a \;\; _{1} \;\; \tau^{-1}Z^{-2} \;\; \tau + \;\; _{3}, \qquad = \;\; _{1}\tau^{-2 \;\; 3}Z^{-2}, \qquad \text{where} \quad A = \;\; \tfrac{4}{3}a^{-3} \;\; ^{3}.$$

7. = $^{-1} \;\; ^{2}$.

Solution in parametric form:

$$= a \;\; _{1} \;\; Z \;\; \tau + \;\; _{3}, \qquad = \;\; ^{2}_{1}\tau^{2 \;\; 3}Z^{2}, \qquad \text{where} \quad A = \mp\tfrac{4}{3}a^{-3} \;\; ^{3} \;\; ^{2}.$$

8. = $^{-5} \;\; ^{4}$.

This is a special case of equation 3.2.4.171 with $\;\; = 0$.

3.2.3. Equations of the Form =

For = 0, see Subsection 3.2.2.

1. = .

Solution: $= A \; (\;) + \;_2 \;^2 + \;_1 \; + \;_0$, where

$$(\;) = \begin{cases} \dfrac{\;^{+3}}{(\; +1)(\; +2)(\; +3)} & \text{if } \; \neq -1,\ -2,\ -3; \\ \frac{1}{2} \;^2 \ln|\;| - \frac{3}{4} \;^2 & \text{if } \; = -1; \\ - \; \ln|\;| + & \text{if } \; = -2; \\ \frac{1}{2} \ln|\;| & \text{if } \; = -3. \end{cases}$$

2. = .

See equation 3.1.2.7.

3. = $^{3 \;\; -7 \;\; 2}$.

Solution in parametric form:

$$= a \;_1^3 \quad ^{-3\;\;2} \; \tau + \;_3 \quad ^{-1}, \qquad = \;_1^4 \;^{-1} \quad ^{-3\;\;2} \; \tau + \;_3 \quad ^{-2},$$

where $= \;_1 e^{2\sigma\tau} + \;_2 e^{-\sigma\tau} \sin\big(\;\overline{3}\,\sigma\tau \;\;$, $A = 8a^{-6\;\;9\;\;2}\sigma^3$.

4. = $^{-5\;\;2}$.

Solution in parametric form:

$$= a \;_1^7 \quad (\tau^3 - 3\tau + \;_2)^{-3\;\;2} \; \tau + \;_3 \quad ^{-1},$$

$$= \;_1^8 (\tau^3 - 3\tau + \;_2)^{-1} \quad (\tau^3 - 3\tau + \;_2)^{-3\;\;2} \; \tau + \;_3 \quad ^{-2},$$

where $A = 6a^{-4\;\;7\;\;2}$.

In the solutions of equations 5 and 6, the following notation is used:

$$R = \quad \overline{(4\tau^3 - 1)}, \qquad = \; \tau R^{-1} \; \tau + \;_2.$$

5. = $^{-4\;\;3\;\;-4\;\;3}$.

Solution in parametric form:

$$= a \;_1^7 \quad R^{-1}(2\tau \;\; \mp R)^2 \; \tau + \;_3 \quad ^{-1}, \qquad = \;_1^5 (2\tau \;\; \mp R)^3 \quad R^{-1}(2\tau \;\; \mp R)^2 \; \tau + \;_3 \quad ^{-2},$$

where $A = \mp 18a^{-5\;\;3\;\;7\;\;3}$.

6. = $^{-5\;\;3\;\;-7\;\;6}$.

Solution in parametric form:

$$= a \;_1^{13} \quad R^{-1}(2\tau \;\; \mp R)^{-5\;\;2} \; \tau + \;_3 \quad ^{-1},$$

$$= \;_1^8 (2\tau \;\; \mp R)^{-3} \quad R^{-1}(2\tau \;\; \mp R)^{-5\;\;2} \; \tau + \;_3 \quad ^{-2},$$

where $A = \mp 18a^{-4\;\;3\;\;13\;\;6}$.

7. $= \ ^{-3\ 2\ -5\ 4}.$

Solution in parametric form:

$$= a \ _1^3 \quad \tau^{-1\ 2} z^{-1\ 2\ \ 3\ 4} \ \tau + \ _3 \ ^{-1}, \qquad = \ _1^2 \quad \tau^{-1\ 2} z^{-1\ 2\ \ 3\ 4} \ \tau + \ _3 \ ^{-2},$$

where $z = \ _2 + \frac{1}{4}\tau^2 + 4B\tau^{1\ 2}$, $\ = \exp \quad z^{-1\ 2} \ \tau$, $A = \frac{1}{2} B a^{-3\ 2\ 9\ 4}$.

8. $= \ ^{-3\ -1\ 2}.$

Solution in parametric form:

$$= \ _1 \quad Z \ \tau + \ _3 \ ^{-1}, \qquad = \tau^{2\ 3} Z^2 \quad Z \ \tau + \ _3 \ ^{-2},$$

where

$$A = \ \tfrac{4}{3} \ ^{3\ 2}, \qquad Z = \begin{cases} \ _1 \ _{1\ 3}(\tau) + \ _2 \ _{1\ 3}(\tau) & \text{for the upper sign,} \\ \ _1 \ _{1\ 3}(\tau) + \ _2 \ _{1\ 3}(\tau) & \text{for the lower sign,} \end{cases}$$

$_{1\ 3}(\tau)$ and $_{1\ 3}(\tau)$ are the Bessel functions, and $_{1\ 3}(\tau)$ and $_{1\ 3}(\tau)$ are the modified Bessel functions.

9. $= \ ^{-3\ 2\ -1\ 2}.$

Solution in parametric form:

$$= a \ _3 \exp \ 2 \quad \tau \ , \qquad = \ _3 \ ^2 \exp \ 2 \quad \tau \ .$$

Here, $\ = \ (\tau, \ _1, \ _2)$ is the general solution of the second Painlevé transcendent: $''_{\tau\tau} = \ \tau \ + 2 \ ^3$, and $A = \ \frac{1}{4} a^{-3\ 2\ \ 3\ 2}$.

3.2.4. Equations with $| \ | + | \ | \neq 0$

1. $= \ (\ ) \ (\ ) \ , \quad \gamma \neq -1, \quad \neq 2.$

Solution in parametric form:

$$= a \ _1^{\ + \ -1} \ \tau^{-1\ 2} \ 1 \ \ \tau^{\frac{+1}{2}} \ ^{\frac{1}{-2}} \ \tau + \ _3, \qquad = \ _1^{\ +2\ -3} \ \ 1 \ \ \tau^{\frac{+1}{2}} \ ^{\frac{1}{-2}} \ \tau + \ _2,$$

where $A = \dfrac{\ + 1}{2 - \ } 2 \ ^{-2} a^{\ +2\ -3\ \ 1-\ -\ }$.

2. $= \ \ (\ ) \ , \quad \neq -1, \quad \neq 1.$

Solution in parametric form:

$$= a \ _1^{\ +} \quad \big(1 \ \ \tau^{\ +1} \ ^{\frac{1}{1-}} \ \tau + \ _2 \ ^{-1\ 2} \ \tau + \ _3, \qquad = \ _1^2 \ ^{-2} \tau,$$

where $A = \dfrac{\beta + 1}{1 - \ } 2 \ ^{-1} a^{2\ -2\ -\ -}$.

In the solutions of equations 3–10, the following notation is used:

$$R = \sqrt{1 \ \ \tau^{\ +1}}, \qquad = \ (1 \ \ \tau^{\ +1})^{-1\ 2} \ \tau + \ _2, \qquad = R \ - \tau.$$

3. $= \ (\ ) \ , \quad \gamma \neq -1.$

Solution in parametric form:

$$= a \ _1 \quad \tau^{-1\ 2} R^{-1} \ \tau + \ _3, \qquad = \ _1 \ ^{-1} \ ,$$

where $\ = \dfrac{\ - 1}{2}$, $A = \dfrac{\ + 1}{4} a^{2\ -2\ -2}$.

4. $= \quad (\quad)\,, \qquad \neq 2.$

Solution in parametric form:

$$= a \ {}^{3}_{1} \ {}^{+1} \quad {}^{-1} \ {}^{2}R^{-1} \ \tau + \ {}_{3}, \qquad = \ {}^{2}_{1} \ {}^{+2}R,$$

where $= \dfrac{1}{-2}, \ A = -8 \quad a^{2} \ {}^{-3} \ \dfrac{2a^{2}}{(\quad +1)} \ {}^{1} \quad .$

5. $= \quad (\quad)^{-2 \ \ -5}, \qquad \neq -2.$

Solution in parametric form:

$$= a \ {}_{1} \ {}^{-3} \quad \tau^{-1 \ \ 2} \ \ {}^{-3 \ \ 2}R^{-1} \ \tau + \ {}_{3}, \qquad = \ {}^{2}_{1} \ {}^{-2} \ \ {}^{-1},$$

where $= -\beta - 3, \ A = \ \tfrac{1}{4}(-1)^{-2} \ \ (\quad +1)a^{2} \ {}^{-2 \ \ 3-} \quad .$

6. $= \quad (\quad)^{-2 \ \ -5}(\quad)^{3}, \qquad \neq -2.$

Solution in parametric form:

$$= a \ {}_{1} \ {}^{+3} \quad {}^{-3 \ \ 2}R^{-1} \quad \tau + \ {}_{3}, \qquad = \ {}^{2}_{1} \ {}^{+2}\tau \ \ {}^{-1},$$

where $= \beta, \ A = \mp 2(\quad +1)a^{-2} \ \ {}^{-2} \quad {}^{+3}.$

7. $= \quad (\quad) \ (\quad)^{\frac{3 \ \ +7}{2 \ \ +4}}, \qquad \gamma \neq -2.$

Solution in parametric form:

$$= a \ {}_{1} \ {}^{2+ \ \ +2} \quad {}^{-} \quad {}^{2}R^{-1} \ \tau + \ {}_{3}, \qquad = \ {}^{4}_{1} \ \ ,$$

where $= -\dfrac{2(\ +2)}{+1}, \ A = -\dfrac{2 \quad {}^{-1}}{+2} \ \dfrac{(\quad +1)}{2a} \ {}^{\frac{2}{+2}} \ \dfrac{4a^{2}}{(\quad +1)(\quad +2)} \ {}^{\frac{1}{}} \quad .$

8. $= \quad (\quad)^{\frac{3 \ \ +4}{2 \ \ +3}}, \qquad \neq -3 \ \ 2.$

Solution in parametric form:

$$= a \ {}_{1} \ {}^{2+2 \ \ -1} \quad \tau \quad {}^{- \ \ -2}R^{-1} \ \tau + \ {}_{3}, \qquad = \ {}^{(\ \ +1)^{2}}_{1}\tau \ \ {}^{+1 \ \ - \ \ -1},$$

where $= -\dfrac{\beta}{\beta+1}, \ A = (\quad +3)a^{-1} \ \ {}^{\overline{\ \ +1}} \quad \dfrac{2a^{2}}{(\quad +1)^{2}} \ {}^{\frac{1}{+3}}.$

9. $= \quad (\quad)^{3}(\quad)^{\frac{3 \ \ +4}{2 \ \ +3}}, \qquad \neq -3 \ \ 2.$

Solution in parametric form:

$$= a \ {}_{1} \ {}^{2+ \ \ -1} \quad {}^{+1 \ \ -1 \ \ 2}R^{-1} \ \tau + \ {}_{3}, \qquad = \ {}^{(\ \ -1)(\ \ +2)}_{1} \quad {}^{+2},$$

where $= -\dfrac{2\beta+3}{\beta+1}, \ A = \dfrac{}{(\quad +2)^{3}}a^{2} \ {}^{-\frac{2 \ \ +1}{+1}} \ \dfrac{(\quad +1)(\quad +2)}{4a^{2}} \ {}^{\frac{1}{}} \quad .$

---

10. $\quad = \quad ^{-1}\ ^{2}(\quad)\ (\quad)^{\frac{3\ +7}{2\ +4}}, \qquad \gamma \neq -2.$

Solution in parametric form:

$$= a \ _1 \ ^{2+2 \ -7} \quad \tau^{\frac{-1}{2}} R^{-1} \quad ^{\frac{+3}{2}} \quad \tau + \ _3, \qquad = \ _1^{-8} \ ^2,$$

where $\quad = \frac{1-}{1+}, \ A = -\frac{2(\quad +3)}{+1} \ _1 \ _2 \quad \frac{a}{(\quad +1)} \ ^{-\frac{2}{+1}} \ \frac{(\quad +1)^2}{2a^2} \ ^{-\frac{1}{+3}}.$

11. $\quad = \quad (\quad)^{-1}(\quad)\ , \qquad \neq 2.$

Solution in parametric form:

$$= a \ _1 \quad \tau^{\overline{\ -2}} \exp(\mp\tau^2) \ \ \tau + \ _2, \qquad = \ _1^2 \quad \tau^{\overline{\ -2}} \exp(\mp 2\tau^2) \ \ \tau + \ _3,$$

where $A = \mp\frac{4 \ ^2}{(2-\quad)a^4} \ \mp\frac{a^2}{2} \quad .$

12. $\quad = \quad (\quad)\ (\quad)^2, \qquad \gamma \neq -1.$

Solution in parametric form:

$$= a \ _1 \quad \tau^{\frac{1-}{1+}} \exp(\mp\tau^2) \ \ \tau + \ _2, \qquad = \ _1 \quad \tau^{\frac{3-}{1+}} \exp(\mp\tau^2) \ \ \tau + \ _3,$$

where $A = \ (\quad +1)a \ ^{+1 \ - \ -1}.$

13. $\quad = \quad ^{-1} \quad (\quad)\ , \qquad \neq 1.$

Solution in parametric form:

$$= a \ _1 \quad \tau \exp(\mp\tau^2) \quad \tau^{\frac{3-}{1-}} \exp(\mp\tau^2) \ \ \tau + \ _2 \quad ^{-1 \ 2} \quad \tau + \ _3, \qquad = \ _1^2 \exp(\mp\tau^2),$$

where $A = \frac{1}{1-}(\mp 1)^{-} \ a^{2 \ -2 \ 1-} \ .$

14. $\quad = \qquad , \qquad \neq -1.$

Solution in parametric form:

$$= \ _1 \quad \tau^{\frac{1-}{1+}} \quad \tau^{\frac{1-}{1+}} \exp(\mp\tau^2) \ \ \tau + \ _2 \quad ^{-1 \ 2} \quad \tau + \ _3, \qquad = \tau^{\frac{2}{1+}},$$

where $A = \mp(\beta+1) \ ^{-1-} \ .$

15. $\quad = \quad (\quad)\ (\quad)\ .$

Solution in parametric form:

$$= a \ _1^{\ + \ -1} X(\tau), \qquad = \ _1^{\ +2 \ - \ -3} \quad (\tau)\frac{X(\tau)}{\tau} \ \ \tau + \ _3.$$

Here, $X = X(\tau), \quad = \ (\tau)$ is the general solution of the generalized Emden–Fowler equation:

$$'' \ = BX \quad (\ ') \ , \qquad \text{where} \quad A = Ba^{\ +2 \ - \ -3 \ 1- \ -} \ .$$

16. $\quad = \quad (\quad)^{-1}(\quad)^2.$

Solution:
$$= \begin{cases} \frac{1-A}{(2-A) \ _1}(\ _1 \ + \ _2)^{\frac{2-}{1-}} + \ _3 & \text{if } A \neq 1, \ A \neq 2; \\ \frac{2}{_1}\exp(\ _1 \ ) + \ _3 & \text{if } A = 1; \\ \frac{1}{_1}\ln(\ _1 \ + \ _2) + \ _3 & \text{if } A = 2. \end{cases}$$

17. $= \;^{-1}\;.$

Solution: $=\begin{cases}(\;_1\;^{+1} + \;_2)^{-1\;2}\; + \;_3 & \text{if } A \neq -1;\\ (\;_1 \ln \; + \;_2)^{-1\;2}\; + \;_3 & \text{if } A = -1.\end{cases}$

18. $= \;(\;)^{-1}.$

Solution in parametric form:

$$= a\;_1\; \exp(\mp\tfrac{1}{2}\tau^2)\;\tau + \;_2, \qquad = \;^2_1\; \exp(\mp\tau^2)\;\tau + \;_3, \qquad \text{where} \quad A = \mp a^{-4}\;^2.$$

19. $= \;(\;)^3(\;)^2.$

Solution in parametric form:

$$= a\;_1\; \tau^{-1\;2} \exp(\mp\tau^2)\;\tau + \;_2, \qquad = \;_1\; \exp(\mp\tau^2)\;\tau + \;_3, \qquad \text{where} \quad A = \;4a^4\;^{-4}.$$

In the solutions of equations 20–25, the following notation is used:

$$= \;\exp(\mp\tau^2)\;\tau + \;_2, \qquad = 2\tau\; \exp(\mp\tau^2).$$

20. $= \;^{-1}\;(\;)^3.$

Solution in parametric form:

$$= a\;_1\; \tau \exp(\mp\tau^2)\;^{-1\;2}\;\tau + \;_3, \qquad = \;^2_1 \exp(\mp\tau^2), \qquad \text{where} \quad A = \;\tfrac{1}{2}a^4\;^{-2}.$$

21. $= \;.$

Solution in parametric form:

$$= \;_1\;^{-1\;2}\;\tau + \;_3, \qquad = \;\tau, \qquad \text{where} \quad A = \mp 2\;^{-2}.$$

22. $= \;^{-2}(\;)^{-1}.$

Solution in parametric form:

$$= a\;_1\;^{-3\;2} \exp\bigl(\mp\tfrac{1}{2}\tau^2\;\tau + \;_3, \qquad = \;_1\;^{-1}, \qquad \text{where} \quad A = \mp a^{-4}\;^4.$$

23. $= \;^{-1}(\;)^{-3}(\;)^3.$

Solution in parametric form:

$$= \;_1\;^{-3\;2}\; \exp(\mp\tau^2)\;\tau + \;_3, \qquad = \;^{-1}\exp(\mp\tau^2), \qquad \text{where} \quad A = \mp 8\;^2.$$

24. $= \;(\;)^{-3}\;.$

Solution in parametric form:

$$= a\;_1\;^{1\;2}\;\tau + \;_3, \qquad = \;^2_1\;, \qquad \text{where} \quad A = \mp 8a^{-4}\;^2.$$

25. $= \;^{-2}(\;)^3(\;)^2.$

Solution in parametric form:

$$= a\;_1\;^{-1\;2} \exp(\mp\tau^2)\;\tau + \;_3, \qquad = \;^2_1\;, \qquad \text{where} \quad A = \;a^4\;^{-2}.$$

In the solutions of equations 26–33, the following notation is used:

$$= \overline{\tau(\tau+1)} - \ln\big(\ \bar{\tau} + \ \overline{\tau+1}\ \ + \ _2, \qquad R = \sqrt{\frac{\tau+1}{\tau}}, \qquad = R \quad - \tau.$$

26. $=\ (\quad)^{-3}$**.**

Solution in parametric form:

$$= 2a\ {}^{2}_{1}\ \ \overline{\tau+1} + \ _3, \qquad = \ {}^{3}_{1}\ \ , \qquad \text{where} \quad A = -\tfrac{1}{4}a^{-6\ \ 4}.$$

27. $=\ {}^{-2}\ \ (\quad)^{3}$**.**

Solution in parametric form:

$$= a\ \ _1\ \ \ {}^{-1\ \ 2}\ \ \tau + \ _3, \qquad = \ {}^{4}_{1}\tau, \qquad \text{where} \quad A = 2a^{4\ \ -1}.$$

28. $=\ \ (\quad)^{3\ \ 2}$**.**

Solution in parametric form:

$$= a\ {}^{5}_{1}\ \ \tau^{-2}R^{-1}\ \ {}^{-1\ \ 2}\ \ \tau + \ _3, \qquad = \ {}^{2}_{1}R, \qquad \text{where} \quad A = -8a(-\ )^{-5\ \ 2}.$$

29. $=\ (\quad)^{3}(\quad)^{3\ \ 2}$**.**

Solution in parametric form:

$$= a\ {}^{7}_{1}\ \ R^{-3\ \ 2}\ \ \tau + \ _3, \qquad = \ {}^{6}_{1}\ \ , \qquad \text{where} \quad A = 4a^{3}(-\ )^{-7\ \ 2}.$$

30. $=\ {}^{-1}(\quad)^{-3}$**.**

Solution in parametric form:

$$= a\ {}^{5}_{1}\ \ \tau^{-1\ \ 2}R^{-1}\ \ {}^{-3\ \ 2}\ \ \tau + \ _3, \qquad = \ {}^{6}_{1}\ \ {}^{-1}, \qquad \text{where} \quad A = -\tfrac{1}{4}a^{-6\ \ 5}.$$

31. $=\ {}^{-2}(\quad)^{-1}(\quad)^{3}$**.**

Solution in parametric form:

$$= a\ {}^{-1}_{1}\ \ R^{-1}\ \ {}^{-3\ \ 2}\ \ \tau + \ _3, \qquad = \ {}^{2}_{1}\tau\ \ {}^{-1}, \qquad \text{where} \quad A = 2a^{2}\ \ .$$

32. $=\ {}^{-1\ \ 2}(\quad)^{-3}\qquad$**.**

Solution in parametric form:

$$= a\ {}^{7}_{1}\ \ \tau^{-3\ \ 2}R^{-1}\ \ {}^{1\ \ 2}\ \ \tau + \ _3, \qquad = \ {}^{8}_{1}\ \ {}^{2}, \qquad \text{where} \quad A = a^{-4\ \ 7\ \ 2}.$$

33. $=\ {}^{-2}(\quad)^{2}$**.**

Solution in parametric form:

$$= a\ {}^{-1}_{1}\ \ \tau^{-2}R^{-1}\ \ \tau + \ _3, \qquad = \ _1\tau^{-1}\ \ , \qquad \text{where} \quad A = 2a\ \ .$$

34. $=\ {}^{-1\ \ 2}\ \ (\quad)^{3}$**.**

Solution in parametric form:

$$= \ a\ {}^{5}_{1}\ \ \tau(\tau^2-1)(\tau^3 - 3\tau + \ _2)^{-1\ \ 2}\ \ \tau + \ _3, \qquad = \ {}^{8}_{1}(\tau^2-1)^2,$$

where $A = \mp\frac{1}{144}a^{4\ \ -5\ \ 2}$.

35. **= .**

Solution in parametric form:

$$= a\ {}_1^{-1}\ (\tau^3 - 3\tau + \ {}_2)^{-1\ 2}\ \tau + \ {}_3, \qquad = \ {}_1^{2}\tau, \qquad \text{where} \quad A = 3a^{-2\ -1}.$$

36. **= ( )^3( )^3.**

Solution in parametric form:

$$= \tfrac{2}{5}a\ {}_1^{5}\tau^{1\ 2}(\tau^2 - 5) + \ {}_3, \qquad = \ {}_1^{6}(\tau^3 - 3\tau + \ {}_2), \qquad \text{where} \quad A = -\tfrac{8}{243}a^{6\ -5}.$$

37. **= $^{-1\ 2}$( )$^{-4}$( )^3.**

Solution in parametric form:

$$= a\ {}_1^{5}\ (\tau^2 - 1)(\tau^3 - 3\tau + \ {}_2)^{-3\ 2}(\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)\ \ \tau + \ {}_3,$$
$$= \ {}_1^{2}(\tau^2 - 1)^2(\tau^3 - 3\tau + \ {}_2)^{-1},$$

where $A = \mp\tfrac{16}{9}a^{-1\ 5\ 2}$.

38. **= ( )$^{-7\ 3}$.**

Solution in parametric form:

$$= a\ {}_1^{7}\ (\tau^3 - 3\tau + \ {}_2)^{1\ 4}\ \tau + \ {}_3, \qquad = \ {}_1^{16}(\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3),$$

where $A = \ 72a^{-5\ 2}(4\ \ a)^{1\ 3}$.

39. **= $^{-5\ 3}$( )^3( )^3.**

Solution in parametric form:

$$= \ a\ {}_1^{5}\ (\tau^2 - 1)(\tau^3 - 3\tau + \ {}_2)^{1\ 2}[\ (\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)]^{-1\ 2}\ \tau + \ {}_3,$$
$$= \ {}_1^{9}(\tau^3 - 3\tau + \ {}_2)^{3\ 2},$$

where $A = \mp 8 \times 9^{-5}a^{6\ -10\ 3}$.

40. **= ( )$^{7\ 5}$.**

Solution in parametric form:

$$= a\ {}_1^{-7}\ (\tau^3 - 3\tau + \ {}_2)^{-3\ 2}\ \tau + \ {}_3, \qquad = \ {}_1(\tau^2 - 1)(\tau^3 - 3\tau + \ {}_2)^{-1\ 2},$$

where $A = \dfrac{15}{2}a^{-1\ -1}\ \dfrac{a^2}{2}^{\ 2\ 5}.$

41. **= $^{-1\ 2}$( )^3( )$^{8\ 5}$.**

Solution in parametric form:

$$= \ a\ {}_1^{31}\ [\ (\tau^2 - 1)]^{-1\ 2}(\tau^3 - 3\tau + \ {}_2)^{5\ 4}(\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)\ \ \tau + \ {}_3,$$
$$= \ {}_1^{32}(\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)^2,$$

where $A = -15 \times 2^{-10}a^{2\ -5\ 2}\ \dfrac{a^2}{2}^{\ 3\ 5}.$

42. $=\quad ^{-7\ 3}(\quad)^{-7\ 3}.$

Solution in parametric form:

$$= a\ {}_1^{17}\quad (\tau^3 - 3\tau + \ {}_2)^{1\ 4}[\ (\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)]^{-3\ 2}\quad \tau + \ \ {}_3,$$
$$= \quad {}_1^{16}(\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)^{-1},$$

where $A = \ 72a^{-5\ \ 17\ \ 3}(a\ \ 4)^{-1\ \ 3}.$

43. $=\quad ^{-5\ 2}(\quad)^{10\ 7}.$

Solution in parametric form:

$$= \ a\ {}_1^{29}\quad (\tau^3 - 3\tau + \ \ {}_2)^{-3\ 2}(\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)^{-1\ 3}\quad \tau + \ \ {}_3,$$
$$= \quad {}_1^{2}(\tau^3 - 3\tau + \ \ {}_2)^{-1}(\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)^{2\ 3},$$

where $A = -\frac{28}{3}a^{-1\ \ 5\ \ 2}\ \ \frac{2a^2}{3}\ \ ^{3\ \ 7}.$

In the solutions of equations 44–47, the following notation is used:

$${}_6(\tau) = \ (\tau^6 - 15\tau^4 + 20\ \ {}_2\tau^3 - 45\tau^2 + 12\ \ {}_2\tau + 27 - 8\ \ {}_2^2).$$

44. $=\quad ^{-5\ 3}(\quad)^{-11\ 3}(\quad)^{3}.$

Solution in parametric form:

$$= a\ \ {}_1^{5}\quad (\tau^3 - 3\tau + \ \ {}_2)^{1\ 2}[\ (\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)]^{-3\ 2}\ \ {}_6(\tau)\ \ \tau + \ \ {}_3,$$
$$= \quad {}_1(\tau^3 - 3\tau + \ \ {}_2)^{3\ 2}(\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)^{-1},$$

where $A = \mp\frac{9}{16}\ \ ^{10\ \ 3}(2a)^{-2\ \ 3}.$

45. $=\quad (\quad)^{-5\ 2}(\quad)^{7\ 5}.$

Solution in parametric form:

$$= a\ \ {}_1^{11}\quad (\tau^3 - 3\tau + \ \ {}_2)^{-3\ 2}(\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)^{4\ 3}\quad \tau + \ \ {}_3,$$
$$= \quad {}_1^{27}(\tau^3 - 3\tau + \ \ {}_2)^{-1\ 2}\ \ {}_6(\tau),$$

where $A = \frac{405}{8}a^{-3}\quad \frac{}{2a}\ \ ^{1\ \ 2}\ \ \frac{a^2}{12}\ \ ^{2\ \ 5}.$

46. $=\quad ^{-7\ 4}(\quad)^{3}(\quad)^{8\ 5}.$

Solution in parametric form:

$$= \ a\ \ {}_1^{37}\quad (\tau^3 - 3\tau + \ \ {}_2)^{5\ 4}(\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)^{1\ 3}[\ \ {}_6(\tau)]^{-1\ 2}\ \ \tau + \ \ {}_3,$$
$$= \quad {}_1^{64}(\tau^4 - 6\tau^2 + 4\ \ {}_2\tau - 3)^{4\ 3},$$

where $A = -45 \times 2^{-13}a^{2\ \ -5\ \ 4}\ \ \frac{a^2}{12}\ \ ^{3\ \ 5}.$

47. $= \;\; ^{-1\ 2}(\quad)^{-4}(\quad)^{11\ 7}$**.**

Solution in parametric form:

$$= a \;\; _1^{55} \;\; (\tau^3 - 3\tau + \;\; _2)^{-3\ 2}(\tau^4 - 6\tau^2 + 4 \;\; _2\tau - 3)^{5\ 3} \;\; _6(\tau) \;\; \tau + \;\; _3,$$

$$= \;\; _1^{54}(\tau^3 - 3\tau + \;\; _2)^{-1}[\;\; _6(\tau)]^2,$$

where $A = \mp 28 \times 3^7 a^{-5\ \ 9\ 2} \;\; \frac{2a^2}{3} \;\; ^{4\ 7}$.

48. $= \;(\quad)^{-7\ 3}$**.**

Solution in parametric form:

$$= a \;\; _1^{5} \;\; (\tau^2 \quad 1)^{1\ 4} \;\; \tau + \;\; _3, \qquad = \;\; _1^{8}(\tau^3 \quad 3\tau + \;\; _2), \quad \text{where } A = \tfrac{81}{2} a^{-5\ 3}(3 \quad a)^{1\ 3}.$$

49. $= \;\; ^{-5\ 3} \;\; (\quad)^3$**.**

Solution in parametric form:

$$= a \;\; _1 \;\; \tau(\tau^2 \quad 1)^{1\ 2}(\tau^3 \quad 3\tau + \;\; _2)^{-1\ 2} \;\; \tau + \;\; _3, \qquad = \;\; _1^{3}(\tau^2 \quad 1)^{3\ 2},$$

where $A = \mp\tfrac{4}{243} a^4 \;\; ^{-4\ 3}$.

50. $= \quad (\quad)^{7\ 5}$**.**

Solution in parametric form:

$$= a \;\; _1^{3} \;\; (\tau^2 \quad 1)^{-3\ 2}(\tau^3 \quad 3\tau + \;\; _2)^{-1\ 2} \;\; \tau + \;\; _3, \qquad = \;\; _1\tau(\tau^2 \quad 1)^{-1\ 2},$$

where $A = \;\; 5 \;\; ^{-2} \;\; \frac{2a^2}{3} \;\; ^{2\ 5}$.

51. $= \;(\quad)^3(\quad)^{8\ 5}$**.**

Solution in parametric form:

$$= a \;\; _1^{9} \;\; \tau^{-1\ 2}(\tau^2 \quad 1)^{5\ 4} \;\; \tau + \;\; _3, \qquad = \;\; _1^{8}(\tau^3 \quad 3\tau + \;\; _2),$$

where $A = \mp\frac{4}{27} a^2 \;\; ^{-3} \;\; \frac{2a^2}{3} \;\; ^{3\ 5}$.

52. $= \;\; ^{-4\ 3}(\quad)^{-7\ 3}$**.**

Solution in parametric form:

$$= a \;\; _1^{7} \;\; (\tau^2 \quad 1)^{1\ 4}(\tau^3 \quad 3\tau + \;\; _2)^{-3\ 2} \;\; \tau + \;\; _3, \qquad = \;\; _1^{8}(\tau^3 \quad 3\tau + \;\; _2)^{-1},$$

where $A = \;\; \tfrac{81}{2} a^{-5\ \ 13\ 3}(3 \quad a)^{1\ 3}$.

53. $= \;\; ^{-5\ 3}(\quad)^{-5\ 3}(\quad)^3$**.**

Solution in parametric form:

$$= a \;\; _1^{-1} \;\; (\;\; \tau^2 + \;\; _2\tau - 1)(\tau^2 \quad 1)^{1\ 2}(\tau^3 \quad 3\tau + \;\; _2)^{-3\ 2} \;\; \tau + \;\; _3,$$

$$= \;\; _1(\tau^2 \quad 1)^{3\ 2}(\tau^3 \quad 3\tau + \;\; _2)^{-1},$$

where $A = \mp\tfrac{4}{27} a^2 \;\; ^{2\ 3}(3 \quad a)^{2\ 3}$.

54. $= \quad (\quad)^{-7}(\quad)^{7\ 5}$**.**

Solution in parametric form:

$$= a \ {}_1^{7} \quad (\tau^2 \quad 1)^{-3\ 2}(\tau^3 \quad 3\tau + \ {}_2)^{5\ 6} \quad \tau + \ {}_3, \qquad = \ {}_1^{9}(\ \tau^2 + \ {}_2\tau - 1)(\tau^2 \quad 1)^{-1\ 2},$$

where $A = \quad 5a^{-8\ 6}(2a^2 \quad)^{2\ 5}$.

55. $= \quad {}^{-4}(\quad)^{3}(\quad)^{8\ 5}$**.**

Solution in parametric form:

$$= a \ {}_1^{-1} \quad (\tau^2 \quad 1)^{5\ 4}(\ \tau^2 + \ {}_2\tau - 1)^{-1\ 2}(\tau^3 \quad 3\tau + \ {}_2)^{-2\ 3} \quad \tau + \ {}_3,$$
$$= \ {}_1^{8}(\tau^3 \quad 3\tau + \ {}_2)^{1\ 3},$$

where $A = \mp 5a^2 \ (2a^2 \quad)^{3\ 5}$.

56. $= \quad {}^{-5\ 2}(\quad)^{7\ 4}$**.**

Solution in parametric form:

$$= a \ {}_1^{-7} \quad (\tau^2 \quad 1)^{-3\ 2}(\tau^3 \quad 3\tau + \ {}_2)^{-1\ 3} \quad \tau + \ {}_3, \qquad = \ {}_1^{2}(\tau^2 \quad 1)^{-1}(\tau^3 \quad 3\tau + \ {}_2)^{2\ 3},$$

where $A = 4a^{-1\ 5\ 2} \ \mp\dfrac{a^2}{2} \ {}^{3\ 4}$.

57. $= \quad {}^{-1\ 2}(\quad)^{-4}(\quad)^{5\ 4}$**.**

Solution in parametric form:

$$= a \ {}_1^{17} \quad (\ \tau^2 + \ {}_2\tau - 1)(\tau^2 \quad 1)^{-3\ 2}(\tau^3 \quad 3\tau + \ {}_2)^{2\ 3} \quad \tau + \ {}_3,$$
$$= \ {}_1^{18}(\ \tau^2 + \ {}_2\tau - 1)^2(\tau^2 \quad 1)^{-1},$$

where $A = -64a^{-5\ 9\ 2} \ \mp\dfrac{a^2}{2} \ {}^{1\ 4}$.

In the solutions of equations 58–69, the following notation is used:

$$_1 = \ {}_1 e^{2\ \tau} + \ {}_2 e^{-\ \tau}\sin(\omega\tau), \qquad \omega = k \ \overline{3},$$
$$_2 = 2k \ {}_1 e^{2\ \tau} + k \ {}_2 e^{-\ \tau}\big[\ \overline{3}\cos(\omega\tau) - \sin(\omega\tau)\big],$$
$$_3 = 4k^2 \ {}_1 e^{2\ \tau} - 2k^2 \ {}_2 e^{-\ \tau}\big[\ \overline{3}\cos(\omega\tau) + \sin(\omega\tau)\big],$$
$$_4 = \ {}_2^2 - 2 \ {}_1 \ {}_3, \qquad {}_5 = 5 \ {}_2 \ {}_4 + 32k^3 \ {}_1^3.$$

58. $= \quad {}^{-1\ 2}(\quad)^{3}(\quad)^{3}$**.**

Solution in parametric form:

$$= a \ {}_1^{3} \quad {}_1^{-1\ 2} \ {}_2 \ {}_3 \quad \tau + \ {}_3, \qquad = \ {}_1^{4} \ {}_2^{2}, \qquad \text{where} \quad A = -a^{6\ -9\ 2}k^3.$$

59. $= \quad {}^{-1\ 2}(\quad)^{-6}(\quad)^{3}$**.**

Solution in parametric form:

$$= a \ {}_1^{3} \quad {}_1^{-3\ 2} \ {}_2 \ {}_4 \quad \tau + \ {}_3, \qquad = \ {}_1^{2} \ {}_1^{-1} \ {}_2^{2}, \qquad \text{where} \quad A = 16a^{-3\ 9\ 2}k^3.$$

60. $= \quad (\quad)^{-9\ 5}$.

Solution in parametric form:

$$= a\ {}^{3}_{1} \quad {}^{3\ 4}_{1}\ \tau + \ {}_{3}, \qquad = \ {}^{8}_{1}\ {}_{4}, \qquad \text{where} \quad A = -160a^{-4}\ k^{6}(16\ k^{3}\ a)^{4\ 5}.$$

61. $= \ {}^{-7\ 5}(\quad)^{3}(\quad)^{3}$.

Solution in parametric form:

$$= a\ {}^{3}_{1} \quad {}^{3\ 2}_{1}\ {}_{2}\ {}^{-1\ 2}_{4}\ \tau + \ {}_{3}, \qquad = \ {}^{5}_{1}\ {}^{5\ 2}_{1}, \qquad \text{where} \quad A = \tfrac{1}{4} \times 5^{-5} a^{6}\ {}^{-18\ 5} k^{-6}.$$

62. $= \quad (\quad)^{9\ 7}$.

Solution in parametric form:

$$= a\ {}^{-3}_{1} \quad {}^{-3\ 2}_{1}\ \tau + \ {}_{3}, \qquad = \ {}_{1}\ {}^{-1\ 2}_{1}\ {}_{2}, \qquad \text{where} \quad A = \frac{7}{2} a^{-1}\ {}^{-1} \quad \frac{a^{2}}{8\ k^{3}} \ {}^{2\ 7}.$$

63. $= \ {}^{-1\ 2}(\quad)^{3}(\quad)^{12\ 7}$.

Solution in parametric form:

$$= a\ {}^{15}_{1} \quad {}^{9\ 4}_{1}\ {}^{-1\ 2}_{2}\ {}_{4}\ \tau + \ {}_{3}, \qquad = \ {}^{16}_{1}\ {}^{2}_{4}, \qquad \text{where} \quad A = 7 \cdot 2^{-16} a^{2}\ {}^{-5\ 2} k^{-9} \quad \frac{a^{2}}{8\ k^{3}} \ {}^{5\ 7}.$$

64. $= \ {}^{-13\ 5}(\quad)^{-9\ 5}$.

Solution in parametric form:

$$= a\ {}^{9}_{1} \quad {}^{3\ 4}_{1}\ {}^{-3\ 2}_{4}\ \tau + \ {}_{3}, \qquad = \ {}^{8}_{1}\ {}^{-1}_{4}, \qquad \text{where} \quad A = -160a^{-4}\ {}^{23\ 5} k^{6}(16\ k^{3}\ a)^{4\ 5}.$$

65. $= \ {}^{-7\ 5}(\quad)^{-21\ 5}(\quad)^{3}$.

Solution in parametric form:

$$= a\ {}^{3}_{1} \quad {}^{3\ 2}_{1}\ {}^{-3\ 2}_{4}\ {}_{5}\ \tau + \ {}_{3}, \qquad = \ {}_{1}\ {}^{5\ 2}_{1}\ {}^{-1}_{4}, \qquad \text{where} \quad A = \frac{5}{512} a^{-1}\ {}^{17\ 5} k^{-6} \quad \frac{}{2a} \ {}^{1\ 5}.$$

66. $= \quad (\quad)^{-9\ 4}(\quad)^{9\ 7}$.

Solution in parametric form:

$$= a\ {}^{9}_{1} \quad {}^{-3\ 2}_{1}\ {}^{6\ 5}_{4}\ \tau + \ {}_{3}, \qquad = \ {}^{25}_{1}\ {}^{-1\ 2}_{1}\ {}_{5}, \qquad \text{where} \quad A = \frac{35}{8} a^{-3} \quad -\frac{5}{2a} \ {}^{1\ 4} \quad \frac{a^{2}}{32\ k^{3}} \ {}^{2\ 7}.$$

67. $= \ {}^{-13\ 8}(\quad)^{3}(\quad)^{12\ 7}$.

Solution in parametric form:

$$= a\ {}^{39}_{1} \quad {}^{9\ 4}_{1}\ {}^{3\ 5}_{4}\ {}^{-1\ 2}_{5}\ \tau + \ {}_{3}, \qquad = \ {}^{64}_{1}\ {}^{8\ 5}_{4},$$

where $A = 175 \times 2^{-22} a^{2}\ {}^{-11\ 8} k^{-9} \quad \frac{a^{2}}{32\ k^{3}} \ {}^{5\ 7}$.

68. $= \;\; ^{-7\;\;2}(\;\;)^{18\;\;13}$**.**

Solution in parametric form:

$$= a \;{}_{1}^{27} \;\; {}_{1}^{-3\;\;2} \;\; {}_{4}^{-3\;\;5} \;\; \tau + \;\; {}_{3}, \qquad = \;\; {}_{1}^{2} \;\; {}_{1}^{-1} \;\; {}_{4}^{2\;\;5}, \quad \text{where} \quad A = -\tfrac{208}{5} a^{-1\;\;7\;\;2} k^{3} (2a^{2} \;\;)^{5\;\;13}.$$

69. $= \;\; ^{-1\;\;2}(\;\;)^{-6}(\;\;)^{21\;\;13}$**.**

Solution in parametric form:

$$= a \;{}_{1}^{51} \;\; {}_{1}^{-3\;\;2} \;\; {}_{4}^{9\;\;5} \;\; {}_{5} \;\; \tau + \;\; {}_{3}, \qquad = \;\; {}_{1}^{50} \;\; {}_{1}^{-1} \;\; {}_{5}^{2},$$

where $A = 208 \times 5^{5} a^{-7\;\;13\;\;2} k^{3} (2a^{2} \;\;)^{8\;\;13}$.

In the solutions of equations 70–81, the following notation is used:

$${}_{1} = \cosh(\tau + \;\; {}_{2}) \cos\tau, \quad {}_{2} = \tanh(\tau + \;\; {}_{2}) + \tan\tau, \quad {}_{3} = \tanh(\tau + \;\; {}_{2}) - \tan\tau, \quad {}_{4} = 3 \;\; {}_{2} \;\; {}_{3} - 4,$$
$$\theta_1 = \cosh\tau - \sin(\tau + \;\; {}_{2}), \quad \theta_2 = \sinh\tau + \cos(\tau + \;\; {}_{2}), \quad \theta_3 = \sinh\tau - \cos(\tau + \;\; {}_{2}), \quad \theta_4 = 3\theta_2\theta_3 - 2\theta_1^2.$$

70. $= \;\; ^{2}(\;\;)^{-7\;\;3}$**.**

1 . Solution in parametric form:

$$= a \;\; {}_{1} \;\; {}_{1}^{1\;\;4} \;\; \tau + \;\; {}_{3}, \qquad = \;\; {}_{1}^{4} \;\; {}_{1} \;\; {}_{2}, \quad \text{where} \quad A = -3a^{-5} \;\; (2 \;\; a)^{1\;\;3}.$$

2 . Solution in parametric form:

$$= a \;\; {}_{1} \;\; \theta_1^{1\;\;4} \;\; \tau + \;\; {}_{3}, \qquad = \;\; {}_{1}^{4}\theta_2, \quad \text{where} \quad A = \tfrac{3}{8} a^{-5} \;\; (\;\; a)^{1\;\;3}.$$

71. $= \;\; ^{-5\;\;3}(\;\;)^{5}(\;\;)^{3}$**.**

1 . Solution in parametric form:

$$= a \;\; {}_{1}^{2} \;\; {}_{1} \;\; {}_{2}^{-1\;\;2} \;\; {}_{3} \;\; \tau + \;\; {}_{3}, \qquad = \;\; {}_{1}^{3} \;\; {}_{1}^{3\;\;2}, \quad \text{where} \quad A = 64 \times 3^{-7} a^{8\;\;-16\;\;3}.$$

2 . Solution in parametric form:

$$= a \;\; {}_{1}^{2} \;\; \theta_1^{1\;\;2} \theta_2^{-1\;\;2} \theta_3 \;\; \tau + \;\; {}_{3}, \qquad = \;\; {}_{1}^{3}\theta_1^{3\;\;2}, \quad \text{where} \quad A = -256 \times 3^{-7} a^{8\;\;-16\;\;3}.$$

72. $= \;\; (\;\;)^{-1\;\;3}(\;\;)^{7\;\;5}$**.**

1 . Solution in parametric form:

$$= a \;\; {}_{1}^{-2} \;\; {}_{1}^{-1} \;\; {}_{2}^{1\;\;2} \;\; \tau + \;\; {}_{3}, \qquad = \;\; {}_{1} \;\; {}_{1}^{1\;\;2} \;\; {}_{3}, \quad \text{where} \quad A = -\frac{5}{2a} \;\; \frac{}{2a} \;\; ^{1\;\;3} \;\; \frac{2a^2}{3} \;\; ^{2\;\;5}.$$

2 . Solution in parametric form:

$$= a \;\; {}_{1}^{-2} \;\; \theta_1^{-3\;\;2} \theta_2^{1\;\;2} \;\; \tau + \;\; {}_{3}, \qquad = \;\; {}_{1}\theta_1^{-1\;\;2}\theta_3, \quad \text{where} \quad A = \frac{5}{2a} \;\; \frac{}{2a} \;\; ^{1\;\;3} \;\; \frac{4a^2}{3} \;\; ^{2\;\;5}.$$

73. $= \;\; ^{-2\;\;3}(\;\;)^{3}(\;\;)^{8\;\;5}$**.**

1 . Solution in parametric form:

$$= a \;\; {}_{1}^{11} \;\; {}_{1}^{11\;\;4} \;\; {}_{2}^{2} \;\; {}_{3}^{-1\;\;2} \;\; \tau + \;\; {}_{3}, \qquad = \;\; {}_{1}^{12} \;\; {}_{1}^{3} \;\; {}_{2}^{3}, \quad \text{where} \quad A = \frac{5}{432} a^{2\;\;-7\;\;3} \;\; \frac{2a^2}{3} \;\; ^{3\;\;5}.$$

2 . Solution in parametric form:

$$= a \;{}_{1}^{11} \quad \theta_1^{5\ 4}\theta_2^2\theta_3^{-1\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1^{12}\theta_2^3, \qquad \text{where} \quad A = -\frac{5}{54}a^{2\ -7\ 3}\ \frac{4a^2}{3}^{\ 3\ 5}.$$

74. $= \quad {}^{-5\ 2}(\quad)^{1\ 2}$**.**

1 . Solution in parametric form:

$$= a \;{}_1^3 \quad {}_1^{-3\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1^2\ {}_1^{-1}, \qquad \text{where} \quad A = 2a^{-2\ 7\ 2}(2\quad)^{1\ 2}.$$

2 . Solution in parametric form:

$$= a \;{}_1^3 \quad \theta_1^{-3\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1^2\theta_1^{-1}, \qquad \text{where} \quad A = -a^{-2\ 7\ 2}(-2\quad)^{1\ 2}.$$

75. $= \quad {}^{-1\ 2}(\quad)^{-4}(\quad)^{5\ 2}$**.**

1 . Solution in parametric form:

$$= a \;{}_1^3 \quad {}_1^{3\ 2}\ {}_2^2\ {}_3 \quad \tau + \quad {}_3, \qquad = \quad {}_1^2\ {}_1\ {}_3^2, \qquad \text{where} \quad A = -32a^{-2\ 7\ 2}(\quad 2)^{-1\ 2}.$$

2 . Solution in parametric form:

$$= a \;{}_1^3 \quad \theta_1^{-3\ 2}\theta_2^2\theta_3 \quad \tau + \quad {}_3, \qquad = \quad {}_1^2\theta_1^{-1}\theta_3^2, \qquad \text{where} \quad A = 16a^{-2\ 7\ 2}(-\quad 2)^{-1\ 2}.$$

76. $= \quad {}^{-10\ 3}(\quad)^{-7\ 3}$**.**

1 . Solution in parametric form:

$$= a \;{}_1^5 \quad {}_1^{-5\ 4}\ {}_2^{-3\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1^4\ {}_1^{-1}\ {}_2^{-1}, \qquad \text{where} \quad A = -3a^{-5\ 20\ 3}(2\ a)^{1\ 3}.$$

2 . Solution in parametric form:

$$= a \;{}_1^5 \quad \theta_1^{1\ 4}\theta_2^{-3\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1^4\theta_2^{-1}, \qquad \text{where} \quad A = \tfrac{3}{8}a^{-16\ 3\ 20\ 3}.$$

77. $= \quad {}^{-5\ 3}(\quad)^{-17\ 3}(\quad)^{3}$**.**

1 . Solution in parametric form:

$$= a \;{}_1^2 \quad {}_1\ {}_2^{-3\ 2}\ {}_4 \quad \tau + \quad {}_3, \qquad = \quad {}_1\ {}_1^{1\ 2}\ {}_2^{-1}, \qquad \text{where} \quad A = \frac{3}{16}a^{-2\ 14\ 3}\ \frac{}{2a}^{\ 2\ 3}.$$

2 . Solution in parametric form:

$$= a \;{}_1^2 \quad \theta_1^{1\ 2}\theta_2^{-3\ 2}\theta_4 \quad \tau + \quad {}_3, \qquad = \quad {}_1\theta_1^{3\ 2}\theta_2^{-1}, \qquad \text{where} \quad A = -\frac{3}{4}a^{-2\ 14\ 3}\ \frac{}{2a}^{\ 2\ 3}.$$

78. $= \quad (\quad)^{-13\ 7}(\quad)^{7\ 5}$**.**

1 . Solution in parametric form:

$$= a \;{}_1^2 \quad {}_1^{1\ 3}\ {}_2^{11\ 6} \quad \tau + \quad {}_3, \qquad = \quad {}_1^9\ {}_1^{3\ 2}\ {}_4, \qquad \text{where} \quad A = -\frac{5}{4}a^{-2}\ \frac{3}{2a}^{\ 6\ 7}\ \frac{2a^2}{7}^{\ 2\ 5}.$$

2 . Solution in parametric form:

$$= a \;{}_1^2 \quad \theta_1^{-3\ 2}\theta_2^{11\ 6} \quad \tau + \quad {}_3, \qquad = \quad {}_1^9\theta_1^{-1\ 2}\theta_4, \qquad \text{where} \quad A = \frac{5}{4}a^{-2}\ \frac{3}{2a}^{\ 6\ 7}\ \frac{4a^2}{7}^{\ 2\ 5}.$$

79. $= \quad {}^{-10\ 7}(\quad)^{3}(\quad)^{8\ 5}$**.**

1 . Solution in parametric form:

$$= a \;{}_1^{19} \quad {}_1^{19\ 12}\ {}_2^{4\ 3}\ {}_4^{-1\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1^{28}\ {}_1^{7\ 3}\ {}_2^{7\ 3},$$

where $A = \frac{45}{16} \times 7^{-3}a^{2\ -11\ 7}\ \frac{2a^2}{7}^{\ 3\ 5}$.

2 . Solution in parametric form:

$$= a \;{}_1^{19} \quad \theta_1^{5\ 4}\theta_2^{4\ 3}\theta_4^{-1\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1^{28}\theta_2^{7\ 3},$$

where $A = -\frac{45}{2} \times 7^{-3}a^{2\ -11\ 7}\ \frac{4a^2}{7}^{\ 3\ 5}$.

80. $= \quad ^{-5/2}(\quad)^{13/10}$.

1 . Solution in parametric form:

$$= a \; _1^{-11} \quad _1^{-11/6} \; _2^{-1/3} \quad \tau + \quad _3, \qquad = \; _1^4 \; _1^{-1/3} \; _2^{2/3}, \qquad \text{where} \quad A = \tfrac{20}{3} a^{-1/5/2} (2a^2 \quad)^{3/10}.$$

2 . Solution in parametric form:

$$= a \; _1^{-11} \quad \theta_1^{-3/2} \theta_2^{-1/3} \quad \tau + \quad _3, \qquad = \; _1^4 \theta_1^{-1} \theta_2^{2/3}, \qquad \text{where} \quad A = \tfrac{10}{3} a^{-1/5/2} (-2a^2 \quad)^{3/10}.$$

81. $= \quad ^{-1/2}(\quad)^{-4}(\quad)^{17/10}$.

1 . Solution in parametric form:

$$= a \; _1^{19} \quad _1^{19/6} \; _2^{8/3} \; _4 \quad \tau + \quad _3, \qquad = \; _1^{18} \; _1^{3} \; _4^{2}, \qquad \text{where} \quad A = -540 a^{-5/9/2} (2a^2 \quad)^{7/10}.$$

2 . Solution in parametric form:

$$= a \; _1^{19} \quad \theta_1^{-3/2} \theta_2^{8/3} \theta_4 \quad \tau + \quad _3, \qquad = \; _1^{18} \theta_1^{-1} \theta_5^{2}, \qquad \text{where} \quad A = -270 a^{-5/9/2} (-2a^2 \quad)^{7/10}.$$

In the solutions of equations 82–84, the following notation is used:

$L_1 = \; _1\tau \; + \; _2\tau^{-}$, $\qquad N_1 = (1+k) \; _1\tau \; + (1-k) \; _2\tau^{-}$,

$L_2 = \; _1 \ln \tau + \; _2$, $\qquad N_2 = \; _1 \ln \tau + \; _1 + \; _2$,

$L_3 = \; _1 \sin(k \ln \tau) + \; _2 \cos(k \ln \tau)$, $\qquad N_3 = (\; _1 - k \; _2) \sin(k \ln \tau) + (\; _2 + k \; _1) \cos(k \ln \tau)$.

82. $= \quad ^{-2}(\quad)^3$.

Solution in parametric form:

$$= \quad \tau^{1/2} L^{-1/2} \quad \tau + \quad _3, \qquad = \tau^2, \qquad \text{where} \quad k = \sqrt{|1+8A|}, \qquad = \begin{cases} 1 & \text{if } A > -1/8, \\ 2 & \text{if } A = -1/8, \\ 3 & \text{if } A < -1/8. \end{cases}$$

83. $= \quad (\quad)^{-3}(\quad)^3$.

Solution in parametric form:

$$= \quad \tau^{-1} N \quad \tau + \quad _3, \qquad = \tau L \; , \qquad \text{where} \quad k = \sqrt{|A-1|}, \qquad = \begin{cases} 1 & \text{if } A < 1, \\ 2 & \text{if } A = 1, \\ 3 & \text{if } A > 1. \end{cases}$$

84. $= \quad ^{-1/2}(\quad)^{3/2}$.

Solution in parametric form:

$$= \mp 4 \quad \tau^2 L_1 \quad \tau + \quad _3, \qquad = \tau^2 L_1^2, \qquad \text{where} \quad k = \sqrt{1 + 8A^{-2}}.$$

In the solutions of equations 85–100, the following notation is used:

$$Z = \begin{cases} _1 \quad (\tau) + \quad _2 \quad (\tau) & \textit{for the upper sign,} \\ _1 \quad (\tau) + \quad _2 \quad (\tau) & \textit{for the lower sign,} \end{cases}$$

$$_1 = \tau Z'_\tau + \quad Z, \qquad _2 = \; _1^2 \quad \tau^2 Z^2, \qquad _3 = \tfrac{2}{3} \tau^2 Z^3 - 2 \; _1 \; _2,$$

where (τ) *and* (τ) *are the Bessel functions, and* (τ) *and* (τ) *are the modified Bessel functions.*

85. $= \quad (\quad)^3, \qquad \neq -2$.

Solution in parametric form:

$$= \; _1 \quad \tau^{\frac{3-2}{2}} Z^{-1/2} \quad \tau + \quad _3, \qquad = \tau^2 \; , \qquad \text{where} \quad = \frac{1}{\beta + 2}, \quad A = \mp \frac{1}{8 \; ^2} \; ^{-} \; ^{-2}.$$

86. $= \quad (\quad) \; (\quad)^3, \qquad \gamma \neq -3.$

Solution in parametric form:

$$= a \;{}_1 \quad \tau^{-1} \;{}_1 \;\; \tau + \;\; {}_3, \qquad = \;{}_1\tau \;\; Z, \qquad \text{where} \qquad = \frac{2}{+3}, \;\; A = \;\; \frac{1}{2}a^{\;+3\; - \;-3}.$$

87. $= \quad ^{-1\;\;2}(\qquad) \;, \qquad \neq 3 \;\; 2.$

Solution in parametric form:

$$= a \;{}_1 \quad \tau^3 \;{}^{-1} Z \;\; \tau + \;\; {}_3, \qquad = \;{}_1^2\tau^2 \;\; Z^2, \qquad \text{where} \qquad = \frac{1-}{3-2}, \;\; A = \mp\frac{4a^{-3\;\;3\;\;2}}{3-2} \;\; \mp\frac{a^2}{2} \qquad .$$

88. $= \quad ^{-3} \;(\qquad)^3.$

Solution in parametric form:

$$= a \;{}_1 \quad Z \;\; \tau + \;\; {}_3, \qquad = \;{}_1^2\tau^{-1\;\;3} \;\; \tau Z^2 - \;{}_1 \quad Z \;\; \tau - \;\; {}_3 \;\; {}_1 \;\; , \qquad \text{where} \qquad = \tfrac{1}{3}, \;\; A = \tfrac{9}{4}a^{6\;\;-3}.$$

89. $= \quad ^{-1\;\;2}(\qquad)^3.$

Solution in parametric form:

$$= a \;{}_1 \quad \tau Z \;\; \tau + \;\; {}_3, \qquad = \;{}_1^2\tau^{4\;\;3} Z^2, \qquad \text{where} \qquad = \tfrac{2}{3}, \;\; A = -\tfrac{1}{6}a^{3\;\;-3\;\;2}.$$

90. $= \quad ^{-1\;\;2}(\quad)^{-3}(\qquad)^3.$

Solution in parametric form:

$$= \;{}_1 \quad \tau^{-2} Z^{-2} \;\; {}_1 \;\; {}_2 \;\; \tau + \;\; {}_3, \qquad = \;\tau^{-4\;\;3} Z^{-2} \;\; {}_1^2, \qquad \text{where} \qquad = \tfrac{1}{3}, \;\; A = \;\; \tfrac{4}{3} \;\; {}^{3\;\;2}.$$

91. $= \quad (\quad)^{-3}.$

Solution in parametric form:

$$= a \;{}_1 \quad Z \;\; \tau + \;\; {}_3, \qquad = \;{}_1^2\tau^{-2\;\;3} \;\; {}_2, \qquad \text{where} \qquad = \tfrac{1}{3}, \;\; A = -\tfrac{16}{81}a^{-6\;\;3}.$$

92. $= \quad ^{-2}(\quad)^3(\qquad)^3.$

Solution in parametric form:

$$= a \;{}_1 \quad Z \;\; {}_1 \;\; {}_2^{-1\;\;2} \;\; \tau + \;\; {}_3, \qquad = \;{}_1^2\tau^{2\;\;3} Z^2, \qquad \text{where} \qquad = \tfrac{1}{3}, \;\; A = \tfrac{9}{32}a^{6\;\;-3}.$$

93. $= \quad (\qquad)^{3\;\;2}.$

Solution in parametric form:

$$= \;{}_1 \quad \tau^{-1} Z^{-2} \;\; \tau + \;\; {}_3, \qquad = \;\tau^{-2\;\;3} Z^{-1} \;\; {}_1, \qquad \text{where} \qquad = \tfrac{1}{3}, \;\; A = 2 \;\; {}^{-1}(\mp 6 \quad)^{1\;\;2}.$$

94. $= \quad ^{-1\;\;2}(\quad)^3(\qquad)^{3\;\;2}.$

Solution in parametric form:

$$= a \;{}_1 \quad Z^{5\;\;2} \;\; {}_1^{-1\;\;2} \;\; {}_2 \;\; \tau + \;\; {}_3, \qquad = \;{}_1\tau^{-4\;\;3} \;\; {}_2^2,$$

where $= \tfrac{1}{3}, \;\; A = \mp\tfrac{27}{32}a^{3\;\;-5\;\;2}(\mp 6 \quad)^{1\;\;2}.$

95. $= \quad ^{-2}(\quad)^{-3}.$

Solution in parametric form:

$$= a \;{}_1 \quad \tau Z \;\; {}_2^{-3\;\;2} \;\; \tau + \;\; {}_3, \qquad = \;{}_1\tau^{2\;\;3} \;\; {}_2^{-1}, \qquad \text{where} \qquad = \tfrac{1}{3}, \;\; A = -\tfrac{16}{81}a^{-6\;\;6}.$$

96. $\quad = \quad {}^{-2}(\quad)^{-3}(\quad)^{3}.$

Solution in parametric form:

$$= \quad {}_1 \quad Z \quad {}_2^{-3\ 2} \quad {}_3 \quad \tau + \quad {}_3, \qquad = \tau^{4\ 3} Z^2 \quad {}_2^{-1}, \qquad \text{where} \qquad = \tfrac{1}{3}, \ \ A = 18 \quad {}^{-3}.$$

97. $\quad = \quad (\quad)^{-3}(\quad)^{3\ 2}.$

Solution in parametric form:

$$= a \quad {}_1 \quad \tau^{-2} Z^{-2} \quad {}_2^{3\ 2} \quad \tau + \quad {}_3, \qquad = \ {}_1^2 \tau^{-4\ 3} Z^{-1} \quad {}_3, \qquad \text{where} \qquad = \tfrac{1}{3}, \ \ A = -8a^{-3\ 2}(\ 6\ )^{1\ 2}.$$

98. $\quad = \quad {}^{-2}(\quad)^{3}(\quad)^{3\ 2}.$

Solution in parametric form:

$$= a \quad {}_1 \quad \tau Z^{5\ 2} \quad {}_3^{-1\ 2} \quad \tau + \quad {}_3, \qquad = \ {}_1^2 \tau^{-2\ 3} \quad {}_2, \qquad \text{where} \qquad = \tfrac{1}{3}, \ \ A = \ \tfrac{27}{8} a^{3\ -1}(\ 6\ )^{1\ 2}.$$

99. $\quad = \quad {}^{-2}(\quad)^{3\ 2}.$

Solution in parametric form:

$$= \quad {}_1 \quad \tau^{-1} Z^{-2} \quad \tau + \quad {}_3, \qquad = \tau^{-4\ 3} Z^{-2} \quad {}_2, \qquad \text{where} \qquad = \tfrac{1}{3}, \ \ A = \ \tfrac{4}{3} \ {}^2(2\ )^{-1\ 2}.$$

100. $\quad = \quad {}^{-1\ 2}(\quad)^{-3}(\quad)^{3\ 2}.$

Solution in parametric form:

$$= a \quad {}_1 \quad \tau^{-3} Z^{-2} \quad {}_2^{3\ 2} \quad {}_3 \quad \tau + \quad {}_3, \qquad = \quad {}_1 \tau^{-8\ 3} Z^{-2} \quad {}_3^2,$$

where $\quad = \tfrac{1}{3}, \ \ A = \mp \tfrac{256}{3} a^{-3\ \ 7\ \ 2}(2\ )^{-1\ 2}.$

In the solutions of equations 101–138, the following notation is used:

$$\tau = \quad \frac{\wp}{\overline{\quad (4\wp^3 - 1)}} - \quad {}_2, \qquad = \quad \overline{(4\wp^3 - 1)}.$$

The function $\wp = \wp(\tau)$ is defined implicitly by the above elliptic integral of the first kind. For the upper sign, the function $\wp$ coincides with the classical elliptic Weierstrass function $\wp = \wp(\tau + \quad {}_2, 0, 1)$. In the solution given below, we can take $\wp$ to be the parameter instead of τ and use the explicit dependence $\tau = \tau(\wp)$.

101. $\quad = \quad (\quad)^{5}.$

Solution in parametric form:

$$= a \quad {}_1^2 \quad \wp^{-1\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1 \tau, \qquad \text{where} \quad A = \quad 3a^{2\ -4}.$$

102. $\quad = \quad {}^{2} \quad (\quad)^{3}.$

Solution in parametric form:

$$= a \quad {}_1^5 \quad \tau^{-1\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1^4 \wp, \qquad \text{where} \quad A = \mp 24 a^{4\ -5}.$$

103. $\quad = \quad (\quad)^{5\ 2}.$

Solution in parametric form:

$$= a \quad {}_1^7 \quad \tau^{-1\ 2} \wp^2 \quad \tau + \quad {}_3, \qquad = \quad {}_1^6 \quad , \qquad \text{where} \quad A = -\tfrac{1}{9} a^{3\ -3}(\ 3\ )^{-1\ 2}.$$

104. $= \quad (\quad)^{3}(\quad)^{1 \; 2}.$

Solution in parametric form:

$$= a \; {}^{5}_{1} \quad {}^{-1 \; 2} \quad \tau + \quad {}_{3}, \qquad = \; {}^{2}_{1}\tau, \qquad \text{where} \quad A = \; 6a^{\;-2}(\; 3 \;)^{-1 \; 2}.$$

105. $= \quad {}^{-5}(\quad)^{5}.$

Solution in parametric form:

$$= a \; {}^{-1}_{1} \quad \tau^{-3 \; 2}\wp^{-1 \; 2} \quad \tau + \quad {}_{3}, \qquad = \; {}^{2}_{1}\tau^{-1}, \qquad \text{where} \quad A = \; 3a^{2} \; .$$

106. $= \quad {}^{2}(\quad)^{-9}(\quad)^{3}.$

Solution in parametric form:

$$= a \; {}^{5}_{1} \quad \tau^{-3 \; 2}(\tau \quad - \wp) \quad \tau + \quad {}_{3}, \qquad = \; {}^{6}_{1}\tau^{-1}\wp, \qquad \text{where} \quad A = \mp 24a^{-6 \; 5}.$$

107. $= \quad (\quad)^{-3 \; 2}(\quad)^{5 \; 2}.$

Solution in parametric form:

$$= a \; {}^{2}_{1} \quad \tau^{-1}\wp^{2} \quad \tau + \quad {}_{3}, \qquad = \quad {}_{1}(\tau \quad - \wp), \qquad \text{where} \quad A = \mp \tfrac{1}{2}a \; (\; 2 \; a)^{1 \; 2}.$$

108. $= \quad {}^{-5 \; 4}(\quad)^{3}(\quad)^{1 \; 2}.$

Solution in parametric form:

$$= a \; {}^{5}_{1} \quad \tau^{3}(\tau \quad - \wp)^{-1 \; 2} \quad \tau + \quad {}_{3}, \qquad = \; {}^{4}_{1}\tau^{4}, \qquad \text{where} \quad A = \; \tfrac{3}{16}a^{\;-3 \; 4}(\; 3 \;)^{-1 \; 2}.$$

109. $= \quad {}^{-2 \; 3}(\quad)^{6 \; 5}.$

Solution in parametric form:

$$= a \; {}^{7}_{1} \quad \tau^{-4}\wp^{2} \quad \tau + \quad {}_{3}, \qquad = \; {}^{9}_{1}\tau^{-3}\wp^{3}, \qquad \text{where} \quad A = \; 5a^{-1 \;\; -2 \; 3} \quad \frac{a^{2}}{18}^{\;\;1 \; 5}.$$

110. $= \quad {}^{-1 \; 2}(\quad)^{-1 \; 3}(\quad)^{9 \; 5}.$

Solution in parametric form:

$$= a \; {}^{-1}_{1} \quad \tau^{5 \; 2}\wp^{1 \; 2}(\tau \quad - \wp) \quad \tau + \quad {}_{3}, \qquad = \; {}^{8}_{1}(\tau \quad - \wp)^{2},$$

where $A = \mp 5a^{-1 \;\; 1 \; 2} \quad \frac{12}{a}^{\;\;1 \; 3} \quad \frac{a^{2}}{18}^{\;\;4 \; 5}.$

111. $= \quad {}^{-15 \; 7}(\quad)^{5}.$

Solution in parametric form:

$$= a \; {}^{13}_{1} \quad \tau^{11 \; 2}(\tau^{2}\wp \mp 1)^{-1 \; 2} \quad \tau + \quad {}_{3}, \qquad = \; {}^{14}_{1}\tau^{7}, \qquad \text{where} \quad A = \; 3 \times 7^{-4}a^{2 \;\; -13 \; 7}.$$

112. $= \quad {}^{2}(\quad)^{-23 \; 7}(\quad)^{3}.$

Solution in parametric form:

$$= a \; {}^{-5}_{1} \quad \tau^{-7 \; 2}(\tau^{3} \quad + 3\tau^{2}\wp \mp 1) \quad \tau + \quad {}_{3}, \qquad = \; {}^{2}_{1}\tau(\tau^{2}\wp \mp 1), \qquad \text{where} \quad A = \mp \tfrac{24}{49}a^{-2 \;\; 7 \;\; -5 \;\; 7}.$$

113. $= \quad (\quad)^{-11 \; 4}(\quad)^{5 \; 2}.$

Solution in parametric form:

$$= a \quad {}_{1} \quad \tau^{-3}(\tau^{2}\wp \mp 1)^{2} \quad \tau + \quad {}_{3}, \qquad = \; {}^{3}_{1}\tau^{-6}(\tau^{3} \quad + 3\tau^{2}\wp \mp 1),$$

where $A = \; \tfrac{3}{2}(\; 6 \quad a)^{3 \; 4}(\mp 6 \;)^{-1 \; 2}.$

114. $=\ {}^{-15\ 8}(\quad)^3(\quad)^{1\ 2}$**.**

Solution in parametric form:

$$=a\ {}^{5}_{1}\quad \tau^{-6}(\tau^3\ \ +3\tau^2\wp\mp1)^{-1\ 2}\quad \tau+\ \ {}_3, \qquad = \ {}^{8}_{1}\tau^{-8}, \qquad \text{where}\quad A=\ \tfrac{3}{64}a\ {}^{-1\ 8}(\mp6\ )^{-1\ 2}.$$

115. $=\ {}^{-2\ 3}(\quad)^{22\ 15}$**.**

Solution in parametric form:

$$=a\ {}^{3}_{1}\quad \tau^{8}(\tau^2\wp\mp1)^2\quad \tau+\ \ {}_3, \qquad = \ {}_1\tau^3(\tau^2\wp\mp1)^3, \qquad \text{where}\quad A=\mp15a^{-1\ 15\ 2\ 3}(18\ )^{-7\ 15}.$$

116. $=\ {}^{-1\ 2}(\quad)^{-1\ 3}(\quad)^{23\ 15}$**.**

Solution in parametric form:

$$=a\ {}^{9}_{1}\quad \tau^{-29\ 2}(\tau^3\ \ +3\tau^2\wp\mp1)(\tau^2\wp\mp1)^{1\ 2}\quad \tau+\ \ {}_3, \qquad = \ {}^{8}_{1}\tau^{-12}(\tau^3\ \ +3\tau^2\wp\mp1)^2,$$

where $A=\ \ 15a^{-1\ \ 1\ 2}\ \dfrac{12}{a}^{\ 1\ 3}\ \dfrac{a^2}{18}^{\ 8\ 15}.$

117. $=\ {}^{-20\ 7}(\quad)^5$**.**

Solution in parametric form:

$$=a\ {}^{4}_{1}\quad \tau^{-5}(\tau^2\wp\mp1)^{-1\ 2}\quad \tau+\ \ {}_3, \qquad = \ {}^{7}_{1}\tau^{-7}, \qquad \text{where}\quad A=\ \ 3\times7^{-4}a^{2\ \ -8\ 7}.$$

118. $=\ {}^{2}(\quad)^{-33\ 7}(\quad)^3$**.**

Solution in parametric form:

$$=a\ {}^{5}_{1}\quad \tau^{-7\ 2}(\tau^3\ \ -4\tau^2\wp\ \ 6)\ \ \tau+\ \ {}_3, \qquad = \ {}^{12}_{1}\tau^{-6}(\tau^2\wp\mp1), \qquad \text{where}\quad A=\mp\tfrac{24}{49}a^{-12\ 7\ \ 5\ 7}.$$

119. $=\ \ (\quad)^{-27\ 13}(\quad)^{5\ 2}$**.**

Solution in parametric form:

$$=a\ {}^{-11}_{1}\quad \tau^{-13\ 2}(\tau^2\wp\mp1)^2\quad \tau+\ \ {}_3, \qquad = \ {}^{2}_{1}\tau(\tau^3\ \ -4\tau^2\wp\ \ \ 6),$$

where $A=-\tfrac{24}{13}(6\ \ \ a)^{1\ 13}(\ \ 39\ )^{-1\ 2}.$

120. $=\ {}^{-20\ 13}(\quad)^3(\quad)^{1\ 2}$**.**

Solution in parametric form:

$$=a\ {}^{25}_{1}\quad \tau^{23\ 2}(\tau^3\ \ -4\tau^2\wp\ \ \ 6)^{-1\ 2}\quad \tau+\ \ {}_3, \qquad = \ {}^{26}_{1}\tau^{13},$$

where $A=\ \ \tfrac{6}{169}a\ {}^{-6\ 13}(\ \ 39\ )^{-1\ 2}.$

121. $=\ {}^{-2\ 3}(\quad)^{27\ 20}$**.**

Solution in parametric form:

$$=a\ {}^{19}_{1}\quad \tau^{-20}(\tau^2\wp\mp1)^2\quad \tau+\ \ {}_3, \qquad = \ {}^{18}_{1}\tau^{-18}(\tau^2\wp\mp1)^3,$$

where $A=20a^{-1\ \ 2\ 3}\ \ \dfrac{a^2}{18}^{\ 7\ 20}.$

122. $= \ ^{-1\ 2}(\quad)^{-1\ 3}(\quad)^{33\ 20}$**.**

Solution in parametric form:

$$= a \ {}_1^{11} \quad \tau^{10}(\tau^3 \quad -4\tau^2\wp \quad 6)(\tau^2\wp \mp 1)^{1\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1^2\tau^2(\tau^3 \quad -4\tau^2\wp \quad 6)^2,$$

where $A = \mp 20a^{-1\ \ 1\ 2} \ \dfrac{12}{a}^{\ 1\ 3} \ \dfrac{a^2}{18}^{\ 13\ 20}.$

123. $= \ (\quad)^{-4}$**.**

Solution in parametric form:

$$= a \ {}_1^5 \quad \wp^{-2} \quad \tau + \quad {}_3, \qquad = \quad {}_1^7\wp^{-2}(\quad 2\tau\wp^2), \qquad \text{where} \quad A = -192a^{-7\ \ 5}.$$

124. $= \ ^{-5\ 2} \quad (\quad)^3$**.**

Solution in parametric form:

$$= a \ {}_1 \quad \wp^{-2}(\quad 2\tau\wp^2)^{-1\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1^8\wp^{-2}, \qquad \text{where} \quad A = \ \tfrac{3}{4}a^{4\ \ -1\ 2}.$$

125. $= \quad (\quad)^{8\ 5}$**.**

Solution in parametric form:

$$= a \ {}_1^{13} \quad \wp^3(\quad 2\tau\wp^2)^{-1\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1^6 \ , \qquad \text{where} \quad A = \mp\frac{5}{6}^{\ -2} \ \frac{a^2}{6}^{\ 3\ 5}.$$

126. $= \ (\quad)^3(\quad)^{7\ 5}$**.**

Solution in parametric form:

$$= a \ {}_1^{17} \quad \wp^{-3\ \ -1\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1^{14}\wp^{-2}(\quad 2\tau\wp^2), \qquad \text{where} \quad A = \mp\frac{5}{8}a^{2\ \ -3} \ \frac{a^2}{6}^{\ 2\ 5}.$$

127. $= \ ^{-1\ 2}(\quad)^{-4}$**.**

Solution in parametric form:

$$= a \ {}_1^{11} \quad \frac{\wp \quad \tau}{(\quad 2\tau\wp^2)^{3\ 2}} + \quad {}_3, \qquad = \quad {}_1^{14}\frac{\wp^2}{\quad 2\tau\wp^2}, \qquad \text{where} \quad A = 192a^{-7\ \ 11\ 2}.$$

128. $= \ ^{-5\ 2}(\quad)^3$**.**

Solution in parametric form:

$$= a \ {}_1^{-1} \quad (\quad 2\tau\wp^2)^{-3\ 2}(\tau \quad +2\wp) \quad \tau + \quad {}_3, \qquad = \quad {}_1^6(\quad 2\tau\wp^2)^{-1}, \qquad \text{where} \ A = -\tfrac{3}{16}a^{3\ \ 1\ 2}.$$

129. $= \ (\quad)^3(\quad)^{8\ 5}$**.**

Solution in parametric form:

$$= a \ {}_1^{23} \quad \wp^{-1\ 2}(\quad 2\tau\wp^2)^{5\ 4} \quad \tau + \quad {}_3, \qquad = \quad {}_1^{16}(\tau \quad +2\wp), \qquad \text{where} \ A = \frac{10}{27}a^{2\ \ -4} \ \frac{2a^2}{3}^{\ 3\ 5}.$$

130. $= \ (\quad)^3(\quad)^{7\ 5}$**.**

Solution in parametric form:

$$= a \ {}_1^{11} \quad (\quad 2\tau\wp^2)^{-3\ 2}(\tau \quad +2\wp)^{-1\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1^7\wp(\quad 2\tau\wp^2)^{-1\ 2},$$

where $A = -10a^{2\ \ -4} \ \dfrac{2a^2}{3}^{\ 2\ 5}.$

131. $= \quad {}^{-1\ 2}(\quad)^{-7\ 3}$.

Solution in parametric form:

$$= a \ {}_{1}^{23} \quad (\quad 2\tau\wp^2)^{1\ 4}(\tau \quad + 2\wp) \ \ \tau + \quad {}_3, \qquad = \ {}_1^{32}(\tau \quad + 2\wp)^2,$$

where $A = -648a^{-5\ \ 7\ \ 2}(6 \quad a)^{1\ \ 3}$.

132. $= \quad {}^{-5\ 3}(\quad)^{3}$.

Solution in parametric form:

$$= a \ {}_1 \quad \wp(\quad 2\tau\wp^2)^{1\ \ 2} \ \ \tau + \quad {}_3, \qquad = \ {}_1^{9}(\quad 2\tau\wp^2)^{3\ \ 2}, \qquad \text{where} \quad A = \tfrac{1}{324}a^{3\ \ -1\ \ 3}.$$

133. $= \quad {}^{-5\ 6}(\quad)^{-7\ 3}$.

Solution in parametric form:

$$= a \ {}_1^{25} \quad (\quad 2\tau\wp^2)^{1\ \ 4}(\tau \quad + 2\wp)^{-2} \ \ \tau + \quad {}_3, \qquad = \ {}_1^{32}(\tau \quad + 2\wp)^{-2},$$

where $A = -648a^{-5\ \ 23\ \ 6}(6 \quad a)^{1\ \ 3}$.

134. $= \quad {}^{-5\ 3}(\quad)^{-2\ 3}(\quad)^{3}$.

Solution in parametric form:

$$= a \ {}_1^{-1} \quad (\tau^2\wp \mp 1)(\quad 2\tau\wp^2)^{1\ \ 2}(\tau \quad + 2\wp)^{-2} \ \ \tau + \quad {}_3, \qquad = \ {}_1^{7}(\quad 2\tau\wp^2)^{3\ \ 2}(\tau \quad + 2\wp)^{-2},$$

where $A = -\frac{1}{324}a^{3\ \ -1\ \ 3}(6 \quad a)^{2\ \ 3}$.

135. $= \quad (\quad)^{11}(\quad)^{7\ 5}$.

Solution in parametric form:

$$= a \ {}_1^{31} \quad (\quad 2\tau\wp^2)^{-3\ \ 2}(\tau \quad + 2\wp)^{13\ \ 6} \ \ \tau + \quad {}_3, \qquad = \ {}_1^{27}(\tau^2\wp \mp 1)(\quad 2\tau\wp^2)^{-1\ \ 2},$$

where $A = -20a^{10\ \ -12}(2a^2 \quad)^{2\ \ 5}$.

136. $= \quad {}^{5}(\quad)^{3}(\quad)^{8\ 5}$.

Solution in parametric form:

$$= a \ {}_1^{43} \quad (\tau^2\wp \mp 1)^{-1\ \ 2}(\quad 2\tau\wp^2)^{5\ \ 4}(\tau \quad + 2\wp)^{-4\ \ 3} \ \ \tau + \quad {}_3, \qquad = \ {}_1^{16}(\tau \quad + 2\wp)^{-1\ \ 3},$$

where $A = 20a^{2\ \ -8}(2a^2 \quad)^{3\ \ 5}$.

137. $= \quad {}^{-1\ 2}(\quad)^{-4}(\quad)^{4\ 5}$.

Solution in parametric form:

$$= a \ {}_1^{47} \quad (\tau^2\wp \mp 1)(\quad 2\tau\wp^2)^{-3\ \ 2}(\tau \quad + 2\wp)^{4\ \ 3} \ \ \tau + \quad {}_3, \qquad = \ {}_1^{54}(\tau^2\wp \mp 1)^2(\quad 2\tau\wp^2)^{-1},$$

where $A = -320a^{-7\ \ 11\ \ 2} \ \frac{a^2}{4}^{\ 4\ \ 5}$.

138. $= \quad {}^{-5\ 2}(\quad)^{11\ 5}$**.**

Solution in parametric form:

$$= a \quad {}_1^{-13} \quad (\quad 2\tau\wp^2)^{-3\ 2}(\tau \quad +2\wp)^{1\ 3} \quad \tau + \quad {}_3, \qquad = \quad {}_1^{14}(\quad 2\tau\wp^2)^{-1}(\tau \quad +2\wp)^{4\ 3},$$

where $A = \frac{5}{4}a \quad {}^{3\ 2} \quad \frac{a^2}{4} \quad {}^{1\ 5}$.

In the solutions of equations 139 and 140, the following notation is used:

$$= \frac{\tau \ {}^{-1} \quad \tau}{z(\tau)}, \qquad z = \begin{cases} \frac{1}{k+1}\tau \quad {}^{+1} + \frac{1}{k}\tau \quad + \quad {}_2 & \text{if } k \neq 0,\ k \neq -1; \\ \tau + \ln|\tau| + \quad {}_2 & \text{if } k = 0; \\ \ln|\tau| - \frac{1}{\tau} + \quad {}_2 & \text{if } k = -1. \end{cases}$$

139. $= \quad (\quad)^{-1}(\quad)^2, \qquad \neq 0$**.**

Solution in parametric form:

$$= \quad {}_1 \quad \tau^{\overline{1-}} \quad \exp(-\tfrac{1}{2} \quad) \quad \tau + \quad {}_3, \qquad = \tau^1 \quad, \qquad \text{where} \quad k = 1 \quad \beta, \quad A = -2 \ {}^{-} \quad .$$

140. $= \quad {}^{-1}(\quad) \qquad , \qquad \gamma \neq 1$**.**

Solution in parametric form:

$$= a \quad {}_1 \quad \tau^{\frac{2-}{-1}} z^{-1} e \quad \tau + \quad {}_3, \qquad = e \quad , \qquad \text{where} \quad k = \frac{2}{\quad -1}, \quad A = a \quad {}^{-1\ \ 1-} \quad .$$

In the solutions of equations 141–170, the following notation is used:

$$R = \overline{\quad (4\tau^3 - 1)}, \qquad {}_1 = 2\tau \quad \mp R, \qquad {}_2 = \tau^{-1}(R \quad {}_1 - 1),$$
$$ {}_3 = 4\tau \quad {}_1^2 \mp \quad {}_2^2, \qquad {}_4 = \quad {}_2 \quad {}_3 - 8 \quad {}_1^2, \qquad {}_5 = 2R \quad - \tau^2,$$

where $= \frac{\tau \quad \tau}{R} + \quad {}_2$ *is the incomplete elliptic integral of the second kind in the form of Weierstrass.*

141. $= \quad (\quad)^{-7}$**.**

Solution in parametric form:

$$= a \quad {}_1^4 \quad \tau^{-3\ 2} R^{-1} \quad \tau + \quad {}_3, \qquad = \quad {}_1^5 \tau^{-1} \quad {}_1, \qquad \text{where} \quad A = \mp 3a^{-10\ \ 8}.$$

142. $= \quad {}^{-4} \quad (\quad)^3$**.**

Solution in parametric form:

$$= a \quad {}_1^{-1} \quad \tau^{-3\ 2} \quad {}_1^{-1\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1^4 \tau^{-1}, \qquad \text{where} \quad A = \quad 24a^4 \quad .$$

143. $= \qquad (\quad)^{7\ 4}$**.**

Solution in parametric form:

$$= a \quad {}_1^{11} \quad \tau^{5\ 2} \quad {}_1^{-1\ 2} R^{-1} \quad \tau + \quad {}_3, \qquad = \quad {}_1^6 R, \qquad \text{where} \quad A = \mp\frac{2}{3} \quad {}^{-2} \quad \mp\frac{a^2}{3} \quad {}^{3\ 4}.$$

144. $= \ (\quad)^3(\quad)^{5\ \ 4}$.

Solution in parametric form:

$$= a\ {}_1^{13}\quad \tau^{-2}R^{-3\ \ 2}\quad \tau +\quad {}_3, \qquad = \ {}_1^{10}\tau^{-1}\ {}_1, \qquad \text{where}\quad A = -4a^2\ {}^{-3}\ \mp\frac{a^2}{3}\ {}^{1\ \ 4}.$$

145. $= \quad (\quad)^{-7}$.

Solution in parametric form:

$$= a\ {}_1^{7}\quad {}_1^{-3\ \ 2}R^{-1}\quad \tau +\quad {}_3, \qquad = \ {}_1^{10}\tau\ {}_1^{-1}, \qquad \text{where}\quad A = \mp 3a^{-10\ \ 7}.$$

146. $= \quad {}^{-4}(\quad)^3(\quad)^3$.

Solution in parametric form:

$$= a\ {}_1\quad \tau^{-1\ \ 2}\ {}_1^{-3\ \ 2}\ {}_2R^{-1}\quad \tau +\quad {}_3, \qquad = \ {}_1^{6}\ {}_1^{-1}, \qquad \text{where}\quad A = \ \ 24a^{6\ \ -1}.$$

147. $= \quad (\quad)^{7\ \ 4}$.

Solution in parametric form:

$$= a\ {}_1^{7}\quad {}_1^{2}R^{-1}\quad \tau +\quad {}_3, \qquad = \ {}_1^{2}\ {}_2, \qquad \text{where}\quad A = -4a^{-1}\ {}^{-1}\quad \frac{a^2}{6}\ {}^{3\ \ 4}.$$

148. $= \quad {}^{-1\ \ 2}(\quad)^3(\quad)^{5\ \ 4}$.

Solution in parametric form:

$$= a\ {}_1^{11}\quad \tau\ {}_1^{-3}\ {}_2^{-1\ \ 2}R^{-1}\quad \tau +\quad {}_3, \qquad = \ {}_1^{10}\tau^2\ {}_1^{-2}, \qquad \text{where}\quad A = \frac{1}{2}a^2\ {}^{-5\ \ 2}\quad \frac{a^2}{6}\ {}^{5\ \ 4}.$$

149. $= \quad {}^{-1\ \ 2}(\quad)^{-5\ \ 3}(\quad)^3$.

Solution in parametric form:

$$= a\ {}_1^{-1}\quad \tau\ {}_1^{-1\ \ 2}\ {}_2R^{-1}\quad \tau +\quad {}_3, \qquad = \ {}_1^{8}\ {}_2^{2}, \qquad \text{where}\quad A = \mp\tfrac{1}{27}a^2\ {}^{-1\ \ 2}(12\quad a)^{2\ \ 3}.$$

150. $= \quad {}^{-1\ \ 2}(\quad)^{-4\ \ 3}(\quad)^3$.

Solution in parametric form:

$$= a\ {}_1\quad {}_1^{-5\ \ 2}\ {}_2\ {}_3R^{-1}\quad \tau +\quad {}_3, \qquad = \ {}_1^{10}\ {}_1^{-3}\ {}_2^{2}, \qquad \text{where}\quad A = \mp\tfrac{16}{27}a^2\ {}^{-1\ \ 2}(\ 3\quad a)^{1\ \ 3}.$$

151. $= \quad {}^{-1\ \ 2}(\quad)^3(\quad)^{8\ \ 7}$.

Solution in parametric form:

$$= a\ {}_1^{37}\quad {}_1^{7\ \ 4}\ {}_2^{-1\ \ 2}\ {}_3R^{-1}\quad \tau +\quad {}_3, \qquad = \ {}_1^{32}\ {}_3^{2}, \qquad \text{where}\quad A = \mp 7\times 2^{-10}a^2\ {}^{-5\ \ 2}\quad \frac{a^2}{6}\ {}^{1\ \ 7}.$$

152. $= \quad (\quad)^{13\ \ 7}$.

Solution in parametric form:

$$= a\ {}_1^{13}\quad {}_1^{-5\ \ 2}R^{-1}\quad \tau +\quad {}_3, \qquad = \ {}_1^{5}\ {}_1^{-3\ \ 2}\ {}_2, \qquad \text{where}\quad A = \frac{7}{2}a^{-1}\ {}^{-1}\quad \frac{a^2}{6}\ {}^{6\ \ 7}.$$

153. $=\quad(\quad)^{-13}$.

Solution in parametric form:

$$=a\ {}_1^{13}\quad {}_1^{3\ 4}R^{-1}\ \ \tau+\ \ {}_3,\qquad =\ {}_1^{16}\ {}_3,\qquad \text{where}\quad A=\ \ 3\times 2^{25}a^{-16\ \ 13}.$$

154. $=\quad{}^{-7}(\quad)^3(\quad)^3$.

Solution in parametric form:

$$=a\ {}_1^{-1}\quad {}_1^{-1\ \ 2}\ {}_2\ {}_3^{-1\ \ 2}R^{-1}\ \ \tau+\ \ {}_3,\qquad =\ {}_1^{3}\ {}_1^{1\ \ 2},\qquad \text{where}\quad A=\mp 12a^{6\ \ 2}.$$

155. $=\quad{}^3(\quad)^{-13}$.

Solution in parametric form:

$$=a\ {}_1^{11}\quad {}_1^{3\ 4}\ {}_3^{-3\ \ 2}R^{-1}\ \ \tau+\ \ {}_3,\qquad =\ {}_1^{16}\ {}_3^{-1},\qquad \text{where}\quad A=\ \ 3\times 2^{25}a^{-16\ \ 11}.$$

156. $=\quad{}^{-7}(\quad)^7(\quad)^3$.

Solution in parametric form:

$$=a\ {}_1\quad {}_1^{-1\ \ 2}\ {}_3^{-3\ \ 2}\ {}_4R^{-1}\ \ \tau+\ \ {}_3,\qquad =\ {}_1^{5}\ {}_1^{1\ \ 2}\ {}_3^{-1},\qquad \text{where}\quad A=\mp 192a^{10\ \ -2}.$$

157. $=\quad{}^{-7\ \ 6}(\quad)^{2\ \ 3}$.

Solution in parametric form:

$$=a\ {}_1^{9}\quad {}_1^{-5\ \ 2}\ {}_3^{5}R^{-1}\ \ \tau+\ \ {}_3,\qquad =\ {}_1^{10}\ {}_1^{-3}\ {}_3^{6},\qquad \text{where}\quad A=\ \ 54a^{-3\ \ 13\ \ 6}\ \ \frac{2a^2}{9}^{\ \ 2\ \ 3}.$$

158. $=\quad{}^{-1\ \ 2}(\quad)^{-4\ \ 3}(\quad)^{7\ \ 3}$.

Solution in parametric form:

$$=a\ {}_1^{3}\quad {}_1^{-5\ \ 2}\ {}_3^{-1}\ {}_4R^{-1}\ \ \tau+\ \ {}_3,\qquad =\ {}_1^{2}\ {}_1^{-3}\ {}_4^{2},\qquad \text{where}\quad A=8^{\ \ 1\ \ 2}(2a\ \ 3)^{1\ \ 3}.$$

159. $=\quad{}^{-3\ \ 4}(\quad)^3(\quad)^{8\ \ 7}$.

Solution in parametric form:

$$=a\ {}_1^{67}\quad {}_1^{7\ 4}\ {}_3^{-5}\ {}_4^{-1\ \ 2}R^{-1}\ \ \tau+\ \ {}_3,\qquad =\ {}_1^{64}\ {}_3^{-4},\qquad \text{where}\quad A=\mp 7\cdot 2^{-13}a^{2\ \ -9\ \ 4}\ \ \frac{a^2}{12}^{\ \ 1\ \ 7}.$$

160. $=\quad(\quad)^{-1\ \ 2}(\quad)^{13\ \ 7}$.

Solution in parametric form:

$$=a\ {}_1^{19}\quad {}_1^{-5\ \ 2}\ {}_3^{4}R^{-1}\ \ \tau+\ \ {}_3,\qquad =\ {}_1^{3}\ {}_1^{-3\ \ 2}\ {}_4,\qquad \text{where}\quad A=\frac{7}{2}a^{-1\ \ -1}\ \ \frac{3}{a}^{\ \ 1\ \ 2}\ \ \frac{a^2}{12}^{\ \ 6\ \ 7}.$$

161. $=\quad(\quad)^2$.

Solution in parametric form:

$$=a\ {}_1^{-1}\quad R^{-1}\ \ \tau+\ \ {}_3,\qquad =\ \ {}_1\ \ \tau R^{-1}\ \ \tau+\ \ {}_2,\qquad \text{where}\quad A=\ \ 6a^{-1\ \ -1}.$$

162. $=\quad{}^{1\ \ 2}\quad(\quad)^3$.

Solution in parametric form:

$$=a\ {}_1^{7}\quad \tau\ \ {}^{-1\ \ 2}\ \ \tau+\ \ {}_3,\qquad =\ {}_1^{8}\tau^2,\qquad \text{where}\quad A=\mp 24a^{4\ \ -7\ \ 2}.$$

163. **= ( )^4.**

Solution in parametric form:

$$= a\ {}_1^5\quad \tau^2\ {}^{-1\ 2}R^{-1}\quad \tau +\quad {}_3, \qquad = \ {}_1^6 R, \qquad \text{where}\quad A = -\tfrac{1}{162}a^6\ {}^{-5}.$$

164. **= ( )^3( )$^{-1}$.**

Solution in parametric form:

$$= a\ {}_1^{-1}\quad \tau R^{-3\ 2}\quad \tau +\quad {}_3, \qquad = \ {}_1^2\quad \tau R^{-1}\quad \tau +\quad {}_2, \qquad \text{where}\quad A = 9a^{-2\ \ -1}.$$

165. **= $^{-7\ 2}$( )^2.**

Solution in parametric form:

$$= a\ {}_1^5\qquad {}^{-3\ 2}R^{-1}\quad \tau +\quad {}_3, \qquad = \ {}_1^2\ {}^{-1}, \qquad \text{where}\quad A = \mp 6a^{-1\ \ 5\ 2}.$$

166. **= $^{1\ 2}$( )$^{-6}$( )^3.**

Solution in parametric form:

$$= a\ {}_1^7\quad \tau\ {}^{-3\ 2}\ {}_5R^{-1}\quad \tau +\quad {}_3, \qquad = \ {}_1^6\tau^2\ {}^{-1}, \qquad \text{where}\quad A = \mp 48a^{-3\ \ 7\ 2}.$$

167. **= ( )$^{-9\ 5}$( )^4.**

Solution in parametric form:

$$= a\ {}_1^{11}\quad \tau^2\ {}^{-1\ 4}R^{-1}\quad \tau +\quad {}_3, \qquad = \ {}_1^{16}\ {}_5, \qquad \text{where}\quad A = -\tfrac{2}{1125}a^4\ {}^{-3}(12\quad a)^{4\ 5}.$$

168. **= $^{-7\ 5}$( )^3( )$^{-1}$.**

Solution in parametric form:

$$= a\ {}_1\quad \tau\ {}^{3\ 2}\ {}_5^{-1\ 2}R^{-1}\quad \tau +\quad {}_3, \qquad = \ {}_1\ {}^{5\ 2}, \qquad \text{where}\quad A = \tfrac{36}{5}a^{-2\ \ 2\ 5}.$$

169. **= $^{-1\ 3}$( )$^{9\ 7}$.**

Solution in parametric form:

$$= a\ {}_1\quad \tau^2\ {}^{-5\ 2}R^{-1}\quad \tau +\quad {}_3, \qquad = \ {}_1^9\tau^3\ {}^{-3\ 2}, \qquad \text{where}\quad A = \frac{7}{2}a^{-1\ \ 1\ 3}\quad \frac{a^2}{18}\quad {}^{2\ 7}.$$

170. **= $^{-1\ 2}$( )$^{1\ 3}$( )$^{12\ 7}$.**

Solution in parametric form:

$$= a\ {}_1^{23}\quad \tau^{1\ 2\ \ 7\ 4}\ {}_5R^{-1}\quad \tau +\quad {}_3, \qquad = \ {}_1^{32}\ {}_5^2,$$

where $A = -\dfrac{7}{4}a^{-1\ \ 1\ 2}\quad \dfrac{a}{3}\quad {}^{1\ 3}\quad \dfrac{a^2}{18}\quad {}^{5\ 7}.$

In the solutions of equations 171 and 172, the following notation is used:

$$= \exp\quad \frac{\tau}{z}\Big), \qquad z = \begin{cases} {}_2 + \dfrac{1}{4}\tau^2 + \dfrac{2B}{k+1}\tau^{\ +1} & \textit{if}\ \ k \neq -1, \\ {}_2 + \dfrac{1}{4}\tau^2 + 2B\ln|\tau| & \textit{if}\ \ k = -1. \end{cases}$$

171. **= $-\frac{+5}{4}$( ) .**

Solution in parametric form:

$$= a\ {}_1^3\quad \tau^{-1\ 2\ \ 3\ 4}z^{-1\ 2}\quad \tau +\quad {}_3, \qquad = \ {}_1^4\ , \qquad \text{where}\quad k = \tfrac{1}{2}(\quad -1),\ A = \frac{1}{2}Ba^{2(\ \ -1)\ \ \frac{3}{2}(1-\ \ )}.$$

172. $= \quad {}^{-\ -2}(\quad)\ (\quad)^3.$

Solution in parametric form:

$$= a \quad (\tau z^{-1\ \ 2} + 2) \quad \tau + \quad {}_3, \qquad = {}_1\tau^{\ 1\ 2}, \qquad \text{where} \quad k = - \quad -2, \ \ A = -2^{3-} \ a^{1-} \ B.$$

173. $= \quad \frac{-1}{2}(\quad)\ (\quad)^{3\ 2}.$

Solution in parametric form:

$$= a \ {}^3_1 \quad \frac{- \quad \overline{\ \ {}^2 + 4}}{(\tau \quad)^{3\ 4} \quad \overline{\ \ {}^2 + 4}} \quad \tau + \quad {}_3, \qquad = {}^2_1\tau^{1\ 2} \quad {}^{1\ 2},$$

where

$$= \exp \quad \frac{\tau}{\tau \quad \overline{\ \ {}^2 + 4}}\Big), \qquad = \begin{cases} \tau^{-1\ 2} \quad {}_2 + \dfrac{B}{+1}\tau^{\frac{+1}{2}} & \text{if} \quad \neq -1, \\ \tau^{-1\ 2}(\quad {}_2 + \frac{1}{2}B \ln|\tau|) & \text{if} \quad = -1, \end{cases}$$

$$A = 2^{3\ 2} a^{\ -\frac{}{2} - 1} B(-2a \quad)^{\ -1}.$$

174. $= \quad (\quad)\ (\quad)^{\frac{+4\ \ +5}{+2\ \ +3}}, \qquad \neq -1, \ \ \gamma \neq -1.$

Solution in parametric form:

$$= a \quad {}_1^{\ +\ \ +2} \quad \tau^{-3\ 2} \quad {}^{-\frac{+4\ \ +5}{2(\ \ +1)}} z^{-1} \quad \tau + \quad {}_3, \qquad = \quad {}_1^{\ +1} \quad ,$$

where $A = a^{\ -1\ -\ -} \quad \frac{2a^2}{} \quad {}^{\frac{2(\ \ +1)}{+2\ \ +3}} B$, $\ = \exp \quad \frac{\tau}{\tau z}$; $z = z(\tau)$ is the solution of the transcendental equation

$$(z + k - 1)(z + k)^{\overline{1-}} = \quad {}_2 + \frac{2B}{+2\beta + 3}\tau^{\frac{+1}{2}} \quad \tau^{\frac{1}{-1}}, \qquad k = -\frac{2(\beta + 1)}{+1}.$$

175. $= \quad {}^{-1}(\quad)^{-1}(\quad)\ , \qquad \neq 1, \quad \neq 2.$

Solution in parametric form:

$$= a \quad {}_1^{\ -3} \quad \tau^{\ -1} \quad {}^{\frac{2-}{2}} z^{-1} \quad \tau + \quad {}_3, \qquad = {}^2_1 \ {}^{-4}k(kz - \tau)^{-1} \quad {}^1 \quad ,$$

where $k = \frac{-1}{-2}$, $A = \frac{1-k}{2}a^{-4\ \ 3} \quad -\frac{2a^2}{} \quad B$, $\ = \exp \quad \frac{\tau}{z}$; $z = z(\tau)$ is the solution of the transcendental equation

$$\ln|kz - \tau| - \frac{\tau}{kz - \tau} = \frac{1}{k}\tau \quad + \quad {}_2.$$

176. $= \quad {}^{-1}(\quad)^{-1}(\quad)^2.$

Solution in parametric form:

$$= \quad {}_1 \quad e^{\tau} z^{-1\ \ 2} \quad {}^{-1\ \ 2} \quad \tau + \quad {}_3, \quad = \tfrac{1}{2}e^{\tau}, \quad \text{where} \quad z = \mp A\tau + e^{\tau} + \quad {}_2, \quad = \exp \quad A \quad \frac{\tau}{z} \quad .$$

177. $= \quad {}^{-1}(\quad)^{-1} \quad .$

Solution in parametric form:

$$= \quad {}_1 \quad e^{\tau\ \ 2} \quad \tau + \quad {}_3, \qquad = {}_1 z \quad , \qquad \text{where} \quad z = \quad A\tau + e^{\tau} + \quad {}_2, \qquad = \exp \ \mp \quad \frac{\tau}{z} \quad .$$

3.2.5. Some Transformations

Let us consider some transformations of the equation

$$''' = A \quad (\,'\,)\;(\,''\,)\,.$$

1 . In the special case $\quad = \quad = 0$, the transformation $\quad = 1 \quad , \quad = \quad ^2$ reduces the equation

$$''' = A$$

to an equation of similar form (with other parameters):

$$''' = -A^{\;-\;\;-2\;\;-4}\quad .$$

2 . In the special case $\quad = \quad = 0$, the transformation $\quad = -\dfrac{\tau}{[z(\tau)]^{3\;\;2}}, \quad = \dfrac{1}{z(\tau)}$ reduces the equation

$$''' = A \quad (\,'\,)$$

to an equation of similar form (with other parameters):

$$z'''_{\tau\tau\tau} = Az^{-\frac{2\;\;+\;\;+5}{2}}(\;\;'_{\tau})\;\;.$$

3 . In the special case $\beta = 0$, the substitution $\;(\;) = \;\;'$ brings the equation

$$''' = A \quad (\,'\,)\;(\,''\,)$$

to the generalized Emden–Fowler equation:

$$'' = A \quad (\,'\,)\,,$$

which is discussed in Sections 2.3 and 2.5.

4 . In the special case $\quad = 0$, the substitution $v(\;) = (\;\,'\,)^2$ reduces the equation

$$''' = A \quad (\,'\,)\;(\,''\,)$$

to the generalized Emden–Fowler equation:

$$v'' = A \times 2^{1-} \quad v^{\frac{-1}{2}}(v')\,,$$

which is discussed in Sections 2.3 and 2.5.

3.3. Equations of the Form $y \quad = f(y)\;(y\;)\;(y\quad)$

3.3.1. Equations Containing Power Functions

1. $\quad = (\quad^2 + \quad + \;)^{-5\;\;4}$.

This is a special case of equation 3.5.2.29 with $\;(\;) = 1$.

2. $\quad = (\quad + B \quad)\quad$.

This is a special case of equation 3.5.2.1 with $\;(\;) = A \quad + B \quad$.

3. $\quad = (\quad + B \quad)[\;(\;\;)^3 + \quad]$.

This is a special case of equation 3.5.2.3 with $\;(\;) = \;(A \quad + B \quad), g(\;) = a(A \quad + B \quad)$.

4. $= {}^{-2} \; -\dfrac{(\quad +1)}{(\quad +3)^2}(\quad)^3 + \;(\quad)^2 \;\; {}^{+1}\;, \qquad \neq -3, \quad \neq -1.$

The substitution $(\;) = (\;'\,)^2$ leads to a second-order equation of the form 2.4.2.4: $'' = {}^{-2} \; -\dfrac{2(\quad +1)}{(\quad +3)^2} \quad +2A \qquad .$

5. $= {}^{-2}\big[\frac{15}{8}(\quad)^3 + \;(\quad)^{-13}\big].$

The substitution $(\;) = (\;'\,)^2$ leads to a second-order equation of the form 2.4.2.35: $'' = {}^{-2}(\frac{15}{4} \quad + 2A \quad {}^{-7}).$

6. $= {}^{-2}\big[3(\quad)^3 + \;(\quad)^{-7}\big].$

The substitution $(\;) = (\;'\,)^2$ leads to a second-order equation of the form 2.4.2.31: $'' = {}^{-2}(6 \quad + 2A \quad {}^{-4}).$

7. $= {}^{-2}\big[6(\quad)^3 + \;(\quad)^{-4}\big].$

The substitution $(\;) = (\;'\,)^2$ leads to a second-order equation of the form 2.4.2.64: $'' = {}^{-2}(12 \quad + 2A \quad {}^{-5\;\;2}).$

8. $= {}^{-2}\big[(\quad)^3 + \;(\quad)^{-3}\big].$

The substitution $(\;) = (\;'\,)^2$ leads to a second-order equation of the form 2.4.2.6: $'' = {}^{-2}(2 \quad + 2A \quad {}^{-2}).$

9. $= {}^{-2}\big[-\frac{3}{32}(\quad)^3 + \;(\quad)^{-7\;\;3}\big].$

The substitution $(\;) = (\;'\,)^2$ leads to a second-order equation of the form 2.4.2.26: $'' = {}^{-2}(-\frac{3}{16} \quad + 2A \quad {}^{-5\;\;3}).$

10. $= {}^{-2}\big[-\frac{9}{200}(\quad)^3 + \;(\quad)^{-7\;\;3}\big].$

The substitution $(\;) = (\;'\,)^2$ leads to a second-order equation of the form 2.4.2.10: $'' = {}^{-2}(-\frac{9}{100} \quad + 2A \quad {}^{-5\;\;3}).$

11. $= {}^{-2}\big[\frac{3}{8}(\quad)^3 + \;(\quad)^{-7\;\;3}\big].$

The substitution $(\;) = (\;'\,)^2$ leads to a second-order equation of the form 2.4.2.12: $'' = {}^{-2}(\frac{3}{4} \quad + 2A \quad {}^{-5\;\;3}).$

12. $= {}^{-2}\big[\frac{63}{8}(\quad)^3 + \;(\quad)^{-7\;\;3}\big].$

The substitution $(\;) = (\;'\,)^2$ leads to a second-order equation of the form 2.4.2.66: $'' = {}^{-2}(\frac{63}{4} \quad + 2A \quad {}^{-5\;\;3}).$

13. $= {}^{-2}\big[-\frac{5}{72}(\quad)^3 + \;(\quad)^{-9\;\;5}\big].$

The substitution $(\;) = (\;'\,)^2$ leads to a second-order equation of the form 2.4.2.29: $'' = {}^{-2}(-\frac{5}{36} \quad + 2A \quad {}^{-7\;\;5}).$

14. $= {}^{-2}\big[-\frac{1}{9}(\quad)^3 + \quad\big].$

The substitution $(\;) = (\;'\,)^2$ leads to a second-order equation of the form 2.4.2.14: $'' = {}^{-2}(-\frac{2}{9} \quad + 2A \quad {}^{-1\;\;2}).$

15. $= {}^{-2}\big[-\frac{2}{25}(\quad)^3 + \quad\big].$

The substitution $(\;) = (\;'\,)^2$ leads to a second-order equation of the form 2.4.2.8: $'' = {}^{-2}(-\frac{4}{25} \quad + 2A \quad {}^{-1\;\;2}).$

16. $\quad = {}^{-2}\big[10(\quad)^3 + \quad\big].$

The substitution $(\ ) = (\ ')^2$ leads to a second-order equation of the form 2.4.2.33: $'' = {}^{-2}(20 \quad + 2A \quad {}^{-1\ 2})$.

17. $\quad = {}^{-2}\big[-\frac{6}{49}(\quad)^3 + \quad (\quad)^2\big].$

The substitution $(\ ) = (\ ')^2$ leads to a second-order equation of the form 2.4.2.37: $'' = {}^{-2}(-\frac{12}{49} \quad + 2A \quad {}^{1\ 2})$.

18. $\quad = {}^{-2}\big[\quad (\quad)^5 - \frac{3}{25}(\quad)^3\big].$

The substitution $(\ ) = (\ ')^2$ leads to a second-order equation of the form 2.4.2.60: $'' = {}^{-2}(2A \quad {}^2 - \frac{6}{25} \quad)$.

19. $\quad = {}^{-2}\big[\quad (\quad)^5 + \frac{3}{25}(\quad)^3\big].$

The substitution $(\ ) = (\ ')^2$ leads to a second-order equation of the form 2.4.2.62: $'' = {}^{-2}(2A \quad {}^2 + \frac{6}{25} \quad)$.

20. $\quad = {}^{-4\ 3}(\quad + B).$

The substitution $(\ ) = (\ ')^2$ leads to a second-order equation of the form 2.4.2.40: $'' = {}^{-4\ 3}(2A + 2B \quad {}^{-1\ 2})$.

21. $\quad = (\quad {}^4 + B \quad {}^3)(\quad)^{-13}.$

The substitution $(\ ) = (\ ')^2$ leads to a second-order equation of the form 2.4.2.39: $'' = (2A \quad {}^4 + 2B \quad {}^3) \quad {}^{-7}$.

22. $\quad = (\quad {}^2 + B)(\quad)^{-9}.$

The substitution $(\ ) = (\ ')^2$ leads to a second-order equation of the form 2.4.2.16: $'' = (2A \quad {}^2 + 2B) \quad {}^{-5}$.

23. $\quad = (\quad {}^{-1} + B \quad {}^{-2})(\quad)^{-3}.$

The substitution $(\ ) = (\ ')^2$ leads to a second-order equation of the form 2.4.2.28: $'' = (2A \quad {}^{-1} + 2B \quad {}^{-2}) \quad {}^{-2}$.

24. $\quad = (\quad {}^{-7\ 3} + B \quad {}^{-10\ 3})(\quad)^{-7\ 3}.$

The substitution $(\ ) = (\ ')^2$ leads to a second-order equation of the form 2.4.2.48: $'' = (2A \quad {}^{-7\ 3} + 2B \quad {}^{-10\ 3}) \quad {}^{-5\ 3}$.

25. $\quad = (\quad {}^{-4\ 3} + B \quad {}^{-10\ 3})(\quad)^{-7\ 3}.$

The substitution $(\ ) = (\ ')^2$ leads to a second-order equation of the form 2.4.2.49: $'' = (2A \quad {}^{-4\ 3} + 2B \quad {}^{-10\ 3}) \quad {}^{-5\ 3}$.

26. $\quad = (\quad {}^{-4\ 3} + B \quad {}^{-7\ 3})(\quad)^{-7\ 3}.$

The substitution $(\ ) = (\ ')^2$ leads to a second-order equation of the form 2.4.2.24: $'' = (2A \quad {}^{-4\ 3} + 2B \quad {}^{-7\ 3}) \quad {}^{-5\ 3}$.

27. $\quad = (\quad {}^{-2\ 3} + B \quad {}^{-4\ 3})(\quad)^{-7\ 3}.$

The substitution $(\ ) = (\ ')^2$ leads to a second-order equation of the form 2.4.2.90: $'' = (2A \quad {}^{-2\ 3} + 2B \quad {}^{-4\ 3}) \quad {}^{-5\ 3}$.

28. $= (\quad + B \quad ^{-2\ 3})(\quad)^{-7\ 3}$**.**

The substitution $(\ ) = (\ ')^2$ leads to a second-order equation of the form 2.4.2.89: $'' \ = (2A + 2B \quad ^{-2\ 3}) \quad ^{-5\ 3}$.

29. $= (\quad ^2 + B)(\quad)^{-7\ 3}$**.**

The substitution $(\ ) = (\ ')^2$ leads to a second-order equation of the form 2.4.2.47: $'' \ = (2A \quad ^2 + 2B) \quad ^{-5\ 3}$.

30. $= (\quad ^2 + B \quad)(\quad)^{-7\ 3}$**.**

The substitution $(\ ) = (\ ')^2$ leads to a second-order equation of the form 2.4.2.46: $'' \ = (2A \quad ^2 + 2B \quad) \quad ^{-5\ 3}$.

31. $= (\quad + B \quad ^k) \quad (\quad) \quad$**.**

This is a special case of equation 3.5.4.12 with $(\ ) = A \quad + B \quad$.

3.3.2. Equations Containing Exponential Functions

Tables 29–31 present the equations whose solutions are given in this subsection.

TABLE 29
Solvable equations of the form
$''' \quad = Ae \ (\ ') \ (\ '' \ )$

		Equation
arbitrary ($\neq 1$)	1	3.3.2.1
0	3	3.3.2.9
1	1	3.3.2.2
$\frac{3}{2}$	0	3.3.2.3
$\frac{3}{2}$	3	3.3.2.7
2	arbitrary ($\neq -1$)	3.3.2.11
2	-1	3.3.2.13

TABLE 30
Solvable equations of the form
$''' \quad = A \quad ' \exp[(\ ')^2](\ '' \ )$

	β	Equation
arbitrary ($\neq 2$)	0	3.3.2.4
1	arbitrary ($\beta \neq -1$)	3.3.2.12
1	-1	3.3.2.14
$\frac{3}{2}$	$-\frac{1}{2}$	3.3.2.5
$\frac{3}{2}$	1	3.3.2.8
2	0	3.3.2.6
3	1	3.3.2.10

TABLE 31
Other solvable equations of the type considered

Form of equation	Equation
$''' \quad = Ae \quad ' \exp[(\ ')^2](\ '' \ )$	3.3.2.20
$''' \quad = A(\ ') \ \exp(\ '' \ ), \quad \neq -1$	3.3.2.15
$''' \quad = A(\ ')^{-1} \exp(\ '' \ )$	3.3.2.16
$''' \quad = A \quad ' \exp(\ '' \ ), \quad \beta \neq -1$	3.3.2.17
$''' \quad = A \ ^{-1} \ ' \exp(\ '' \ )$	3.3.2.18
$''' \quad = Ae \quad ' \exp[(\ ')^2 + \ '' \ ]$	3.3.2.19

In the solutions of equations 1–6, the following notation is used:

$$= \exp(\tau^2) \quad \tau + \quad {}_2, \qquad = (1 \quad \tau) \quad \frac{\tau}{\tau} + \quad {}_2, \qquad = 2\tau \quad - \exp(\tau^2), \qquad = \exp(\mp\tau)\frac{\tau}{\tau} + \quad {}_2.$$

1. $\quad = \quad (\quad) \,, \quad \neq 1.$

Solution in parametric form:

$$= a \quad {}_1 \quad \tau^{-1} \quad {}^{-1\ 2} \quad \tau + \quad {}_3, \qquad = \ln(\quad {}_1^2 \ {}^{-2}\tau),$$

where $k = \dfrac{1}{1-}$, $A = \dfrac{1}{2(1- \quad)} a^{-2} \ {}^{-1}(2a^2)$.

2. $\quad = \quad .$

Solution in parametric form:

$$= \quad {}_1 \quad \tau^{-1} \quad {}^{-1\ 2} \quad \tau + \quad {}_3, \qquad = \ln \ \mp\frac{\tau}{A} \quad .$$

3. $\quad = \quad (\quad)^{3\ 2}.$

Solution in parametric form:

$$= \quad {}_1 \quad {}^{-1} \quad \tau + \quad {}_3, \qquad = \tau^2 + \ln\big(\quad \overline{2}\, A^{-1} \quad {}^{-1} \quad .$$

4. $\quad = \quad \mathbf{exp}\big[(\quad)^2\big](\quad) \,, \quad \neq 2.$

Solution in parametric form:

$$= a \quad {}_1 \quad \tau^{-1}(1 \quad \tau)^{\frac{1}{-2}} [\ln(\quad {}_1 \ {}^{-2}\tau)]^{-1\ 2} \quad \tau + \quad {}_3, \qquad = a \quad {}_1 \quad ,$$

where $k = \dfrac{1}{-2}$, $A = \dfrac{1}{2-} a \ {}^{-3} \ {}^{-1}$.

5. $\quad = \quad {}^{-1\ 2} \quad \mathbf{exp}\big[(\quad)^2\big](\quad)^{3\ 2}.$

Solution in parametric form:

$$= 4 \quad {}_1 \qquad [\tau^2 + \ln(a \quad {}^{-1})]^{-1\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1 \quad {}^2, \qquad \text{where} \quad A = - \quad \overline{2}\, a^{-1}.$$

6. $\quad = \quad \mathbf{exp}\big[(\quad)^2\big](\quad)^2.$

Solution in parametric form:

$$= \quad {}_1 \quad \tau^{-1} \exp(\mp\tau) \ \ln \quad \frac{\tau}{A} \quad {}^{-1\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1 \quad .$$

In the solutions of equations 7 and 8, the following notation is used:

$$= \overline{\tau(\tau+1)} - \ln\big[\quad {}_2\big(\quad \overline{\tau} + \quad \overline{\tau+1} \quad \big], \quad R = \sqrt{\frac{\tau+1}{\tau}}, \qquad = 1 - \sqrt{\frac{\tau+1}{\tau}} \ln\big[\quad {}_2\big(\quad \overline{\tau} + \quad \overline{\tau+1} \quad \big].$$

7. $\quad = \quad (\quad)^3(\quad)^{3\ 2}.$

Solution in parametric form:

$$= a \quad {}_1 \quad R^{-1} \quad {}^{-1} \quad {}^{-1\ 2} \quad \tau + \quad {}_3, \qquad = - \ln(\quad {}_1^{-3} \quad), \qquad \text{where} \quad A = 2^{3\ 2} a^3 \quad .$$

8. $\quad = \quad \mathbf{exp}\big[(\quad)^2\big](\quad)^{3\ 2}.$

Solution in parametric form:

$$= -\tfrac{1}{2} \quad {}_1 \quad \tau^{-2} R^{-1} \quad [\ln(a \quad {}_1^{-3\ 2} \quad {}^{-1})]^{-1\ 2} \quad \tau + \quad {}_3, \qquad = \quad {}_1 \quad , \qquad \text{where} \quad A = -4a^{-1} \quad {}^{-3\ 2}.$$

In the solutions of equations 9 and 10, the following notation is used:

$$Z = \begin{cases} {}_1\ {}_0(\tau) + {}_2\ {}_0(\tau) & \textit{for the upper sign,} \\ {}_1\ {}_0(\tau) + {}_2\ {}_0(\tau) & \textit{for the lower sign,} \end{cases}$$

where $_0(\tau)$ *and* $_0(\tau)$ *are the Bessel functions, and* $_0(\tau)$ *and* $_0(\tau)$ *are the modified Bessel functions.*

9. $\quad = \quad (\quad)^3.$

Solution in parametric form:

$$= 2 \ {}_1 \quad \tau^{-1} Z^{-1\ 2} \quad \tau + \ {}_3, \qquad = \ln(\mp \tfrac{1}{8} A^{-1} \tau^2).$$

10. $\quad = \quad \exp\big[(\quad)^2\big](\quad)^3.$

Solution in parametric form:

$$= \ {}_1 \quad Z'_\tau \ \ln \quad \frac{\tau^2}{A} \quad {}^{-1\ 2} \quad \tau + \ {}_3, \qquad = \ {}_1 Z.$$

11. $\quad = \quad (\quad) \ (\quad)^2, \qquad \gamma \neq -1.$

Solution in parametric form:

$$= a \quad \tau^{-1\ 2\ \ -1} \exp \quad \frac{}{+1} \quad \tau + \ {}_3, \qquad = \quad ,$$

where $A = \frac{1}{2} a^{\ +1} k, \quad = \quad \frac{\tau}{} + \ {}_1; \quad = \ (\tau)$ is the solution of the transcendental equation

$$\ln(\lambda \quad - \tau) - \frac{\tau}{\lambda \quad - \tau} = \frac{k}{\lambda} \tau^\lambda + \ {}_2, \qquad \lambda = \frac{+1}{2}.$$

12. $\quad = \quad \exp\big[(\quad)^2\big] \quad , \qquad \neq -1.$

Solution in parametric form:

$$= a \quad {}^{-1} \quad - \frac{\tau}{\beta + 1} \quad {}^{-1\ 2} \exp \ - \frac{}{\beta + 1} \quad \tau + \ {}_3, \qquad = a\tau \exp \ - \frac{}{\beta + 1} \quad ,$$

where $A = a^{- \ -1} k, \quad = \quad \frac{\tau}{} + \ {}_1; \quad = \ (\tau)$ is the solution of the transcendental equation

$$\ln(\lambda \quad - \tau) - \frac{\tau}{\lambda \quad - \tau} = - \frac{k}{\lambda} \tau^\lambda + \ {}_2, \qquad \lambda = \beta + 1.$$

13. $\quad = \quad (\quad)^{-1} (\quad)^2.$

Solution in parametric form:

$$= \ {}_1 \quad {}^{-1\ 2} \quad \tau + \ {}_3, \qquad = \tau, \qquad \text{where} \qquad = \exp \quad \frac{\tau}{\tau - 2Ae^\tau + \ {}_2} \quad .$$

14. $\quad = \quad {}^{-1} \quad \exp\big[(\quad)^2\big] \quad .$

Solution in parametric form:

$$= \ {}_1 \quad \tau^{-1\ 2} (\tau + Ae^\tau + \ {}_2)^{-1} \quad \tau + \ {}_3, \qquad = \ {}_1 \quad , \qquad \text{where} \qquad = \exp \quad \frac{\tau}{\tau + Ae^\tau + \ {}_2} \quad .$$

In the solutions of equations 15–19, the following notation is used:

$$= \quad {}_1 - \frac{1}{\lambda}(\quad +1)(\tau+1)e^{-\tau}, \qquad = \quad {}_1 - \frac{1}{\lambda}(\tau+2)e^{-\tau \;\; 2},$$

$$= \begin{cases} {}_2 - \quad \ln \quad {}_1 - \dfrac{\lambda}{+1}\tau^{\;\;+1} \quad \tau & \textit{if} \quad \neq -1, \\ {}_2 - \quad \ln(\quad {}_1 - \lambda \ln|\tau|) \quad \tau & \textit{if} \quad = -1, \end{cases}$$

$$N = \ln \quad -\frac{1}{2\lambda} \quad e^{-\tau \;\; 2} \quad {}^{-1} \quad \tau - \frac{1}{2} + \quad {}_2.$$

15. $\quad = \quad (\quad) \; \mathbf{exp}(\quad), \qquad \gamma \neq -1.$

Solution in parametric form:

$$= \frac{1}{\lambda} \quad e^{-\tau} \quad -\frac{2 \quad +1}{2 \quad +2} \quad \tau + \quad {}_3, \qquad = \frac{2}{\lambda} \quad e^{-\tau} \quad -\frac{}{+1} \quad \tau + \quad {}_2,$$

where $\quad = \frac{1}{2}(\quad -1), \; A = 2^{-} \quad \lambda.$

16. $\quad = \quad (\quad)^{-1} \; \mathbf{exp}(\quad).$

Solution in parametric form:

$$= \frac{1}{\lambda} \quad \exp\bigl(-\tau + \tfrac{1}{2} \qquad \tau + \quad {}_3, \qquad = \frac{2}{\lambda} \quad \exp\bigl(-\tau + \qquad \tau + \quad {}_2, \qquad \text{where} \qquad = 0, \;\; A = 2\lambda.$$

17. $\quad = \qquad \mathbf{exp}(\quad), \qquad \neq -1.$

Solution in parametric form:

$$= \qquad {}^{-1 \;\; 2} \quad \tau + \quad {}_3, \qquad = 2\tau, \qquad \text{where} \qquad = \beta, \;\; A = 2^{- \quad -1}\lambda.$$

18. $\quad = \quad {}^{-1} \quad \mathbf{exp}(\quad).$

Solution in parametric form:

$$= \qquad {}^{-1 \;\; 2} \quad \tau + \quad {}_3, \qquad = 2\tau, \qquad \text{where} \qquad = -1, \;\; A = \lambda.$$

19. $\quad = \qquad \mathbf{exp}\bigl[(\quad)^2 + \qquad \bigr].$

Solution in parametric form:

$$= \frac{1}{2\lambda} \quad e^{-\tau \;\; 2} \quad {}^{-1}N^{-1 \;\; 2} \quad \tau + \quad {}_3, \qquad = \frac{1}{2\lambda} \quad e^{-\tau \;\; 2} \quad {}^{-1} \quad \tau + \quad {}_2, \qquad \text{where} \quad A = \lambda.$$

20. $\quad = \qquad \mathbf{exp}[(\quad)^2](\quad) \; .$

Solution in parametric form:

$$= \quad \tau^{-} \; z^{-1} \; \ln \frac{z}{A} - \quad - \quad {}_1 \quad {}^{-1 \;\; 2} \quad \tau + \quad {}_3, \qquad = \quad ,$$

where

$$= \frac{\tau}{z\tau}, \qquad z = \begin{cases} \dfrac{1}{2-}\tau^{2-} \quad + \dfrac{1}{1-}\tau^{1-} \quad + \quad {}_2 & \text{if} \quad \neq 2, \quad \neq 1; \\ \tau + \ln|\tau| + \quad {}_2 & \text{if} \quad = 1; \\ \ln|\tau| - \dfrac{1}{\tau} + \quad {}_2 & \text{if} \quad = 2. \end{cases}$$

3.3.3. Other Equations

1. $= \{\cosh[\ (\ )^2]\}^{-2}$ **.**

The substitution $' = \overline{\ \ }(\ )$ leads to a second-order equation of the form 2.7.4.1: $'' = A\ [\cosh(\lambda\ )]^{-2}\ '$.

2. $= \{\sinh[\ (\ )^2]\}^{-2}$ **.**

The substitution $' = \overline{\ \ }(\ )$ leads to a second-order equation of the form 2.7.4.2: $'' = A\ [\sinh(\lambda\ )]^{-2}\ '$.

3. $= \cosh[\ (\ )^2]\,(\ )^{3\ 2}$**.**

The substitution $' = \overline{\ \ }(\ )$ leads to a second-order equation of the form 2.7.4.3: $'' = \frac{\overline{2}}{2}A\ \cosh(\lambda\ )(\ ')^{3\ 2}$.

4. $= \sinh[\ (\ )^2]\,(\ )^{3\ 2}$**.**

The substitution $' = \overline{\ \ }(\ )$ leads to a second-order equation of the form 2.7.4.4: $'' = \frac{\overline{2}}{2}A\ \sinh(\lambda\ )(\ ')^{3\ 2}$.

5. $= \cosh(\ )\,(\ )^3(\ )^{3\ 2}$**.**

The substitution $' = \overline{\ \ }(\ )$ leads to a second-order equation of the form 2.7.4.5: $'' = \frac{\overline{2}}{2}A\cosh(\lambda\ )\ (\ ')^{3\ 2}$.

6. $= \sinh(\ )\,(\ )^3(\ )^{3\ 2}$**.**

The substitution $' = \overline{\ \ }(\ )$ leads to a second-order equation of the form 2.7.4.6: $'' = \frac{\overline{2}}{2}A\sinh(\lambda\ )\ (\ ')^{3\ 2}$.

7. $= [\cosh(\ )]^{-2}(\ )^3(\ )^2$**.**

The substitution $' = \overline{\ \ }(\ )$ leads to a second-order equation of the form 2.7.4.7: $'' = \frac{1}{2}A[\cosh(\lambda\ )]^{-2}\ (\ ')^2$.

8. $= [\sinh(\ )]^{-2}(\ )^3(\ )^2$**.**

The substitution $' = \overline{\ \ }(\ )$ leads to a second-order equation of the form 2.7.4.8: $'' = \frac{1}{2}A[\sinh(\lambda\ )]^{-2}\ (\ ')^2$.

9. $= \cosh\ (\ )\ (\ )$ **.**

This is a special case of equation 3.5.4.12 with $(\ ) = A\cosh\ (\lambda\ )$.

10. $= \sinh\ (\ )\ (\ )$ **.**

This is a special case of equation 3.5.4.12 with $(\ ) = A\sinh\ (\lambda\ )$.

11. $= \tanh\ (\ )\ (\ )$ **.**

This is a special case of equation 3.5.4.12 with $(\ ) = A\tanh\ (\lambda\ )$.

12. $= \coth\ (\ )\ (\ )$ **.**

This is a special case of equation 3.5.4.12 with $(\ ) = A\coth\ (\lambda\ )$.

13. $= \ln\ (\ )\ (\ )$ **.**

This is a special case of equation 3.5.4.12 with $(\ ) = A\ln\ (\lambda\ )$.

14. $= \quad \{\cos[\ (\ )^2]\}^{-2} \quad .$

The substitution $' = \overline{\quad}(\ )$ leads to a second-order equation of the form 2.7.5.1: $'' = A\ [\cos(\lambda\ )]^{-2}\ '$.

15. $= \quad \{\sin[\ (\ )^2]\}^{-2} \quad .$

The substitution $' = \overline{\quad}(\ )$ leads to a second-order equation of the form 2.7.5.2: $'' = A\ [\sin(\lambda\ )]^{-2}\ '$.

16. $= \ [\cos(\ )]^{-2}(\ )^3(\ )^2.$

The substitution $' = \overline{\quad}(\ )$ leads to a second-order equation of the form 2.7.5.3: $'' = \frac{1}{2}A[\cos(\lambda\ )]^{-2}\ (\ ')^2$.

17. $= \ [\sin(\ )]^{-2}(\ )^3(\ )^2.$

The substitution $' = \overline{\quad}(\ )$ leads to a second-order equation of the form 2.7.5.4: $'' = \frac{1}{2}A[\sin(\lambda\ )]^{-2}\ (\ ')^2$.

18. $= \quad \cos[\ (\ )^2]\,(\ )^{3\ 2}.$

The substitution $' = \overline{\quad}(\ )$ leads to a second-order equation of the form 2.7.5.5: $'' = \frac{\bar{2}}{2}A\ \ \cos(\lambda\ )(\ ')^{3\ 2}$.

19. $= \quad \sin[\ (\ )^2]\,(\ )^{3\ 2}.$

The substitution $' = \overline{\quad}(\ )$ leads to a second-order equation of the form 2.7.5.6: $'' = \frac{\bar{2}}{2}A\ \ \sin(\lambda\ )(\ ')^{3\ 2}$.

20. $= \ \cos(\ )\,(\ )^3(\ )^{3\ 2}.$

The substitution $' = \overline{\quad}(\ )$ leads to a second-order equation of the form 2.7.5.7: $'' = \frac{\bar{2}}{2}A\cos(\lambda\ )\ (\ ')^{3\ 2}$.

21. $= \ \sin(\ )\,(\ )^3(\ )^{3\ 2}.$

The substitution $' = \overline{\quad}(\ )$ leads to a second-order equation of the form 2.7.5.8: $'' = \frac{\bar{2}}{2}A\sin(\lambda\ )\ (\ ')^{3\ 2}$.

22. $= \ \cos\ (\ )\ \ (\ )\ .$

This is a special case of equation 3.5.4.12 with $(\ ) = A\cos\ (\lambda\ )$.

23. $= \ \sin\ (\ )\ \ (\ )\ .$

This is a special case of equation 3.5.4.12 with $(\ ) = A\sin\ (\lambda\ )$.

24. $= \ \tan\ (\ )\ \ (\ )\ .$

This is a special case of equation 3.5.4.12 with $(\ ) = A\tan\ (\lambda\ )$.

25. $= \ \cot\ (\ )\ \ (\ )\ .$

This is a special case of equation 3.5.4.12 with $(\ ) = A\cot\ (\lambda\ )$.

26. $= \ (\arcsin\ )\ \ (\ )\ .$

This is a special case of equation 3.5.4.12 with $(\ ) = A(\arcsin\ )\ $.

27. $= \ (\arctan\ )\ \ (\ )\ .$

This is a special case of equation 3.5.4.12 with $(\ ) = A(\arctan\ )\ $.

3.4. Nonlinear Equations with Arbitrary Parameters

3.4.1. Equations Containing Power Functions

3.4.1-1. Equations of the form $(\, , \,) \; ''' \; = g(\, , \,)$.

1. $\; = \;$ **.**

See Subsection 3.2.2. The substitution $(\,) = (\, ')^2$ leads to the Emden–Fowler equation $'' = 2a \; ^{-1} \; ^{2}$, which is discussed in Section 2.3.

2. $\; = \; ^{-1}$**.**

This is a special case of equation 3.5.1.2 with $(\,) = a$. On integrating the equation, we have $'' - \frac{1}{2}(\, ')^2 = \frac{a}{+1} \; ^{+1} + \; .$

3. $\; = \;$ **.**

See Subsections 3.2.3 and 3.2.5 (Item 1). The transformation $z = \; ^{+3} \; ^{-1}, \; = \; '$ leads to a second-order equation.

4. $\; = \; ^{-5 \; 2} + \; ^{-7 \; 2}$**.**

Using the transformation given in 3.5.2.15 (Item 2), we reduce this equation to a constant coefficient nonhomogeneous linear equation.

Solution in parametric form ($\neq 0$):

$$= \frac{\tau}{[\; (\tau)]^{3 \; 2}} + \; _3, \qquad = \frac{1}{(\tau)},$$

where $(\tau) = -\frac{a}{} + \; _1 e^{- \; \tau} + \; _2 e^{\; \tau \; 2} \sin \frac{k\tau \; \overline{3}}{2}$, $k = \; ^{1 \; 3}$.

5. $\; = \; ^{-5 \; 2} + \; ^{3 \; -7 \; 2}$**.**

The transformation $= 1 \; , \; = \; ^{2}$ leads to an autonomous equation of the form 3.4.1.4: $''' = -a \; ^{-5 \; 2} - \; ^{-7 \; 2}$.

6. $\; = (\; ^{2} + \; + \;)^{-5 \; 4}$**.**

This is a special case of equation 3.5.2.29 with $(\,) = k$.

7. $\; = (\; ^{2} + \; ^{2} + \; ^{4})^{-5 \; 4}$**.**

This is a special case of equation 3.5.2.30 with $(\xi) \equiv 1$.

8. $\; = (\; + \; ^{2} + \; + \;)$ **.**

The substitution $z = \; + a \; ^{2} + \; + \;$ leads to an equation $z''' = kz$, whose solvable cases are outlined in Subsection 3.2.2.

9. $\; = \; (\; ^{2} + \; + \;)^{- \; -2}$**.**

This is a special case of equation 3.5.1.13 with $(\,) = k$.

10. $\; = (\; + \;) \; (\; + \;)^{- \; -2 \; -4}$ **.**

The transformation $\xi = \frac{a \; +}{+}$, $= \frac{}{(\; + \;)^2}$ leads to a simpler equation: $''' = \Delta^{-3} \xi$, where $\Delta = a \; - \;$ (see Subsections 3.2.2 and 3.2.3).

11. $= \; ^{2} \; ^{-1}(\; - \;)^{-3} \; ^{-} \; .$

This is a special case of equation 3.5.1.11 with $(\xi) = \; \xi^{-}$.

12. $(\; + \; ^{2} + \; + \;) \; = \; .$

The substitution $= \; + a \; ^{2} + \; +$ leads to an equation of the form 3.4.1.2: $''' \; = k$.

3.4.1-2. Equations of the form $''' = (\; , \; , \; ')$.

13. $= \; + \; .$

Integrating yields a second-order equation: $'' = \frac{a}{+1} \; ^{+1} + \frac{}{+1} \; ^{+1} + \; .$

14. $= \; ^{-} \; ^{-2} \; - \; ^{-} \; ^{-3} \; ^{+1}.$

This is a special case of equation 3.5.2.5 with $(\xi) = a\xi$.

15. $= \; ^{-2} \; ^{-4} \; - 2 \; ^{-2} \; ^{-5} \; ^{+1}.$

This is a special case of equation 3.5.2.8 with $(\xi) = a\xi$.

16. $= \; ^{-3} \; + \; ^{-5} \; ^{2} + \; ^{-7} \; ^{2}.$

The transformation $= [\; (\tau)]^{-3} \; ^{2} \; \tau, \; = [\; (\tau)]^{-1}$ leads to a constant coefficient linear equation: $'''_{\tau\tau\tau} - \lambda \; '_{\tau} + \; + a = 0.$

17. $= - \; ^{-2} \; + \; ^{-3} \; + \; ^{1} \; ^{2} \; ^{-5} \; ^{2}.$

This is a special case of equation 3.5.2.31 with $(\xi) = a.$

18. $= - \; ^{-2} \; + \; ^{-3} \; + \; ^{-3} \; ^{4} \; ^{-5} \; ^{4}.$

This is a special case of equation 3.5.2.32 with $(\xi) = a.$

19. $= \; + \; (\;)^{3}.$

This is a special case of equation 3.5.2.3 with $(\;) = a$ and $g(\;) =$.

20. $= \; (\;)^{3} + (\;)^{-5}.$

This is a special case of equation 3.5.2.4 with $(\;) =$.

21. $= (\; ^{2} + \; + \;)^{-\frac{+5}{4}} (\;) \; .$

This is a special case of equation 3.5.2.29 with $(\xi) = \xi$.

22. $= - \; ^{3} \; + (\; + \;) \; .$

The substitution $= \; ' + a$ leads to a second-order autonomous equation: $'' - a \; ' + a^{2} \; = \; .$

23. $= \; (\; - \;) \; .$

This is a special case of equation 3.5.2.18 with $(\xi) = a\xi$.

24. $= \; ^{-} \; ^{-2}(\; - \;) \; .$

This is a special case of equation 3.5.2.23 with $(\xi) = a\xi$.

25. $= \; (\; - \;) \; .$

The substitution $(\;) = \; ' -$ leads to a second-order generalized homogeneous equation: $(\; ' \;)' = a \; .$

26. $= \quad (\quad - \ ) \ + \quad {}^{k}$.

The substitution $(\ ) = \ ' -$ leads to a second-order equation: $(\ ' \ )' = a \qquad + \quad .$

27. $= \quad {}^{-5} \ - \ (\quad - \ )^3$.

This is a special case of equation 3.5.2.6 with $(\xi) = a\xi^{-}$.

28. $= \ - \ {}^{-4}(\quad -2 \ ) \ .$

This is a special case of equation 3.5.2.24 with $(\xi) = a\xi$.

29. $= \quad (\quad -2 \ ) \ + \quad {}^{k}$.

The substitution $(\ ) = \ ' - 2$ leads to a second-order equation: $'' = a \quad {}^{+1} \quad + \quad {}^{+1}$.

30. $= \ {}^{2} \ {}^{-7} \ - \ (\quad -2 \ )^3$.

This is a special case of equation 3.5.2.9 with $(\xi) = a\xi^{-}$.

31. $= \qquad (\quad -2 \ ) \ .$

The transformation $= 1 \quad , z = \quad {}^{2}$ leads to the equation $z''' = -a(-1) \ {}^{-} \ {}^{-2} \ {}^{-} {}^{-4} z \ (z')$, which is discussed in Section 3.2.

See also equations 3.3.1.2–3.3.1.30.

3.4.1-3. Equations of the form $(\ , \ , \ ') \ ''' + g(\ , \ , \ ') \ '' + (\ , \ , \ ') = 0$.

32. $+ \qquad - \ (\ )^2 = 0.$

1 . The substitution $(\ ) = (\ ')^2$ leads to a second-order generalized homogeneous equation: $'' + a \quad ' - 2a \quad = 0.$

2 . Particular solutions:

$$= \ {}_1 \exp(\ {}_2 \ ) - a^{-1} \ {}_2,$$
$$= 6(a \ + \ {}_1)^{-1}.$$

33. $+ \qquad + \qquad - \ (\ )^2 + \qquad = 0.$

Particular solution: $= \ {}_1 \exp(\ {}_2 \ ) - \dfrac{{}^2_2 + \quad {}_2 +}{a \quad {}_2}.$

34. $+ \ {}_2 \quad + \ {}_1 \quad + \ {}_0 \ = \qquad - \ (\ )^2 + \ .$

Particular solutions: $= \ e^{\lambda} \ + \dfrac{k}{a_0}$, where is an arbitrary constant and $\lambda = \lambda$ are roots of the cubic equation $\lambda^3 + \ a_2 - \dfrac{k}{a_0} \ \lambda^2 + a_1\lambda + a_0 = 0.$

35. $= \qquad + \quad (\quad - \ )^k + \quad {}^{s}.$

The substitution $(\ ) = \ ' -$ leads to a second-order equation: $(\ ' \ )' = a \quad {}^{-1} \ ' + \quad + \quad .$

36. $= \qquad - \ (\ )^3 + \quad .$

1 . Particular solution:

$$= \ {}_1 \exp(\ {}_3 \ ) + \ {}_2 \exp(- \ {}_3 \ ),$$

where the constants ${}_1$, ${}_2$, and ${}_2$ are related by the constraint ${}^2_3 - 4a \ {}_1 \ {}_2 \ {}^2_3 - \ = 0.$

2 . Particular solution:

$$= \ {}_1 \cos(\ {}_3 \ ) + \ {}_2 \sin(\ {}_3 \ ),$$

where the constants ${}_1$, ${}_2$, and ${}_2$ are related by the constraint ${}^2_3 - a(\ {}^2_1 + \ {}^2_2) \ {}^2_3 + \ = 0.$

37. $+ 3 \quad = \quad .$

The substitution $(\ ) =$ leads to the equation $''' \ = a$, which is discussed in Subsection 3.2.2.

38. $- \quad - 2 \quad + \ ^2 + \ = 0.$

Integrating yields a second-order equation: $'' = a\ ^2 + \ + \ .$ The transformation $= k^3$, $= kz - —$, where $k = \dfrac{6}{a}\ ^{1\ 3}$, leads to the first Painlevé transcendent: $'' = \ ^2 + 6z$ (see Paragraph 2.8.2-2).

39. $+ (\ + 2) \quad = (\quad + \quad)\ .$

The substitution $= \ ' + a$ leads to a second-order autonomous equation of the form 2.9.1.1: $'' = \quad .$

40. $= -\frac{3}{2} \quad + \ ^{- \ -2\ \ 2}\ (2 \quad - \ ).$

This is a special case of equation 3.5.3.37 with $(\xi) = a\xi^2$.

41. $= -\frac{3}{2} \quad + \ ^{- \ -3\ \ 2}\ (2 \quad - \ )^3.$

This is a special case of equation 3.5.3.39 with $(\xi) = a\xi^2$.

42. $+ \quad = \ ^{- \ -3}(\quad - \ )\ .$

This is a special case of equation 3.5.3.47 with $(\xi) = a\xi$.

43. $+ (1 - \ ) \quad = \ ^2\ (\quad - \ )\ .$

This is a special case of equation 3.5.3.42 with $(\xi) = \ \xi$.

44. $^2 \quad + 6 \quad + 6 \quad = \ ^2 \quad .$

The substitution $(\ ) = \ ^2$ leads to the equation $''' \ = a$, which is discussed in Subsection 3.2.2.

45. $+ \frac{1}{2} \quad = \quad + \ .$

The transformation $= (\ ),\ \ = (\ ')^2$ leads to a fourth-order constant coefficient nonhomogeneous linear equation of the form 4.1.2.2: $2\ '''' = a \ + \ .$

46. $- \frac{1}{3} \quad = \quad + \ .$

1 . On integrating the equation, we obtain $'' - \frac{2}{3}(\ ')^2 = \frac{1}{2}a\ ^2 + \ + \ .$ The substitution $= \ ^3$ leads to a solvable equation of the form 2.8.1.5: $'' = \frac{1}{3}\left(\frac{1}{2}a\ ^2 + \ + \ \right)\ ^{-5}.$

2 . Particular solution:

$$= \ _1\ ^3 + \ _2\ ^2 + \ _3\ + \ _4,$$

where the constants $_1$, $_2$, $_3$, and $_4$ are related by two constraints

$$4\ _1\ _3 - \tfrac{4}{3}\ _2^2 = a,$$
$$6\ _1\ _4 - \tfrac{2}{3}\ _2\ _3 = \ .$$

47. $+ \frac{1}{2} \quad = \ ^{-5\ 3}.$

The transformation $= (\ ),\ \ = (\ ')^2$ leads to an equation of the form 4.2.1.1: $2\ '''' = A\ ^{-5\ 3}.$

48. $\quad = \quad + \quad .$

Integrating yields a second-order constant coefficient linear equation: $\;'' = \quad - a.$

Solutions:

$$= {}_1 \sinh(\;_3\;) + \;_2 \cosh(\;_3\;) + a \;_3^{-2},$$
$$= {}_1 \sin(\;_3\;) + \;_2 \cos(\;_3\;) - a \;_3^{-2},$$
$$= -\tfrac{1}{2} a \;^2 + \;_1 \; + \;_2.$$

49. $\quad - 2 \quad + 2 \quad - \quad = 0.$

Integrating yields a second-order equation: $\;'' = \;^2 + a\;$. The transformation

$$= \frac{1}{k^2}\;, \qquad = kz, \quad \text{where} \quad k = \frac{6}{a}\;^{1\;5},$$

leads to the first Painlevé transcendent: $\;'' = \;^2 + 6z$ (see Paragraph 2.8.2-2).

There is also the trivial solution $\; = 0$.

50. $\quad + 3 \quad + 4 \quad + \;^2 = 0.$

Multiplying by $\;^2$, we arrive at an exact differential equation. Integrating it yields Yermakov's equation 2.9.1.2: $\;'' + a \; = \;^{-3}$.

There is also the trivial solution $\; = 0$.

51. $\quad + \frac{1}{2} \quad = \quad \overline{} \quad + \quad + \quad \overline{} + \quad + \; .$

The transformation $\; = \;(\;),\; = (\;')^2$ leads to a fourth-order constant coefficient nonhomogeneous linear equation:

$$2\;'''' = 2k\;''' + 2\;'' \quad a\;' + \quad + \;.$$

Here, the plus sign corresponds to $\;' > 0$ and the minus sign to $\;' < 0$.

52. $\quad + 3 \quad + \quad = \quad .$

This is a special case of equation 3.5.3.10 with $\;(\;) = a\;$ and $g(\;) = \;$.

53. $\quad + 3 \quad + \;[\quad + (\;)^2] = \quad .$

This is a special case of equation 3.5.3.13 with $\;(\;) = \;$.

54. $\quad - \quad = \;[\quad - (\;)^2] + \quad + \; .$

This is a special case of equation 3.5.3.16 with $\;(\;) = a$ and $g(\;) = \; + \;$. The substitution $\; = \;'' - (\;')^2$ leads to a first-order linear equation: $\;' = a\; + \; + \;$.

55. $\quad + (3 \quad + 2 \quad) \quad + 2\;(\;)^2 + \;^2 \quad = \quad .$

This is a special case of equation 3.5.3.29 with $\;(\;) = e\;$ and $g(\;) = \; e\;$.

56. $\quad + (3 \quad + \quad) \quad + \quad (\;)^2 = 0.$

This is a special case of equation 3.5.3.17 with $\;(\;) = a\;$.

57. $(\; + \;) \quad + \quad + \quad = 0.$

This is a special case of equation 3.5.3.21 with $\;(\;) = \;$.

58. $(\; + \;) \quad - \quad + \quad = 0.$

A solution of this equation is any function that solves the first-order linear equation $\;' = \;_1\; + (a\;_1 + \;)\;$.

Particular solution: $\; = \;_2 \exp(\;_1\;) - \dfrac{a\;_1 +}{\;_1^2}(\;_1\; + 1).$

59. $(\; + \;) \quad = (\; + \;) \quad + \quad - \; .$

1 . Integrating yields a second-order constant coefficient equation: $\;'' + \; = -(a \; + \;) \; .$

2 . Solutions:

$$= \;_1 \sin(\;_3 \;) + \;_2 \cos(\;_3 \;) - (a + \; _3^{-2}) \qquad \text{if} \quad = \; _3^2 > 0,$$
$$= \;_1 \sinh(\;_3 \;) + \;_2 \cosh(\;_3 \;) - (a - \; _3^{-2}) \qquad \text{if} \quad = - \; _3^2 < 0,$$
$$= -\tfrac{1}{6} \; ^3 + \;_1 \; + \;_2 \qquad \text{if} \quad = 0.$$

60. $(\; + \; + \;) \quad - \tfrac{1}{3} \quad + \quad = \quad + s.$

Particular solution:

$$= \;_1 \; ^3 + \;_2 \; ^2 + \;_3 \; + \;_4,$$

where the constants $_1$, $_2$, $_3$, and $_4$ are related by two constraints

$$4 \;_1 \;_3 - \tfrac{4}{3} \; _2^2 + 6(a + \;) \;_1 = k,$$
$$6 \;_1 \;_4 - \tfrac{2}{3} \;_2 \;_3 + 6 \;_1 + 2 \;_2 = \; .$$

61. $(\; + \; + \;) \quad + 3(\; + \;) \quad = \quad .$

This is a special case of equation 3.5.3.23 with $(\;) = \quad .$

62. $\quad = \quad + \quad .$

Integrating yields a second-order linear equation of the form 2.1.2.7: $\;'' = \quad .$

63. $\quad = (\; + \quad) \quad .$

Integrating yields the Emden–Fowler equation: $\;'' = \quad ^b$ (see Section 2.3).

64. $(\quad + 3 \quad) + \left[\quad + (\;)^2 \right] = \quad .$

This is a special case of equation 3.5.3.25 with $(\;) = \quad .$

65. $^2 \quad + (3 \quad + 2 \;) \quad + 2 \; (\;)^2 + (\; - 1) \quad = \quad .$

This is a special case of equation 3.5.3.29 with $(\;) = \quad$ and $g(\;) = \quad ^{+ \; -2}$.

66. $^2 \quad - 3 \quad + 2(\;)^3 = \quad ^3.$

This is a special case of equation 3.5.3.26 with $(\;) = a \quad .$

67. $^2 \quad + 3 \quad + (\; - 1)(\;)^3 = \quad ^{k \; 2-} \; .$

This is a special case of equation 3.5.3.27 with $(\;) = a \quad$ and $= \; + 1.$

68. $\quad - (\;)^2 = \quad - (\;)^2 + \; .$

1 . Particular solution:

$$= \;_1 \exp(\;_3 \;) + \;_2 \exp(- \;_3 \;),$$

where the constants $_1$, $_2$, and $_3$ are related by the constraint $4 \;_1 \;_2(\; _3^4 + a \; _3^2) + \; = 0.$

2 . Particular solution:

$$= \;_1 \cos(\;_3 \;) + \;_2 \sin(\;_3 \;),$$

where the constants $_1$, $_2$, and $_3$ are related by the constraint $(\; _1^2 + \; _2^2)(\; _3^4 - a \; _3^2) + \; = 0.$

3 . Particular solutions: $= \quad \overline{\quad a +} \; .$

3.4.1-4. Other equations.

69. **2 − ( )^2 = ( )^2 + ^{2} + + .**

Differentiating with respect to and dividing by ′, we arrive at a fourth-order constant coefficient linear equation: 2 ′′′′ = 2λ ′′ + 2a + .

70. **2 − 3( )^2 = $^{4-2}$ ( ) .**

This is a special case of equation 3.5.4.14 with (ξ) = aξ .

71. **2 − 3( )^2 = ^{2} $^{-8}$ $^{4-2}$ ( ) .**

This is a special case of equation 3.5.4.16 with (ξ) = aξ .

72. **2 − 3( )^2 = $^{-4}$ $^{4-2}$ ( ) .**

This is a special case of equation 3.5.4.15 with (ξ) = aξ .

73. **2 − 3( )^2 = ( )^2 + ( )^4.**

This is a special case of equation 3.5.4.7 with ( ) = a and g( ) = .

74. **2 − 3( )^2 = ( )^4 + $^{-1}$ ( )^7 ^{2}.**

This is a special case of equation 3.5.4.9 with ( ) = a and g( ) = .

75. **2 − 3( )^2 = ( )^2 + $^{-1}$( )^5 ^{2}.**

This is a special case of equation 3.5.4.8 with ( ) = a and g( ) = .

76. **2 − ( )^2 = ( )^2 + ^{2} + 2 + .**

This is a special case of equation 3.5.4.5 with ( ) = −λ .

77. **− 3 ( )^2 + 3 = ( )^4 + ( )^5.**

This is a special case of equation 3.5.4.11 with ( ) = a and g( ) = .

78. **= $^{-2}$ $^{-5}$(−) ( )^3.**

This is a special case of equation 3.5.4.18 with (ξ) = aξ .

79. **= $^{-4}$ $^{-5}$(−) ( )^3.**

This is a special case of equation 3.5.4.19 with (ξ) = aξ .

80. **= [$^{-5}$ + ^{3}(−)]( )^3.**

This is a special case of equation 3.5.4.17 with (ξ) = ξ .

81. **= [( ) + ( ) + ( )^k]( )^3 + s( ) ( )^2.**

This is a special case of equation 3.5.4.20 with (ξ) = aξ , g(ξ) = ξ , (ξ) = ξ , and (ξ) = ξ .

82. **= ( ) + ^{k}(−) + ^{s}.**

The substitution ( ) = ′ − leads to a second-order equation.

83. **+ = (−) ( ) .**

This is a special case of equation 3.5.5.5 with (z) = az and g(ξ) = ξ .

84. $= \quad (\ ) \ (\quad - \ ) (\quad)^{k}$**.**

The Legendre transformation $= \ '$, $= \ ' -$ $(\ ' = \ )$ leads to the equation $''' = -A \quad (\ ') \ (\ '')^{3-}$, which is discussed in Section 3.2.

85. $= \ (\quad - \ ) \ (\quad)^{2} + \quad (\quad - \ ) \ (\quad)^{k}$**.**

This is a special case of equation 3.5.4.21 with $(\xi) = a\xi$ and $g(\xi) = \ \xi$.

86. $= \quad + \ ^{- -} \ (\ ) \ (\ ) \ [\quad - (\ )^{2}]$**.**

This is a special case of equation 3.5.5.7 with $(\xi) = a\xi$ and $g(\xi) = \xi$.

87. $(\quad)^{2} = \ (\ ^{2} \quad -2 \quad +2 \ ) + \quad + \ .$

Differentiating with respect to , we obtain $''' \ (2 \ '''' \quad -a \ ^{2} - \ ) = 0$. Equating the second factor to zero and integrating, we find the solution:

$$= \tfrac{1}{720} a \ ^{6} + \tfrac{1}{48} \ ^{4} + \ _{3} \ ^{3} + \ _{2} \ ^{2} + \ _{1} \ + \ _{0}.$$

The integration constants and the parameters a, , and are related by:

$$36 \ _{3}^{2} = 2a \ _{0} + 2 \ _{2} + \ .$$

This constraint is obtained by substituting the above solution into the original equation. In addition, to the first factor there corresponds the solution $= \ _{2} \ ^{2} + \ _{1} \ + \ _{0}$, where the constants are related by the constraint $2a \ _{0} + 2 \ _{2} + \ = 0$.

3.4.2. Equations Containing Exponential Functions

3.4.2-1. Equations of the form $''' = \ (\ ,\ ,\ ')$.

1. $= \ ^{\lambda}$ **.**

Autonomous equation. This is a special case of equation 3.5.1.1 with $(\ ) = ae^{\lambda}$. The substitution $(\ ) = (\ ')^{2}$ leads to a second-order equation: $'' = 2ae^{\lambda} \ ^{-1\ 2}$. The transformation $z = e^{\lambda} \ ^{-3\ 2}$, $= \ '$ leads to a first-order equation: $z\left(\lambda - \tfrac{3}{2}\right) \ ' = 2az - \ ^{2}$.

2. $= \ ^{\lambda \ +}$ **.**

The substitution $= \ + (\beta \ \lambda)$ leads to an autonomous equation of the form 3.4.2.1: $''' = ae^{\lambda w}$.

3. $= \ ^{\lambda}$ **.**

The transformation $z = e^{\lambda} \ ^{-1}$, $= \ '$ leads to a second-order equation.

4. $= \ ^{\lambda}$ **.**

The transformation $z = \ ^{+3} e^{\lambda}$, $= \ '$ leads to a second-order equation.

5. $= (\ + \quad + \ ) \ -$ **.**

The substitution $= \ + \ e \ +$ leads to the equation $''' = a$, whose solvable cases are outlined in Subsection 3.2.2.

6. $= \ ^{\lambda}$ **.**

Solution: $_{3} \quad = \left(\ _{2} \ + \ _{1} + 2a\lambda^{-2} e^{\lambda} \ ^{-1\ 2} \right.$.

7. $= \quad ^{\lambda} \quad + \quad .$

This is a special case of equation 3.5.2.2 with $(\ ) = ae^{\lambda}$ and $g(\ ) = \ e$. Integrating yields a second-order equation: $'' = \dfrac{a}{\lambda} e^{\lambda} \ + \text{—} e \ + \ .$

8. $= \quad ^{\lambda} \quad + \quad (\ \ )^3.$

This is a special case of equation 3.5.2.3 with $(\ ) = ae^{\lambda}$ and $g(\ ) = \ e$.

9. $= \quad ^{\lambda} \quad + \quad (\ \ )^3.$

This is a special case of equation 3.5.2.3 with $(\ ) = ae^{\lambda}$ and $g(\ ) =$.

10. $= \quad + \ ^{\lambda} (\ \ )^3.$

This is a special case of equation 3.5.2.3 with $(\ ) = a$ and $g(\ ) = \ e^{\lambda}$.

11. $= \ ^{\lambda} (\ \ )^3 + (\ \ )^{-5}.$

This is a special case of equation 3.5.2.4 with $(\ ) = \ e^{\lambda}$.

12. $= 2 \ ^2(\ \ )^3 + \ ^{\lambda} \ (\ \ ) \ ^{-5}.$

This is a special case of equation 3.5.2.34 with $(\xi) = a\xi \ ^{-6}$.

13. $= \ ^3 + \ ^{\lambda} (\ \ - \ \ ) .$

The substitution $= \ ' - a$ leads to a second-order equation: $'' + a \ ' + a^2 \ = e^{\lambda}$.

14. $= \ ^{\lambda} (\ \ - \ ) .$

The substitution $= \ ' -$ leads to a second-order equation: $(\ '\ \ )' = ae^{\lambda}$.

15. $= \ ^{\lambda} (\ \ - 2 \ ) .$

The substitution $= \ ' - 2$ leads to a second-order equation: $'' = a \ e^{\lambda}$.

3.4.2-2. Other equations.

16. $= -3 \quad + \quad + \quad ^{+1} + 2 \ .$

This is a special case of equation 3.5.3.33 with $(\xi) = a\xi$.

17. $+ 3 \quad + \ ^2(\ \ )^3 = \ ^{-\lambda} .$

This is a special case of equation 3.5.3.1 with $(\ ) = ae$.

18. $+ 3 \quad + \ ^2(\ \ )^3 = \ ^{-\lambda} .$

This is a special case of equation 3.5.3.1 with $(\ ) = a$.

19. $+ (1 - \ \ ) \quad = \ ^2 \ (\ \ - \ ) .$

This is a special case of equation 3.5.3.44 with $(\xi) = \ \xi$.

20. $+ 3 \quad = \ ^{\lambda} .$

Solution: $^2 = \ _2 \ ^2 + \ _1 \ + \ _0 + 2a\lambda^{-3} e^{\lambda}$.

21. $+ 3 \quad = \ ^{\lambda} + \ .$

This is a special case of equation 3.5.3.8 with $(\ ) = ae^{\lambda} \ + \ .$

22. $+3 \quad + \;\lambda \quad = \quad .$

This is a special case of equation 3.5.3.10 with $(\;) = ae^{\lambda}$ and $g(\;) = \; e \;$.

23. $+3 \quad + \;\lambda \quad = \quad .$

This is a special case of equation 3.5.3.10 with $(\;) = ae^{\lambda}$ and $g(\;) = \quad$.

24. $+3 \quad + \quad = \;\lambda\; .$

This is a special case of equation 3.5.3.10 with $(\;) = a$ and $g(\;) = \; e^{\lambda}$.

25. $+3 \quad + \big[\quad + (\;)^2 \big] = \;\lambda\; .$

This is a special case of equation 3.5.3.13 with $(\;) = \; e^{\lambda}$.

26. $- \quad = \big[\quad - (\;)^2 \big] + \;\lambda\; + \; .$

This is a special case of equation 3.4.3.16 with $(\;) = a$ and $g(\;) = \; e^{\lambda} \; + \;$. The substitution $= \;'' - (\;')^2$ leads to a first-order linear equation: $' = a \; + \; e^{\lambda} \; + \;$.

27. $+(3 \quad +2 \quad) \quad +2\;(\;)^2 + \;^2 \quad = \;\lambda\; .$

This is a special case of equation 3.5.3.29 with $(\;) = e$ and $g(\;) = \; e^{(\lambda+\;)}$.

28. $+(3 \quad + \;\lambda\;) \quad + \;\lambda\;(\;)^2 = 0.$

This is a special case of equation 3.5.3.17 with $(\;) = ae^{\lambda}$.

29. $(\;+\;) \quad + \quad + \;\lambda\; = 0.$

This is a special case of equation 3.5.3.21 with $(\;) = \; e^{\lambda}$.

30. $(\;+\quad+\;) \quad +3(\quad+\;) \quad = \;\lambda\; .$

Solution: $(\;+a\;+\;)^2 = \;_2\;^2 + \;_1\; + \;_0 + 2k\lambda^{-3}e^{\lambda}$.

31. $^2 \quad -(\;)^3 + \;^2 \quad = \;\lambda\; .$

Integrating yields a second-order equation: $^2\;'' - \;(\;')^2 + \frac{1}{3}a\;^3 = \;\lambda^{-1}e^{\lambda}\; + \;$. For $\; = 0$, we have an equation of the form 2.8.3.57 with $k = -1$: $\;'' - (\;')^2 + \frac{1}{3}a\;^2 = \;\lambda^{-1}e^{\lambda}\;^{-1}$.

32. $^2 \quad -(\;)^3 + \;^2 \quad = \; \exp(\;^2).$

Integrating yields a second-order equation: $^2\;'' - \;(\;')^2 + \frac{1}{3}a\;^3 = \frac{1}{2}\;\lambda^{-1}\exp(\lambda\;^2) + \;$. For $\; = 0$, we have an equation of the form 2.8.3.5.7 with $k = -1$: $\;'' - (\;')^2 + \frac{1}{3}a\;^2 = \frac{1}{2}\;\lambda^{-1}\exp(\lambda\;^2)\;^{-1}$.

33. $^2 \quad -3 \quad +2(\;)^3 = \;\lambda\;^3.$

Solution: $\ln|\;| = \;_2\;^2 + \;_1\; + \;_0 + a\lambda^{-3}e^{\lambda}$.

34. $^2 \quad +3 \quad + \;(\;-1)(\;)^3 = \;\lambda\;^{2-}\; .$

This is a special case of equation 3.5.3.27 with $(\;) = ae^{\lambda}$ and $\; = \; + 1$.

35. $^2 \quad + \;(3 \quad +2 \quad) \quad +2\;(\;)^2 + \;(\;-1) \quad = \;\lambda\; .$

This is a special case of equation 3.5.3.29 with $(\;) = \;$ and $g(\;) = \;^{-2}e^{\lambda}$.

36. $2 \quad -(\;)^2 = \;\lambda\;(\;)^2 + \;^2 + 2\; + \; .$

This is a special case of equation 3.5.4.5 with $(\;) = -ke^{\lambda}$.

37. $\mathbf{2 \qquad - 3(\quad)^2 = \quad ^{\lambda} \, (\quad)^2 + \qquad (\quad)^4.}$

This is a special case of equation 3.5.4.7 with $(\ ) = ae^{\lambda}$ and $g(\ ) = \ e \ $.

38. $\mathbf{2 \qquad - 3(\quad)^2 = \quad ^{\lambda} \, (\quad)^2 + \qquad (\quad)^4.}$

This is a special case of equation 3.5.4.7 with $(\ ) = ae^{\lambda}$ and $g(\ ) = \qquad$.

39. $\mathbf{2 \qquad - 3(\quad)^2 = \qquad (\quad)^2 + \qquad (\quad)^4.}$

This is a special case of equation 3.5.4.7 with $(\ ) = a \qquad$ and $g(\ ) = \ e \ $.

40. $\mathbf{2 \qquad - 3(\quad)^2 = \quad ^{\lambda} \, (\quad)^2 + \qquad ^{-1}(\quad)^{5 \ 2}.}$

This is a special case of equation 3.5.4.8 with $(\ ) = ae^{\lambda}$ and $g(\ ) = \ e \ $.

41. $\mathbf{2 \qquad - 3(\quad)^2 = \quad ^{\lambda} \, (\quad)^2 + \qquad ^{-1}(\quad)^{5 \ 2}.}$

This is a special case of equation 3.5.4.8 with $(\ ) = ae^{\lambda}$ and $g(\ ) = \qquad$.

42. $\mathbf{2 \qquad - 3(\quad)^2 = \qquad (\quad)^2 + \quad ^{\lambda} \quad ^{-1}(\quad)^{5 \ 2}.}$

This is a special case of equation 3.5.4.8 with $(\ ) = a \qquad$ and $g(\ ) = \ e^{\lambda} \ $.

43. $\mathbf{2 \qquad - 3(\quad)^2 = \quad ^{\lambda} \, (\quad)^4 + \quad ^{-1} \quad (\quad)^{7 \ 2}.}$

This is a special case of equation 3.5.4.9 with $(\ ) = ae^{\lambda}$ and $g(\ ) = \ e \ $.

44. $\mathbf{2 \qquad - 3(\quad)^2 = \quad ^{\lambda} \, (\quad)^4 + \quad ^{-1} \quad (\quad)^{7 \ 2}.}$

This is a special case of equation 3.5.4.9 with $(\ ) = ae^{\lambda}$ and $g(\ ) = \qquad$.

45. $\mathbf{2 \qquad - 3(\quad)^2 = \qquad (\quad)^4 + \quad ^{-1} \ ^{\lambda} \, (\quad)^{7 \ 2}.}$

This is a special case of equation 3.5.4.9 with $(\ ) = a \qquad$ and $g(\ ) = \ e^{\lambda} \ $.

3.4.3. Equations Containing Hyperbolic Functions

3.4.3-1. Equations with hyperbolic sine.

1. $\mathbf{\qquad = \ \sinh(\quad) \quad .}$

Solution: $\quad _3 \quad = \left[\ _2 \ + \ \ _1 + 2a\lambda^{-2} \sinh(\lambda \ ) \right]^{-1 \ 2} \quad .$

2. $\mathbf{\qquad = \ \sinh(\quad) \quad + \ \sinh(\quad).}$

This is a special case of equation 3.5.2.2 with $(\ ) = a \sinh(\lambda \ )$ and $g(\ ) = \ \sinh(\quad)$.

Integrating yields a second-order equation: $\quad '' \ = \dfrac{a}{\lambda} \cosh(\lambda \ ) + \text{—} \cosh(\quad) + \quad .$

3. $\mathbf{\qquad = \ \sinh \ (\quad) \quad + \ \sinh(\quad)(\quad)^3.}$

This is a special case of equation 3.5.2.3 with $(\ ) = a \sinh \ (\lambda \ )$ and $g(\ ) = \ \sinh(\quad)$.

4. $\mathbf{\qquad = \tfrac{1}{2} \ ^2(\quad)^3 + \ (\sinh \quad)^{-} \ ^{-3}(\quad)^{2 \ \ +1}.}$

This is a special case of equation 3.5.2.36 with $(\xi) = a\xi^2 \quad .$

5. $\mathbf{\qquad + 3 \qquad = \ \sinh(\quad).}$

Solution: $\quad ^2 = \ _2 \ ^2 + \ _1 \ + \ _0 + 2a\lambda^{-3} \cosh(\lambda \ ).$

6. $\mathbf{\qquad + 3 \qquad = \ \sinh \ (\quad) + \ .}$

This is a special case of equation 3.5.3.6 with $(\ ) = a \sinh \ (\lambda \ ) + \ .$

7. $+3 \quad = \quad \mathbf{sinh} \; (\quad) + \; .$

This is a special case of equation 3.5.3.8 with $(\ ) = a \sinh \ (\lambda \ ) + \ .$

8. $+3 \quad + \quad = \quad \mathbf{sinh} \; (\quad).$

This is a special case of equation 3.5.3.10 with $(\ ) = a$ and $g(\ ) = \ \sinh \ (\lambda \ ).$

9. $+3 \quad + \big[\quad + (\quad)^2 \big] = \quad \mathbf{sinh} \; (\quad).$

This is a special case of equation 3.5.3.13 with $(\ ) = \ \sinh \ (\lambda \ ).$

10. $+(3 \quad + \quad \mathbf{sinh} \quad) \quad + \quad \mathbf{sinh} \quad (\quad)^2 = 0.$

This is a special case of equation 3.5.3.17 with $(\ ) = a \sinh \quad .$

11. $(\ + \ ) \quad + \quad + \ \mathbf{sinh} \; (\quad) \quad = 0.$

This is a special case of equation 3.5.3.21 with $(\ ) = \ \sinh \ (\lambda \ ).$

12. $^2 \quad +3 \quad + \quad (\quad -1)(\quad)^3 = \quad \mathbf{sinh}^k(\quad) \quad ^{2-} \quad .$

This is a special case of equation 3.5.3.27 with $(\ ) = a \sinh \ (\lambda \ )$ and $= \quad +1.$

3.4.3-2. Equations with hyperbolic cosine.

13. $= \quad \mathbf{cosh}(\quad) \quad .$

Solution: $_3 \quad = \big[\ _2 \ + \ _1 + 2a\lambda^{-2} \cosh(\lambda \ ) \big]^{-1 \ 2} \quad .$

14. $= \quad \mathbf{cosh}(\quad) \quad + \quad \mathbf{cosh}(\quad).$

This is a special case of equation 3.5.2.2 with $(\ ) = a \cosh(\lambda \ )$ and $g(\ ) = \ \cosh(\quad).$
Integrating yields a second-order equation: $'' \ = \dfrac{a}{\lambda} \sinh(\lambda \ ) + \text{—} \sinh(\quad) + \quad .$

15. $= \quad \mathbf{cosh} \; (\quad) \quad + \quad \mathbf{cosh}(\quad)(\quad)^3.$

This is a special case of equation 3.5.2.3 with $(\ ) = a \cosh \ (\lambda \ )$ and $g(\ ) = \ \cosh(\quad).$

16. $= \quad + \quad \mathbf{cosh}(\quad)(\quad)^3.$

This is a special case of equation 3.5.2.3 with $(\ ) = a$ and $g(\ ) = \ \cosh(\quad).$

17. $= \quad \mathbf{cosh}(\quad)(\quad)^3 + \quad (\quad)^{-5}.$

This is a special case of equation 3.5.2.4 with $(\ ) = \ \cosh(\lambda \ ).$

18. $= \tfrac{1}{2} \ ^2(\quad)^3 + \quad (\mathbf{cosh} \quad)^{-} \ ^{-3}(\quad)^2 \ ^{+1}.$

This is a special case of equation 3.5.2.35 with $(\xi) = a\xi^2 \quad .$

19. $+3 \quad = \quad \mathbf{cosh}(\quad).$

Solution: $^2 = \ _2 \ ^2 + \ _1 \ + \ _0 + 2a\lambda^{-3} \sinh(\lambda \ ).$

20. $+3 \quad = \quad \mathbf{cosh} \; (\quad).$

This is a special case of equation 3.5.3.6 with $(\ ) = a \cosh \ (\lambda \ ).$

21. $+3 \quad = \quad \mathbf{cosh} \; (\quad) + \; .$

This is a special case of equation 3.5.3.8 with $(\ ) = a \cosh \ (\lambda \ ) + \ .$

22. $+3 \quad + \quad = \ \cosh\ (\ \ )$.

This is a special case of equation 3.5.3.10 with $(\ ) = a$ and $g(\ ) = \ \cosh\ (\lambda\ )$.

23. $+3 \quad + \ \big[\quad + (\ \ )^2\big] = \ \cosh\ (\ \ )$.

This is a special case of equation 3.5.3.13 with $(\ ) = \ \cosh\ (\lambda\ )$.

24. $+(3 \quad + \quad \cosh \quad) \quad + \ \cosh \quad (\ \ )^2 = 0$.

This is a special case of equation 3.5.3.17 with $(\ ) = a \cosh$.

25. $(\ + \ ) \quad + \quad + \ \cosh\ (\ \ ) \quad = 0$.

This is a special case of equation 3.5.3.21 with $(\ ) = \ \cosh\ (\lambda\ )$.

26. $^2 \quad +3 \quad + \ (\ \ -1)(\ \ )^3 = \ \cosh^k(\ \ )\ ^{2-}$.

This is a special case of equation 3.5.3.27 with $(\ ) = a \cosh\ (\lambda\ )$ and $= \ +1$.

27. $2 \quad -3(\ \ )^2 = \ \cosh\ (\ \ )(\ \ )^2 + \quad (\ \ )^4$.

This is a special case of equation 3.5.4.7 with $(\ ) = a \cosh\ (\lambda\ )$ and $g(\ ) =$.

28. $2 \quad -3(\ \ )^2 = \quad (\ \ )^2 + \ \cosh\ (\ \ )(\ \ )^4$.

This is a special case of equation 3.5.4.7 with $(\ ) = a$ and $g(\ ) = \ \cosh\ (\lambda\ )$.

29. $2 \quad -3(\ \ )^2 = \ \cosh\ (\ \ )(\ \ )^2 + \ \cosh\ (\ \ )^{-1}(\ \ )^{5\ 2}$.

This is a special case of equation 3.5.4.8 with $(\ ) = a \cosh\ (\lambda\ )$ and $g(\ ) = \ \cosh\ (\ \ )$.

30. $2 \quad -3(\ \ )^2 = \ \cosh\ (\ \ )(\ \ )^2 + \quad ^{-1}(\ \ )^{5\ 2}$.

This is a special case of equation 3.5.4.8 with $(\ ) = a \cosh\ (\lambda\ )$ and $g(\ ) =$.

31. $2 \quad -3(\ \ )^2 = \quad (\ \ )^2 + \ \cosh\ (\ \ )^{-1}(\ \ )^{5\ 2}$.

This is a special case of equation 3.5.4.8 with $(\ ) = a$ and $g(\ ) = \ \cosh\ (\lambda\ )$.

32. $2 \quad -3(\ \ )^2 = \ \cosh\ (\ \ )(\ \ )^4 + \ ^{-1}\cosh\ (\ \ )(\ \ )^{7\ 2}$.

This is a special case of equation 3.5.4.9 with $(\ ) = a \cosh\ (\lambda\ )$ and $g(\ ) = \ \cosh\ (\ \ )$.

33. $2 \quad -3(\ \ )^2 = \ \cosh\ (\ \ )(\ \ )^4 + \ ^{-1}\ (\ \ )^{7\ 2}$.

This is a special case of equation 3.5.4.9 with $(\ ) = a \cosh\ (\lambda\ )$ and $g(\ ) =$.

34. $2 \quad -3(\ \ )^2 = \quad (\ \ )^4 + \ ^{-1}\cosh\ (\ \ )(\ \ )^{7\ 2}$.

This is a special case of equation 3.5.4.9 with $(\ ) = a$ and $g(\ ) = \ \cosh\ (\lambda\ )$.

3.4.3-3. Equations with hyperbolic tangent.

35. $= \ \tanh(\ \ )$.

This is a special case of equation 3.5.2.1 with $(\ ) = a \tanh(\lambda\ )$.

36. $= \ \tanh(\ \ ) \ + \ \tanh(\ \ )$.

This is a special case of equation 3.5.2.2 with $(\ ) = a \tanh(\lambda\ )$ and $g(\ ) = \ \tanh(\ \ )$.

37. $+3 \quad = \ \tanh\ (\ \ ) + $.

This is a special case of equation 3.5.3.6 with $(\ ) = a \tanh\ (\lambda\ ) +$.

38. $+ 3 \quad = \quad \tanh (\quad) + \; .$

This is a special case of equation 3.5.3.8 with ( ) = a tanh (λ) + .

39. $+ 3 \quad + \quad = \quad \tanh (\quad).$

This is a special case of equation 3.5.3.10 with ( ) = a and g( ) = tanh (λ).

40. $+ 3 \quad + \left[\quad + (\quad)^2 \right] = \quad \tanh (\quad).$

This is a special case of equation 3.5.3.13 with ( ) = tanh (λ).

41. $+ (3 \quad + \quad \tanh \quad) \quad + \quad \tanh \quad (\quad)^2 = 0.$

This is a special case of equation 3.5.3.17 with ( ) = a tanh .

42. $(\; + \;) \quad + \quad + \quad \tanh (\quad) \quad = 0.$

This is a special case of equation 3.5.3.21 with ( ) = tanh (λ).

43. $^2 \quad + 3 \quad + \quad (\quad - 1)(\quad)^3 = \quad \tanh^k (\quad) \quad ^{2-} \; .$

This is a special case of equation 3.5.3.27 with ( ) = a tanh (λ) and = + 1.

44. $2 \quad - 3(\quad)^2 = \quad \tanh (\quad)(\quad)^2 + \quad \tanh (\quad)(\quad)^4.$

This is a special case of equation 3.5.4.7 with ( ) = a tanh (λ) and g( ) = tanh ( ).

45. $2 \quad - 3(\quad)^2 = \quad \tanh (\quad)(\quad)^2 + \quad (\quad)^4.$

This is a special case of equation 3.5.4.7 with ( ) = a tanh (λ) and g( ) = .

46. $2 \quad - 3(\quad)^2 = \quad (\quad)^2 + \quad \tanh (\quad)(\quad)^4.$

This is a special case of equation 3.5.4.7 with ( ) = a and g( ) = tanh (λ).

47. $2 \quad - 3(\quad)^2 = \quad \tanh (\quad)(\quad)^2 + \quad \tanh (\quad) \quad ^{-1}(\quad)^{5 \; 2}.$

This is a special case of equation 3.5.4.8 with ( ) = a tanh (λ) and g( ) = tanh ( ).

48. $2 \quad - 3(\quad)^2 = \quad \tanh (\quad)(\quad)^2 + \quad ^{-1}(\quad)^{5 \; 2}.$

This is a special case of equation 3.5.4.8 with ( ) = a tanh (λ) and g( ) = .

49. $2 \quad - 3(\quad)^2 = \quad (\quad)^2 + \quad \tanh (\quad) \quad ^{-1}(\quad)^{5 \; 2}.$

This is a special case of equation 3.5.4.8 with ( ) = a and g( ) = tanh (λ).

50. $2 \quad - 3(\quad)^2 = \quad \tanh (\quad)(\quad)^4 + \quad ^{-1} \tanh (\quad)(\quad)^{7 \; 2}.$

This is a special case of equation 3.5.4.9 with ( ) = a tanh (λ) and g( ) = tanh ( ).

51. $2 \quad - 3(\quad)^2 = \quad \tanh (\quad)(\quad)^4 + \quad ^{-1} \quad (\quad)^{7 \; 2}.$

This is a special case of equation 3.5.4.9 with ( ) = a tanh (λ) and g( ) = .

52. $2 \quad - 3(\quad)^2 = \quad (\quad)^4 + \quad ^{-1} \tanh (\quad)(\quad)^{7 \; 2}.$

This is a special case of equation 3.5.4.9 with ( ) = a and g( ) = tanh (λ).

3.4.3-4. Equations with hyperbolic cotangent.

53. = coth( ) .

This is a special case of equation 3.5.2.1 with ( ) = a coth(λ).

54. = coth( ) + coth( ).

This is a special case of equation 3.5.2.2 with ( ) = a coth(λ) and g( ) = coth( ).

55. + 3 = coth ( ) + .

This is a special case of equation 3.5.3.6 with ( ) = a coth (λ) + .

56. + 3 = coth ( ) + .

This is a special case of equation 3.5.3.8 with ( ) = a coth (λ) + .

57. + 3 + = coth ( ).

This is a special case of equation 3.5.3.10 with ( ) = a and g( ) = coth (λ).

58. + 3 + [+ ( )^2] = coth ( ).

This is a special case of equation 3.5.3.13 with ( ) = coth (λ).

59. + (3 + coth) + coth ( )^2 = 0.

This is a special case of equation 3.5.3.17 with ( ) = a coth .

60. (+) + + coth ( ) = 0.

This is a special case of equation 3.5.3.21 with ( ) = coth (λ).

61. ^{2} + 3 + (− 1)( )^3 = coth^k( ) $^{2-}$.

This is a special case of equation 3.5.3.27 with ( ) = a coth (λ) and = + 1.

62. 2 − 3( )^2 = coth ( )( )^2 + coth ( )( )^4.

This is a special case of equation 3.5.4.7 with ( ) = a coth (λ) and g( ) = coth ( ).

63. 2 − 3( )^2 = coth ( )( )^2 + ( )^4.

This is a special case of equation 3.5.4.7 with ( ) = a coth (λ) and g( ) = .

64. 2 − 3( )^2 = ( )^2 + coth ( )( )^4.

This is a special case of equation 3.5.4.7 with ( ) = a and g( ) = coth (λ).

65. 2 − 3( )^2 = coth ( )( )^2 + coth ( ) $^{-1}$( )$^{5\ 2}$.

This is a special case of equation 3.5.4.8 with ( ) = a coth (λ) and g( ) = coth ( ).

66. 2 − 3( )^2 = coth ( )( )^2 + $^{-1}$( )$^{5\ 2}$.

This is a special case of equation 3.5.4.8 with ( ) = a coth (λ) and g( ) = .

67. 2 − 3( )^2 = ( )^2 + coth ( ) $^{-1}$( )$^{5\ 2}$.

This is a special case of equation 3.5.4.8 with ( ) = a and g( ) = coth (λ).

68. 2 − 3( )^2 = coth ( )( )^4 + $^{-1}$ coth ( )( )$^{7\ 2}$.

This is a special case of equation 3.5.4.9 with ( ) = a coth (λ) and g( ) = coth ( ).

69. 2 − 3( )^2 = coth ( )( )^4 + $^{-1}$ ( )$^{7\ 2}$.

This is a special case of equation 3.5.4.9 with ( ) = a coth (λ) and g( ) = .

70. 2 − 3( )^2 = ( )^4 + $^{-1}$ coth ( )( )$^{7\ 2}$.

This is a special case of equation 3.5.4.9 with ( ) = a and g( ) = coth (λ).

3.4.4. Equations Containing Logarithmic Functions

3.4.4-1. Equations of the form ″′ = (, , ′).

1. = (+ ln).

This is a special case of equation 3.5.1.16 with (*z*) = *a* ln *z*.

2. = $^{-3}$(+ ln).

This is a special case of equation 3.5.1.17 with (*z*) = *a* ln *z*.

3. = $^{-4}$(ln − 2 ln).

This is a special case of equation 3.5.1.7 with (*z*) = *a* ln *z*.

4. = ln( ) .

This is a special case of equation 3.5.2.1 with ( ) = *a* ln(λ).

5. = ln( ) + ln( ).

This is a special case of equation 3.5.2.2 with ( ) = *a* ln(λ) and *g*( ) = ln( ).

6. = ln ( ) + ln ( )( )^3.

This is a special case of equation 3.5.2.3 with ( ) = *a* ln (λ) and *g*( ) = ln ( ).

7. = ln ( ) + ( )^3.

This is a special case of equation 3.5.2.3 with ( ) = *a* ln (λ) and *g*( ) = .

8. = + ln ( )( )^3.

This is a special case of equation 3.5.2.3 with ( ) = *a* and *g*( ) = ln (λ).

9. = $^{-3}$(ln − ln)(−).

This is a special case of equation 3.5.2.5 with (ξ) = *a* ln ξ.

10. = $^{-5}$(ln − 2 ln)(− 2).

This is a special case of equation 3.5.2.8 with (ξ) = *a* ln ξ.

11. = $^{-5}$ ^{2}(2 ln − ln).

This is a special case of equation 3.5.2.27 with (ξ) = 2*a* ln ξ.

12. = $^{-5}$ ^{4}(4 ln − ln).

This is a special case of equation 3.5.2.28 with (ξ) = 4*a* ln ξ.

13. = ^{3} + ln (−) .

This is a special case of equation 3.5.2.16 with (,) = ln .

14. = ln (−) .

This is a special case of equation 3.5.2.20 with (,) = *a* ln .

15. = ln (− 2) .

This is a special case of equation 3.5.2.21 with (,) = *a* ln .

3.4.4-2. Other equations.

16. $= -3 \quad + \ (\ + \ln \ ) \ (\quad + \ ) + 2 \ .$

This is a special case of equation 3.5.3.33 with $(\xi) = a \ln \ \xi$.

17. $= \ (\quad - \ + \ \ln \ ) \quad .$

This is a special case of equation 3.5.3.45 with $(\xi) = \ \xi$.

18. $+ 3 \quad = \ \ln \ (\quad).$

This is a special case of equation 3.5.3.6 with $(\ ) = a \ln \ (\quad)$.

19. $+ 3 \quad = \ \ln \ (\quad).$

This is a special case of equation 3.5.3.8 with $(\ ) = a \ln \ (\quad)$.

20. $+ 3 \quad + \quad = \ \ln \ (\quad).$

This is a special case of equation 3.5.3.10 with $(\ ) = a \quad$ and $g(\ ) = \ \ln \ (\lambda \ )$.

21. $+ 3 \quad + \ \left[\quad + (\quad)^2 \right] = \ \ln \ (\quad).$

This is a special case of equation 3.5.3.13 with $(\ ) = \ \ln \ (\lambda \ )$.

22. $(\ + \ ) \quad + \quad + \ \ln \ (\quad) \quad = 0.$

This is a special case of equation 3.5.3.21 with $(\ ) = \ \ln \ (\lambda \ )$.

23. $^2 \quad + 3 \quad + \ (\quad - 1)(\quad)^3 = \ \ln^k(\quad) \ ^{2-} \ .$

This is a special case of equation 3.5.3.27 with $(\ ) = a \ln \ (\quad)$ and $= \quad + 1$.

24. $2 \quad - 3(\quad)^2 = \quad ^4(\ln \quad - 2 \ln \ ).$

This is a special case of equation 3.5.4.14 with $(\xi) = a \ln \xi$.

25. $2 \quad - 3(\quad)^2 = \ \ln \ (\quad)(\quad)^2 + \ \ln \ (\quad)(\quad)^4.$

This is a special case of equation 3.5.4.7 with $(\ ) = a \ln \ (\lambda \ )$ and $g(\ ) = \ \ln \ (\quad)$.

26. $2 \quad - 3(\quad)^2 = \ \ln \ (\quad)(\quad)^2 + \quad (\quad)^4.$

This is a special case of equation 3.5.4.7 with $(\ ) = a \ln \ (\lambda \ )$ and $g(\ ) = \quad .$

27. $2 \quad - 3(\quad)^2 = \quad (\quad)^2 + \ \ln \ (\quad)(\quad)^4.$

This is a special case of equation 3.5.4.7 with $(\ ) = a \quad$ and $g(\ ) = \ \ln \ (\lambda \ )$.

28. $2 \quad - 3(\quad)^2 = \ \ln \ (\quad)(\quad)^2 + \ \ln \ (\quad) \ ^{-1}(\quad)^{5 \ 2}.$

This is a special case of equation 3.5.4.8 with $(\ ) = a \ln \ (\lambda \ )$ and $g(\ ) = \ \ln \ (\quad)$.

29. $2 \quad - 3(\quad)^2 = \ \ln \quad (\quad)^2 + \quad ^{-1}(\quad)^{5 \ 2}.$

This is a special case of equation 3.5.4.8 with $(\ ) = a \ln \quad$ and $g(\ ) = \quad .$

30. $2 \quad - 3(\quad)^2 = \quad (\quad)^2 + \ \ln \ (\quad) \ ^{-1}(\quad)^{5 \ 2}.$

This is a special case of equation 3.5.4.8 with $(\ ) = a \quad$ and $g(\ ) = \ \ln \ (\lambda \ )$.

31. $2 \quad - 3(\quad)^2 = \ \ln \ (\quad)(\quad)^4 + \quad ^{-1} \ln \ (\quad)(\quad)^{7 \ 2}.$

This is a special case of equation 3.5.4.9 with $(\ ) = a \ln \ (\lambda \ )$ and $g(\ ) = \ \ln \ (\quad)$.

32. $2 \quad - 3(\quad)^2 = \quad (\quad)^4 + \quad ^{-1} \ln \ (\quad)(\quad)^{7\ 2}.$

This is a special case of equation 3.5.4.9 with $(\ ) = a \qquad$ and $g(\ ) = \ \ln \ (\lambda \ )$.

33. $2 \quad - 3(\quad)^2 = \ \ln \ (\quad)(\quad)^4 + \quad ^{-1} \quad (\quad)^{7\ 2}.$

This is a special case of equation 3.5.4.9 with $(\ ) = a \ln \ (\lambda \ )$ and $g(\ ) = \qquad$.

34. $\quad \ln \ - \quad \ln \quad + \quad = 0.$

Integrating yields a second-order autonomous equation of the form 2.9.1.1: $\quad '' \ = e \qquad$.

3.4.5. Equations Containing Trigonometric Functions

3.4.5-1. Equations with sine.

1. $\quad = \ \sin(\quad) \quad .$

Solution: $\quad _3 \qquad = \left[\ _2 \ + \ \ _1 - 2a\lambda^{-2} \sin(\lambda \ ) \right]^{-1\ 2} \qquad .$

2. $\quad = \ \sin(\quad) \quad + \ \sin(\quad).$

This is a special case of equation 3.5.2.2 with $(\ ) = a \sin(\lambda \ )$ and $g(\ ) = \ \sin(\quad)$. Integrating yields a second-order equation: $\quad '' \ = -\dfrac{a}{\lambda} \cos(\lambda \ ) - \text{—} \cos(\quad) + \quad .$

3. $\quad = \ \sin \ (\quad) \quad + \ \sin(\quad)(\quad)^3.$

This is a special case of equation 3.5.2.3 with $(\ ) = a \sin \ (\lambda \ )$ and $g(\ ) = \ \sin(\quad)$.

4. $\quad = \frac{1}{2} \ ^2(\quad)^3 + \ (\sin \quad)^{-} \ ^{-3}(\quad)^2 \ ^{+1}.$

This is a special case of equation 3.5.2.36 with $(\xi) = a\xi^2 \quad$.

5. $\quad + 3 \qquad = \ \sin(\quad).$

Solution: $\quad ^2 = \ _2 \ ^2 + \ _1 \ + \ _0 + 2a\lambda^{-3} \cos(\lambda \ ).$

6. $\quad + 3 \qquad = \ \sin \ (\quad) + \ .$

This is a special case of equation 3.5.3.6 with $(\ ) = a \sin \ (\lambda \ ) + \ .$

7. $\quad + 3 \qquad = \ \sin \ (\quad) + \ .$

This is a special case of equation 3.5.3.8 with $(\ ) = a \sin \ (\lambda \ ) + \ .$

8. $\quad + 3 \qquad + \qquad = \ \sin \ (\quad).$

This is a special case of equation 3.5.3.10 with $(\ ) = a \qquad$ and $g(\ ) = \ \sin \ (\lambda \ )$.

9. $\quad + 3 \qquad + \left[\qquad + (\quad)^2 \right] = \ \sin \ (\quad).$

This is a special case of equation 3.5.3.13 with $(\ ) = \ \sin \ (\lambda \ )$.

10. $\quad + (3 \quad + \quad \sin \quad) \quad + \ \sin \quad (\quad)^2 = 0.$

This is a special case of equation 3.5.3.17 with $(\ ) = a \sin \quad$.

11. $(\ + \ ) \quad + \qquad + \ \sin \ (\quad) \quad = 0.$

This is a special case of equation 3.5.3.21 with $(\ ) = \ \sin \ (\lambda \ )$.

12. $\ ^2 \quad + 3 \qquad + \ (\quad - 1)(\quad)^3 = \ \sin^k(\quad) \ ^{2-} \quad .$

This is a special case of equation 3.5.3.27 with $(\ ) = a \sin \ (\lambda \ )$ and $\quad = \quad + 1.$

3.4.5-2. Equations with cosine.

13. **= cos( ) .**

Solution: $_3$ = $\left[\ _2 + \ _1 - 2a\lambda^{-2}\cos(\lambda\ )\right]^{-1\ 2}$.

14. **= cos( ) + cos( ).**

This is a special case of equation 3.5.2.2 with ( ) = $a\cos(\lambda\ )$ and $g(\ ) =$ cos( ). Integrating, we obtain a second-order equation: $'' = \frac{a}{\lambda}\sin(\lambda\ ) + \text{—}\sin(\ ) +$.

15. **= cos ( ) + cos( )( )^3.**

This is a special case of equation 3.5.2.3 with ( ) = $a\cos\ (\lambda\ )$ and $g(\ ) =$ cos( ).

16. **= + cos( )( )^3.**

This is a special case of equation 3.5.2.3 with ( ) = a and $g(\ ) =$ cos( ).

17. **= cos( )( )^3 + ( )$^{-5}$.**

This is a special case of equation 3.5.2.4 with ( ) = cos(λ).

18. **= $\frac{1}{2}$ ^{2}( )^3 + (cos)$^{-}$ $^{-3}$( )^2 $^{+1}$.**

This is a special case of equation 3.5.2.35 with $(\xi) = a\xi^2$.

19. **+ 3 = cos( ).**

Solution: $^2 = \ _2\ ^2 + \ _1 + \ _0 - 2a\lambda^{-3}\sin(\lambda\ )$.

20. **+ 3 = cos ( ).**

This is a special case of equation 3.5.3.6 with ( ) = $a\cos\ (\lambda\ )$.

21. **+ 3 = cos ( ) + .**

This is a special case of equation 3.5.3.8 with ( ) = $a\cos\ (\lambda\ ) +$.

22. **+ 3 + = cos ( ).**

This is a special case of equation 3.5.3.10 with ( ) = a and $g(\ ) =$ cos (λ).

23. **+ 3 + [+ ( )^2] = cos ( ).**

This is a special case of equation 3.5.3.13 with ( ) = cos (λ).

24. **+ (3 + cos) + cos ( )^2 = 0.**

This is a special case of equation 3.5.3.17 with ( ) = a cos .

25. **(+) + + cos ( ) = 0.**

This is a special case of equation 3.5.3.21 with ( ) = cos (λ).

26. **^{2} + 3 + (− 1)( )^3 = cos^k( ) $^{2-}$.**

This is a special case of equation 3.5.3.27 with ( ) = $a\cos\ (\lambda\ )$ and = + 1.

27. **2 − 3( )^2 = cos ( )( )^2 + ( )^4.**

This is a special case of equation 3.5.4.7 with ( ) = $a\cos\ (\lambda\ )$ and $g(\ ) =$.

28. $\mathbf{2 \qquad -3(\quad)^2 = \quad (\quad)^2 + \ \cos\ (\quad)(\quad)^4.}$
This is a special case of equation 3.5.4.7 with $(\ ) = a \quad$ and $g(\ ) = \ \cos\ (\lambda\ )$.

29. $\mathbf{2 \qquad -3(\quad)^2 = \ \cos\ (\quad)(\quad)^2 + \ \cos\ (\quad)\ ^{-1}(\quad)^{5\ 2}.}$
This is a special case of equation 3.5.4.8 with $(\ ) = a\cos\ (\lambda\ )$ and $g(\ ) = \ \cos\ (\quad)$.

30. $\mathbf{2 \qquad -3(\quad)^2 = \ \cos\ (\quad)(\quad)^2 + \qquad ^{-1}(\quad)^{5\ 2}.}$
This is a special case of equation 3.5.4.8 with $(\ ) = a\cos\ (\lambda\ )$ and $g(\ ) = \qquad$.

31. $\mathbf{2 \qquad -3(\quad)^2 = \quad (\quad)^2 + \ \cos\ (\quad)\ ^{-1}(\quad)^{5\ 2}.}$
This is a special case of equation 3.5.4.8 with $(\ ) = a \quad$ and $g(\ ) = \ \cos\ (\lambda\ )$.

32. $\mathbf{2 \qquad -3(\quad)^2 = \ \cos\ (\quad)(\quad)^4 + \quad ^{-1}\cos\ (\quad)(\quad)^{7\ 2}.}$
This is a special case of equation 3.5.4.9 with $(\ ) = a\cos\ (\lambda\ )$ and $g(\ ) = \ \cos\ (\quad)$.

33. $\mathbf{2 \qquad -3(\quad)^2 = \ \cos\ (\quad)(\quad)^4 + \quad ^{-1}\quad (\quad)^{7\ 2}.}$
This is a special case of equation 3.5.4.9 with $(\ ) = a\cos\ (\lambda\ )$ and $g(\ ) = \qquad$.

34. $\mathbf{2 \qquad -3(\quad)^2 = \quad (\quad)^4 + \quad ^{-1}\cos\ (\quad)(\quad)^{7\ 2}.}$
This is a special case of equation 3.5.4.9 with $(\ ) = a \quad$ and $g(\ ) = \ \cos\ (\lambda\ )$.

3.4.5-3. Equations with tangent.

35. $\mathbf{\qquad = \ \tan(\quad)\quad .}$
This is a special case of equation 3.5.2.1 with $(\ ) = a\tan(\lambda\ )$.

36. $\mathbf{\qquad = \ \tan(\quad)\quad + \ \tan(\quad).}$
This is a special case of equation 3.5.2.2 with $(\ ) = a\tan(\lambda\ )$ and $g(\ ) = \ \tan(\quad)$.

37. $\mathbf{\qquad + 3 \qquad = \ \tan\ (\quad) + \ .}$
This is a special case of equation 3.5.3.6 with $(\ ) = a\tan\ (\lambda\ ) + \ $.

38. $\mathbf{\qquad + 3 \qquad = \ \tan\ (\quad) + \ .}$
This is a special case of equation 3.5.3.8 with $(\ ) = a\tan\ (\lambda\ ) + \ $.

39. $\mathbf{\qquad + 3 \qquad + \qquad = \ \tan\ (\quad).}$
This is a special case of equation 3.5.3.10 with $(\ ) = a \quad$ and $g(\ ) = \ \tan\ (\lambda\ )$.

40. $\mathbf{\qquad + 3 \qquad + \left[\qquad + (\quad)^2\right] = \ \tan\ (\quad).}$
This is a special case of equation 3.5.3.13 with $(\ ) = \ \tan\ (\lambda\ )$.

41. $\mathbf{\qquad + (3 \quad + \quad \tan \quad)\quad + \ \tan\quad (\quad)^2 = 0.}$
This is a special case of equation 3.5.3.17 with $(\ ) = a\tan \quad$.

42. $\mathbf{(\ + \ )\qquad + \qquad + \ \tan\ (\quad)\quad = 0.}$
This is a special case of equation 3.5.3.21 with $(\ ) = \ \tan\ (\lambda\ )$.

43. $\mathbf{\ ^2 \qquad + 3 \qquad + \ (\quad - 1)(\quad)^3 = \ \tan^k(\quad)\ ^{2-}\quad .}$
This is a special case of equation 3.5.3.27 with $(\ ) = a\tan\ (\lambda\ )$ and $\quad = \quad + 1$.

44. **2 $-$ 3( )^2 = tan ( )( )^2 + tan ( )( )^4.**

This is a special case of equation 3.5.4.7 with ( ) = a tan (λ) and g( ) = tan ( ).

45. **2 $-$ 3( )^2 = tan ( )( )^2 + ( )^4.**

This is a special case of equation 3.5.4.7 with ( ) = a tan (λ) and g( ) = .

46. **2 $-$ 3( )^2 = ( )^2 + tan ( )( )^4.**

This is a special case of equation 3.5.4.7 with ( ) = a and g( ) = tan (λ).

47. **2 $-$ 3( )^2 = tan ( )( )^2 + tan ( ) $^{-1}$( )$^{5\ 2}$.**

This is a special case of equation 3.5.4.8 with ( ) = a tan (λ) and g( ) = tan ( ).

48. **2 $-$ 3( )^2 = tan ( )( )^2 + $^{-1}$( )$^{5\ 2}$.**

This is a special case of equation 3.5.4.8 with ( ) = a tan (λ) and g( ) = .

49. **2 $-$ 3( )^2 = ( )^2 + tan ( ) $^{-1}$( )$^{5\ 2}$.**

This is a special case of equation 3.5.4.8 with ( ) = a and g( ) = tan (λ).

50. **2 $-$ 3( )^2 = tan ( )( )^4 + $^{-1}$ tan ( )( )$^{7\ 2}$.**

This is a special case of equation 3.5.4.9 with ( ) = a tan (λ) and g( ) = tan ( ).

51. **2 $-$ 3( )^2 = tan ( )( )^4 + $^{-1}$ ( )$^{7\ 2}$.**

This is a special case of equation 3.5.4.9 with ( ) = a tan (λ) and g( ) = .

52. **2 $-$ 3( )^2 = ( )^4 + $^{-1}$ tan ( )( )$^{7\ 2}$.**

This is a special case of equation 3.5.4.9 with ( ) = a and g( ) = tan (λ).

3.4.5-4. Equations with cotangent.

53. **= cot( ) .**

This is a special case of equation 3.5.2.1 with ( ) = a cot(λ).

54. **= cot( ) + cot( ).**

This is a special case of equation 3.5.2.2 with ( ) = a cot(λ) and g( ) = cot( ).

55. **+ 3 = cot ( ) + .**

This is a special case of equation 3.5.3.6 with ( ) = a cot (λ) + .

56. **+ 3 = cot ( ) + .**

This is a special case of equation 3.5.3.8 with ( ) = a cot (λ) + .

57. **+ 3 + = cot ( ).**

This is a special case of equation 3.5.3.10 with ( ) = a and g( ) = cot (λ).

58. **+ 3 + [+ ( )^2] = cot ( ).**

This is a special case of equation 3.5.3.13 with ( ) = cot (λ).

59. **+ (3 + cot) + cot ( )^2 = 0.**

This is a special case of equation 3.5.3.17 with ( ) = a cot .

60. $(\ \ +\ \ )\quad +\quad +\ \mathbf{cot}\ (\quad)\quad = 0.$

This is a special case of equation 3.5.3.21 with $(\ ) =\ \cot\ (\lambda\ ).$

61. $^{2}\quad + 3\quad +\ (\ \ - 1)(\ \ )^3 =\ \mathbf{cot}^k(\ \ )\ ^{2-}\ .$

This is a special case of equation 3.5.3.27 with $(\ ) = a\cot\ (\lambda\ )$ and $=\ \ + 1.$

62. $2\quad - 3(\quad)^2 =\ \mathbf{cot}\ (\quad)(\quad)^2 +\ \mathbf{cot}\ (\quad)(\quad)^4.$

This is a special case of equation 3.5.4.7 with $(\ ) = a\cot\ (\lambda\ )$ and $g(\ ) =\ \cot\ (\quad).$

63. $2\quad - 3(\quad)^2 =\ \mathbf{cot}\ (\quad)(\quad)^2 +\quad (\quad)^4.$

This is a special case of equation 3.5.4.7 with $(\ ) = a\cot\ (\lambda\ )$ and $g(\ ) =\quad .$

64. $2\quad - 3(\quad)^2 =\quad (\quad)^2 +\ \mathbf{cot}\ (\quad)(\quad)^4.$

This is a special case of equation 3.5.4.7 with $(\ ) = a\quad$ and $g(\ ) =\ \cot\ (\lambda\ ).$

65. $2\quad - 3(\quad)^2 =\ \mathbf{cot}\ (\quad)(\quad)^2 +\ \mathbf{cot}\ (\quad)\ ^{-1}(\quad)^{5\ 2}.$

This is a special case of equation 3.5.4.8 with $(\ ) = a\cot\ (\lambda\ )$ and $g(\ ) =\ \cot\ (\quad).$

66. $2\quad - 3(\quad)^2 =\ \mathbf{cot}\ (\quad)(\quad)^2 +\quad ^{-1}(\quad)^{5\ 2}.$

This is a special case of equation 3.5.4.8 with $(\ ) = a\cot\ (\lambda\ )$ and $g(\ ) =\quad .$

67. $2\quad - 3(\quad)^2 =\quad (\quad)^2 +\ \mathbf{cot}\ (\quad)\ ^{-1}(\quad)^{5\ 2}.$

This is a special case of equation 3.5.4.8 with $(\ ) = a\quad$ and $g(\ ) =\ \cot\ (\lambda\ ).$

68. $2\quad - 3(\quad)^2 =\ \mathbf{cot}\ (\quad)(\quad)^4 +\ ^{-1}\mathbf{cot}\ (\quad)(\quad)^{7\ 2}.$

This is a special case of equation 3.5.4.9 with $(\ ) = a\cot\ (\lambda\ )$ and $g(\ ) =\ \cot\ (\quad).$

69. $2\quad - 3(\quad)^2 =\ \mathbf{cot}\ (\quad)(\quad)^4 +\ ^{-1}\quad (\quad)^{7\ 2}.$

This is a special case of equation 3.5.4.9 with $(\ ) = a\cot\ (\lambda\ )$ and $g(\ ) =\quad .$

70. $2\quad - 3(\quad)^2 =\quad (\quad)^4 +\ ^{-1}\mathbf{cot}\ (\quad)(\quad)^{7\ 2}.$

This is a special case of equation 3.5.4.9 with $(\ ) = a\quad$ and $g(\ ) =\ \cot\ (\lambda\ ).$

3.5. Nonlinear Equations Containing Arbitrary Functions

3.5.1. Equations of the Form $F(\ ,\ )\quad + G(\ ,\ ) = 0$

3.5.1-1. Arguments of the arbitrary functions are or .

1. $\quad = f(\ ).$

The substitution $(\ ) = (\ '\ )^2$ leads to a second-order equation: $'' =\ 2\ (\ )\ ^{-1\ 2}$. In particular, with $(\ ) = a\quad$ the obtained equation is an Emden–Fowler equation; see Section 2.3.

2. $\quad = f(\ )\ ^{-1}.$

1 . On integrating the equation, we have $''\ - \frac{1}{2}(\ '\ )^2 =\quad (\ )\quad +\ $. The substitution $=\ ^{2}$ reduces the latter equation to the form $'' = \frac{1}{2}\quad (\ )\quad +\quad ^{-3}.$

2 . The transformation $= 1\ ,\quad =\quad ^{2}$ leads to an equation of the same form: $''' = -\ ^{-2}\ (1\ \ )\ ^{-1}.$

3. $= {}^{-3} f(\ )$.

The substitution $= \ln|\ |$ leads to an autonomous equation of the form 3.5.5.9: $''' - 3\ '' + 2\ ' = (\ )$.

4. $(\ +\ {}^{2} +\ +\ )\ = f(\ )$.

The substitution $= +a\ {}^{2} +\ +$ leads to an equation of the form 3.5.1.2: $''' = (\ )$.

5. $(\ +\ )\ +\ = f(\ )$.

Integrating yields a second-order equation:

$$(a\ + e\ )\ '' - \tfrac{1}{2}a(\ ')^2 - e\ ' + e\ = (\ )\ + \ .$$

3.5.1-2. Arguments of the arbitrary functions depend on and .

6. $= {}^{-2} f(\ {}^{-1})$.

The transformation $= \ln|\ |,\ = {}^{-1}$ leads to an autonomous equation of the form 3.5.5.9: $''' - \ ' = (\ )$.

7. $= {}^{-4} f(\ {}^{-2})$.

The transformation $= {}^{-1},\ = {}^{-2}$ leads to an autonomous equation of the form 3.5.1.1: $''' = -(\ )$.

8. $= {}^{-k-3} f(\ {}^{k}\ )$.

Generalized homogeneous equation.

1 . The transformation $= \ln\ , z =$ leads to an autonomous equation.

2 . The transformation $z = \ ,\ = \ '$ leads to a second-order equation.

9. $= {}^{-3} f(\ )$.

Generalized homogeneous equation. The transformation $z = \ ,\ = \ '$ leads to a second-order equation.

10. $= f(\ +\ {}^{3} +\ {}^{2} +\ +\ )$.

The substitution $= +a\ {}^{3} +\ {}^{2} +\ +k$ leads to an autonomous equation of the form 3.5.1.1: $''' = (\ ) + 6a$.

11. $(\ -\ )^3\ = f(\ {}^{-2}),\ \neq 0$.

The transformation $\xi = \ln \frac{-a}{},\ = \frac{}{\ {}^{2}}$ leads to an autonomous equation of the form 3.5.5.9: $''' - 3\ '' + 2\ ' = a^{-3}(\ )$.

12. $= (\ +\ +\ )^{-2} f\left(\frac{+\ +}{+\ +\gamma}\right)$.

This is a special case of equation 5.2.6.19 with $= 3$.

13. $(\ {}^{2} +\ +\ )^2\ = f\left(\frac{}{{}^{2} +\ +}\right)$.

The transformation $\xi = \frac{}{a\ {}^{2} +\ +},\ = \frac{}{a\ {}^{2} +\ +}$ leads to an autonomous equation of the form 3.5.5.9: $''' + (4a\ -\ {}^{2})\ ' = (\ )$.

14. $= {}^{-2} f \dfrac{}{{}^{2} + \quad +}$.

Setting $(\ ) = {}^{2}{}_{1}(\ )$, we have equation 3.5.1.13 with the function ${}_{1}$ (instead of).

15. $= {}^{\lambda} f({}^{-\lambda}).$

This is a special case of equation 3.5.3.32 with $a = \ = \ = 0$. The substitution $(\ ) = e^{-\lambda}$ leads to an autonomous equation.

16. $= f({}^{\lambda}).$

This is a special case of equation 3.5.5.21. The transformation $z = e^{\lambda}$, $(z) = {}'$ leads to a second-order equation.

17. $= {}^{-3} f({}^{\lambda}).$

The transformation $z = e^{\lambda}$, $(z) = {}'$ leads to a second-order equation.

18. $= f(+ {}^{\lambda}) - {}^{3\ \lambda}$.

The substitution $= + ae^{\lambda}$ leads to an autonomous equation of the form 3.5.1.1: ${}''' = (\ )$.

19. $= (\ ,\ ).$

The transformation $= 1\ ,\ = {}^{2}$ leads to an equation of the same form: ${}''' = - {}^{-4} (1\ ,\ {}^{2})$.

3.5.2. Equations of the Form $F(\ ,\ ,\ ) + G(\ ,\ ,\ ) = 0$

3.5.2-1. Arguments of the arbitrary functions depend on and .

1. $= f(\ )$.

Solution: ${}_{3} = {}_{2} + {}_{1} + 2 \quad (\ ) \quad {}^{-1\ 2}$, where $(\ ) = (\ )$.

2. $= f(\ ) + (\ ).$

Integrating yields a second-order equation: ${}'' = (\ ) + g(\ ) + $.

3. $= f(\ ) + (\ )(\ )^{3}.$

The substitution $z(\ ) = (\ ')^{2}$ leads to a second-order linear equation: $z'' = 2g(\ )z + 2\ (\ )$.

4. $= f(\ )(\ )^{3} + (\ )^{-5}.$

The substitution $z(\ ) = (\ ')^{2}$ leads to Yermakov's equation 2.9.1.2: $z'' = 2\ (\ )z + 2az^{-3}$.

5. $= {}^{-3} f \ — (\ \ - \ ).$

The transformation $z = \ ,\ = {}^{-2}(\ ' - \ )^{2}$ leads to a second-order linear equation: ${}'' = 2\ (z) + 2$. Integrating the latter equation twice, we arrive at a first-order homogeneous equation for $(\ )$:

$${}' = z \quad z^{2} + {}_{2}z + {}_{1} + 2 \ {}_{0}(z - \ )\ (\ ) \quad {}^{1\ 2}, \quad \text{where} \quad z = —, \ z_0 \text{ is an arbitrary number.}$$

6. $= {}^{-5} f \ — (\ \ - \ )^{3}.$

The transformation $z = \ ,\ = {}^{-2}(\ ' - \ )^{2}$ leads to a second-order linear equation: ${}'' = 2\ (z) \ + 2$.

7. $= {}^{-3}f \;-\; (\quad - \;) + \;{}^{-5} \quad - \;(\quad - \;)^3.$

This is a special case of equation 3.5.3.38 with $k = -1$. The transformation $= \ln \;,$ $z = \quad,$ followed by the substitution $(z) = (z')^2$, leads to a second-order linear equation: $'' = 2g(z) \quad + 2 \;(z) + 2.$

8. $= {}^{-5}f \;\frac{}{2}\; (\quad - 2 \;).$

The transformation $= 1 \quad , z = \quad {}^2$ leads to an autonomous equation: $z''' = \;(z)z'$. The substitution $(z) = (z')^2$ then yields the second-order linear equation $'' = 2 \;(z)$, whose solution is given by:

$$= \;{}_2 z + \;{}_1 + 2 \;\int_{0} (z - \xi) \;(\xi) \;\xi, \qquad z_0 \text{ is an arbitrary number.}$$

9. $= {}^{-7}f \;\frac{}{2}\; (\quad - 2 \;)^3.$

The transformation $= 1 \quad , z = \quad {}^2$, followed by the substitution $(z) = (z')^2$, leads to a second-order linear equation: $'' = 2 \;(z) \quad .$

10. $= {}^{-5}f \;\frac{}{2}\; (\quad - 2 \;) + \;{}^{-7} \;\frac{}{2}\; (\quad - 2 \;)^3.$

The transformation $= 1 \quad , z = \quad {}^2$, followed by the substitution $(z) = (z')^2$, leads to a second-order linear equation: $'' = 2g(z) \quad + 2 \;(z).$

11. $+f \quad + \quad = -3 \quad {}^{-4}(\quad)^2 - 3(\quad {}^3 + 2 \;{}^2) \;{}^{-6} \quad - \;{}^3 \;{}^{-8} - 3 \quad {}^{-5} - (f \quad + \quad) \;{}^{-2}.$

Here, $= \;(\;)$, $g = g(\;)$, and $= \;(\;)$ are arbitrary functions.

Solution: $= \quad + 3 \;\frac{(\;)}{3}\;^{1\;3}$, where $= \;(\;)$ is the general solution of the linear equation: $''' \;+ \;(\;) \;' + g(\;) \quad = 0.$

12. $[\quad + f(\;)] \quad + \;(\;) \quad + f \quad (\;) \;+ \;(\;) = 0.$

The equation admits a first integral:

$$[a \;+ \;(\;)] \;'' - \tfrac{1}{2}a(\;')^2 - \;'(\;) \;' + \;''(\;) \;+ \quad g(\;) \quad + \quad (\;) \quad = \quad .$$

$\boxed{\text{3.5.2-2. Arguments of the arbitrary functions depend on } \;, \;, \text{ and } \;'.}$

13. $= f(\quad).$

Solution in parametric form:

$$= \;\int^{\tau}_{2} \frac{\tau}{(\tau)}, \qquad = \;\int^{\tau}_{3} \frac{\tau \;\tau}{(\tau)}, \qquad \text{where} \quad = \quad {}_1 + 2 \quad (\tau) \;\tau \;^{1\;2}.$$

14. $= f(\;, \quad).$

The substitution $(\;) = \;'$ leads to a second-order equation: $'' \;= \;(\;, \;).$

15. $= f(\;, \quad).$

Autonomous equation (it is a special case of equation 3.5.5.9).

1 . The substitution $(\;) = (\;')^2$ leads to a second-order equation: $'' \;= \;2 \;{}^{-1\;2} \;(\;, \quad {}^{1\;2}).$

2 . The transformation

$$= [\ (\tau)]^{-3\ 2}\ \tau, \qquad = [\ (\tau)]^{-1} \qquad (1)$$

leads to an analogous equation with respect to $= (\tau)$:

$$'''_{\tau\tau\tau} = - \ ^{-5\ 2}\ (\ ^{-1}, -\ ^{-1\ 2}\ '_{\tau}).$$

Note two important cases of transforming equations of special form:

$$''' = (\ ) \quad \xrightarrow{\text{transformation (1)}} \quad '''_{\tau\tau\tau} = - \ ^{-5\ 2}\ (\ ^{-1}),$$

$$''' = A \quad \xrightarrow{\text{transformation (1)}} \quad '''_{\tau\tau\tau} = -A \ ^{-\ -5\ 2}.$$

16. $= \ ^{3} + f(\ , \ - \ ).$

The substitution $= \ ' - a$ leads to a second-order equation: $'' + a\ ' + a^2 = (\ ,\ ).$

17. $= (3\ ^{2} - \ ^{3}\ ^{3}) + f(\ , \ + \ ).$

The substitution $= \ ' + a$ leads to a second-order equation:

$$'' - a\ ' + (a^2\ ^2 - 2a) = (\ ,\ ).$$

18. $= f(\ - \ ).$

The substitution $z = \ ' -$ leads to a second-order equation of the form 2.9.2.20 with $= 1$: $z'' = z' + \ ^{3}\ (z).$

19. $= \ ^{-3} f(\ - \ ).$

The transformation $= \ln|\ |$, $z = \ ' -$ leads to the second-order autonomous equation $z'' - 2z' = (z)$, which is reduced, with the aid of the substitution $(z) = \frac{1}{2}z'$, to the Abel equation $' - = \frac{1}{4}\ (z)$ (for some functions , solutions of this Abel equation are given in Subsection 1.3.1).

20. $= f(\ , \ - \ ).$

The substitution $= \ ' -$ leads to a second-order equation: $(\ '\ )' = (\ ,\ ).$

21. $= f(\ , \ - 2\ ).$

The substitution $= \ ' - 2$ leads to a second-order equation: $'' = (\ ,\ ).$

22. $^{3} = f(\ , \ + \ ) - (\ + 1)(\ + 2)\ .$

The substitution $= \ ' + a$ leads to a second-order equation:

$$^{2}\ '' - (a+2)\ ' + (a+1)(a+2) = (\ ,\ ).$$

23. $= \ ^{-2} \ - - \ .$

The substitution $z = \ ' -$ leads to the second-order equation $z'' = z' + (z\ )$, which is a special case of the equation 2.9.4.22 with $= -1$, $= 1$, $k = -1$, $(\xi) = \xi\ (\xi).$

24. $= \ ^{-4} f \ - 2 - \ .$

The transformation $= 1\ $, $z = \ ^{2}$ yields $z''' = -(-z')$. The substitution $= -z'$ leads to a second-order autonomous equation of the form 2.9.1.1: $'' = (\ ).$

25. $= \frac{1}{4} f \quad \frac{}{2}, \quad - 2 - \;.$

The transformation $= 1 \;,\; z = \;\; {}^2$ leads to an equation of the form 3.5.2.15: $z''' = - \;(z, -z')$, which admits, with the aid of the substitution $(z) = (z')^2$, reduction of its order: $'' = 2 \;\; {}^{-1\;2} \;(z, \;\; {}^{1\;2})$.

26. $= \;\; {}^{-3} f(\qquad).$

The transformation $z = \;\; ' \;\;,\;\; = \;\; {}^2 \;''$ leads to a first-order equation: $(\; + z - z^2) \;' = 2 \;\; - z \;\; + \;(z)$.

27. $= \;\; {}^{-5\;2} f \;\; \frac{}{=} \Big).$

The substitution $(\;) = (\;')^2$ leads to an equation of the form 2.9.1.8: $'' = \;\; {}^{-3} \;(\qquad)$, where $(\xi) = \; 2\xi^{-1\;2} \;(\;\xi^{1\;2})$.

28. $= \;\; {}^{-5\;4} f \;\; \frac{}{1\;4} \Big).$

The substitution $(\;) = (\;')^2$ leads to an equation of the form 2.9.1.9: $'' = \;\; {}^{-3\;2} \;\Big(\;\; {}^{-1\;2} \;,$ where $(\xi) = \; \xi^{-1\;2} \;(\;\xi^{1\;2})$.

29. $= (\;\; {}^2 + \;\; + \;)^{-5\;4} f \;\; \frac{}{(\;\; {}^2 + \;\; + \;)^{1\;4}} \Big).$

The substitution $(\;) = (\;')^2$ leads to an equation of the form 2.9.1.21:

$$'' = \;\; {}^{-3} \;\; \frac{}{a \;\; {}^2 + \;\; +} \Big), \qquad \text{where} \qquad (\xi) = \; 2\xi^{5\;2} \;(\;\xi^{1\;2}).$$

30. $= \;(\;\; {}^2 + \;\; {}^2 \; + \;\; {}^4)^{-5\;4} f \;\; \frac{-2}{(\;\; {}^2 + \;\; {}^2 \; + \;\; {}^4)^{1\;4}} \Big).$

The transformation $= 1 \;,\; z = \;\; {}^2$ leads to an equation of the form 3.5.2.29:

$$z''' = -(az^2 + \;z + \;)^{-5\;4} \;\; \frac{-z'}{(az^2 + \;z + \;)^{1\;4}} \Big).$$

31. $= - \;\; {}^{-2} \;\; + \;\; {}^{-3} \; + \;\; {}^{1\;2} \;\; {}^{-5\;2} f \;\; \frac{-}{=} \Big).$

The transformation $= \ln \;,\; z = \;\;$, followed by the substitution $(z) = (z')^2$, leads to a second-order equation of the form 2.9.1.8: $'' = z^{-3} \;(\;\; z)$, where $(\xi) = \; 2\xi^{-1\;2} \;\Big(\;\; \overline{\xi} \;.$

32. $= - \;\; {}^{-2} \;\; + \;\; {}^{-3} \; + \;\; {}^{-3\;4} \;\; {}^{-5\;4} f \;\; \frac{-}{3\;4 \;\; 1\;4} \Big).$

The transformation $= \ln \;,\; z = \;\;$, followed by the substitution $(z) = (z')^2$, leads to a second-order equation of the form 2.9.1.9: $'' = z^{-3\;2} \;(\;\; z^{-1\;2})$, where $(\xi) = \; 2\xi^{-1\;2} \;(\;\; \overline{\xi} \;.$

33. $= \;\; f(\;\; {}^2 + \;\; \;.$

The substitution $(\;) = (\;')^2 + a$ leads to a second-order autonomous equation of the form 2.9.1.1: $'' = 2 \;(\;)$.

34. $= 2\ \ ^2(\quad)^3 + \ ^{6\lambda}\ f(\ ^{\lambda}\quad)\ .$

This is a special case of equation 3.5.2.48 with $(\ ) = e^{-2\lambda}$.

35. $= \frac{1}{2}\ \ ^2(\quad)^3 + (\cosh\quad)^{-3} f\left(\frac{}{\overline{\cosh}}\right)\ .$

This is a special case of equation 3.5.2.48 with $(\ ) = \cosh\lambda$.

36. $= \frac{1}{2}\ \ ^2(\quad)^3 + (\sinh\quad)^{-3} f\left(\frac{}{\overline{\sinh}}\right)\ .$

This is a special case of equation 3.5.2.48 with $(\ ) = \sinh\lambda$.

37. $= \quad \tanh\quad + f(\ ,\quad - \quad \tanh\quad).$

This is a special case of equation 3.5.2.47 with $(\ ) = \cosh$.

38. $= (\sinh\ )^{-2}(\quad)^3 + (\tanh\ )^3 f\big(\quad \overline{\tanh}\quad\ .$

This is a special case of equation 3.5.2.48 with $(\ ) = \coth$.

39. $= \quad \coth\quad + f(\ ,\quad - \quad \coth\quad).$

This is a special case of equation 3.5.2.47 with $(\ ) = \sinh$.

40. $= -(\cosh\ )^{-2}(\quad)^3 + (\coth\ )^3 f\big(\quad \overline{\coth}\quad\ .$

This is a special case of equation 3.5.2.48 with $(\ ) = \tanh$.

41. $= -\frac{1}{2}\ \ ^2(\quad)^3 + (\cos\quad)^{-3} f\left(\frac{}{\overline{\cos}}\right)\ .$

This is a special case of equation 3.5.2.48 with $(\ ) = \cos\lambda$.

42. $= -\frac{1}{2}\ \ ^2(\quad)^3 + (\sin\quad)^{-3} f\left(\frac{}{\overline{\sin}}\right)\ .$

This is a special case of equation 3.5.2.48 with $(\ ) = \sin\lambda$.

43. $= \quad \tan\quad + f(\ ,\quad + \quad \tan\quad).$

This is a special case of equation 3.5.2.47 with $(\ ) = \cos$.

44. $= (\sin\ )^{-2}(\quad)^3 + (\tan\ )^3 f\big(\quad \overline{\tan}\quad\ .$

This is a special case of equation 3.5.2.48 with $(\ ) = \cot$.

45. $= -\quad \cot\quad + f(\ ,\quad - \quad \cot\quad).$

This is a special case of equation 3.5.2.47 with $(\ ) = \sin$.

46. $= (\cos\ )^{-2}(\quad)^3 + (\cot\ )^3 f\big(\quad \overline{\cot}\quad\ .$

This is a special case of equation 3.5.2.48 with $(\ ) = \tan$.

47. $= \frac{}{} + f\quad ,\quad - \frac{}{}\quad ,\qquad = (\ ).$

The substitution $= \ ' - \frac{'}{}$ leads to a second-order equation:

$$'' + \frac{'}{}\ ' + 2\frac{''}{} - \frac{(\ ')^2}{2} = (\ ,\ ).$$

48. $= \frac{1}{2} \frac{}{}(\ \ )^3 + {}^{-3} f \ \frac{}{} \Big)$, $=$ ( ).

The substitution $z = (\ ')^2$ leads to a second-order equation of the form 2.9.1.46:

$$z'' = \frac{''}{} z + 2 \ {}^{-3} \ \sqrt{\frac{z}{}} \Big).$$

49. $=$ (, ,).

Let $\neq$ ( ) $'$ + ( ) $+\chi($), i.e., the equation is nonlinear. Then its order can be reduced by one if the right-hand side of the equation has the following form:

$$(\ ,\ ,\ ') = {}^{-2} \quad (\ ,\ ) + \ [2 \ ''' \ +(\ {}^2 \ '''' \ + 2 \ ' \ ''' \ )$$

$$-(2 \ ''' \ g + \ g''' \ ) \ {}^{-1} - (2 \ ' \ ''' \ + \ {}^2 \ '''' \ - 2k \ ''' \ ) \ ] \quad ,$$

where

$$= \exp \ -k \ \frac{}{} \ , \qquad = \frac{g}{2}, \qquad = \frac{}{} + \ , \qquad = \frac{' - \ ' + g}{} - k \ ;$$

$=$ (,), $=$ ( ), and $g = g($) are arbitrary functions; k is an arbitrary constant.

In this case, the transformation $= \ {}^{-1}$, $= \ {}^{-1} \ {}^{-1} +$, followed by the substitution $z(\ ) = \ '$, leads to a second-order equation: $z^2 z'' + z(z')^2 - 3kzz' + 3k^2 z - k^3 = (\ , z - k\ )$.

3.5.3. Equations of the Form $F(\ ,\ ,\ ) \ + G(\ ,\ ,\ ) \ + \ (\ ,\ ,\ ) = 0$

3.5.3-1. The arbitrary functions depend on or .

1. $+3 \ + \ {}^2(\ \ )^3 = f(\ ) \ {}^{-\lambda}$.

Solution: $e^{\lambda} = {}_2 \ {}^2 + {}_1 + {}_0 + \frac{\lambda}{2} \int_0 (\ - \ )^2 (\ )$, where ${}_0$ is an arbitrary number.

2. $= f(\ )$.

Integrating yields a second-order autonomous equation of the form 2.9.1.1: $'' = (\ )$, where $(\ ) = \ \exp \int (\ )$.

3. $= [f(\ ) \ + \ (\ )]$.

Integrating yields a second-order equation: $'' = \ \exp \ (\ ) \ + \ g(\ )$.

4. ${}^3 \ + \ {}^2 \ + \ = f(\ )$.

The substitution $= \ln|\ |$ leads to an autonomous equation of the form 3.5.5.9:

$$''' + (a-3) \ '' + (\ - a + 2) \ ' = (\ ).$$

5. $(\ \ + 3 \ + 2 \ {}^2 \ ) = f(\ )$.

Integrating yields a second-order equation:

$$2 \ '' + 2a \ ' - (\ ')^2 = e^{-2} \ 2 \ e^2 \ (\ ) \ + \ .$$

6. $+3 \ = f(\ )$.

Solution: ${}^2 = {}_2 \ {}^2 + {}_1 + {}_0 + \int_0 (\ - \ )^2 (\ )$, where ${}_0$ is an arbitrary number.

7. $+ \qquad = f(\ ).$

Integrating yields a second-order equation: $'' + \frac{1}{2}(a-1)(\ ')^2 = \quad (\ ) \quad + \ .$

8. $+3 \qquad = f(\ ).$

The substitution $= \ ^2$ leads to an autonomous equation of the form 3.5.1.1: $''' =$ $2 \ (\quad \overline{\ } \ .$

9. $- \qquad = f(\ )\ ^2.$

Integrating yields a second-order linear equation: $'' = \quad (\ ) \quad + \quad .$

10. $+3 \qquad + f(\ ) \qquad = (\ ).$

The substitution $= \ '$ leads to a second-order linear nonhomogeneous equation: $'' +$ $(\ ) \ = g(\ ).$

11. $+3 \qquad + 4f(\ ) \qquad + f\ (\ )\ ^2 = 0.$

Multiplying by ^{2}, we arrive at an exact differential equation. Integrating it yields Yermakov's equation 2.9.1.2: $'' + \ (\ ) \ = \quad ^{-3}.$

There is also the trivial solution $= 0$.

12. $+ \qquad = f(\ ) \quad + (\ ).$

Integrating yields a second-order equation: $'' + \frac{1}{2}(a-1)(\ ')^2 = \quad (\ ) \quad + \ g(\ ) \quad + \ .$

13. $+3 \qquad + \ \big[\qquad + (\quad)^2\big] = f(\ ).$

Solution:

$$^2 = \ _3 e^{-} \quad + \ _2 \ + \ _1 + 2 \int_0 (\ - \ )e^{-} \quad (\ ) \quad , \quad \text{where} \quad (\ ) = \quad e \quad (\ ) \quad ,$$

$_0$ is an arbitrary number.

14. $+ (3 \quad + 2 \quad) \qquad + 2\ (\quad)^2 + \ ^2 \qquad = f(\ ).$

Integrating the equation twice, we arrive at a first-order separable equation:

$$e \qquad ' = \ _2 \ + \ _1 + \int_0 (\ - \ )e \quad (\ ) \quad .$$

15. $= \qquad + f(\ ) \qquad .$

Integrating yields a second-order linear equation: $'' = \ \exp \quad (\ ) \quad .$

16. $- \qquad = f(\ )\big[\qquad - (\quad)^2\big] + (\ ).$

The substitution $= \ '' - (\ ')^2$ leads to a first-order linear equation: $' = (\ ) \ + g(\ ).$

17. $+ [3 \quad + f(\ )\ ] \qquad + f(\ )(\quad)^2 = 0.$

Solution: $^2 = \ _3 \ + \ _2 + \ _1 \int_0 (\ - \ )e^{-F(\ )} \quad$, where $(\ ) = \quad (\ ) \quad .$

18. $+ [3 \quad + f(\ )\ ] \qquad + f(\ )(\quad)^2 + (\ ) \qquad + (\ ) = 0.$

The substitution $= \ '$ leads to a second-order linear nonhomogeneous equation: $'' + \ (\ )\ ' + g(\ ) \ + \ (\ ) = 0.$

19. $+ (f-1) \quad + f \quad + \quad {}^{2} = 0, \quad f = f(\;), \quad = (\;).$

A solution of this equation is any function that solves the second-order linear equation $'' + g(\;) = 0$.

20. $(\; + \;) \quad + \quad = f(\;).$

Having integrated the equation, we obtain $(\; + a)\;'' + \frac{1}{2}(\; - 1)(\;')^2 = \quad (\;) \quad + \;$. For $\neq -1$, the substitution $= \quad \frac{2}{b+1} - a$ leads to the equation:

$$'' = \frac{+1}{2} \quad (\;) \quad + \quad \frac{b-3}{b+1}$$

(with $= 0$ and $(\;) = \lambda$, see Section 2.3).

21. $(\; + \;) \quad + \quad + f(\;) \quad = 0.$

Having integrated the equation, we obtain a second-order autonomous equation:

$$(\; + a)\;'' + \tfrac{1}{2}(\; - 1)(\;')^2 + \quad (\;) \quad = \;,$$

which is reduced with the aid of the substitution $(\;) = (\;')^2$ to a first-order linear equation:

$$(\; + a)\;' + (\; - 1) \; + 2 \quad (\;) \quad = 2 \;.$$

22. $(\; + \;) \quad + \quad + f(\;) \quad = (\;).$

Having integrated the equation, we obtain a second-order equation:

$$(\; + a)\;'' + \tfrac{1}{2}(\; - 1)(\;')^2 + \quad (\;) \quad = \quad g(\;) \quad + \;.$$

23. $(\; + \quad + \;) \quad + 3(\quad + \;) \quad = f(\;).$

Solution: $(\; + a \; + \;)^2 = \;_2\;^2 + \;_1 \; + \;_0 + \int_0 (\; - \;)^2 \;(\;)\;$, where $_0$ is an arbitrary number.

24. $[\; + f(\;)] \quad = [\quad + f\;(\;)] \quad + \; f(\;) \quad - \; f\;(\;)\;.$

Integrating yields a second-order constant coefficient linear equation of the form 2.1.9.1: $'' + \quad = -(a + \;)\;(\;)$. There is also the trivial solution $= 0$.

25. $(\quad + 3 \quad) + \big[\quad + (\quad)^2\big] = f(\;).$

Solution:

$${}^2 = \;_3\;{}^{2-} + \;_2 + \;_1 + 2 \int_0 (\; - \;)^{-} \;(\;)\;, \quad \text{where} \quad (\;) = \quad {}^{-1}\;(\;)\;;$$

$_0$ is an arbitrary number.

26. ${}^2 \quad - 3 \quad + 2(\quad)^3 = f(\;)\;{}^3.$

Solution: $\ln|\;| = \;_2\;^2 + \;_1 \; + \;_0 + \frac{1}{2}\int_0 (\; - \;)^2 \;(\;)\;$, where $_0$ is an arbitrary number.

27. ${}^2 \quad + 3(\; - 1) \quad + (\; - 1)(\; - 2)(\quad)^3 = f(\;)\;{}^{3-}\;.$

Solution for $\neq 0$:

$$= \;_2\;^2 + \;_1 \; + \;_0 + \frac{}{2}\int_0 (\; - \;)^2 \;(\;)\;, \quad \text{where } \;_0 \text{ is an arbitrary number.}$$

For the case $= 0$, see equation 3.5.3.26.

28. $\big(f \quad + \tfrac{3}{2}f \quad + \tfrac{1}{2}f \quad = \ (\ ), \qquad f = f(\ ).$

Having integrated the equation, we obtain a second-order equation:

$$2 \quad '' + \ ' \quad ' - \ (\ ')^2 = 2 \quad g(\ ) \quad + \ .$$

29. $f \qquad + (3f \quad + 2f \quad) \quad + 2f\ (\quad)^2 + f \qquad = \ (\ ), \qquad f = f(\ ).$

Integrating the equation twice, we arrive at a first-order separable equation: $(\ ) \quad ' =$ $_2 \ + \ _1 + \ _0 \ (\ -\ )g(\ ) \ .$

30. $\quad + f \quad + \quad = -(\quad + 2) \quad ^{-1} \quad - (\quad - 1)(\quad + 1) \quad ^{-2}(\quad)^2$
$- [(2 \quad + 1) \quad + 3 \quad ^2 \quad ^{-1}] \quad ^{-1} \quad - \quad ^3 \ ^3 \ ^{-2} - 3 \quad ^2 \ ^{-1} - (f \quad + \quad) \quad .$

Here, $\ = \ (\ )$, $g = g(\ )$, and $\ = \ (\ )$ are arbitrary functions.

Solution: $\ = \quad + (1 - \ ) \quad (\ ) \quad ^{-1} \quad ^{\frac{1}{1-}}$, where $\ = \ (\ )$ is the general solution of the linear equation: $\ ''' \ + \ (\ ) \ ' + g(\ ) \ = 0.$

31. $\quad + f(\ ) \quad + \ (\ ) \quad + \ (\ )(\quad)^2 + \big[\ (\ ) + f(\ )\ (\ )\big] \quad + \ ^2(\ ) \ ^2 = 0.$

The solution satisfies the second-order linear equation $\ '' - z(\ ,\ ) \ ' + g(\ ) \ = 0$, where $z = z(\ ,\ )$ is the general solution of the Riccati equation $z' + z^2 + \ (\ )z - g(\ ) + \ (\ ) = 0.$

$\boxed{\text{3.5.3-2. Arguments of arbitrary functions depend on \quad and \quad .}}$

32. $\quad + \quad + \quad + \quad = \ ^{\lambda} f(\quad ^{-\lambda}\ ).$

The substitution $\ (\ ) = \ e^{-\lambda}$ leads to an autonomous equation of the form 3.5.5.9:

$$\ ''' + (3\lambda + a) \ '' + (3\lambda^2 + 2a\lambda + \ ) \ ' + (\lambda^3 + a\lambda^2 + \ \lambda + \ ) \ = \ (\ ).$$

33. $\quad = -3 \quad + 2 \ + f(\quad)(\quad + \ ).$

The transformation $z = e \ $, $\ = e^2 \ (\ ' + \ )^2$ leads to a second-order linear equation: $\ '' = 2 \ (z) + 6$. Integrating the latter, we find the solution:

$$\frac{z}{\sqrt{3z^2 + \ _2 z + \ _1 + 2 \ (z)}} = \quad + \ _3, \qquad \text{where} \quad z = e \ , \quad (z) = \qquad (z) \ z \quad z.$$

34. $\quad + 3 \quad = f(\quad).$

The substitution $\ (\ ) = \quad$ leads to an autonomous equation of the form 3.5.1.1: $\ ''' = \ (\ ).$

35. $\ ^2 \quad + 6 \quad + 6 \quad = f(\ ^2\ ).$

The substitution $\ (\ ) = \ ^2$ leads to an autonomous equation of the form 3.5.1.1: $\ ''' = \ (\ ).$

36. $\ ^3 \quad + \ ^2 \quad + \quad = f(\quad ^{\lambda}\ ).$

The transformation $\ = \ln \ $, $\lambda \ = \lambda \ + \quad$ leads to an autonomous equation of the form 3.5.5.9: $\ ''' + (a - 3) \ '' + (\ - a + 2) \ ' = \ (e^{\lambda w}) + \frac{}{\lambda}(\ - a + 2).$

37. $^{3} \quad = -\frac{3}{2} \; ^{2} \quad + f \; \frac{}{} \; (2 \quad - \;)$.

The transformation $= \frac{}{},\ z = \frac{1}{}(\;' - \frac{1}{2} \;^{2}$ leads to the second-order linear equation $2z'' = 8\;(\;) + 1$, whose solution is given by:

$$z = \tfrac{1}{4}\;^{2} + \;_{2} + \;_{1} + 4 \int_{0} (\; - \xi)\;(\xi)\;\xi, \qquad _{0} \text{ is an arbitrary number.}$$

Passing on to the variables $\;,\;\; = \;^{-1}\;^{2}$, we obtain a separable equation.

38. $^{3} \quad = -3(\; +1)\;^{2} \quad + \;(\; +1)(2\; +1)\; + f(\;^{k}\;)(\quad + \;) + \;^{2k}\;(\;^{k}\;)(\quad + \;)^{3}$.
The transformation $= \ln\;,\ z = \quad$, followed by the substitution $(z) = (z')^{2}$, leads to a second-order linear equation: $'' = 2g(z)\; + 2\;(z) + 6k^{2} + 6k + 2$.

39. $^{4} \quad = -\frac{3}{2}\;^{3} \quad + f \; \frac{}{} \; (2 \quad - \;)^{3}$.

The transformation $= \frac{}{},\ z = \frac{1}{}(\;' - \frac{1}{2}\;)^{2}$ leads to a second-order linear equation: $2z'' = 16\;(\;)z + \frac{1}{2}$.

40. $^{2} \quad = -3\;^{2} \quad + 2\;^{3} + \;^{2}f(\quad)(\; + \;) + \;(\quad)(\; + \;)^{3}$.
The substitution $z(\;) = e \;$, followed by reduction of the equation order and the substitution $(z) = (z')^{2}$, leads to a second-order linear equation: $'' = 2z^{-2}g(z)\; + 2\;(z) + 6$.

3.5.3-3. Arguments of arbitrary functions depend on $\;,\;$, and $'$.

41. $= f(\quad - \;)\quad$.
The substitution $z = \;' - \;$ leads to a second-order equation of the form 2.9.2.21: $z'' = [\;(z) + 1]z'$.

42. $+ (1 - \;)\quad = \;^{2}\; f(\quad - \;)$.
The substitution $z = \;' - \;$ leads to a second-order equation of the form 2.9.2.20: $z'' = az' + \;^{2\;+1}\;(z)$.

43. $+ (\; + 2)\quad = f(\;,\quad + \;)$.
The substitution $= \;' + a$ leads to a second-order equation: $'' = \;(\;,\;)$.

44. $+ (1 - \quad)\quad = \;^{2}\; f(\quad - \;)$.
The substitution $z = \;' - \;$ leads to a second-order equation of the form 2.9.2.17: $z'' - az' = e^{2}\;(z)$.

45. $= f(\quad - \; + \; \ln\;)\quad$.
The substitution $z = \;' - \;$ leads to a second-order equation of the form 2.9.2.39: $z'' = \left[\;\left(\ln(\; e\;)\right) + 1\right]z'$.

46. $^{3} \quad + \;^{2} \quad = f(\quad - \;)$.
The transformation $= \ln|\;|,\ z = \;' - \;$ leads to an autonomous equation of the form 2.9.6.2: $z'' - z' = \;(z)$, which is reduced, with the aid of the substitution $= z'$, to the Abel equation $\;' - \; = \;(\;)$ (see Subsection 1.3.1).

47. $^{4} \quad + \;^{3} \quad = f \quad - \frac{}{}\;$.

The substitution $(\;) = \;' - \;$ leads to an equation of the form 2.9.1.8: $'' = \;^{-3}\;(\quad)$.

3.5.4. Equations of the Form $F(\ ,\ ,\ )\qquad +\sum G\ (\ ,\ ,\ )(\quad)\ = 0$

3.5.4-1. Arbitrary functions depend on or .

1. $\qquad + (\quad)^2 - \frac{1}{4}(\quad)^2 - \frac{1}{4}\qquad - f(\ )\ ^2 = 0.$

This is a special case of equation 3.5.4.4. The solution satisfies the second-order linear equation $\ '' + \frac{1}{2}\ ' - z(\ ,\ )\ = 0$, where $z = z(\ ,\ )$ is the general solution of the Riccati equation $z' + z^2 - \frac{1}{2}z = \ (\ )$.

2. $\qquad + (\quad)^2 - \qquad - f(\ )\ ^2 = 0.$

The solution satisfies the second-order linear equation $\ '' - z(\ ,\ )\ = 0$, where $z = z(\ ,\ )$ is the general solution of the Riccati equation $z' + z^2 = \ (\ )$.

3. $\qquad + (\quad)^2 + [2\ (\ ) - 1]\qquad + [f(\ ) + \ (\ )]$

$\qquad + \ (\ )[\ (\ ) - 1](\quad)^2 + [\quad(\ ) + f(\ )\ (\ )]\qquad = 0.$

This is a special case of equation 3.5.4.4 with $g(\ ) = 0$. The solution satisfies the second-order linear equation $\ '' + \ (\ )\ ' - z(\ ,\ )\ = 0$, where $z(\ ,\ ) = \ (\ )\quad + \quad(\ )\qquad^{-1}$,

$(\ ) = \exp -\quad(\ )\quad .$

4. $\qquad + (\quad)^2 + [2\ (\ ) - 1]\qquad + [f(\ ) + \ (\ )]$

$\qquad + \ (\ )[\ (\ ) - 1](\quad)^2 + [\quad(\ ) + f(\ )\ (\ )]\qquad = \ (\ )\ ^2.$

The solution satisfies the second-order linear equation $\ '' + \ (\ )\ ' - z(\ ,\ )\ = 0$, where $z = z(\ ,\ )$ is the general solution of the Riccati equation $z' + z^2 + \ (\ )z = g(\ )$.

5. $2\qquad - (\quad)^2 + f(\ )(\quad)^2 = \quad ^2 + 2\quad + \ .$

Differentiating both sides of the equation with respect to and dividing by $'$, we arrive at a fourth-order linear equation: $\ '''' + \quad '' + \frac{1}{2}\ '\ ' = a\ + \ .$

6. $2\qquad - (\quad)^2 - \qquad + \ (\ )(\quad)^2 = \ ^{\lambda}\ (\quad ^2 + 2\quad + \ ).$

Multiplying both sides by $e^{-\lambda}$, we arrive at an equation of the form 3.5.4.13 with $(\ ) = e^{-\lambda}$ and $g(\ ) = e^{-\lambda}\quad(\ )$.

7. $2\qquad - 3(\quad)^2 = f(\ )(\quad)^2 + \ (\ )(\quad)^4.$

Solution:

$$\frac{}{^2(\ )} = \frac{}{^2(\ )} + \ ,$$

where $= \ (\ )$ and $= \ (\ )$ are the general solutions of the second-order linear equations:

$$4\ '' - g(\ )\ = 0 \quad\text{and}\quad 4\ '' + \ (\ )\ = 0.$$

8. $2\qquad - 3(\quad)^2 = f(\ )(\quad)^2 + \ (\ )\ ^{-1}(\quad)^{5\ 2}.$

The substitution $(\ ) = \frac{}{\overline{\ '}}$ leads to a second-order nonhomogeneous linear equation: $4\ '' + \ (\ )\ + g(\ ) = 0.$

9. $2\qquad - 3(\quad)^2 = f(\ )(\quad)^4 + \ ^{-1}\ (\ )(\quad)^{7\ 2}.$

Taking to be the independent variable, we obtain an equation of the form 3.5.4.8 for $= \ (\ )$: $2\ '\ ''' - 3(\ '')^2 = -\ (\ )(\ ')^2 - g(\ )\ ^{-1}(\ ')^{5\ 2}.$

10. $2 \quad - \; (\quad)^2 + \quad + \; (\;)(\quad)^2 = \; ^{1-} \; (\quad ^2 + 2 \quad + \;).$

Multiplying both sides by $\quad ^{-1}$, we arrive at an equation of the form 3.5.4.13 with $\; (\;) =$ and $g(\;) = \quad ^{-1} \; (\;)$.

11. $\quad - 3 \; (\quad)^2 + 3 \quad = f(\;)(\quad)^4 + \; (\;)(\quad)^5.$

Taking to be the independent variable, we obtain an equation of the form 3.5.3.10 for $\; = \; (\;)$: $\quad ''' \; + 3 \; ' \; '' \; = -g(\;) - \; (\;) \quad '$.

12. $\quad = f(\;) \quad (\quad) \; .$

Solution for $\; \neq 1$:

$$_3 \; = \; 2 \; (1 - \;) \; (\;) + \; _2 \; ^{\frac{1}{1-}} \; + \; _1 \; ^{-\frac{1}{2}} \; , \quad \text{where} \quad (\;) = \; (\;) \; .$$

Solution for $\; = 1$:

$$_3 \; = \; _2 \; e^{F(\;)} \; + \; _1 \; ^{-\frac{1}{2}} \; , \quad \text{where} \quad (\;) = \; (\;) \; .$$

13. $2f \quad - f(\quad)^2 + f \quad + \; (\;)(\quad)^2 = \; ^2 + 2 \quad + \; , \quad f = f(\;).$

Differentiating both sides of the equation with respect to and dividing by $\; '$, we arrive at a fourth-order linear equation: $\quad '''' \; + \frac{3}{2} \; ' \; ''' \; + (g + \frac{1}{2} \; '' \;) \; '' \; + \frac{1}{2} g' \; ' = a \; + \; .$

3.5.4-2. Arguments of arbitrary functions depend on , , and $'$.

14. $2 \quad - 3(\quad)^2 = \; ^4 f \; \frac{\quad}{2} \; .$

The substitution $\; (\;) = \; (\; ')^{-1 \; 2}$ leads to a second-order autonomous equation of the form 2.9.1.1: $\; '' \; = \; (\;)$, where $\; (\;) = -\frac{1}{4} \; ^5 \; (\; ^{-2})$.

15. $2 \quad - 3(\quad)^2 = \; ^{-4} \; ^4 f \; \frac{\quad}{2} \; .$

The substitution $\; (\;) = \; (\; ')^{-1 \; 2}$ leads to a second-order equation of the form 2.9.1.9: $\; '' \; = \; ^{-3 \; 2} \; (\quad ^{-1 \; 2})$, where $\; (\xi) = -\frac{1}{4} \xi^5 \; (\xi^{-2})$.

16. $2 \quad - 3(\quad)^2 = \; ^{-8} \; ^4 f \; \frac{\quad 2}{\quad 2} \; .$

The substitution $\; (\;) = \; (\; ')^{-1 \; 2}$ leads to a second-order equation of the form 2.9.1.8: $\; '' \; = \; ^{-3} \; (\quad ^{-1})$, where $\; (\xi) = -\frac{1}{4} \xi^5 \; (\xi^{-2})$.

17. $\quad = \left[\; ^3 f(\quad - \;) + \quad ^{-5} \right] (\quad)^3.$

The Legendre transformation $\; = \; '$, $\; = \; ' - \;$ leads to an equation of the form 3.5.2.4: $\; ''' \; = - \; (\;)(\; ')^3 - a(\; ')^{-5}$.

18. $\quad = \; ^{-5} f \; \frac{\quad - \quad}{\quad 2} \; (\quad)^3.$

The Legendre transformation $\; = \; '$, $\; = \; ' - \;$ leads to an equation of the form 3.5.2.27: $\; ''' \; = \; ^{-5 \; 2} \; (\; ' \; ^{-1 \; 2})$, where $\; (\xi) = -\xi^{-5} \; (\xi^{-2})$.

19. $= {}^{-5} f \, \dfrac{-}{4} \; (\quad)^3.$

The Legendre transformation $= {}'$, $= {}' -$ leads to an equation of the form 3.5.2.28: $''' = {}^{-5\;4} \; (\; ' \; {}^{-1\;4})$, where $(\xi) = -\xi^{-5} \; (\xi^{-4})$.

20. $= [\; f(\quad) + \; (\quad) + \; (\quad)](\quad)^3 + \; (\quad)(\quad)^2.$

The Legendre transformation $= {}'$, $= {}' -$ leads to a linear equation:

$$''' = - \; (\,) \; '' - [\; (\,) + \; g(\,)] \; ' + g(\,) \; - \; (\,).$$

21. $= f(\quad - \;)(\quad)^2 + \; (\quad - \;)(\quad)^k.$

The Legendre transformation $= {}'$, $= {}' -$ leads to the equation $''' = - \; (\,) \; ' \; '' - g(\,) \; '(\; '')^{3-}$. Lowering its order with the substitution $z(\,) = {}''$ $(z'_w = {}''' \; ')$, we have a Bernoulli equation: $z'_w = - \; (\,)z - g(\,)z^{3-}$.

3.5.5. Other Equations

3.5.5-1. Equations of the form $(\, , \, , \, ' , \, '' \,) \, ''' + (\, , \, , \, ' , \, '' \,) = 0.$

1. $= f(\quad).$

Solution in parametric form:

$$= \; {}_1 \, \frac{1}{(\;_1)}, \qquad = \; {}_2 \, \frac{1}{(\;_1)} \; {}^1_3 \, \frac{2 \;\; 2}{(\;_2)}.$$

2. $= f(\,) \; (\quad).$

Integrating the equation and substituting $(\,) = \frac{1}{2}(\, ')^2$, we arrive at a first-order equation:

$$\frac{\xi}{g(\xi)} = \; (\,) \; + \; , \qquad \text{where} \quad \xi = {}'.$$

Solving this equation for $'$, we obtain a separable equation.

3. $= f(\,) \; (\quad) \; (\quad).$

The substitution $(\,) = \frac{1}{2}(\, ')^2$ leads to a second-order equation:

$$'' \; = \; (\,) \; (\,) \; (\, '), \qquad \text{where} \quad (\,) = \frac{g(\; \overline{2}\;}{\overline{2}},$$

whose solvable cases for some functions $, g$, and are outlined in Section 2.7.

4. $= \; f(\quad - \;) \; (\quad).$

The Legendre transformation $= {}'$, $= {}' -$, where $= (\,)$, leads to an equation of the form 3.5.5.2: $''' = - \; (\,) \; ' g(1 \; '')(\, '')^3$.

5. $+ \quad = f(\quad - \;) \; (\quad).$

The substitution $(\,) = {}' -$ leads to an equation of the form 2.9.4.36: $'' \; = \; (\,) g(\, ')$.

6. $= f(\,) \; (\, {}^2 \quad - 2 \quad + 2 \;).$

The substitution $(\,) = {}^2 \; '' \; - 2 \; ' + 2$ leads to a first-order separable equation: $' = {}^2 \; (\,) g(\,)$.

7. $= \text{———} + \quad - \frac{(\ )^2}{\text{———}} f \ \text{—}\big) \quad \text{—}\big).$

The transformation $= {}'$, $= {}''$ leads to a first-order separable equation: $' = (\ )g(\ )$.

8. $= (\ ,\ ,\ )$.

The substitution $(\ ) = {}'$ leads to a second-order equation: $'' = (\ ,\ ,\ ')$.

9. $= (\ ,\ ,\ )$.

Autonomous equation. The substitution $(\ ) = (\ ')^2$ leads to a second-order equation: $'' = \frac{2}{=} \big(\ ,\ \ \overline{\ }, \frac{1}{2}\ '\ .$

10. $= (\ \ ,\ \ )$.

This is a special case of equation 3.5.5.9. The transformation $= {}'$, $= {}''$ leads to a first-order equation: $(\ -\ ^2)\ ' = -\ \ +\ (\ ,\ )$.

11. $= {}^{-k-3} (\ ^{k}\ ,\ \ ^{k+1}\ ,\ \ ^{k+2}\ )$.

Generalized homogeneous equation. The transformation $= \ln$, $z =$, followed by the substitution $(z) = (z')^2$, leads to a second-order equation:

$$'' = 3(k+1)\ \ ^{-1}\ ^{2}\ \ ' - 6k^2 - 12k - 4 \quad 2k(k+1)(k+2)z\ \ ^{-1}\ ^{2}$$
$$2\ \ ^{-1}\ ^{2}\ \big(z,\ \ ^{1}\ ^{2} - kz, \tfrac{1}{2}\ \ ' \mp (2k+1)\ \ ^{1}\ ^{2} + k(k+1)z\ .$$

12. $= {}^{-3} (\ ^{k}\ ,\ \ ,\ ^{2}\ )$.

Generalized homogeneous equation. The transformation $=$, $z = {}'$ leads to a second-order equation.

13. $= {}^{-3} (\ \ ,\ ^{2}\ )$.

This is a special case of equation 3.5.5.12. The transformation $z = {}'$, $= {}^{2}\ ''$ leads to a first-order equation: $(\ + z - z^2)\ ' = 2\ \ - z\ \ +\ (z,\ )$.

14. $= (\ ,\ \ -\ ,\ \ )$.

The substitution $z = {}' -$ leads to a second-order equation: $z'' = z' +\ ^{2}\ (\ , z, z'\ )$.

15. $= f(\ ,\ \ +\ \ +\ ) +\ .$

The substitution $= {}'' + {}' +$ leads to a first-order equation: $' = (\ ,\ ) +\ .$

16. $= f(\ ,\ \ -\ \ +\ ) -\ .$

The substitution $= {}'' - {}' +$ leads to a first-order equation: $' = (\ ,\ ) -\ .$

17. $= (\ ,\ ,\ ,\ )$.

The Legendre transformation $= {}'$, $= {}' -$ leads to the equation

$$''' = -\ \big(\ ',\ \ ' -\ ,\ ,\ 1\ \ ''\ (\ '')^3.$$

18. $= f(\ \ -\ \ ^{2})$.

1 . Particular solution:

$$= {}_1 \exp(\ _3\ ) + {}_2 \exp(-\ _3\ ),$$

where the constants $_1$, $_2$, and $_3$ are related by the constraint $\frac{2}{3} = \big(4\ _1\ _2\ \frac{2}{3}\ .$

2 . Particular solution:

$$= {}_1 \cos(\ _3\ ) + {}_2 \sin(\ _3\ ),$$

where the constants $_1$, $_2$, and $_3$ are related by the constraint $\frac{2}{3} + \big(-(\ \frac{2}{1} + \ \frac{2}{2})\ \frac{2}{3}\ = 0.$

19. $-(\quad)^2 = f(\quad - \quad ^2).$

1 . Particular solution:

$$= \ _1\exp(\ _3\ ) + \ _2\exp(-\ _3\ ),$$

where the constants $_1$, $_2$, and $_3$ are related by the constraint $4\ _1\ _2\ ^4_3 + \ (4\ _1\ _2\ ^2_3) = 0.$

2 . Particular solution:

$$= \ _1\cos(\ _3\ ) + \ _2\sin(\ _3\ ),$$

where $_1$, $_2$, and $_3$ are related by the constraint $(\ ^2_1 + \ ^2_2)\ ^4_3 + \ \big(-(\ ^2_1 + \ ^2_2)\ ^2_3\ = 0.$

20. $= \ ^{-\lambda}\ (\ ^{\lambda}\ ,\ ^{\lambda}\ ,\ ^{\lambda}\ ).$

The substitution $z = e^{\lambda}$ leads to an autonomous equation of the form 3.5.5.9:

$$z''' - 3\lambda z'' + 3\lambda^2 z' - \lambda^3 z = \ (z,\ z' - \lambda z,\ z'' - 2\lambda z' + \lambda^2 z).$$

21. $= \ (\ ^{\lambda}\ ,\ ,\ ).$

Equation invariant under "translation–dilatation" transformation. The transformation $z = e^{\lambda}\ ,\ = \ '$ leads to a second-order equation:

$$z^2(\ +\lambda)^2\ ''\ + z^2(\ +\lambda)(\ ')^2 + z(\ +\lambda)(4\ +\lambda)\ ' + \ ^3 = \ \big(z,\ ,\ z(\ +\lambda)\ ' + \ ^2\ .$$

22. $= \ ^{-3}\ (\ ,\ ,\ ^2\ ).$

Equation invariant under "dilatation–translation" transformation. The transformation $z = \ e\ ,\ = \ ' + $ leads to a second-order equation:

$$z^2\ ^2\ '' + z^2\ (\ ')^2 + z\ ^2\ ' - 3z\ ' + 2\ - 2\ = \ (z,\ -\ ,\ z\ ' - \ + \ ).$$

$\boxed{\text{3.5.5-2. Equations of the form } (\ ,\ ,\ ',\ '',\ ''') = 0.}$

23. $-\frac{1}{3}\ = f(\ ) + \ (\ ).$

Particular solution:

$$= \tfrac{1}{6}\ _1\ ^3 + \tfrac{1}{2}\ _2\ ^2 + \ _3\ + \ _4,$$

where the constants $_1$, $_2$, $_3$, and $_4$ are related by two constraints

$$2\ _1\ _3 - \ ^2_2 = 3\ (\ _1),$$
$$3\ _1\ _4 - \ _2\ _3 = 3g(\ _1).$$

Here, $_3$ and $_4$ are defined in terms of two arbitrary constants $_1$ and $_2$.

24. $-\frac{1}{3}\ = \ f(\ ) + \ (\ ) + \ (\ ).$

Particular solution:

$$= \tfrac{1}{6}\ _1\ ^3 + \tfrac{1}{2}\ _2\ ^2 + \ _3\ + \ _4,$$

where the constants $_1$, $_2$, $_3$, and $_4$ are related by two constraints

$$\tfrac{2}{3}\ _1\ _3 - \tfrac{1}{3}\ ^2_2 = \ _1\ (\ _1) + g(\ _1),$$
$$_1\ _4 - \tfrac{1}{3}\ _2\ _3 = \ _2\ (\ _1) + \ (\ _1).$$

Here, $_3$ and $_4$ are defined in terms of two arbitrary constants $_1$ and $_2$.

25. $(\ ,\ \ -\ ^2,\ \ -\ \ ) = 0.$

The substitution $= \ '' - (\ ')^2$ leads to a first-order equation: $(\ ,\ ,\ ') = 0.$

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26. $\Big(\quad , \quad -(\ )^2, \quad , \quad -(\ )^2 \quad = 0.$

1 . Particular solution:

$$= {}_1 \exp(\ {}_3\ ) + {}_2 \exp(-\ {}_3\ ),$$

where the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint

$$(\ {}^2_3, 4\ {}_1\ {}_2\ {}^2_3,\ {}^2_3, -4\ {}_1\ {}_2\ {}^4_3 \quad = 0.$$

2 . Particular solution:

$$= {}_1 \cos(\ {}_3\ ) + {}_2 \sin(\ {}_3\ ),$$

where the constants ${}_1$, ${}_2$, and ${}_3$ are related by the constraint

$$\big(-\ {}^2_3, -(\ {}^2_1 + {}^2_2)\ {}^2_3, -\ {}^2_3, -(\ {}^2_1 + {}^2_2)\ {}^4_3 \quad = 0.$$

27. $\frac{}{}, \quad - \frac{}{}, \frac{}{} \quad = 0.$

Particular solution:

$$= {}_1 \exp(\ {}_2\ ) + {}_3,$$

where ${}_1$ is an arbitrary constant and the constants ${}_2$ and ${}_2$ are related by the constraint $(\ {}_2, -\ {}_2\ {}_3,\ {}^2_2) = 0.$

28. $\frac{}{} + \quad , \quad {}^{+1}\frac{}{} \quad = 0.$

A solution of this equation is any function that solves the following second-order autonomous equation of the form 2.9.1.1:

$$'' = {}_1\ {}^{-} + {}_2,$$

where the constants ${}_1$ and ${}_2$ are related by the constraint $(a\ {}_2, -a\ {}_1) = 0.$

29. $\frac{}{} + \quad , \quad \frac{}{} \quad = 0.$

A solution of this equation is any function that solves the following second-order autonomous equation of the form 2.9.1.1:

$$'' = {}_1 e^{-} + {}_2,$$

where the constants ${}_1$ and ${}_2$ are related by the constraint $(\ {}_2, -\ {}_1) = 0.$

30. $\frac{1}{}\frac{}{}, \quad - \frac{}{}\frac{}{} \quad = 0, \quad = (\ ).$

A solution of this equation is any function that solves the following second-order autonomous equation of the form 2.9.1.1:

$$'' = {}_1\ (\ ) + {}_2,$$

where the constants ${}_1$ and ${}_2$ are related by the constraint $(\ {}_1,\ {}_2) = 0.$

31. $(\quad , \quad - \quad , 2 \quad - \ {}^2, \quad) = 0.$

Particular solution:

$$= {}_1\ {}^2 + {}_2\ + {}_3,$$

where the constants ${}_1$, ${}_2$, and ${}_3$ are related by $(2\ {}_1, -\ {}_2, 4\ {}_1\ {}_3 - {}^2_2, 0) = 0.$

32. $(\quad , \quad - \quad , {}^2 \quad -2 \quad +2 \quad , {}^3 \quad -3\ {}^2 \quad +6 \quad -6\ ) = 0.$

Solution:

$$= {}_1\ {}^3 + {}_2\ {}^2 + {}_3\ + {}_4,$$

where the constants ${}_1$, ${}_2$, ${}_3$, and ${}_4$ are related by $(6\ {}_1, -2\ {}_2, 2\ {}_3, -6\ {}_4) = 0.$

Chapter 4

Fourth-Order Differential Equations

4.1. Linear Equations

4.1.1. Preliminary Remarks

1 . A nonhomogeneous linear equation of the fourth order has the form

$$ {}_4 \;'''' + {}_3 \;''' + {}_2 \;'' + {}_1 \;' + {}_0 \; = g(\;), \qquad = \;(\;). \tag{1}$$

Let ${}_0 = {}_0(\;)$ be a nontrivial particular solution of the corresponding homogeneous equation (with $g \equiv 0$). Then the substitution

$$ = {}_0(\;) \quad z(\;) \tag{2}$$

leads to a third-order linear equation:

$$ {}_4\, {}_0 z''' + (4\, {}_4\, {}_0' + {}_3\, {}_0)z'' + (6\, {}_4\, {}_0'' + 3\, {}_3\, {}_0' + {}_2\, {}_0)z' + (4\, {}_4\, {}_0''' + 3\, {}_3\, {}_0'' + 2\, {}_2\, {}_0' + {}_1\, {}_0)z = g,$$

where the prime denotes differentiation with respect to .

2 . Let ${}_1 = {}_1(\;)$ and ${}_2 = {}_2(\;)$ be two nontrivial linearly independent particular solutions of equation (1) with $g \equiv 0$. Then the substitution

$$ = {}_1 \quad {}_2 \quad - {}_2 \quad {}_1$$

leads to a second-order linear equation:

$$ {}_4\Delta_1 \;'' + (3\, {}_4\Delta_2 + {}_3\Delta_1)\;' + [\, {}_4(3\Delta_3 + 2\varepsilon) + 2\, {}_3\Delta_2 + {}_2\Delta_1] \; = g,$$

where

$$\Delta_1 = {}_1'\, {}_2 - {}_1\, {}_2', \quad \Delta_2 = {}_1''\, {}_2 - {}_1\, {}_2'', \quad \Delta_3 = {}_1'''\, {}_2 - {}_1\, {}_2''', \quad \varepsilon = {}_1''\, {}_2' - {}_1'\, {}_2''.$$

See also Subsections 0.4.1 and 0.4.2.

4.1.2. Equations Containing Power Functions

4.1.2-1. Equations of the form ${}_4(\;)\;'''' + {}_0(\;)\; = g(\;)$.

1. $\quad + \quad = 0.$

1 . Solution for $a = 0$:

$$ = {}_1 + {}_2 \; + {}_3 \;^2 + {}_4 \;^3.$$

2 . Solution for $a = 4k^4 > 0$:

$$ = {}_1 \cosh k \;\; \cos k \; + {}_2 \cosh k \;\; \sin k \; + {}_3 \sinh k \;\; \cos k \; + {}_4 \sinh k \;\; \sin k \; .$$

3 . Solution for $a = -k^4 < 0$:

$$ = {}_1 \cos k \; + {}_2 \sin k \; + {}_3 \cosh k \; + {}_4 \sinh k \; .$$

2. $\quad + \quad = \;^3 + \;^2 + \; + s, \qquad \neq 0.$

Solution: $ = \frac{1}{\lambda}(a\;^3 + \;^2 + \; + \;) + \;(\;)$, where $\;(\;)$ is the general solution of equation 4.1.2.1: $\;'''' + \lambda\; = 0$.

3. $= \quad + \;.$

This is a special case of equation 5.1.2.3 with $= 4$.

4. $= \quad .$

This is a special case of equation 5.1.2.4 with $= 4$. For $\beta = -2, -4, -6, -8$, and -9, see equations 4.1.2.5, 4.1.2.6, 4.1.2.7, 4.1.2.8, and 4.1.2.12, respectively.

The transformation $= {}^{-1}, \; = {}^{-3}$ leads to an equation of the same form: $'''' = a \; {}^{-} \; {}^{-8} \;.$

5. ${}^{2} \quad = \quad .$

This is a special case of equation 5.1.2.6 with $= 2$.

6. ${}^{4} \quad = \quad .$

Solution:

$$= {}_{1} \; {}^{1} + {}_{2} \; {}^{2} + {}_{3} \; {}^{3} + {}_{4} \; {}^{4}, \quad k_{1,2} = \tfrac{3}{2} \; \left(\tfrac{5}{4} + \overline{a+1} \; {}^{1 \; 2}, \quad k_{3,4} = \tfrac{3}{2} \; \left(\tfrac{5}{4} - \overline{a+1} \; {}^{1 \; 2}.$$

7. ${}^{6} \quad = \quad .$

This is a special case of equation 5.1.2.7 with $= 2$.

8. ${}^{8} \quad = \quad .$

The transformation $= {}^{-1}, \; = {}^{-3}$ leads to a constant coefficient linear equation of the form 4.1.2.1: $'''' = a \;.$

9. $(\; + \;)^4 (\; + \;)^4 \quad = \quad .$

The transformation $\xi = \ln \dfrac{a \; +}{+}, \quad = \dfrac{}{(\; + \;)^3}$ leads to a constant coefficient linear equation.

10. $(\; {}^2 + \; + \;)^4 \quad = \quad .$

The transformation $\xi = \dfrac{}{a \; {}^2 + \; +}, \quad = \dfrac{}{(a \; {}^2 + \; + \;)^{3 \; 2}}$ leads to a constant coefficient linear equation: $'''' - \tfrac{5}{2} \; '' + \left(\tfrac{9}{16} \; {}^2 - k \; = 0$, where $= {}^2 - 4a \;.$

11. $(\; + \;)^2 (\; + \;)^6 \quad = \quad .$

The transformation $\xi = \dfrac{a \; +}{+}, \quad = \dfrac{}{(\; + \;)^3}$ leads to an equation of the form 4.1.2.5: $\xi^2 \; '''' = k\Delta^{-4}$, where $\Delta = a \; - \;.$

12. ${}^{9} \quad = \; + \; {}^{4}.$

The transformation $= {}^{-1}, \; = {}^{-3}$ leads to an equation of the form 4.1.2.3: $4 \quad = a \quad + \;.$

13. $(\; + \;)^9 \quad = (\; + \;) \;.$

The transformation $\xi = \dfrac{+}{a \; +}, \quad = \dfrac{}{(a \; + \;)^3}$ leads to an equation of the form 4.1.2.3: $'''' = \Delta^{-4}\xi$, where $\Delta = a \; - \;.$

$\boxed{\text{4.1.2-2. Equations of the form } {}_{4}(\;) \; '''' \; + {}_{1}(\;) \; ' + {}_{0}(\;) \; = g(\;).}$

14. $+ \quad + \quad = 0.$

This is a special case of equation 4.1.2.41 with $a_2 = a_3 = 0$.

15. $+2 \quad - \;^{2}\;^{2} \quad = 0.$

This is a special case of equation 4.1.2.25 with $= 1$.

16. $+4 \quad + (2 \; - \;^{2}\;^{4}) \quad = 0.$

This is a special case of equation 4.1.2.25 with $= 2$.

17. $+ (\;_{1} \; + \;_{1}) \quad + (\;_{2} \; + \;_{2}) \quad = 0.$

This is a special case of equation 5.1.2.35 with $= 4$.

18. $+ \quad (2 \; -3 \; - \;^{2}\;^{2}) \quad + \; (2 \; - \; + \;^{2}\;^{2}) \quad = 0.$

The substitution $= \;'' - a \;' + \;$ leads to a second-order equation of the form 2.1.2.31: $'' + a \;' + (2a - \; + a^{2}\;^{2}) \; = 0$.

19. $+ \;^{k} \; - \;^{k-1} \; = \; .$

For $= 0$, a particular solution is: $_{0} = \;$. The substitution $z = \;' - \;$ leads to a third-order linear equation.

20. $+ \;^{k} \; - 2 \;^{k-1} \; = \; .$

For $= 0$, a particular solution is: $_{0} = \;^{2}$. The substitution $z = \;' - 2 \;$ leads to a third-order linear equation.

21. $+ \;^{k} \; - 3 \;^{k-1} \; = \; .$

For $= 0$, a particular solution is: $_{0} = \;^{3}$. The substitution $z = \;' - 3 \;$ leads to a third-order linear equation: $z''' + a \; z = \;^{+1}$ (for $= 0$, see 3.1.2.7).

22. $+ \;^{k} \; + \;^{k-1} \; = \; .$

Integrating yields a third-order linear equation: $''' + a \; = \frac{}{+1} \;^{+1} + \; .$

23. $+ \;^{k} \; + \; (\; + 3) \;^{k-1} \; = 0.$

The transformation $= \;^{-1}, \quad = \;^{-3}$ leads to an equation of the form 4.1.2.22 with $= 0$: $'''' + \; ' + \;^{-1} \; = 0$, where $= -a, \quad = -k - 6$.

24. $+ \;^{k} \; - \; (\;^{3} + \;^{k}) \; = 0.$

This is a special case of equation 4.1.6.4 with $= \;$.

25. $+2 \;^{-1} \; + \; [\; (\; -1) \;^{-2} - \;^{2} \;] \; = 0.$

The substitution $= \;'' + a \;$ leads to a second-order equation of the form 2.1.2.7: $'' - a \; = 0$.

26. $+ (\; + \;) \;^{k} \; - \;^{k} \; = 0.$

Particular solution: $_{0} = a \; + \; .$

27. $+ (\; + \;) \;^{k} \; -2 \;^{k} \; = 0.$

Particular solution: $_{0} = (a \; + \;)^{2}.$

28. $+ (\; + \;) \;^{k} \; -3 \;^{k} \; = 0.$

Particular solution: $_{0} = (a \; + \;)^{3}.$

29. $+ (\;^{k} + \;^{3}) \; + \;^{k} \; = 0.$

Particular solution: $_{0} = e^{-b} \; .$

30. $+ \;^{k+1} \; - [\; (\; + 1) \;^{k} + \; + 4] \; = 0.$

Particular solution: $_{0} = \; e \; .$

4.1.2-3. Equations of the form $_4(\ )\ ''''\ +\ _2(\ )\ ''\ +\ _1(\ )\ '\ +\ _0(\ )\ = g(\ )$.

31. $\quad + 2 \quad + \ ^2 \ = 0.$

Solution: $\ = \begin{cases} (\ _1 + \ _2\ )\cos(k\ ) + (\ _3 + \ _4\ )\sin(k\ ) & \text{if } a = k^2 > 0, \\ (\ _1 + \ _2\ )\exp(k\ ) + (\ _3 + \ _4\ )\exp(-k\ ) & \text{if } a = -k^2 < 0. \end{cases}$

32. $\quad + (\ + \ ) \quad + \quad = 0.$

The case $a = \ $ is given in 4.1.2.31. Let $a \neq \ $.

1 . Solution for $a = \ ^2 > 0,\ \ = \beta^2 > 0$:

$$= \ _1\cos(\ ) + \ _2\sin(\ ) + \ _3\cos(\beta\ ) + \ _4\sin(\beta\ ).$$

2 . Solution for $a = \ ^2 > 0,\ \ = -\beta^2 < 0$:

$$= \ _1\cos(\ ) + \ _2\sin(\ ) + \ _3\exp(\beta\ ) + \ _4\exp(-\beta\ ).$$

3 . Solution for $a = -\ ^2 < 0,\ \ = \beta^2 > 0$:

$$= \ _1\exp(\ ) + \ _2\exp(-\ ) + \ _3\cos(\beta\ ) + \ _4\sin(\beta\ ).$$

4 . Solution for $a = -\ ^2 < 0,\ \ = -\beta^2 < 0$:

$$= \ _1\exp(\ ) + \ _2\exp(-\ ) + \ _3\exp(\beta\ ) + \ _4\exp(-\beta\ ).$$

33. $\quad + \quad + \quad + \quad ^{-1} \ = s \quad .$

Integrating yields a third-order linear equation: $\ ''' \ + a\ ' + \quad = \frac{}{+1}\ ^{+1} + \ .$

34. $\quad - 2\ ^2 \quad + \ ^4 \ - \ (\quad - \ )(\quad - \ ^2\ ) = 0.$

This equation arises in the turbulence theory. Setting $z(\ ) = \ '' \ - a^2\ $, one obtains a second-order linear equation of the form 2.1.2.12:

$$z'' - a^2 z - \lambda(a\ \ -\ \ )z = 0. \tag{1}$$

Let the following boundary conditions be given:

$$(0) = \ '(0) = 0, \qquad (1) = \ '(1) = 0, \tag{2}$$

The solution of the original equation satisfying the first two conditions in (2) can represented as:

$$2a\ \ = e\ \ \int_0 e^{-}\ \ z\ \ - e^{-}\ \int_0 e\ \ z\ \ .$$

To meet the last two conditions in (2), one should take the solution of (1) that satisfies the integral relations $\int_0^1 e^{-}\ \ z\ \ = \int_0^1 e\ \ z\ \ = 0.$

35. $\quad + (\ ^2 + \ ) \quad - 2 \quad = 0.$

Particular solution: $\ _0 = a\ ^2 + \ .$

36. $\quad + \quad + \ (\quad - \ ) \ = 0.$

1 . Particular solutions with $\ > 0$: $\ _1 = \cos(\ \ \bar{}\ ,\ \ _2 = \sin(\ \ \bar{}\ .$

2 . Particular solutions with $\ < 0$: $\ _1 = \exp(-\ \ \overline{-}\ ,\ \ _2 = \exp(\ \ \overline{-}\ .$

The substitution $\ = \ '' + \ $ leads to a second-order linear equation: $\ '' + (a\ \ - \ )\ = 0.$

37. $+ \quad ^{+1} \quad -4 \quad +6 \quad ^{-1} \quad = 0.$

Particular solutions: $_1 = \ ^2, \quad _2 = \ ^3$. The substitution $= \ ^2 \ '' - 4 \ ' + 6$ leads to a second-order linear equation of the form 2.1.2.7: $'' + a \ ^{+1} \ = 0$.

38. $+ 10 \quad + 10 \quad ^{-1} \quad + [3 \ (\ - 1) \ ^{-2} + 9 \ ^2 \ ^2 \ ] \ = 0.$

This is a special case of equation 4.1.6.25 with $= a$.

39. $+ (\quad + \ ) \quad + \quad = 0.$

1 . Particular solutions with > 0: $_1 = \cos(\ \ -$, $_2 = \sin(\ \ -$.

2 . Particular solutions with < 0: $_1 = \exp(- \ \ -$, $_2 = \exp(\ \ -$.

The substitution $= \ '' + $ leads to a second-order linear equation of the form 2.1.2.7: $'' + a \quad = 0$.

40. $^2 \quad - 2(\ ^2 + 6) \quad + (\ ^2 + 4) \ = 0.$

Particular solutions: $_1 = \ ^{-1} \ ^2 \ _{1\ 2}(\ \overline{a}$, $_2 = \ ^{-1} \ ^2 \ _{1\ 2}(\ \overline{a}$, where $_{1\ 2}(z)$ and $_{1\ 2}(z)$ are the modified Bessel functions.

4.1.2-4. Other equations.

41. $+ \ _3 \quad + \ _2 \quad + \ _1 \quad + \ _0 \ = 0.$

A fourth-order constant coefficient linear equation. For $a_0 = 0$, the substitution $(\ ) = \ '$ leads to a third-order equation. Let $a_0 \neq 0$ and $(\lambda) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ be the characteristic polynomial.

1 . Let be factorizable, so that $(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$, where $\lambda_1, \lambda_2, \lambda_3$, and λ_4 are real numbers. The following cases are possible:

a) λ are all different, then

$$= \ _1 e^{\lambda_1} \ + \ _2 e^{\lambda_2} \ + \ _3 e^{\lambda_3} \ + \ _4 e^{\lambda_4} \ ;$$

b) $\lambda_1 = \lambda_2$; λ_3 and λ_4 are different and not equal to λ_1, then

$$= (\ _1 + \ _2 \ ) e^{\lambda_1} \ + \ _3 e^{\lambda_3} \ + \ _4 e^{\lambda_4} \ ;$$

c) $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$, then

$$= (\ _1 + \ _2 \ + \ _3 \ ^2) e^{\lambda_1} \ + \ _4 e^{\lambda_4} \ ;$$

d) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, then

$$= (\ _1 + \ _2 \ + \ _3 \ ^2 + \ _4 \ ^3) e^{\lambda_1} \ .$$

2 . Let $(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda^2 + 2 \ _1\lambda + \ _0)$, where λ_1 and λ_2 are real numbers, and $^2_1 - \ _0 < 0$. The following cases are possible:

a) $\lambda_1 \neq \lambda_2$, then

$$= \ _1 e^{\lambda_1} \ + \ _2 e^{\lambda_2} \ + e^{-b_1} \ [\ _3 \cos(\ ) + \ _4 \sin(\ )], \qquad = \sqrt{\ _0 - \ ^2_1};$$

b) $\lambda_1 = \lambda_2$, then

$$= (\ _1 + \ _2 \ ) e^{\lambda_1} \ + e^{-b_1} \ [\ _3 \cos(\ ) + \ _4 \sin(\ )], \qquad = \sqrt{\ _0 - \ ^2_1}.$$

3 . Let us assume that $(\lambda) = (\lambda^2 + 2\ {}_1\lambda + {}_0)(\lambda^2 + 2\beta_1\lambda + \beta_0)$, where ${}^2_1 - {}_0 < 0$ and $\beta_1^2 - \beta_0 < 0$. The following cases are possible:

a) $({}_1 - \beta_1)^2 + ({}_0 - \beta_0)^2 \neq 0$, then

$$= e^{-b_1}\ [\ {}_1 \cos(\ ) + {}_2 \sin(\ )] + e^{-\ 1}\ [\ {}_3 \cos(\ ) + {}_4 \sin(\ )],$$

where $= \sqrt{{}_0 - {}^2_1}$, $= \sqrt{\beta_0 - \beta_1^2}$;

b) ${}_1 = \beta_1$, ${}_0 = \beta_0$, then

$$= e^{-b_1}\ [(\ {}_1 + {}_2\ ) \cos(\ ) + (\ {}_3 + {}_4\ ) \sin(\ )], \qquad = \sqrt{{}_0 - {}^2_1}.$$

42. $+ 4 \quad + 6\ {}^2\ {}^2 \quad + 4\ {}^3\ {}^3 \quad + \ {}^4\ {}^4 = 0.$

Solution: $= \sum_{=1}^{4} \exp(\lambda \ - \tfrac{1}{2}a\ {}^2)$, where the λ are roots of the biquadratic equation $\lambda^4 - 6a\lambda^2 + 3a^2 = 0$.

43. $+ (\ + \ ) \quad + [\ (\ + \ ) + \ ] \quad + \ {}^2 \quad - \ {}^2 = 0.$

Particular solutions: ${}_1 =\ $, ${}_2 = e^{-b}\ $.

44. $= \quad + \quad - \quad .$

Particular solutions: $= \exp(\lambda\ )$ $(k = 1, 2, 3)$, where the λ are roots of the cubic equation $\lambda^3 - \ = 0$.

45. $+ \ {}^{+3} \quad - 3 \ {}^{+2} \quad + 6 \ {}^{+1} \quad - 6 \quad = 0.$

Particular solutions: ${}_1 =\ $, ${}_2 = {}^2$, ${}_3 = {}^3$. The substitution $= {}^3\ ''' - 3\ {}^2\ '' + 6\ ' - 6$ leads to a first-order linear equation: $' + a\ {}^{+3} \ = 0$.

46. $+ \quad + \ {}^{+1} \quad - 2 \quad + 2 \ {}^{-1} = 0.$

Particular solutions: ${}_1 =\ $, ${}_2 = {}^2$. The substitution $= {}^2\ '' - 2\ ' + 2$ leads to a second-order linear equation: $'' + (a\ {}^{+1} - 2)\ ' + \ {}^{+2} = 0$.

47. $+ \quad + \quad + \quad + (\ - \ ) = 0.$

1 . Particular solutions with > 0: ${}_1 = \cos(\ \overline{\ }$, ${}_2 = \sin(\ \overline{\ }$.

2 . Particular solutions with < 0: ${}_1 = \exp(-\ \overline{\ }$, ${}_2 = \exp(\ \overline{\ }$.

The substitution $= '' +$ leads to a second-order equation: $'' + a\ ' + (\ - \ ) = 0$.

48. $+ \quad + (\ + \ ) \quad + \quad + \quad = 0.$

1 . Particular solutions with > 0: ${}_1 = \cos(\ \overline{\ }$, ${}_2 = \sin(\ \overline{\ }$.

2 . Particular solutions with < 0: ${}_1 = \exp(-\ \overline{\ }$, ${}_2 = \exp(\ \overline{\ }$.

The substitution $= '' +$ leads to a second-order linear equation: $'' + a\ ' + \ = 0$.

49. $+ 4 \quad + \quad = 0.$

The substitution $(\ ) =$ leads to a constant coefficient linear equation of the form 4.1.2.1: $'''' + a\ = 0$.

50. $- 4 \quad + \quad = 0, \qquad = 1, 2, 3, \ldots$

Solution: $= {}^4\ {}^{+3}(\ {}^{-3}\ )\ (\ {}^{-3}\ )$, where $= —$ and $= (\ )$ is the general solution of a linear constant coefficient equation of the form 4.1.2.1: $'''' + a\ = 0$.

51. $\;^{2}\quad + 6 \quad + 6 \quad - \;^{2} = 0.$

Equation of transverse vibrations of a pointed bar.

Solution: $= \frac{1}{}\big[\;_1\;_1(2\;\overline{a}\; + \;_2\;_1(2\;\overline{a}\; + \;_3\;_1(2\;\overline{a}\; + \;_4\;_1(2\;\overline{a}\;\big],$

where $_1(z)$ and $_1(z)$ are the Bessel functions, and $_1(z)$ and $_1(z)$ are the modified Bessel functions.

52. $\;^{2}\quad + 2(\;+2)\quad + (\;+1)(\;+2)\quad - \;^{4} = 0.$

Solution: $= \;^{-}\;^{2}\big[\;_1\;(2\;^{-} + \;_2\;(2\;^{-} + \;_3\;(2\;^{-} + \;_4\;(2\;^{-}\big]$, where (z) and (z) are the Bessel functions, and (z) and (z) are the modified Bessel functions.

53. $\;^{2}\quad + 8 \quad + 12 \quad + \;^{2} = 0.$

The substitution $(\;) = \;^{2}$ leads to a constant coefficient linear equation of the form 4.1.2.1: $'''' + a = 0.$

54. $\;^{2}\quad + 8 \quad + 12 \quad = \;^{3} + \;.$

The substitution $(\;) = \;^{2}$ leads to an equation of the form 4.1.2.3: $'''' = a \; + \;.$

55. $\;^{2}\quad + \quad + (\;^{+1} + \;)\quad + (\;-4)\quad + (\;-2\;+6)\;^{-1} = 0.$

The substitution $(\;) = \;^{2}\;'' + (a-4)\;' + (\;-2a+6)$ leads to a second-order equation of the form 2.1.2.7: $'' + \;^{-1} = 0.$

56. $\;^{3}\quad + 2\;^{2}\quad - \quad + \quad - \;^{4}\;^{3} = 0.$

Solution: $= \;_1\;_0(a\;) + \;_2\;_0(a\;) + \;_3\;_0(a\;) + \;_4\;_0(a\;)$, where $_0(z)$ and $_0(z)$ are the Bessel functions, and $_0(z)$ and $_0(z)$ are the modified Bessel functions.

57. $\;^{4}\quad + \;_3\;^{3}\quad + \;_2\;^{2}\quad + \;_1\quad + \;_0 = 0.$

The Euler equation. The substitution $= \ln|\;|$ leads to a constant coefficient linear equation of the form 4.1.2.41: $'''' + (A_3-6)\;''' + (11-3A_3+A_2)\;'' + (2A_3-A_2+A_1-6)\;' + A_0 = 0.$

58. $\;^{4}\quad + 2\;^{3}\quad - (2\;^{2}+1)\;^{2}\quad + (2\;^{2}+1)\quad - \big[\;^{4}\;^{4} - \;^{2}(\;^{2}-4)\big] = 0.$

This equation governs free transverse vibration modes of a thin round elastic plate. The equation arises from separation of variables in the two-dimensional equation $\Delta\Delta\; - \;^{4} = 0$, where Δ is the Laplace operator written in the polar coordinate system, with being the polar radius.

Solution: $= \;_1\;(\;) + \;_2\;(\;) + \;_3\;(\;) + \;_4\;(\;)$, where (z) and (z) are the Bessel functions, and (z) and (z) are the modified Bessel functions. In applications, one usually sets $a =$, where $= 0, 1, 2,$

The solution is specified by Popov (1998).

59. $\;^{4}\quad - 2\;(\;+1)\;^{2}\quad + 4\;(\;+1)\quad + [\;^{4} + \;(\;+1)(\;+3)(\;-2)] = 0.$

Here, is a positive integer and $a \neq 0$ (for $a = 0$, we have the Euler equation 4.1.2.57).

Solution: $= \;^{-}\sum_{=1}^{4} \exp(\lambda\;)\;(\;)$, where the λ are four different roots of the equation $\lambda^4 + a = 0$, and $(\;)$ is some definite polynomial of degree ≤ 4 .

60. $\;^{4}\quad + 2(2-\;)\;^{3}\quad + (1-\;)(2-\;)\;^{2}\quad - \;^{4}\;^{2} = 0.$

Solution: $= \;^{-}\big[\;_1\;_1(\xi) + \;_2\;_1(\xi) + \;_3\;_1(\xi) + \;_4\;_1(\xi)\big]$, where $\xi = 2(a\;)\;^{2}$; (ξ) and (ξ) are the Bessel functions, and (ξ) and (ξ) are the modified Bessel functions.

61. $\;^{4}\quad +6\;^{3}\quad +[4\;^{4}+(7-\;^{2}-\;^{2})\;^{2}]\quad +\;(16\;^{2}+1-\;^{2}-\;^{2})\quad +(8\;^{2}+\;^{2}\;^{2})\;=0.$

Solution for $a\neq 0$: $\;=\;_{1}\;(\;)\;\;(\;)+\;_{2}\;(\;)\;\;(\;)+\;_{3}\;(\;)\;\;(\;)+\;_{4}\;(\;)\;\;(\;)$, where $\;(\;)$ and $\;(\;)$ are the Bessel functions; $\;=\frac{1}{2}(a+\;)$ and $\;=\frac{1}{2}(a-\;)$.

62. $\;^{8}\quad +4\;^{7}\quad =\quad.$

The substitution $\;(\;)=\;$ leads to an equation of the form 4.1.2.8: $\;^{8}\;''''\;=a\;.$

4.1.3. Equations Containing Exponential and Hyperbolic Functions

4.1.3-1. Equations with exponential functions.

1. $\quad+\;^{3}\quad+\quad(\;^{2}-\quad)\;=0.$

The substitution $\;=\;''+a\;'+\;e\;$ leads to a second-order linear equation of the form 2.1.3.10: $\;''\;-a\;'\;+(a^{2}-\;e\;)\;=0.$

2. $\quad+\;^{\lambda}\quad+\;^{\lambda}\quad=\quad.$

Integrating yields a third-order linear equation: $\;'''\;+ae^{\lambda}\;=\;^{-1}e\;+\;.$

3. $\quad+\;^{\lambda}\quad-(\;^{\lambda}+\;^{4})\;=0.$

Particular solution: $\;_{0}=e^{b}\;.$

4. $\quad+2\;^{\lambda}\quad+\;(\;^{2}\;^{\lambda}-\;^{2\lambda}\;)\;=0.$

The substitution $\;=\;''+ae^{\lambda}\;$ leads to a second-order linear equation of the form 2.1.3.1: $\;''\;-ae^{\lambda}\;=0.$

5. $\quad+(\;^{\lambda}+\;^{3})\quad+\;^{\lambda}\quad=0.$

Particular solution: $\;_{0}=e^{-b}\;.$

6. $\quad+(\;+\;)\;^{\lambda}\quad-\;^{\lambda}\quad=0.$

Particular solution: $\;_{0}=a\;+\;.$

7. $\quad+(\;+\;)\;^{\lambda}\quad-2\;^{\lambda}\quad=0.$

Particular solution: $\;_{0}=(a\;+\;)^{2}.$

8. $\quad+(\;+\;)\;^{\lambda}\quad-3\;^{\lambda}\quad=0.$

Particular solution: $\;_{0}=(a\;+\;)^{3}.$

9. $\quad+\;^{\lambda}\quad-\;(\;^{\lambda}+\;)\;=0.$

1 . Particular solutions with $\;>0$: $\;_{1}=\exp\bigl(-\;\overline{\;}\;,\;\;_{2}=\exp\bigl(\;\overline{\;}\;.$

2 . Particular solutions with $\;<0$: $\;_{1}=\cos\bigl(\;\overline{-}\;,\;\;_{2}=\sin\bigl(\;\overline{-}\;.$

The substitution $\;=\;''-\;$ leads to a second-order linear equation of the form 2.1.3.2: $\;''\;+(ae^{\lambda}\;+\;)\;=0.$

10. $\quad+(\;+\;^{\lambda}\;)\quad+\;^{\lambda}\quad=0.$

1 . Particular solutions with $a>0$: $\;_{1}=\cos\bigl(\;\overline{a}\;,\;\;_{2}=\sin\bigl(\;\overline{a}\;.$

2 . Particular solutions with $a<0$: $\;_{1}=\exp\bigl(-\;\overline{-a}\;,\;\;_{2}=\exp\bigl(\;\overline{-a}\;.$

The substitution $\;=\;''+a\;$ leads to a second-order linear equation of the form 2.1.3.1: $\;''\;+\;e^{\lambda}\;=0.$

11. $+ 10 \quad ^{\lambda} \quad + 10 \quad ^{\lambda} \quad + (3 \quad ^{2} \ ^{\lambda} + 9 \ ^{2} \ ^{2\lambda}) \quad = 0.$

This is a special case of equation 4.1.6.25 with $(\ ) = ae^{\lambda}$.

12. $+ \quad + \quad ^{\lambda} \quad + \quad ^{\lambda} \quad = 0.$

Particular solution: $_0 = e^{-}$.

13. $= \quad ^{\lambda} \quad + \quad - \quad ^{\lambda} \quad .$

Particular solutions: $= e$ $(k = 1, 2, 3)$, where the β are roots of the cubic equation $\beta^3 - \ = 0$.

14. $+ \quad ^{\lambda} \quad + \quad + \quad ^{\lambda} \quad + (\quad - \ ) \quad = 0.$

1 . Particular solutions with > 0: $_1 = \cos(\ ^{-}$, $_2 = \sin(\ ^{-}$.

2 . Particular solutions with < 0: $_1 = \exp(- \ \overline{-}$, $_2 = \exp(\ \overline{-}$.

The substitution $= '' +$ leads to a second-order equation: $'' + ae^{\lambda} \ ' + (\ e \ - \ ) \ = 0$.

15. $+ \quad ^{\lambda} \quad + (\quad + \ ) \quad + \quad ^{\lambda} \quad + \quad = 0.$

1 . Particular solutions with > 0: $_1 = \cos(\ ^{-}$, $_2 = \sin(\ ^{-}$.

2 . Particular solutions with < 0: $_1 = \exp(- \ \overline{-}$, $_2 = \exp(\ \overline{-}$.

The substitution $= '' +$ leads to a second-order equation: $'' + ae^{\lambda} \ ' + \ e \ = 0$.

16. $+ \quad ^{3} \ ^{\lambda} \quad - 3 \quad ^{2} \ ^{\lambda} \quad + 6 \quad ^{\lambda} \quad - 6 \quad ^{\lambda} \quad = 0.$

Particular solutions: $_1 = \ $, $_2 = \ ^2$, $_3 = \ ^3$. The substitution $= \ ^3 \ ''' - 3 \ ^2 \ '' + 6 \ ' - 6$ leads to a first-order linear equation: $' + a \ ^3 e^{\lambda} \ = 0$.

17. $+ \quad ^{\lambda} \quad - [\ (\ + 1) \ ^{\lambda} \ + \ + 4] \quad = 0.$

Particular solution: $_0 = \ e$.

18. $(\quad + \ ) \quad = \quad .$

Particular solution: $_0 = ae \ + \ $.

19. $(\quad + \quad + \ ) \quad = \quad , \quad = 1, 2, 3.$

Particular solution: $_0 = a \quad + \ e \ + \ $.

20. $(\quad + \ ) \quad = \quad , \quad = 0, 1, 2, 3.$

Particular solution: $_0 = a \quad + \ e^{-}$.

21. $+ \ \exp(\quad) \quad + [\ \exp(\quad) - \ ] \quad = 0.$

This is a special case of equation 4.1.6.20 with $(\ ) = \ \exp(\lambda \ )$.

22. $+ [\ + \ \exp(\quad)] \quad + \quad \exp(\quad) \quad = 0.$

This is a special case of equation 4.1.6.21 with $(\ ) = \ \exp(\lambda \ )$.

$\boxed{\text{4.1.3-2. Equations with hyperbolic functions.}}$

23. $+ \ \sinh(\quad) \quad + [\ \sinh(\quad) - \ ^{3}] \quad = 0.$

Particular solution: $_0 = e^{-b}$.

24. **+ [sinh ( ) + ^{3}] + sinh ( ) = 0.**

Particular solution: $_0 = e^{-b}$.

25. **+ (+) sinh ( ) − sinh ( ) = 0.**

Particular solution: $_0 = a$ + .

26. **+ (+) sinh ( ) − 2 sinh ( ) = 0.**

Particular solution: $_0 = (a$ + $)^2$.

27. **+ (+) sinh ( ) − 3 sinh ( ) = 0.**

Particular solution: $_0 = (a$ + $)^3$.

28. **+ sinh ( ) + [sinh ( ) −] = 0.**

This is a special case of equation 4.1.6.20 with ( ) = sinh (λ).

29. **+ [+ sinh ( )] + sinh ( ) = 0.**

This is a special case of equation 4.1.6.21 with ( ) = sinh (λ).

30. **(+ sinh) = sinh , = 1, 2, 3.**

Particular solution: $_0 = a$ + sinh .

31. **+ cosh ( ) + [cosh ( ) − ^{3}] = 0.**

Particular solution: $_0 = e^{-b}$.

32. **+ [cosh ( ) + ^{3}] + cosh ( ) = 0.**

Particular solution: $_0 = e^{-b}$.

33. **+ (+) cosh ( ) − cosh ( ) = 0.**

Particular solution: $_0 = a$ + .

34. **+ (+) cosh ( ) − 2 cosh ( ) = 0.**

Particular solution: $_0 = (a$ + $)^2$.

35. **+ (+) cosh ( ) − 3 cosh ( ) = 0.**

Particular solution: $_0 = (a$ + $)^3$.

36. **+ cosh ( ) + [cosh ( ) −] = 0.**

This is a special case of equation 4.1.6.20 with ( ) = cosh (λ).

37. **+ [+ cosh ( )] + cosh ( ) = 0.**

This is a special case of equation 4.1.6.21 with ( ) = cosh (λ).

38. **(+ cosh) = cosh , = 1, 2, 3.**

Particular solution: $_0 = a$ + cosh .

39. **= + (cosh − sinh).**

The substitution = $'$ cosh − sinh leads to a third-order linear equation.

40. **= + (sinh − cosh).**

The substitution = $'$ sinh − cosh leads to a third-order linear equation.

41. $+ \ \tanh(\ ) \ + [\ \tanh(\ ) - {}^{3}] = 0.$

Particular solution: ${}_0 = e^{-b}$.

42. $+ [\ \tanh(\ ) + {}^{3}] \ + \ \tanh(\ ) = 0.$

Particular solution: ${}_0 = e^{-b}$.

43. $+ (\ + \ ) \tanh(\ ) - \ \tanh(\ ) = 0.$

Particular solution: ${}_0 = a \ + \ .$

44. $+ (\ + \ ) \tanh(\ ) - 2\ \tanh(\ ) = 0.$

Particular solution: ${}_0 = (a \ + \ )^2.$

45. $+ (\ + \ ) \tanh(\ ) - 3\ \tanh(\ ) = 0.$

Particular solution: ${}_0 = (a \ + \ )^3.$

46. $+ \ \tanh(\ ) \ + [\ \tanh(\ ) - \ ] = 0.$

This is a special case of equation 4.1.6.20 with $(\ ) = \ \tanh(\lambda\ )$.

47. $+ [\ + \ \tanh(\ )] \ + \ \tanh(\ ) = 0.$

This is a special case of equation 4.1.6.21 with $(\ ) = \ \tanh(\lambda\ )$.

48. $+ \ \coth(\ ) \ + [\ \coth(\ ) - {}^{3}] = 0.$

Particular solution: ${}_0 = e^{-b}$.

49. $+ [\ \coth(\ ) + {}^{3}] \ + \ \coth(\ ) = 0.$

Particular solution: ${}_0 = e^{-b}$.

50. $+ \ \coth(\ ) \ + [\ \coth(\ ) - \ ] = 0.$

This is a special case of equation 4.1.6.20 with $(\ ) = \ \coth(\lambda\ )$.

51. $+ [\ + \ \coth(\ )] \ + \ \coth(\ ) = 0.$

This is a special case of equation 4.1.6.21 with $(\ ) = \ \coth(\lambda\ )$.

4.1.4. Equations Containing Logarithmic Functions

1. $+ \ \ln^k \ - (\ \ln^k \ + {}^4) = 0.$

Particular solution: ${}_0 = e^{b}$.

2. $+ (\ + \ ) \ln^k(\ ) - \ \ln^k(\ ) = 0.$

Particular solution: ${}_0 = a \ + \ .$

3. $+ (\ + \ ) \ln^k(\ ) - 2\ \ln^k(\ ) = 0.$

Particular solution: ${}_0 = (a \ + \ )^2.$

4. $+ (\ + \ ) \ln^k(\ ) - 3\ \ln^k(\ ) = 0.$

Particular solution: ${}_0 = (a \ + \ )^3.$

5. ${}^2 \ + 2 \ - [1 + {}^2 \ln^2(\ )] = 0.$

The substitution $= {}'' + a\ln(\ )$ leads to a second-order equation: ${}'' - a\ln(\ ) = 0.$

6. $\quad + \quad \ln^k(\quad) \quad + [\quad \ln^k(\quad) - \quad] \quad = 0.$

This is a special case of equation 4.1.6.20 with $(\;) = \quad \ln (\lambda \;)$.

7. $\quad + [\quad + \quad \ln^k(\quad)] \quad + \quad \ln^k(\quad) \quad = 0.$

This is a special case of equation 4.1.6.21 with $(\;) = \quad \ln (\lambda \;)$.

8. $\quad + \ln^k(\quad)(\;^2 \quad - 2 \quad + 2 \;) = 0.$

Particular solutions: $\;_1 = \;, \quad _2 = \;^2.$

9. $\quad + \ln^k(\quad)(\;^2 \quad - 4 \quad + 6 \;) = 0.$

Particular solutions: $\;_1 = \;^2, \quad _2 = \;^3.$

10. $\quad + \;^2 \ln^k(\quad) \quad - 2 \quad \ln^k(\quad) \quad = 0.$

Particular solution: $\;_0 = \;^2.$

11. $\quad + \quad + \quad \ln^k(\quad) \quad + \quad \ln^k(\quad) \quad = 0.$

Particular solution: $\;_0 = e^{-} \quad .$

4.1.5. Equations Containing Trigonometric Functions

4.1.5-1. Equations with sine and cosine.

1. $\quad + \quad \sin (\quad) \quad + [\quad \sin (\quad) - \;^3] \quad = 0.$

Particular solution: $\;_0 = e^{-b} \quad .$

2. $\quad + [\quad \sin (\quad) + \;^3] \quad + \quad \sin (\quad) \quad = 0.$

Particular solution: $\;_0 = e^{-b} \quad .$

3. $\quad + (\quad + \;) \sin (\quad) \quad - \quad \sin (\quad) \quad = 0.$

Particular solution: $\;_0 = a \quad + \quad .$

4. $\quad + (\quad + \;) \sin (\quad) \quad - 2 \quad \sin (\quad) \quad = 0.$

Particular solution: $\;_0 = (a \quad + \;)^2.$

5. $\quad + (\quad + \;) \sin (\quad) \quad - 3 \quad \sin (\quad) \quad = 0.$

Particular solution: $\;_0 = (a \quad + \;)^3.$

6. $\quad + \quad \sin (\quad) \quad + [\quad \sin (\quad) - \quad] \quad = 0.$

The substitution $\quad = \;'' + \quad$ leads to a second-order equation: $\quad'' + [a \sin \;(\lambda \;) - \quad] \quad = 0.$

7. $\quad + [\quad + \quad \sin (\quad)] \quad + \quad \sin (\quad) \quad = 0.$

The substitution $\quad = \;'' + a \quad$ leads to a second-order linear equation: $\quad'' + \quad \sin \;(\lambda \;) \quad = 0.$

8. $\quad = \quad \sin (\quad) \quad + \quad - \quad \sin (\quad) \quad .$

Particular solutions: $\quad = e \quad$ ($k = 1, 2, 3$), where the $\beta \quad$ are roots of the cubic equation $\beta^3 - \quad = 0.$

9. $\quad + \quad \sin (\quad)(\;^3 \quad - 3 \;^2 \quad + 6 \quad - 6 \;) = 0.$

This is a special case of equation 4.1.6.33 with $(\;) = a \sin \;(\lambda \;)$.

10. ^{2} + sin ( )(^{2} − 4 + 6) = 0.

The substitution = ^{2} ″ −4 ′ +6 leads to a second-order equation: ″ + a sin (λ) = 0.

11. (sin +) = sin .

Particular solution: $_{0}$ = a sin + .

12. (+ sin) = sin .

Particular solution: $_{0}$ = a + sin .

13. (+ sin) = sin , = 2, 3.

Particular solution: $_{0}$ = a + sin .

14. + cos ( ) + [cos ( ) − ^{3}] = 0.

Particular solution: $_{0}$ = e^{-b} .

15. + [cos ( ) + ^{3}] + cos ( ) = 0.

Particular solution: $_{0}$ = e^{-b} .

16. + (+) cos ( ) − cos ( ) = 0.

Particular solution: $_{0}$ = a + .

17. + (+) cos ( ) − 2 cos ( ) = 0.

Particular solution: $_{0}$ = (a +)^{2}.

18. + (+) cos ( ) − 3 cos ( ) = 0.

Particular solution: $_{0}$ = (a +)^{3}.

19. + cos ( ) + [cos ( ) −] = 0.

The substitution = ″ + leads to a second-order equation: ″ + [a cos (λ) −] = 0.

20. + [+ cos ( )] + cos ( ) = 0.

The substitution = ″ + a leads to a second-order linear equation: ″ + cos (λ) = 0.

21. = cos ( ) + − cos ( ) .

Particular solutions: = e (k = 1, 2, 3), where the β are roots of the cubic equation β^{3} − = 0.

22. + cos ( )(^{3} − 3 ^{2} + 6 − 6) = 0.

This is a special case of equation 4.1.6.33 with ( ) = a cos (λ).

23. ^{2} + cos ( )(^{2} − 4 + 6) = 0.

The substitution = ^{2} ″ −4 ′ +6 leads to a second-order equation: ″ + a cos (λ) = 0.

24. (cos +) = cos .

Particular solution: $_{0}$ = a cos + .

25. (+ cos) = cos .

Particular solution: $_{0}$ = a + cos .

26. $(\quad + \quad \cos \quad) \qquad = \quad \cos \qquad , \qquad = 2,\ 3.$

Particular solution: $\ _0 = a \quad + \quad \cos \quad .$

27. $\qquad + 2 \quad \cos(\quad) \quad - \ [\ ^2 \sin(\quad) + \quad \sin^2(\quad)] \quad = 0.$

The substitution $\quad = \ '' + a \sin(\quad)$ leads to a second-order linear equation of the form 2.1.6.2: $\ '' - a \sin(\quad) \quad = 0.$

28. $\sin^4 \qquad + 2 \sin^3 \quad \cos \qquad + \sin^2 \quad (\sin^2 \quad - 3)$

$\qquad + \sin \quad \cos \quad (2 \sin^2 \quad + 3) \qquad + (\ ^4 \sin^4 \quad - 3) \quad = 0.$

Equation of a loaded rigid spherical shell. If $a^4 = 1 - \lambda^2$, the equation can be rewritten as

$$\mathbf{LL}(\quad) - \lambda^2 \quad = 0, \qquad \text{where} \quad \mathbf{L} \equiv \frac{^2}{\quad ^2} + \cot \quad \frac{}{} - \cot^2 \quad .$$

This equation falls into two second-order equations:

$$\mathbf{L}(\quad) + \lambda \quad = 0, \qquad \mathbf{L}(\quad) - \lambda \quad = 0,$$

which differ only in the sign of the parameter λ. The transformation $\xi = \sin^2 \quad , \quad = \quad \sin$ reduces the latter equations to the hypergeometric equations 2.1.2.171:

$$\xi(\xi - 1) \quad '' + (\tfrac{5}{2}\xi - 2) \quad ' + \tfrac{1}{4}(1 \mp \lambda) \quad = 0.$$

29. $\qquad = \quad + \ (\quad \sin \quad - \quad \cos \quad).$

The substitution $\quad = \ ' \sin \quad - \quad \cos$ leads to a third-order linear equation.

30. $\qquad = \quad + \ (\quad \cos \quad + \quad \sin \quad).$

The substitution $\quad = \ ' \cos \quad + \quad \sin$ leads to a third-order linear equation.

4.1.5-2. Equations with tangent and cotangent.

31. $\qquad + \qquad + (\quad \tan \quad - 1) \quad = 0.$

Particular solution: $\ _0 = \cos \quad .$

32. $\qquad + (\quad + \quad) \tan (\quad) \quad - \quad \tan (\quad) \quad = 0.$

Particular solution: $\ _0 = a \quad + \quad .$

33. $\qquad + (\quad + \quad) \tan (\quad) \quad - 2 \quad \tan (\quad) \quad = 0.$

Particular solution: $\ _0 = (a \quad + \quad)^2.$

34. $\qquad + (\quad + \quad) \tan (\quad) \quad - 3 \quad \tan (\quad) \quad = 0.$

Particular solution: $\ _0 = (a \quad + \quad)^3.$

35. $\qquad + \quad \tan (\quad) \qquad + [\quad \tan (\quad) - \quad ^3] \quad = 0.$

Particular solution: $\ _0 = e^{-b} \quad .$

36. $\qquad + [\quad \tan (\quad) + \quad ^3] \qquad + \quad \tan (\quad) \quad = 0.$

Particular solution: $\ _0 = e^{-b} \quad .$

37. $\qquad + \quad \tan (\quad) \qquad + [\quad \tan (\quad) - \quad] \quad = 0.$

This is a special case of equation 4.1.6.20 with $\ (\quad) = \quad \tan (\lambda \quad).$

38. $+ [\ + \ \tan\ (\ \ )] \quad + \quad \tan\ (\ \ ) \ = 0.$

This is a special case of equation 4.1.6.21 with $(\ ) = \ \tan\ (\lambda\ )$.

39. $= \ \tan\ (\ \ ) \quad + \quad - \quad \tan\ (\ \ )\ .$

Particular solutions: $= e$ $(k = 1, 2, 3)$, where the β are roots of the cubic equation $\beta^3 - \ = 0$.

40. $+ \quad - (1 + \ \cot\ ) \ = 0.$

Particular solution: $_0 = \sin\ .$

41. $+ (\ \ + \ ) \cot\ (\ \ ) \quad - \quad \cot\ (\ \ ) \ = 0.$

Particular solution: $_0 = a \ + \ .$

42. $+ (\ \ + \ ) \cot\ (\ \ ) \quad - 2 \ \cot\ (\ \ ) \ = 0.$

Particular solution: $_0 = (a \ + \ )^2.$

43. $+ (\ \ + \ ) \cot\ (\ \ ) \quad - 3 \ \cot\ (\ \ ) \ = 0.$

Particular solution: $_0 = (a \ + \ )^3.$

44. $+ \ \cot\ (\ \ ) \quad + [\ \cot\ (\ \ ) - \ ^3] \ = 0.$

Particular solution: $_0 = e^{-b}\ .$

45. $+ [\ \cot\ (\ \ ) + \ ^3] \quad + \quad \cot\ (\ \ ) \ = 0.$

Particular solution: $_0 = e^{-b}\ .$

46. $+ \ \cot\ (\ \ ) \quad + [\ \cot\ (\ \ ) - \ ] \ = 0.$

This is a special case of equation 4.1.6.20 with $(\ ) = \ \cot\ (\lambda\ )$.

47. $+ [\ + \ \cot\ (\ \ )] \quad + \quad \cot\ (\ \ ) \ = 0.$

This is a special case of equation 4.1.6.21 with $(\ ) = \ \cot\ (\lambda\ )$.

48. $= \ \cot\ (\ \ ) \quad + \quad - \quad \cot\ (\ \ )\ .$

Particular solutions: $= e$ $(k = 1, 2, 3)$, where the β are roots of the cubic equation $\beta^3 - \ = 0$.

4.1.6. Equations Containing Arbitrary Functions

4.1.6-1. Equations of the form $_4(\ )\ '''' \ + \ _1(\ )\ ' + \ _0(\ )\ = g(\ )$.

1. $= f(\ )\ .$

The transformation $= \ ^{-1}, \ = \ ^{-3}$ leads to an equation of the same form: $'''' = \ ^{-8}\ (1\ \ )\ .$

2. $$= f\ \frac{+}{+}\ \frac{}{(\ \ +\ )^8}.$$

The transformation $z = \dfrac{a\ \ +}{+}$, $= \dfrac{}{(\ \ +\ )^3}$ leads to a simpler equation: $'''' \ = \Delta^{-4}\ (z)\ $, where $\Delta = a \ - \ .$

3. $f \quad - f \quad = 0, \qquad f = f(\ ).$

Particular solution: $_0 = \ (\ ).$

4. $\;\; + f \quad - \; (f + \;\;^3) \;\; = 0, \qquad f = f(\;\;).$

Particular solution: $\;_0 = e \quad .$

5. $\;\; + (f + \;\;^3) \quad + \;\; f \;\; = 0, \qquad f = f(\;\;).$

Particular solution: $\;_0 = e^{-} \quad .$

6. $\;\; + (\quad + \;\;)f(\;\;) \quad - \;\; f(\;\;) \;\; = 0.$

Particular solution: $\;_0 = a \;\; + \;\; .$

7. $\;\; + (\quad + \;\;)f(\;\;) \quad - 2 \;\; f(\;\;) \;\; = 0.$

Particular solution: $\;_0 = (a \;\; + \;\;)^2.$

8. $\;\; + (\quad + \;\;)f(\;\;) \quad - 3 \;\; f(\;\;) \;\; = 0.$

Particular solution: $\;_0 = (a \;\; + \;\;)^3$. The substitution $z = (a \;\; + \;\;)\;\;' - 3a \quad$ leads to a third-order linear equation: $z''' \quad + (a \;\; + \;\;)\;\;(\;\;)z = 0.$

9. $\;\; + f(\;\;) \quad + f\;\;(\;\;) \;\; = \;\;(\;\;).$

Integrating yields a third-order linear equation: $\;\;''' \quad + \;\;(\;\;) \;\; = \quad g(\;\;) \quad + \quad .$

10. $\;\; + 2f \quad + (f \quad - f^2) \;\; = 0, \qquad f = f(\;\;).$

The substitution $\;\; = \;\;'' + \;\;(\;\;)$ leads to a second-order linear equation: $\;\;'' - \;\;(\;\;) \;\; = 0.$

11. $\;\; + \;\; f(\;\;) \quad - [(\;\; + 1)f(\;\;) + \quad + 4] \;\; = 0.$

Particular solution: $\;_0 = \;\; e \;\; .$

12. $\;\; + f(\;\;) \quad + \;\;(\;\;) \;\; + \;\;(\;\;) = 0.$

The transformation $\;\; = \;\;^{-1}, \quad = \quad^{-3}$ leads to an equation of the same form:

$$\;\;'''' - \;\;^{-6}\;\;(1\;\;)\;\;' + \left[3\;\;^{-7}\;\;(1\;\;) + \;\;^{-8}g(1\;\;)\right] \;\; + \;\;^{-5}\;\;(1\;\;) = 0.$$

13. $\;\; = \;\; + f(\;\;)(\;\; \cosh \;\; - \;\; \sinh \;\;).$

The substitution $\;\; = \;\;' \cosh \;\; - \;\; \sinh \;\;$ leads to a third-order linear equation.

14. $\;\; = \;\; + f(\;\;)(\;\; \sinh \;\; - \;\; \cosh \;\;).$

The substitution $\;\; = \;\;' \sinh \;\; - \;\; \cosh \;\;$ leads to a third-order linear equation.

15. $\;\; = \;\; + f(\;\;)(\;\; \sin \;\; - \;\; \cos \;\;).$

The substitution $\;\; = \;\;' \sin \;\; - \;\; \cos \;\;$ leads to a third-order linear equation.

16. $\;\; = \;\; + f(\;\;)(\;\; \cos \;\; + \;\; \sin \;\;).$

The substitution $\;\; = \;\;' \cos \;\; + \;\; \sin \;\;$ leads to a third-order linear equation.

17. $\;\; + f \quad + (f \tan \;\; - 1) \;\; = 0, \qquad f = f(\;\;).$

Particular solution: $\;_0 = \cos \;\; .$

18. $\;\; + f \quad - (1 + f \cot \;\;) \;\; = 0, \qquad f = f(\;\;).$

Particular solution: $\;_0 = \sin \;\; .$

19. $\;\; = \frac{\quad}{\quad} \;\; + f(\;\;) \quad - \frac{\quad}{\quad} \quad , \qquad \;\; = \;\;(\;\;).$

The substitution $\;\; = \;\;' - \frac{\;\;'}{\quad}$ leads to a third-order linear equation.

4.1.6-2. Equations of the form $\ _4(\ )\ ''''\ +\ _2(\ )\ ''\ +\ _1(\ )\ '\ +\ _0(\ )\ \ = g(\ )$.

20. $\quad + f \quad + \ (f - \ ) \ \ = 0, \qquad f = f(\ ).$

1 . Particular solutions with $a > 0$: $\ _1 = \cos\big(\ \ \overline{a}\ ,\ \ _2 = \sin\big(\ \ \overline{a}\ .$

2 . Particular solutions with $a < 0$: $\ _1 = \exp\big(-\ \ \overline{-a}\ ,\ \ _2 = \exp\big(\ \ \overline{-a}\ .$

The substitution $\ = \ '' + a\ $ leads to a second-order linear equation: $\ '' + (\ - a)\ \ = 0.$

21. $\quad + (f + \ ) \quad + \ f \quad = 0, \qquad f = f(\ ).$

1 . Particular solutions with $a > 0$: $\ _1 = \cos\big(\ \ \overline{a}\ ,\ \ _2 = \sin\big(\ \ \overline{a}\ .$

2 . Particular solutions with $a < 0$: $\ _1 = \exp\big(-\ \ \overline{-a}\ ,\ \ _2 = \exp\big(\ \ \overline{-a}\ .$

The substitution $\ = \ '' + a\ $ leads to a second-order linear equation: $\ '' + \ (\ )\ \ = 0.$

22. $\quad + f(\ )(\ ^2 \quad - 2 \quad + 2\ ) = 0.$

Particular solutions: $\ _1 = \ ,\ \ _2 = \ ^2$. The substitution $z = \ ^2\ '' - 2\ \ ' + 2\ $ leads to a second-order linear equation: $z'' - 2z' + \ ^3\ (\ )z = 0.$

23. $\quad + f(\ )(\ ^2 \quad - 4 \quad + 6\ ) = 0.$

Particular solutions: $\ _1 = \ ^2,\ \ _2 = \ ^3$. The substitution $\ = \ ^2\ '' - 4\ \ ' + 6\ $ leads to a second-order linear equation: $\ '' + \ ^2\ (\ )\ \ = 0.$

24. $\quad + (\ \ ^2 + \ \ + \ )f(\ ) \quad - 2\ f(\ )\ \ = 0.$

Particular solution: $\ _0 = a\ ^2 + \ \ + \ .$

25. $\quad + 10f \quad + 10f \quad + (3f \quad + 9f^2)\ \ = 0, \qquad f = f(\ ).$

Solution:

$$= \ _1\ _1^3 + \ _2\ _1^2\ _2 + \ _3\ _1\ _2^2 + \ _4\ _2^3,$$

where $\ _1$ and $\ _2$ are nontrivial linearly independent solutions of the second-order linear equation: $\ '' + \ \ = 0.$

26. $\quad + (f + \ ) \quad + 2f \quad + (f \quad + f\ )\ \ = 0, \qquad f = f(\ ), \quad = (\ ).$

The substitution $\ = \ '' + \ $ leads to a second-order linear equation: $\ '' + g\ \ = 0.$

4.1.6-3. Other equations.

27. $\quad + f(\ ) \quad + \ (\ ) \quad - 2\ (\ )\ \ = 0.$

Particular solution: $\ _0 = \ ^2.$

28. $\quad + f(\ ) \quad - 2\ ^2 \quad - \ ^2 f(\ ) \quad + \ ^4 \ = 0.$

Particular solutions: $\ _1 = e^{-}\ ,\ \ _2 = e\ .$

29. $\quad + f \quad + \quad + f \quad + (\ - \ )\ = 0, \qquad f = f(\ ), \quad = (\ ).$

1 . Particular solutions with $a > 0$: $\ _1 = \cos\big(\ \ \overline{a}\ ,\ \ _2 = \sin\big(\ \ \overline{a}\ .$

2 . Particular solutions with $a < 0$: $\ _1 = \exp\big(-\ \ \overline{-a}\ ,\ \ _2 = \exp\big(\ \ \overline{-a}\ .$

The substitution $\ = \ '' + a\ $ leads to a second-order linear equation: $\ '' + \ \ ' + (g - a)\ \ = 0.$

30. $+f \quad +(\; + \;) \quad + \; f \quad + \quad = 0, \quad f = f(\;), \quad = \; (\;).$

1 . Particular solutions with $a > 0$: $\;_1 = \cos\big(\quad \overline{a} \;,\; _2 = \sin\big(\quad \overline{a}$.

2 . Particular solutions with $a < 0$: $\;_1 = \exp\big(- \quad \overline{-a} \;,\; _2 = \exp\big(\quad \overline{-a}$.

The substitution $\; = \; '' \; + a \;$ leads to a second-order linear equation: $\; '' \; + \quad ' + g \quad = 0.$

31. $+f(\;) \quad + \; (\;) \quad + \quad (\;) \quad - \; (\;) \quad = 0.$

Particular solution: $\;_0 = \;$.

32. $+f(\;) \quad + \; (\;)(\; ^2 \quad -2 \quad +2 \;) = 0.$

Particular solutions: $\;_1 = \;,\; _2 = \; ^2$. The substitution $z = \; ^2 \; '' \; -2 \quad ' + 2 \;$ leads to a second-order linear equation: $z'' \; + [\quad (\;) - 2]z' + \; ^3 g(\;)z = 0.$

33. $+f(\;)(\; ^3 \quad -3 \; ^2 \quad +6 \quad -6 \;) = 0.$

Particular solutions: $\;_1 = \;,\; _2 = \; ^2,\; _3 = \; ^3$. The substitution $\; = \; ^3 \; ''' \; -3 \; ^2 \; '' \; +6 \quad ' -6$ leads to a first-order linear equation: $\; ' + \; ^3 \; (\;) \quad = 0.$

34. $= f(\;) \quad + \quad - \; f(\;) \; .$

Particular solutions: $\; = e^{\lambda} \quad (k = 1, 2, 3)$, where the λ are roots of the cubic equation $\lambda^3 - a = 0$.

35. $= (f - \;) \quad + (\; f - \;) \quad + (\; f - \;) \quad + \; f \; , \quad f = f(\;).$

Particular solutions: $\; = e^{\lambda} \quad (k = 1, 2, 3)$, where the λ are roots of the cubic equation $\lambda^3 + a\lambda^2 + \; \lambda + \; = 0$.

36. $+(f + \;) \quad +(\; f + \; + \quad) \quad + \; ^2 \quad - \; ^2 \quad = 0, \quad f = f(\;), \quad = \; (\;).$

Particular solutions: $\;_1 = \;,\; _2 = e^{-} \;$.

37. $+(f_3 + \;) \quad +(f_2 + \; f_3) \quad +(f_1 + \; f_2) \quad + \; f_1 \quad = 0, \quad f \; = f \; (\;).$

Particular solution: $\;_0 = e^{-} \;$.

38. $+4 \quad + \quad = f(\;).$

The substitution $\; (\;) = \;$ leads to a constant coefficient nonhomogeneous linear equation: $\; '''' \; + a \quad = \; (\;).$

39. $^2 \quad + \quad +(\; ^2 f + \;) \quad +(\; -4) \; f \quad +(\; -2 \; +6)f \quad = 0, \quad f = f(\;).$

The substitution $\; = \; ^2 \; '' \; +(a-4) \quad ' +(\; -2a+6) \;$ leads to a second-order linear equation: $\; '' \; + \quad = 0.$

40. $^4 \quad + \; ^3 \quad + \; f(\;) \quad +(\; -3)f(\;) \quad = 0.$

Particular solution: $\;_0 = \; ^{3-} \;$.

41. $+6f \quad +(4f \; +11f^2 + 10 \;) \quad +(f \quad +7ff \; +6f^3 + 30f \; +10 \quad)$

$+3(2f \quad +5f \quad +6f^2 \quad + \quad +3 \; ^2) \quad = 0, \quad f = f(\;), \quad = \; (\;).$

Solution:

$$= \;_1 \; ^3_1 + \;_2 \; ^2_1 \; _2 + \;_3 \; _1 \; ^2_2 + \;_4 \; ^3_2,$$

where $\;_1$ and $\;_2$ are nontrivial linearly independent solutions of the second-order linear equation: $\; '' \; + \quad ' + g \quad = 0.$

42. $(f \quad) \quad = 0, \quad f = f(\;).$

Equation of transverse vibrations of a bar. Solution: $\; = \;_1 + \;_2 + \; \int_0 \frac{-}{(\;)}(\;_3 + \;_4 \;) \;$.

4.2. Nonlinear Equations

4.2.1. Equations Containing Power Functions

4.2.1-1. Equations of the form $'''' = (\, , \,).$

1. $= {}^{-5\;3}.$

Multiply both sides of the equation by $^{5\;3}$ and differentiate the resulting expression with respect to to obtain

$$3\;{}^{(5)} + 5\;'\;'''' \;= 0.$$

Integrating this equation three times, we arrive at the chain of equalities:

$$3\;'''' \;+ 2\;'\;''' \;- (\;''\;)^2 = 2\;{}_2, \qquad (1)$$

$$3\;''' \;-\;'\;'' \;= 2\;{}_2\; + \;{}_1, \qquad (2)$$

$$3\;'' \;- 2(\;'\;)^2 = \;{}_2\;{}^2 + \;{}_1\; + \;{}_0, \qquad (3)$$

where ${}_0$, ${}_1$, and ${}_2$ are arbitrary constants. By eliminating the highest derivatives from (1)–(3) with the help of the original equation, we obtain a first-order equation:

$$(2\;\;' - 3\;\;'\;)^2 = 9(\;{}_1^2 - 4\;{}_0\;{}_2)\;{}^2 - 2\;{}^3 + 54A\;{}^{4\;3},$$

where $= \;{}_2\;{}^2 + \;{}_1\; + \;{}_0$. The substitution $= (\quad)^{3\;2}$ leads to a separable equation, the integration of which finally yields:

$$\int \left[9(\;{}_1^2 - 4\;{}_0\;{}_2) + 54A\; - 2\;{}^3\right]^{-1\;2} \overline{\qquad} \quad \frac{}{3} = \;{}_3.$$

2. $= \quad .$

This is a special case of equation 4.2.6.1 with $(\;) = A$.

1 . By integrating, we obtain

$$2\;'\;''' \;- (\;''\;)^2 = \frac{2A}{+1}\;{}^{+1} + \frac{4}{3}\;,$$

where is an arbitrary constant ($\neq -1$). The substitution $(\;) = (\;'\;)^{3\;2}$ leads to a second-order equation:

$$'' \;= \frac{3A}{2\;\;+2}\;{}^{+1} + \;{}^{-5\;3}.$$

The value $= 0$ corresponds to the Emden–Fowler equation, whose integrable cases are specified in Section 2.3 for some values of (to those cases there correspond three-parameter families of particular solutions of the original equation).

2 . Particular solution: $= \left[\dfrac{8(\;+1)(\;+3)(3\;+1)}{A(\;-1)^4}\right]^{\frac{1}{-1}} \left(\; + \;\right)^{\frac{4}{1-}}.$

3. $= {}^{-3\;\;-5}\quad .$

The transformation $= {}^{-1}$, $= {}^{-3}\;(\;)$ leads to an equation of the form 4.2.1.2: $'''' = a$.

4. $= {}^{-\frac{3\;\;+5}{2}}\quad .$

This is a special case of equation 4.2.6.5 with $(\;) = a$.

5. $=$ **.**

Generalized homogeneous equation.

1 . The transformation $= \;^{+4} \;^{-1}$, $= \;'$ leads to a third-order equation.

2 . The transformation $= z^{-1}$, $= z^{-3}\ (z)$ leads to an equation of the same form: $'''' = z^{-}\;^{-3}\;^{-5}$.

6. $= (\ + \ ^{k})\ ,$ $= 0,\ 1,\ 2,\ 3.$

The substitution $a \ = a\ +$ leads to an equation of the form 4.2.1.2: $'''' = a$.

7. $^{3}\;^{+1}(\ +\ )^{4}$ $=$ **.**

This is a special case of equation 4.2.6.10 with $(\ ) =$.

8. $= (\ ^{2} + \ + \ )^{-\frac{3\ +5}{2}}$ **.**

This is a special case of equation 4.2.6.12 with $(\ ) =$.

4.2.1-2. Equations of the form $'''' = (\ ,\ ,\ ')$.

9. $=$ $+ \ ^{k}$**.**

By integrating, we find $''' = \frac{a}{+1}\ ^{+1} + \frac{}{k+1}\ ^{+1} +$. For $= 0$, the order of this equation can be reduced by one with the help of the substitution $(\ ) = \ '$.

10. $= \ ^{-4}(\ - \ )$ **.**

The transformation $= \ln|\ |$, $= \ ' -$ leads to a third-order autonomous equation: $''' - 5\ '' + 6\ ' = a$.

11. $= \ (\ - \ )^{k}$**.**

This is a special case of equation 4.2.6.23 with $(\ ,\ ) = a$. The substitution $= \ ' -$ leads to a third-order generalized homogeneous equation: $(\ '\ )'' = a$.

12. $= \ (\ - 2\ )^{k}$**.**

This is a special case of equation 4.2.6.24 with $(\ ,\ ) = a$. The substitution $= \ ' - 2$ leads to a third-order generalized homogeneous equation: $''' - \ '' = a\ ^{+2}$.

13. $= \ (\ - 3\ )^{k}$**.**

This is a special case of equation 4.2.6.25 with $(\ ,\ ) = a$. The substitution $= \ ' - 3$ leads to a third-order generalized homogeneous equation: $''' = a\ ^{+1}$.

14. $= \ ^{4} + \ (\ - \ )^{k}$**.**

The substitution $= \ ' - a$ leads to a third-order equation: $''' + a\ '' + a^{2}\ ' + a^{3} =$.

4.2.1-3. Equations of the form $'''' = (\ ,\ ,\ ',\ '')$.

15. $+ \ = \ + \$ **.**

This is a special case of equation 4.2.6.33 with $(\ ) = \ + \$.

16. $- \frac{5}{2} \ + \frac{9}{16}\ ^{2} = \ ^{-5}\ ^{3}$**.**

The transformation $\xi = e^{-}$, $(\xi) = \xi^{3}\ ^{2}$ leads to an autonomous equation of the form 4.2.1.1: $'''' = a^{-2}\ ^{-5}\ ^{3}$.

17. $+ \quad + \quad = \quad - (\quad)^2 + \quad .$

1 . Particular solution:

$$= \ _1 \sinh(\ _4\ ) + \ _2 \cosh(\ _4\ ) + \ _3,$$

where the constants $_1$, $_2$, $_3$, and $_4$ are related by two constraints

$$_4^4 + (a - \ _3)\ _4^2 + \ = 0,$$
$$(\ _2^2 - \ _1^2)\ _4^2 - \ _3 + k = 0.$$

2 . Particular solution:

$$= \ _1 \sin(\ _4\ ) + \ _2 \cos(\ _4\ ) + \ _3,$$

where the constants $_1$, $_2$, $_3$, and $_4$ are related by two constraints

$$_4^4 - (a - \ _3)\ _4^2 + \ = 0,$$
$$(\ _1^2 + \ _2^2)\ _4^2 + \ _3 - k = 0.$$

18. $+ \quad + \quad - (\quad)^2 + \quad = 0.$

Particular solution: $= \ _1 \exp(\ _2\ ) - \dfrac{_2^3 + \ _2 +}{a\ _2}.$

19. $= \ ^2 \quad - \quad (\quad)^2 + \quad .$

1 . Particular solution:

$$= \ _1 \exp(\ _3\ ) + \ _2 \exp(- \ _3\ ),$$

where the constants $_1$, $_2$, and $_2$ are related by the constraint $_3^4 - 4a\ _1\ _2\ _3^2 - \ = 0.$

2 . Particular solution:

$$= \ _1 \cos(\ _3\ ) + \ _2 \sin(\ _3\ ),$$

where the constants $_1$, $_2$, and $_2$ are related by the constraint $_3^4 + a(\ _1^2 + \ _2^2)\ _3^2 - \ = 0.$

3 . There are also solutions $= \ \overline{\quad a} + \ $ and $= 0.$

20. $+ (\quad) \quad = \ ^k + \ .$

This is a special case of equation 4.2.6.34 with $(\ ) = a$ and $g(\ ) = \quad + \ .$

21. $= \ ^{-2}(\quad - \ ) \quad .$

This is a special case of equation 4.2.6.35 with $(\ ) = a \quad .$

22. $- (\quad)^2 = 0.$

1 . Particular solutions:

$$= \ _1 \ + \ _2,$$
$$= \ _1(\ + \ _2)^{-3\ 2},$$
$$= \ _1 \exp(\ _3\ ) + \ _2 \exp(- \ _3\ ),$$
$$= \ _1 \cos(\ _3\ ) + \ _2 \sin(\ _3\ ).$$

2 . Integrating the equation twice, we arrive at a second-order equation:

$$'' - (\ ')^2 = \ _1 \ + \ _2.$$

The substitution $z = \ _1 \ + \ _2$ leads to a generalized homogeneous equation.

23. $\quad - (\quad)^2 + \quad + \quad = 0.$

Particular solutions:

$$= {}_1 \exp(\lambda \ ) + \ {}_2 \exp(-\lambda \ ) - \ a, \quad \lambda = (a^2 \ )^{1\ 4};$$
$$= {}_1 \sin(\lambda \ ) + \ {}_2 \cos(\lambda \ ) - \ a, \quad \lambda = (a^2 \ )^{1\ 4}.$$

24. $\quad - (\quad)^2 + \quad = 0.$

1 . Integrating the equation two times, we obtain a second-order equation: $'' - (\ ')^2 + a \ = {}_1 \ + \ {}_2.$

2 . Particular solutions:

$$= {}_1 \exp(\ {}_3 \ ) + \ {}_2 \exp(- \ {}_3 \ ) - a \ {}_3^{-2},$$
$$= {}_1 \sin(\ {}_3 \ ) + \ {}_2 \cos(\ {}_3 \ ) + a \ {}_3^{-2},$$
$$= {}_1 \ + \ {}_2.$$

25. $\quad - (\quad)^2 = \ [\quad - (\quad)^2] + \ .$

1 . The substitution $(\ ) = \ '' - (\ ')^2$ leads to a second-order constant coefficient linear equation of the form 2.1.9.1: $'' = a \ + \ .$

2 . Particular solutions:

$$= {}_1 \exp\big(\ {}_2 \ - \frac{}{4a \ {}_1 \ {}_2^2} \exp\big(- \ {}_2 \qquad \text{if } a \neq 0,$$
$$= {}_1 \exp\big(\ \overline{a} \ - \frac{}{4a^2 \ {}_1} \big(- \ \overline{a} \ + \ {}_2 \qquad \text{if } a > 0,$$
$$= {}_1 \sin\big(\lambda \ + \ {}_2 \cos\big(\lambda \ , \quad \lambda^2 = \frac{}{a(\ {}_1^2 + \ {}_2^2)} \qquad \text{if } a \ > 0,$$
$$= \frac{\overline{-}}{a} \sin\big(\ \overline{-a} + \ {}_1 \ + \ {}_2 \qquad \text{if } a < 0, \ < 0,$$
$$= \ \overline{a} + \ {}_1 \qquad \text{if } a \ > 0.$$

26. $\quad - (\quad)^2 = \ \big[\quad - (\quad)^2\big] + \ {}^{k} + \ .$

The substitution $(\ ) = \ '' - (\ ')^2$ leads to a second-order constant coefficient nonhomogeneous linear equation of the form 2.1.9.1: $'' = a \ + \ + \ .$

27. $\quad - \frac{1}{6}(\quad)^2 = \ {}^2 + \ + \ .$

Particular solution:

$$= \tfrac{1}{24} \ {}_1 \ {}^4 + \tfrac{1}{6} \ {}_2 \ {}^3 + \tfrac{1}{2} \ {}_3 \ {}^2 + \ {}_4 \ + \ {}_5,$$

where the constants ${}_1$, ${}_2$, ${}_3$, ${}_4$, and ${}_5$ are related by three constraints

$$\tfrac{1}{3} \ {}_1 \ {}_3 - \tfrac{1}{6} \ {}_2^2 = a,$$
$${}_1 \ {}_4 - \tfrac{1}{3} \ {}_2 \ {}_3 = \ ,$$
$${}_1 \ {}_5 - \tfrac{1}{6} \ {}_3^2 = \ .$$

28. $\quad = \ {}^2 + (\quad + \quad)^k.$

The substitution $= \ '' + a$ leads to a second-order autonomous equation of the form 2.9.1.1: $'' = a \ + \ .$

29. $\quad = \ \big[\quad - (\quad)^2\big] \ .$

This is a special case of equation 4.2.6.42 with $(\ ) = 0$ and $g(\ ) = a \ .$

4.2.1-4. Equations of the form $'''' = (\; , \; , \; ', \; '', \; ''' \;)$.

30. $+ \;_3 \qquad + \;_2 \qquad + \;_1 \qquad + \;_0 \; = \qquad - \;(\quad)^2 + \; .$

Particular solutions: $= \quad \exp(\lambda \quad) + ka_0^{-1}$, where is an arbitrary constant and $\lambda = \lambda$ are roots of the algebraic equation $\lambda^4 + a_3\lambda^3 + \left(a_2 - \dfrac{k}{a_0} \right) \lambda^2 + a_1\lambda + a_0 = 0$.

31. $+ \qquad = \quad .$

Integrating, we arrive at a third-order equation: $''' \quad + a \quad '' - \frac{1}{2} a(\; ')^2 = \dfrac{}{+1} \quad ^{+1} + \quad .$

32. $+ \qquad - \qquad = 0.$

This equation arises in hydrodynamics.

1 . Particular solutions:

$$= \;_1 + \;_2,$$
$$= \;_1 \exp(\;_2 \;) - a^{-1} \;_2,$$
$$= 6(a \quad + \;_1)^{-1}.$$

2 . Integrating, we arrive at a third-order autonomous equation: $''' \quad + a \quad '' - a(\; ')^2 = \quad .$

Reference: A. D. Polyanin and V. F. Zaitsev (2002).

33. $+ \qquad - \qquad = \quad .$

Particular solutions:

$$= \;_1 \exp(\lambda \;) + \;_2 \exp(-\lambda \;), \quad \lambda = \; ^{1 \; 4},$$
$$= \;_1 \sin(\lambda \;) + \;_2 \cos(\lambda \;), \qquad \lambda = \; ^{1 \; 4}.$$

34. $+ \qquad + \; (\quad) \qquad = \quad ^k + \; .$

This is a special case of equation 4.2.6.50 with $(\;) =$ and $g(\;) = \quad + \; .$

35. $= \qquad .$

This is a special case of equation 4.2.6.51 with $(\;) = a \quad .$

36. $+ 4 \qquad = \quad ^{-5 \; 3} \; ^{-5 \; 3}.$

The substitution $(\;) =$ leads to an equation of the form 4.2.1.1: $'''' \quad = a \quad ^{-5 \; 3}.$

37. $+ 4 \qquad = (\quad)^k.$

The substitution $(\;) =$ leads to an equation of the form 4.2.1.2: $'''' \quad = a \quad .$

38. $+ 2 \qquad = (\qquad - \quad)^k.$

The substitution $(\;) = \quad ' -$ leads to a third-order equation: $''' \quad = a$ (Section 3.2 presents its solutions for $k = -\frac{7}{2}, -\frac{5}{2}, -2, -\frac{4}{3}, -\frac{7}{6}, -\frac{1}{2}$, 0, and 1).

39. $+ (\quad + 3) \qquad = \qquad (\qquad + \quad)^k.$

The substitution $= \quad ' + a$ leads to a third-order generalized homogeneous equation: $''' \quad = \qquad .$

40. $^2 \qquad + 8 \qquad + 12 \qquad = \quad ^{-10 \; 3} \; ^{-5 \; 3}.$

The substitution $(\;) = \quad ^2$ leads to an equation of the form 4.2.1.1: $'''' \quad = a \quad ^{-5 \; 3}.$

41. $^{4} \quad + 6 \ ^{3} \quad + 7 \ ^{2} \quad + \quad = \quad ^{-5\ 3}$.

The substitution $= \ln|\ |$ leads to an equation of the form 4.2.1.1: $'''' \ = a \ ^{-5\ 3}$.

42. $=$.

Having integrated this equation, we obtain the third-order equation $''' \ =$, whose solvable cases are specified in Subsection 3.2.2.

43. $- \quad = \quad ^{2}$.

Integrating yields a third-order linear equation: $''' \ = \dfrac{a}{\ + 1} \ ^{+1} +$.

44. $- \quad =$.

Integrating yields a third-order constant coefficient nonhomogeneous linear equation: $''' \ = \ - a$.

45. $+ 4 \quad + 3(\quad)^2 =$.

This is a special case of equation 4.2.6.58 with $(\ ) = a$.

46. $+ 4 \quad + 3(\quad)^2 = \ ^{-10\ 3}$.

The substitution $= \ ^{2}$ leads to an equation of the form 4.2.1.1: $'''' \ = 2a \ ^{-5\ 3}$.

47. $+ 4 \quad + 3(\quad)^2 =$.

The substitution $= \ ^{2}$ leads to an equation of the form 4.2.1.2: $'''' \ = 2a \ \ ^{2}$.

48. $+ \frac{2}{3} \quad - \frac{1}{3}(\quad)^2 =$.

This is a special case of equation 4.2.6.59 with $a = \frac{2}{3}$ and $(\ ) = a$. Integrating the equation twice, we arrive at a second-order equation of the form 2.8.1.54:

$$3 \ '' - 2(\ ')^2 = \tfrac{3}{2} a \ ^{2} + \ _{1} + \ _{2}.$$

49. $+ \frac{3}{2} \quad + \frac{1}{2}(\quad)^2 =$.

Integrating the equation twice, we arrive at a second-order equation of the form 2.8.1.53:

$$'' - \tfrac{1}{4}(\ ')^2 = \tfrac{1}{2} a \ ^{2} + \ _{1} + \ _{2}.$$

50. $+ \frac{3}{2} \quad + \frac{1}{2}(\quad)^2 = (\quad + \ ) \ ^{-1\ 2}$.

The transformation $= (\ ), \quad = (\ ')^2$ leads to a constant coefficient linear equation: $2 \ ^{(5)} = a \ + \ $.

51. $- \quad =$.

Integrating yields a third-order linear equation: $''' \ = \ \exp \dfrac{a}{\ + 1} \ ^{+1}$.

52. $- \ (\quad)^2 =$.

Integrating the equation twice, we arrive at a second-order equation:

$$'' - (\ ')^2 = a \ \ln|\ | + \ _{1} + \ _{2}.$$

53. $= \quad + \quad$.

Integrating yields a third-order linear equation of the form 3.1.2.7: $''' \ =$.

54. $^{3} \quad = 4\ ^{2} \quad + 3\ ^{2}(\quad)^{2} - 6(\quad)^{4}.$

This is a special case of equation 4.2.6.67 with $\equiv 0$.

Solution in parametric form:

$$= \frac{}{2\xi^4 + \ _2\xi + \ _1} + \ _3, \qquad = \ _4 \exp \frac{\xi \ \xi}{2\xi^4 + \ _2\xi + \ _1}\Big).$$

55. $= (\quad)^{2}.$

Solution: $= \begin{cases} _0 + \ _1 \ + (\ _2 + \ _3\ )^{\frac{3-2}{1-}} & \text{if } a \neq 1, \\ _0 + \ _1 \ + \ _2 \exp(\ _3\ ) & \text{if } a = 1. \end{cases}$

56. $- \frac{1}{2}(\quad)^{2} = (\quad - \ ) + \quad + \gamma.$

Differentiating with respect to yields

$$'' \big[\ ^{(5)} - \quad - \beta\big] = 0. \tag{1}$$

Equating the second factor in (1) with zero and integrating it, we find the solution:

$$= \frac{^6}{6!} + \beta \frac{^5}{5!} + \ _4\ ^4 + \ _3\ ^3 + \ _2\ ^2 + \ _1\ + \ _0.$$

The constants and parameters , β, and are related by the constraint

$$48\ _2\ _4 - 18\ _3^2 = - \ _0 + \beta\ _1 + \ ,$$

obtained by means of substituting the solution into the original equation.

The other solution, which corresponds to setting the first factor in (1) to zero, is given by:

$$= \ _1\ + \ _0, \qquad \text{where} \qquad _0 - \beta\ _1 - \ = 0.$$

57. $= \ ^{k}\ (\quad)^{s}.$

This is a special case of equation 4.2.6.72 with $(\ ) = a$ and $g(\ ) =$. For $k = -1$ and $= 1$, see equation 4.2.1.42.

The first integral has the form:

$$\frac{1}{1-}(\ '''\ )^{1-} - \frac{a}{k+1}\ ^{+1} = \qquad \text{if } k \neq -1, \quad \neq 1; \tag{1}$$

$$\ln|\ '''\ | - \frac{a}{k+1}\ ^{+1} = \qquad \text{if } k \neq -1, \quad = 1; \tag{2}$$

$$\frac{1}{1-}(\ '''\ )^{1-} - a \ln|\ | = \qquad \text{if } k = -1, \quad \neq 1. \tag{3}$$

For $= 0$, equality (1) is changing to the equation

$$''' = \frac{a(1-\ )}{k+1}^{\frac{1}{1-}}\ ^{\frac{+1}{1-}},$$

which is discussed in Subsection 3.2.2 (the solutions given there generate 3-parametric families of particular solutions of the original equation for $k = (1-\ )\beta - 1$, where $\beta = -\frac{7}{2}, -\frac{5}{2}, -2, -\frac{4}{3}, -\frac{7}{6}, -\frac{1}{2}$, 0, and 1).

58. $_1(\quad)^{2} + (\ _2\ + \ _3\ + 6\ _4\ + \ _5\ + \ _6)$
$+ \ _1(\quad)^{2} + (\ _2 + \ _3) \quad - \ _4(\quad)^{2} + \ _4\ + \ _5\ ^{2} + \ _6\ + \ _7 = 0.$

There are particular solutions of the form $= \ _1\ ^4 + \ _2\ ^3 + \ _3\ ^2 + \ _4\ + \ _5$, where the five constants $_1$, $_2$, $_3$, $_4$, and $_5$ are related by three constraints.

4.2.2. Equations Containing Exponential Functions

4.2.2-1. Equations of the form $'''' = (\ , \ ).$

1. $= \ ^{\lambda} + .$

This is a special case of equation 4.2.6.1 with $(\ ) = ae^{\lambda} + .$

2. $= \ ^{\lambda \ +} + .$

The substitution $= \ + (\beta \ \lambda)$ leads to an autonomous equation of the form 4.2.6.1: $'''' = ae^{\lambda w} + .$

3. $= \ ^{-4 \ \lambda} .$

This is a special case of equation 4.2.6.2 with $(\ ) = ae^{\lambda}$. The substitution $= \ln|\ |$ leads to an autonomous equation.

4. $= \ ^{k \ \lambda} .$

This is a special case of equation 4.2.6.15 with $(\ ) = a$ and $= k + 4.$

5. $= \ ^{\lambda} .$

This is a special case of equation 4.2.6.14 with $(\ ) = a$ and $= \ -1.$

6. $= \exp(\ + \ ^{2}) + .$

The substitution $= \ + (\beta \ \lambda) \ ^{2}$ leads to an autonomous equation of the form 4.2.6.1: $'''' = ae^{\lambda w} + .$

7. $= (\ + \ )^{-5 \ 3} - .$

The substitution $= \ + e$ leads to an equation of the form 4.2.1.1: $'''' = a \ ^{-5 \ 3}.$

8. $= (\ + \ ) \ - .$

The substitution $= \ + e$ leads to an equation of the form 4.2.1.2: $'''' = a \ .$

4.2.2-2. Other equations.

9. $= \ ^{\lambda} + .$

Integrating yields a third-order equation: $''' = \frac{a}{\lambda} e^{\lambda} + \frac{}{\beta} e \ + .$

10. $= \ ^{4} + \ ^{\lambda} (\ - \ )^{k}.$

This is a special case of equation 4.2.6.27 with $(\ ,\ ) = e^{\lambda} .$

11. $= \ ^{\lambda} (\ - \ )^{k}.$

This is a special case of equation 4.2.6.23 with $(\ ,\ ) = e^{\lambda} .$

12. $= \ ^{\lambda} (\ - 2 \ )^{k}.$

This is a special case of equation 4.2.6.24 with $(\ ,\ ) = e^{\lambda} .$

13. $= \ ^{\lambda} (\ - 3 \ )^{k}.$

This is a special case of equation 4.2.6.25 with $(\ ,\ ) = e^{\lambda} .$

14. $\quad - (\quad)^2 = \quad \lambda \quad .$

1 . Integrating the equation twice, we arrive at a second-order equation:

$$'' \;-(\;'\;)^2 = a\lambda^{-2}e^{\lambda} \;+\; {}_1 \;+\; {}_2.$$

For $\;_1 = \;_2 = 0,$ it is an equation of the form 2.8.3.47.

2 . Particular solution: $\quad = \quad \exp(\lambda\;) + \dfrac{a}{\lambda^4}.$

15. $\quad - (\quad)^2 - \quad + \quad \lambda \quad = 0.$

1 . Integrating yields a third-order equation: $\;''' \;-\; '\;'' \;- a \;+\; \lambda^{-1}e^{\lambda} \;= \;.$

2 . Particular solutions:

$$= \quad \exp(\lambda\;) + \frac{a\quad \lambda -}{\lambda^4},$$

$$= \frac{}{2a\lambda}\exp(\lambda\;) + \quad \exp(-\lambda\;) - \frac{a}{\lambda^3}.$$

16. $\quad - (\quad)^2 - \quad + \quad \lambda \quad = 0.$

1 . Integrating the equation two times, we obtain a second-order equation: $\;'' \;-(\;'\;)^2 - a \;+\; {}_1 \;+\; {}_2 + \lambda^{-2}e^{\lambda} \;= 0.$

2 . Particular solution: $\quad = \quad \exp(\lambda\;) + \dfrac{a\quad \lambda^2 -}{\lambda^4}.$

17. $\quad - (\quad)^2 = \;[\quad - (\quad)^2] + \quad \lambda \quad .$

1 . The substitution $\;(\;) = \;'' \;-(\;'\;)^2$ leads to a second-order constant coefficient linear equation of the form 2.1.9.1: $\;'' \;= a \;+\; e^{\lambda}\;.$

2 . Particular solution: $\quad = \quad \exp(\lambda\;) + \dfrac{}{\lambda^2(\lambda^2 - a)}.$

18. $\quad - \quad - (\quad)^2 + \quad \lambda \quad = 0.$

1 . Integrating the equation two times, we obtain a second-order equation: $\;'' \;-(\;'\;)^2 - a\;' \;+\; {}_1 \;+\; {}_2 + \lambda^{-2}e^{\lambda} \;= 0.$

2 . Particular solutions:

$$= \quad \exp(\lambda\;) + \frac{a\quad \lambda^3 -}{\lambda^4},$$

$$= \frac{}{2a\lambda^3}\exp(\lambda\;) + \quad \exp(-\lambda\;) - \frac{a}{\lambda}.$$

19. $\quad = \quad \lambda \quad .$

This is a special case of equation 4.2.6.51 with $\;(\;) = ae^{\lambda}\;.$

20. $\quad = (\quad \lambda \quad + \quad) \quad .$

This is a special case of equation 5.2.6.58 with $\; = 4,\;\; (\;) = ae^{\lambda}\;$, and $g(\;) = \; e \;.$

21. $\quad - 4 \quad + 6\;^2 \quad - 4\;^3 \quad + \;^4 \quad = \exp\left(\frac{8}{3}\right. \quad {}^{-5\;3}.$

The substitution $\;(\;) = \; e^{-\lambda}\;$ leads to an equation of the form 4.2.1.1: $\;'''' \;= a \;{}^{-5\;3}.$

22. $\quad - 4 \quad + 6\;^2 \quad - 4\;^3 \quad + \;^4 \quad = \quad {}^{\lambda(1-\;)} \quad .$

The substitution $\;(\;) = \; e^{-\lambda}\;$ leads to an equation of the form 4.2.1.2: $\;'''' \;= a \;\; .$

23. $\quad + 4 \quad + 3(\quad)^2 = \quad \lambda \quad .$

Solution: $\;^2 = \;_3\;^3 + \;_2\;^2 + \;_1 \;+\; _0 + 2a\lambda^{-4}e^{\lambda}\;.$

24. $\quad + 4 \quad + 3(\quad)^2 = \quad \lambda \quad + \;.$

This is a special case of equation 4.2.6.60 with $\;(\;) = ae^{\lambda}\; + \;.$

4.2.3. Equations Containing Hyperbolic Functions

4.2.3-1. Equations with hyperbolic sine.

1. $= \ \sinh \ (\quad) + \ .$

This is a special case of equation 4.2.6.1 with $(\ ) = a \sinh \ (\lambda \ ) + \ .$

2. $= \ \sinh(\quad + \quad) + \ .$

The substitution $= \ + (\beta \ \lambda)$ leads to an autonomous equation of the form 4.2.6.1: $'''' \ = a \sinh(\lambda \ ) + \ .$

3. $= \ (\ + \ \sinh \ )^2 - \ \sinh \ .$

The substitution $= \ + \ \sinh$ leads to an autonomous equation of the form 4.2.6.1: $'''' \ = a \ ^2.$

4. $= \ ^{-4} \sinh \ (\quad).$

This is a special case of equation 4.2.6.2 with $(\ ) = a \sinh \ (\lambda \ ).$

5. $= \ \sinh(\quad) \quad + \ \sinh(\quad).$

Integrating yields a third-order equation: $''' \quad = \dfrac{a}{\lambda} \cosh(\lambda \ ) + \dfrac{}{\beta} \cosh(\beta \ ) + \quad .$

6. $= \ ^4 \ + \ \sinh(\quad)(\quad - \quad)^k.$

This is a special case of equation 4.2.6.27 with $(\ , \ ) = \ \sinh(\lambda \ ) \quad .$

7. $= \ \sinh(\quad)(\quad - \ )^k.$

This is a special case of equation 4.2.6.23 with $(\ , \ ) = \ \sinh(\lambda \ ) \quad .$

8. $= \ \sinh(\quad)(\quad - 2 \ )^k.$

This is a special case of equation 4.2.6.24 with $(\ , \ ) = \ \sinh(\lambda \ ) \quad .$

9. $= \ \sinh(\quad)(\quad - 3 \ )^k.$

This is a special case of equation 4.2.6.25 with $(\ , \ ) = \ \sinh(\lambda \ ) \quad .$

10. $- (\quad)^2 = \ \sinh(\quad).$

1 . Integrating the equation twice, we arrive at a second-order equation: $'' \ - (\ ')^2 = a\lambda^{-2} \sinh(\lambda \ ) + \ _1 \ + \ _2.$

2 . Particular solution: $= \ \sinh(\lambda \ ) + \dfrac{a}{\lambda^4}.$

11. $- (\quad)^2 - \quad + \ \sinh(\quad) = 0.$

1 . Integrating yields a third-order equation: $''' \ - \ ' \ '' - a \ + \ \lambda^{-1} \cosh(\lambda \ ) = \quad .$

2 . Particular solution: $= \dfrac{}{\lambda(a^2 - \ ^2\lambda^6)} \left[\ \lambda^3 \sinh(\lambda \ ) + a \cosh(\lambda \ ) \right] + \quad .$

12. $- (\quad)^2 - \quad + \ \sinh(\quad) = 0.$

1 . Integrating the equation two times, we obtain a second-order equation: $'' \ - (\ ')^2 - a \ + \ _1 \ + \ _2 + \ \lambda^{-2} \sinh(\lambda \ ) = 0.$

2 . Particular solution: $= \ \sinh(\lambda \ ) + \dfrac{a \ \lambda^2 -}{\lambda^4}.$

13. $- (\quad)^2 = \ [\quad - (\quad)^2] + \ \sinh(\quad) + \ .$

The substitution $(\ ) = \ '' - (\ ')^2$ leads to a second-order constant coefficient nonhomogeneous linear equation of the form 2.1.9.1: $'' = a \quad + \ \sinh(\lambda\ ) + \ .$

14. $- \quad - (\quad)^2 + \ \sinh(\quad) = 0.$

1 . Integrating the equation two times, we obtain a second-order equation: $'' - (\ ')^2 - a\ ' + \ _1 + \ _2 + \ \lambda^{-2} \sinh(\lambda\ ) = 0.$

2 . Particular solution: $= \dfrac{}{\lambda^3(a^2 - \ ^2\lambda^2)} \big[\ \lambda \sinh(\lambda\ ) + a\cosh(\lambda\ )\big] + \ .$

15. $= \ \sinh^k(\quad) \quad .$

This is a special case of equation 4.2.6.51 with $(\ ) = a \sinh\ (\lambda\ ).$

16. $- \quad = \ \sinh(\quad)\ ^2.$

This is a special case of equation 4.2.6.57 with $(\ ) = a\sinh(\lambda\ ).$ Integrating yields a third-order linear equation: $''' \ = \dfrac{a}{\lambda}\cosh(\lambda\ ) + \quad .$

17. $+ 4 \quad + 3(\quad)^2 = \ \sinh(\quad).$

Solution: $^2 = \ _3\ ^3 + \ _2\ ^2 + \ _1\ + \ _0 + 2a\lambda^{-4}\sinh(\lambda\ ).$

18. $+ 4 \quad + 3(\quad)^2 = \ \sinh^k(\quad) + \ .$

This is a special case of equation 4.2.6.60 with $(\ ) = a \sinh\ (\lambda\ ) + \ .$

4.2.3-2. Equations with hyperbolic cosine.

19. $= \ \cosh\ (\quad) + \ .$

This is a special case of equation 4.2.6.1 with $(\ ) = a\cosh\ (\lambda\ ) + \ .$

20. $= \ \cosh(\quad + \quad) + \ .$

The substitution $= \ + (\beta\ \lambda)$ leads to an autonomous equation of the form 4.2.6.1: $'''' \ = a\cosh(\lambda\ ) + \ .$

21. $= \ (\ + \ \cosh\ )^2 - \ \cosh\ .$

The substitution $= \ + \ \cosh$ leads to an autonomous equation of the form 4.2.6.1: $'''' \ = a\ ^2.$

22. $= \ ^{-4} \cosh\ (\quad).$

This is a special case of equation 4.2.6.2 with $(\ ) = a\cosh\ (\lambda\ ).$

23. $= \ \cosh(\quad) \quad + \ \cosh(\quad).$

Integrating yields a third-order equation: $''' \ = \dfrac{a}{\lambda}\sinh(\lambda\ ) + \dfrac{}{\beta}\sinh(\beta\ ) + \ .$

24. $= \ ^4 \ + \ \cosh(\quad)(\quad - \quad)^k.$

This is a special case of equation 4.2.6.27 with $(\ ,\ ) = \ \cosh(\lambda\ ) \quad .$

25. $= \ \cosh(\quad)(\quad - \ )^k.$

This is a special case of equation 4.2.6.23 with $(\ ,\ ) = \ \cosh(\lambda\ ) \quad .$

26. $= \cosh(\quad)(\quad -2\ )^k$.

This is a special case of equation 4.2.6.24 with $(\ ,\ ) = \ \cosh(\lambda\ )\quad$.

27. $= \cosh(\quad)(\quad -3\ )^k$.

This is a special case of equation 4.2.6.25 with $(\ ,\ ) = \ \cosh(\lambda\ )\quad$.

28. $-(\quad)^2 = \ \cosh(\quad)$.

1 . Integrating the equation twice, we arrive at a second-order equation: $'' - (\ ')^2 = a\lambda^{-2}\cosh(\lambda\ ) + \ _1\ + \ _2$.

2 . Particular solution: $= \ \cosh(\lambda\ ) + \dfrac{a}{\lambda^4}$.

29. $-(\quad)^2 - \quad + \ \cosh(\quad) = 0$.

1 . Integrating yields a third-order equation: $''' - \ ' \ '' - a\ + \ \lambda^{-1}\sinh(\lambda\ ) = \quad$.

2 . Particular solution: $= \dfrac{}{\lambda(a^2 - \ ^2\lambda^6)}\left[\ \lambda^3\cosh(\lambda\ ) + a\sinh(\lambda\ )\right] + \quad$.

30. $-(\quad)^2 - \quad + \ \cosh(\quad) = 0$.

1 . Integrating the equation two times, we obtain a second-order equation: $'' - (\ ')^2 - a\ + \ _1\ + \ _2 + \ \lambda^{-2}\cosh(\lambda\ ) = 0$.

2 . Particular solution: $= \ \cosh(\lambda\ ) + \dfrac{a\ \ \lambda^2 - }{\lambda^4}$.

31. $-(\quad)^2 = \ [\quad -(\quad)^2] + \ \cosh(\quad) + \ $.

The substitution $(\ ) = \ '' - (\ ')^2$ leads to a second-order constant coefficient nonhomogeneous linear equation of the form 2.1.9.1: $'' = a\ + \ \cosh(\lambda\ ) + \ $.

32. $- \quad -(\quad)^2 + \ \cosh(\quad) = 0$.

1 . Integrating the equation two times, we obtain a second-order equation: $'' - (\ ')^2 - a\ ' + \ _1\ + \ _2 + \ \lambda^{-2}\cosh(\lambda\ ) = 0$.

2 . Particular solution: $= \dfrac{}{\lambda^3(a^2 - \ ^2\lambda^2)}\left[\ \lambda\cosh(\lambda\ ) + a\sinh(\lambda\ )\right] + \quad$.

33. $= \ \cosh^k(\quad)\quad$.

This is a special case of equation 4.2.6.51 with $(\ ) = a\cosh\ (\lambda\ )$.

34. $- \quad = \ \cosh(\quad)\ ^2$.

This is a special case of equation 4.2.6.57 with $(\ ) = a\cosh(\lambda\ )$. Integrating yields a third-order linear equation: $''' = \dfrac{a}{\lambda}\sinh(\lambda\ ) + \quad$.

35. $+4 \quad + 3(\quad)^2 = \ \cosh(\quad)$.

Solution: $^2 = \ _3\ ^3 + \ _2\ ^2 + \ _1\ + \ _0 + 2a\lambda^{-4}\cosh(\lambda\ )$.

36. $+4 \quad + 3(\quad)^2 = \ \cosh^k(\quad) + \ $.

This is a special case of equation 4.2.6.60 with $(\ ) = a\cosh\ (\lambda\ ) + \ $.

4.2.3-3. Equations with hyperbolic tangent.

37. $= \tanh\ (\quad) + \ .$

This is a special case of equation 4.2.6.1 with $(\ ) = a \tanh\ (\lambda\ ) +\ $.

38. $= \tanh(\quad + \quad) + \ .$

The substitution $=\ +(\beta\ \lambda)$ leads to an autonomous equation of the form 4.2.6.1: $'''' \ = a \tanh(\lambda\ ) +\ $.

39. $= \ ^{-4} \tanh\ (\quad).$

This is a special case of equation 4.2.6.2 with $(\ ) = a \tanh\ (\lambda\ )$.

40. $= \tanh(\quad) \quad + \tanh(\quad).$

This is a special case of equation 4.2.6.21 with $(\ ) = a \tanh(\lambda\ )$ and $g(\ ) = \ \tanh(\beta\ )$.

41. $= \ ^4 + \tanh(\quad)(\quad - \quad)^k.$

This is a special case of equation 4.2.6.27 with $(\ ,\ ) = \ \tanh(\lambda\ )$.

42. $= \tanh(\quad)(\quad - \ )^k.$

This is a special case of equation 4.2.6.23 with $(\ ,\ ) = \ \tanh(\lambda\ )$.

43. $= \tanh(\quad)(\quad - 2\ )^k.$

This is a special case of equation 4.2.6.24 with $(\ ,\ ) = \ \tanh(\lambda\ )$.

44. $= \tanh(\quad)(\quad - 3\ )^k.$

This is a special case of equation 4.2.6.25 with $(\ ,\ ) = \ \tanh(\lambda\ )$.

45. $= \ + (\quad - \ \tanh\ )^k.$

This is a special case of equation 5.2.6.32 with $(\ ,\ ) = a$ and $(\ ) = \cosh$.

46. $= \tanh^k(\quad) \qquad .$

This is a special case of equation 4.2.6.51 with $(\ ) = a \tanh\ (\lambda\ )$.

47. $- \qquad = \tanh(\quad)\ ^2.$

This is a special case of equation 4.2.6.57 with $(\ ) = a \tanh(\lambda\ )$.

48. $+ 4 \qquad + 3(\quad)^2 = \ \tanh^k(\quad) + \ .$

This is a special case of equation 4.2.6.58 with $(\ ) = a \tanh\ (\lambda\ ) +\ $.

49. $+ 4 \qquad + 3(\quad)^2 = \ \tanh^k(\quad) + \ .$

This is a special case of equation 4.2.6.60 with $(\ ) = a \tanh\ (\lambda\ ) +\ $.

4.2.3-4. Equations with hyperbolic cotangent.

50. $= \coth\ (\quad) + \ .$

This is a special case of equation 4.2.6.1 with $(\ ) = a \coth\ (\lambda\ ) +\ $.

51. $= \coth(\quad + \quad) + \ .$

The substitution $=\ +(\beta\ \lambda)$ leads to an autonomous equation of the form 4.2.6.1: $'''' \ = a \coth(\lambda\ ) +\ $.

52. $= \quad {}^{-4}\,\mathbf{coth} \quad (\quad).$

This is a special case of equation 4.2.6.2 with $(\ ) = a\coth \ (\lambda \ )$.

53. $= \ \mathbf{coth}(\quad) \ + \ \mathbf{coth}(\quad).$

This is a special case of equation 4.2.6.21 with $(\ ) = a\coth(\lambda \ )$ and $g(\ ) = \ \coth(\beta \ )$.

54. $= \ {}^{4} \ + \ \mathbf{coth}(\quad)(\quad - \quad)^{k}.$

This is a special case of equation 4.2.6.27 with $(\ , \ ) = \ \coth(\lambda \ ) \quad .$

55. $= \ \mathbf{coth}(\quad)(\quad - \quad)^{k}.$

This is a special case of equation 4.2.6.23 with $(\ , \ ) = \ \coth(\lambda \ ) \quad .$

56. $= \ \mathbf{coth}(\quad)(\quad -2 \ )^{k}.$

This is a special case of equation 4.2.6.24 with $(\ , \ ) = \ \coth(\lambda \ ) \quad .$

57. $= \ \mathbf{coth}(\quad)(\quad -3 \ )^{k}.$

This is a special case of equation 4.2.6.25 with $(\ , \ ) = \ \coth(\lambda \ ) \quad .$

58. $= \ + \ (\quad - \ \mathbf{coth} \quad)^{k}.$

This is a special case of equation 5.2.6.32 with $(\ , \ ) = a \quad$ and $(\ ) = \sinh \ .$

59. $= \ \mathbf{coth}^{k}(\quad) \qquad .$

This is a special case of equation 4.2.6.51 with $(\ ) = a\coth \ (\lambda \ )$.

60. $- \qquad = \ \mathbf{coth}(\quad) \ {}^{2}.$

This is a special case of equation 4.2.6.57 with $(\ ) = a\coth(\lambda \ )$.

61. $+4 \qquad +3(\quad)^{2} = \ \mathbf{coth}^{k}(\quad) + \ .$

This is a special case of equation 4.2.6.58 with $(\ ) = a\coth \ (\lambda \ ) + \ .$

62. $+4 \qquad +3(\quad)^{2} = \ \mathbf{coth}^{k}(\quad) + \ .$

This is a special case of equation 4.2.6.60 with $(\ ) = a\coth \ (\lambda \ ) + \ .$

4.2.4. Equations Containing Logarithmic Functions

4.2.4-1. Equations of the form $'''' \quad = \ (\ , \ ).$

1. $= \ \mathbf{ln} \ (\quad) + \ .$

This is a special case of equation 4.2.6.1 with $(\ ) = a\ln \ (\lambda \ ) + \ .$

2. $= \ \mathbf{ln}(\quad + \quad) + \ .$

The substitution $= \quad + (\beta \ \lambda)$ leads to an autonomous equation of the form 4.2.6.1: $'''' \quad = a\ln(\lambda \ ) + \ .$

3. $= \ \mathbf{ln}(\quad + \quad {}^{2}) + \ .$

The substitution $= \quad + (\beta \ \lambda) \ {}^{2}$ leads to an autonomous equation of the form 4.2.6.1: $'''' \quad = a\ln(\lambda \ ) + \ .$

4. $= \quad {}^{-4}\,\mathbf{ln} \quad (\quad).$

This is a special case of equation 4.2.6.2 with $(\ ) = a\ln \ (\lambda \ )$.

5. $= \quad (\quad + \quad \ln \quad).$

This is a special case of equation 4.2.6.14 with $(\quad) = a \ln \quad .$

6. $= \quad ^{-4}(\quad + \quad \ln \quad).$

This is a special case of equation 4.2.6.15 with $(\quad) = a \ln \quad .$

7. $= \quad ^{-3}(\ln \quad - \ln \quad).$

This is a special case of equation 4.2.6.3 with $(\quad) = a \ln \quad .$

8. $= \quad ^{-5}(\ln \quad - 3 \ln \quad).$

This is a special case of equation 4.2.6.4 with $(\quad) = a \ln \quad .$

9. $= \quad ^{-5} \quad ^{2}(2 \ln \quad - 3 \ln \quad).$

This is a special case of equation 4.2.6.5 with $(\quad) = 2a \ln \quad .$

10. $= \quad ^{-4}(\ln \quad - \quad \ln \quad).$

This is a special case of equation 4.2.6.6 with $(\quad) = a \ln \quad$ and $k = - \quad .$

4.2.4-2. Other equations.

11. $= \quad ^{4} + \quad \ln(\quad)(\quad - \quad)^{k}.$

This is a special case of equation 4.2.6.27 with $(\; , \;) = \quad \ln(\lambda \quad) \quad .$

12. $= \quad \ln(\quad)(\quad - \quad)^{k}.$

This is a special case of equation 4.2.6.23 with $(\; , \;) = \quad \ln(\lambda \quad) \quad .$

13. $= \quad \ln(\quad)(\quad - 2 \quad)^{k}.$

This is a special case of equation 4.2.6.24 with $(\; , \;) = \quad \ln(\lambda \quad) \quad .$

14. $= \quad \ln(\quad)(\quad - 3 \quad)^{k}.$

This is a special case of equation 4.2.6.25 with $(\; , \;) = \quad \ln(\lambda \quad) \quad .$

15. $- (\quad)^{2} = \quad \ln(\quad).$

This is a special case of equation 4.2.6.36 with $(\quad) = a \ln(\lambda \quad).$

16. $+ 4 \quad = \quad (\ln \quad + \ln \quad).$

The substitution $(\quad) = \quad$ leads to an equation of the form 4.2.6.1: $'''' \quad = a \ln \quad .$

17. $= \quad \ln(\quad) \quad .$

This is a special case of equation 4.2.6.51 with $(\quad) = a \ln(\lambda \quad).$

18. $- \quad = \quad \ln(\quad) \quad ^{2}.$

Integrating yields a third-order linear equation: $''' \quad = \left[a \quad \ln(\lambda \quad) - a \quad + \quad \right] \quad .$

19. $+ 4 \quad + 3(\quad)^{2} = \quad \ln \quad (\quad) + \quad .$

This is a special case of equation 4.2.6.58 with $(\quad) = a \ln \quad (\lambda \quad) + \quad .$

20. $+ 4 \quad + 3(\quad)^{2} = \quad \ln \quad (\quad) + \quad .$

This is a special case of equation 4.2.6.60 with $(\quad) = a \ln \quad (\lambda \quad) + \quad .$

4.2.5. Equations Containing Trigonometric Functions

4.2.5-1. Equations with sine.

1. $= \sin\ (\ ) + .$

This is a special case of equation 4.2.6.1 with $(\ ) = a \sin\ (\lambda\ ) + .$

2. $= \sin(\ + \ ) + .$

The substitution $= + (\beta\ \lambda)$ leads to an autonomous equation of the form 4.2.6.1: $'''' = a \sin(\lambda\ ) + .$

3. $= (\ + \sin\ )^2 - \sin\ .$

The substitution $= + \sin$ leads to an autonomous equation of the form 4.2.6.1: $'''' = a\ ^2.$

4. $= {}^{-4} \sin\ (\ ).$

This is a special case of equation 4.2.6.2 with $(\ ) = a \sin\ (\lambda\ ).$

5. $= \sin(\ ) + \sin(\ ).$

Integrating yields a third-order equation: $''' = -\frac{a}{\lambda} \cos(\lambda\ ) - \frac{}{\beta} \cos(\beta\ ) + .$

6. $= {}^4 + \sin(\ )(\ - \ )^k.$

This is a special case of equation 4.2.6.27 with $(\ ,\ ) = \sin(\lambda\ )\ .$

7. $= \sin(\ )(\ - \ )^k.$

This is a special case of equation 4.2.6.23 with $(\ ,\ ) = \sin(\lambda\ )\ .$

8. $= \sin(\ )(\ - 2\ )^k.$

This is a special case of equation 4.2.6.24 with $(\ ,\ ) = \sin(\lambda\ )\ .$

9. $= \sin(\ )(\ - 3\ )^k.$

This is a special case of equation 4.2.6.25 with $(\ ,\ ) = \sin(\lambda\ )\ .$

10. $- (\ )^2 = \sin(\ ).$

1 . Integrating the equation twice, we arrive at a second-order equation: $'' - (\ ')^2 = -a\lambda^{-2} \sin(\lambda\ ) + {}_1 + {}_2.$

2 . Particular solution: $= \sin(\lambda\ ) + \frac{a}{\lambda^4}.$

11. $- (\ )^2 - \ + \sin(\ ) = 0.$

1 . Integrating yields a third-order equation: $''' - \ '\ '' - a\ - \lambda^{-1} \cos(\lambda\ ) = .$

2 . Particular solution: $= -\frac{}{\lambda(a^2 + \ ^2\lambda^6)} \left[a \cos(\lambda\ ) + \lambda^3 \sin(\lambda\ )\right] + .$

12. $- (\ )^2 - \ + \sin(\ ) = 0.$

1 . Integrating the equation two times, we obtain a second-order equation: $'' - (\ ')^2 - a\ + {}_1 + {}_2 - \lambda^{-2} \sin(\lambda\ ) = 0.$

2 . Particular solution: $= \sin(\lambda\ ) - \frac{+ a\ \lambda^2}{\lambda^4}.$

13. $\quad - (\quad)^2 = \quad [\quad - (\quad)^2] + \quad \sin(\quad) + \quad .$

The substitution $(\ ) = \ '' - (\ ')^2$ leads to a second-order constant coefficient nonhomogeneous linear equation of the form 2.1.9.1: $'' = a \quad + \quad \sin(\lambda\ ) + \ .$

14. $\quad - \quad - (\quad)^2 + \quad \sin(\quad) = 0.$

1 . Integrating the equation two times, we obtain a second-order equation: $'' - (\ ')^2 - a\ ' + \ _1 + \ _2 - \ \lambda^{-2} \sin(\lambda\ ) = 0.$

2 . Particular solution: $= \dfrac{}{\lambda^3(a^2 + \ ^2\lambda^2)} \big[a\cos(\lambda\ ) - \ \lambda \sin(\lambda\ )\big] + \ .$

15. $\quad = \quad \sin^k(\quad) \quad .$

This is a special case of equation 4.2.6.51 with $(\ ) = a \sin\ (\lambda\ ).$

16. $\quad - \quad = \quad \sin(\quad)\ ^2.$

This is a special case of equation 4.2.6.57 with $(\ ) = a\sin(\lambda\ ).$ Integrating yields a third-order linear equation: $''' = \quad - \dfrac{a}{\lambda}\cos(\lambda\ ) \quad .$

17. $\quad + 4 \quad + 3(\quad)^2 = \quad \sin(\quad).$

Solution: $^2 = \ _3\ ^3 + \ _2\ ^2 + \ _1\ + \ _0 + 2a\lambda^{-4}\sin(\lambda\ ).$

18. $\quad + 4 \quad + 3(\quad)^2 = \quad \sin^k(\quad) + \ .$

This is a special case of equation 4.2.6.60 with $(\ ) = a\sin\ (\lambda\ ) + \ .$

4.2.5-2. Equations with cosine.

19. $\quad = \quad \cos\ (\quad) + \ .$

This is a special case of equation 4.2.6.1 with $(\ ) = a\cos\ (\lambda\ ) + \ .$

20. $\quad = \quad \cos(\quad + \quad) + \ .$

The substitution $= \ + (\beta\ \lambda)$ leads to an autonomous equation of the form 4.2.6.1: $'''' = a\cos(\lambda\ ) + \ .$

21. $\quad = (\ + \ \cos\ )^2 - \ \cos\ .$

The substitution $= \ + \ \cos$ leads to an autonomous equation of the form 4.2.6.1: $'''' = a\ ^2.$

22. $\quad = \quad ^{-4}\cos\ (\quad).$

This is a special case of equation 4.2.6.2 with $(\ ) = a\cos\ (\lambda\ ).$

23. $\quad = \quad \cos(\quad) \quad + \quad \cos(\quad).$

Integrating yields a third-order equation: $''' = \dfrac{a}{\lambda}\sin(\lambda\ ) + \dfrac{}{\beta}\sin(\beta\ ) + \ .$

24. $\quad = \ ^4 \ + \ \cos(\quad)(\quad - \quad)^k.$

This is a special case of equation 4.2.6.27 with $(\ ,\ ) = \ \cos(\lambda\ ) \quad .$

25. $\quad = \quad \cos(\quad)(\quad - \quad)^k.$

This is a special case of equation 4.2.6.23 with $(\ ,\ ) = \ \cos(\lambda\ ) \quad .$

26. $= \cos(\quad)(\quad - 2\ )^k.$

This is a special case of equation 4.2.6.24 with $(\ ,\ ) = \ \cos(\lambda\ )\quad .$

27. $= \cos(\quad)(\quad - 3\ )^k.$

This is a special case of equation 4.2.6.25 with $(\ ,\ ) = \ \cos(\lambda\ )\quad .$

28. $-(\quad)^2 = \ \cos(\quad).$

1 . Integrating the equation twice, we arrive at a second-order equation: $'' - (\ ')^2 = -a\lambda^{-2}\cos(\lambda\ ) + \ _1\ + \ _2.$

2 . Particular solution: $= \ \cos(\lambda\ ) + \dfrac{a}{\lambda^4}.$

29. $-(\quad)^2 - \quad + \ \cos(\quad) = 0.$

1 . Integrating yields a third-order equation: $''' - \ '\ '' - a\ + \ \lambda^{-1}\sin(\lambda\ ) = \ .$

2 . Particular solution: $= \dfrac{}{\lambda(a^2 + \ ^2\lambda^6)}\big[a\sin(\lambda\ ) - \ \lambda^3\cos(\lambda\ )\big] + \ .$

30. $-(\quad)^2 - \quad + \ \cos(\quad) = 0.$

1 . Integrating the equation two times, we obtain a second-order equation: $'' - (\ ')^2 - a\ + \ _1\ + \ _2 - \ \lambda^{-2}\cos(\lambda\ ) = 0.$

2 . Particular solution: $= \ \cos(\lambda\ ) - \dfrac{a\ \ \lambda^2 +}{\lambda^4}.$

31. $-(\quad)^2 = \ [\quad - (\quad)^2] + \ \cos(\quad) + \ .$

The substitution $(\ ) = \ '' - (\ ')^2$ leads to a second-order constant coefficient nonhomogeneous linear equation of the form 2.1.9.1: $'' = a\ + \ \cos(\lambda\ ) + \ .$

32. $- \quad - (\quad)^2 + \ \cos(\quad) = 0.$

1 . Integrating the equation two times, we obtain a second-order equation: $'' - (\ ')^2 - a\ ' + \ _1\ + \ _2 - \ \lambda^{-2}\cos(\lambda\ ) = 0.$

2 . Particular solution: $= -\dfrac{}{\lambda^3(a^2 + \ ^2\lambda^2)}\big[\ \lambda\cos(\lambda\ ) + a\sin(\lambda\ )\big] + \ .$

33. $= \ \cos^k(\quad)\quad .$

This is a special case of equation 4.2.6.51 with $(\ ) = a\cos\ (\lambda\ ).$

34. $- \quad = \ \cos(\quad)\ ^2.$

This is a special case of equation 4.2.6.57 with $(\ ) = a\cos(\lambda\ )$. Integrating yields a third-order linear equation: $''' = \dfrac{a}{\lambda}\sin(\lambda\ ) + \quad .$

35. $+ 4\quad + 3(\quad)^2 = \ \cos(\quad).$

Solution: $^2 = \ _3\ ^3 + \ _2\ ^2 + \ _1\ + \ _0 + 2a\lambda^{-4}\cos(\lambda\ ).$

36. $+ 4\quad + 3(\quad)^2 = \ \cos^k(\quad) + \ .$

This is a special case of equation 4.2.6.60 with $(\ ) = a\cos\ (\lambda\ ) + \ .$

4.2.5-3. Equations with tangent.

37. $\quad = \ \mathbf{tan} \ (\quad) + \ .$

This is a special case of equation 4.2.6.1 with $(\ ) = a \tan \ (\lambda \ ) + \ .$

38. $\quad = \ \mathbf{tan}(\quad + \quad) + \ .$

The substitution $\quad = \quad + (\beta \ \lambda)$ leads to an autonomous equation of the form 4.2.6.1: $'''' \quad = a \tan(\lambda \ ) + \ .$

39. $\quad = \ ^{-4} \, \mathbf{tan} \ (\quad).$

This is a special case of equation 4.2.6.2 with $(\ ) = a \tan \ (\lambda \ ).$

40. $\quad = \ \mathbf{tan}(\quad) \quad + \ \mathbf{tan}(\quad).$

This is a special case of equation 4.2.6.21 with $(\ ) = a \tan(\lambda \ )$ and $g(\ ) = \ \tan(\beta \ ).$

41. $\quad = \ ^{4} \ + \ \mathbf{tan}(\quad)(\quad - \quad)^{k}.$

This is a special case of equation 4.2.6.27 with $(\ ,\ ) = \ \tan(\lambda \ ) \quad .$

42. $\quad = \ \mathbf{tan}(\quad)(\quad - \ )^{k}.$

This is a special case of equation 4.2.6.23 with $(\ ,\ ) = \ \tan(\lambda \ ) \quad .$

43. $\quad = \ \mathbf{tan}(\quad)(\quad -2 \ )^{k}.$

This is a special case of equation 4.2.6.24 with $(\ ,\ ) = \ \tan(\lambda \ ) \quad .$

44. $\quad = \ \mathbf{tan}(\quad)(\quad -3 \ )^{k}.$

This is a special case of equation 4.2.6.25 with $(\ ,\ ) = \ \tan(\lambda \ ) \quad .$

45. $\quad = \ + \ (\quad + \ \mathbf{tan} \ )^{k}.$

This is a special case of equation 5.2.6.32 with $(\ ,\ ) = a \quad$ and $(\ ) = \cos \ .$

46. $\quad = \ \mathbf{tan}^{k}(\quad) \qquad .$

This is a special case of equation 4.2.6.51 with $(\ ) = a \tan \ (\lambda \ ).$

47. $\quad - \qquad = \ \mathbf{tan}(\quad) \ ^{2}.$

This is a special case of equation 4.2.6.57 with $(\ ) = a \tan(\lambda \ ).$

48. $\quad + 4 \qquad + 3(\quad)^{2} = \ \mathbf{tan}^{k}(\quad) + \ .$

This is a special case of equation 4.2.6.58 with $(\ ) = a \tan \ (\lambda \ ) + \ .$

49. $\quad + 4 \qquad + 3(\quad)^{2} = \ \mathbf{tan}^{k}(\quad) + \ .$

This is a special case of equation 4.2.6.60 with $(\ ) = a \tan \ (\lambda \ ) + \ .$

4.2.5-4. Equations with cotangent.

50. $\quad = \ \mathbf{cot} \ (\quad) + \ .$

This is a special case of equation 4.2.6.1 with $(\ ) = a \cot \ (\lambda \ ) + \ .$

51. $\quad = \ \mathbf{cot}(\quad + \quad) + \ .$

The substitution $\quad = \quad + (\beta \ \lambda)$ leads to an autonomous equation of the form 4.2.6.1: $'''' \quad = a \cot(\lambda \ ) + \ .$

52. $= \quad {}^{-4}\cot \quad (\quad).$

This is a special case of equation 4.2.6.2 with $(\ ) = a \cot \ (\lambda \ )$.

53. $= \quad \cot(\quad) \quad + \quad \cot(\quad).$

This is a special case of equation 4.2.6.21 with $(\ ) = a \cot(\lambda \ )$ and $g(\ ) = \ \cot(\beta \ )$.

54. $= \ {}^{4} \ + \ \cot(\quad)(\quad - \quad)^{k}.$

This is a special case of equation 4.2.6.27 with $(\ , \ ) = \ \cot(\lambda \ ) \quad .$

55. $= \ \cot(\quad)(\quad - \ )^{k}.$

This is a special case of equation 4.2.6.23 with $(\ , \ ) = \ \cot(\lambda \ ) \quad .$

56. $= \ \cot(\quad)(\quad - 2 \ )^{k}.$

This is a special case of equation 4.2.6.24 with $(\ , \ ) = \ \cot(\lambda \ ) \quad .$

57. $= \ \cot(\quad)(\quad - 3 \ )^{k}.$

This is a special case of equation 4.2.6.25 with $(\ , \ ) = \ \cot(\lambda \ ) \quad .$

58. $= \ + \ (\quad - \ \cot \ )^{k}.$

This is a special case of equation 5.2.6.32 with $(\ , \ ) = a$ and $(\ ) = \sin \ .$

59. $= \ \cot^{k}(\quad) \qquad .$

This is a special case of equation 4.2.6.51 with $(\ ) = a \cot \ (\lambda \ )$.

60. $- \qquad = \ \cot(\quad) \ {}^{2}.$

This is a special case of equation 4.2.6.57 with $(\ ) = a \cot(\lambda \ )$.

61. $+ 4 \qquad + 3(\quad)^2 = \ \cot^{k}(\quad) + \ .$

This is a special case of equation 4.2.6.58 with $(\ ) = a \cot \ (\lambda \ ) + \ .$

62. $+ 4 \qquad + 3(\quad)^2 = \ \cot^{k}(\quad) + \ .$

This is a special case of equation 4.2.6.60 with $(\ ) = a \cot \ (\lambda \ ) + \ .$

4.2.6. Equations Containing Arbitrary Functions

4.2.6-1. Equations of the form $'''' \ = \ (\ , \ ).$

1. $= f(\ ).$

Autonomous equation. By integrating, we obtain $2 \ ' \ ''' \ - (\ '' \ )^2 = 2 \qquad (\ ) \quad + 2 \ .$ The substitution $(\ ) = | \ ' |^{3 \ 2}$ leads to a second-order equation: $'' \ = \frac{3}{2} \qquad (\ ) \quad + \qquad {}^{-5 \ 3}.$

2. $= \ {}^{-4} f(\ ).$

This is a special case of equation 4.2.6.55 with $a_1 = a_2 = a_3 = 0$. The substitution $= \ln | \ |$ leads to an autonomous equation.

3. $= \ {}^{-3} f(\quad).$

Homogeneous equation. The transformation $= \ln \ , \quad = \qquad$ leads to an autonomous equation of the form 4.2.6.79.

4. $\quad = \ {}^{-5} f(\quad {}^{-3})$.

The transcription $\quad = {}^{-1}, \quad = \quad {}^{-3}$ leads to an autonomous equation of the form 4.2.6.1: $''''\ = \ (\ )$.

5. $\quad = \ {}^{-5\ 2} f(\quad {}^{-3\ 2})$.

The transformation $\ = e\ ,\ \ = \ {}^{3\ 2}$ leads to an autonomous equation of the form 4.2.6.33: $'''' - \frac{5}{2}\ '' = -\frac{9}{16}\ \ + \ (\ )$.

6. $\quad = \ {}^{-k-4} f(\ {}^{k}\ )$.

Generalized homogeneous equation.

1 . The transformation $\ = \ln\ ,\ z = $ leads to an autonomous equation.

2 . The transformation $z = \ ,\ \ = \ '$ leads to a third-order equation.

7. $\quad = \ {}^{-4} f(\ {}^{k}\ )$.

Generalized homogeneous equation. The transformation $z = \ ,\ \ = \ '$ leads to a third-order equation.

8. $\quad = f(\ + \ {}_{3}\ {}^{3} + \ {}_{2}\ {}^{2} + \ {}_{1}\ + \ {}_{0})$.

The substitution $\ = \ + a_3\ {}^{3} + a_2\ {}^{2} + a_1\ + a_0$ leads to an equation of the form 4.2.6.1: $''''\ = \ (\ )$.

9. $\quad = f(\ + \ {}^{4})$.

The substitution $\ = \ + a\ {}^{4}$ leads to an equation of the form 4.2.6.1: $''''\ = \ (\ ) + 24a$.

10. $(\ \ + \ )^{4} \quad = f(\ {}^{-3})$.

The transformation $\xi = \ln \frac{a\ +\ }{\ }$, $\ = \frac{\ }{\ {}^{3}}$ leads to an autonomous equation of the form 4.2.6.79.

11. $\quad = (\ \ + \ \ + \ )^{-3} f\left(\frac{\ +\ \ +\ }{\ +\ \ + \gamma} \right)$.

This is a special case of equation 5.2.6.19 with $\ = 4$.

12. $\quad = (\ {}^{2} + \ \ + \ )^{-5\ 2} f\left(\frac{\ }{(\ {}^{2} + \ \ + \ )^{3\ 2}} \right)$.

1 . The transformation $\xi = \frac{\ }{a\ {}^{2} + \ \ + \ }$, $\ = \frac{\ }{(a\ {}^{2} + \ \ + \ )^{3\ 2}}$ leads to an autonomous equation of the form 4.2.6.33 with respect to $\ = \ (\xi)$:

$$'''' - \tfrac{5}{2}\Delta\ '' + \tfrac{9}{16}\Delta^2\ \ = \ (\ ), \qquad \text{where} \quad \Delta = \ {}^{2} - 4a\ .$$

Therefore, having integrated the latter equation, we obtain

$$'\ ''' - \tfrac{1}{2}(\ '')^2 - \tfrac{5}{4}\Delta(\ ')^2 = -\tfrac{9}{32}\Delta^2\ {}^{2} + \ \ (\ )\ \ + \ .$$

The substitution $z(\ ) = |\ '|^{3\ 2}$ leads to a second-order equation:

$$z''_{ww} = \tfrac{15}{8}\Delta z^{-1\ 3} + \tfrac{3}{2}\ -\tfrac{9}{32}\Delta^2\ {}^{2} + \ \ (\ )\ \ + \ \ z^{-5\ 3}.$$

2 . The first integral of the original equation has the form:

$$(\ \ ' - \tfrac{3}{2}\ \ '\ )\ ''' - \tfrac{1}{2}\ \ (\ '')^2 + \tfrac{1}{2}\ \ '\ '\ '' + 3a\ \ '' - 2a(\ ')^2 = \ \ (\ )\ \ + \ ,$$

where $\ = a\ {}^{2} + \ \ + \ ,\ \ = \ {}^{-3\ 2}$.

13. $= {}^{\lambda} f(\;\; ^{-\lambda})$.

This is a special case of equation 4.2.6.47 with $a = \;\; = \;\; = 0$. The substitution $(\;) = \;\; e^{-\lambda}$ leads to an autonomous equation.

14. $= f(\; ^{\lambda} \;\;)$.

The transformation $z = e^{\lambda}$, $(z) = \; '$ leads to a third-order equation.

15. $= {}^{-4} f(\;\; ^{\lambda})$.

The transformation $z = \;\; e^{\lambda}$, $(z) = \;\; '$ leads to a third-order equation.

16. $= f(\; + \;\;) - \;\;$.

The substitution $= \; + ae$ leads to an equation of the form 4.2.6.1: $'''' \; = (\;)$.

17. $= f(\; + \; \cosh \;) - \; \cosh \;$.

The substitution $= \; + a \cosh$ leads to an autonomous equation of the form 4.2.6.1: $'''' \; = (\;)$.

18. $= f(\; + \; \sinh \;) - \; \sinh \;$.

The substitution $= \; + a \sinh$ leads to an autonomous equation of the form 4.2.6.1: $'''' \; = (\;)$.

19. $= f(\; + \; \cos \;) - \; \cos \;$.

The substitution $= \; + a \cos$ leads to an autonomous equation of the form 4.2.6.1: $'''' \; = (\;)$.

20. $= f(\; + \; \sin \;) - \; \sin \;$.

The substitution $= \; + a \sin$ leads to an autonomous equation of the form 4.2.6.1: $'''' \; = (\;)$.

4.2.6-2. Equations of the form $'''' \; = (\;, \;, \; ')$.

21. $= f(\;) \;\; + \; (\;)$.

By integrating, we find $''' \; = \;\; (\;) \;\; + \;\; g(\;) \;\; + \;$. For $g(\;) \equiv 0$, the order of this equation can reduced by one with the help of the substitution $(\;) = \; '$.

22. $= {}^{-4} f(\;\; - \;)$.

The transformation $= \ln|\;|$, $= \; ' - \;$ leads to a third-order autonomous equation: $''' - 5 \; '' + 6 \; ' = (\;)$.

23. $= f(\;, \;\; - \;)$.

The substitution $= \; ' - \;$ leads to a third-order equation: $(\; ' \;\;)'' \; = (\;, \;)$.

24. $= f(\;, \;\; - 2 \;)$.

The substitution $= \; ' - 2 \;$ leads to a third-order equation: $''' \; - \; '' \; = {}^{2} (\;, \;)$.

25. $= f(\;, \;\; - 3 \;)$.

The substitution $= \; ' - 3 \;$ leads to a third-order equation: $''' \; = \;\; (\;, \;)$.

26. $= \;\; {}^{-4} f(\;\;\;)$.

The transformation $z = \; '$, $= {}^{2} \; ''$ leads to a second-order equation.

27. $= {}^{4} + f(\ , \ - \ ).$

The substitution $= {}' - a$ leads to a third-order equation: $''' + a{}'' + a^2{}' + a^3 =$ $(\ , \ ).$

28. $= f(\ , \ \sinh \ - \ \cosh \ ) + \ .$

The substitution $= {}' \sinh - \cosh$ leads to a third-order equation.

29. $= f(\ , \ \cosh \ - \ \sinh \ ) + \ .$

The substitution $= {}' \cosh - \sinh$ leads to a third-order equation.

30. $= f(\ , \ \sin \ - \ \cos \ ) + \ .$

The substitution $= {}' \sin - \cos$ leads to a third-order equation.

31. $= f(\ , \ \cos \ + \ \sin \ ) + \ .$

The substitution $= {}' \cos + \sin$ leads to a third-order equation.

32. $= \overline{} + f \ , \ - \overline{} \ , \ = (\ ).$

The substitution $= {}' - \frac{'}{}$ leads to a third-order equation.

4.2.6-3. Equations of the form $'''' = (\ , \ , \ ', \ '').$

33. $+ \ = f(\ ).$

Having integrated this equation, we obtain $2{}'{}''' - ({}'')^2 + a({}')^2 = 2 \ (\ ) \ + 2 \ ,$ where is an arbitrary constant. The substitution $(\ ) = |{}'|^{3 \ 2}$ leads to a second-order equation:

$$'' = -\tfrac{3}{4}a \ {}^{-1 \ 3} + \tfrac{3}{2} \ (\ ) \ + \ {}^{-5 \ 3}.$$

34. $+ f(\ ) \ = (\ ).$

Having integrated this equation, we obtain a third-order autonomous equation:

$$2{}'{}''' - ({}'')^2 + 2 \ ({}') = 2 \ g(\ ) \ + 2 \ , \qquad \text{where} \quad (\ ) = \ (\ ) \ .$$

The substitution $(\ ) = {}'$ leads to a second-order equation.

35. $= {}^{-2} f(\ - \ ) \ .$

The transformation $= \ln| \ |, \ = {}' -$ leads to a third-order equation:

$$''' - 5{}'' + 6{}' = (\ ){}'.$$

Integrating it, we obtain a second-order autonomous equation:

$$'' - 5{}' + 6 \ = \ (\ ) \ + \ .$$

The substitution $z(\ ) = \frac{1}{5}{}'$ leads to an Abel equation of the second kind:

$$zz'_w - z = \tfrac{1}{25} \ {-6} \ + \ (\ ) \ +$$

(see Subsection 1.3.1).

36. $-(\ \ )^2 = f(\ )$.

Integrating the equation twice, we arrive at a second-order equation:

$$'' - (\ ')^2 = {}_0 (\ -\ )\ (\ )\ +\ _1 +\ _2.$$

37. $-(\ \ )^2 = f(\ )[\ \ -(\ \ )^2] + (\ )$.

This is a special case of equation 4.2.6.93. The substitution $(\ ) =\ '' - (\ ')^2$ leads to a second-order linear equation: $'' = (\ )\ + g(\ )$.

38. $= {}^2 + f(\ \ +\ )$.

The substitution $=\ '' + a$ leads to a second-order autonomous equation of the form 2.9.1.1: $'' = a\ + (\ )$.

39. $= f(\ ,\ \ )$.

The substitution $(\ ) = (\ ')^2$ leads to a third-order equation: $''' + \frac{1}{2}\ '\ '' = 2\ (\ ,\ \frac{1}{2}\ '$.

40. $= {}^2 + f(\ ,\ \ +\ )$.

The substitution $=\ '' + a$ leads to a second-order equation: $'' = a\ + (\ ,\ )$.

41. $= f(\ \ -\ {}^2$.

This is a special case of equation 4.2.6.42.

42. $= f(\ \ -\ {}^2 + (\ \ -\ {}^2$.

1 . Particular solution:

$$= {}_1 \exp(\ _3\ ) + {}_2 \exp(-\ _3\ ),$$

where the constants $_1$, $_2$, and $_2$ are related by the constraint

$$^4_3 - {}^2_3\ (4\ _1\ _2\ ^2_3) - g(4\ _1\ _2\ ^2_3) = 0.$$

2 . Particular solution:

$$= {}_1 \cos(\ _3\ ) + {}_2 \sin(\ _3\ ),$$

where the constants $_1$, $_2$, and $_2$ are related by the constraint

$$^4_3 + {}^2_3\ (-\ ^2_1\ ^2_3 -\ ^2_2\ ^2_3) - g(-\ ^2_1\ ^2_3 -\ ^2_2\ ^2_3) = 0.$$

43. $= f(\ ^2\ \ -2\ \ +2\ )$.

The substitution $= {}^2\ '' - 2\ \ ' + 2$ leads to a second-order equation: $'' - 2\ ' = {}^{+3}\ (\ )$. For $= -4$, the substitution $z(\ ) = \frac{1}{3}\ '$ leads to an Abel equation of the second kind: $zz'_w - z = \frac{1}{9}\ (\ )$ (see Subsection 1.3.1).

44. $= f(\ ,\ \ -\ ,\ \ )$.

The substitution $(\ ) =\ ' -$ leads to a third-order equation.

45. $= f(\ ,\ ^2\ \ -2\ \ +2\ )$.

The substitution $= {}^2\ '' - 2\ \ ' + 2$ leads to a second-order equation: $'' - 2\ ' = {}^3\ (\ ,\ )$.

46. $= f\ \frac{}{},\ \ -\ \frac{}{}$.

Particular solution: $= {}_1 \exp(\ _2\ ) +\ _3$, where $_1$ is an arbitrary constant and the constants $_2$ and $_2$ are related by the constraint $^3_2 = (\ _2, -\ _2\ _3)$.

4.2.6-4. Equations of the form $'''' = (, , ', '', ''')$.

47. $+ \quad + \quad + \quad = {}^{\lambda} f(\quad ^{-\lambda}).$

The substitution $( ) = e^{-\lambda}$ leads to an autonomous equation:

$$'''' + (4\lambda + a) ''' + (6\lambda^2 + 3a\lambda +) '' + (4\lambda^3 + 3a\lambda^2 + 2 \lambda +) ' + (\lambda^4 + a\lambda^3 + \lambda^2 + \lambda) = ( ),$$

which can be reduced to a third-order equation by means of the substitution $z( ) = '$. For $a = -4\lambda$ and $= 8\lambda^3 - 2 \lambda$, the above equation coincides, up to notation, with equation 4.2.6.33 and can be reduced to a second-order equation.

48. $+ \quad = f( ).$

Integrating, we arrive at a third-order equation: $''' + a '' - \frac{1}{2}a(')^2 = ( ) + .$

49. $+ \quad - \quad = f( ).$

Integrating, we arrive at a third-order equation: $''' + a '' - a(')^2 = ( ) + .$

50. $+ \quad + f( ) = ( ).$

Integrating, we arrive at a third-order equation:

$$''' + a '' - \tfrac{1}{2}a(')^2 + (') = g( ) + , \qquad \text{where} \quad ( ) = ( ) .$$

51. $= f( ) \quad .$

Integrating, we arrive at a third-order autonomous equation of the form 3.5.1.1: $''' = \exp ( ) .$

52. $+ 4 \quad = f( ).$

The substitution $( ) =$ leads to an equation of the form 4.2.6.1: $'''' = ( ).$

53. $+ (+ 3) \quad = f(, \quad +).$

The substitution $= ' + a$ leads to a third-order equation: $''' = (,).$

54. $^2 \quad + 8 \quad + 12 \quad = f(^2).$

The substitution $( ) = {}^2$ leads to an autonomous equation of the form 4.2.6.1: $'''' = ( ).$

55. $^4 \quad + {}_3 {}^3 \quad + {}_2 {}^2 \quad + {}_1 \quad = f( ).$

The substitution $= \ln| |$ leads to an autonomous equation:

$$'''' + (a_3 - 6) ''' + (11 - 3a_3 + a_2) '' + (2a_3 - a_2 + a_1 - 6) ' = ( ), \tag{1}$$

the order of which can be lowered with the help of the substitution $( ) = '$. For $a_3 = 6$ and $a_1 = a_2 - 6$, equation (1) coincides, up to notation, with equation 4.2.6.33 and can be reduced to a second-order equation.

56. $^4 \quad + {}^3 \quad + {}^2 \quad + \quad + s = {}^{-k} f(\quad ^k).$

The transformation $= \ln , \quad =$ leads to an autonomous equation of the form 4.2.6.79.

57. $- \quad = f(\;)\;^2.$

Integrating yields a third-order linear equation: $''' \; = \; (\;) \; + \; .$

58. $+4 \quad +3(\quad)^2 = f(\;).$

Solution: $^2 = \;_3\;^3 + \;_2\;^2 + \;_1\; + \;_0 + \frac{1}{3}\;_0\;(\;-\;)^3\;(\;)\;.$

59. $+ \quad + (\;-1)(\quad)^2 = f(\;).$

Integrating the equation two times, we obtain a second-order equation:

$$'' + \frac{a-2}{2}(\;'\;)^2 = \;_1\; + \;_0 + \;_0\;(\;-\;)\;(\;)\;.$$

60. $+4 \quad +3(\quad)^2 = f(\;).$

The substitution $= \;^2$ leads to an equation of the form 4.2.6.1: $'''' \; = 2\;(\quad \overline{\quad}\;.$

61. $- \quad = f(\;) \quad .$

Integrating yields a third-order linear equation: $''' \; = \; \exp \; (\;) \quad .$

62. $+ \quad = f(\;) \quad .$

Integrating yields a third-order equation of the form 3.5.1.2: $''' \; = \; \exp \; (\;) \quad .$

63. $+(f-1) \quad +f \quad + \quad ^2 = 0, \quad f = f(\;), \quad = \;(\;).$

The functions that solve the third-order linear equation $''' \; + g(\;) \; = 0$ are solutions of the given equation.

64. $+(4 \quad + f\;) \quad +3(\quad)^2 + 3f \quad + \;(\;) = 0, \quad f = f(\;).$

The substitution $= (\;\;'\;)''$ leads to a first-order linear equation: $' + \quad + g = 0.$

Solution:

$$^2 = \;_2\;^2 + \;_1\; + \;_0 + \;_0\;(\;-\;)^2\;(\;)\;,$$

where $(\;) = e^{-F(\;)}\;\;_3 - \;\; e^{F(\;)}g(\;)\;,\quad (\;) = \;(\;)\;;\;\;_0$ is an arbitrary number.

65. $+(4 \quad + f\;) \quad +3(\quad)^2 + (3f \quad + \quad) \quad + \;(\quad)^2 + \quad + s = 0.$

Here, $= \;(\;),\; g = g(\;),\; = \;(\;),\; = \;(\;).$ The substitution $= \;'$ leads to a third-order nonhomogeneous linear equation: $''' + \;'' + g\;' + \; + \; = 0.$

66. $(\; + \; + \;) \quad +4(\; + \;) \quad +3(\quad)^2 = f(\;).$

Solution: $(\; + a\; + \;)^2 = \;_3\;^3 + \;_2\;^2 + \;_1\; + \;_0 + \frac{1}{3}\;_0\;(\;-\;)^3\;(\;)\;.$

67. $= 4 \quad + 3(\quad)^2 - 6\frac{(\quad)^4}{2} + [\quad - (\quad)^2]\,f \;\overline{\quad}\;.$

The transformation $\xi = \frac{'}{\;}, \quad = \frac{''}{\;} - \frac{'}{\;}^2$ leads to a second-order linear equation with respect to ^{2}: $(\;^2)'' = 24\xi^2 + 2\;(\xi)$. Integrating it, we obtain

$$^2 = \;_2\xi + \;_1 + 2\xi^4 + 2\;_0\;(\xi - \;)\;(\;)\;.$$

Taking into account that $\xi' = \quad , \quad ' = \xi \quad , \quad ' = \xi \qquad$, we find the solution in parametric form:

$$= \frac{\xi}{\quad} + \ _3, \qquad = \ _4 \exp \frac{\xi \ \xi}{\quad},$$

where $= \ _2\xi + \ _1 + 2\xi^4 + 2 \int_0 (\xi - \ ) \ (\ ) \qquad {}^{1\ 2}.$

68. $\ ^2 \qquad -2 \qquad + f(\ ) \ ^2 \qquad + 2(\quad)^2 \qquad - f(\ )$
$+ 2f \ (\ ) \ ^2 \qquad + 2f(\ )(\quad)^3 + [f^2(\ ) - 2f \ (\ )] \ (\quad)^2 + f \quad (\ ) \ ^2 \qquad = 0.$

The solution satisfies the second-order linear equation $'' + \ (\ ) \ ' - z(\ , \ _1, \ _2) \ = 0$, where $z = z(\ , \ _1, \ _2)$ is the Weierstrass elliptic function determined by the second-order autonomous equation $z'' + z^2 = 0$.

69. $\ ^2 \qquad -2 \qquad + f(\ ) \ ^2 \qquad + 2(\quad)^2 \qquad - f(\ )$
$+ 2f \ (\ ) \ ^2 \qquad + 2f(\ )(\quad)^3 + [f^2(\ ) - 2f \ (\ )] \ (\quad)^2 + f \quad (\ ) \ ^2 \qquad = \qquad ^3.$

The solution satisfies the second-order linear equation $'' + \ (\ ) \ ' - z(\ , \ _1, \ _2) \ = 0$, where $z = z(\ , \ _1, \ _2)$ is the solution of the first Painlevé transcendent $z'' + z^2 = A \ $.

70. $\ ^2 \qquad -2 \qquad + [\quad + f(\ )] \ ^2 \qquad - \ (\quad)^2 - [3 \quad + f(\ )]$
$+ 2(\quad)^2 \qquad + [\ f(\ ) + \ (\ )] \ ^2 \qquad + 2 \ (\quad)^3 - \ f(\ ) \ (\quad)^2 + \quad (\ ) \ ^2 \qquad = \ (\ ) \ ^3.$

The solution satisfies the second-order linear equation $'' + a \ ' - z(\ , \ _1, \ _2) \ = 0$, where $z = z(\ , \ _1, \ _2)$ is the solution of the second-order linear equation $z'' + \ (\ )z' + g(\ )z = \ (\ )$.

71. $\qquad - 3(\quad)^2 = f(\qquad - \ )(\quad)^5.$

The Legendre transformation $= \ ', \quad = \ ' - \ $ leads to an equation of the form 4.2.6.1: $'''' \ = - \ (\ )$.

72. $\qquad = f(\ ) \quad (\qquad).$

Integrating yields a third-order autonomous equation:

$$\frac{\quad}{g(\ )} = \quad (\ ) \quad + \ , \qquad \text{where} \qquad = \ ''' \ ,$$

the order of which can be lowered by means of the substitution $z(\ ) = \ '$.

73. $\qquad + 2 \qquad = (\qquad)^{-5} f \ \frac{\qquad}{\overline{\qquad} - } \ .$

The substitution $(\ ) = \ ' - \ $ leads to a third-order equation of the form 3.5.2.27:

$$''' \ = \ ^{-5\ 2} \ \frac{'}{=} , \qquad \text{where} \qquad (\xi) = \xi^{-5} \ (\xi).$$

74. $\ ^2 \qquad + 2 \qquad = f(\ ^2 \qquad - 2 \qquad + 2 \ ) \ (\ ^2 \qquad).$

The substitution $(\ ) = \ ^2 \ '' - 2 \ ' + 2 \ $ leads to a second-order equation of the form 2.9.4.36: $'' \ = \ (\ )g(\ ')$.

75. $\qquad = f(\ ) \ (\ ^3 \qquad - 3 \ ^2 \qquad + 6 \qquad - 6 \ ).$

The substitution $(\ ) = \ ^3 \ ''' - 3 \ ^2 \ '' + 6 \ ' - 6 \ $ leads to a first-order separable equation: $' = \ ^3 \ (\ )g(\ )$.

4.2.6-5. Other equations.

76. $-\frac{1}{6}(\quad)^2 = {}^2 f_1(\quad) + f_2(\quad) + f_3(\quad).$

Particular solution:

$$= \tfrac{1}{24}\ {}_1\ {}^4 + \tfrac{1}{6}\ {}_2\ {}^3 + \tfrac{1}{2}\ {}_3\ {}^2 + {}_4 + {}_5,$$

where the constants ${}_1$, ${}_2$, ${}_3$, ${}_4$, and ${}_5$ are related by three constraints

$$\tfrac{1}{3}\ {}_1\ {}_3 - \tfrac{1}{6}\ {}_2^2 = {}_1(\ {}_1),$$
$${}_1\ {}_4 - \tfrac{1}{3}\ {}_2\ {}_3 = {}_2(\ {}_1),$$
$${}_1\ {}_5 - \tfrac{1}{6}\ {}_3^2 = {}_3(\ {}_1).$$

77. $-\frac{1}{6}(\quad)^2 = f_1(\quad) + f_2(\quad) + {}^2 f_3(\quad) + f_4(\quad) + f_5(\quad).$

Particular solution:

$$= \tfrac{1}{24}\ {}_1\ {}^4 + \tfrac{1}{6}\ {}_2\ {}^3 + \tfrac{1}{2}\ {}_3\ {}^2 + {}_4 + {}_5,$$

where the constants ${}_1$, ${}_2$, ${}_3$, ${}_4$, and ${}_5$ are related by three constraints

$$\tfrac{1}{3}\ {}_1\ {}_3 - \tfrac{1}{6}\ {}_2^2 = \tfrac{1}{2}\ {}_1\ {}_2(\ {}_1) + {}_3(\ {}_1),$$
$${}_1\ {}_4 - \tfrac{1}{3}\ {}_2\ {}_3 = {}_1\ {}_1(\ {}_1) + {}_2\ {}_2(\ {}_1) + {}_4(\ {}_1),$$
$${}_1\ {}_5 - \tfrac{1}{6}\ {}_3^2 = {}_2\ {}_1(\ {}_1) + {}_3\ {}_2(\ {}_1) + {}_5(\ {}_1).$$

78. $= (\ ,\ ,\ ,\ ).$

The substitution $(\ ) = {}'$ leads to a third-order equation: ${}''' = (\ ,\ ,\ {}',\ {}'').$

79. $= (\ ,\ ,\ ,\ ).$

Autonomous equation. The substitution $(\ ) = (\ {}')^2$ leads to a third-order equation:

$${}''' + \tfrac{1}{2}\ {}'\ {}'' = 2\ (\ ,\ \overline{},\ \tfrac{1}{2}\ {}',\ \tfrac{1}{2}\ \overline{}\ {}'').$$

80. $= {}^{-3}(\ ,\ ,\ ,\ {}^2\ ).$

Homogeneous equation. The transformation $= \ln\ ,\ =$ leads to an autonomous equation of the form 4.2.6.79.

81. $= {}^{-k-4}(\ {}^k,\ {}^{k+1},\ {}^{k+2},\ {}^{k+3}\ ).$

Generalized homogeneous equation. The transformation $= \ln\ ,\ =$ leads to an autonomous equation of the form 4.2.6.79.

82. $= (\ ,\ -\ ,\ ,\ ).$

This is a special case of equation 5.2.6.78 with $= 4$. The substitution $= {}' -$ leads to a third-order equation.

83. $= (\ ,\ -2\ ,\ ).$

The substitution $= {}' - 2$ leads to a third-order equation: ${}' = (\ ,\ ,\ )$, where $= {}''$.

84. $= (\ ,\ {}^2\ -2\ +2\ ,\ ).$

The substitution $(\ ) = {}^2\ {}'' - 2\ {}' + 2$ leads to a second-order equation: $(\ {}^{-2}\ {}')' = (\ ,\ ,\ {}^{-2}\ {}').$

85. $= \quad \frac{}{}, \frac{}{}, \frac{}{}$.

The transformation $\xi = \frac{'}{}$, $= \frac{''}{} - \frac{'}{}^{2}$ leads to a second-order equation:

$$^{2}\ '' + \ (\ ')^2 + 4\xi \quad ' + 3 \quad ^2 + 6\xi^2 \quad + \xi^4 = \quad (\xi, \quad + \xi^2, \quad ' + 3\xi \quad + \xi^3).$$

86. $= \ ^{-4} \quad ^{k} \quad , \frac{}{}, \frac{^{2}}{}, \frac{^{3}}{}$.

Generalized homogeneous equation. The transformation $=$, $z = \frac{'}{}$ leads to a third-order equation.

87. $= \ ^{-4} \quad \frac{}{}, \frac{^{2}}{}, \frac{^{3}}{}$.

The transformation $z = \frac{'}{}$, $= \frac{^{2}\ ''}{}$ leads to a second-order equation.

88. $= \quad \frac{}{}, \ - \frac{}{}, \frac{}{}$.

Autonomous equation. This is a special case of equation 4.2.6.79.

Particular solution:

$$= \ _1 \exp(\ _2 \ ) + \ _3,$$

where $_1$ is an arbitrary constant and the constants $_2$ and $_2$ are related by the constraint $^3_2 = \ (\ _2, - \ _2 \ _3, \ ^2_2)$.

89. $= \ ^{-} \ (\quad , \quad , \quad , \quad$.

Equation invariant under "translation–dilatation" transformation. The substitution $= e$ leads to an autonomous equation of the form 4.2.6.79.

90. $= \ ^{-4} \ (\quad , \quad , \ ^{2} \quad , \ ^{3} \quad)$.

The transformation $z = \ e \ ,$ $= \ '$ leads to a third-order equation.

91. $= \quad , \frac{}{}, \frac{}{}, \frac{}{}$.

The transformation $z = e \quad , \quad = \ '$ leads to a third-order equation.

92. $(\quad , \quad - \ ^{2}, \quad , \quad) = 0.$

Autonomous equation. This is a special case of equation 4.2.6.79.

1 . Particular solution:

$$= \ _1 \exp(\ _3 \ ) + \ _2 \exp(- \ _3 \ ),$$

where the constants $_1$, $_2$, $_2$ are related by the constraint $(\ ^2_3, 4 \ _1 \ _2 \ ^2_3, \ ^2_3, \ ^4_3 = 0.$

2 . Particular solution:

$$= \ _1 \cos(\ _3 \ ) + \ _2 \sin(\ _3 \ ),$$

where $_1$, $_2$, and $_2$ are related by the constraint $(- \ ^2_3, -(\ ^2_1 + \ ^2_2) \ ^2_3, - \ ^2_3, \ ^4_3 = 0.$

93. $(\ , \quad - (\quad)^2, \quad - \quad , \quad - (\quad)^2 = 0.$

The substitution $(\ ) = \ '' - (\ ')^2$ leads to a second-order equation: $(\ , \ , \ ', \ '' \ ) = 0.$

94. $\quad \text{———}, \quad \text{———} - \quad = 0.$

A solution of this equation is any function that solves the third-order linear equation:

$$''' \;=\; {}_1 + {}_2,$$

where the constants ${}_1$ and ${}_2$ are related by the constraint $({}_1, -{}_2) = 0$.

95. $(\;,\quad + \quad,\quad - {}^2\;,\quad + \quad) = 0.$

The substitution $= '' + a$ leads to a second-order equation: $(\;,\;,\; '' - a\;,\; '') = 0.$

Chapter 5

Higher-Order Differential Equations

5.1. Linear Equations

5.1.1. Preliminary Remarks

In this Chapter, we denote higher derivatives by $^{(\ )}$ to mean ——.

1 . The general solution of a homogeneous linear equation of the th-order

$$(\ )^{(\ )} + {}_{-1}(\ )^{(\ -1)} + \quad + {}_1(\ )\,' + {}_0(\ ) \ = 0 \tag{1}$$

has the form:

$$= {}_1\,{}_1(\ ) + {}_2\,{}_2(\ ) + \quad + \quad (\ ). \tag{2}$$

Here, ${}_1(\ )$, ${}_2(\ )$, , $(\ )$ make up a fundamental set of solutions (the are linearly independent solutions; 0); ${}_1$, ${}_2$, , are arbitrary constants.

2 . Let ${}_0 = {}_0(\ )$ be a nontrivial particular solution of equation (1). Then the substitution

$$= {}_0(\ ) \int z(\ )$$

leads to a linear equation of the (− 1)st-order for $z(\ )$.

Let ${}_1 = {}_1(\ )$ and ${}_2 = {}_2(\ )$ be two nontrivial linearly independent particular solutions of equation (1) with $g \equiv 0$. Then the substitution

$$= {}_1 \quad {}_2 \quad - {}_2 \quad {}_1$$

leads to a linear equation of the (− 2)nd-order for $= (\ )$.

3 . Some additional information about higher-order linear equations can be found in Subsections 0.4.1–0.4.3.

5.1.2. Equations Containing Power Functions

5.1.2-1. Equations of the form $(\ )^{(\ )} + {}_0(\ ) \ = g(\ )$.

1. $^{(6)} + \quad = 0.$

1 . Solution for $a = 0$:

$$= {}_1 + {}_2 + {}_3{}^2 + {}_4{}^3 + {}_5{}^4 + {}_6{}^5.$$

2 . Solution for $a = k^6 > 0$:

$$= {}_1 \cos k \ + {}_2 \sin k \ + \cos\left(\tfrac{1}{2}k\right)\left({}_3 \cosh\xi + {}_4 \sinh\xi \right.$$
$$\left. + \sin\left(\tfrac{1}{2}k\right)\left({}_5 \cosh\xi + {}_6 \sinh\xi\right), \quad \text{where} \quad \xi = \tfrac{3}{2}k .$$

3 . Solution for $a = -k^6 < 0$:

$$= \ _1 \cosh k \ + \ _2 \sinh k \ + \cosh\left(\tfrac{1}{2}k\right)\left(\ _3 \cos\xi + \ _4 \sin\xi\right)$$
$$+ \sinh\left(\tfrac{1}{2}k\right)\left(\ _5 \cos\xi + \ _6 \sin\xi\right), \quad \text{where} \quad \xi = \tfrac{3}{2}k \ .$$

2. $\quad {}^{(2\ )} = \ ^2 \quad .$

Solution:

$$= \ _1 e \ + \ _2 e^{-} \ + \sum_{=1}^{-1} e \ (A \ \cos\theta \ + B \ \sin\theta \ ),$$

where $= a \ \cos \frac{k}{}$, $\theta \ = a \ \sin \frac{k}{}$; $\ _1$, $\ _2$, A , B $(k = 1, 2, \ , \ - 1)$ are arbitrary constants.

3. $\quad {}^{(\ )} = \quad + \ , \quad > 0.$

Solution:

$$= \sum_{=0} \varepsilon \ \ _0 \ \exp \varepsilon \ - \frac{+1}{a(\ + 1)} \ , \quad \varepsilon \ = \exp \frac{2}{+1} \ ,$$

where $\sum_{=0} \ = \frac{}{a}$ and $^2 = -1$.

4. $\quad {}^{(\ )} = \quad .$

For specific β, see equations 5.1.2.2, 5.1.2.3 (with $= 0$), 5.1.2.5–5.1.2.9, and 5.1.2.10 (with $= 0$).

1 . Let ≥ 2, $\beta > -$, and $(\ + \beta)(\ + 1) \neq 1, 2, \ , \ - 1$, where $= 0, 1,$ Then the equation has solutions that can be represented as:

$$(\ ) = \ ^{-1} \ _{,1+ \ ,(\ + -1)} \left(a \ ^{+} \ , \qquad = 1, 2, \ , \ . \tag{1}$$

Here, $\ _{, \ ,}(z)$ is a Mittag-Leffler type special function defined by:

$$\ _{, \ ,}(z) = 1 + \sum_{=1} \ z \ ,$$
$$= \prod_{=0}^{-1} \frac{\Gamma\left(\ (\ + l) + 1\right)}{\Gamma\left(\ (\ + l + 1) + 1\right)} = \prod_{=0}^{-1} \frac{1}{[\ (\ + l) + 1] \ [\ (\ + l) + \ ]}, \tag{2}$$

where $\Gamma(\xi)$ is the gamma function, l is an arbitrary number, and > 0.

If $\beta \geq 0$, solutions (1) are linearly independent. Series expansions of (1) are convenient for small .

2 . Let ≥ 2, $\beta < -$, and $(\ + \beta)(\ + 1) \neq -1, -2, \ , -(\ - 1)$, where $= 0, 1,$ Then the equation in question has solutions that can be represented as:

$$(\ ) = \ ^{-1} \ _{,-1- \ ,-1-(\ +\ )} \left(a(-1) \ ^{+} \ , \qquad = 1, 2, \ , \ , \tag{3}$$

where $\ _{, \ ,}(z)$ is the Mittag-Leffler type special function defined by (2). If $\beta \leq -2$, solutions (3) are linearly independent. Series expansions of (3) are convenient for large .

3 . The transformation $= \ ^{-1}$, $= \ ^{1-}$ leads to an equation of similar form:

$$^{(\ )} = a(-1) \ ^{+1 \ -2 \ -} \quad .$$

Reference: M. Saigo and A. A. Kilbas (2000).

5. $\quad {}^{2}\ ({\ }) = \ .$

The transformation $= {}^{-1}, \quad = {}^{1-}$ leads to a constant coefficient linear equation: ${}^{(\ )} = (-1)\ a\ .$

6. $\quad {}^{(2\ )} = \ .$

Solution:

$$= {}^{2} \underset{=1}{} \Big[\ {}_{1}\ \big(2\beta \ \overline{\ }\ + \ {}_{2}\ \big(2\beta \ \overline{\ }\ \Big],$$

where (z) and (z) are the modified Bessel functions; $\beta_1, \beta_2, \quad , \beta$ are roots of the equation $\beta = \overline{a}$.

7. $\quad {}^{3}\ {}^{(2\ )} = \ .$

The transformation $= {}^{-1}, \quad = {}^{1-2}$ leads to an equation of the form 5.1.2.6: ${}^{(2\ )} = a\ .$

8. $\quad {}^{+1\ 2\ \ (2\ +1)} = \ .$

Solution:

$$= {}^{(2\ +1)\ 4} \underset{=0}{\overset{2}{}} \Big[\ {}_{-\ -1\ 2}\big(2\beta \ \overline{\ }\ + \ {}_{+1\ 2}\big(2\beta \ \overline{\ }\ \Big],$$

where (z) are the Bessel functions; $\beta_0, \beta_1, \quad , \beta_2$ are roots of the equation $\beta^{2\ +1} = -a$; ${}^{2} = -1$.

9. $\quad {}^{3\ +3\ 2\ \ (2\ +1)} = \ .$

The transformation $= {}^{-1}, \quad = {}^{-2}$ leads to a linear equation of the form 5.1.2.8: ${}^{+1\ 2\ \ (2\ +1)} = -a\ .$

10. $\quad {}^{2\ +1\ \ (\ )} = \ + \ .$

The transformation $= {}^{-1}, \quad = {}^{1-}$ leads to a linear equation of the form 5.1.2.3: ${}^{(\ )} = (-1)\ (a\ \ +\ ).$

11. $\quad (\ \ +\ )^{2\ +1}\ {}^{(\ )} = (\ \ +\ )\ .$

The transformation $\xi = \dfrac{\ +\ }{a\ +\ }, \quad = \dfrac{}{(a\ +\ )^{\ -1}}$ leads to an equation of the form 5.1.2.3: ${}^{(\ )} = \Delta^{-}\ \xi\ $, where $\Delta = \ -a\ .$

12. $\quad (\ \ +\ )\ (\ \ +\ )\ {}^{(\ )} = \ .$

1 . The transformation $\xi = \ln \dfrac{a\ +\ }{\ +\ }, \quad = \dfrac{}{(\ \ +\ )^{\ -1}}$ leads to a constant coefficient linear equation.

2 . The transformation $= \dfrac{a\ +\ }{\ +\ }, \quad = \dfrac{}{(\ \ +\ )^{\ -1}}$ leads to the Euler equation 5.1.2.39: ${}^{(\ )} = k\Delta^{-}\ $, where $\Delta = a\ -\ .$

13. $\quad (\ {}^{2} +\ \ +\ )\ {}^{(\ )} = \ .$

The transformation $\xi = \dfrac{}{a\ {}^{2} +\ \ +\ }, \quad = (a\ {}^{2} +\ \ +\ )^{\frac{1-}{2}}$ leads to a constant coefficient linear equation.

14. $\quad (\ \ +\ )\ (\ \ +\ )^{3}\ {}^{(2\ )} = \ .$

The transformation $\xi = \dfrac{a\ +\ }{\ +\ }, \quad = \dfrac{}{(\ \ +\ )^{2\ -1}}$ leads to an equation of the form 5.1.2.6: $\xi\ {}^{(2\ )} = k\Delta^{-2}\ $, where $\Delta = a\ -\ .$

15. $(\quad + \quad)^{\quad +1 \; 2}(\quad + \quad)^{3 \quad +3 \; 2 \quad (2 \quad +1)} = \quad .$

The transformation $\xi = \dfrac{a \quad +}{\quad +}$, $\quad = \dfrac{}{(\quad + \quad)^2}$ leads to an equation of the form 5.1.2.8: $\xi^{\quad +1 \; 2 \quad (2 \quad +1)} = k\Delta^{-2 \quad -1}$, where $\Delta = a \quad - \quad .$

5.1.2-2. Equations of the form $\quad (\;)^{(\;)} + \; _1(\;) \; ' + \; _0(\;) \quad = g(\;).$

16. $^{(\;)} + \quad ^{k} \quad + \quad ^{k-1} \quad = 0.$

Integrating yields an $(\quad - 1)$st-order linear equation: $^{(\quad -1)} + a \quad = \quad .$

17. $^{(\;)} + \quad ^{k+1} \quad - \; (\quad - 1) \quad ^{k} \quad = 0.$

The substitution $z = \quad ' - (\quad - 1)$ leads to an $(\quad - 1)$st-order equation: $z^{(\quad -1)} + a \quad ^{+1} z = 0.$

18. $^{(\;)} + \quad ^{k+1} \quad + \; (\quad + \quad) \quad ^{k} \quad = 0.$

The transformation $\quad = \quad ^{-1}, \quad = \quad ^{1-}$ leads to an equation of the form 5.1.2.16: $^{(\;)} + \quad ' + \quad ^{-1} = 0$, where $\quad = a(-1) \quad ^{+1}, \quad = 1 - k - 2 \quad .$

19. $^{(\;)} + (\quad + \quad) \quad ^{k} \quad - \quad ^{k} \quad = 0.$

Particular solution: $\; _0 = a \quad + \quad .$

20. $^{(\;)} + (\quad + \quad) \quad ^{k} \quad - 2 \quad ^{k} \quad = 0.$

Particular solution: $\; _0 = (a \quad + \quad)^2.$

21. $^{(\;)} + (\quad + \quad) \quad ^{k} \quad - 3 \quad ^{k} \quad = 0.$

Particular solution: $\; _0 = (a \quad + \quad)^3.$

22. $^{(\;)} + (\quad + \quad) \quad ^{k} \quad - \; (\quad - 1) \quad ^{k} \quad = 0.$

Particular solution: $\; _0 = (a \quad + \quad) \quad ^{-1}$. The substitution $z = (a \quad + \quad) \quad ' - a(\quad - 1)$ leads to an $(\quad - 1)$st-order linear equation: $z^{(\quad -1)} + (a \quad + \quad) \quad z = 0.$

23. $^{(\;)} + \quad ^{k+1} \quad - \quad ^{k} \quad = 0, \qquad = 1, 2, \ldots, \quad - 1.$

Particular solution: $\; _0 = \quad .$ The substitution $z = \quad ' - \quad$ leads to an $(\quad - 1)$st-order linear equation:

$$- \quad -1 \; \frac{z^{(\;)}}{} + a \quad z = 0, \qquad \text{where} \qquad = \frac{}{}.$$

5.1.2-3. Other equations.

24. $^{(2 \quad)} = \quad + \quad ^{k}(\quad - \quad).$

This is a special case of equation 5.1.6.18 with $\quad (\;) = \quad .$ The substitution $\quad = \quad '' - a$ leads to an $(2 \quad - 2)$nd-order linear equation: $^{(2 \quad -2)} + a \quad ^{(2 \quad -4)} + \quad + a \quad ^{-1} \quad = \quad .$

25. $^{(\;)} + \quad _{-1} \quad ^{(\quad -1)} + \cdots + \quad _1 \quad + \quad _0 \quad = 0.$

Constant coefficient homogeneous linear equation. To solve this equation, determine the roots of the characteristic equation:

$$\lambda \quad + a \quad _{-1} \lambda \quad ^{-1} + \quad + a_1 \lambda + a_0 = 0.$$

If the roots $\lambda_1, \lambda_2, \quad , \lambda$ are all real and different, then the general solution of the original equation is:

$$= \quad _1 \exp(\lambda_1 \quad) + \quad _2 \exp(\lambda_2 \quad) + \quad + \quad \exp(\lambda \quad).$$

The general case, which involves the cases of multiple and/or complex roots of the characteristic equation, is discussed in Subsection 0.4.1.

26. $\quad ^{(\ )} + \quad ^{k}\ ^{(\ )} - (\quad ^{k} + \quad) \quad = 0.$

Particular solution: $\quad {}_0 = e^{b}\quad .$

27. $\quad ^{(\ )} + (\quad ^{k} - \quad ^{-} \ ) \ ^{(\ )} - \quad ^{k} \quad = 0.$

Particular solution: $\quad {}_0 = e^{b}\quad .$

28. $\quad ^{(\ )} + \quad ^{(\ -1)} + \quad + \quad = 0.$

Particular solution: $\quad {}_0 = e^{-}\quad .$

29. $\quad ^{(\ )} - \quad ^{(\ -1)} + \quad = 0, \qquad = 2, 3, 4, \ldots, \qquad = 1, 2, 3, \ldots$

Solution:

$$= \ ^{(\ +1)\ -1} \quad ^{1-} \ \frac{}{} \ (\ ^{1-} \quad),$$

where is the general solution of the constant coefficient linear equation: $\ ^{(\ )} + a \quad = 0.$

30. $\quad ^{(\ )} + \quad ^{(\ -1)} = \quad + \ .$

The substitution $\ = \ $ leads to a constant coefficient linear equation: $\ ^{(\ )} = a \quad + \ .$

31. $\quad ^{(\ )} + \quad ^{(\ -1)} = \quad ^{2} \quad + \ .$

The substitution $\ = \ $ leads to an equation of the form 5.1.2.3: $\ ^{(\ )} = a \quad + \ .$

32. $\quad ^{(\ )} + (\quad - \quad - 1) \ ^{(\ -1)} + \quad ^{k} \quad - \quad ^{k-1} \quad = 0.$

Particular solution: $\quad {}_0 = \quad .$

33. $\quad ^{(\ )} + \quad ^{k}\ ^{(\ )} - (\quad ^{k} + \quad ^{k-1} + \quad + \quad) \quad = 0.$

Particular solution: $\quad {}_0 = \quad e \quad .$

34. $\quad ^{(\ )} = \sum_{\nu=0}^{\ -1} [(\quad _{\nu+1} - \quad _{\nu}) \quad + \quad _{\nu+1}] \ ^{(\nu)}.$

Here, $A \quad = 1$, $A_0 = 0$; a and A are arbitrary numbers ($= 1, 2, \quad , \quad - 1$).

Denote $\ (\lambda) = \sum_{=0}^{-1} A \ _{+1}\lambda\ $. Let the roots $\lambda_1, \lambda_2, \quad , \lambda \ _{-1}$ of the algebraic equation $\ (\lambda) = 0$ be all different, and $\ (a) \neq 0$. Then the solution is as follows:

$$= \ _1 e^{\lambda_1} \ + \ _2 e^{\lambda_2} \ + \quad + \ _{-1} e^{\lambda \ -1} \ + \quad e \qquad - \frac{\ '(a)}{\ (a)} \ .$$

35. $\quad \sum_{\nu=0} (\ _{\nu} \ + \ _{\nu}) \ ^{(\nu)} = 0.$

The Laplace equation. Particular solutions:

$$= \quad \frac{1}{(\ )} \exp \quad + \frac{Q(\ )}{(\ )} \qquad ,$$

where $\ (\ ) = \sum_{=0} a \quad$, $Q(\ ) = \sum_{=0} \quad$; and β are found from the condition

$$\exp \quad + \frac{Q(\ )}{(\ )} \qquad = 0.$$

In many cases, the path of integration has to be chosen on the complex plane.

36. $^{2}\ ^{(\ )} + 2 \quad ^{(\ -1)} + \ (\ -1) \ ^{(\ -2)} = \quad ^{2} \ + \ .$

The substitution $= \ ^{2}$ leads to a constant coefficient linear equation: $^{(\ )} = a \ + \ .$

37. $^{2}\ ^{(\ )} + 2 \quad ^{(\ -1)} + \ (\ -1) \ ^{(\ -2)} = \quad ^{3} \ + \ .$

The substitution $= \ ^{2}$ leads to an equation of the form 5.1.2.3: $^{(\ )} = a \quad + \ .$

38. $(\ + \ ) \ ^{(\ )} + \ (\ ^{k} - \ - \ ) \ ^{(\ )} - \ (\ + \ ) \ ^{k} = 0.$

Particular solution: $_{0} = \ e \ .$

39. $^{(\ )} + \ _{-1} \ ^{-1} \ ^{(\ -1)} + \cdots + \ _{1} \ + \ _{0} = 0.$

Euler equation. If all roots λ $(k = 1, 2, \ , \ )$ of the algebraic equation

$$\sum_{=1} a \ \lambda(\lambda-1) \ (\lambda - \ +1) = -a_0$$

are different, the general solution of the original differential equation is given by:

$$= \ _1| \ |^{\lambda_1} + \ _2| \ |^{\lambda_2} + \ + \ | \ |^{\lambda} \ .$$

In the general case, the substitution $= \ln| \ |$ leads to a constant coefficient linear equation of the form 5.1.2.25:

$$\sum_{=1} a \ (\ -1) \ (\ - \ +1) \ = -a_0 \ , \qquad \text{where} \quad = \text{—}.$$

40. $^{2 \ +1} \ ^{(\ )} + \ ^{2} \ ^{(\ -1)} = \ .$

The substitution $=$ leads to an equation of the form 5.1.2.5: $^{2} \ ^{(\ )} = a \ .$

41. $^{2 \ +1} \ ^{(\ )} + \ ^{2} \ ^{(\ -1)} = \ .$

The substitution $=$ leads to an equation of the form 5.1.2.10: $^{2 \ +1} \ ^{(\ )} = a \ .$

42. $^{(2 \ )} + 2 \ ^{-1} \ ^{(2 \ -1)} = \ .$

The substitution $=$ leads to an equation of the form 5.1.2.6: $^{(2 \ )} = a \ .$

43. $^{3} \ ^{(2 \ )} + 2 \ ^{3 \ -1} \ ^{(2 \ -1)} = \ .$

The substitution $=$ leads to an equation of the form 5.1.2.7: $^{3} \ ^{(2 \ )} = a \ .$

44. $^{+1} \ ^{(2 \ +1)} + (2 \ +1) \ ^{(2 \ )} = \ \text{—} \ .$

The substitution $=$ leads to an equation of the form 5.1.2.8: $^{+1 \ 2} \ ^{(2 \ +1)} = a \ .$

45. $^{3 \ +3 \ 2} \ ^{(2 \ +1)} + (2 \ +1) \ ^{3 \ +1 \ 2} \ ^{(2 \ )} = \ .$

The substitution $=$ leads to an equation of the form 5.1.2.9: $^{3 \ +3 \ 2} \ ^{(2 \ +1)} = a \ .$

46. $_{-1}(\ ) \ ^{(\ )} + \ _{-2}(\ ) \ ^{(\ -1)} + \cdots + \ _{1}(\ ) \ + (\ _1 \ + \ _1) \ - \ _1 = 0.$

Here, the $(\ )$ are polynomials of degree $\le$, is a positive integer, $a_1 \neq 0$.

A particular solution of this equation is the polynomial of degree that can be written as:

$$_0 = \sum_{=0} -\frac{1}{a_1} \ [\ ^{- \ -1}(\ _{-1} \ + \ + \ _1 \ ^{2} + \ _1 \ )] \ ,$$

where $= \text{—}, \quad = \dfrac{^{+1}}{+1}$ with $\neq -1.$

Reference: E. Kamke (1977).

47. $[\quad + \quad_{-1}(\ )]\ ^{(\ )} + \cdots + [\ _{1} \quad + \quad_{0}(\ )] \quad + \ _{0} \quad = 0.$

Here, the $\quad(\ )$ are polynomials of degree $\le \ $.

Assume that for some integer $\quad \ge 0$:

$$\quad !\,a \quad = 0, \qquad \text{where} \qquad = \frac{!}{!\,(\quad - \quad)!},$$
$$_{=0}$$

and is the least of the numbers satisfying this condition. Then there exists a solution in the form of a polynomial of degree such that no polynomial of a smaller degree satisfies the original equation.

Reference: E. Kamke (1977).

48. $\Big[\quad(\ ) - Q(\ )\Big] \quad = 0, \qquad \equiv \quad \text{——}.$

Here, $\quad = \quad(z)$ and $Q = Q(z)$ are arbitrary polynomials of degree and , respectively.

Suppose $Q(z+1) = (z+1)Q_1(z+1)$, where the polynomial $Q_1(z+1)$ is such that $\quad(z)$ and $Q_1(z+1)$ do not have common factors. Then the original equation admits a formal solution in the power series form:

$$_{0} = \quad A \quad, \qquad \text{where} \quad \frac{A_{\ +1}}{A} = \frac{(\ )}{Q(\ +1)}.$$
$$_{=0}$$

Reference: H. Bateman and A. Erdélyi (1953, Vol. 1).

5.1.3. Equations Containing Exponential and Hyperbolic Functions

5.1.3-1. Equations with exponential functions.

1. $^{(\ )} + (\quad + \ )\ ^{\lambda} \quad - \quad ^{\lambda} \quad = 0.$

Particular solution: $_{0} = a \quad + \ .$

2. $^{(\ )} + (\quad + \ )\ ^{\lambda} \quad - 2 \quad ^{\lambda} \quad = 0.$

Particular solution: $_{0} = (a \quad + \ )^2.$

3. $^{(\ )} + (\quad + \ )\ ^{\lambda} \quad - 3 \quad ^{\lambda} \quad = 0.$

Particular solution: $_{0} = (a \quad + \ )^3.$

4. $^{(\ )} + (\quad + \ )\ ^{\lambda} \quad - (\quad - 1) \quad ^{\lambda} \quad = 0.$

Particular solution: $_{0} = (a \quad + \ )\ ^{-1}$. The substitution $z = (a \quad + \ )\ ' - a(\quad - 1)$ leads to an $(\quad - 1)$st-order linear equation: $z^{(\ -1)} + (a \quad + \ )e^{\lambda}\ z = 0.$

5. $^{(\ )} + \quad ^{\lambda} \quad - \quad ^{\lambda} \quad = 0, \qquad = 1, 2, \ldots, \quad - 1.$

Particular solution: $_{0} = \quad .$

6. $^{(2\ )} = \quad + \ ^{\lambda}\ (\quad - \quad).$

This is a special case of equation 5.1.6.18 with $(\ ) = \ e^{\lambda}$. The substitution $\quad = \ '' - a$ leads to an $(2 \quad - 2)$nd-order linear equation: $^{(2\ -2)} + a \ ^{(2\ -4)} + \quad + a \ ^{-1} \quad = e^{\lambda} \quad .$

7. $^{(\ )} + (\ ^{\lambda} \ - \ ^{-}\ )\ ^{(\ )} - \quad ^{\lambda} \quad = 0.$

Particular solution: $_{0} = e^{b}\ .$

8. $(\ ) + \quad (\ -1) + \quad \lambda \quad + \quad \lambda \quad = 0.$

Particular solution: $_0 = e^{-}$.

9. $(\ ) + \quad \lambda \quad (\ ) - (\quad \lambda \quad + \quad) \quad = 0.$

Particular solution: $_0 = e^{b}$.

10. $(\ ) = \sum_{k=0}^{-1} (\quad _{k+1} \ \lambda \quad + \quad _{k+1} - \quad _k) \ ^{(k)}.$

Here, $A \quad = 1$, $A_0 = 0$; and A are arbitrary numbers ($k = 1, 2, \quad , \quad - 1$).

Particular solutions: $= e^{\ m}$, where the are roots of the polynomial equation $\sum_{=0}^{-1} A_{+1} \quad = 0.$

11. $(\ ) + \quad \lambda \quad (\ ) - [\ (\ + \ ) \ \lambda \ + \ + \ ] \ = 0.$

Particular solution: $_0 = \ e$.

12. $(\ ) + (\ - \ - 1) \ (\ -1) + \quad \lambda \quad - \quad \lambda \quad = 0.$

Particular solution: $_0 =$.

13. $(\ + \ ) \ (\ ) + \ (\ \lambda \ - \ - \ ) \ (\ ) - \ (\ + \ ) \ \lambda \quad = 0.$

Particular solution: $_0 = \ e$.

14. $(\quad + \quad + \ ) \ (\ ) = \quad , \quad = 0, 1, \dots, \quad - 1.$

Particular solution: $_0 = a \quad + \ e \ + \ $.

15. $(\quad + \ ) \ (\ ) = (-1) \quad , \quad = 0, 1, \dots, \quad - 1.$

Particular solution: $_0 = a \quad + \ e^{-}$.

16. $+ \sum_{k=0}^{-1} \ _k \ ^k \ (\ ) = \quad .$

Particular solution: $_0 = ae \ + \sum_{=0}^{-1}$.

5.1.3-2. Equations with hyperbolic functions.

17. $^{(2 \ )} = \quad + \ \sinh^k(\ )(\quad - \quad).$

This is a special case of equation 5.1.6.18 with $(\ ) = \ \sinh \ (\lambda \ )$.

18. $(\ ) + \ \sinh^k \quad (\ ) - (\quad \sinh^k \quad + \quad) \ = 0.$

Particular solution: $_0 = e^{b}$.

19. $(\ ) + (\ \sinh^k \ - \ ^{-} \ ) \ (\ ) - \quad \sinh^k \quad = 0.$

Particular solution: $_0 = e^{b}$.

20. $(\ ) + (\ + \ ) \sinh \ (\ ) \ - \ \sinh \ (\ ) \ = 0.$

Particular solution: $_0 = a \ + \ $.

21. $(\ ) + \quad \sinh^k \quad (\ ) - [\ (\ + \ ) \sinh^k \ + \ + \ ] \ = 0.$

Particular solution: $_0 = \ e$.

22. $^{(2\ \ )} = \quad + \ \cosh^k(\quad)(\quad - \quad).$

This is a special case of equation 5.1.6.18 with $(\ ) = \ \cosh\ (\lambda\ ).$

23. $^{(\ )} + \ \cosh^k \quad ^{(\ )} - (\quad \cosh^k \quad + \quad) \ = 0.$

Particular solution: $_0 = e^b\ .$

24. $^{(\ )} + (\ \cosh^k \quad - \quad ^{-}\ )\ ^{(\ )} - \quad \cosh^k \quad = 0.$

Particular solution: $_0 = e^b\ .$

25. $^{(\ )} + (\quad + \ )\cosh\ (\quad) \quad - \ \cosh\ (\quad) \ = 0.$

Particular solution: $_0 = a\ + \ .$

26. $^{(\ )} + \quad \cosh^k \quad ^{(\ )} - [\ (\ + \quad)\cosh^k \quad + \quad + \ ] \ = 0.$

Particular solution: $_0 = \ e\ .$

27. $^{(2\ \ )} = \ + \ (\quad \cosh \ - \ \sinh \ ).$

The substitution $= {}' \cosh \ - \ \sinh$ leads to an $(2\ \ -1)$st-order linear equation.

28. $^{(2\ \ )} = \ + \ (\quad \sinh \ - \ \cosh \ ).$

The substitution $= {}' \sinh \ - \ \cosh$ leads to an $(2\ \ -1)$st-order linear equation.

29. $^{(\ )} + \ \tanh^k \quad ^{(\ )} - (\quad \tanh^k \quad + \quad) \ = 0.$

Particular solution: $_0 = e^b\ .$

30. $^{(\ )} + (\ \tanh^k \quad - \quad ^{-}\ )\ ^{(\ )} - \quad \tanh^k \quad = 0.$

Particular solution: $_0 = e^b\ .$

31. $^{(\ )} + (\quad + \ )\tanh\ (\quad) \quad - \ \tanh\ (\quad) \ = 0.$

Particular solution: $_0 = a\ + \ .$

32. $^{(\ )} + \quad \tanh^k \quad ^{(\ )} - [\ (\ + \quad)\tanh^k \quad + \quad + \ ] \ = 0.$

Particular solution: $_0 = \ e\ .$

33. $^{(\ )} + \ \coth^k \quad ^{(\ )} - (\quad \coth^k \quad + \quad) \ = 0.$

Particular solution: $_0 = e^b\ .$

34. $^{(\ )} + (\ \coth^k \quad - \quad ^{-}\ )\ ^{(\ )} - \quad \coth^k \quad = 0.$

Particular solution: $_0 = e^b\ .$

35. $^{(\ )} + (\quad + \ )\coth\ (\quad) \quad - \ \coth\ (\quad) \ = 0.$

Particular solution: $_0 = a\ + \ .$

36. $^{(\ )} + \quad \coth^k \quad ^{(\ )} - [\ (\ + \quad)\coth^k \quad + \quad + \ ] \ = 0.$

Particular solution: $_0 = \ e\ .$

5.1.4. Equations Containing Logarithmic Functions

1. $^{(2\ )} = \quad + \ \ln\ (\quad - \quad).$

This is a special case of equation 5.1.6.18 with $(\ ) = \ \ln\ $.

2. $^{(\ )} + \ \ln^k \quad ^{(\ )} - (\quad \ln^k \quad + \quad) \quad = 0.$

Particular solution: $_0 = e^b$.

3. $^{(\ )} + (\ \ln^k \quad - \quad ^- \ )\ ^{(\ )} - \quad \ln^k \quad = 0.$

Particular solution: $_0 = e^b$.

4. $^{(\ )} + \quad ^{(\ -1)} + \ \ln^k(\quad) \quad + \quad \ln^k(\quad) \quad = 0.$

Particular solution: $_0 = e^-$.

5. $^{(\ )} + (\quad + \quad) \ln^k(\quad) \quad - \quad \ln^k(\quad) \quad = 0.$

Particular solution: $_0 = a \quad + \ $.

6. $^{(\ )} + (\quad + \quad) \ln^k(\quad) \quad - 2 \quad \ln^k(\quad) \quad = 0.$

Particular solution: $_0 = (a \quad + \quad)^2$.

7. $^{(\ )} + (\quad + \quad) \ln^k(\quad) \quad - 3 \quad \ln^k(\quad) \quad = 0.$

Particular solution: $_0 = (a \quad + \quad)^3$.

8. $^{(\ )} + (\quad + \quad) \ln^k(\quad) \quad - \quad (\quad - 1) \ln^k(\quad) \quad = 0.$

Particular solution: $_0 = (a \quad + \quad)^{\ -1}$.

9. $^{(\ )} + \quad \ln^k(\quad) \quad - \quad \ln^k(\quad) \quad = 0, \qquad = 1, 2, \ldots, \quad - 1.$

Particular solution: $_0 = \quad$.

10. $^{(\ )} + \quad \ln^k(\quad)\ ^{(\ )} - [\ (\ + \quad) \ln^k(\quad) + \quad + \quad] \quad = 0.$

Particular solution: $_0 = \quad e\ $.

5.1.5. Equations Containing Trigonometric Functions

5.1.5-1. Equations with sine and cosine.

1. $^{(\ )} + \ \sin^k \quad ^{(\ )} - (\quad \sin^k \quad + \quad) \quad = 0.$

Particular solution: $_0 = e^b$.

2. $^{(\ )} + (\ \sin^k \quad - \quad ^- \ )\ ^{(\ )} - \quad \sin^k \quad = 0.$

Particular solution: $_0 = e^b$.

3. $^{(\ )} + \quad ^{(\ -1)} + \ \sin\ (\quad) \quad + \quad \sin\ (\quad) \quad = 0.$

Particular solution: $_0 = e^-$.

4. $^{(\ )} + (\quad + \quad) \sin\ (\quad) \quad - \quad \sin\ (\quad) \quad = 0.$

Particular solution: $_0 = a \quad + \ $.

5. $^{(\ )} + (\quad + \quad) \sin\ (\quad) \quad - 2 \quad \sin\ (\quad) \quad = 0.$

Particular solution: $_0 = (a \quad + \quad)^2$.

6. $^{(\ )} + (\quad + \quad) \sin \ (\quad) \quad - 3 \quad \sin \ (\quad) \quad = 0.$

Particular solution: $_0 = (a \quad + \quad)^3.$

7. $^{(2\ \ )} = \qquad + \quad \sin^k(\quad)(\quad - \quad).$

This is a special case of equation 5.1.6.18 with $(\ ) = \ \sin\ (\lambda\ \ ).$

8. $^{(\ )} + \quad \sin^k(\quad)\ ^{(\ )} - [\ (\quad + \quad) \sin^k(\quad) + \quad + \quad] \quad = 0.$

Particular solution: $_0 = \quad e\ .$

9. $(\qquad + \quad \sin \quad)\ ^{(\ )} = \quad \sin\Big(\quad + \tfrac{1}{2}\qquad\qquad, \qquad\qquad = 0,\ 1,\ \ldots,\quad - 1.$

Particular solution: $_0 = a \qquad + \quad \sin\ .$

10. $\sin \quad + \sum_{k=0}^{-1} \ _k \ ^k \ ^{(\ )} = \quad \sin\Big(\quad + \tfrac{1}{2}\qquad\qquad.$

Particular solution: $_0 = a \sin \quad + \sum_{=0}^{-1} \qquad .$

11. $^{(\ )} + \quad \cos^k \qquad ^{(\ )} - (\qquad \cos^k \quad + \quad) \quad = 0.$

Particular solution: $_0 = e^b\ .$

12. $^{(\ )} + (\ \cos^k \quad - \quad ^{-}\ )\ ^{(\ )} - \qquad \cos^k \qquad = 0.$

Particular solution: $_0 = e^b\ .$

13. $^{(\ )} + \qquad ^{(\ -1)} + \quad \cos\ (\quad) \quad + \quad \cos\ (\quad) \quad = 0.$

Particular solution: $_0 = e^{-}\ .$

14. $^{(\ )} + (\quad + \quad) \cos\ (\quad) \quad - \quad \cos\ (\quad) \quad = 0.$

Particular solution: $_0 = a \quad + \ .$

15. $^{(\ )} + (\quad + \quad) \cos\ (\quad) \quad - 2 \quad \cos\ (\quad) \quad = 0.$

Particular solution: $_0 = (a \quad + \quad)^2.$

16. $^{(\ )} + (\quad + \quad) \cos\ (\quad) \quad - 3 \quad \cos\ (\quad) \quad = 0.$

Particular solution: $_0 = (a \quad + \quad)^3.$

17. $^{(2\ \ )} = \qquad + \quad \cos^k(\quad)(\quad - \quad).$

This is a special case of equation 5.1.6.18 with $(\ ) = \ \cos\ (\lambda\ \ ).$

18. $^{(\ )} + \quad \cos^k(\quad)\ ^{(\ )} - [\ (\quad + \quad) \cos^k(\quad) + \quad + \quad] \quad = 0.$

Particular solution: $_0 = \quad e\ .$

19. $(\qquad + \quad \cos \quad)\ ^{(\ )} = \quad \cos\Big(\quad + \tfrac{1}{2}\qquad\qquad, \qquad\qquad = 0,\ 1,\ \ldots,\quad - 1.$

Particular solution: $_0 = a \qquad + \quad \cos\ .$

20. $\cos \quad + \sum_{k=0}^{-1} \ _k \ ^k \ ^{(\ )} = \quad \cos\Big(\quad + \tfrac{1}{2}\qquad\qquad.$

Particular solution: $_0 = a \sin \quad + \sum_{=0}^{-1} \qquad .$

21. $\quad {}^{(2\ \ )} = (-1) \quad + \ (\quad \sin \quad - \quad \cos \quad).$

The substitution $\quad = \ {}' \sin \quad - \quad \cos \quad$ leads to an $(2 \quad - 1)$st-order linear equation.

22. $\quad {}^{(2\ \ )} = (-1) \quad + \ (\quad \cos \quad + \quad \sin \quad).$

The substitution $\quad = \ {}' \cos \quad + \quad \sin \quad$ leads to an $(2 \quad - 1)$st-order linear equation.

5.1.5-2. Equations with tangent and cotangent.

23. $\quad {}^{(\ )} + \quad \tan^k \quad {}^{(\ )} - (\quad \tan^k \quad + \quad) \quad = 0.$

Particular solution: $\quad {}_0 = e^{b} \quad .$

24. $\quad {}^{(\ )} + (\ \tan^k \quad - \quad {}^{-} \ ) \ {}^{(\ )} - \quad \tan^k \quad = 0.$

Particular solution: $\quad {}_0 = e^{b} \quad .$

25. $\quad {}^{(\ )} + \quad {}^{(\ -1)} + \quad \tan \ (\quad) \quad + \quad \tan \ (\quad) \quad = 0.$

Particular solution: $\quad {}_0 = e^{-} \quad .$

26. $\quad {}^{(\ )} + (\quad + \quad) \tan \ (\quad) \quad - \quad \tan \ (\quad) \quad = 0.$

Particular solution: $\quad {}_0 = a \quad + \quad .$

27. $\quad {}^{(\ )} + (\quad + \quad) \tan \ (\quad) \quad - 2 \quad \tan \ (\quad) \quad = 0.$

Particular solution: $\quad {}_0 = (a \quad + \quad)^2.$

28. $\quad {}^{(\ )} + (\quad + \quad) \tan \ (\quad) \quad - 3 \quad \tan \ (\quad) \quad = 0.$

Particular solution: $\quad {}_0 = (a \quad + \quad)^3.$

29. $\quad {}^{(\ )} + \quad \tan^k (\quad) \ {}^{(\ )} - [\ (\quad + \quad) \tan^k (\quad) + \quad + \quad] \quad = 0.$

Particular solution: $\quad {}_0 = \quad e \quad .$

30. $\quad {}^{(\ )} + \quad \cot^k \quad {}^{(\ )} - (\quad \cot^k \quad + \quad) \quad = 0.$

Particular solution: $\quad {}_0 = e^{b} \quad .$

31. $\quad {}^{(\ )} + (\ \cot^k \quad - \quad {}^{-} \ ) \ {}^{(\ )} - \quad \cot^k \quad = 0.$

Particular solution: $\quad {}_0 = e^{b} \quad .$

32. $\quad {}^{(\ )} + \quad {}^{(\ -1)} + \quad \cot \ (\quad) \quad + \quad \cot \ (\quad) \quad = 0.$

Particular solution: $\quad {}_0 = e^{-} \quad .$

33. $\quad {}^{(\ )} + (\quad + \quad) \cot \ (\quad) \quad - \quad \cot \ (\quad) \quad = 0.$

Particular solution: $\quad {}_0 = a \quad + \quad .$

34. $\quad {}^{(\ )} + (\quad + \quad) \cot \ (\quad) \quad - 2 \quad \cot \ (\quad) \quad = 0.$

Particular solution: $\quad {}_0 = (a \quad + \quad)^2.$

35. $\quad {}^{(\ )} + (\quad + \quad) \cot \ (\quad) \quad - 3 \quad \cot \ (\quad) \quad = 0.$

Particular solution: $\quad {}_0 = (a \quad + \quad)^3.$

36. $\quad {}^{(\ )} + \quad \cot^k (\quad) \ {}^{(\ )} - [\ (\quad + \quad) \cot^k (\quad) + \quad + \quad] \quad = 0.$

Particular solution: $\quad {}_0 = \quad e \quad .$

5.1.6. Equations Containing Arbitrary Functions

5.1.6-1. Equations of the form $\;(\;)^{(\;)}+\;_1(\;)\;'+\;_0(\;)\;=g(\;)$.

1. $^{(\;)}=f(\;)$**.**

Solution: $=\sum_{=0}^{-1}\;+\int_0\frac{(\;-\;)^{-1}}{(\;-1)!}(\;)\;$, where $_0$ is an arbitrary number.

2. $^{(\;)}=f(\;)\;$**.**

The transformation $=\;^{-1}$, $=\;^{1-}$ leads to an equation of similar form: $^{(\;)}=(-1)\;^{-2}\;(1\;)\;$.

3. $^{(\;)}=(\;+\;)^{-2}\,f\left(\frac{\;+\;}{\;+\;}\right)$**.**

The transformation $\xi=\frac{a\;+\;}{\;+\;}$, $=\frac{}{(\;+\;)^{-1}}$ leads to a simpler equation: $^{(\;)}=\Delta^{-}\;(\xi)\;$, where $\Delta=a\;-\;$.

4. $f\;^{(\;)}-f^{(\;)}\;=0,\quad f=f(\;)$**.**

Particular solution: $_0=\;(\;)$.

5. $f\;^{(2\;+1)}+f^{(2\;+1)}\;=\;(\;),\quad f=f(\;)$**.**

First integral: $\sum_{=0}^{2}(-1)\;^{(2\;-\;)\;(\;)}=\int g(\;)\;+\;$.

6. $^{(\;)}+(\;+\;)f(\;)\;-\;f(\;)\;=0$**.**

Particular solution: $_0=a\;+\;$.

7. $^{(\;)}+(\;+\;)f(\;)\;-2\;f(\;)\;=0$**.**

Particular solution: $_0=(a\;+\;)^2$.

8. $^{(\;)}+(\;+\;)f(\;)\;-3\;f(\;)\;=0$**.**

Particular solution: $_0=(a\;+\;)^3$.

9. $^{(\;)}+(\;+\;)f(\;)\;-(\;-1)\;f(\;)\;=0$**.**

Particular solution: $_0=(a\;+\;)^{-1}$. The substitution $z=(a\;+\;)\;'-a(\;-1)\;$ leads to an $(\;-1)$st-order linear equation: $z^{(\;-1)}+(a\;+\;)\;(\;)z=0$.

10. $^{(\;)}+\;f(\;)\;-\;f(\;)\;=0,\qquad =1,\,2,\,\ldots,\;-1$**.**

Particular solution: $_0=\;$. The substitution $z=\;'-\;$ leads to an $(\;-1)$st-order equation:

$$-\;^{-1}\frac{z^{(\;)}}{}+\;(\;)z=0,\quad\text{where}\quad =\frac{}{}.$$

11. $^{(\;)}+f(\;)\;+\;(\;)\;+\;(\;)=0$**.**

The transformation $=\;^{-1}$, $=\;^{1-}$ leads to an equation of similar form:

$$^{(\;)}+(-1)\;^{-2}\;-\;^{2}\;(1\;)\;'+\big[(\;-1)\;(1\;)+g(1\;)\big]\;+\;^{-1}\;(1\;)\;=0.$$

12. $^{(\;)}+f(\;)\;+f\;(\;)\;=\;(\;)$**.**

Integrating yields an $(\;-1)$st-order linear equation: $^{(\;-1)}+\;(\;)\;=\int g(\;)\;+\;$.

13. $^{(2\ \ )} =\ \ + f(\ \ )(\ \ \cosh\ \ -\ \ \sinh\ \ ).$

The substitution $\ = {}'\cosh\ \ -\ \ \sinh\ \ $ leads to an (2 − 1)st-order linear equation.

14. $^{(2\ \ )} =\ \ + f(\ \ )(\ \ \sinh\ \ -\ \ \cosh\ \ ).$

The substitution $\ = {}'\sinh\ \ -\ \ \cosh\ \ $ leads to an (2 − 1)st-order linear equation.

15. $^{(2\ \ )} = (-1)\ \ + f(\ \ )(\ \ \sin\ \ -\ \ \cos\ \ ).$

The substitution $\ = {}'\sin\ \ -\ \ \cos\ \ $ leads to an (2 − 1)st-order linear equation.

16. $^{(2\ \ )} = (-1)\ \ + f(\ \ )(\ \ \cos\ \ +\ \ \sin\ \ ).$

The substitution $\ = {}'\cos\ \ +\ \ \sin\ \ $ leads to an (2 − 1)st-order linear equation.

17. $^{(\ \ )} = \frac{(\ \ )}{\ \ } + f(\ \ )\ \ - \frac{\ }{\ }\ \ ,\qquad = (\ \ ).$

The substitution $\ = {}' - \frac{'}{\ }$ leads to an (− 1)st-order linear equation.

5.1.6-2. Other equations.

18. $^{(2\ \ )} =\ \ + f(\ \ )(\ \ -\ \ ).$

The substitution $\ = {}'' - a\ \ $ leads to an (2 − 2)nd-order linear equation:

$$^{(2\ \ -2)} + a\ \ ^{(2\ \ -4)} +\ \ + a\ \ ^{-1}\ \ = (\ \ )\ \ .$$

19. $^{(\ \ )} + f(\ \ )(\ ^{2}\ \ -2\ \ +2\ \ ) = 0.$

Particular solutions: $\ _1 =\ \ ,\ \ _2 =\ ^{2}$. The substitution $z =\ ^{2}\ ''-2\ \ '+2\ \ $ leads to a linear equation of the (− 2)nd-order.

20. $^{(\ \ +2)} + f(\ \ )[\ ^{2}\ \ -2\ \ +\ (\ \ +1)\ \ ] = 0.$

The substitution $\ (\ \ ) =\ ^{2}\ ''-2\ \ '+\ (\ \ +1)\ \ $ leads to an th-order linear equation: $\ ^{(\ \ )} +\ ^{2}\ (\ \ )\ \ = 0.$

21. $^{(2\ \ )} =\ ^{2}\ + f(\ \ )[\ ^{(\ \ )} +\ \ ].$

The substitution $\ =\ ^{(\ \ )} + a\ \ $ leads to an th-order linear equation: $\ ^{(\ \ )} = [\ (\ \ ) + a]\ \ .$

22. $^{(\ \ )} + f(\ \ )\ ^{(\ \ )} - [\ \ +\ \ f(\ \ )]\ \ = 0.$

Particular solution: $\ _0 = e\ \ .$

23. $^{(\ \ )} + (f -\ \ ^{-}\ )\ ^{(\ \ )} -\ \ f\ \ = 0,\qquad f = f(\ \ ).$

Particular solution: $\ _0 = e\ \ .$

24. $^{(\ \ )} +\ \ ^{(\ \ -1)} + f\ \ +\ f\ \ = 0,\qquad f = f(\ \ ).$

Particular solution: $\ _0 = e^{-}\ \ .$

25. $^{(\ \ )} + f(\ \ )\ ^{(\ \ -1)} +\ (\ \ )\ ^{(\ \ -2)} +\ (\ \ ) = 0.$

The substitution $\ (\ \ ) =\ ^{(\ \ -2)}$ leads to a second-order linear equation: $\ '' +\ (\ \ )\ ' + g(\ \ )\ \ +\ (\ \ ) = 0.$

26. $\mathbf{{}^{(\ )} + {}_{-1}\,{}^{(\ -1)} + \cdots + {}_{1}\ \ + {}_{0}\ \ = f(\ ).}$

Constant coefficient nonhomogeneous linear equation. The general solution of this equation is the sum of the general solution of the corresponding homogeneous equation (see 5.1.2.25) and any particular solution of the nonhomogeneous equation.

If the roots $\lambda_1, \lambda_2, \quad , \lambda$ of the characteristic equation

$$\lambda \ \ + a_{\ -1}\lambda^{\ -1} + \quad + a_1\lambda + a_0 = 0$$

are all different, the original equation has the general solution:

$$= \sum_{=1} \quad e^{\lambda} \quad + \sum_{=1} \frac{e^{\lambda}}{{}_{\lambda}'(\lambda\ )} \int \quad (\ )e^{-\lambda}$$

(with complex roots, the real part should be taken).

Paragraph 0.4.1-2 lists the forms of particular solutions corresponding to some special forms of the right-hand side function of the nonhomogeneous linear equation.

27. $\mathbf{{}^{(\ )} + f(\ )\sum_{k=0}^{-1}(-1)^k\ !\ \ {}^{k}_{-1}\ \ {}^{-k-1}\ {}^{(\ -k-1)} = 0.}$

Here, $\quad = \dfrac{!}{k!\,(\ \ - k)!}$ are binomial coefficients.

Particular solutions: $\quad = \quad$, where $\quad = 1, 2, \quad , \quad - 1$.

The substitution $z = \sum_{=0}^{-1}(-1)\ \ k!\ \ \ {}_{-1}\ \ {}^{-\ -1}\ {}^{(\ -\ -1)}$ leads to a first-order linear equation: $z' + \quad {}^{-1}\ (\ )z = 0$. Having solved this equation, we obtain an $(\ \ - 1)$st-order linear equation of the form 5.1.6.34 for $\ (\ )$.

28. $\mathbf{{}^{(\ )} = \sum_{k=0}^{-1}(\ {}_{k+1}f - \ {}_{k})\ {}^{(k)}.}$

Here, $\quad = \ (\ );\ a \ \ = 1, a_0 = 0;\ a \ \ $ are arbitrary numbers $(k = 1, 2, \quad , \quad - 1)$.

Particular solutions: $\quad = e^{\lambda} \quad (k = 1, 2, \quad , \quad - 1)$, where the λ are roots of the polynomial equation $\sum_{=0}^{-1} a_{\ +1}\lambda \ \ = 0$.

29. $\mathbf{{}^{(\ )} + (\ \ + \ \ - 1)\ {}^{(\ -1)} = f(\ )(\quad + \quad).}$

The substitution $\quad = \ ' + a \ $ leads to an $(\ \ - 1)$st-order linear equation: ${}^{(\ -1)} = \ (\ )\ .$

30. $\mathbf{{}^{(\ )} + \ f\ {}^{(\ )} - [(\ + \ )f + \ \ + \ \ ]\ = 0, \qquad f = f(\ ).}$

Particular solution: ${}_{0} = \ e\ .$

31. $\mathbf{{}^{(\ )} + \ \ {}^{(\ -1)} = \ {}^{1-2}\ f(1\ \ ) + \ {}^{-\ -1}\ (1\ \ ).}$

The transformation $\quad = {}^{-1}, \quad = \ {}^{2-}$ leads to an th-order linear equation: ${}^{(\ )} = (-1)\ \ [\ \ (\ )\ \ + g(\ )]$.

32. $\mathbf{{}^{2}\ {}^{(\ +2)} + \ \ {}^{(\ +1)} + \ {}^{(\ )} + f(\ )[\ {}^{2}\ \ + (\ -2\ )\ \ + (\ - \ \ + \ {}^{2} + \ )\ ] = 0.}$

The substitution $\ (\ ) = \ {}^{2}\ {}'' + (\ -2\ )\ \ {}' + (\beta - \ \ + \ {}^{2} + \ )$ leads to an th-order linear equation: ${}^{(\ )} + \ (\ )\ \ = 0.$

33. $\mathbf{(\ + \ )\ {}^{(\ )} + \ (f - \ - \ )\ {}^{(\ )} - (\ + \ )f\ \ = 0, \qquad f = f(\ ).}$

Particular solution: ${}_{0} = \ e\ .$

34. $\;^{(\;)} + \;_{-1}\;^{-1}\;^{(\;-1)} + \cdots + \;_{1}\; + \;_{0}\; = f(\;).$

Nonhomogeneous Euler equation. The substitution $\; = ae\;$ $(a \neq 0)$ leads to a constant coefficient nonhomogeneous linear equation of the form 5.1.6.26.

35. $\;^{(\;)} + (\;-\;-1)\;^{-1}\;^{(\;-1)} + \;f\;-\;f\; = 0, \qquad f = f(\;).$

Particular solution: $\;_0 = \;.$

36. $\;^{(\;)} + \;f\;^{(\;)} - (\;!\;+\;!\;f)\; = 0.$

Here, $\; = (\;)$, $\; = \dfrac{\Gamma(a+1)}{!\,\Gamma(a-\;+1)}$ are binomial coefficients, and $\Gamma(a)$ is the gamma function.

Particular solution: $\;_0 = \;.$

37. $\;^{(\;)} = \sum_{k=0}^{-1}[\;(\;_{k+1}f - \;_k) + \;_{k+1}]\;^{(k)}.$

Here, $\; = (\;)$; $a\; = 1$, $a_0 = 0$; $\;$ and $a\;$ are arbitrary numbers $(k = 1, 2, \;, \;-1)$.

Particular solutions: $\; = e^{\lambda}\;$ $(k = 1, 2, \;, \;-1)$, where the $\lambda\;$ are roots of the polynomial equation $\sum_{=0}^{-1} a\;_{+1}\lambda\; = 0.$

38. $\sin\;^{(\;)} + \sin\;f(\;)\;^{(\;)} - \left[\sin\left(\;+\tfrac{1}{2}\;\right) + f(\;)\sin\left(\;+\tfrac{1}{2}\;\right)\right] = 0.$

Particular solution: $\;_0 = \sin\;.$

39. $\cos\;^{(\;)} + \cos\;f(\;)\;^{(\;)} - \left[\cos\left(\;+\tfrac{1}{2}\;\right) + f(\;)\cos\left(\;+\tfrac{1}{2}\;\right)\right] = 0.$

Particular solution: $\;_0 = \cos\;.$

40. $\sum_{k=2} f_k(\;)\;^{(k)} = (\;)(\;-\;).$

Particular solution: $\;_0 = \;$. The substitution $\;(\;) = \;'-\;$ leads to an $(\;-1)$st-order linear equation.

41. $\sum_{k=\;+1} f_k(\;)\;^{(k)} = (\;)(\;-\;), \qquad = 1, 2, \ldots, \;-1.$

Particular solution: $\;_0 = \;$. The substitution $\;(\;) = \;'-\;$ leads to an $(\;-1)$st-order linear equation.

42. $\sum_{k=3} f_k(\;)\;^{(k)} = (\;)(\;^2\;-2\;+2\;).$

Particular solutions: $\;_1 = \;$, $\;_2 = \;^2$. The substitution $\;(\;) = \;^2\;''-2\;'+2\;$ leads to an $(\;-2)$nd-order linear equation.

43. $\sum_{k=4} f_k(\;)\;^{(k)} = (\;)(\;^3\;-3\;^2\;+6\;-6\;).$

Particular solutions: $\;_1 = \;$, $\;_2 = \;^2$, $\;_3 = \;^3$. The substitution $\;(\;) = \;^3\;'''-3\;^2\;''+6\;'-6\;$ leads to an $(\;-3)$rd-order linear equation.

44. $\sum_{k=\;+1} f_k(\;)\;^{(k)} + (\;)\sum_{k=0}(-1)^k\;!\;^{k}\;^{-k}\;^{(\;-k)} = 0.$

Here, $\; = \dfrac{!}{k!\,(\;-k)!}$ are binomial coefficients.

Particular solutions: $\; = \;$, where $\; = 1, 2, \;, \;.$

The substitution $z = \quad (-1) \; k! \quad\quad - \; (\; - \;)$ leads to an $(\quad - \quad)$th-order linear equation:
$\quad =0$

$$\quad (\;) \quad - \; -1 \big(\; - \; z' \quad + g(\;)z = 0, \text{ where } \quad = \quad .$$
$= \; +1$

45. $\quad (f_k - \quad f_{k+1}) \quad ^{(k)} = 0.$
$k=0$

Here, $\quad = \quad (\;) \; (k = 1, 2, \quad , \;); \quad +1 \equiv \quad 0 \equiv 0.$

Particular solution: $\quad 0 = e \quad .$

46. $\quad ^k[f_k + (\quad - \quad)f_{k+1}] \quad ^{(k)} = 0.$
$k=0$

Here, $\quad = \quad (\;) \; (k = 1, 2, \quad , \;); \quad +1 \equiv \quad 0 \equiv 0.$

Particular solution: $\quad 0 = \quad .$

5.2. Nonlinear Equations

5.2.1. Equations Containing Power Functions

5.2.1-1. Fifth- and sixth-order equations.

1. $\quad ^{(5)} = \quad\quad - \; (\quad)^2 + \quad + \; .$

This is a special case of equation 5.2.6.1 with $\; (\;) = a \quad + \; .$

2. $\quad ^{(5)} = \quad + \; .$

This is a special case of equation 5.2.6.17 with $\quad = 2$ and $\; (\;) = a \quad + \; .$

3. $\quad ^{(5)} = \quad\quad .$

1 . For $a \neq -1$, integrating the equation two times, we arrive at a third-order autonomous equation: $\; ' \; ''' \quad - \frac{1}{2}(\; '' \;)^2 = \quad 1 \quad ^{+1} + \quad 2.$ The substitution $\; (\;) = \frac{1}{2}(\; ' \;)^2$ leads to a second-order equation:

$$'' \; - \frac{1}{4}(\; ' \;)^2 = \frac{1}{2} \quad 1 \quad ^{+1} + \frac{1}{2} \quad 2.$$

For $a = 1$, this is a solvable equation of the form 2.8.1.53.

2 . For $a = -1$, integrating the equation two times, we arrive at a third-order autonomous equation: $\; ' \; ''' \quad - \frac{1}{2}(\; '' \;)^2 = \quad 1 \ln| \quad | + \quad 2.$

3 . Particular solution: $\quad = \quad 1 \quad ^3 + \quad 2 \quad ^2 + \quad 3 \quad + \quad 4.$

4. $\quad 3 \quad ^{(5)} + 5 \quad\quad\quad = 0.$

This is a special case of equation 5.2.1.3 with $a = -\frac{5}{3}$. Integrating the equation three times, we arrive at a second-order equation: $3 \quad '' \; - 2(\; ' \;)^2 = \quad 1 \quad ^2 + \quad 2 \quad + \quad 3.$ The substitution $\quad = \quad ^3$ leads to a solvable equation of the form 2.8.1.5: $\quad '' \; = \frac{1}{9}(\quad 1 \quad ^2 + \quad 2 \quad + \quad 3) \quad ^{-5}.$

5. $\quad 2 \quad ^{(5)} + 5 \quad\quad\quad + 5 \quad\quad\quad = 0.$

This is a special case of equation 5.2.6.4 with $a = \frac{5}{2}$ and $\; (\;) = 0$. Integrating the equation three times, we arrive at a second-order equation of the form 2.8.1.53: $\quad '' \; - \frac{1}{4}(\; ' \;)^2 = \quad 1 \quad ^2 + \quad 2 \quad + \quad 3.$

6. $\quad {}^{(5)} + \qquad + (3 \quad - 5) \qquad = 0.$

Integrating the equation three times, we obtain a second-order equation: $\quad '' + \frac{1}{2}(a-3)(\ ')^2 = {}_1\ {}^2 + \ {}_2 \ + \ {}_3.$

7. $\quad {}^{(5)} + \qquad + \qquad = 0.$

Integrating yields a fourth-order equation: $\quad '''' \quad + (a-1)\ ' \ ''' \quad + \frac{1}{2}(1 - a + \ )(\ '' \ )^2 = \quad .$

8. $\quad {}^{(5)} + 5 \qquad + 10 \qquad = \qquad .$

This is a special case of equation 5.2.6.2 with $\ (\ ) = a \quad .$

9. $\quad {}^{(5)} + 5 \qquad + 10 \qquad = \qquad .$

This is a special case of equation 5.2.6.3 with $\ (\ ) = a \quad .$

10. $\quad {}^{(6)} = \qquad {}^{-7}\ {}^{5}.$

Multiplying by $\ {}^{7}\ {}^{5}$ and differentiating with respect to $\ $, we obtain $5 \quad {}^{(7)} + 7\ ' \quad {}^{(6)} = 0.$ Having integrated this equation three times, we arrive at the chain of equations:

$$5 \quad {}^{(6)} + 2\ ' \ {}^{(5)} - 2\ '' \quad '''' \quad + (\ ''' \ )^2 = 2 \ {}_2, \tag{1}$$

$$5 \quad {}^{(5)} - 3\ ' \quad '''' \quad + \ '' \quad ''' \quad = 2 \ {}_2 \ + \ {}_1, \tag{2}$$

$$5 \quad '''' \quad - 8\ ' \quad ''' \quad + \tfrac{9}{2}(\ '' \ )^2 = \ {}_2\ {}^2 + \ {}_1 \ + \ {}_0, \tag{3}$$

where ${}_0$, ${}_1$, and ${}_2$ are arbitrary constants. Eliminating the highest derivatives from (1)–(3), with the aid of the original equation, we obtain a third-order equation that can be reduced to a second-order equation (see equation 5.2.1.12 with $\ = 3$).

11. $\quad {}^{(6)} + 6 \qquad {}^{(5)} + 15 \qquad\qquad + 10(\qquad)^2 = \qquad .$

This is a special case of equation 5.2.6.6 with $\ (\ ) = a \quad .$

5.2.1-2. Equations of the form $\ {}^{(\ )} = \ (\ ,\ ).$

12. $\quad {}^{(2\ \ )} = \qquad \frac{1+2}{1-2}.$

Multiply both sides by $\ \frac{2\ \ +1}{2\ \ -1}$ and differentiate with respect to $\ $. As a result, we obtain

$$(2 \quad - 1) \quad {}^{(2\ \ +1)} + (2 \quad + 1)\ ' \quad {}^{(2\ \ )} = 0.$$

Three integrals containing arbitrary constants ${}_0$, ${}_1$, and ${}_2$ are presented in 5.2.6.62, where one should let $\ \equiv 0$. Eliminating the highest derivatives from those integrals and the original equation, one can always obtain an $(2 \quad - 3)$rd-order equation. With the aid of the transformation

$$= \ \text{—}, \qquad = \ \frac{1-2}{2}, \qquad \text{where} \qquad = \ {}_2\ {}^2 + \ {}_1 \ + \ {}_0,$$

this equation can be reduced to the autonomous form 5.2.6.77. Therefore, the substitution $z(\ ) = \ '$ finally leads to an $(2 \quad - 4)$th-order equation with respect to $z = z(\ )$.

13. $\quad {}^{(2\ \ )} = \qquad {}^{k} + \ , \qquad \neq -1.$

This is a special case of equation 5.2.6.8. Integrating yields an $(2 \quad - 1)$st-order equation:

$$\sum_{=1}^{-1} (-1) \quad {}^{(\ )} \ {}^{(2\ -\ \ )} + \frac{1}{2}(-1) \ \left[\ {}^{(\ )}\right]^2 = -\frac{a}{k+1} \quad {}^{+1} - \quad + \quad ,$$

where $\ $ is an arbitrary constant. Furthermore, the order of the obtained autonomous equation can be reduced by one by the substitution $\ (\ ) = \ '$.

14. $\;^{(\ )} = \quad ^{-} \quad .$

This is a special case of equation 5.2.6.11 with $(\ ) = a \quad .$

15. $\;^{(\ )} = \quad ^{k} \quad .$

1 . The transformation $\quad = \quad ^{-1}, \quad = \quad ^{1-} \quad (\ )$ leads to an equation of the same form: $^{(\ )} = (-1) \quad A \quad ^{- \ -(\ -1) \ - \ -1} \quad .$

2 . The transformation $\xi = \quad ^{+} \quad ^{-1}$, $z = \quad ' \quad$ leads to an $(\quad - 1)$st-order equation.

16. $\;^{(2 \ \ +1)} = \quad ^{k} + \ .$

This is a special case of equation 5.2.6.17 with $(\ ) = a \quad + \ .$

17. $\;^{(\ )} = \quad ^{-} \quad ^{- \ -1}(\quad + \quad ^{-1}) \ .$

This is a special case of equation 5.2.6.13 with $(\ ) = (a \quad + \ ) \ .$

18. $\;^{(2 \ )} = \dfrac{^{-2} \quad ^{-2 \ -1}}{2} \quad + \dfrac{^{2 \ -1}}{2} \quad .$

This is a special case of equation 5.2.6.14 with $(\ ) = (a \quad + \ ) \ .$

19. $\;^{(\ )} = (\quad + \quad ^{k}) \ ; \qquad = 1, 2, \ldots, \quad - 1.$

The substitution $a \quad = a \ +$ leads to an autonomous equation of the form 5.2.6.8: $^{(\ )} = a$ (see also 5.2.1.12 and 5.2.1.13 with $\ = 0$).

20. $\;^{(\ )} = (\quad ^{2} + \quad + \ )^{\frac{- \quad - \ -1}{2}} \quad .$

This is a special case of equation 5.2.6.22 with $(\ ) = \quad .$

21. $\;^{(\ )} = (\quad + \ )^{-} \ (\quad + \ ) \ ^{-} \quad ^{-1} \quad .$

This is a special case of equation 5.2.6.21 with $(\ ) = \quad .$

5.2.1-3. Equations of the form $^{(\ )} = \ (\ , \ , \ ', \ '' \ ).$

22. $\;^{(\ )} = \quad ^{k} \quad + \quad .$

This is a special case of equation 5.2.6.34 with $(\ ) = a$ and $g(\ ) = \quad .$ Integrating yields an $(\quad - 1)$st-order equation: $^{(\ -1)} = \dfrac{a}{k+1} \quad ^{+1} + \dfrac{}{\ +1} \quad ^{+1} + \quad .$

23. $\;^{(\ )} = \quad + \ (\quad - \quad)^{k}.$

This is a special case of equation 5.2.6.38 with $(\ , \ ) = \quad .$ The substitution $\quad = \ ' - a$ leads to an $(\quad - 1)$st-order autonomous equation: $^{(\ -1)} + a \ ^{(\ -2)} + \quad + a \ ^{-1} \quad = \quad .$

24. $\;^{(\ )} = \quad ^{-} \quad ^{1-} \ (\ ) \ .$

This is a special case of equation 5.2.6.37 with $(\ ) = a \quad .$

25. $\;^{(\ )} = \quad (\quad - \ )^{k}.$

This is a special case of equation 5.2.6.39 with $(\ , \ ) = a \quad .$ The substitution $\quad = \ ' -$ leads to an $(\quad - 1)$st-order equation.

26. $\;^{(\ )} = \quad ^{k}(\quad - \quad) \ .$

Here, is a positive integer and $\ \geq \quad + 1$. The substitution $\quad = \ ' - \quad$ leads to an $(\quad - 1)$st-order equation: $^{(\ - \ -1)} = a \quad$, where $\quad = \ ^{(\ )} \quad .$

27. $^{(2\ )} + \quad + \quad = \quad - \ (\quad)^2 + \quad .$

1 . Particular solution:

$$= \ {}_1 \sinh(\ {}_4 \ ) + \ {}_2 \cosh(\ {}_4 \ ) + \ {}_3,$$

where the constants ${}_1$, ${}_2$, ${}_3$, and ${}_4$ are related by two constraints

$$\begin{gathered} {}^2_4 \ + (a - \ {}_3) \ {}^2_4 + \ = 0, \\ (\ {}^2_2 - \ {}^2_1) \ {}^2_4 - \ {}_3 + k = 0. \end{gathered}$$

2 . Particular solution:

$$= \ {}_1 \sin(\ {}_4 \ ) + \ {}_2 \cos(\ {}_4 \ ) + \ {}_3,$$

where the constants ${}_1$, ${}_2$, ${}_3$, and ${}_4$ are related by two constraints

$$\begin{gathered} {}^2_4 \ - (a - \ {}_3) \ {}^2_4 + \ = 0, \\ (\ {}^2_1 + \ {}^2_2) \ {}^2_4 + \ {}_3 - k = 0. \end{gathered}$$

28. $^{(\ )} + \quad - \ (\quad)^2 + \quad + \quad = 0.$

Particular solution: $\ = \ {}_1 \exp(\ {}_2 \ ) - \dfrac{{}_2^{\ -1} + \ {}_2 +}{a \ {}_2}.$

29. $^{(2\ )} = \quad + \ (\quad - \quad)^k.$

This is a special case of equation 5.2.6.50 with $(\ , \ ) = \quad$. The substitution $\ = \ '' - a$ leads to an (2 $\ $ − 2)nd-order autonomous equation: $^{(2\ -2)} + a \ ^{(2\ -4)} + \quad + a \ ^{-1} \ = \quad .$

30. $^{(\ )} = \quad (\quad - \quad)^k (\quad) .$

The substitution $(\ ) = \ ' - \ $ leads to an ($\ $ − 1)st-order equation:

$$\frac{\ ^{-2}}{\ ^{-2}} \ \frac{\ '}{\ } \ = a \ \ ^{-} \ (\ ') .$$

31. $^{(2\ )} = \quad (\quad - \ {}^2 \ {}^k.$

This is a special case of equation 5.2.6.52 with $(\ ) = 0$ and $g(\ ) = a \quad$.

5.2.1-4. Other equations.

32. $^{(\ )} = \quad - \ (\quad)^2.$

1 . Integrating the equation two times, we obtain an ($\ $ − 2)nd-order equation:

$$^{(\ -2)} = a \ \ '' \ - a(\ ')^2 + \ {}_1 \ + \ {}_2.$$

2 . Particular solutions:

$= \ {}_1 \exp(\ {}_3 \ ) + \ {}_2 \exp(- \ {}_3 \ ) + a^{-1} \ {}_3^{\ -4}$	if $\ $ is even number,
$= \ {}_1 \sin(\ {}_3 \ ) + \ {}_2 \cos(\ {}_3 \ ) + (-1) \ \ ^2 a^{-1} \ {}_3^{\ -4}$	if $\ $ is even number,
$= \ {}_1 \exp(\ {}_2 \ ) + a^{-1} \ {}_2^{\ -4}$	if $\ $ is odd number,
$= \ {}_1 \ + \ {}_2$	if $\ \geq 2$ is any number.

33. $^{(\ )} = \quad - \ (\quad)^2 + \quad + \quad .$

This is a special case of equation 5.2.6.54 with $(\ ) = \quad + k$. Integrating the equation two times, we obtain an ($\ $ −2)nd-order equation: $^{(\ -2)} = a \ \ '' \ - a(\ ')^2 + \frac{1}{6} \ \ ^3 + \frac{1}{2} k \ \ ^2 + \ {}_1 \ + \ {}_2.$

34. $^{(2\ )} = \ ^{2} + [\ ^{(\ )} + \ ]^{k}.$

The substitution $= {}^{(\ )} + a$ leads to an th-order autonomous equation: $^{(\ )} = a + $.

35. $^{(\ )} + \ ^{(\ -1)} + 2 + \ ^{2} + = 0.$

The functions that solve the ($-$ 1)st-order autonomous equation $^{(\ -1)} = - \ ^{2} - \ a$ are solutions of the original equation.

36. $^{(\ )} + \ ^{(\ -1)} + + \ ^{2} + = 0.$

The functions that solve the ($-$ 1)st-order constant coefficient nonhomogeneous linear equation $^{(\ -1)} + = - \ a$ are solutions of the original equation.

37. $^{(\ )} + \ ^{(\ -1)} = $.

This is a special case of equation 5.2.6.59 with $(\ ) = a$.

38. $^{(\ )} + (+ - 1)\ ^{(\ -1)} = (+)^{k}.$

This is a special case of equation 5.2.6.60 with $(\ ,\ ) =$. The substitution $= \ ' + a$ leads to an ($-$ 1)st-order autonomous equation: $^{(\ -1)} =$.

39. $^{2}\ ^{(\ )} + 2 \ ^{(\ -1)} + (\ -1)\ ^{(\ -2)} = \ ^{2}$.

This is a special case of equation 5.2.6.61 with $(\ ) = a$.

40. $^{(2\ +1)} = \ ^{(2\ )}.$

The equation admits two different (with $a \neq -1$) first integrals:

$$^{(2\ )} = \ _{1}\ ,$$

$$^{(2\ )} + (a+1) \sum_{=1}^{-1} (-1)\ ^{(\ )}\ ^{(2\ -\ )} + \tfrac{1}{2}(-1)\ (a+1)\big[\ ^{(\ )}\big]^{2} = \ _{2},$$

where $_{1}$ and $_{2}$ are arbitrary constants. Eliminating the highest derivative from the first integrals, we arrive at an (2 $-$ 1)st-order autonomous equation:

$$\sum_{=1}^{-1} (-1)\ ^{(\ )}\ ^{(2\ -\ )} + \tfrac{1}{2}(-1)\ \big[\ ^{(\ )}\big]^{2} = \ _{1}\ ^{+1} + \ _{2},$$

where $_{1} = -\frac{1}{a+1}$, $_{2} = \frac{2}{a+1}$. The order of the obtained equation next can be lowered by the standard substitution $(\ ) = \ '$.

41. $(2\ -1)\ ^{(2\ +1)} + (2\ +1)\ ^{(2\ )} = $.

This is a special case of equation 5.2.6.62 with $(\ ) = a$.

42. $^{(\ )} - \ ^{(\ -1)} = \ ^{2}.$

Integrating yields an ($-$ 1)st-order linear equation: $^{(\ -1)} = (a + \ )$. The transformation $z = + \ a$ brings it to an equation of the form 5.1.2.3 with $= 0$.

43. $^{(\ )} = \ ^{(\ -1)} + $.

Integrating yields an ($-$ 1)st-order constant coefficient nonhomogeneous linear equation: $^{(\ -1)} = \ - a.$

44. $\sum_{k=0} {}_k {}^{(k)} = - (\ )^2 + .$

Particular solutions: $= e^{\lambda} + k a_0^{-1}$, where is an arbitrary constant and λ are roots of the algebraic equation $a_0 \sum_{=0} a\ \lambda = k\lambda^2$.

45. $^{(\ )} = (\ + \ )\ ^{(\ -1)}.$

Integrating yields an $(\ -1)$st-order linear equation of the form 5.1.2.4: $^{(\ -1)} =$.

46. $(\ + \ ^{-1})\ ^{(\ )} - \ ^{(\ )}\ ^{(\ -\ )} + \ ^{-1}\ ^{(\ )} = 0, \quad > \ .$

The functions that solve the $(\ -\ )$th-order linear equation

$$^{(\ -\ )} = \ + (a\ + \ )\ ^{-1}$$

are solutions of the original equation.

47. $^{(\ -2)}\ ^{(\ )} = \left[\ ^{(\ -1)}\right]^2.$

Solution: $= \begin{cases} {}_0 + {}_1 + \ + {}_{-3}\ {}^{-3} + (\ {}_{-2} + \ {}_{-1}\ )^{-2+\frac{1}{1-}} & \text{if } a \neq 1, \\ {}_0 + {}_1 + \ + {}_{-3}\ {}^{-3} + {}_{-2} \exp(\ {}_{-1}\ ) & \text{if } a = 1. \end{cases}$

48. $^{(\ )} = \ {}^{k} \left[\ ^{(\ -1)}\right] .$

This is a special case of equation 5.2.6.73 with $(\ ) = \ , g(\ ) = a$.

49. $^{(\ )} = \ {}^{1}\ {}^{2}(\ )\ {}^{3} \cdots (\ ^{(\ -1)})\ {}^{+1}.$

Generalized homogeneous equation. The transformation $\xi = \ ^{\lambda}\ , \ = \ '\ $, where

$$\lambda = \ + {}_1 - {}_3 - 2\ {}_4 - \ - (\ -1)\ {}_{+1}, \qquad = {}_2 + {}_3 + \ + {}_{+1} - 1,$$

leads to an $(\ -1)$st-order equation.

50. $- \frac{}{}\ {}^{-1}(\ ) = \ + .$

The transformation $= (\ ), \ = (\ ')^2$ leads to a constant coefficient linear equation: $2\ ^{(\ +1)} = a\ + .$

51. $2 \sum_{=1}^{-1} (-1)\ ^{(\ )}\ ^{(2\ -\ )} + (-1) \left[\ ^{(\ )}\right]^2 + (\ )^2 = \ ^2 + \ + .$

Differentiating both sides with respect to and dividing by $'$, we arrive at a constant coefficient linear equation: $2\ ^{(2\ )} - 2\lambda\ '' + 2a\ + \ = 0.$

52. $2 \sum_{=1}^{-1} (-1)\ ^{(\ )}\ ^{(2\ -\ )} + (-1) \left[\ ^{(\ )}\right]^2 = (\ -\ ) + \ + \gamma.$

Differentiating both sides of the equation with respect to , we have

$$''\left[2\ ^{(2\ -1)} - \ - \beta\right] = 0. \qquad (1)$$

Equate the second factor to zero to obtain:

$$= \frac{\ ^2}{2(2\ )!} + \frac{\beta\ ^{2\ -1}}{2(2\ -1)!} + \sum_{=0}^{2\ -2} \ .$$

The integration constants and parameters , β, and are related by

$$2 \sum_{=2}^{-1} (-1) \quad !\,(2 \quad - \quad)! \qquad {}_{2\,-} \quad + (-1) \; (\;\, !)^2 \quad {}^2 = \beta \quad {}_1 - \quad {}_0 + \quad ,$$

which is obtained as a result of substituting the above solution into the original equation.

In addition, there is a solution corresponding to equating the first factor in (1) to zero: $= \quad {}_1 \quad + \quad {}_0$, where $\beta \quad {}_1 - \quad {}_0 + \quad = 0$.

53. $\mathbf{2 \sum_{=1}^{-1} (-1) \quad ^{(\;\;)} \quad ^{(2\;-\;\;)} + (-1) \; \left[\;\; ^{(\;\;)} \right]^2 + s(\qquad)^2 = \quad (\qquad - \quad) + \qquad + \gamma, \qquad \geq 3.}$

For the case $= 0$, see equation 5.2.1.52. Let now $\neq 0$. Differentiating the equation with respect to , we have

$$'' \left[2 \quad ^{(2\;\;-1)} + 2 \quad ''' \quad - \qquad - \beta \right] = 0.$$

Equate the second factor to zero and integrate to obtain:

$$= \frac{\quad ^4}{48} + \frac{\beta \quad ^3}{12} + \quad {}_2 \quad ^2 + \quad {}_1 \quad + \quad {}_0 + \qquad\qquad ,$$

where $= \quad (\;\;)$ is the general solution of a linear constant coefficient linear equation of the form 5.1.2.2: $^{(2\;\;-4)} + \qquad = 0$. The constants of integration are related by the constraint that results from substituting the obtained solution into the original equation.

In addition, there is the solution $= \quad {}_1 \quad + \quad {}_0$, where the constants of integration are related by $\beta \quad {}_1 - \quad {}_0 + \quad = 0$.

54. $\mathbf{\sum_{=1} \quad 2 \sum_{\nu=1}^{-1} (-1)^{\nu} \quad ^{(\nu)} \quad ^{(2\;\;\;-\nu)} + (-1) \quad \left[\;\; ^{(\;\;\;)} \right]^2 \quad = \qquad ^2 + 2 \qquad + \gamma.}$

Differentiating with respect to , we arrive at a constant coefficient linear equation:

$$\sum_{=1} a \quad ^{(2\;\;)} + \qquad + \beta = 0.$$

5.2.2. Equations Containing Exponential Functions

5.2.2-1. Fifth- and sixth-order equations.

1. $\mathbf{^{(5)} = \qquad\qquad - \; (\qquad)^2 + \qquad ^{\lambda} \quad .}$

1 . This is a special case of equation 5.2.6.1 with $(\;\;) = \quad e^{\lambda}$. Integrating the equation two times, we obtain a third-order equation: $''' \quad = a \qquad '' \quad - a(\;\; ')^2 + \quad {}_1 \quad + \quad {}_2 + \quad \lambda^{-2} e^{\lambda}$.

2 . Particular solutions:

$$= \qquad \exp(\lambda \quad) + \frac{\lambda^5 -}{a \quad \lambda^4},$$

$$= \frac{}{2\lambda^5} \exp(\lambda \quad) + \quad \exp(-\lambda \quad) - \frac{\lambda}{a}.$$

2. $\mathbf{^{(5)} = \qquad ^{\lambda} \qquad\qquad\qquad .}$

Integrating yields a fourth-order autonomous equation of the form 5.2.6.8 with $=4$: $'''' \qquad = \quad \exp \left(\frac{a}{\lambda} e^{\lambda} \qquad \right)$.

3. $\mathbf{^{(5)} = \qquad ^{\lambda} \quad + \quad .}$

This is a special case of equation 5.2.6.17 with $= 2$ and $(\;\;) = ae^{\lambda} \quad + \quad$.

4. $^{(5)} + 5 \qquad + 10 \qquad = \quad \lambda$ **.**

Solution: $^2 = \ _4\ ^4 + \ _3\ ^3 + \ _2\ ^2 + \ _1\ + \ _0 + 2a\lambda^{-5}e^{\lambda}$.

5. $^{(5)} + 5 \qquad + 10 \qquad = \quad \lambda$ **.**

This is a special case of equation 5.2.6.3 with $(\ ) = ae^{\lambda}$.

6. $^{(6)} = \quad \lambda \ + $ **.**

This is a special case of equation 5.2.6.8 with $= 6$ and $(\ ) = ae^{\lambda} \ + $.

7. $^{(6)} + 6 \qquad ^{(5)} + 15 \qquad + 10(\qquad)^2 = \quad \lambda$ **.**

This is a special case of equation 5.2.6.6 with $(\ ) = ae^{\lambda}$.

5.2.2-2. Equations of the form $^{(\ )} = (\ ,\ )$.

8. $^{(\ )} = \quad \lambda \ + $ **.**

This is a special case of equation 5.2.6.8 with $(\ ) = ae^{\lambda} \ + $.

9. $^{(\ )} = \quad ^{\lambda\ +} \ + $ **.**

This is a special case of equation 5.2.6.9 with $= 1$. The substitution $= \ + (\beta\ \ \lambda)$ leads to an autonomous equation of the form 5.2.6.8: $^{(\ )} = ae^{\lambda w} + $.

10. $^{(\ )} = \quad ^{-\ \ \lambda}$ **.**

This is a special case of equation 5.2.6.11 with $(\ ) = ae^{\lambda}$.

11. $^{(\ )} = \quad ^{k}$ **.**

This is a special case of equation 5.2.6.26 with $(\ ) = a$ and $= k + $.

12. $^{(\ )} = $ **.**

This is a special case of equation 5.2.2.23 with $= \ _1$ and $_2 = \ _3 = \ = \ = 0$.

13. $^{(2\ \ +1)} = \quad \lambda \ + $ **.**

This is a special case of equation 5.2.6.17 with $(\ ) = ae^{\lambda} \ + $.

14. $^{(\ )} = $ **exp(** $\ + \ $ **)** $+ \ ,$ $\qquad = 1,\ 2,\ \ldots,\ \ - 1$**.**

The substitution $= \ + (\beta\ \ \lambda)$ leads to an autonomous equation of the form 5.2.6.8: $^{(\ )} = ae^{\lambda w} + $.

5.2.2-3. Other equations.

15. $^{(\ )} = \quad \lambda \qquad + $ **.**

Integrating yields an $(\ - 1)$st-order equation: $^{(\ -1)} = \dfrac{a}{\lambda} e^{\lambda} \ + \dfrac{}{\beta} e \ \ + $.

16. $^{(\ )} = \qquad - \ (\qquad)^2 + \quad \lambda$ **.**

1 . This is a special case of equation 5.2.6.54 with $(\ ) = \ e^{\lambda}$. Integrating the equation two times, we obtain an $(\ - 2)$nd-order equation: $^{(\ -2)} = a \quad '' - a(\ ')^2 + \ _1\ + \ _2 + \ \lambda^{-2} e^{\lambda}$.

2 . Particular solutions:

$$= \quad \exp(\lambda \quad) + \frac{\lambda \quad -}{a \quad \lambda^4} \qquad\qquad (\quad \text{is any number}),$$

$$= \frac{}{2\lambda} \exp(\lambda \quad) + \quad \exp(-\lambda \quad) - \frac{\lambda \quad ^{-4}}{a} \qquad (\quad \text{is odd number}).$$

17. $^{(\ )} = \quad + \quad ^{\lambda} \ (\quad - \quad)^{k}$.

This is a special case of equation 5.2.6.38 with $(\ , \ ) = \ e^{\lambda}$.

18. $^{(\ )} = \quad ^{\lambda} \ (\quad - \quad)^{k}$.

This is a special case of equation 5.2.6.39 with $(\ , \ ) = \ e^{\lambda}$.

19. $(2 \quad - 1) \quad ^{(2 \quad +1)} + (2 \quad + 1) \quad ^{(2 \ )} = \quad ^{\lambda}$.

This is a special case of equation 5.2.6.62 with $(\ ) = ae^{\lambda}$.

20. $^{(\ )} = \quad ^{\lambda} \quad ^{(\ -1)}$.

This is a special case of equation 5.2.6.57 with $(\ ) = ae^{\lambda}$.

21. $^{(\ )} = (\quad ^{\lambda} \quad + \quad) \ ^{(\ -1)}$.

This is a special case of equation 5.2.6.58 with $(\ ) = ae^{\lambda}$ and $g(\ ) = \ e$.

22. $^{(\ )} = \quad ^{\lambda} \quad \left[\ ^{(\ -1)} \right]$.

This is a special case of equation 5.2.6.73 with $(\ ) = ae^{\lambda}$ and $g(\ ) =$.

23. $^{(\ )} = \quad ^{1}(\quad) \ ^{2} \ldots (\ ^{(\ -1)}$.

The substitution $(\ ) = \ e$, where $\beta = \dfrac{}{_1 + \ _2 + \quad + \quad - 1}$, leads to an autonomous equation of the form 5.2.6.77.

24. $^{(\ )} = \quad ^{1}(\quad) \ ^{2}(\quad) \ ^{3} \ldots (\ ^{(\ -1)}$.

The transformation $z = \ ^{\sigma}e$, $=$ $'$, where $\sigma = \ + \ _1 - \ _2 - 2 \ _3 - 3 \ _4 - \quad -(\ -1)$, leads to an $(\ - 1)$st-order equation.

5.2.3. Equations Containing Hyperbolic Functions

5.2.3-1. Equations with hyperbolic sine.

1. $^{(5)} = \quad - \ (\quad)^2 + \ \mathbf{sinh}(\quad)$.

1 . This is a special case of equation 5.2.6.1 with $(\ ) = \ \sinh(\lambda \ )$. Integrating the equation two times, we obtain a third-order equation: $''' \ = a \quad '' - a(\ ')^2 + \ _1 \ + \ _2 + \ \lambda^{-2} \sinh(\lambda \ )$.

2 . Particular solution: $= \dfrac{}{\lambda^4(\lambda^2 - a^2 \ ^2)} \left[a \quad \sinh(\lambda \ ) + \lambda \cosh(\lambda \ ) \right] + \quad$.

2. $^{(5)} + 5 \quad + 10 \quad = \ \mathbf{sinh}(\quad)$.

Solution: $^2 = \ _4 \ ^4 + \ _3 \ ^3 + \ _2 \ ^2 + \ _1 \ + \ _0 + 2a\lambda^{-5} \cosh(\lambda \ )$.

3. $^{(5)} + 5 \quad + 10 \quad = \ \mathbf{sinh} \ (\quad) + \ $.

This is a special case of equation 5.2.6.2 with $(\ ) = a \sinh \ (\lambda \ ) + \ $.

4. $^{(5)} + 5 \qquad + 10 \qquad = \ \sinh \ (\quad) + \ .$

This is a special case of equation 5.2.6.3 with $(\ ) = a \sinh \ (\lambda \ ) + \ .$

5. $^{(6)} + 6 \quad ^{(5)} + 15 \qquad + 10(\quad)^2 = \ \sinh \ (\quad).$

This is a special case of equation 5.2.6.6 with $(\ ) = a \sinh \ (\lambda \ ).$

6. $^{(\ )} = \ \sinh \ (\quad) + \ .$

This is a special case of equation 5.2.6.8 with $(\ ) = a \sinh \ (\lambda \ ) + \ .$

7. $^{(\ )} = \ ^{-} \sinh \ (\quad).$

This is a special case of equation 5.2.6.11 with $(\ ) = a \sinh \ (\lambda \ ).$

8. $^{(2 \ +1)} = \ \sinh \ (\quad) + \ .$

This is a special case of equation 5.2.6.17 with $(\ ) = a \sinh \ (\lambda \ ) + \ .$

9. $^{(\ )} = \qquad - \ (\quad)^2 + \ \sinh(\quad).$

1 . This is a special case of equation 5.2.6.54 with $(\ ) = \ \sinh(\lambda \ )$. Integrating the equation two times, we obtain an $(\ -2)$nd-order equation: $^{(\ -2)} = a \quad '' - a(\ ')^2 + \ _1 \ + \ _2 + \ \lambda^{-2} \sinh(\lambda \ ).$

2 . Particular solutions:

$$= \ \sinh(\lambda \ ) + \frac{\lambda \ -}{a \ \lambda^4} \qquad \text{if is even number,}$$

$$= \frac{}{\lambda^2 \ ^{-4} - a^2 \ ^2\lambda^4}\left[a \ \sinh(\lambda \ ) + \lambda \ ^{-4}\cosh(\lambda \ )\right] + \qquad \text{if is odd number.}$$

10. $(2 \ -1) \ ^{(2 \ +1)} + (2 \ +1) \ ^{(2 \ )} = \ \sinh \ (\quad) + \ .$

This is a special case of equation 5.2.6.62 with $(\ ) = a \sinh \ (\lambda \ ) + \ .$

11. $^{(\ )} = \ \sinh^k(\quad) \ ^{(\ -1)}.$

This is a special case of equation 5.2.6.57 with $(\ ) = a \sinh \ (\lambda \ ).$

12. $^{(\ )} - \ ^{(\ -1)} = \ \sinh(\quad) \ ^2.$

This is a special case of equation 5.2.6.64 with $(\ ) = a \sinh(\lambda \ )$. Integrating yields an $(\ -1)$st-order linear equation: $^{(\ -1)} = \dfrac{a}{\lambda}\cosh(\lambda \ ) + \qquad .$

5.2.3-2. Equations with hyperbolic cosine.

13. $^{(5)} = \qquad - \ (\quad)^2 + \ \cosh(\quad).$

1 . This is a special case of equation 5.2.6.1 with $(\ ) = \ \cosh(\lambda \ )$. Integrating the equation twice, we obtain a third-order equation: $''' = a \quad '' - a(\ ')^2 + \ _1 \ + \ _2 + \ \lambda^{-2}\cosh(\lambda \ ).$

2 . Particular solution: $= \dfrac{}{\lambda^4(\lambda^2 - a^2 \ ^2)}\left[a \ \cosh(\lambda \ ) + \lambda \sinh(\lambda \ )\right] + \ .$

14. $^{(5)} + 5 \qquad + 10 \qquad = \ \cosh(\quad).$

Solution: $^2 = \ _4 \ ^4 + \ _3 \ ^3 + \ _2 \ ^2 + \ _1 \ + \ _0 + 2a\lambda^{-5}\sinh(\lambda \ ).$

15. $^{(5)} + 5 \qquad + 10 \qquad = \ \cosh \ (\quad) + \ .$

This is a special case of equation 5.2.6.2 with $(\ ) = a \cosh \ (\lambda \ ) + \ .$

16. $\quad ^{(5)} + 5 \qquad + 10 \qquad = \quad \cosh \quad (\quad) + \quad .$

This is a special case of equation 5.2.6.3 with $(\ ) = a \cosh \ (\lambda \ ) + \ .$

17. $\quad ^{(6)} + 6 \quad ^{(5)} + 15 \qquad + 10(\quad)^2 = \quad \cosh \quad (\quad).$

This is a special case of equation 5.2.6.6 with $(\ ) = a \cosh \ (\lambda \ ).$

18. $\quad ^{(\ )} = \quad \cosh \quad (\quad) + \quad .$

This is a special case of equation 5.2.6.8 with $(\ ) = a \cosh \ (\lambda \ ) + \ .$

19. $\quad ^{(\ )} = \quad ^{-} \cosh \quad (\quad).$

This is a special case of equation 5.2.6.11 with $(\ ) = a \cosh \ (\lambda \ ).$

20. $\quad ^{(2 \ +1)} = \quad \cosh \quad (\quad) + \quad .$

This is a special case of equation 5.2.6.17 with $(\ ) = a \cosh \ (\lambda \ ) + \ .$

21. $\quad ^{(\ )} = \qquad - \ (\quad)^2 + \quad \cosh(\quad).$

1 . This is a special case of equation 5.2.6.54 with $(\ ) = \ \cosh(\lambda \ )$. Integrating the equation twice, we obtain an (−2)nd-order equation: $^{(\ -2)} = a \quad '' - a(\ ')^2 + \ _1 \ + \ _2 + \lambda^{-2} \cosh(\lambda \ ).$

2 . Particular solutions:

$$= \quad \cosh(\lambda \ ) + \frac{\lambda \ -}{a \ \lambda^4} \qquad \text{if is even number,}$$

$$= \frac{}{\lambda^2 \ ^{-4} - a^2 \ ^2 \lambda^4} \left[a \quad \cosh(\lambda \ ) + \lambda \ ^{-4} \sinh(\lambda \ ) \right] + \qquad \text{if is odd number.}$$

22. $(2 \quad - 1) \quad ^{(2 \ +1)} + (2 \quad + 1) \quad ^{(2 \ )} = \quad \cosh \quad (\quad) + \quad .$

This is a special case of equation 5.2.6.62 with $(\ ) = a \cosh \ (\lambda \ ) + \ .$

23. $\quad ^{(\ )} = \quad \cosh^k (\quad) \quad ^{(\ -1)}.$

This is a special case of equation 5.2.6.57 with $(\ ) = a \cosh \ (\lambda \ ).$

24. $\quad ^{(\ )} - \quad ^{(\ -1)} = \quad \cosh(\quad) \ ^2.$

This is a special case of equation 5.2.6.64 with $(\ ) = a \cosh(\lambda \ )$. Integrating yields an (− 1)st-order linear equation: $^{(\ -1)} = \frac{a}{\lambda} \sinh(\lambda \ ) + \qquad .$

5.2.3-3. Equations with hyperbolic tangent.

25. $\quad ^{(5)} = \qquad - \ (\quad)^2 + \quad \tanh(\quad) + \quad .$

This is a special case of equation 5.2.6.1 with $(\ ) = \ \tanh(\lambda \ ) + \ .$

26. $\quad ^{(5)} + 5 \qquad + 10 \qquad = \quad \tanh \quad (\quad) + \quad .$

This is a special case of equation 5.2.6.2 with $(\ ) = a \tanh \ (\lambda \ ) + \ .$

27. $\quad ^{(5)} + 5 \qquad + 10 \qquad = \quad \tanh \quad (\quad) + \quad .$

This is a special case of equation 5.2.6.3 with $(\ ) = a \tanh \ (\lambda \ ) + \ .$

28. $\quad ^{(6)} + 6 \quad ^{(5)} + 15 \qquad + 10(\quad)^2 = \quad \tanh \quad (\quad).$

This is a special case of equation 5.2.6.6 with $(\ ) = a \tanh \ (\lambda \ ).$

29. $^{(\ )} = \ \mathbf{tanh}\ (\ \ ) + \ .$

This is a special case of equation 5.2.6.8 with $(\ ) = a \tanh\ (\lambda\ ) + \ .$

30. $^{(\ )} = \ ^{-}\ \mathbf{tanh}\ (\ \ ).$

This is a special case of equation 5.2.6.11 with $(\ ) = a \tanh\ (\lambda\ ).$

31. $^{(2\ +1)} = \ \mathbf{tanh}\ (\ \ ) + \ .$

This is a special case of equation 5.2.6.17 with $(\ ) = a \tanh\ (\lambda\ ) + \ .$

32. $^{(2\ )} = \ + \ (\ \ - \ \mathbf{tanh}\ \ )^{k}.$

This is a special case of equation 5.2.6.47 with $(\ ,\ ) = a$ and $(\ ) = \cosh\ .$

33. $^{(2\ +1)} = \ \mathbf{tanh}\ \ + \ (\ \ - \ \mathbf{tanh}\ \ )^{k}.$

This is a special case of equation 5.2.6.47 with $(\ ,\ ) = a$ and $(\ ) = \cosh\ .$

34. $(2\ \ - 1)\ \ ^{(2\ +1)} + (2\ \ + 1)\ \ ^{(2\ )} = \ \mathbf{tanh}\ (\ \ ) + \ .$

This is a special case of equation 5.2.6.62 with $(\ ) = a \tanh\ (\lambda\ ) + \ .$

35. $^{(\ )} = \ \mathbf{tanh}^{k}(\ \ )\ \ ^{(\ -1)}.$

This is a special case of equation 5.2.6.57 with $(\ ) = a \tanh\ (\lambda\ ).$

36. $^{(\ )} - \ \ ^{(\ -1)} = \ \mathbf{tanh}(\ \ )\ ^{2}.$

This is a special case of equation 5.2.6.64 with $(\ ) = a \tanh(\lambda\ ).$

5.2.3-4. Equations with hyperbolic cotangent.

37. $^{(5)} = \ \ - \ (\ \ )^{2} + \ \mathbf{coth}(\ \ ) + \ .$

This is a special case of equation 5.2.6.1 with $(\ ) = \ \coth(\lambda\ ) + \ .$

38. $^{(5)} + 5\ \ + 10\ \ = \ \mathbf{coth}\ (\ \ ) + \ .$

This is a special case of equation 5.2.6.2 with $(\ ) = a \coth\ (\lambda\ ) + \ .$

39. $^{(5)} + 5\ \ + 10\ \ = \ \mathbf{coth}\ (\ \ ) + \ .$

This is a special case of equation 5.2.6.3 with $(\ ) = a \coth\ (\lambda\ ) + \ .$

40. $^{(6)} + 6\ \ ^{(5)} + 15\ \ + 10(\ \ )^{2} = \ \mathbf{coth}\ (\ \ ).$

This is a special case of equation 5.2.6.6 with $(\ ) = a \coth\ (\lambda\ ).$

41. $^{(\ )} = \ \mathbf{coth}\ (\ \ ) + \ .$

This is a special case of equation 5.2.6.8 with $(\ ) = a \coth\ (\lambda\ ) + \ .$

42. $^{(\ )} = \ ^{-}\ \mathbf{coth}\ (\ \ ).$

This is a special case of equation 5.2.6.11 with $(\ ) = a \coth\ (\lambda\ ).$

43. $^{(2\ +1)} = \ \mathbf{coth}\ (\ \ ) + \ .$

This is a special case of equation 5.2.6.17 with $(\ ) = a \coth\ (\lambda\ ) + \ .$

44. $^{(2\ )} = \ + \ (\ \ - \ \mathbf{coth}\ \ )^{k}.$

This is a special case of equation 5.2.6.47 with $(\ ,\ ) = a$ and $(\ ) = \sinh\ .$

45. $^{(2\quad+1)} = \quad \coth \quad + \ (\quad - \quad \coth \quad)^k$.

This is a special case of equation 5.2.6.47 with $(\ ,\ ) = a$ and $(\ ) = \sinh$.

46. $(2\quad - 1)\quad ^{(2\quad +1)} + (2\quad + 1)\quad ^{(2\quad)} = \quad \coth\ (\quad) + \ $.

This is a special case of equation 5.2.6.62 with $(\ ) = a \coth\ (\lambda\ ) + \ $.

47. $^{(\ )} = \quad \coth^k(\quad)\quad ^{(\ -1)}$.

This is a special case of equation 5.2.6.57 with $(\ ) = a \coth\ (\lambda\ )$.

48. $^{(\ )} - \quad ^{(\ -1)} = \quad \coth(\quad)\ ^2$.

This is a special case of equation 5.2.6.64 with $(\ ) = a \coth(\lambda\ )$.

5.2.4. Equations Containing Logarithmic Functions

5.2.4-1. Equations of the form $^{(\ )} = \ (\ ,\ )$.

1. $^{(\ )} = \quad \ln\ (\quad) + \ $.

This is a special case of equation 5.2.6.8 with $(\ ) = a \ln\ (\quad) + \ $.

2. $^{(2\quad +1)} = \quad \ln\ (\quad)$.

This is a special case of equation 5.2.6.17 with $(\ ) = a \ln\ (\quad)$.

3. $^{(\ )} = \ (\quad + \quad \ln \quad + \ )$.

This is a special case of equation 5.2.6.25 with $(\ ) = \ln \quad + \ $.

4. $^{(\ )} = \ ^{-}\ (\quad + \quad \ln \quad + \ )$.

This is a special case of equation 5.2.6.26 with $(\ ) = \ln \quad + \ $.

5. $^{(\ )} = \quad ^{-}\ \ln\ (\quad)$.

This is a special case of equation 5.2.6.11 with $(\ ) = a \ln\ (\quad)$.

6. $^{(\ )} = \quad ^{-\ -1}[\ln \quad + (1 - \quad) \ln \quad]$.

This is a special case of equation 5.2.6.13 with $(\ ) = a \ln \quad $.

7. $^{(\ )} = \quad ^{-\ -k}(\ln \quad + \quad \ln \quad)$.

This is a special case of equation 5.2.6.15 with $(\ ) = a \ln \quad $.

8. $^{(\ )} = \quad ^{-}\ (\quad \ln \quad + \quad \ln \quad)$.

This is a special case of equation 5.2.6.16 with $(\ ) = a \ln \quad $.

9. $^{(2\quad)} = \quad ^{-\frac{2\quad +1}{2}}[2 \ln \quad + (1 - 2\quad) \ln \quad]$.

This is a special case of equation 5.2.6.14 with $(\ ) = 2a \ln \quad $.

10. $^{(\ )} = (\quad ^2 + \quad)^{-\frac{\quad +1}{2}}[2 \ln \quad + (1 - \quad) \ln(\quad ^2 + \quad)]$.

This is a special case of equation 5.2.6.22 with $\ = 0$ and $(\ ) = 2 \ln \quad $.

11. $^{(\ )} = \quad (\ln \quad - \quad)$.

This is a special case of equation 5.2.6.24 with $(\ ) = \quad \ln \quad $.

5.2.4-2. Other equations.

12. $^{(\ )} =\quad \ln\quad + \ln\quad .$

This is a special case of equation 5.2.6.34 with $(\ ) = a\ln$ and $g(\ ) = \ln$.

13. $^{(\ )} =\quad + \ln\ (\quad -\quad)^k.$

This is a special case of equation 5.2.6.38 with $(\ ,\ ) = \quad \ln$.

14. $^{(\ )} = \ln\ (\quad -\ )^k.$

This is a special case of equation 5.2.6.39 with $(\ ,\ ) = a\quad \ln$.

15. $^{(\ )} = \ln\ (\quad - 2\ )^k.$

This is a special case of equation 5.2.6.40 with $= 2$ and $(\ ,\ ) = a\quad \ln$.

16. $^{(\ )} =\quad - (\quad)^2 + \ln\quad + .$

This is a special case of equation 5.2.6.54 with $(\ ) = \ln\ + .$

17. $^{(\ )} + \quad ^{(\ -1)} = \ln\quad + \ln\quad .$

This is a special case of equation 5.2.6.59 with $(\ ) = a\ln$.

18. $^{2\ (\ )} + 2\quad ^{(\ -1)} + (\ -1)\ ^{(\ -2)} = 2\ \ln\quad + \ln\quad .$

This is a special case of equation 5.2.6.61 with $(\ ) = a\ln$.

19. $^{(\ )} - \quad ^{(\ -1)} = \ ^2\ln\quad .$

This is a special case of equation 5.2.6.64 with $(\ ) = a\ln$.

20. $(2\ -1)\quad ^{(2\ +1)} + (2\ +1)\quad ^{(2\ )} = \ln\ (\quad) + .$

This is a special case of equation 5.2.6.62 with $(\ ) = a\ln\ (\quad) + .$

21. $^{(\ )} = \ln^k(\ )\quad ^{(\ -1)}.$

This is a special case of equation 5.2.6.57 with $(\ ) = a\ln\ (\quad).$

22. $^{(\ )} =\quad \ln\ ^{(\ -1)}.$

This is a special case of equation 5.2.6.73 with $(\ ) = a$ and $g(\ ) = \ln$.

5.2.5. Equations Containing Trigonometric Functions

5.2.5-1. Equations with sine.

1. $^{(5)} =\quad - (\quad)^2 + \sin(\quad).$

1 . This is a special case of equation 5.2.6.1 with $(\ ) = \sin(\lambda\ )$. Integrating the equation twice, we obtain a third-order equation: $''' = a\ '' - a(\ ')^2 + \ _1\ + \ _2 - \lambda^{-2}\sin(\lambda\ ).$

2 . Particular solution: $= -\dfrac{}{\lambda^4(a^2\ ^2 + \lambda^2)}\left[a\quad \sin(\lambda\ ) + \lambda\cos(\lambda\ )\right] + .$

2. $^{(5)} + 5\quad + 10\quad = \sin(\quad).$

Solution: $^2 = \ _4\ ^4 + \ _3\ ^3 + \ _2\ ^2 + \ _1\ + \ _0 - 2a\lambda^{-5}\cos(\lambda\ ).$

3. $^{(5)} + 5\quad + 10\quad = \sin\ (\quad) + .$

This is a special case of equation 5.2.6.2 with $(\ ) = a\sin\ (\lambda\ ) + .$

4. $\quad ^{(5)} + 5 \qquad + 10 \qquad = \quad \sin \ (\quad) + \ .$

This is a special case of equation 5.2.6.3 with $\ (\ ) = a \sin \ (\lambda \ ) + \ .$

5. $\quad ^{(6)} + 6 \quad ^{(5)} + 15 \qquad + 10(\quad)^2 = \quad \sin \ (\quad).$

This is a special case of equation 5.2.6.6 with $\ (\ ) = a \sin \ (\lambda \ ).$

6. $\quad ^{(\ )} = \quad \sin \ (\quad) + \ .$

This is a special case of equation 5.2.6.8 with $\ (\ ) = a \sin \ (\lambda \ ) + \ .$

7. $\quad ^{(\ )} = \quad ^{-} \sin \ (\quad).$

This is a special case of equation 5.2.6.11 with $\ (\ ) = a \sin \ (\lambda \ ).$

8. $\quad ^{(2 \ +1)} = \quad \sin \ (\quad) + \ .$

This is a special case of equation 5.2.6.17 with $\ (\ ) = a \sin \ (\lambda \ ) + \ .$

9. $\quad ^{(\ )} = \qquad - \ (\quad)^2 + \ \sin(\quad).$

1 . This is a special case of equation 5.2.6.54 with $\ (\ ) = \ \sin(\lambda \ )$. Integrating the equation two times, we obtain an $(\ - 2)$nd-order equation: $\ ^{(\ -2)} = a \ '' \ - a(\ ')^2 + \ _1 \ + \ _2 - \ \lambda^{-2} \sin(\lambda \ ).$

2 . Particular solutions:

$$= \quad \sin(\lambda \ ) + \frac{(-1) \ ^2 \ \lambda \ -}{a \ \lambda^4} \qquad \text{if is even number,}$$

$$= -\frac{}{\lambda^2 \ ^{-4} + a^2 \ ^2\lambda^4} \left[a \quad \sin(\lambda \ ) + (-1)^{\frac{-1}{2}} \lambda \ ^{-4} \cos(\lambda \ )\right] + \qquad \text{if is odd number.}$$

10. $\quad (2 \ - 1) \quad ^{(2 \ +1)} + (2 \ + 1) \quad ^{(2 \ )} = \quad \sin \ (\quad) + \ .$

This is a special case of equation 5.2.6.62 with $\ (\ ) = a \sin \ (\lambda \ ) + \ .$

11. $\quad ^{(\ )} = \quad \sin^k(\quad) \quad ^{(\ -1)}.$

This is a special case of equation 5.2.6.57 with $\ (\ ) = a \sin \ (\lambda \ ).$

12. $\quad ^{(\ )} - \quad ^{(\ -1)} = \quad \sin(\quad) \ ^2.$

This is a special case of equation 5.2.6.64 with $\ (\ ) = a \sin(\lambda \ )$. Integrating yields an $(\ - 1)$st-order linear equation: $\ ^{(\ -1)} = \qquad - \frac{a}{\lambda} \cos(\lambda \ ) \quad .$

5.2.5-2. Equations with cosine.

13. $\quad ^{(5)} = \qquad - \ (\quad)^2 + \ \cos(\quad).$

1 . This is a special case of equation 5.2.6.1 with $\ (\ ) = \ \cos(\lambda \ )$. Integrating the equation twice, we obtain a third-order equation: $\ ''' \ = a \ '' \ - a(\ ')^2 + \ _1 \ + \ _2 - \ \lambda^{-2} \cos(\lambda \ ).$

2 . Particular solution: $\quad = -\frac{}{\lambda^4(a^2 \ ^2 + \lambda^2)} \left[a \quad \cos(\lambda \ ) - \lambda \sin(\lambda \ )\right] + \quad .$

14. $\quad ^{(5)} + 5 \qquad + 10 \qquad = \quad \cos(\quad).$

Solution: $\ ^2 = \ _4 \ ^4 + \ _3 \ ^3 + \ _2 \ ^2 + \ _1 \ + \ _0 + 2a\lambda^{-5} \sin(\lambda \ ).$

15. $\quad ^{(5)} + 5 \qquad + 10 \qquad = \quad \cos \ (\quad) + \ .$

This is a special case of equation 5.2.6.2 with $\ (\ ) = a \cos \ (\lambda \ ) + \ .$

16. $^{(5)} + 5 \quad + 10 \quad = \quad \cos \quad (\quad) + \quad .$

This is a special case of equation 5.2.6.3 with $(\ ) = a \cos \ (\lambda \ ) + \ .$

17. $^{(6)} + 6 \quad ^{(5)} + 15 \quad + 10(\quad)^2 = \quad \cos \quad (\quad).$

This is a special case of equation 5.2.6.6 with $(\ ) = a \cos \ (\lambda \ ).$

18. $^{(\ )} = \quad \cos \quad (\quad) + \quad .$

This is a special case of equation 5.2.6.8 with $(\ ) = a \cos \ (\lambda \ ) + \ .$

19. $^{(\ )} = \quad ^{-} \cos \quad (\quad).$

This is a special case of equation 5.2.6.11 with $(\ ) = a \cos \ (\lambda \ ).$

20. $^{(2 \ +1)} = \quad \cos \quad (\quad) + \quad .$

This is a special case of equation 5.2.6.17 with $(\ ) = a \cos \ (\lambda \ ) + \ .$

21. $^{(\ )} = \quad - \ (\quad)^2 + \quad \cos(\quad).$

1 . This is a special case of equation 5.2.6.54 with $(\ ) = \ \cos(\lambda \ )$. Integrating the equation two times, we obtain an $(\ - 2)$nd-order equation: $^{(\ -2)} = a \quad '' \ - a(\ ')^2 + \quad _1 \ + \quad _2 - \quad \lambda^{-2} \cos(\lambda \ ).$

2 . Particular solutions:

$$= \quad \cos(\lambda \ ) + \frac{(-1) \quad ^2 \quad \lambda \quad -}{a \quad \lambda^4} \qquad \text{if is even number,}$$

$$= -\frac{}{\lambda^2 \ ^{-4} + a^2 \ \ ^2\lambda^4} \left[a \quad \cos(\lambda \ ) + (-1)^{\frac{+1}{2}} \lambda \ ^{-4} \sin(\lambda \ ) \right] + \qquad \text{if is odd number.}$$

22. $(2 \ -1) \quad ^{(2 \ +1)} + (2 \ +1) \quad ^{(2 \ )} = \quad \cos \quad (\quad) + \quad .$

This is a special case of equation 5.2.6.62 with $(\ ) = a \cos \ (\lambda \ ) + \ .$

23. $^{(\ )} = \quad \cos^k(\quad) \quad ^{(\ -1)}.$

This is a special case of equation 5.2.6.57 with $(\ ) = a \cos \ (\lambda \ ).$

24. $^{(\ )} - \quad ^{(\ -1)} = \quad \cos(\quad) \ ^2.$

This is a special case of equation 5.2.6.64 with $(\ ) = a \cos(\lambda \ )$. Integrating yields an $(\ - 1)$st-order linear equation: $^{(\ -1)} = \dfrac{a}{\lambda} \sin(\lambda \ ) + \quad .$

5.2.5-3. Equations with tangent.

25. $^{(5)} = \quad - \ (\quad)^2 + \quad \tan(\quad) + \quad .$

This is a special case of equation 5.2.6.1 with $(\ ) = \ \tan(\lambda \ ) + \ .$

26. $^{(5)} + 5 \quad + 10 \quad = \quad \tan \quad (\quad) + \quad .$

This is a special case of equation 5.2.6.2 with $(\ ) = a \tan \ (\lambda \ ) + \ .$

27. $^{(5)} + 5 \quad + 10 \quad = \quad \tan \quad (\quad) + \quad .$

This is a special case of equation 5.2.6.3 with $(\ ) = a \tan \ (\lambda \ ) + \ .$

28. $^{(6)} + 6 \quad ^{(5)} + 15 \quad + 10(\quad)^2 = \quad \tan \quad (\quad).$

This is a special case of equation 5.2.6.6 with $(\ ) = a \tan \ (\lambda \ ).$

29. $^{(\ )} = \ \mathbf{tan}\ (\ \ ) + \ .$

This is a special case of equation 5.2.6.8 with $(\ ) = a \tan\ (\lambda\ ) + \ .$

30. $^{(\ )} = \ ^{-}\ \mathbf{tan}\ (\ \ ).$

This is a special case of equation 5.2.6.11 with $(\ ) = a \tan\ (\lambda\ ).$

31. $^{(2\ +1)} = \ \mathbf{tan}\ (\ \ ) + \ .$

This is a special case of equation 5.2.6.17 with $(\ ) = a \tan\ (\lambda\ ) + \ .$

32. $^{(2\ )} = (-1)\ \ + \ (\ \ + \ \mathbf{tan}\ )^k.$

This is a special case of equation 5.2.6.47 with $(\ ,\ ) = a$ and $(\ ) = \cos\ .$

33. $^{(2\ +1)} = (-1)\ ^{+1}\ \mathbf{tan}\ \ + \ (\ \ + \ \mathbf{tan}\ )^k.$

This is a special case of equation 5.2.6.47 with $(\ ,\ ) = a$ and $(\ ) = \cos\ .$

34. $(2\ \ -1)\ ^{(2\ +1)} + (2\ \ +1)\ ^{(2\ )} = \ \mathbf{tan}\ (\ \ ) + \ .$

This is a special case of equation 5.2.6.62 with $(\ ) = a \tan\ (\lambda\ ) + \ .$

35. $^{(\ )} = \ \mathbf{tan}^k(\ \ )\ ^{(\ -1)}.$

This is a special case of equation 5.2.6.57 with $(\ ) = a \tan\ (\lambda\ ).$

36. $^{(\ )} - \ ^{(\ -1)} = \ \mathbf{tan}(\ \ )\ ^2.$

This is a special case of equation 5.2.6.64 with $(\ ) = a \tan(\lambda\ ).$

5.2.5-4. Equations with cotangent.

37. $^{(5)} = \ \ - \ (\ \ )^2 + \ \mathbf{cot}(\ \ ) + \ .$

This is a special case of equation 5.2.6.1 with $(\ ) = \ \cot(\lambda\ ) + \ .$

38. $^{(5)} + 5\ \ + 10\ \ = \ \mathbf{cot}\ (\ \ ) + \ .$

This is a special case of equation 5.2.6.2 with $(\ ) = a \cot\ (\lambda\ ) + \ .$

39. $^{(5)} + 5\ \ + 10\ \ = \ \mathbf{cot}\ (\ \ ) + \ .$

This is a special case of equation 5.2.6.3 with $(\ ) = a \cot\ (\lambda\ ) + \ .$

40. $^{(6)} + 6\ ^{(5)} + 15\ \ + 10(\ \ )^2 = \ \mathbf{cot}\ (\ \ ).$

This is a special case of equation 5.2.6.6 with $(\ ) = a \cot\ (\lambda\ ).$

41. $^{(\ )} = \ \mathbf{cot}\ (\ \ ) + \ .$

This is a special case of equation 5.2.6.8 with $(\ ) = a \cot\ (\lambda\ ) + \ .$

42. $^{(\ )} = \ ^{-}\ \mathbf{cot}\ (\ \ ).$

This is a special case of equation 5.2.6.11 with $(\ ) = a \cot\ (\lambda\ ).$

43. $^{(2\ +1)} = \ \mathbf{cot}\ (\ \ ) + \ .$

This is a special case of equation 5.2.6.17 with $(\ ) = a \cot\ (\lambda\ ) + \ .$

44. $^{(2\ )} = (-1)\ \ + \ (\ \ - \ \mathbf{cot}\ )^k.$

This is a special case of equation 5.2.6.47 with $(\ ,\ ) = a$ and $(\ ) = \sin\ .$

45. $^{(2\ \ +1)} = (-1)\quad \cot\quad + (\quad - \quad\cot\quad)^k$.

This is a special case of equation 5.2.6.47 with $(\ ,\ ) = a$ and $(\ ) = \sin$.

46. $(2\quad - 1)\qquad ^{(2\ \ +1)} + (2\quad + 1)\qquad ^{(2\ \ )} = \quad \cot\ (\quad) + \ $.

This is a special case of equation 5.2.6.62 with $(\ ) = a \cot\ (\lambda\ ) + \ $.

47. $^{(\ )} = \quad \cot^k(\quad)\qquad ^{(\ \ -1)}$.

This is a special case of equation 5.2.6.57 with $(\ ) = a \cot\ (\lambda\ )$.

48. $^{(\ )} - \qquad ^{(\ \ -1)} = \quad \cot(\quad)\ ^2$.

This is a special case of equation 5.2.6.64 with $(\ ) = a \cot(\lambda\ )$.

5.2.6. Equations Containing Arbitrary Functions

5.2.6-1. Fifth- and sixth-order equations.

1. $^{(5)} = \qquad\qquad - (\quad)^2 + f(\ )$.

Integrating the equation two times, we obtain a third-order equation:

$$\ ''' \quad = a \quad '' - a(\ ')^2 + \ _1 \ + \ _2 + \ _0 (\ -\ )\ (\ )\ ,\quad \text{where}\quad _0 \text{ is an arbitrary number.}$$

2. $^{(5)} + 5 \qquad\qquad + 10 \qquad\qquad = f(\ )$.

Solution:

$$^2 = \ _4\ ^4 + \ _3\ ^3 + \ _2\ ^2 + \ _1\ + \ _0 + \frac{1}{12}\ _0 (\ -\ )^4\ (\ )\ ,\quad \text{where}\quad _0 \text{ is an arbitrary number.}$$

3. $^{(5)} + 5 \qquad\qquad + 10 \qquad\qquad = f(\ )$.

The substitution $\ = \ ^2$ leads to an autonomous equation of the form 5.2.6.8: $^{(5)} = 2\ (\ \overline{\ }\ $.

4. $^{(5)} + \qquad\qquad + (3\quad - 5) \qquad\qquad = f(\ )$.

Integrating the equation three times, we obtain a second-order equation:

$$'' + \frac{a-3}{2}(\ ')^2 = \ _2\ ^2 + \ _1\ + \ _0 + \frac{1}{2}\ _0 (\ -\ )^2\ (\ )\ ,\quad \text{where}\quad _0 \text{ is an arbitrary number.}$$

5. $(\ + \ )\ ^{(5)} + \qquad\qquad + \qquad\qquad = f(\ )$.

Integrating yields a fourth-order equation:

$$(a + \ )\ '''' \quad + (\ - 1)\ '\quad ''' \ + \frac{1}{2}(1 - \ + \ )(\ '')^2 = \quad (\ ) \quad + \ .$$

6. $^{(6)} + 6 \quad ^{(5)} + 15 \qquad\qquad + 10(\qquad)^2 = f(\ )$.

Solution: $\ ^2 = \ _5\ ^5 + \ _4\ ^4 + \ _3\ ^3 + \ _2\ ^2 + \ _1\ + \ _0 + \frac{1}{60}\ _0 (\ -\ )^5\ (\ )\ $.

7. $^{(6)} = (\quad ^2 + \quad + \ )^{-7\ 2} f\big(\ (\quad ^2 + \quad + \ )^{-5\ 2}\ $.

This is a special case of equation 5.2.6.22 with $\ = 6$.

5.2.6-2. Equations of the form $^{(\ )} = \ (\ ,\ )$.

8. $^{(\ )} = f(\ )$.

Autonomous equation. This is a special case of equation 5.2.6.77.

1 . The substitution $(\ ) = \ '$ leads to an $(\ -1)$st-order equation.

2 . For even $= 2$, the first integral of the equation is:

$$\sum_{=1}^{-1} (-1) \ ^{(\ )} \ ^{(2\ -\ )} + \tfrac{1}{2}(-1) \ \left[\ ^{(\ )}\right]^2 + \ \ (\ ) \ \ = \ .$$

Furthermore, the order of the obtained equation can be reduced by one by the substitution $(\ ) = \ '$.

9. $^{(\ )} = f(\ + \ )$, $= 0, 1, \ldots, \ -1$.

The substitution $= \ + a$ leads to an autonomous equation of the form 5.2.6.8: $^{(\ )} = \ (\ )$.

10. $^{(\ )} = f\bigl(\ + \ + \ _{-1} \ ^{-1} + \ldots + \ _0$.

The substitution $= \ + a \ + a_{\ -1} \ ^{-1} + \ + a_0$ leads to an autonomous equation of the form 5.2.6.8: $^{(\ )} = a \ ! + \ (\ )$.

11. $^{(\ )} = \ ^{-} \ f(\ )$.

The substitution $= \ln|\ |$ leads to an autonomous equation of the form 5.2.6.77.

12. $^{(\ )} = \ ^{1-} \ f(\ )$.

Homogeneous equation. This is a special case of equation 5.2.6.83. The transformation $= \ln \ ,\ =$ leads to an autonomous equation of the form 5.2.6.77.

13. $^{(\ )} = \ ^{-\ -1} f(\ ^{1-}\ )$.

The transformation $= \ ^{-1}$, $= \ ^{1-}$ leads to an autonomous equation of the form 5.2.6.8: $^{(\ )} = (-1) \ (\ )$.

14. $^{(2\ )} = \ ^{-\frac{2\ +1}{2}} f \ \ ^{\frac{1-2}{2}}$.

The transformation $= e \ ,\ = \ ^{\frac{2\ -1}{2}} \ (\ )$ leads to an autonomous equation of the form 5.2.6.68, whose order can be reduced by two.

15. $^{(\ )} = \ ^{-\ -k} f(\ ^{k})$.

This is a special case of equation 5.2.6.86.

1 . The transformation $= \ln \ ,\ z =$ leads to an autonomous equation of the form 5.2.6.77.

2 . The transformation $z = \ ,\ = \ '$ leads to an $(\ -1)$st-order equation.

16. $^{(\ )} = \ ^{-} \ f(\ ^{k} \ )$.

This is a special case of equation 5.2.6.89. The transformation $= \ ,\ = \ '$ leads to an $(\ -1)$st-order equation.

17. $\quad {}^{(2\;\;+1)} = f(\;\;).$

Integrating yields a 2 th-order equation:

$$2\sum_{=0}^{-1}(-1)\quad{}^{(\;)}\;{}^{(2\;-\;)}+(-1)\;\left[\;{}^{(\;)}\right]^2=2\quad(\;)\quad+\;,$$

where the notation ${}^{(0)}\equiv$ is used.

18. $\quad {}^{(\;)} = f(\;\;,\;\;).$

The transformation $\;=z^{-1},\;\;=z^{1-}\;(z)$ leads to an equation of the same form: ${}^{(\;)}=(-1)\;z^{-\;-1}\;(z^{-1},z^{1-}\;\;).$

19. $\quad {}^{(\;)} = (\quad+\quad+\;)^{1-}\;f\left(\dfrac{\;+\;\;+\;}{\;+\;\;+\gamma}\right).$

1 . For $a\beta-\quad=0$, the substitution $\;=a\;+\;\;+\;$ leads to an autonomous equation of the form 5.2.6.8.

2 . For $a\beta-\quad\neq 0$, the transformation

$$z=\;-\;{}_0,\qquad =\;-\;{}_0,$$

where ${}_0$ and ${}_0$ are the constants which are determined by solving the linear algebraic system

$$a\;{}_0+\;{}_0+\;=0,\qquad {}_0+\beta\;{}_0+\;=0,$$

leads to a homogeneous equation of the form 5.2.6.12:

$${}^{(\;)}=z^{1-}\quad\frac{}{z}\;,\qquad\text{where}\qquad(\xi)=(a+\;\xi)^{1-}\quad\frac{a+\;\xi}{\;+\beta\xi}\;.$$

20. $\quad {}^{(\;)} = (\;{}_1\;+\;{}_1\;+\;{}_1)^{1-}\;f\left(\dfrac{\;{}_2\;+\;{}_2\;+\;{}_2}{\;{}_3\;+\;{}_3\;+\;{}_3}\right).$

Let the following condition hold: $\begin{vmatrix} a_1 & {}_1 & {}_1 \\ a_2 & {}_2 & {}_2 \\ a_3 & {}_3 & {}_3 \end{vmatrix}=0.$

For $a_2\;{}_3-a_3\;{}_2\neq 0$, the transformation

$$z=\;-\;{}_0,\qquad =\;-\;{}_0,$$

where ${}_0$ and ${}_0$ are the constants determined by the linear algebraic system

$$a_2\;{}_0+\;{}_2\;{}_0+\;{}_2=0,\qquad a_3\;{}_0+\;{}_3\;{}_0+\;{}_3=0,$$

leads to a homogeneous equation of the form 5.2.6.12:

$${}^{(\;)}=z^{1-}\quad\frac{}{z}\;,\qquad\text{where}\qquad(\xi)=(a_1+\;{}_1\xi)^{1-}\quad\frac{a_2+\;{}_2\xi}{a_3+\;{}_3\xi}\;.$$

21. $\quad (\quad+\;)\;(\quad+\;)\;{}^{(\;)} = f\left(\dfrac{}{(\quad+\;)^{\;-1}}\right).$

The transformation $\xi=\ln\dfrac{a\;\;+\;}{\;+\;}\,,\quad\;=\dfrac{}{(\quad+\;)^{\;-1}}$ leads to an autonomous equation of the form 5.2.6.77.

22. $^{( )} = (\quad ^2 + \quad + \quad)^{-\frac{1+}{2}} f \quad (\quad ^2 + \quad + \quad)^{\frac{1-}{2}}$ **.**

1 . The transformation

$$= \frac{}{a \quad ^2 + \quad +}, \qquad = (a \quad ^2 + \quad + \quad)^{\frac{1-}{2}} \tag{1}$$

leads to an autonomous equation with respect to $=( )$, which admits reduction of order by the substitution $z(\ ) = \ '$.

2 . Let $= 2$ be an even integer ($= 1, 2, 3,$). In this case, transformation (1) yields an equation of the form 5.2.6.68, whose order can be reduced by two.

Setting $= a \quad ^2 + \quad + \quad , \quad = \frac{2 \quad -1}{2}$ and multiplying both sides of the original equation by $' = \quad ^{-\frac{1+2}{2}} \quad ' + \frac{1-2}{2} \quad '$, we obtain

$$' + \frac{1-2}{2} \quad ' \quad ^{(2 \quad)} = (\quad) \quad ' .$$

Integrating both sides of this equality with respect to (the left-hand side is integrated by parts), we have

$$\sum_{=0}^{-2} (-1) \quad ^{(\ )} \quad ^{(2 \quad -1- \quad)} + (-1) \quad ^{-1} \quad ^{(\quad -1)} \quad ^{(\quad +1)} = \quad (\quad) \quad + \quad , \tag{2}$$

where

$$^{(\ )} = \frac{}{} \quad ' + \frac{1-2}{2} \quad ' \quad = \quad ^{(\quad +1)} + \quad k - \quad + \frac{1}{2} \quad ' \quad ^{(\ )} + ak(k-2 \quad) \quad ^{(\quad -1)}$$

(remember that $= 2$). It can be shown that the integrand on the left-hand side of (2) is a total differential. Finally, we arrive at the first integral:

$$\sum_{=0}^{-2} (-1) \quad ^{(\quad +1)} + \big(k - \quad + \tfrac{1}{2} \quad ' \quad ^{(\ )} + ak(k-2 \quad) \quad ^{(\quad -1)} \quad ^{(2 \quad -1- \quad)}$$

$$+ (-1) \quad ^{-1} \quad \tfrac{1}{2} \quad \big[\ ^{(\ )} \big]^2 - \tfrac{1}{2} \quad ' \quad ^{(\quad -1)} \quad ^{(\ )} + a(1 - \quad ^2) \quad ^{(\quad -2)} \quad ^{(\ )} + \tfrac{1}{2} a \quad ^2 \big[\ ^{(\quad -1)} \big]^2$$

$$= \quad (\quad) \quad + \quad .$$

23. $^{( )} = \ ^{\frac{1+}{1-}} f \quad (\quad ^2 + \quad + \quad)^{\frac{1-}{2}}$ **.**

1 . Setting $(\ ) = \ ^{\frac{+1}{-1}} \ _1(\ )$, we have equation 5.2.6.22 with the function $_1$ (instead of).

2 . The transformation $= z^{-1}, \quad = z^{1-} \quad (z)$ leads to an equation of similar form: $^{( )} = (-1) \quad ^{\frac{1+}{1-}} \quad (\ z^2 + \ z + a)^{\frac{1-}{2}}$.

24. $^{( )} =$ ***f*** $(\quad ^{-} \quad)$**.**

The substitution $(\ ) = \ e^{-}$ leads to an autonomous equation of the form 5.2.6.77.

25. $^{( )} =$ ***f*** $(\qquad)$**.**

The transformation $z = e \quad , \quad (z) = \quad '$ leads to an $(\quad - 1)$st-order equation.

26. $^{( )} = \ ^{-}$ ***f*** $(\qquad)$**.**

The transformation $z = \quad e \quad , \quad (z) = \quad '$ leads to an $(\quad - 1)$st-order equation.

27. $^{(\ )} = f(\ \ +\ \ ^{\lambda}\ ) -\ \ ^{\lambda}$ **.**

The substitution $(\ ) = \ + ae^{\lambda}$ leads to an autonomous equation of the form 5.2.6.8: $^{(\ )} = \ (\ )$.

28. $^{(2\ \ )} = f(\ \ +\ \ \cosh\ \ ) -\ \ \cosh$ **.**

The substitution $(\ ) = \ + a\cosh$ leads to an autonomous equation of the form 5.2.6.8: $^{(2\ \ )} = \ (\ )$.

29. $^{(2\ \ )} = f(\ \ +\ \ \sinh\ \ ) -\ \ \sinh$ **.**

The substitution $(\ ) = \ + a\sinh$ leads to an autonomous equation of the form 5.2.6.8: $^{(2\ \ )} = \ (\ )$.

30. $^{(2\ \ +1)} = f(\ \ +\ \ \cosh\ \ ) -\ \ \sinh$ **.**

The substitution $(\ ) = \ + a\cosh$ leads to an autonomous equation of the form 5.2.6.8: $^{(2\ \ +1)} = \ (\ )$.

31. $^{(2\ \ +1)} = f(\ \ +\ \ \sinh\ \ ) -\ \ \cosh$ **.**

The substitution $(\ ) = \ + a\sinh$ leads to an autonomous equation of the form 5.2.6.8: $^{(2\ \ +1)} = \ (\ )$.

32. $^{(\ )} = f(\ \ +\ \ \cos\ \ ) -\ \ \cos(\ \ + \tfrac{1}{2}$ **.**

The substitution $(\ ) = \ + a\cos$ leads to an autonomous equation of the form 5.2.6.8: $^{(\ )} = \ (\ )$.

33. $^{(\ )} = f(\ \ +\ \ \sin\ \ ) -\ \ \sin(\ \ + \tfrac{1}{2}$ **.**

The substitution $(\ ) = \ + a\sin$ leads to an autonomous equation of the form 5.2.6.8: $^{(\ )} = \ (\ )$.

5.2.6-3. Equations of the form $^{(\ )} = \ (\ ,\ ,\ ')$.

34. $^{(\ )} = f(\ )\ \ +\ (\ )$**.**

Integrating yields an $(\ -1)$st-order equation: $^{(\ -1)} = \ \ (\ )\ \ +\ \ g(\ )\ \ +\ \ .$

35. $^{(\ )} = f(\ ,\ \ )$**.**

The substitution $(\ ) = \ '$ leads to an $(\ -1)$st-order equation: $^{(\ -1)} = \ (\ ,\ \ )$.

36. $^{(\ )} = f(\ ,\ \ )$**.**

Autonomous equation. This is a special case of equation 5.2.6.77.

The substitution $(\ ) = \ '$ leads to an $(\ -1)$st-order equation.

37. $^{(\ )} = \ \ ^{-}\ f(\ \ \ \ )$**.**

The transformation $z = \ \ '\ \ ,\ \ =\ \ ^{2}\ ''$ leads to an $(\ -2)$nd-order equation.

38. $^{(\ )} = \ \ \ + f(\ ,\ \ -\ \ )$**.**

The substitution $=\ ' - a$ leads to an $(\ -1)$st-order equation:

$$^{(\ -1)} + a\ ^{(\ -2)} + \ \ \ + a\ ^{-1}\ \ = \ (\ ,\ \ ).$$

39. $^{(\ )} = f(\ ,\ \ \ \ -\ \ )$**.**

The substitution $=\ ' -$ leads to an $(\ -1)$st-order equation: $\frac{-2}{-2}\ \frac{'}{}\ = \ (\ ,\ \ )$.

40. $^{(\ )} = f(\ ,\quad -\quad .$

Here, is a positive integer and $\geq\ +1$. The substitution $=\ '-$ leads to an $(\ -1)$st-order equation: $^{(\ -\ -1)} = (\ ,\ )$, where $=\ ^{(\ )}$.

41. $^{(\ )} = f(\ ,\quad +\quad) - (\ )\quad .$

Here, $(a)\ = a(a+1)\quad (a+\ -1)$ is the Pochhammer symbol. The substitution $=\ '+a$ leads to an $(\ -1)$st-order equation.

42. $^{(\ )} = f(\ ,\quad -\quad ,\quad = \sum_{k=0} {}_k\ ^k,\quad = \sum_{k=0} {}_k\ ^{k-1},\quad >\ .$

The substitution $=\ '-\ '$ leads to an $(\ -1)$st-order equation.

43. $^{(2\ )} =\ + f(\ ,\ \cosh\ -\ \sinh\ ).$

The substitution $=\ '\cosh\ -\ \sinh$ leads to an $(2\ -1)$st-order equation.

44. $^{(2\ )} =\ + f(\ ,\ \sinh\ -\ \cosh\ ).$

The substitution $=\ '\sinh\ -\ \cosh$ leads to an $(2\ -1)$st-order equation.

45. $^{(2\ )} = (-1)\ + f(\ ,\ \sin\ -\ \cos\ ).$

The substitution $=\ '\sin\ -\ \cos$ leads to an $(2\ -1)$st-order equation.

46. $^{(2\ )} = (-1)\ + f(\ ,\ \cos\ +\ \sin\ ).$

The substitution $=\ '\cos\ +\ \sin$ leads to an $(2\ -1)$st-order equation.

47. $^{(\ )} = \frac{(\ )}{\ } + f\ ,\ -\frac{\ }{\ }\ ,\quad = (\ ).$

The substitution $=\ '-\frac{'}{\ }$ leads to an $(\ -1)$st-order equation.

5.2.6-4. Equations of the form $^{(\ )} = (\ ,\ ,\ ',\ '')$.

48. $^{(\ )} = f(\ ,\quad -\ ,\quad).$

This is a special case of equation 5.2.6.78. The substitution $(\ ) =\ '-$ leads to an $(\ -1)$st-order equation.

49. $^{(\ )} = f(\ ,\ ^2\quad -2\quad +2\ ).$

This is a special case of equation 5.2.6.81. The substitution $(\ ) =\ ^2\ ''\ -2\ '+2$ leads to an $(\ -2)$nd-order equation.

50. $^{(2\ )} =\quad + f(\ ,\quad -\quad).$

The substitution $=\ ''-a$ leads to an $(2\ -2)$nd-order equation:

$$^{(2\ -2)} + a\ ^{(2\ -4)} +\quad + a\ ^{-1}\ = (\ ,\ ).$$

51. $^{(2\ )} =\ f(\quad -\ ^2\ .$

This is a special case of equation 5.2.6.52.

52. $^{(2\ )} = \quad f(\quad - \ ^2 + \quad (\quad - \ ^2 .$

1 . Particular solution:

$$= \ _1 \exp(\ _3 \ ) + \ _2 \exp(- \ _3 \ ),$$

where the constants $_1$, $_2$, and $_2$ are related by the constraint

$$\ ^2_3 - \ ^2_3 \ (4 \ _1 \ _2 \ ^2_3) - g(4 \ _1 \ _2 \ ^2_3) = 0.$$

2 . Particular solution:

$$= \ _1 \cos(\ _3 \ ) + \ _2 \sin(\ _3 \ ),$$

where the constants $_1$, $_2$, and $_2$ are related by the constraint

$$(-1) \quad ^2_3 + \ ^2_3 \ (- \ ^2_1 \ ^2_3 - \ ^2_2 \ ^2_3) - g(- \ ^2_1 \ ^2_3 - \ ^2_2 \ ^2_3) = 0.$$

53. $^{(\ )} = \quad f \ \text{——}, \quad - \ \text{——} \ .$

Particular solution: $= \ _1 \exp(\ _2 \ ) + \ _3$, where $_1$ is an arbitrary constant and the constants $_2$ and $_2$ are related by the constraint $_2^{-1} = \ (\ _2, - \ _2 \ _3)$.

$\boxed{\text{5.2.6-5. Equations of the form } \ (\ , \ ) \ ^{(\ )} + g(\ , \ , \ ') \ ^{(\ -1)} = \ (\ , \ , \ ', \ \ , \ ^{(\ -2)} \ .}$

54. $^{(\ )} = \quad - \ (\quad)^2 + f(\ ).$

Integrating the equation two times, we obtain an ($-$ 2)nd-order equation:

$$^{(\ -2)} = a \quad '' - a(\ ')^2 + \ _1 \ + \ _2 + \ _0 \ (\ - \ ) \ (\ ) \ , \quad \text{where } _0 \text{ is an arbitrary number.}$$

55. $^{(2\ )} = \ ^2 + f(\ , \ ^{(\ )} + \quad .$

The substitution $= \ ^{(\ )} + a$ leads to an th-order equation: $^{(\ )} = a \ + \ (\ , \ )$.

56. $^{(\ )} = f(\ ^{(\ -2)} \ .$

Having set $(\ ) = \ ^{(\ -2)}$, we obtain a second-order equation $'' = \ (\ )$, whose solution has the form:

$$= \frac{}{(\ )} + \ _2, \quad \text{where} \quad (\ ) = \ _1 + 2 \ (\ ) \ ^{1\ 2} .$$

Expressing in terms of and integrating the resulting relation $- 2$ times, we find .

Solution in parametric form:

$$= \ _2 \frac{}{(\ )}, \quad = \ _3 \frac{1}{(\ _1)} \ ^1_4 \frac{2}{(\ _2)} \quad ^{-4}_{-1} \frac{-3}{(\ -3)} \quad ^{-3} \frac{-2 \quad -2}{(\ -2)}.$$

57. $^{(\ )} = f(\ ) \quad ^{(\ -1)}.$

Integrating yields an ($-$ 1)st-order autonomous equation of the form 5.2.6.8:

$$^{(\ -1)} = \ (\ ), \quad \text{where} \quad (\ ) = \ \exp \quad (\ ) \quad .$$

58. $^{(\ )} = \left[f(\ ) \quad + \ (\ ) \right] \ ^{(\ -1)}.$

Integrating yields an ($-$ 1)st-order equation: $^{(\ -1)} = \ \exp \quad (\ ) \quad + \ g(\ ) \quad .$

59. $^{(\ )} + \quad ^{(\ -1)} = f(\quad).$

The substitution $(\ ) =$ leads to an autonomous equation of the form 5.2.6.8: $^{(\ )} = \ (\ )$.

60. $\;{}^{(\ )} + (\ \ + \ \ - 1)\ {}^{(\ -1)} = f(\ ,\ \ + \ \ ).$

The substitution $\ = \ {}' + a$ leads to an $(\ -1)$st-order equation: $\ {}^{(\ -1)} = \ (\ ,\ \ )$.

61. $\;{}^{2}\ {}^{(\ )} + 2\ \ {}^{(\ -1)} + \ (\ \ - 1)\ {}^{(\ -2)} = f(\ {}^{2}\ ).$

The substitution $\ (\ ) = \ {}^{2}$ leads to an autonomous equation of the form 5.2.6.8: $\ {}^{(\ )} = \ (\ )$.

62. $(2\ \ - 1)\ \ {}^{(2\ +1)} + (2\ \ + 1)\ \ {}^{(2\ )} = f(\ ).$

Having integrated the equation, we obtain

$$(2\ \ - 1)\ \ {}^{(2\ )} + 2 \sum_{=1}^{-1} (-1)^{\ +1}\ {}^{(\ )}\ {}^{(2\ -\ )} + (-1)^{\ +1}\left[\ {}^{(\ )}\right]^2 = \int \ (\ )\ \ + 2\ {}_2.$$

The second integration leads to an $(2\ \ - 1)$st-order equation:

$$\sum_{=0}^{-1} (2\ \ - 1 - 2k)(-1)\ {}^{(\ )}\ {}^{(2\ -1-\ )} = 2\ {}_2\ \ + \ {}_1 + \int_0 (\ -\ )\ (\ )\ \ .$$

The third integration leads to an $(2\ \ - 2)$nd-order equation:

$$\sum_{=0}^{-2} (k+1)(2\ \ - k - 1)(-1)\ {}^{(\ )}\ {}^{(2\ -2-\ )} + \frac{1}{2}(-1)^{\ -1}\ {}^{2}\left[\ {}^{(\ -1)}\right]^2$$
$$= \ {}_2\ {}^{2} + \ {}_1\ \ + \ {}_0 + \frac{1}{2}\int_0 (\ -\ )^2\ (\ )\ \ .$$

63. $(2\ \ - 1)\ \ {}^{(2\ +1)} + (2\ \ + 1)\ \ {}^{(2\ )} = f(\ )\ \ + \ (\ ).$

Integrating yields an $(\ -1)$st-order equation:

$$(2\ \ - 1)\ \ {}^{(2\ )} + 2 \sum_{=1}^{-1} (-1)^{\ +1}\ {}^{(\ )}\ {}^{(2\ -\ )} + (-1)^{\ +1}\left[\ {}^{(\ )}\right]^2 = \int \ (\ )\ \ + \int g(\ )\ \ + \ .$$

64. $\;{}^{(\ )} - \ \ {}^{(\ -1)} = f(\ )\ {}^{2}.$

Integrating yields an $(\ -1)$st-order linear equation: $\ {}^{(\ -1)} = \int \ (\ )\ \ + \ \ .$

65. $\;{}^{(\ )} = \ \ {}^{(\ -1)} + f(\ )\ \ {}^{(\ -1)}.$

Integrating yields an $(\ -1)$st-order linear equation: $\ {}^{(\ -1)} = \ \exp\ \int \ (\ )\ \ .$

66. $\;{}^{(\ )} + (f - 1)\ \ {}^{(\ -1)} + f\ \ \ + \ \ {}^{2} = 0,\qquad f = f(\ ),\qquad = \ (\ ).$

This equation is solved by the functions that are solutions of the $(\ -1)$st-order linear equation $\ {}^{(\ -1)} + g(\ )\ \ = 0$.

67. $[\ \ + f(\ )]\ {}^{(\ )} = [\ \ + f\ (\ )]\ {}^{(\ -1)} + \ f(\ )\ \ - \ f\ (\ )\ .$

Integrating yields an $(\ -1)$st-order constant coefficient nonhomogeneous linear equation: $\ {}^{(\ -1)} - \ \ = (\ \ - a)\ (\ )$. There is also the trivial solution $\ = 0$.

68. $\displaystyle\sum_{=1}\ \ {}^{(2\ )} = f(\ ).$

The first integral has the form:

$$\sum_{=1} a\ \sum_{=1}^{-1} (-1)\ {}^{(\ )}\ {}^{(2\ -\ )} + \frac{1}{2}(-1)\ \left[\ {}^{(\ )}\right]^2 + \int \ (\ )\ \ = \ ,$$

where $\ $ is an arbitrary constant. Furthermore, the order of the obtained equation can be reduced by one by the substitution $\ (\ ) = \ {}'$.

69. $\quad {}^{(\ )} = f(\ ).$
$\quad_{=1}$

The substitution $= \ln|\ |$ leads to an autonomous equation of the form 5.2.6.77.

70. $\quad {}^{(2\ \ +1)} = f(\ ).$
$\quad_{=0}$

Integrating yields a 2 th-order equation:

$$a_{=0} \quad 2 \sum_{=0}^{-1} (-1) \ {}^{(\ )} \ {}^{(2\ \ -\ )} + (-1) \ \left[{}^{(\ )} \right]^2 = 2 \quad (\ ) \quad + \ ,$$

where ${}^{(0)}$ stands for .

71. $\quad {}^{(\ )} \ {}^{(2\ \ +1-\ \ )} = f(\ ).$
$\quad_{=0}$

The first integral has the form:

$$2 \sum_{=0}^{-1} A \quad {}^{(\ )} \ {}^{(2\ -\ \ )} + A \ \left[{}^{(\ )} \right]^2 = 2 \quad (\ ) \quad + \ ,$$

where $A \ = \sum_{=0} (-1)^{\ +} \ a \ = a \ - a_{\ -1} + a_{\ -2} -$. If the condition $A \ = 2 \sum_{=0}^{-1} (-1)^{\ -1+} \ A$ is satisfied, the obtained equation can be integrated two times more (see, in particular, equation 5.2.6.62).

5.2.6-6. Equations of the form ${}^{(\ )} = \ (\ ,\ ,\ ',\ \ ,\ {}^{(\ -1)}).$

72. $\quad {}^{(\ )} = f\big({}^{(\ -1)}\big).$

Having set $(\ ) = {}^{(\ -1)}$, we obtain a first-order equation $' = \ (\ )$. Further, find from the relation $= \dfrac{}{(\ )} + \ _1$. Then the $(\ \ -1)$-fold integration yields .

Solution in parametric form:

$$= \int \frac{}{_1 \ (\ )}, \qquad = \int \frac{1}{_2 \ (\ _1)} \int^{1} \frac{2}{_3 \ (\ _2)} \cdots \int^{-3} \frac{-2}{_{-1} \ (\ \ _{-2})} \int^{-2} \frac{-1 \quad -1}{(\ \ _{-1})}.$$

73. $\quad {}^{(\ )} = f(\ ) \quad \big({}^{(\ -1)}\big).$

Integrating yields an $(\ \ -1)$st-order equation:

$$\int \frac{}{g(\ )} = \int (\ ) \quad + \ , \qquad \text{where} \qquad = {}^{(\ -1)}.$$

Furthermore, the order of this equation can be reduced by one by the substitution $z(\ ) = \ '$.

74. $\quad {}^{(\ )} = \big[f(\ ) \quad + \ (\ )\big] \ \big({}^{(\ -1)}\big).$

Integrating yields an $(\ \ -1)$st-order equation:

$$\int \frac{}{(\ )} = \int (\ ) \quad + \int g(\ ) \quad + \ , \qquad = {}^{(\ -1)}.$$

75. $\quad {}^{(\ )} = f\big(\ ,\ {}^{(\ -2)},\ {}^{(\ -1)}\big).$

The substitution $(\ ) = {}^{(\ -2)}$ leads to a second-order equation: $'' = \ (\ ,\ ,\ ')$.

5.2.6-7. Equations of the general form $(, , ', \quad , ^{( )} = 0.$

76. $(\; , \quad , \quad , \dots , \; ^{( )} \; = 0.$

The equation does not depend on explicitly. Hence, the substitution $( ) = \; '$ leads to an $(\; - 1)$st-order equation:

$$(\; , \; , \; ', \; , \; ^{(-1)} = 0.$$

77. $(\; , \quad , \quad , \dots , \; ^{( )} \; = 0.$

Autonomous equation. It does not depend on explicitly. The substitution $( ) = \; '$ leads to an $(\; - 1)$st-order equation. The derivatives of the original equation and the transformed one are related by

$$'' = \; ', \qquad ''' = \; ^2 \; '' + (\; ')^2, \qquad , \qquad ^{( )} = (\; ^{(-1)} \;)'.$$

78. $(\; , \quad - \; , \quad , \quad , \dots , \; ^{( )} \; = 0.$

The substitution $( ) = \; ' - $ leads to an $(\; - 1)$st-order equation:

$$(\; , \; , \; , \; ', \; , \; ^{(-2)} = 0, \qquad \text{where} \qquad = \; ' \; .$$

79. $(\; , \quad - 2 \; , \quad , \quad , \dots , \; ^{( )} \; = 0.$

The substitution $= \; ' - 2$ leads to an $(\; - 1)$st-order equation:

$$(\; , \; , \; , \; ', \; , \; ^{(-3)} = 0, \qquad \text{where} \qquad = \; '' \; .$$

80. $(\; , \quad - \quad , \; ^{(+1)}, \; ^{(+2)}, \dots , \; ^{( )} \; = 0, \qquad \geq \; + 1.$

The substitution $= \; ' - $ leads to an $(\; - 1)$st-order equation:

$$(\; , \; , \; , \; ', \; , \; ^{(- -1)} = 0, \qquad \text{where} \qquad = \; ^{( )} \; .$$

81. $(\; , \; ^2 \quad - 2 \quad + 2 \; , \quad , \dots , \; ^{( )} \; = 0.$

The substitution $( ) = \; ^2 \; '' - 2 \; ' + 2$ leads to an $(\; - 2)$nd-order equation:

$$(\; , \; , \; , \; ', \; , \; ^{(-3)} = 0, \qquad \text{where} \qquad = \; ^{-2} \; '.$$

82. $\sum_{k=0} (-1)^k \; ! \; ^k \quad ^{-k} \; ^{(-k)} = (\; , \; ^{(+1)}, \dots , \; ^{( )}).$

Here, $= \dfrac{!}{k!\,(\; - k)!}$ are binomial coefficients.

The substitution $( ) = \sum_{=0} (-1) \; k! \quad ^{-} \; ^{(-)}$ leads to an $(\; - \;)$th-order equation; the derivatives on the right-hand side are calculated in consecutive manner using the formula $^{(+1)} = \; ^{-} \; '$.

83. $\; —, \quad , \quad , \dots , \; ^{-1} \; ^{( )} \; = 0.$

Homogeneous equation. The transformation $= \ln \; , \quad =$ leads to an autonomous equation of the form 5.2.6.77.

84. $\displaystyle \frac{+\quad+}{+\quad+\gamma},\quad ,\ldots,(\quad+\quad+\ )^{-1}\ (\ )$ **= 0.**

1 . For $a\beta -\quad = 0$, the substitution $\quad = a\quad +\quad +$ leads to an autonomous equation of the form 5.2.6.77.

2 . For $a\beta -\quad \neq 0$, the transformation

$$z = \quad - \quad _0, \qquad = \quad - \quad _0,$$

where $_0$ and $_0$ are the constants determined by the linear algebraic system

$$a\quad _0 + \quad _0 + \quad = 0, \qquad _0 + \beta\quad _0 + \quad = 0,$$

leads to a homogeneous equation of the form 5.2.6.83:

$$\frac{a + \quad z}{+\beta\quad z},\quad ',\quad ,(a + \quad z)^{-1} z^{-1}\ (\ )\quad = 0.$$

85. $\displaystyle \frac{_1 + _1 + _1}{_2 + _2 + _2},\quad ,\ldots,(\ _3 + \ _3 + \ _3)^{-1}\ (\ )$ **= 0.**

Let the following condition hold: $\begin{vmatrix} a_1 & _1 & _1 \\ a_2 & _2 & _2 \\ a_3 & _3 & _3 \end{vmatrix} = 0.$

For $a_1\ _2 - a_2\ _1 \neq 0$, the transformation

$$z = \quad - \quad _0, \qquad = \quad - \quad _0,$$

where $_0$ and $_0$ are the constants determined by the linear algebraic system

$$a_1\ _0 + \ _1\ _0 + \ _1 = 0, \qquad a_2\ _0 + \ _2\ _0 + \ _2 = 0,$$

leads to a homogeneous equation of the form 5.2.6.83:

$$\frac{a_1 + \ _1\quad z}{a_2 + \ _2\quad z},\quad ',\quad ,(a_3 + \ _3\quad z)^{-1} z^{-1}\ (\ )\quad = 0.$$

86. $(\ ^k,\quad ^{k+1}\quad ,\ldots,\quad ^{k+}\quad (\ )$ **= 0.**

Generalized homogeneous equation. The transformation $\quad = \ln\quad ,\quad =$ leads to an autonomous equation of the form 5.2.6.77.

87. $\displaystyle \frac{}{},\ \frac{^2}{},\ldots,\ \frac{(\ )}{}\Big)$ **= 0.**

Generalized homogeneous equation. The transformation $z = \quad '\quad ,\quad = \ ^2\ ''$ leads to an $(\quad - 2)$nd-order equation.

88. $\displaystyle -\ \frac{}{},\ \frac{}{},\ \frac{}{},\ldots,\ \frac{(\ )}{}\Big)$ **= 0.**

Autonomous equation. Particular solution: $\quad = \ _1 \exp(\ _2\ ) + \ _3$, where $_1$ is an arbitrary constant and the constants $_2$ and $_3$ are related by $(-\ _2\ _3,\ _2,\ _2^2,\quad ,\ _2^{\,-1}) = 0$.

89. $\displaystyle ^k\quad ,\ \frac{}{},\ \frac{^2}{},\ldots,\ \frac{(\ )}{}\Big)$ **= 0.**

Generalized homogeneous equation. The transformation $\quad = \quad ,\ z = \quad '$ leads to an $(\quad - 1)$st-order equation.

90. $\dfrac{(\ )}{\quad},\ \dfrac{(\ )}{\quad} - \ (\ -1)\Big) = 0.$

A solution of this equation is any function that satisfies the (-1)st-order constant coefficient linear equation $^{(\ -1)} = \ _1 + \ _2$, where the constants $_1$ and $_2$ are related by the constraint ($_1, -\ _2) = 0$.

91. $\dfrac{(\ )}{(k)},\ ^{1-k}\ \dfrac{(\ )}{(k)} - \ ^{1-k}\ (\ -k)\Big) = 0, \quad > \ .$

A solution of this equation is any function that satisfies the ($-k$)th-order linear equation $^{(\ -\ )} = \ _1 + \ _2\ \ ^{-1}$, where the constants $_1$ and $_2$ are related by ($_1, -\ _2) = 0$.

92. $(\ ,\ ^{(\ )} -\ ,\ ^{(\ )} - \quad = 0.$

The substitution $= \ ' -$ reduces the order of the equation by one.

93. $(\ ^{(\ )} -\ ,\ ^{(2\ )} -\ ,\ ^{(2\ )} -\ ^{(\ )} \quad = 0.$

The substitution $= \ ^{(\ )} -$ leads to an th-order autonomous equation $(\ ,\ ^{(\ )} +\ ,\ ^{(\ )} = 0$.

94. $(\ ,\ ^{(\ )} + \quad ,\ ^{(2\ )} - \ ^{2}\ ,\ ^{(2\ )} + \quad ^{(\ )} \quad = 0.$

The substitution $= \ ^{(\ )} + a$ leads to an th-order equation $(\ ,\ ,\ ^{(\ )} - a\ ,\ ^{(\ )} \quad = 0$.

95. $(\quad ,\quad ,\quad ,\ \ldots,\quad ^{(\ )} \quad = 0.$

Equation invariant under “translation–dilatation” transformation. The substitution $= e$ leads to an autonomous equation of the form 5.2.6.77.

96. $,\ \dfrac{}{\quad},\ \dfrac{}{\quad},\ \ldots,\ \dfrac{(\ )}{\quad}\Big) = 0.$

Equation invariant under “translation–dilatation” transformation. The transformation $z = e \quad ,\ = \ '$ leads to an (-1)st-order equation. See also Paragraph 0.5.2-7.

97. $(\quad ,\quad ,\ ^{2}\quad ,\ \ldots,\quad ^{(\ )} \quad = 0.$

Equation invariant under “dilatation–translation” transformation. The transformation $z = \quad e \quad ,\ = \ '$ leads to an (-1)st-order equation. See also Paragraph 0.5.2-8.

Supplements

S.1. Elementary Functions and Their Properties

Throughout Section S.1 it is assumed that   is a positive integer unless otherwise specified.

S.1.1. Trigonometric Functions

S.1.1-1. Simplest relations.

$$
\begin{aligned}
&\sin^2 \quad + \cos^2 \quad = 1, && \tan \quad \cot \quad = 1,\\
&\sin(- \quad) = -\sin \quad, && \cos(- \quad) = \cos \quad,\\
&\tan \quad = \frac{\sin}{\cos}, && \cot \quad = \frac{\cos}{\sin},\\
&\tan(- \quad) = -\tan \quad, && \cot(- \quad) = -\cot \quad,\\
&1 + \tan^2 \quad = \frac{1}{\cos^2}, && 1 + \cot^2 \quad = \frac{1}{\sin^2}.
\end{aligned}
$$

S.1.1-2. Relations between trigonometric functions of single argument.

$$
\begin{aligned}
\sin \quad &= \quad \overline{1-\cos^2} \quad = \frac{\tan}{\overline{1+\tan^2}} = \frac{1}{\overline{1+\cot^2}},\\
\cos \quad &= \quad \overline{1-\sin^2} \quad = \frac{1}{\overline{1+\tan^2}} = \frac{\cot}{\overline{1+\cot^2}},\\
\tan \quad &= \frac{\sin}{\overline{1-\sin^2}} = \frac{\overline{1-\cos^2}}{\cos} = \frac{1}{\cot},\\
\cot \quad &= \frac{\overline{1-\sin^2}}{\sin} = \frac{\cos}{\overline{1-\cos^2}} = \frac{1}{\tan}.
\end{aligned}
$$

S.1.1-3. Reduction formulas.

$$
\begin{aligned}
&\sin(\qquad) = (-1) \quad \sin \quad, && \cos(\qquad) = (-1) \quad \cos \quad,\\
&\sin \quad \frac{2 \quad +1}{2} \quad = (-1) \quad \cos \quad, && \cos \quad \frac{2 \quad +1}{2} \quad = \mp(-1) \quad \sin \quad,\\
&\tan(\qquad) = \tan \quad, && \cot(\qquad) = \cot \quad,\\
&\tan \quad \frac{2 \quad +1}{2} \quad = -\cot \quad, && \cot \quad \frac{2 \quad +1}{2} \quad = -\tan \quad,\\
&\sin \quad \frac{\quad}{4} \quad = \frac{\overline{2}}{2}(\sin \qquad \cos \quad), && \cos \quad \frac{\quad}{4} \quad = \frac{\overline{2}}{2}(\cos \quad \mp \sin \quad),\\
&\tan \quad \frac{\quad}{4} \quad = \frac{\tan \qquad 1}{1 \mp \tan}, && \cot \quad \frac{\quad}{4} \quad = \frac{\cot \quad \mp 1}{1 \quad \cot}.
\end{aligned}
$$

S.1.1-4. Addition and subtraction of trigonometric functions.

$$\sin\quad +\sin\quad = 2\sin\frac{\quad+\quad}{2}\cos\frac{\quad-\quad}{2},$$
$$\sin\quad -\sin\quad = 2\sin\frac{\quad-\quad}{2}\cos\frac{\quad+\quad}{2},$$
$$\cos\quad +\cos\quad = 2\cos\frac{\quad+\quad}{2}\cos\frac{\quad-\quad}{2},$$
$$\cos\quad -\cos\quad = -2\sin\frac{\quad+\quad}{2}\sin\frac{\quad-\quad}{2},$$
$$\sin^2\quad -\sin^2\quad = \cos^2\quad -\cos^2\quad = \sin(\quad+\quad)\sin(\quad-\quad),$$
$$\sin^2\quad -\cos^2\quad = -\cos(\quad+\quad)\cos(\quad-\quad),$$
$$\tan\quad\quad\tan\quad = \frac{\sin(\quad\quad)}{\cos\quad\cos\quad},\quad \cot\quad\quad\cot\quad = \frac{\sin(\quad\quad)}{\sin\quad\sin\quad},$$
$$a\cos\quad+\quad\sin\quad = \quad\sin(\quad+\quad) = \quad\cos(\quad-\quad).$$

Here, $\quad = \sqrt{a^2+\quad^2}$, $\sin\quad = a\quad$, $\cos\quad = \quad$, $\sin\quad = \quad$, and $\cos\quad = a\quad$.

S.1.1-5. Products of trigonometric functions.

$$\sin\quad\sin\quad = \tfrac{1}{2}[\cos(\quad-\quad)-\cos(\quad+\quad)],$$
$$\cos\quad\cos\quad = \tfrac{1}{2}[\cos(\quad-\quad)+\cos(\quad+\quad)],$$
$$\sin\quad\cos\quad = \tfrac{1}{2}[\sin(\quad-\quad)+\sin(\quad+\quad)].$$

S.1.1-6. Powers of trigonometric functions.

$$\cos^2\quad = \tfrac{1}{2}\cos 2\quad + \tfrac{1}{2}, \qquad \sin^2\quad = -\tfrac{1}{2}\cos 2\quad + \tfrac{1}{2},$$
$$\cos^3\quad = \tfrac{1}{4}\cos 3\quad + \tfrac{3}{4}\cos\quad, \qquad \sin^3\quad = -\tfrac{1}{4}\sin 3\quad + \tfrac{3}{4}\sin\quad,$$
$$\cos^4\quad = \tfrac{1}{8}\cos 4\quad + \tfrac{1}{2}\cos 2\quad + \tfrac{3}{8}, \qquad \sin^4\quad = \tfrac{1}{8}\cos 4\quad - \tfrac{1}{2}\cos 2\quad + \tfrac{3}{8},$$
$$\cos^5\quad = \tfrac{1}{16}\cos 5\quad + \tfrac{5}{16}\cos 3\quad + \tfrac{5}{8}\cos\quad, \qquad \sin^5\quad = \tfrac{1}{16}\sin 5\quad - \tfrac{5}{16}\sin 3\quad + \tfrac{5}{8}\sin\quad,$$
$$\cos^{2\quad}\quad = \frac{1}{2^{2\quad-1}}\ {}^{\quad-1}_{\quad=0}\ {}_{2\quad}\cos[2(\quad-k)\quad] + \frac{1}{2^{2\quad}}\ {}_{2\quad},$$
$$\cos^{2\quad+1}\quad = \frac{1}{2^{2\quad}}\ {}_{\quad=0}\ {}_{2\quad+1}\cos[(2\quad-2k+1)\quad],$$
$$\sin^{2\quad}\quad = \frac{1}{2^{2\quad-1}}\ {}^{\quad-1}_{\quad=0}(-1)^{\quad-\quad}\ {}_{2\quad}\cos[2(\quad-k)\quad] + \frac{1}{2^{2\quad}}\ {}_{2\quad},$$
$$\sin^{2\quad+1}\quad = \frac{1}{2^{2\quad}}\ {}_{\quad=0}(-1)^{\quad-\quad}\ {}_{2\quad+1}\sin[(2\quad-2k+1)\quad].$$

Here, $\quad = \dfrac{\quad!}{k!\,(\quad-k)!}$ are binomial coefficients $(0! = 1)$.

S.1.1-7. Addition formulas.

$$\sin(\quad) = \sin\quad\cos\qquad\cos\quad\sin\quad, \qquad \cos(\quad) = \cos\quad\cos\quad\mp\sin\quad\sin\quad,$$

$$\tan(\quad) = \frac{\tan\qquad\tan}{1\mp\tan\quad\tan}, \qquad \cot(\quad) = \frac{1\mp\tan\quad\tan}{\tan\qquad\tan}.$$

S.1.1-8. Trigonometric functions of multiple arguments.

$$\cos 2\quad = 2\cos^2\quad - 1 = 1 - 2\sin^2\quad, \qquad \sin 2\quad = 2\sin\quad\cos\quad,$$

$$\cos 3\quad = -3\cos\quad + 4\cos^3\quad, \qquad \sin 3\quad = 3\sin\quad - 4\sin^3\quad,$$

$$\cos 4\quad = 1 - 8\cos^2\quad + 8\cos^4\quad, \qquad \sin 4\quad = 4\cos\quad(\sin\quad - 2\sin^3\quad),$$

$$\cos 5\quad = 5\cos\quad - 20\cos^3\quad + 16\cos^5\quad, \qquad \sin 5\quad = 5\sin\quad - 20\sin^3\quad + 16\sin^5\quad,$$

$$\cos(2\quad) = 1 + \sum_{=1} (-1) \frac{{}^2(\,{}^2-1)\quad[\,{}^2-(k-1)^2]}{(2k)!} 4\quad \sin^2\quad,$$

$$\cos[(2\quad+1)\quad] = \cos\qquad 1 + \sum_{=1} (-1) \frac{[(2\quad+1)^2-1][(2\quad+1)^2-3^2]\quad[(2\quad+1)^2-(2k-1)^2]}{(2k)!} \sin^2\qquad,$$

$$\sin(2\quad) = 2\quad\cos\quad\sin\quad + \sum_{=1} (-4) \frac{(\,{}^2-1)(\,{}^2-2^2)\quad(\,{}^2-k^2)}{(2k-1)!} \sin^{2\quad-1}\quad,$$

$$\sin[(2\quad+1)\quad] = (2\quad+1)\quad\sin\quad + \sum_{=1} (-1) \frac{[(2\quad+1)^2-1][(2\quad+1)^2-3^2]\quad[(2\quad+1)^2-(2k-1)^2]}{(2k+1)!} \sin^{2\quad+1}\quad,$$

$$\tan 2\quad = \frac{2\tan}{1-\tan^2}, \qquad \tan 3\quad = \frac{3\tan\quad - \tan^3}{1-3\tan^2}, \qquad \tan 4\quad = \frac{4\tan\quad - 4\tan^3}{1-6\tan^2\quad + \tan^4}.$$

S.1.1-9. Trigonometric functions of half argument.

$$\sin^2\frac{}{2} = \frac{1-\cos}{2}, \qquad \cos^2\frac{}{2} = \frac{1+\cos}{2},$$

$$\tan\frac{}{2} = \frac{\sin}{1+\cos} = \frac{1-\cos}{\sin}, \qquad \cot\frac{}{2} = \frac{\sin}{1-\cos} = \frac{1+\cos}{\sin},$$

$$\sin\quad = \frac{2\tan\frac{}{2}}{1+\tan^2\frac{}{2}}, \qquad \cos\quad = \frac{1-\tan^2\frac{}{2}}{1+\tan^2\frac{}{2}}, \qquad \tan\quad = \frac{2\tan\frac{}{2}}{1-\tan^2\frac{}{2}}.$$

S.1.1-10. Euler and de Moivre formulas. Relationship with hyperbolic functions.

$$e^{\quad+}\quad = e\quad(\cos\quad + \quad\sin\quad), \qquad (\cos\quad + \quad\sin\quad)\quad = \cos(\quad) + \quad\sin(\quad), \qquad {}^2 = -1,$$

$$\sin(\quad) = \quad\sinh\quad, \qquad \cos(\quad) = \cosh\quad, \qquad \tan(\quad) = \quad\tanh\quad, \qquad \cot(\quad) = -\quad\coth\quad.$$

S.1.1-11. Differentiation formulas.

$$\frac{\sin}{} = \cos\quad, \qquad \frac{\cos}{} = -\sin\quad, \qquad \frac{\tan}{} = \frac{1}{\cos^2}, \qquad \frac{\cot}{} = -\frac{1}{\sin^2}.$$

S.1.1-12. Expansion into power series.

$$\cos \quad = 1 - \frac{\;^{2}}{2!} + \frac{\;^{4}}{4!} - \frac{\;^{6}}{6!} + \qquad (|\;| < \quad),$$

$$\sin \quad = \quad - \frac{\;^{3}}{3!} + \frac{\;^{5}}{5!} - \frac{\;^{7}}{7!} + \qquad (|\;| < \quad),$$

$$\tan \quad = \quad + \frac{\;^{3}}{3} + \frac{2\;^{5}}{15} + \frac{17\;^{7}}{315} + \qquad (|\;| < \quad 2),$$

$$\cot \quad = \frac{1}{\;} - \frac{\;}{3} - \frac{\;^{3}}{45} - \frac{2\;^{5}}{945} - \qquad (|\;| < \;).$$

S.1.2. Hyperbolic Functions

S.1.2-1. Definitions.

$$\sinh \quad = \frac{e \;\; - e^{-}}{2}, \quad \cosh \quad = \frac{e \;\; + e^{-}}{2}, \quad \tanh \quad = \frac{e \;\; - e^{-}}{e \;\; + e^{-}}, \quad \coth \quad = \frac{e \;\; + e^{-}}{e \;\; - e^{-}}.$$

S.1.2-2. Simplest relations.

$$\cosh^2 \quad - \sinh^2 \quad = 1, \qquad \tanh \quad \coth \quad = 1,$$

$$\sinh(- \;) = -\sinh \;, \qquad \cosh(- \;) = \cosh \;,$$

$$\tanh \quad = \frac{\sinh}{\cosh}, \qquad \coth \quad = \frac{\cosh}{\sinh},$$

$$\tanh(- \;) = -\tanh \;, \qquad \coth(- \;) = -\coth \;,$$

$$1 - \tanh^2 \quad = \frac{1}{\cosh^2}, \qquad \coth^2 \quad - 1 = \frac{1}{\sinh^2}.$$

S.1.2-3. Relations between hyperbolic functions of single argument ($\;\geq 0$).

$$\sinh \quad = \overline{\cosh^2 \quad - 1} = \frac{\tanh}{\overline{\overline{1 - \tanh^2}}} = \frac{1}{\overline{\overline{\coth^2 \quad - 1}}},$$

$$\cosh \quad = \overline{\sinh^2 \quad + 1} = \frac{1}{\overline{\overline{1 - \tanh^2}}} = \frac{\coth}{\overline{\overline{\coth^2 \quad - 1}}},$$

$$\tanh \quad = \frac{\sinh}{\overline{\overline{\sinh^2 \quad + 1}}} = \frac{\overline{\cosh^2 \quad - 1}}{\cosh} = \frac{1}{\coth},$$

$$\coth \quad = \frac{\overline{\sinh^2 \quad + 1}}{\sinh} = \frac{\cosh}{\overline{\overline{\cosh^2 \quad - 1}}} = \frac{1}{\tanh}.$$

S.1.2-4. Addition formulas.

$$\sinh(\qquad) = \sinh \quad \cosh \qquad \sinh \quad \cosh \quad , \qquad \cosh(\qquad) = \cosh \quad \cosh \qquad \sinh \quad \sinh \quad ,$$

$$\tanh(\qquad) = \frac{\tanh \qquad \tanh}{1 \quad \tanh \quad \tanh}, \qquad \coth(\qquad) = \frac{\coth \quad \coth \qquad 1}{\coth \qquad \coth}.$$

S.1.2-5. Addition and subtraction of hyperbolic functions.

$$\sinh \quad \sinh \quad = 2\sinh \frac{}{2} \quad \cosh \frac{\mp}{2} \quad ,$$
$$\cosh \quad + \cosh \quad = 2\cosh \frac{+}{2} \quad \cosh \frac{-}{2} \quad ,$$
$$\cosh \quad - \cosh \quad = 2\sinh \frac{+}{2} \quad \sinh \frac{-}{2} \quad ,$$
$$\sinh^2 \quad - \sinh^2 \quad = \cosh^2 \quad - \cosh^2 \quad = \sinh(\quad + \quad)\sinh(\quad - \quad),$$
$$\sinh^2 \quad + \cosh^2 \quad = \cosh(\quad + \quad)\cosh(\quad - \quad),$$
$$\tanh \quad \tanh \quad = \frac{\sinh(\quad)}{\cosh \quad \cosh}, \quad \coth \quad \coth \quad = \frac{\sinh(\quad)}{\sinh \quad \sinh}.$$

S.1.2-6. Products of hyperbolic functions.

$$\sinh \quad \sinh \quad = \tfrac{1}{2}[\cosh(\quad + \quad) - \cosh(\quad - \quad)],$$
$$\cosh \quad \cosh \quad = \tfrac{1}{2}[\cosh(\quad + \quad) + \cosh(\quad - \quad)],$$
$$\sinh \quad \cosh \quad = \tfrac{1}{2}[\sinh(\quad + \quad) + \sinh(\quad - \quad)].$$

S.1.2-7. Powers of hyperbolic functions.

$$\cosh^2 \quad = \tfrac{1}{2}\cosh 2 \quad + \tfrac{1}{2}, \qquad \sinh^2 \quad = \tfrac{1}{2}\cosh 2 \quad - \tfrac{1}{2},$$
$$\cosh^3 \quad = \tfrac{1}{4}\cosh 3 \quad + \tfrac{3}{4}\cosh \quad , \qquad \sinh^3 \quad = \tfrac{1}{4}\sinh 3 \quad - \tfrac{3}{4}\sinh \quad ,$$
$$\cosh^4 \quad = \tfrac{1}{8}\cosh 4 \quad + \tfrac{1}{2}\cosh 2 \quad + \tfrac{3}{8}, \qquad \sinh^4 \quad = \tfrac{1}{8}\cosh 4 \quad - \tfrac{1}{2}\cosh 2 \quad + \tfrac{3}{8},$$
$$\cosh^5 \quad = \tfrac{1}{16}\cosh 5 \quad + \tfrac{5}{16}\cosh 3 \quad + \tfrac{5}{8}\cosh \quad , \qquad \sinh^5 \quad = \tfrac{1}{16}\sinh 5 \quad - \tfrac{5}{16}\sinh 3 \quad + \tfrac{5}{8}\sinh \quad ,$$

$$\cosh^2 \quad = \frac{1}{2^{2 \quad -1}} \sum_{=0}^{-1} {}_2 \quad \cosh[2(\quad - k) \quad] + \frac{1}{2^2} \quad {}_2 \quad ,$$
$$\cosh^{2 \quad +1} \quad = \frac{1}{2^2} \sum_{=0} {}_{2 \quad +1} \cosh[(2 \quad - 2k + 1) \quad],$$
$$\sinh^2 \quad = \frac{1}{2^{2 \quad -1}} \sum_{=0}^{-1} (-1) \quad {}_2 \quad \cosh[2(\quad - k) \quad] + \frac{(-1)}{2^2} \quad {}_2 \quad ,$$
$$\sinh^{2 \quad +1} \quad = \frac{1}{2^2} \sum_{=0} (-1) \quad {}_{2 \quad +1} \sinh[(2 \quad - 2k + 1) \quad].$$

Here, are binomial coefficients.

S.1.2-8. Hyperbolic functions of multiple arguments.

$$\cosh 2 \quad = 2\cosh^2 \quad - 1, \qquad \sinh 2 \quad = 2\sinh \quad \cosh \quad ,$$
$$\cosh 3 \quad = -3\cosh \quad + 4\cosh^3 \quad , \qquad \sinh 3 \quad = 3\sinh \quad + 4\sinh^3 \quad ,$$
$$\cosh 4 \quad = 1 - 8\cosh^2 \quad + 8\cosh^4 \quad , \qquad \sinh 4 \quad = 4\cosh \quad (\sinh \quad + 2\sinh^3 \quad),$$
$$\cosh 5 \quad = 5\cosh \quad - 20\cosh^3 \quad + 16\cosh^5 \quad , \qquad \sinh 5 \quad = 5\sinh \quad + 20\sinh^3 \quad + 16\sinh^5 \quad .$$

$$\cosh(\ \ ) = 2^{\ -1}\cosh \ \ + \frac{\ }{2}\sum_{=0}^{[\ \ 2]} \frac{(-1)^{\ +1}}{k+1}\ {}^{-2}_{-\ -2} 2^{\ -2\ -2}(\cosh\ \ )^{\ -2\ -2},$$

$$\sinh(\ \ ) = \sinh \ \ \sum_{=0}^{[(\ -1)\ 2]} 2^{\ -\ -1}\ {}_{-\ -1}(\cosh\ \ )^{\ -2\ -1}.$$

Here, are binomial coefficients and $[A]$ stands for the integer part of a number A.

S.1.2-9. Relationship with trigonometric functions.

$$\sinh(\ \ ) = \ \sin\ \ , \quad \cosh(\ \ ) = \cos\ \ , \quad \tanh(\ \ ) = \ \tan\ \ , \quad \coth(\ \ ) = -\ \cot\ \ , \qquad {}^2 = -1.$$

S.1.2-10. Differentiation formulas.

$$\frac{\sinh}{\ } = \cosh\ \ , \quad \frac{\cosh}{\ } = \sinh\ \ , \quad \frac{\tanh}{\ } = \frac{1}{\cosh^2}, \quad \frac{\coth}{\ } = -\frac{1}{\sinh^2}.$$

S.1.2-11. Expansion into power series.

$$\begin{aligned}
\cosh\ \ &= 1 + \frac{\ ^2}{2!} + \frac{\ ^4}{4!} + \frac{\ ^6}{6!} + \qquad && (|\ \ | < \ \ ),\\
\sinh\ \ &= \ \ + \frac{\ ^3}{3!} + \frac{\ ^5}{5!} + \frac{\ ^7}{7!} + && (|\ \ | < \ \ ),\\
\tanh\ \ &= \ \ - \frac{\ ^3}{3} + \frac{2\ ^5}{15} - \frac{17\ ^7}{315} + && (|\ \ | < \ \ 2),\\
\coth\ \ &= \frac{1}{\ } + \frac{\ }{3} - \frac{\ ^3}{45} + \frac{2\ ^5}{945} - && (|\ \ | < \ \ ).
\end{aligned}$$

S.1.3. Inverse Trigonometric Functions

S.1.3-1. Definitions and some properties.

$$\sin(\arcsin\ \ ) = \ \ , \quad \cos(\arccos\ \ ) = \ \ ,$$
$$\tan(\arctan\ \ ) = \ \ , \quad \cot(\operatorname{arccot}\ \ ) = \ \ .$$

Principal values of inverse trigonometric functions are defined by the inequalities:

$$-\tfrac{\pi}{2} \le \arcsin\ \ \le \tfrac{\pi}{2}, \quad 0 \le \arccos\ \ \le \qquad (-1 \le \ \ \le 1),$$
$$-\tfrac{\pi}{2} < \arctan\ \ < \tfrac{\pi}{2}, \quad 0 < \operatorname{arccot}\ \ < \qquad (-\ \ < \ \ < \ \ ).$$

S.1.3-2. Simplest formulas.

$$\arcsin(-\ ) = -\arcsin\ , \qquad \arccos(-\ ) = \ -\arccos\ ,$$
$$\arctan(-\ ) = -\arctan\ , \qquad \operatorname{arccot}(-\ ) = \ -\operatorname{arccot}\ ,$$

$$\arcsin(\sin\ ) = \begin{cases} -2 & \text{if } 2\ \ -\tfrac{\pi}{2} \le \ \le 2\ \ +\tfrac{\pi}{2}, \\ -\ \ +2(\ \ +1) & \text{if } (2\ \ +1)\ -\tfrac{\pi}{2} \le \ \le 2(\ \ +1)\ \ +\tfrac{\pi}{2}, \end{cases}$$

$$\arccos(\cos\ ) = \begin{cases} -2 & \text{if } 2\ \ \le \ \le (2\ \ +1)\ , \\ -\ \ +2(\ \ +1) & \text{if } (2\ \ +1)\ \le \ \le 2(\ \ +1)\ , \end{cases}$$

$$\arctan(\tan\ ) = \ - \quad \text{if} \quad -\tfrac{\pi}{2} < \ < \ \ +\tfrac{\pi}{2},$$
$$\operatorname{arccot}(\cot\ ) = \ - \quad \text{if} \quad < \ < (\ \ +1)\ .$$

S.1.3-3. Relations between inverse trigonometric functions.

$$\arcsin\ +\arccos\ = \tfrac{\pi}{2}, \qquad \arctan\ +\operatorname{arccot}\ = \tfrac{\pi}{2};$$

$$\arcsin\ = \begin{cases} \arccos\sqrt{1-\ ^2} & \text{if } 0 \le \ \le 1, \\ -\arccos\sqrt{1-\ ^2} & \text{if } -1 \le \ \le 0, \\ \arctan\dfrac{}{\sqrt{1-\ ^2}} & \text{if } -1 < \ < 1, \\ \operatorname{arccot}\dfrac{\sqrt{1-\ ^2}}{}- & \text{if } -1 \le \ < 0; \end{cases} \qquad \arccos\ = \begin{cases} \arcsin\sqrt{1-\ ^2} & \text{if } 0 \le \ \le 1, \\ -\arcsin\sqrt{1-\ ^2} & \text{if } -1 \le \ \le 0, \\ \arctan\dfrac{\sqrt{1-\ ^2}}{} & \text{if } 0 < \ \le 1, \\ \operatorname{arccot}\dfrac{}{\sqrt{1-\ ^2}} & \text{if } -1 < \ < 1; \end{cases}$$

$$\arctan\ = \begin{cases} \arcsin\dfrac{}{\sqrt{1+\ ^2}} & \text{for any}\ , \\ \arccos\dfrac{1}{\sqrt{1+\ ^2}} & \text{if}\ \ge 0, \\ -\arccos\dfrac{1}{\sqrt{1+\ ^2}} & \text{if}\ \le 0, \\ \operatorname{arccot}\dfrac{1}{} & \text{if}\ > 0; \end{cases} \qquad \operatorname{arccot}\ = \begin{cases} \arcsin\dfrac{1}{\sqrt{1+\ ^2}} & \text{if}\ > 0, \\ -\arcsin\dfrac{1}{\sqrt{1+\ ^2}} & \text{if}\ < 0, \\ \arctan\dfrac{1}{} & \text{if}\ > 0, \\ +\arctan\dfrac{1}{} & \text{if}\ < 0. \end{cases}$$

S.1.3-4. Addition and subtraction of inverse trigonometric functions.

$$\arcsin\ +\arcsin\ = \arcsin\left(\ \sqrt{1-\ ^2}+\ \sqrt{1-\ ^2}\right) \quad \text{for}\ \ ^2+\ ^2 \le 1,$$
$$\arccos\ \ \arccos\ = \arccos\left[\ \ \mp\sqrt{(1-\ ^2)(1-\ ^2)}\right] \quad \text{for}\ \ \ \ge 0,$$
$$\arctan\ +\arctan\ = \arctan\frac{\ +\ }{1-\ } \quad \text{for}\ \ < 1,$$
$$\arctan\ -\arctan\ = \arctan\frac{\ -\ }{1+\ } \quad \text{for}\ \ > -1.$$

S.1.3-5. Differentiation formulas.

$$\frac{}{}\arcsin\ = \frac{1}{\sqrt{1-\ ^2}}, \qquad \frac{}{}\arccos\ = -\frac{1}{\sqrt{1-\ ^2}},$$
$$\frac{}{}\arctan\ = \frac{1}{1+\ ^2}, \qquad \frac{}{}\operatorname{arccot}\ = -\frac{1}{1+\ ^2}.$$

S.1.3-6. Expansion into power series.

$$\arcsin \quad = \quad + \frac{1}{2}\frac{{}^{3}}{3} + \frac{1\times 3}{2\times 4}\frac{{}^{5}}{5} + \frac{1\times 3\times 5}{2\times 4\times 6}\frac{{}^{7}}{7} + \qquad (|\ | < 1),$$

$$\arctan \quad = \quad - \frac{{}^{3}}{3} + \frac{{}^{5}}{5} - \frac{{}^{7}}{7} + \qquad (|\ | \le 1),$$

$$\arctan \quad = \frac{}{2} - \frac{1}{} + \frac{1}{3\ {}^{3}} - \frac{1}{5\ {}^{5}} + \qquad (|\ | > 1).$$

The expansions for arccos and arccot can be obtained with the aid of the formulas arccos $= \frac{\pi}{2} - \arcsin$ and arccot $= \frac{\pi}{2} - \arctan$.

S.1.4. Inverse Hyperbolic Functions

S.1.4-1. Relationships with logarithmic functions.

$$\operatorname{arcsinh} \ = \ln\big(\ + \sqrt{{}^{2}+1}\ \big), \qquad \operatorname{arctanh} \ = \frac{1}{2}\ln\frac{1+}{1-},$$

$$\operatorname{arccosh} \ = \ln\big(\ + \sqrt{{}^{2}-1}\ \big), \qquad \operatorname{arccoth} \ = \frac{1}{2}\ln\frac{1+}{-1},$$

$$\operatorname{arcsinh}(-\ ) = -\operatorname{arcsinh}\ , \qquad \operatorname{arctanh}(-\ ) = -\operatorname{arctanh}\ ,$$

$$\operatorname{arccosh}(-\ ) = \operatorname{arccosh}\ , \qquad \operatorname{arccoth}(-\ ) = -\operatorname{arccoth}\ .$$

S.1.4-2. Relations between inverse hyperbolic functions.

$$\operatorname{arcsinh} \ = \operatorname{arccosh}\sqrt{{}^{2}+1} = \operatorname{arctanh}\frac{}{\sqrt{{}^{2}+1}},$$

$$\operatorname{arccosh} \ = \operatorname{arcsinh}\sqrt{{}^{2}-1} = \operatorname{arctanh}\frac{\sqrt{{}^{2}-1}}{},$$

$$\operatorname{arctanh} \ = \operatorname{arcsinh}\frac{}{\sqrt{1-{}^{2}}} = \operatorname{arccosh}\frac{1}{\sqrt{1-{}^{2}}} = \operatorname{arccoth}\frac{1}{}.$$

S.1.4-3. Addition and subtraction of inverse hyperbolic functions.

$$\operatorname{arcsinh} \quad \operatorname{arcsinh} \ = \operatorname{arcsinh}\big(\quad \sqrt{1+{}^{2}} \quad \sqrt{1+{}^{2}}\ \big),$$

$$\operatorname{arccosh} \quad \operatorname{arccosh} \ = \operatorname{arccosh}\big[\quad \sqrt{({}^{2}-1)({}^{2}-1)}\big],$$

$$\operatorname{arcsinh} \quad \operatorname{arccosh} \ = \operatorname{arcsinh}\big[\quad \sqrt{({}^{2}+1)({}^{2}-1)}\big],$$

$$\operatorname{arctanh} \quad \operatorname{arctanh} \ = \operatorname{arctanh}\frac{}{1}, \qquad \operatorname{arctanh} \quad \operatorname{arccoth} \ = \operatorname{arctanh}\frac{1}{}.$$

S.1.4-4. Differentiation formulas.

$$\frac{}{}\operatorname{arcsinh} \ = \frac{1}{\sqrt{{}^{2}+1}}, \qquad \frac{}{}\operatorname{arccosh} \ = \frac{1}{\sqrt{{}^{2}-1}},$$

$$\frac{}{}\operatorname{arctanh} \ = \frac{1}{1-{}^{2}} \quad ({}^{2} < 1), \qquad \frac{}{}\operatorname{arccoth} \ = \frac{1}{1-{}^{2}} \quad ({}^{2} > 1).$$

S.1.4-5. Expansion into power series.

$$\operatorname{arcsinh} \quad = \quad -\frac{1}{2}\frac{^3}{3}+\frac{1\times 3}{2\times 4}\frac{^5}{5}-\frac{1\times 3\times 5}{2\times 4\times 6}\frac{^7}{7}+ \qquad (|\;|<1),$$

$$\operatorname{arctanh} \quad = \quad +\frac{^3}{3}+\frac{^5}{5}+\frac{^7}{7}+ \qquad (|\;|<1).$$

References for Section S.1: H. B. Dwight (1961), M. Abramowitz and I. A. Stegun (1964), G. A. Korn and T. M. Korn (1968), W. H. Beyer (1991).

S.2. Special Functions and Their Properties

Throughout Section S.2 it is assumed that   is a positive integer unless otherwise specified.

S.2.1. Some Symbols and Coefficients

S.2.1-1. Factorials.

Definitions and some properties:

$$0! = 1! = 1, \qquad ! = 1\times 2\times 3 \quad (\;-1)\;, \qquad = 2, 3, \quad ,$$
$$(2\;)!! = 2\times 4\times 6 \quad (2\;-2)(2\;) = 2\;\;!,$$
$$(2\;+1)!! = 1\times 3\times 5 \quad (2\;-1)(2\;+1) = \frac{2^{\;+1}}{\;}\Gamma\left(\;+\frac{3}{2}\right),$$
$$!! = \begin{cases} (2k)!! & \text{if } \;= 2k, \\ (2k+1)!! & \text{if } \;= 2k+1, \end{cases} \qquad 0!! = 1.$$

S.2.1-2. Binomial coefficients.

Definition:

$$= \frac{!}{k!(\;-k)!}, \quad \text{where } k = 1, \;,\;,$$
$$= (-1)\;\frac{(-a)}{k!} = \frac{a(a-1)\quad(a-k+1)}{k!}, \quad \text{where } k = 1, 2,$$

General case:

$$^b = \frac{\Gamma(a+1)}{\Gamma(\;+1)\Gamma(a-\;+1)}, \quad \text{where } \Gamma(\;) \text{ is the gamma function.}$$

Properties:

$$^0 = 1, \qquad = 0 \quad \text{for } k = -1, -2, \quad \text{or } k > \;,$$
$$^{b+1} = \frac{a}{\;+1}\;{}^b_{-1} = \frac{a-}{\;+1}\;{}^b, \qquad {}^b + {}^{b+1} = {}^{b+1}_{+1},$$
$${}_{-1\;2} = \frac{(-1)}{2^2}\;{}_2 = (-1)\;\frac{(2\;-1)!!}{(2\;)!!},$$
$${}_{1\;2} = \frac{(-1)^{\;-1}}{2^{2\;-1}}\;{}^{-1}_{2\;-2} = \frac{(-1)^{\;-1}}{}\frac{(2\;-3)!!}{(2\;-2)!!},$$
$${}^{2\;+1}_{+1\;2} = (-1)\;2^{-4\;-1}\;{}_2\;, \qquad {}_{2\;+1\;2} = 2^{-2}\;{}^2_{4\;+1},$$
$${}_{1\;2} = \frac{2^{2\;+1}}{\;2}, \qquad {}_2 = \frac{2^2}{\;}\;{}_{(\;-1)\;2}.$$

S.2.1-3. Pochhammer symbol.

Definition and some properties ($k = 1, 2, \quad$):

$$(a) \quad = a(a+1) \quad (a+ \quad -1) = \frac{\Gamma(a+ \quad)}{\Gamma(a)} = (-1) \quad \frac{\Gamma(1-a)}{\Gamma(1-a- \quad)},$$

$$(a)_0 = 1, \quad (a) \quad _+ \quad = (a) \quad (a+ \quad) \quad, \quad (\quad) \quad = \frac{(\quad +k-1)!}{(\quad -1)!},$$

$$(a)_- \quad = \frac{\Gamma(a- \quad)}{\Gamma(a)} = \frac{(-1)}{(1-a)}, \quad \text{where } a \neq 1, \quad , \quad ;$$

$$(1) \quad = \quad !, \quad (1 \quad 2) \quad = 2^{-2} \quad \frac{(2 \quad)!}{!}, \quad (3 \quad 2) \quad = 2^{-2} \quad \frac{(2 \quad +1)!}{!},$$

$$(a+ \quad k) \quad = \frac{(a) \quad _+}{(a)}, \quad (a+ \quad) \quad = \frac{(a)_2}{(a)}, \quad (a+ \quad) \quad = \frac{(a) \quad (a+k)}{(a)}.$$

S.2.2. Error Functions and Exponential Integral

S.2.2-1. Error function and complementary error function.

Definitions:

$$\text{erf} \quad = \frac{2}{} \int_0 \quad \exp(- \quad ^2) \quad, \qquad \text{erfc} \quad = 1 - \text{erf} \quad = \frac{2}{} \int \quad \exp(- \quad ^2) \quad .$$

Expansion of erf into series in powers of as 0:

$$\text{erf} \quad = \frac{2}{} \quad _{=0} (-1) \quad \frac{ \quad ^{2 \quad +1}}{k!\,(2k+1)} = \frac{2}{} \exp\bigl(- \quad ^2 \quad _{=0} \frac{2 \quad ^{2 \quad +1}}{(2k+1)!!}.$$

Asymptotic expansion of erfc as :

$$\text{erfc} \quad = \frac{1}{} \exp\bigl(- \quad ^2 \quad _{=0}^{M-1} (-1) \quad \frac{(\frac{1}{2}}{ \quad ^{2 \quad +1}} + \quad (| \quad |^{-2M-1} \quad , \qquad = 1, 2,$$

S.2.2-2. Exponential integral.

Definition:

$$\text{Ei}(\quad) = \int_- \quad \frac{e}{} \qquad \text{for} \quad < 0,$$

$$\text{Ei}(\quad) = \lim_{+0} \Bigl(\int_-^- \frac{e}{} \quad + \int \quad \frac{e}{} \quad \Bigr) \qquad \text{for} \quad > 0.$$

Other integral representations:

$$\text{Ei}(- \quad) = -e^- \int_0 \quad \frac{\sin \quad + \quad \cos}{ \quad ^2 + \quad ^2} \qquad \text{for} \quad > 0,$$

$$\text{Ei}(- \quad) = e^- \int_0 \quad \frac{\sin \quad - \quad \cos}{ \quad ^2 + \quad ^2} \qquad \text{for} \quad < 0,$$

$$\text{Ei}(- \quad) = - \int_1 \quad e^- \quad \ln \qquad \text{for} \quad > 0.$$

Expansion into series in powers of as 0:

$$\mathrm{Ei}(\ ) = \begin{cases} \mathcal{C} + \ln(-\ ) + \ \ \ _{=1} \ \dfrac{}{k \times k!} & \text{if } \ < 0, \\ \mathcal{C} + \ln \ + \ \ _{=1} \ \dfrac{}{k \times k!} & \text{if } \ > 0, \end{cases}$$

where $\mathcal{C} = 0.5772$ is the Euler constant.

Asymptotic expansion as :

$$\mathrm{Ei}(-\ ) = e^{-} \ \ _{=1} (-1) \ \frac{(k-1)!}{} + R \ , \qquad R \ < \frac{!}{}.$$

S.2.2-3. Logarithmic integral.

Definition:

$$\mathrm{li}(\ ) = \begin{cases} \displaystyle\int_0 \ \frac{}{\ln} = \mathrm{Ei}(\ln \ ) & \text{if } 0 < \ < 1, \\ \displaystyle\lim_{+0} \Big(\int_0^{1-} \frac{}{\ln} + \int_{1+} \frac{}{\ln} \Big) & \text{if } \ > 1. \end{cases}$$

For small ,

$$\mathrm{li}(\ ) \approx \frac{}{\ln(1 \ \ )}.$$

Asymptotic expansion as 1:

$$\mathrm{li}(\ ) = \mathcal{C} + \ln|\ln \ | + \ \ _{=1} \ \frac{\ln}{k \times k!}.$$

S.2.3. Gamma and Beta Functions

S.2.3-1. Gamma function.

The gamma function, $\Gamma(z)$, is an analytic function of the complex argument z everywhere, except for the points $z = 0, -1, -2,$

For $\mathrm{Re}\, z > 0$,

$$\Gamma(z) = \int_0 \ \ ^{-1} e^{-} \ \ .$$

For $-(\ +1) < \mathrm{Re}\, z < -\ $, where $\ = 0,\ 1,\ 2,\ \ ,$

$$\Gamma(z) = \int_0 \ \ e^{-} - \ \ _{=0} \ \frac{(-1)}{!} \ \ ^{-1} \ \ .$$

Euler formula:

$$\Gamma(z) = \lim \ \frac{!}{z(z+1) \ \ (z+\ )} \qquad (z \neq 0, -1, -2, \ \ ).$$

Simplest properties:

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(\ +1) = \ !, \quad \Gamma(1) = \Gamma(2) = 1.$$

Symmetry formulas:

$$\Gamma(z)\Gamma(-z)=-\frac{\quad}{z\sin(\ \ z)},\qquad \Gamma(z)\Gamma(1-z)=\frac{\quad}{\sin(\ \ z)},$$

$$\Gamma\Big(\frac{1}{2}+z\Big)\Gamma\Big(\frac{1}{2}-z\Big)=\frac{\quad}{\cos(\ \ z)}.$$

Multiple argument formulas:

$$\Gamma(2z)=\frac{2^{2\ \ -1}}{\ }\Gamma(z)\Gamma\Big(z+\frac{1}{2}\Big),$$

$$\Gamma(3z)=\frac{3^{3\ \ -1\ \ 2}}{2}\Gamma(z)\Gamma\Big(z+\frac{1}{3}\Big)\Gamma\Big(z+\frac{2}{3}\Big),$$

$$\Gamma(\ \ z)=(2\ \ )^{(1-\ \ )\ 2}\ \ \ ^{-1\ \ 2}\prod_{=0}^{\ -1}\Gamma\Big(z+\frac{k}{\ }\Big).$$

Fractional values of the argument:

$$\Gamma\Big(\frac{1}{2}\Big)=\sqrt{\ },\qquad \Gamma\Big(\ \ +\frac{1}{2}\Big)=\frac{\sqrt{\ }}{2\ }(2\ \ -1)!!,$$

$$\Gamma\Big(-\frac{1}{2}\Big)=-2\sqrt{\ },\qquad \Gamma\Big(\frac{1}{2}-\ \Big)=(-1)\ \ \frac{2\ \ \sqrt{\ }}{(2\ \ -1)!!}.$$

Asymptotic expansion (Stirling formula):

$$\Gamma(z)=\sqrt{2\ }\ e^{-\ }z^{\ -1\ 2}\left[1+\tfrac{1}{12}z^{-1}+\tfrac{1}{288}z^{-2}+\ \ (z^{-3})\right]\qquad (|\arg z|<\ \ ).$$

S.2.3-2. Logarithmic derivative of the gamma function.

Definition:

$$(z)=\frac{\ln\Gamma(z)}{z}=\frac{\Gamma'(z)}{\Gamma(z)}.$$

Functional relations:

$$(z)-\ \ (1+z)=-\frac{1}{z},$$

$$(z)-\ \ (1-z)=-\ \ \cot(\ \ z),$$

$$(z)-\ \ (-z)=-\ \ \cot(\ \ z)-\frac{1}{z},$$

$$\big(\tfrac{1}{2}+z\big)-\ \ \big(\tfrac{1}{2}-z\big)=\ \ \tan(\ \ z),$$

$$(\ \ z)=\ln\ \ +\frac{1}{\ }\sum_{=0}^{\ -1}\Big(z+\frac{k}{\ }\Big).$$

Integral representations ($\operatorname{Re} z>0$):

$$(z)=\int_0\ \ \left[e^{-\ }-(1+\ \ )^{-\ }\right]\ ^{-1}\ \ ,$$

$$(z)=\ln z+\int_0\ \ \left[\ ^{-1}-(1-e^{-\ })^{-1}\right]e^{-\ }\ \ ,$$

$$(z)=-\mathcal{C}+\int_0^1\frac{1-\ \ ^{-1}}{1-\ }\ \ ,$$

where $\mathcal{C}=-\ \ (1)=0.5772\ldots$ is the Euler constant.

Values for integer argument:

$$(1)=-\mathcal{C},\qquad (\ \ )=-\mathcal{C}+\sum_{=1}^{\ -1}k^{-1}\qquad (\ \ =2,3,\ \ ).$$

S.2.3-3. Beta function.

Definition:

$$B(\ ,\ ) = \int_0^1 \ ^{-1}(1-\ )^{\ -1}\ ,$$

where Re $\ > 0$ and Re $\ > 0$.

Relationship with the gamma function:

$$B(\ ,\ ) = \frac{\Gamma(\ )\Gamma(\ )}{\Gamma(\ +\ )}.$$

S.2.4. Incomplete Gamma and Beta Functions

S.2.4-1. Incomplete gamma function.

Definitions (integral representations):

$$(\ ,\ ) = \int_0 e^{-}\ ^{-1}\ ,\qquad \text{Re}\ > 0,$$

$$\Gamma(\ ,\ ) = \int\ e^{-}\ ^{-1}\ = \Gamma(\ ) - (\ ,\ ).$$

Recurrence formulas:

$$(\ +1,\ ) = \ (\ ,\ ) - \ e^{-}\ ,$$

$$\Gamma(\ +1,\ ) = \ \Gamma(\ ,\ ) + \ e^{-}\ .$$

Asymptotic expansions as $\ 0$:

$$(\ ,\ ) = \sum_{=0} \frac{(-1)\ ^{+}}{!(\ +\ )},$$

$$\Gamma(\ ,\ ) = \Gamma(\ ) - \sum_{=0} \frac{(-1)\ ^{+}}{!(\ +\ )}.$$

Asymptotic expansions as $\ $:

$$(\ ,\ ) = \Gamma(\ ) - \ ^{-1}e^{-} \sum_{=0}^{M-1} \frac{(1-\ )}{(-\ )} + \ (|\ |^{-M}\ ,$$

$$\Gamma(\ ,\ ) = \ ^{-1}e^{-} \sum_{=0}^{M-1} \frac{(1-\ )}{(-\ )} + \ (|\ |^{-M}\ \ (-\tfrac{3}{2}\ < \arg\ < \tfrac{3}{2}\ .$$

Integral functions related to the gamma function:

$$\text{erf}\ = \frac{1}{\ }\ \left(\frac{1}{2},\ ^2\right),\quad \text{erfc}\ = \frac{1}{\ }\Gamma\left(\frac{1}{2},\ ^2\right),\quad \text{Ei}(-\ ) = -\Gamma(0,\ ).$$

S.2.4-2. Incomplete beta function.

Definition:

$$B\ (\ ,\ ) = \int_0 \ ^{-1}(1-\ )^{\ -1}\ ,$$

where Re $\ > 0$ and Re $\ > 0$.

S.2.5. Bessel Functions

S.2.5-1. Definitions and basic formulas.

The Bessel function of the first kind, $J_\nu(x)$, and the Bessel function of the second kind, $Y_\nu(x)$ (also called the Neumann function), are solutions of the Bessel equation

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

and are defined by the formulas:

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)}, \qquad Y_\nu(x) = \frac{J_\nu(x)\cos \pi\nu - J_{-\nu}(x)}{\sin \pi\nu}. \tag{1}$$

The formula for $Y_\nu(x)$ is valid for $\nu \neq 0, \pm1, \pm2, \ldots$ (the cases $\nu \neq 0, \pm1, \pm2, \ldots$ are discussed in the following).

The general solution of the Bessel equation has the form $Z_\nu(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$ and is called the cylindrical function.

Some relations:

$$2\nu Z_\nu(x) = x[Z_{\nu-1}(x) + Z_{\nu+1}(x)],$$

$$\frac{d}{dx} Z_\nu(x) = \frac{1}{2}[Z_{\nu-1}(x) - Z_{\nu+1}(x)] = \pm\left[\frac{\nu}{x} Z_\nu(x) - Z_{\nu\pm1}(x)\right],$$

$$\frac{d}{dx}[x^\nu Z_\nu(x)] = x^\nu Z_{\nu-1}(x), \qquad \frac{d}{dx}[x^{-\nu} Z_\nu(x)] = -x^{-\nu} Z_{\nu+1}(x),$$

$$\left(\frac{1}{x}\frac{d}{dx}\right)^n [x^\nu J_\nu(x)] = x^{\nu-n} J_{\nu-n}(x), \qquad \left(\frac{1}{x}\frac{d}{dx}\right)^n [x^{-\nu} J_\nu(x)] = (-1)^n x^{-\nu-n} J_{\nu+n}(x),$$

$$J_{-n}(x) = (-1)^n J_n(x), \qquad Y_{-n}(x) = (-1)^n Y_n(x), \qquad n = 0, 1, 2, \ldots$$

S.2.5-2. Bessel functions for $\nu = \pm n \pm \frac{1}{2}$, where $n = 0, 1, 2, \ldots$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x,$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{1}{x}\sin x - \cos x\right), \qquad J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(-\frac{1}{x}\cos x - \sin x\right),$$

$$J_{n+1/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\sin\left(x - \frac{n\pi}{2}\right)\sum_{k=0}^{[n/2]} \frac{(-1)^k (n+2k)!}{(2k)!\,(n-2k)!\,(2x)^{2k}} + \cos\left(x - \frac{n\pi}{2}\right)\sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k (n+2k+1)!}{(2k+1)!\,(n-2k-1)!\,(2x)^{2k+1}}\right],$$

$$J_{-n-1/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\cos\left(x + \frac{n\pi}{2}\right)\sum_{k=0}^{[n/2]} \frac{(-1)^k (n+2k)!}{(2k)!\,(n-2k)!\,(2x)^{2k}} - \sin\left(x + \frac{n\pi}{2}\right)\sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k (n+2k+1)!}{(2k+1)!\,(n-2k-1)!\,(2x)^{2k+1}}\right],$$

$$Y_{1/2}(x) = -\sqrt{\frac{2}{\pi x}}\cos x, \qquad Y_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x,$$

$$Y_{n+1/2}(x) = (-1)^{n+1} J_{-n-1/2}(x), \qquad Y_{-n-1/2}(x) = (-1)^n J_{n+1/2}(x),$$

where $[A]$ is the integer part of a number A.

S.2.5-3. Bessel functions for = , where = 0, 1, 2,

Let = be an arbitrary integer. The relations

$$_{-}(\ ) = (-1)\quad(\ ),\qquad _{-}(\ ) = (-1)\quad(\ )$$

are valid. The function ( ) is given by the first formula in (1) with = , and ( ) can be obtained from the second formula in (1) by proceeding to the limit . For nonnegative , the function ( ) can be represented in the form:

$$(\ ) = \frac{2}{\ }\ (\ )\ln\frac{\ }{2} - \frac{1}{\ }\sum_{=0}^{-1}\frac{(\ -k-1)!}{k!}\left(\frac{2}{\ }\right)^{-2} - \frac{1}{\ }\sum_{=0}(-1)\left(\frac{\ }{2}\right)^{+2}\frac{(k+1)+(\ +k+1)}{k!\,(\ +k)!},$$

where $(1) = -\mathcal{C}$, $(\ ) = -\mathcal{C} + \sum_{=1}^{-1} k^{-1}$, $\mathcal{C} = 0.5772$ is the Euler constant, $(\ ) = [\ln\Gamma(\ )]'$ is the logarithmic derivative of the gamma function.

S.2.5-4. Wronskians and similar formulas.

$$(\ ,\ _{-}\ ) = -\frac{2}{\ }\sin(\ ),\qquad (\ ,\ ) = \frac{2}{\ },$$

$$(\ )\ _{-\ +1}(\ ) + \ _{-}(\ )\ _{-1}(\ ) = \frac{2\sin(\ )}{\ },\qquad (\ )\ _{+1}(\ ) - \ _{+1}(\ )\ (\ ) = -\frac{2}{\ }.$$

Here, the notation $(\ ,g) = g' - \ 'g$ is used.

S.2.5-5. Integral representations.

The functions ( ) and ( ) can be represented in the form of definite integrals (for > 0):

$$(\ ) = \int_0^\pi \cos(\ \sin\theta - \ \theta)\ \theta - \sin\ \int_0 \exp(-\ \sinh\ -\ )\ ,$$

$$(\ ) = \int_0^\pi \sin(\ \sin\theta - \ \theta)\ \theta - \int_0 (e\ + e^{-}\ \cos\ )e^{-\ \sinh}\ .$$

For $|\ | < \frac{1}{2}$, > 0,

$$(\ ) = \frac{2^{1+\ -}}{^{1\ 2}\Gamma(\frac{1}{2} - \ )}\int_1 \frac{\sin(\ )}{(\ ^2 - 1)^{+1\ 2}},$$

$$(\ ) = -\frac{2^{1+\ -}}{^{1\ 2}\Gamma(\frac{1}{2} - \ )}\int_1 \frac{\cos(\ )}{(\ ^2 - 1)^{+1\ 2}}.$$

For $> -\frac{1}{2}$,

$$(\ ) = \frac{2(\ 2)}{^{1\ 2}\Gamma(\frac{1}{2} + \ )}\int_0^{\pi\ 2}\cos(\ \cos\ )\sin^2\qquad\text{(Poisson's formula).}$$

For $= 0$, > 0,

$$_0(\ ) = \frac{2}{\ }\int_0 \sin(\ \cosh\ )\ ,\qquad _0(\ ) = -\frac{2}{\ }\int_0 \cos(\ \cosh\ )\ .$$

For integer $=\ = 0, 1, 2,\ ,$

$$(\ ) = \frac{1}{\ }\int_0^\pi \cos(\ -\ \sin\ )\qquad\text{(Bessel's formula),}$$

$$_2(\ ) = \frac{2}{\ }\int_0^{\pi\ 2}\cos(\ \sin\ )\cos(2\ )\ ,$$

$$_{2\ +1}(\ ) = \frac{2}{\ }\int_0^{\pi\ 2}\sin(\ \sin\ )\sin[(2\ +1)\ ]\ .$$

S.2.5-6. Asymptotic expansions.

Asymptotic expansions as $| \ |$:

$$(\ ) = \sqrt{\frac{2}{\ }}\ \cos\frac{4\ -2\ -\ }{4} \sum^{M-1}_{=0} (-1)\ (\ ,2\ )(2\ )^{-2}\ + \ (|\ |^{-2M})$$

$$-\sin\frac{4\ -2\ -\ }{4} \sum^{M-1}_{=0} (-1)\ (\ ,2\ +1)(2\ )^{-2\ -1} + \ (|\ |^{-2M-1})\ ,$$

$$(\ ) = \sqrt{\frac{2}{\ }}\ \sin\frac{4\ -2\ -\ }{4} \sum^{M-1}_{=0} (-1)\ (\ ,2\ )(2\ )^{-2}\ + \ (|\ |^{-2M})$$

$$+\cos\frac{4\ -2\ -\ }{4} \sum^{M-1}_{=0} (-1)\ (\ ,2\ +1)(2\ )^{-2\ -1} + \ (|\ |^{-2M-1})\ ,$$

where $(\ ,\ ) = \frac{1}{2^2\ !}(4\ ^2-1)(4\ ^2-3^2)\ \ [4\ ^2-(2\ -1)^2] = \frac{\Gamma(\frac{1}{2}+\ +\ )}{!\,\Gamma(\frac{1}{2}+\ -\ )}$.

For nonnegative integer and large ,

$$\overline{\ \ }\ _2\ (\ ) = (-1)\ (\cos\ +\sin\ ) + \ (\ ^{-2}),$$

$$\overline{\ \ }\ _{2\ +1}(\ ) = (-1)^{\ +1}(\cos\ -\sin\ ) + \ (\ ^{-2}).$$

Asymptotic behavior for large ( ):

$$(\ )\ \ \frac{1}{\overline{2}}\ \frac{e}{2}\ \ , \qquad (\ )\ \ -\sqrt{\frac{2}{\ }}\ \frac{e}{2}\ ^{-}\ ,$$

where is fixed, and

$$(\ )\ \ \frac{2^{1\ 3}}{3^{2\ 3}\Gamma(2\ 3)}\frac{1}{\ ^{1\ 3}}, \qquad (\ )\ \ -\frac{2^{1\ 3}}{3^{1\ 6}\Gamma(2\ 3)}\frac{1}{\ ^{1\ 3}}.$$

S.2.5-7. Zeros and orthogonality properties of the Bessel functions.

Each of the functions ( ) and ( ) has infinitely many real zeros (for real). All zeros are simple, except possibly for the point $= 0$.

The zeros of $_0(\ )$, i.e., the roots of the equation $_0(\ ) = 0$, are approximately given by:

$$= 2.4 + 3.13\,(\ -1)\qquad (\ = 1, 2,\ ),$$

with a maximum error of 0.2%.

Let $=$ be positive roots of the Bessel function ( ), where > -1 and $= 1, 2, 3,$ Then the set of functions (a) is orthogonal on the interval $0 \le\ \le a$ with weight :

$$\int_0\ \ \overline{\ a\ }\ \ \overline{\ a\ } = \begin{cases} 0 & \text{if } \ne k, \\ \frac{1}{2}a^2\left[\ '(\ )\right]^2 = \frac{1}{2}a^2\ ^2_{\ +1}(\ ) & \text{if } = k. \end{cases}$$

S.2.5-8. Hankel functions (Bessel functions of the third kind).

The Hankel functions of the first kind and the second kind are related to the Bessel functions by:

$$^{(1)}(z) = \ (z) + \ (z), \qquad ^{(2)}(z) = \ (z) - \ (z), \qquad ^2 = -1.$$

Asymptotics for z 0:

$$ {}_0^{(1)}(z) \quad \frac{2}{} \ln z, \qquad {}^{(1)}(z) \quad -\frac{\Gamma(\ )}{(z\ \ 2)} \qquad (\mathrm{Re}\ \ > 0), $$

$$ {}_0^{(2)}(z) \quad -\frac{2}{} \ln z, \qquad {}^{(2)}(z) \quad -\frac{\Gamma(\ )}{(z\ \ 2)} \qquad (\mathrm{Re}\ \ > 0). $$

Asymptotics for $|z|$:

$$ {}^{(1)}(z) \quad \sqrt{\frac{2}{z}} \exp \ \left(z - \tfrac{1}{2} \quad - \tfrac{1}{4}\right. \qquad (- \ < \arg z < 2\ ), $$

$$ {}^{(2)}(z) \quad \sqrt{\frac{2}{z}} \exp - \left(z - \tfrac{1}{2} \quad - \tfrac{1}{4}\right. \qquad (-2\ < \arg z <\ ). $$

S.2.6. Modified Bessel Functions

S.2.6-1. Definitions. Basic formulas.

The modified Bessel functions of the first kind, ( ), and the second kind, ( ) (also called the Macdonald function), of order are solutions of the modified Bessel equation

$$ {}^2\ '' + \ ' - (\ ^2 + \ ^2)\ = 0 $$

and are defined by the formulas:

$$ (\ ) = \sum_{=0} \frac{(\ \ 2)^{2\ +}}{k!\,\Gamma(\ + k + 1)}, \qquad (\ ) = \frac{}{2} \frac{_- \ -}{\sin}, $$

(see below for ( ) with $= 0, 1, 2,$).

The modified Bessel functions possess the properties:

$$ _-(\ ) = \ (\ ); \quad _-(\ ) = (-1)\ \ (\ ), \quad = 0, 1, 2, $$

$$ 2\ \ (\ ) = \ [\ _{-1}(\ ) - \ _{+1}(\ )], \quad 2\ \ (\ ) = -\ [\ _{-1}(\ ) - \ _{+1}(\ )], $$

$$ \frac{}{}\ (\ ) = \frac{1}{2}[\ _{-1}(\ ) + \ _{+1}(\ )], \quad \frac{}{}\ (\ ) = -\frac{1}{2}[\ _{-1}(\ ) + \ _{+1}(\ )]. $$

S.2.6-2. Modified Bessel functions for $=$ $\frac{1}{2}$, where $= 0, 1, 2,$

$$ {}_{1\ 2}(\ ) = \sqrt{\frac{2}{}} \sinh\ , \qquad {}_{-1\ 2}(\ ) = \sqrt{\frac{2}{}} \cosh\ , $$

$$ {}_{3\ 2}(\ ) = \sqrt{\frac{2}{}}\ \ -\frac{1}{} \sinh\ + \cosh\ \Big), \qquad {}_{-3\ 2}(\ ) = \sqrt{\frac{2}{}}\ \ -\frac{1}{} \cosh\ + \sinh\ \Big), $$

$$ {}_{+1\ 2}(\ ) = \frac{1}{\overline{2}}\ e \sum_{=0} \frac{(-1)\ (\ + k)!}{k!\,(\ - k)!\,(2\ )} - (-1)\ e^{-} \sum_{=0} \frac{(\ + k)!}{k!\,(\ - k)!\,(2\ )}\ , $$

$$ {}_{-\ -1\ 2}(\ ) = \frac{1}{\overline{2}}\ e \sum_{=0} \frac{(-1)\ (\ + k)!}{k!\,(\ - k)!\,(2\ )} + (-1)\ e^{-} \sum_{=0} \frac{(\ + k)!}{k!\,(\ - k)!\,(2\ )}\ , $$

$$ {}_{1\ 2}(\ ) = \sqrt{\frac{}{2}}\, e^{-}\ , \qquad {}_{3\ 2}(\ ) = \sqrt{\frac{}{2}}\ \ 1 + \frac{1}{}\ \ e^{-}\ , $$

$$ {}_{+1\ 2}(\ ) = \ {}_{-\ -1\ 2}(\ ) = \sqrt{\frac{}{2}}\, e^{-} \sum_{=0} \frac{(\ + k)!}{k!\,(\ - k)!\,(2\ )}. $$

S.2.6-3. Modified Bessel functions for $\ = \ $, where $\ = 0, 1, 2,$

If $\ = \ $ is a nonnegative integer, then

$$(\ ) = (-1)^{\ +1}\ (\ )\ln\frac{\ }{2} + \frac{1}{2}\ \ _{=0}^{-1}(-1)\ \ \frac{\ }{2}^{2\ -\ }\frac{(\ -\ \ -1)!}{\ !}$$

$$+\frac{1}{2}(-1)\ \ _{=0}\ \frac{\ }{2}^{\ +2}\ \frac{(\ +\ \ +1)+\ (\ \ +1)}{\ !(\ +\ \ )!};\qquad =0,\ 1,\ 2,\ \ ,$$

where $\ (z)$ is the logarithmic derivative of the gamma function; for $\ = 0$, the first sum is dropped.

S.2.6-4. Wronskians and similar formulas.

$$(\ ,\ _{-}) = -\frac{2}{\ }\sin(\ \ ),\qquad (\ ,\ \ ) = -\frac{1}{\ },$$

$$(\ )\ _{-\ +1}(\ ) - \ _{-}(\ )\ \ _{-1}(\ ) = -\frac{2\sin(\ \ )}{\ },\qquad (\ )\ \ _{+1}(\ ) + \ \ _{+1}(\ )\ \ (\ ) = \frac{1}{\ },$$

where $\ (\ ,g) = \ g' - \ 'g$.

S.2.6-5. Integral representations.

The functions $\ (\ )$ and $\ (\ )$ can be represented in terms of definite integrals:

$$(\ ) = \frac{\ }{\ ^{1\ 2}2\ \ \Gamma(\ +\frac{1}{2})}\int_{-1}^{1}\exp(-\ \ )(1-\ ^2)^{\ -1\ 2}\ \qquad (\ >0,\ \ >-\tfrac{1}{2}),$$

$$(\ ) = \int_0\ \exp(-\ \cosh\ )\cosh(\ \ )\ \qquad (\ >0),$$

$$(\ ) = \frac{1}{\cos(\frac{1}{2}\ \ )}\int_0\ \cos(\ \sinh\ )\cosh(\ \ )\ \qquad (\ >0,\ -1<\ \ <1),$$

$$(\ ) = \frac{1}{\sin(\frac{1}{2}\ \ )}\int_0\ \sin(\ \sinh\ )\sinh(\ \ )\ \qquad (\ >0,\ -1<\ \ <1).$$

For integer $\ = \ $,

$$(\ ) = \frac{1}{\ }\int_0^{\pi}\exp(\ \cos\ )\cos(\ \ )\ \qquad (\ =0,1,2,\ \ ),$$

$$_0(\ ) = \int_0\ \cos(\ \sinh\ )\ \ = \int_0\ \frac{\cos(\ \ )}{\ ^2+1}\ \qquad (\ >0).$$

S.2.6-6. Asymptotic expansions as $\ \ $.

$$(\ ) = \frac{e\ }{\ 2\ \ }\ 1 + \ _{=1}^{M}(-1)\ \frac{(4\ ^2-1)(4\ ^2-3^2)\ \ [4\ ^2-(2\ \ -1)^2]}{\ !(8\ \ )\ }\ ,$$

$$(\ ) = \sqrt{\frac{\ }{2\ }}\ e^{-\ }\ 1 + \ _{=1}^{M}\frac{(4\ ^2-1)(4\ ^2-3^2)\ \ [4\ ^2-(2\ \ -1)^2]}{\ !(8\ \ )\ }\ .$$

The terms of the order of $\ (\ ^{-M-1}\ $ are omitted in the braces.

S.2.7. Degenerate Hypergeometric Functions

S.2.7-1. Definitions. The Kummer's series.

The degenerate hypergeometric functions $(a,\ ;\ )$ and $(a,\ ;\ )$ are solutions of the degenerate hypergeometric equation:

$$'' + (\ -\ )\ ' - a\ = 0.$$

In the case $\ \neq 0, -1, -2, -3,\ $, the function $(a,\ ;\ )$ can be represented as Kummer's series:

$$(a,\ ;\ ) = 1 + \sum_{=1} \frac{(a)}{(\ )}\,\frac{}{k!},$$

where $(a)\ = a(a+1)\ \ (a+k-1)$, $(a)_0 = 1$.

Table 14 (see Subsection 2.1.2) presents some special cases where can be expressed in terms of simpler functions.

S.2.7-2. Some transformations and linear relations.

Kummer transformations:

$$(a,\ ;\ ) = e\ \ (\ -a,\ ;-\ ),\qquad (a,\ ;\ ) = \ ^{1-b}\ (1+a-\ ,2-\ ;\ ).$$

Linear relations for :

$$
\begin{aligned}
&(\ -a)\ (a-1,\ ;\ )+(2a-\ +\ )\ (a,\ ;\ )-a\ (a+1,\ ;\ )=0,\\
&(\ -1)\ (a,\ -1;\ )-\ (\ -1+\ )\ (a,\ ;\ )+(\ -a)\ \ (a,\ +1;\ )=0,\\
&(a-\ +1)\ (a,\ ;\ )-a\ (a+1,\ ;\ )+(\ -1)\ (a,\ -1;\ )=0,\\
&\ (a,\ ;\ )-\ \ (a-1,\ ;\ )-\ \ (a,\ +1;\ )=0,\\
&(a+\ )\ (a,\ ;\ )-(\ -a)\ \ (a,\ +1;\ )-a\ \ (a+1,\ ;\ )=0,\\
&(a-1+\ )\ (a,\ ;\ )+(\ -a)\ (a-1,\ ;\ )-(\ -1)\ (a,\ -1;\ )=0.
\end{aligned}
$$

Linear relations for :

$$
\begin{aligned}
&\ (a-1,\ ;\ )-(2a-\ +\ )\ (a,\ ;\ )+a(a-\ +1)\ (a+1,\ ;\ )=0,\\
&(\ -a-1)\ (a,\ -1;\ )-(\ -1+\ )\ (a,\ ;\ )+\ \ (a,\ +1;\ )=0,\\
&\ (a,\ ;\ )-a\ (a+1,\ ;\ )-\ (a,\ -1;\ )=0,\\
&(\ -a)\ (a,\ ;\ )-\ \ (a,\ +1;\ )+\ (a-1,\ ;\ )=0,\\
&(a+\ )\ (a,\ ;\ )+a(\ -a-1)\ (a+1,\ ;\ )-\ \ (a,\ +1;\ )=0,\\
&(a-1+\ )\ (a,\ ;\ )-\ (a-1,\ ;\ )+(a-\ +1)\ (a,\ -1;\ )=0.
\end{aligned}
$$

S.2.7-3. Differentiation formulas and Wronskian.

Differentiation formulas:

$$\frac{}{}\ (a,\ ;\ ) = \frac{a}{}\ (a+1,\ +1;\ ),\qquad \frac{}{}\ (a,\ ;\ ) = \frac{(a)}{(\ )}\ (a+\ ,\ +\ ;\ ),$$

$$\frac{}{}\ (a,\ ;\ ) = -a\ (a+1,\ +1;\ ),\qquad \frac{}{}\ (a,\ ;\ ) = (-1)\ (a)\ \ (a+\ ,\ +\ ;\ ).$$

Wronskian:

$$(\ ,\ ) = \ \ '-\ \ ' \ = -\frac{\Gamma(\ )}{\Gamma(a)}\ ^{-b}e\ .$$

S.2.7-4. Degenerate hypergeometric functions for $\quad = 0, 1, 2,$

$$(a, \quad +1; \quad) = \frac{(-1)^{\quad -1}}{\,!\,\Gamma(a- \quad)} \quad (a, \quad +1; \quad) \ln$$
$$+ \sum_{=0} \frac{(a)}{(\quad +1)} \Big[\quad (a+ \quad) - \quad (1+ \quad) - \quad (1+ \quad + \quad) \Big] \frac{}{!} + \frac{(\quad -1)!}{\Gamma(a)} \sum_{=0}^{-1} \frac{(a- \quad)}{(1- \quad)} \frac{^{-}}{!},$$

where $\quad = 0, 1, 2, \quad$ (the last sum is dropped for $\quad = 0$), $\quad (z) = [\ln \Gamma(z)]'$ is the logarithmic derivative of the gamma function,

$$(1) = -\mathcal{C}, \qquad (\quad) = -\mathcal{C} + \sum_{=1}^{-1} k^{-1},$$

where $\mathcal{C} = 0.5772 \quad$ is the Euler constant.

If $\quad < 0$, then the formula

$$(a, \quad ; \quad) = \quad ^{1-b} \quad (a- \quad +1,\ 2- \quad ; \quad)$$

is valid for any $\quad$.

For $\quad \neq 0, -1, -2, -3, \quad$, the general solution of the degenerate hypergeometric equation can be represented in the form:

$$= \quad _1 \quad (a, \quad ; \quad) + \quad _2 \quad (a, \quad ; \quad),$$

and for $\quad = 0, -1, -2, -3, \quad$, in the form:

$$= \quad ^{1-b} \Big[\quad _1 \quad (a- \quad +1,\ 2- \quad ; \quad) + \quad _2 \quad (a- \quad +1,\ 2- \quad ; \quad) \Big].$$

S.2.7-5. Integral representations.

$$(a, \quad ; \quad) = \frac{\Gamma(\quad)}{\Gamma(a)\,\Gamma(\quad -a)} \int_0^1 e^{\quad} \quad ^{-1}(1- \quad)^{b- \quad -1} \qquad (\text{for} \quad > a > 0),$$
$$(a, \quad ; \quad) = \frac{1}{\Gamma(a)} \int_0 e^{-} \quad ^{-1}(1+ \quad)^{b- \quad -1} \qquad (\text{for } a > 0, \quad > 0),$$

where $\Gamma(a)$ is the gamma function.

S.2.7-6. Asymptotic expansion as $|\quad| \quad$.

$$(a, \quad ; \quad) = \frac{\Gamma(\quad)}{\Gamma(a)} e^{\quad} \quad ^{-b} \sum_{=0} \frac{(\quad -a) \ (1-a)}{!} \quad ^{-} + \varepsilon \quad , \qquad > 0,$$

$$(a, \quad ; \quad) = \frac{\Gamma(\quad)}{\Gamma(\quad -a)} (- \quad)^{-} \sum_{=0} \frac{(a) \ (a- \quad +1)}{!} (- \quad)^{-} + \varepsilon \quad , \qquad < 0,$$

$$(a, \quad ; \quad) = \quad ^{-} \sum_{=0} (-1) \frac{(a) \ (a- \quad +1)}{!} \quad ^{-} + \varepsilon \quad , \quad - \quad < \quad < \quad ,$$

where $\varepsilon = \quad (\quad ^{- \quad -1}$.

S.2.7-7. Whittaker functions.

The Whittaker functions $_{,}(\ )$ and $_{,}(\ )$ are linearly independent solutions of the Whittaker equation:

$$'' + \left[-\tfrac{1}{4} + \tfrac{1}{2}k + \left(\tfrac{1}{4} - \quad ^{2} \quad ^{-2}\right] \quad = 0.$$

The Whittaker functions are expressed in terms of degenerate hypergeometric functions as:

$$_{,}(\ ) = \quad ^{+1\ \ 2} e^{- \quad 2} \quad \left(\tfrac{1}{2} + \quad - k,\ 1 + 2\ ,\quad ,$$

$$_{,}(\ ) = \quad ^{+1\ \ 2} e^{- \quad 2} \quad \left(\tfrac{1}{2} + \quad - k,\ 1 + 2\ ,\quad .$$

S.2.8. Hypergeometric Functions

S.2.8-1. Definition. The hypergeometric series.

The hypergeometric function $(\ ,\beta,\ ;\ )$ is a solution of the Gaussian hypergeometric equation:

$$(\quad - 1)\ '' + [(\quad + \beta + 1) \quad - \quad]\ ' + \quad \beta \quad = 0.$$

For $\neq 0,\ -1,\ -2,\ -3,$, the function $(\ ,\beta,\ ;\ )$ can be expressed in terms of the hypergeometric series:

$$(\ ,\beta,\ ;\ ) = 1 + \sum_{=1} \frac{(\ )\ (\beta)}{(\ )} \frac{}{k!}, \qquad (\ ) \ = \ (\quad + 1) \quad (\quad + k - 1),$$

which, *a fortiori*, is convergent for $|\ | < 1$.

Table 16 (see Subsection 2.1.2) presents some special cases where can be expressed in terms of simpler functions.

S.2.8-2. Basic properties.

The function possesses the following properties:

$$(\ ,\beta,\ ;\ ) = \ (\beta,\ ,\ ;\ ),$$

$$(\ ,\beta,\ ;\ ) = (1 - \ )^{- \ -} \quad (\ - \ ,\quad - \beta,\ ;\ ),$$

$$(\ ,\beta,\ ;\ ) = (1 - \ )^{-} \quad ,\quad - \beta,\ ;\ \frac{}{\ - 1} \ ,$$

$$\frac{}{} \ (\ ,\beta,\ ;\ ) = \frac{(\ )\ (\beta)}{(\ )} \ (\ + \ ,\beta + \ ,\quad + \ ;\ ).$$

If is not an integer, then the general solution of the hypergeometric equation can be written in the form:

$$= \ _{1} \ (\ ,\beta,\ ;\ ) + \ _{2} \ ^{1-} \quad (\ - \quad + 1,\ \beta - \quad + 1,\ 2 - \ ;\ ).$$

S.2.8-3. Integral representations.

For $> \beta > 0$, the hypergeometric function can be expressed in terms of a definite integral:

$$(\ ,\beta,\ ;\ ) = \frac{\Gamma(\ )}{\Gamma(\beta)\,\Gamma(\ - \beta)} \int_0^1 \ ^{-1}(1 - \ )^{\ - \ -1}(1 - \quad)^{-} \quad ,$$

where $\Gamma(\beta)$ is the gamma function.

See M. Abramowitz and I. Stegun (1964) and H. Bateman and A. Erdélyi (1953, Vol. 1) for more detailed information about hypergeometric functions.

S.2.9. Legendre Functions and Legendre Polynomials

S.2.9-1. Definitions. Basic formulas.

The associated Legendre functions (z) and $Q\ (z)$ of the first and the second kind are linearly independent solutions of the Legendre equation:

$$(1-z^2)\ {}''\ -2z\ {}'+[\ (\ +1)-\ {}^2(1-z^2)^{-1}]\ =0,$$

where the parameters and and the variable z can assume arbitrary real or complex values.

For $|1-z|<2$, the formulas

$$(z)=\frac{1}{\Gamma(1-\ )}\ \frac{z+1}{z-1}\ \ {}^{2}\quad -\ ,\ 1+\ ,\ 1-\ ,\ \frac{1-z}{2}\ ,$$

$$Q\ (z)=A\ \frac{z-1}{z+1}\ \ \overline{2}\quad -\ ,\ 1+\ ,\ 1+\ ,\ \frac{1-z}{2}\ +B\ \frac{z+1}{z-1}\ \ \overline{2}\quad -\ ,\ 1+\ ,\ 1-\ ,\ \frac{1-z}{2}\ ,$$

$$A=e\ \ {}^{\pi}\frac{\Gamma(-\ )\Gamma(1+\ +\ )}{2\,\Gamma(1+\ -\ )},\qquad B=e\ \ {}^{\pi}\frac{\Gamma(\ )}{2},\qquad {}^{2}=-1,$$

are valid, where $(a,\ ,\ ;z)$ is the hypergeometric series (see Subsection S.2.8).

For $|z|>1$,

$$(z)=\frac{2^{-\ -1}\Gamma(-\frac{1}{2}-\ )}{\overline{\ }\Gamma(-\ -\ )}z^{-\ +\ -1}(z^2-1)^{-\ \ 2}\quad \frac{1+\ -}{2},\ \frac{2+\ -}{2},\ \frac{2\ +3}{2},\ \frac{1}{z^2}\Big)$$

$$+\frac{2\ \Gamma(\frac{1}{2}+\ )}{\overline{\ }\Gamma(1+\ -\ )}z\ {}^{+}\ (z^2-1)^{-\ \ 2}\quad -\frac{+}{2},\ \frac{1-\ -}{2},\ \frac{1-2}{2},\ \frac{1}{z^2}\Big),$$

$$Q\ (z)=e\ {}^{\pi}\ \frac{\overline{\ }\Gamma(\ +\ +1)}{2\ {}^{+1}\Gamma(\ +\frac{3}{2})}z^{-\ -\ -1}(z^2-1)\ \ {}^{2}\quad \frac{2+\ +}{2},\ \frac{1+\ +}{2},\ \frac{2\ +3}{2},\ \frac{1}{z^2}\Big).$$

The functions $(z)\equiv\ {}^0(z)$ and $Q\ (z)\equiv Q^0(z)$ are called the Legendre functions.

The modified associated Legendre functions, on the cut $z=\ $, $-1<\ <1$, of the real axis, are defined by the formulas:

$$(\ )=\tfrac{1}{2}\big[e^{\frac{1}{2}\ \ \pi}\ \ (\ +\ 0)+e^{-\frac{1}{2}\ \ \pi}\ \ (\ -\ 0)\big],$$

$$Q\ (\ )=\tfrac{1}{2}e^{-\ \ \pi}\big[e^{-\frac{1}{2}\ \ \pi}Q\ (\ +\ 0)+e^{\frac{1}{2}\ \ \pi}Q\ (\ -\ 0)\big].$$

S.2.9-2. Trigonometric expansions.

For $-1<\ <1$, the modified associated Legendre functions can be represented in the form of trigonometric series as:

$$(\cos\theta)=\frac{2\ {}^{+1}}{\overline{\ }}\frac{\Gamma(\ +\ +1)}{\Gamma(\ +\frac{3}{2})}(\sin\theta)\ \ \sum_{=0}\frac{(\frac{1}{2}+\ )\ (1+\ +\ )}{k!\,(\ +\frac{3}{2})}\sin[(2k+\ +\ +1)\theta],$$

$$Q\ (\cos\theta)=\ \overline{\ }2\ \frac{\Gamma(\ +\ +1)}{\Gamma(\ +\frac{3}{2})}(\sin\theta)\ \ \sum_{=0}\frac{(\frac{1}{2}+\ )\ (1+\ +\ )}{k!\,(\ +\frac{3}{2})}\cos[(2k+\ +\ +1)\theta],$$

where $0<\theta<\ $.

S.2.9-3. Some relations.

$$(z) = \ _{-\ -1}(z), \qquad (z) = \frac{\Gamma(\ +\ +1)}{\Gamma(\ -\ +1)}\ ^{-}\ (z), \qquad = 0, 1, 2,$$

$$Q\ (z) = \frac{}{2\sin(\ )} e^{\ \pi} \qquad (z) - \frac{\Gamma(1+\ +\ )}{\Gamma(1+\ -\ )}\ ^{-}\ (z)\ .$$

For $0 < \ < 1$,

$$(-\ ) = \ (\ )\cos[\ (\ +\ )] - 2\ ^{-1} Q\ (\ )\sin[\ (\ +\ )],$$

$$Q\ (-\ ) = -Q\ (\ )\cos[\ (\ +\ )] - \tfrac{1}{2}\ \ (\ )\sin[\ (\ +\ )].$$

For $-1 < \ < 1$,

$$_{+1}(\ ) = \frac{2\ +1}{-\ +1}\ (\ ) - \frac{+}{-\ +1}\ _{-1}(\ ),$$

$$_{+1}(\ ) = \ _{-1}(\ ) - (2\ +1)(1-\ ^2)^{1\ 2}\ ^{-1}(\ ).$$

Wronskians:

$$(\ , Q\ ) = \frac{1}{1-\ ^2}, \qquad (\ , Q\ ) = \frac{k}{1-\ ^2}, \qquad k = 2^2\ \frac{\Gamma\left(\frac{+\ +1}{2}\right)\Gamma\left(\frac{+\ +2}{2}\right)}{\Gamma\left(\frac{-\ +1}{2}\right)\Gamma\left(\frac{-\ +2}{2}\right)}.$$

For $= 0,\ 1,\ 2,\ \ ,$

$$(\ ) = (-1)\ (1-\ ^2)\ ^{\ 2} \frac{}{}\ (\ ), \quad Q\ (\ ) = (-1)\ (1-\ ^2)\ ^{\ 2} \frac{}{} Q\ (\ ).$$

S.2.9-4. Integral representations.

For $= 0,\ 1,\ 2,\ \ ,$

$$(z) = \frac{\Gamma(\ +\ +1)}{\Gamma(\ +1)} \int_0^{\pi} \left(z + \cos\ \ \overline{z^2-1}\ \ \cos(\ )\ \right), \qquad \operatorname{Re} z > 0,$$

$$Q\ (z) = (-1)\ \frac{\Gamma(\ +\ +1)}{2\ ^{+1}\Gamma(\ +1)}(z^2-1)^{-\ \ 2} \int_0^{\pi} (z+\cos\ )^{-\ -1}(\sin\ )^{2\ +1}\ , \qquad \operatorname{Re}\ > -1.$$

Note that $z \neq \ , -1 < \ < 1$, in the latter formula.

S.2.9-5. Legendre polynomials.

The Legendre polynomials $= \ (\ )$ and the Legendre functions $Q\ (\ )$ are solutions of the linear differential equation:

$$(1-\ ^2)\ '' - 2\ \ ' + \ (\ +1)\ = 0.$$

The Legendre polynomials $(\ )$ and the Legendre functions $Q\ (\ )$ are defined by the formulas:

$$(\ ) = \frac{1}{!\,2}\ \frac{}{}(\ ^2-1)\ , \quad Q\ (\ ) = \frac{1}{2}\ (\ )\ln\frac{1+}{1-} - \sum_{=1} \frac{1}{}\ _{-1}(\ )\ _{-}(\ ).$$

The polynomials $= \ (\ )$ can be calculated recursively using the relations:

$$_0(\ ) = 1, \quad _1(\ ) = \ , \quad _2(\ ) = \frac{1}{2}(3\ ^2-1), \quad , \quad _{+1}(\ ) = \frac{2\ +1}{+1}\ (\ ) - \frac{}{+1}\ _{-1}(\ ).$$

The first three functions $Q\ = Q\ (\ )$ are given by:

$$Q_0(\ ) = \frac{1}{2}\ln\frac{1+}{1-}, \quad Q_1(\ ) = \frac{}{2}\ln\frac{1+}{1-} - 1, \quad Q_2(\ ) = \frac{3\ ^2-1}{4}\ln\frac{1+}{1-} - \frac{3}{2}\ .$$

The polynomials $(\ )$ have the explicit representation:

$$(\ ) = 2^{-}\sum_{=0}^{[\ 2]} (-1)\ \ _{2\ -2}\ \ ^{-2}\ ,$$

where $[A]$ is the integer part of a number A.

S.2.9-6. Zeros of the Legendre polynomials and the generating function.

All zeros of $\ (\ )$ are real and lie on the interval $-1 < \ < +1$; the functions $\ (\ )$ form an orthogonal system on the interval $-1 \le \ \le +1$, with

$$\int_{-1}^{+1} \ (\ ) \ (\ ) \ = \begin{cases} 0 & \text{if } \ \neq \ , \\ \dfrac{2}{2\ +1} & \text{if } \ = \ . \end{cases}$$

The generating function is:

$$\frac{1}{\sqrt{1-2\ \ +\ ^2}} = \sum_{=0} \ (\ ) \qquad (|\ | < 1).$$

S.2.9-7. Associated Legendre functions.

The associated Legendre functions $\ (\ )$ of order $\ $ are defined by the formulas:

$$\ (\ ) = (1-\ ^2)^{\ 2} \frac{}{} \ (\ ), \qquad \ = 1,\ 2,\ 3,\ \ , \qquad \ = 0,\ 1,\ 2,$$

It is assumed by definition that $\ ^0(\ ) = \ (\ )$.

The functions $\ (\ )$ form an orthogonal system on the interval $-1 \le \ \le +1$, with

$$\int_{-1}^{+1} \ (\ ) \ (\ ) \ = \begin{cases} 0 & \text{if } \ \neq k, \\ \dfrac{2}{2\ +1} \dfrac{(\ +\ )!}{(\ -\ )!} & \text{if } \ = k. \end{cases}$$

The functions $\ (\ )$ (with $\ \neq 0$) are orthogonal on the interval $-1 \le \ \le +1$ with weight $(1-\ ^2)^{-1}$, that is,

$$\int_{-1}^{+1} \frac{\ (\ ) \ (\ )}{(1-\ ^2)} \ = \begin{cases} 0 & \text{if } \ \neq k, \\ \dfrac{(\ +\ )!}{\ (\ -\ )!} & \text{if } \ = k. \end{cases}$$

S.2.10. Parabolic Cylinder Functions

S.2.10-1. Definitions. Basic formulas.

The Weber parabolic cylinder function $\ (z)$ is a solution of the linear differential equation:

$$\ '' + \left(-\tfrac{1}{4}z^2 + \ + \tfrac{1}{2}\right) \ = 0,$$

where the parameter $\ $ and the variable z can assume arbitrary real or complex values. Another linearly independent solution of this equation is the function $\ _{-\ -1}(\ z)$; if $\ $ is noninteger, then $\ (-z)$ can also be taken as a linearly independent solution.

The parabolic cylinder functions can be expressed in terms of degenerate hypergeometric functions as:

$$\ (z) = 2^{1\ 2} \exp\left(-\tfrac{1}{4}z^2\right) \frac{\Gamma\left(\tfrac{1}{2}\right)}{\Gamma\left(\tfrac{1}{2}-\tfrac{}{2}\right)} \ \left(-\tfrac{}{2}, \tfrac{1}{2}, \tfrac{1}{2}z^2\right) + 2^{-1\ 2} \frac{\Gamma\left(-\tfrac{1}{2}\right)}{\Gamma\left(-\tfrac{}{2}\right)} z \ \left(\tfrac{1}{2}-\tfrac{}{2}, \tfrac{3}{2}, \tfrac{1}{2}z^2\right) .$$

For nonnegative integer $\ = \ $, we have

$$\ (z) = 2^{-\ \ 2} \exp\left(-\tfrac{1}{4}z^2\right) \ \left(2^{-1\ 2} z\right), \qquad \ = 0,\ 1,\ 2,\ \ ;$$

$$\ (z) = (-1)\ \exp\left(z^2\right) \frac{}{z} \exp\left(-z^2\right),$$

where $\ (z)$ is the Hermite polynomial of order $\ $.

S.2.10-2. Integral representations.

$$(z) = \quad \overline{2} \quad \exp\left(\tfrac{1}{4}z^2\right) \int_0 \quad \exp\left(-\tfrac{1}{2} \;^2\right) \cos\left(z \quad - \tfrac{1}{2}\right) \qquad \text{for Re} \quad > -1,$$

$$(z) = \frac{1}{\Gamma(-\ )} \exp\left(-\tfrac{1}{4}z^2\right) \int_0 \quad ^{- \ -1} \exp\left(-z \quad - \tfrac{1}{2} \;^2\right) \qquad \text{for Re} \quad < 0.$$

S.2.10-3. Asymptotic expansion as $|z|$.

$$(z) = z \quad \exp\left(-\tfrac{1}{4}z^2\right) \sum_{=0} \frac{(-2) \ \left(-\frac{}{2}\right) \ \left(\frac{1}{2} - \frac{}{2}\right)}{!} \frac{1}{z^2} + \quad \left(|z|^{-2 \quad -2}\right) \qquad \text{for } |\arg z| < \frac{3}{4},$$

where $(a)_0 = 1$, $(a) \ = a(a+1) \quad (a + \quad - 1)$ for $\ = 1, 2, 3,$

S.2.11. Orthogonal Polynomials

All zeros of each of the orthogonal polynomials $\ (\ )$ considered in this section are real and simple. The zeros of the polynomials $\ (\ )$ and $_{+1}(\ )$ are alternating.

For Legendre polynomials see Subsections S.2.9-5, S.2.9-6, and S.2.9-7.

S.2.11-1. Laguerre polynomials and generalized Laguerre polynomials.

The Laguerre polynomials $L \ = L \ (\ )$ satisfy the linear differential equation

$$'' \ + (1 - \ ) \ ' + \quad = 0$$

and are defined by the formulas:

$$L \ (\ ) = \frac{1}{!} e \ \frac{}{}\left(\ e^{-}\right) = \frac{(-1)}{!} \quad - \ ^2 \quad ^{-1} + \frac{^2(\ -1)^2}{2!} \quad ^{-2} + \quad .$$

The first four polynomials are given by:

$$L_0 = 1, \quad L_1 = - \ + 1, \quad L_2 = \tfrac{1}{2}(\ ^2 - 4 \ + 2), \quad L_3 = \tfrac{1}{6}(- \ ^3 + 9 \ ^2 - 18 \ + 6).$$

To calculate $L \ (\ )$ for $\ \geq 2$, one can use the recurrence formulas:

$$L_{\ +1}(\ ) = \frac{1}{\ +1}\left[(2 \ + 1 - \ )L \ (\ ) - \ L_{\ -1}(\ )\right].$$

The functions $L \ (\ )$ form an orthonormal system on the interval $0 < \ <$ with weight e^{-} :

$$\int_0 \ e^{-} L \ (\ ) L \ (\ ) \quad = \begin{cases} 0 & \text{if} \ \neq \ , \\ 1 & \text{if} \ = \ . \end{cases}$$

The generating function is:

$$\frac{1}{1-} \exp\left(-\frac{}{1-}\right) = \sum_{=0} L \ (\ ) \ , \qquad |\ | < 1.$$

The generalized Laguerre polynomials $L \ = L \ (\ ) \ (\ > -1)$ satisfy the linear differential equation

$$'' \ + (\ + 1 - \ ) \ ' + \quad = 0$$

and are defined by the formulas:

$$L\ (\ ) = \frac{1}{!}\ ^{-}\ e\ \overline{\quad}(\ ^{+}\ e^{-}\ = \ _{=0}\ \ ^{-}_{+}\ \frac{(-\ )}{!}.$$

The first two polynomials have the form:

$$L_0 = 1, \quad L_1 = \quad + 1 - \quad .$$

To calculate $L\ (\ )$ for $\ \geq 2,$ one can use the recurrence formulas:

$$L_{\ +1}(\ ) = \frac{1}{\ +1}\big[(2\ +\ \ +1-\ \ )L\ (\ ) - (\ +\ \ )L_{\ -1}(\ )\big].$$

The functions $L\ (\ )$ form an orthogonal system on the interval $0 < \ < \ $ with weight $\ e^{-}\ :$

$$\int_0 \quad e^{-}\ L\ (\ )L\ (\ ) \quad = \quad \begin{matrix} 0 & \text{if} \ \ \neq \ , \\ \frac{\Gamma(\ +\ \ +1)}{!} & \text{if} \ \ = \ . \end{matrix}$$

The generating function is:

$$(1-\ )^{-\ -1} \exp\ -\frac{}{1-\ } \quad = \quad _{=0}\ L\ (\ )\ \ , \qquad |\ | < 1.$$

S.2.11-2. Chebyshev polynomials.

1 . The Chebyshev polynomials of the first kind $\ = \ (\ )$ satisfy the linear differential equation

$$(1-\ ^2)\ '' - \ \ ' + \ ^2\ = 0$$

and are defined by the formulas:

$$(\ ) = \cos(\ \arccos\ \ ) = \frac{(-2)\ \ !}{(2\ )!}\ \sqrt{1-\ ^2}\ \overline{\quad}\big[(1-\ ^2)^{\ -\frac{1}{2}}\big]$$

$$= \frac{}{2}\ ^{[\ \ 2]}_{=0}\ (-1)\ \ \frac{(\ -\ \ -1)!}{!(\ -2\ \ )!}(2\ )^{\ -2} \qquad (\ = 0, 1, 2, \quad),$$

where $[A]$ stands for the integer part of a number A.

The first five polynomials are:

$$_0 = 1, \quad _1 = \ , \quad _2 = 2\ ^2 - 1, \quad _3 = 4\ ^3 - 3\ , \quad _4 = 8\ ^4 - 8\ ^2 + 1.$$

The recurrence formulas:

$$\begin{aligned} _{+1}(\ ) &= 2\ \ (\ ) - \ _{-1}(\ ), & \geq 2; \\ 2\ \ (\ )\ \ (\ ) &= \ _{+}\ (\ ) + \ _{-}\ (\ ), & \geq \ . \end{aligned}$$

The functions $\ (\ )$ form an orthogonal system on the interval $-1 < \ < +1,$ with

$$\int_{-1}^{+1} \frac{(\ )\ \ (\ )}{\sqrt{1-\ ^2}} \quad = \quad \begin{matrix} 0 & \text{if} \ \ \neq \ , \\ \frac{1}{2} & \text{if} \ \ = \ \neq 0, \\ & \text{if} \ \ = \ = 0. \end{matrix}$$

The generating function is:

$$\frac{1-\ }{1-2\ \ + \ ^2} = \ _{=0}\ \ (\ ) \qquad (|\ | < 1).$$

2 . The Chebyshev polynomials of the second kind $\quad = \quad (\ )$ satisfy the linear differential equation

$$(1- \ ^2)\ '' - 3 \quad ' + \ (\ +2) \ = 0$$

and are defined by the formulas:

$$(\ ) = \frac{\sin[(\ +1)\arccos\ ]}{\sqrt{1- \ ^2}} = \sum_{=0}^{[\ 2]} (-1) \ \frac{(\ - \ )!}{\ !(\ -2\ )!}(2\ )^{\ -2} \qquad (\ = 0,\ 1,\ 2,\ \ ).$$

The recurrence formulas:

$$_{+1}(\ ) = 2 \quad (\ ) - \quad _{-1}(\ ), \qquad \geq 2.$$

The generating function is:

$$\frac{1}{1-2 \quad + \ ^2} = \sum_{=0} \quad (\ ) \qquad (|\ | < 1).$$

S.2.11-3. Hermite polynomials.

The Hermite polynomial $\quad = \quad (\ )$ satisfies the linear differential equation

$$'' - 2 \quad ' + 2 \quad = 0$$

and is defined by the formulas:

$$(\ ) = (-1)\ \exp(\ ^2 \ \frac{}{}\ \exp(-\ ^2 = \sum_{=0}^{[\ 2]} \frac{(-1) \quad !}{\ !(\ -2\ )!}(2\ )^{\ -2} \qquad (\ = 0,\ 1,\ 2,\ \ ),$$

where $[A]$ stands for the integer part of a number A.

The first five polynomials are:

$$_0 = 1, \qquad _1 = \ , \qquad _2 = 4\ ^2 - 2, \qquad _3 = 8\ ^3 - 12\ , \qquad _4 = 16\ ^4 - 48\ ^2 + 12.$$

The recurrence formulas:

$$_{+1}(\ ) = 2 \quad (\ ) - 2 \quad _{-1}(\ ), \qquad \geq 2.$$

The functions $\quad (\ )$ form an orthogonal system on the interval $- \quad < \quad < \quad$ with weight $e^{-\ ^2}$:

$$\int_{-} \exp(-\ ^2 \quad (\ ) \quad (\ ) \quad = \begin{cases} 0 & \text{if } \ \neq \ , \\ \sqrt{\ }\ 2 \quad ! & \text{if } \ = \ . \end{cases}$$

The Hermite functions $\quad (\ )$ are introduced by the formula $\quad (\ ) = \exp(-\frac{1}{2}\ ^2 \quad (\ )$, where $\ = 0,\ 1,\ 2,\ $

The generating function is:

$$\exp(-\ ^2 + 2 \quad = \sum_{=0} \quad (\ )\frac{}{\ !}.$$

S.2.11-4. Gegenbauer polynomials.

The Gegenbauer polynomials $\ ^{(\ )}(\ )$ satisfy the linear differential equation

$$(\ ^2-1)\ ''\ +(2a+1)\ \ '-\ (2a+\ )\ =0$$

and are defined by the formulas:

$$\ ^{(\ )}(\ )=\frac{\Gamma(2a+\ )}{\Gamma(\ +1)\Gamma(2a)}\ \left(2a+\ ,-\ ,a+\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{2}\ \right)=\ _{=0}\ \frac{\Gamma(a+k)\Gamma(2a+\ +k)(\ -1)}{k!\,(\ -k)!\,2\ \Gamma(a)\Gamma(2a+2k)}.$$

The functions $\ ^{(\ )}(\ )$ form an orthogonal system on the interval $-1<\ <+1$, with

$$\int_{-1}^{+1}(1-\ ^2)^{\ -1\ 2}\ ^{(\ )}(\ )\ ^{(\ )}(\ )\ =\begin{cases}0 & \text{if } \ \neq\ ,\\ \dfrac{\Gamma(2a+\ )}{2^{2\ -1}(a+\ )\ !\,\Gamma^2(a)} & \text{if } \ =\ .\end{cases}$$

The generating function is:

$$(1-2\ \ +\ ^2)^{-\ }=\ _{=0}\ \ ^{(\ )}(\ )\qquad(|\ |<1).$$

S.2.11-5. Jacobi polynomials.

The Jacobi polynomials $\ ^{\ ,\ }=\ ^{\ ,\ }(\ )$ satisfy the linear differential equation

$$(1-\ ^2)\ ''\ +\left[\beta-\ \ -(\ +\beta+2)\ \right]\ '+\ (\ +\ +\beta+1)\ =0$$

and are defined by the formulas:

$$\ ^{\ ,\ }=\frac{(-1)}{2\ \ !}(1-\ )^{-}\ (1+\ )^{-}\ \ (1-\ )^{\ +\ }(1+\ )^{\ +\ }\ =2^{-}\ \ _{=0}\ \ _{+}\ \ _{+}^{-}\ (\ -1)^{\ -\ }(\ +1)\ ,$$

where $\ _{b}$ are binomial coefficients.

S.2.12. The Weierstrass Function

S.2.12-1. Definitions.

The Weierstrass function $\wp=\wp(z,g_2,g_3)$ is defined implicitly by the elliptic integral:

$$z=\int\frac{}{\sqrt{4\ ^3-g_2\ -g_3}}$$

and satisfies the first-order nonlinear differential equation:

$$(\wp')^2=4\wp^3-g_2\wp-g_3.$$

S.2.12-2. Some properties.

Below $\wp(z)$ stands for $\wp(z,g_2,g_3)$.

Properties:

$$\wp(z)=\wp(-z),\qquad \wp(z_1+z_2)=-\wp(z_1)-\wp(z_2)+\frac{1}{4}\ \frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\ ^2.$$

In the vicinity of the point $z=0$, the Weierstrass function can be expanded into the series:

$$\wp(z)=\frac{1}{z^2}+\frac{g_2}{20}z^2+\frac{g_3}{28}z^4+\frac{g_2^2}{1200}z^6+\frac{3g_2g_3}{6160}z^8+\ \ .$$

References for Section S.2: H. Bateman and A. Erdélyi (1953, 1955), M. Abramowitz and I. A. Stegun (1964), F. W. J. Olver (1974), A. F. Nikiforov and V. B. Uvarov (1988), N. M. Temme (1996).

S.3. Tables of Indefinite Integrals

Throughout Section S.3 the integration constant is omitted for brevity.

S.3.1. Integrals Containing Rational Functions

S.3.1-1. Integrals containing $a+$.

1. $\int \frac{}{a+} = \frac{1}{}\ln|a+|.$

2. $\int (a+) \quad = \frac{(a+)^{+1}}{(+1)}, \qquad \neq -1.$

3. $\int \frac{}{a+} = \frac{1}{{}^2}\big(a+ \quad - a\ln|a+| \;.$

4. $\int \frac{{}^2}{a+} = \frac{1}{{}^3}\ \frac{1}{2}(a+)^2 - 2a(a+) + a^2\ln|a+| \;.$

5. $\int \frac{}{(a+)} = -\frac{1}{a}\ln\frac{a+}{} \;.$

6. $\int \frac{}{{}^2(a+)} = -\frac{1}{a} + \frac{}{a^2}\ln\frac{a+}{} \;.$

7. $\int \frac{}{(a+)^2} = \frac{1}{{}^2}\ \ln|a+| + \frac{a}{a+} \;.$

8. $\int \frac{{}^2}{(a+)^2} = \frac{1}{{}^3}\ a+ \quad - 2a\ln|a+| - \frac{a^2}{a+} \;.$

9. $\int \frac{}{(a+)^2} = \frac{1}{a(a+)} - \frac{1}{a^2}\ln\frac{a+}{} \;.$

10. $\int \frac{}{(a+)^3} = \frac{1}{{}^2}\ -\frac{1}{a+} + \frac{a}{2(a+)^2} \;.$

S.3.1-2. Integrals containing $a+$ and $+$.

11. $\int \frac{a+}{+} \quad = \ +(a-)\ln|+|.$

12. $\int \frac{}{(a+)(+)} = \frac{1}{a-}\ln\frac{+}{a+}$, $a\neq$. For $a=$, see integral 2 with $=-2.$

13. $\int \frac{}{(a+)(+)} = \frac{1}{a-}\big(a\ln|a+| - \ln|+| \;.$

14. $\int \frac{}{(a+)(+)^2} = \frac{1}{(-a)(+)} + \frac{1}{(a-)^2}\ln\frac{a+}{+} \;.$

15. $\int \frac{}{(a+)(+)^2} = \frac{}{(a-)(+)} - \frac{a}{(a-)^2}\ln\frac{a+}{+} \;.$

16. $\int \frac{{}^2}{(a+)(+)^2} = \frac{{}^2}{(-a)(+)} + \frac{a^2}{(a-)^2}\ln|a+| + \frac{{}^2-2a}{(-a)^2}\ln|+|.$

17. $\int \frac{}{(a+)^2(+)^2} = -\frac{1}{(a-)^2}\ \frac{1}{a+} + \frac{1}{+}\ + \frac{2}{(a-)^3}\ln\frac{a+}{+} \;.$

18. $\int \frac{}{(a+)^2(+)^2} = \frac{1}{(a-)^2}\ \frac{a}{a+} + \frac{}{+}\ + \frac{a+}{(a-)^3}\ln\frac{a+}{+} \;.$

19. $\int \frac{{}^2}{(a+)^2(+)^2} = -\frac{1}{(a-)^2}\ \frac{a^2}{a+} + \frac{{}^2}{+}\ + \frac{2a}{(a-)^3}\ln\frac{a+}{+} \;.$

S.3.1-3. Integrals containing $a^2 + {}^2$.

20. $\displaystyle\int \frac{}{a^2+{}^2} = \frac{1}{a}\arctan\frac{}{a}.$

21. $\displaystyle\int \frac{}{(a^2+{}^2)^2} = \frac{}{2a^2(a^2+{}^2)} + \frac{1}{2a^3}\arctan\frac{}{a}.$

22. $\displaystyle\int \frac{}{(a^2+{}^2)^3} = \frac{}{4a^2(a^2+{}^2)^2} + \frac{3}{8a^4(a^2+{}^2)} + \frac{3}{8a^5}\arctan\frac{}{a}.$

23. $\displaystyle\int \frac{}{(a^2+{}^2)^{\ +1}} = \frac{}{2\ a^2(a^2+{}^2)} + \frac{2\ \ -1}{2\ a^2}\int \frac{}{(a^2+{}^2)};\qquad = 1,\ 2,$

24. $\displaystyle\int \frac{}{a^2+{}^2} = \frac{1}{2}\ln(a^2+{}^2).$

25. $\displaystyle\int \frac{}{(a^2+{}^2)^2} = -\frac{1}{2(a^2+{}^2)}.$

26. $\displaystyle\int \frac{}{(a^2+{}^2)^3} = -\frac{1}{4(a^2+{}^2)^2}.$

27. $\displaystyle\int \frac{}{(a^2+{}^2)^{\ +1}} = -\frac{1}{2\ (a^2+{}^2)};\qquad = 1,\ 2,$

28. $\displaystyle\int \frac{{}^2}{a^2+{}^2} = \ \ - a\arctan\frac{}{a}.$

29. $\displaystyle\int \frac{{}^2}{(a^2+{}^2)^2} = -\frac{}{2(a^2+{}^2)} + \frac{1}{2a}\arctan\frac{}{a}.$

30. $\displaystyle\int \frac{{}^2}{(a^2+{}^2)^3} = -\frac{}{4(a^2+{}^2)^2} + \frac{}{8a^2(a^2+{}^2)} + \frac{1}{8a^3}\arctan\frac{}{a}.$

31. $\displaystyle\int \frac{{}^2}{(a^2+{}^2)^{\ +1}} = -\frac{}{2\ (a^2+{}^2)} + \frac{1}{2}\int \frac{}{(a^2+{}^2)};\qquad = 1,\ 2,$

32. $\displaystyle\int \frac{{}^3}{a^2+{}^2} = \frac{{}^2}{2} - \frac{a^2}{2}\ln(a^2+{}^2).$

33. $\displaystyle\int \frac{{}^3}{(a^2+{}^2)^2} = \frac{a^2}{2(a^2+{}^2)} + \frac{1}{2}\ln(a^2+{}^2).$

34. $\displaystyle\int \frac{{}^3}{(a^2+{}^2)^{\ +1}} = -\frac{1}{2(\ \ -1)(a^2+{}^2)^{\ -1}} + \frac{a^2}{2\ (a^2+{}^2)};\qquad = 2,\ 3,$

35. $\displaystyle\int \frac{}{(a^2+{}^2)} = \frac{1}{2a^2}\ln\frac{{}^2}{a^2+{}^2}.$

36. $\displaystyle\int \frac{}{(a^2+{}^2)^2} = \frac{1}{2a^2(a^2+{}^2)} + \frac{1}{2a^4}\ln\frac{{}^2}{a^2+{}^2}.$

37. $\displaystyle\int \frac{}{(a^2+{}^2)^3} = \frac{1}{4a^2(a^2+{}^2)^2} + \frac{1}{2a^4(a^2+{}^2)} + \frac{1}{2a^6}\ln\frac{{}^2}{a^2+{}^2}.$

38. $\displaystyle\int \frac{}{{}^2(a^2+{}^2)} = -\frac{1}{a^2} - \frac{1}{a^3}\arctan\frac{}{a}.$

39. $\displaystyle\int \frac{}{{}^2(a^2+{}^2)^2} = -\frac{1}{a^4} - \frac{}{2a^4(a^2+{}^2)} - \frac{3}{2a^5}\arctan\frac{}{a}.$

40. $\displaystyle\int \frac{}{{}^3(a^2+{}^2)^2} = -\frac{1}{2a^4\ {}^2} - \frac{1}{2a^4(a^2+{}^2)} - \frac{1}{a^6}\ln\frac{{}^2}{a^2+{}^2}.$

41. $\displaystyle\int \frac{}{{}^2(a^2+{}^2)^3} = -\frac{1}{a^6} - \frac{}{4a^4(a^2+{}^2)^2} - \frac{7}{8a^6(a^2+{}^2)} - \frac{15}{8a^7}\arctan\frac{}{a}.$

42. $\displaystyle\int \frac{}{{}^3(a^2+{}^2)^3} = -\frac{1}{2a^6\ {}^2} - \frac{1}{a^6(a^2+{}^2)} - \frac{1}{4a^4(a^2+{}^2)^2} - \frac{3}{2a^8}\ln\frac{{}^2}{a^2+{}^2}.$

S.3.1-4. Integrals containing $a^2 - \ {}^2$.

43. $\int \frac{}{a^2 - \ {}^2} = \frac{1}{2a} \ln \frac{a+}{a-}$.

44. $\int \frac{}{(a^2 - \ {}^2)^2} = \frac{}{2a^2(a^2 - \ {}^2)} + \frac{1}{4a^3} \ln \frac{a+}{a-}$.

45. $\int \frac{}{(a^2 - \ {}^2)^3} = \frac{}{4a^2(a^2 - \ {}^2)^2} + \frac{3}{8a^4(a^2 - \ {}^2)} + \frac{3}{16a^5} \ln \frac{a+}{a-}$.

46. $\int \frac{}{(a^2 - \ {}^2)^{\ +1}} = \frac{}{2\ a^2(a^2 - \ {}^2)^{\ }} + \frac{2\ -1}{2\ a^2} \int \frac{}{(a^2 - \ {}^2)^{\ }}$; $\quad = 1,\ 2,$

47. $\int \frac{}{a^2 - \ {}^2} = -\frac{1}{2} \ln |a^2 - \ {}^2|$.

48. $\int \frac{}{(a^2 - \ {}^2)^2} = \frac{1}{2(a^2 - \ {}^2)}$.

49. $\int \frac{}{(a^2 - \ {}^2)^3} = \frac{1}{4(a^2 - \ {}^2)^2}$.

50. $\int \frac{}{(a^2 - \ {}^2)^{\ +1}} = \frac{1}{2\ (a^2 - \ {}^2)^{\ }}$; $\quad = 1,\ 2,$

51. $\int \frac{{}^2}{a^2 - \ {}^2} = - \ + \frac{a}{2} \ln \frac{a+}{a-}$.

52. $\int \frac{{}^2}{(a^2 - \ {}^2)^2} = \frac{}{2(a^2 - \ {}^2)} - \frac{1}{4a} \ln \frac{a+}{a-}$.

53. $\int \frac{{}^2}{(a^2 - \ {}^2)^3} = \frac{}{4(a^2 - \ {}^2)^2} - \frac{}{8a^2(a^2 - \ {}^2)} - \frac{1}{16a^3} \ln \frac{a+}{a-}$.

54. $\int \frac{{}^2}{(a^2 - \ {}^2)^{\ +1}} = \frac{}{2\ (a^2 - \ {}^2)^{\ }} - \frac{1}{2\ } \int \frac{}{(a^2 - \ {}^2)^{\ }}$; $\quad = 1,\ 2,$

55. $\int \frac{{}^3}{a^2 - \ {}^2} = -\frac{{}^2}{2} - \frac{a^2}{2} \ln |a^2 - \ {}^2|$.

56. $\int \frac{{}^3}{(a^2 - \ {}^2)^2} = \frac{a^2}{2(a^2 - \ {}^2)} + \frac{1}{2} \ln |a^2 - \ {}^2|$.

57. $\int \frac{{}^3}{(a^2 - \ {}^2)^{\ +1}} = -\frac{1}{2(\ -1)(a^2 - \ {}^2)^{\ -1}} + \frac{a^2}{2\ (a^2 - \ {}^2)^{\ }}$; $\quad = 2,\ 3,$

58. $\int \frac{}{\ (a^2 - \ {}^2)} = \frac{1}{2a^2} \ln \frac{{}^2}{a^2 - \ {}^2}$.

59. $\int \frac{}{\ (a^2 - \ {}^2)^2} = \frac{1}{2a^2(a^2 - \ {}^2)} + \frac{1}{2a^4} \ln \frac{{}^2}{a^2 - \ {}^2}$.

60. $\int \frac{}{\ (a^2 - \ {}^2)^3} = \frac{1}{4a^2(a^2 - \ {}^2)^2} + \frac{1}{2a^4(a^2 - \ {}^2)} + \frac{1}{2a^6} \ln \frac{{}^2}{a^2 - \ {}^2}$.

S.3.1-5. Integrals containing $a^3 + \ {}^3$.

61. $\int \frac{}{a^3 + \ {}^3} = \frac{1}{6a^2} \ln \frac{(a + \ )^2}{a^2 - a\ + \ {}^2} + \frac{1}{a^2\ \overline{3}} \arctan \frac{2\ - a}{a\ \overline{3}}$.

62. $\int \frac{}{(a^3 + \ {}^3)^2} = \frac{}{3a^3(a^3 + \ {}^3)} + \frac{2}{3a^3} \int \frac{}{a^3 + \ {}^3}$.

63. $\int \frac{}{a^3 + \ {}^3} = \frac{1}{6a} \ln \frac{a^2 - a\ + \ {}^2}{(a + \ )^2} + \frac{1}{a\ \overline{3}} \arctan \frac{2\ - a}{a\ \overline{3}}$.

64. $\int \frac{}{(a^3+\ {}^3)^2} = \frac{{}^2}{3a^3(a^3+\ {}^3)} + \frac{1}{3a^3}\int \frac{}{a^3+\ {}^3}.$

65. $\int \frac{{}^2}{a^3+\ {}^3} = \frac{1}{3}\ln|a^3+\ {}^3|.$

66. $\int \frac{}{(a^3+\ {}^3)} = \frac{1}{3a^3}\ln\frac{{}^3}{a^3+\ {}^3}\ .$

67. $\int \frac{}{(a^3+\ {}^3)^2} = \frac{1}{3a^3(a^3+\ {}^3)} + \frac{1}{3a^6}\ln\frac{{}^3}{a^3+\ {}^3}\ .$

68. $\int \frac{}{{}^2(a^3+\ {}^3)} = -\frac{1}{a^3} - \frac{1}{a^3}\int \frac{}{a^3+\ {}^3}.$

69. $\int \frac{}{{}^2(a^3+\ {}^3)^2} = -\frac{1}{a^6} - \frac{{}^2}{3a^6(a^3+\ {}^3)} - \frac{4}{3a^6}\int \frac{}{a^3+\ {}^3}.$

S.3.1-6. Integrals containing $a^3-\ {}^3$.

70. $\int \frac{}{a^3-\ {}^3} = \frac{1}{6a^2}\ln\frac{a^2+a\ \ +\ {}^2}{(a-\ )^2} + \frac{1}{a^2\ \overline{3}}\arctan\frac{2\ \ +a}{a\ \overline{3}}.$

71. $\int \frac{}{(a^3-\ {}^3)^2} = \frac{}{3a^3(a^3-\ {}^3)} + \frac{2}{3a^3}\int \frac{}{a^3-\ {}^3}.$

72. $\int \frac{}{a^3-\ {}^3} = \frac{1}{6a}\ln\frac{a^2+a\ \ +\ {}^2}{(a-\ )^2} - \frac{1}{a\ \overline{3}}\arctan\frac{2\ \ +a}{a\ \overline{3}}.$

73. $\int \frac{}{(a^3-\ {}^3)^2} = \frac{{}^2}{3a^3(a^3-\ {}^3)} + \frac{1}{3a^3}\int \frac{}{a^3-\ {}^3}.$

74. $\int \frac{{}^2}{a^3-\ {}^3} = -\frac{1}{3}\ln|a^3-\ {}^3|.$

75. $\int \frac{}{(a^3-\ {}^3)} = \frac{1}{3a^3}\ln\frac{{}^3}{a^3-\ {}^3}\ .$

76. $\int \frac{}{(a^3-\ {}^3)^2} = \frac{1}{3a^3(a^3-\ {}^3)} + \frac{1}{3a^6}\ln\frac{{}^3}{a^3-\ {}^3}\ .$

77. $\int \frac{}{{}^2(a^3-\ {}^3)} = -\frac{1}{a^3} + \frac{1}{a^3}\int \frac{}{a^3-\ {}^3}.$

78. $\int \frac{}{{}^2(a^3-\ {}^3)^2} = -\frac{1}{a^6} - \frac{{}^2}{3a^6(a^3-\ {}^3)} + \frac{4}{3a^6}\int \frac{}{a^3-\ {}^3}.$

S.3.1-7. Integrals containing $a^4\ \ \ {}^4$.

79. $\int \frac{}{a^4+\ {}^4} = \frac{1}{4a^3\ \overline{2}}\ln\frac{a^2+a\ \ \overline{2}+\ {}^2}{a^2-a\ \ \overline{2}+\ {}^2} + \frac{1}{2a^3\ \overline{2}}\arctan\frac{a\ \ \overline{2}}{a^2-\ {}^2}.$

80. $\int \frac{}{a^4+\ {}^4} = \frac{1}{2a^2}\arctan\frac{{}^2}{a^2}.$

81. $\int \frac{{}^2}{a^4+\ {}^4} = -\frac{1}{4a\ \overline{2}}\ln\frac{a^2+a\ \ \overline{2}+\ {}^2}{a^2-a\ \ \overline{2}+\ {}^2} + \frac{1}{2a\ \overline{2}}\arctan\frac{a\ \ \overline{2}}{a^2-\ {}^2}.$

82. $\int \frac{}{a^4-\ {}^4} = \frac{1}{4a^3}\ln\frac{a+\ }{a-\ }\ + \frac{1}{2a^3}\arctan\frac{}{a}.$

83. $\int \frac{}{a^4-\ {}^4} = \frac{1}{4a^2}\ln\frac{a^2+\ {}^2}{a^2-\ {}^2}\ .$

84. $\displaystyle\int \frac{{}^{2}}{a^4 - {}^{4}} = \frac{1}{4a}\ln\frac{a+}{a-} - \frac{1}{2a}\arctan\frac{}{a}.$

85. $\displaystyle\int \frac{}{(a+\quad)} = \frac{1}{a}\ln\frac{}{a+}\ .$

S.3.2. Integrals Containing Irrational Functions

S.3.2-1. Integrals containing ${}^{1\ 2}$.

1. $\displaystyle\int \frac{{}^{1\ 2}}{a^2 + {}^{2}} = \frac{2}{{}^{2}}\ {}^{1\ 2} - \frac{2a}{{}^{3}}\arctan\frac{{}^{1\ 2}}{a}.$

2. $\displaystyle\int \frac{{}^{3\ 2}}{a^2 + {}^{2}} = \frac{2\ {}^{3\ 2}}{3\ {}^{2}} - \frac{2a^2\ {}^{1\ 2}}{{}^{4}} + \frac{2a^3}{{}^{5}}\arctan\frac{{}^{1\ 2}}{a}.$

3. $\displaystyle\int \frac{{}^{1\ 2}}{(a^2 + {}^{2}\ )^2} = -\frac{{}^{1\ 2}}{{}^{2}(a^2 + {}^{2}\ )} + \frac{1}{a\ {}^{3}}\arctan\frac{{}^{1\ 2}}{a}.$

4. $\displaystyle\int \frac{{}^{3\ 2}}{(a^2 + {}^{2}\ )^2} = \frac{2\ {}^{3\ 2}}{{}^{2}(a^2 + {}^{2}\ )} + \frac{3a^2\ {}^{1\ 2}}{{}^{4}(a^2 + {}^{2}\ )} - \frac{3a}{{}^{5}}\arctan\frac{{}^{1\ 2}}{a}.$

5. $\displaystyle\int \frac{}{(a^2 + {}^{2}\ )\ {}^{1\ 2}} = \frac{2}{a}\arctan\frac{{}^{1\ 2}}{a}.$

6. $\displaystyle\int \frac{}{(a^2 + {}^{2}\ )\ {}^{3\ 2}} = -\frac{2}{a^2\ {}^{1\ 2}} - \frac{2}{a^3}\arctan\frac{{}^{1\ 2}}{a}.$

7. $\displaystyle\int \frac{}{(a^2 + {}^{2}\ )^2\ {}^{1\ 2}} = \frac{{}^{1\ 2}}{a^2(a^2 + {}^{2}\ )} + \frac{1}{a^3}\arctan\frac{{}^{1\ 2}}{a}.$

8. $\displaystyle\int \frac{{}^{1\ 2}}{a^2 - {}^{2}} = -\frac{2}{{}^{2}}\ {}^{1\ 2} + \frac{2a}{{}^{3}}\ln\frac{a + {}^{1\ 2}}{a - {}^{1\ 2}}\ .$

9. $\displaystyle\int \frac{{}^{3\ 2}}{a^2 - {}^{2}} = -\frac{2\ {}^{3\ 2}}{3\ {}^{2}} - \frac{2a^2\ {}^{1\ 2}}{{}^{4}} + \frac{a^3}{{}^{5}}\ln\frac{a + {}^{1\ 2}}{a - {}^{1\ 2}}\ .$

10. $\displaystyle\int \frac{{}^{1\ 2}}{(a^2 - {}^{2}\ )^2} = \frac{{}^{1\ 2}}{{}^{2}(a^2 - {}^{2}\ )} - \frac{1}{2a\ {}^{3}}\ln\frac{a + {}^{1\ 2}}{a - {}^{1\ 2}}\ .$

11. $\displaystyle\int \frac{{}^{3\ 2}}{(a^2 - {}^{2}\ )^2} = \frac{3a^2\ {}^{1\ 2} - 2\ {}^{2}\ {}^{3\ 2}}{{}^{4}(a^2 - {}^{2}\ )} - \frac{3a}{2\ {}^{5}}\ln\frac{a + {}^{1\ 2}}{a - {}^{1\ 2}}\ .$

12. $\displaystyle\int \frac{}{(a^2 - {}^{2}\ )\ {}^{1\ 2}} = \frac{1}{a}\ln\frac{a + {}^{1\ 2}}{a - {}^{1\ 2}}\ .$

13. $\displaystyle\int \frac{}{(a^2 - {}^{2}\ )\ {}^{3\ 2}} = -\frac{2}{a^2\ {}^{1\ 2}} + \frac{}{a^3}\ln\frac{a + {}^{1\ 2}}{a - {}^{1\ 2}}\ .$

14. $\displaystyle\int \frac{}{(a^2 - {}^{2}\ )^2\ {}^{1\ 2}} = \frac{{}^{1\ 2}}{a^2(a^2 - {}^{2}\ )} + \frac{1}{2a^3}\ln\frac{a + {}^{1\ 2}}{a - {}^{1\ 2}}\ .$

S.3.2-2. Integrals containing $(a + \quad)\ {}^{2}$.

15. $\displaystyle\int (a + \quad)\ {}^{2} \quad = \frac{2}{(\ +2)}(a + \quad)^{(\ +2)\ 2}.$

16. $\displaystyle\int \ (a + \quad)\ {}^{2} \quad = \frac{2}{{}^{2}}\ \frac{(a + \quad)^{(\ +4)\ 2}}{+4} - \frac{a(a + \quad)^{(\ +2)\ 2}}{+2}\ .$

17. $\displaystyle\int \ {}^{2}(a + \quad)\ {}^{2} \quad = \frac{2}{{}^{3}}\ \frac{(a + \quad)^{(\ +6)\ 2}}{+6} - \frac{2a(a + \quad)^{(\ +4)\ 2}}{+4} + \frac{a^2(a + \quad)^{(\ +2)\ 2}}{+2}\ .$

$\boxed{\text{S.3.2-3. Integrals containing } (\ {}^2+a^2)^{1\ 2}.}$

18. $\int (\ {}^2+a^2)^{1\ 2} \quad = \frac{1}{2}\ (a^2+\ {}^2)^{1\ 2}+\frac{a^2}{2}\ln\big[\ \ +(\ {}^2+a^2)^{1\ 2}\big].$

19. $\int \quad (\ {}^2+a^2)^{1\ 2} \quad = \frac{1}{3}(a^2+\ {}^2)^{3\ 2}.$

20. $\int (\ {}^2+a^2)^{3\ 2} \quad = \frac{1}{4}\ (a^2+\ {}^2)^{3\ 2}+\frac{3}{8}a^2\ (a^2+\ {}^2)^{1\ 2}+\frac{3}{8}a^4\ln\ \ +(\ {}^2+a^2)^{1\ 2}\ .$

21. $\int \frac{1}{\ }(\ {}^2+a^2)^{1\ 2} \quad = (a^2+\ {}^2)^{1\ 2}-a\ln\ \frac{a+(\ {}^2+a^2)^{1\ 2}}{\ }\ .$

22. $\int \frac{\ }{\overline{\ {}^2+a^2}} = \ln\big[\ \ +(\ {}^2+a^2)^{1\ 2}\big].$

23. $\int \frac{\ }{\overline{\ {}^2+a^2}} = (\ {}^2+a^2)^{1\ 2}.$

24. $\int (\ {}^2+a^2)^{-3\ 2} \quad = a^{-2}\ \ (\ {}^2+a^2)^{-1\ 2}.$

$\boxed{\text{S.3.2-4. Integrals containing } (\ {}^2-a^2)^{1\ 2}.}$

25. $\int (\ {}^2-a^2)^{1\ 2} \quad = \frac{1}{2}\ (\ {}^2-a^2)^{1\ 2}-\frac{a^2}{2}\ln\ \ +(\ {}^2-a^2)^{1\ 2}\ .$

26. $\int \quad (\ {}^2-a^2)^{1\ 2} \quad = \frac{1}{3}(\ {}^2-a^2)^{3\ 2}.$

27. $\int (\ {}^2-a^2)^{3\ 2} \quad = \frac{1}{4}\ (\ {}^2-a^2)^{3\ 2}-\frac{3}{8}a^2\ (\ {}^2-a^2)^{1\ 2}+\frac{3}{8}a^4\ln\ \ +(\ {}^2-a^2)^{1\ 2}\ .$

28. $\int \frac{1}{\ }(\ {}^2-a^2)^{1\ 2} \quad = (\ {}^2-a^2)^{1\ 2}-a\arccos\frac{a}{\ }\ .$

29. $\int \frac{\ }{\overline{\ {}^2-a^2}} = \ln\ \ +(\ {}^2-a^2)^{1\ 2}\ .$

30. $\int \frac{\ }{\overline{\ {}^2-a^2}} = (\ {}^2-a^2)^{1\ 2}.$

31. $\int (\ {}^2-a^2)^{-3\ 2} \quad = -a^{-2}\ \ (\ {}^2-a^2)^{-1\ 2}.$

$\boxed{\text{S.3.2-5. Integrals containing } (a^2-\ {}^2)^{1\ 2}.}$

32. $\int (a^2-\ {}^2)^{1\ 2} \quad = \frac{1}{2}\ (a^2-\ {}^2)^{1\ 2}+\frac{a^2}{2}\arcsin\frac{\ }{a}.$

33. $\int \quad (a^2-\ {}^2)^{1\ 2} \quad = -\frac{1}{3}(a^2-\ {}^2)^{3\ 2}.$

34. $\int (a^2-\ {}^2)^{3\ 2} \quad = \frac{1}{4}\ (a^2-\ {}^2)^{3\ 2}+\frac{3}{8}a^2\ (a^2-\ {}^2)^{1\ 2}+\frac{3}{8}a^4\arcsin\frac{\ }{a}.$

35. $\int \frac{1}{\ }(a^2-\ {}^2)^{1\ 2} \quad = (a^2-\ {}^2)^{1\ 2}-a\ln\ \frac{a+(a^2-\ {}^2)^{1\ 2}}{\ }\ .$

36. $\int \frac{\ }{\overline{a^2-\ {}^2}} = \arcsin\frac{\ }{a}.$

37. $\int \frac{\ }{\overline{a^2-\ {}^2}} = -(a^2-\ {}^2)^{1\ 2}.$

38. $\int (a^2-\ {}^2)^{-3\ 2} \quad = a^{-2}\ \ (a^2-\ {}^2)^{-1\ 2}.$

S.3.2-6. Reduction formulas.

The parameters a, , , , and below can assume arbitrary values, except for those at which denominators vanish in successive applications of a formula. Notation: $= a + $.

39. $\int (a +) = \frac{1}{+ \; +1} \; {}^{+1} + \int {}^{-1} .$

40. $\int (a +) = \frac{1}{(\; +1)} - {}^{+1} \; {}^{+1} + (\; + \; + \; +1) \int {}^{+1} .$

41. $\int (a +) = \frac{1}{(\; +1)} \; {}^{+1} \; {}^{+1} - a(\; + \; + \; +1) \int {}^{+} .$

42. $\int (a +) = \frac{1}{a(\; + \; +1)} \; {}^{- \; +1} \; {}^{+1} - (\; - \; +1) \int {}^{-} .$

S.3.3. Integrals Containing Exponential Functions

1. $\int e = \frac{1}{a} e .$

2. $\int a = \frac{a}{\ln a}.$

3. $\int e = e \left(\frac{}{a} - \frac{1}{a^2} \right).$

4. $\int {}^2 e = e \left(\frac{{}^2}{a} - \frac{2}{a^2} + \frac{2}{a^3} \right).$

5. $\int e = e \left[\frac{1}{a} - \frac{}{a^2} {}^{-1} + \frac{(\; -1)}{a^3} {}^{-2} - + (-1)^{-1} \frac{!}{a} + (-1) \frac{!}{a^{+1}} \right],$
$= 1, 2,$

6. $\int (\;) e = e \sum_{=0} \frac{(-1)}{a^{+1}} (\;)$, where $(\;)$ is an arbitrary polynomial of degree .

7. $\int \frac{}{a + \; e} = \frac{}{a} - \frac{1}{a} \ln |a + \; e \;|.$

8. $\int \frac{}{ae \; + \; e^{-}} = \begin{cases} \frac{1}{\sqrt{a}} \arctan \left(e \sqrt{\frac{a}{}} \right) & \text{if } a > 0, \\ \frac{1}{2\sqrt{-a}} \ln \left(\frac{+ e \sqrt{-a}}{- e \sqrt{-a}} \right) & \text{if } a < 0. \end{cases}$

9. $\int \frac{}{\sqrt{a + \; e}} = \begin{cases} \frac{1}{\sqrt{a}} \ln \frac{\sqrt{a + \; e} - \sqrt{a}}{\sqrt{a + \; e} + \sqrt{a}} & \text{if } a > 0, \\ \frac{2}{\sqrt{-a}} \arctan \frac{\sqrt{a + \; e}}{\sqrt{-a}} & \text{if } a < 0. \end{cases}$

S.3.4. Integrals Containing Hyperbolic Functions

S.3.4-1. Integrals containing cosh .

1. $\int \cosh(a + \;) = \frac{1}{} \sinh(a + \;).$

2. $\int \cosh = \sinh - \cosh .$

3. $\displaystyle \int {}^{2}\cosh \qquad = ({}^{2}+2)\sinh \quad -2\quad \cosh \quad .$

4. $\displaystyle \int {}^{2}\quad \cosh \qquad = (2\quad)! \quad \underset{=1}{} \frac{2}{(2k)!}\sinh \quad - \frac{2\ -1}{(2k-1)!}\cosh \quad .$

5. $\displaystyle \int {}^{2\ +1}\cosh \qquad = (2\quad +1)! \quad \underset{=0}{} \frac{2\ +1}{(2k+1)!}\sinh \quad - \frac{2}{(2k)!}\cosh \quad .$

6. $\displaystyle \int \quad \cosh \quad = \quad \sinh \quad - \quad {}^{-1}\cosh \quad + \ (\ -1)\int {}^{-2}\cosh \quad .$

7. $\displaystyle \int \cosh^{2} \qquad = \tfrac{1}{2} \quad + \tfrac{1}{4}\sinh 2 \quad .$

8. $\displaystyle \int \cosh^{3} \qquad = \sinh \quad + \tfrac{1}{3}\sinh^{3} \quad .$

9. $\displaystyle \int \cosh^{2} \qquad = \quad {}_{2} \frac{}{2^{2}} + \frac{1}{2^{2\ -1}} \underset{=0}{\overset{-1}{}} \ {}_{2} \frac{\sinh[2(\ -k)\ ]}{2(\ -k)}, \qquad = 1,\ 2,$

10. $\displaystyle \int \cosh^{2\ +1} \qquad = \frac{1}{2^{2}} \underset{=0}{} \ {}_{2\ +1} \frac{\sinh[(2\ -2k+1)\ ]}{2\ -2k+1} = \underset{=0}{} \quad \frac{\sinh^{2\ +1}}{2k+1}, \qquad = 1,\ 2,$

11. $\displaystyle \int \cosh \qquad = \frac{1}{}\sinh \quad \cosh^{\ -1} \quad + \frac{-1}{} \int \cosh^{\ -2} \qquad .$

12. $\displaystyle \int \cosh a \quad \cosh \qquad = \frac{1}{a^{2}-\ {}^{2}}(a\cosh \quad \sinh a \ - \ \cosh a \quad \sinh \quad).$

13. $\displaystyle \int \frac{}{\cosh a} = \frac{2}{a}\arctan\big(e \qquad .$

14. $\displaystyle \int \frac{}{\cosh^{2}} = \frac{\sinh}{2\quad -1} \ \frac{1}{\cosh^{2\ -1}} + \underset{=1}{\overset{-1}{}} \frac{2\ (\ -1)(\ -2)\quad (\ -k)}{(2\ -3)(2\ -5)\quad (2\ -2k-1)} \frac{1}{\cosh^{2\ -2\ -1}} \quad ,$
$= 1,\ 2,$

15. $\displaystyle \int \frac{}{\cosh^{2\ +1}} = \frac{\sinh}{2} \ \frac{1}{\cosh^{2}} + \underset{=1}{\overset{-1}{}} \frac{(2\ -1)(2\ -3)\quad (2\ -2k+1)}{2\ (\ -1)(\ -2)\quad (\ -k)} \frac{1}{\cosh^{2\ -2}}$
$\displaystyle + \frac{(2\ -1)!!}{(2\ )!!}\arctan\sinh \quad , \qquad = 1,\ 2,$

16. $$\int \frac{}{a+\quad \cosh} = \begin{cases} -\dfrac{\mathrm{sign}}{\sqrt{{}^{2}-a^{2}}}\arcsin\dfrac{+a\cosh}{a+\quad \cosh} & \text{if } a^{2} < {}^{2}, \\ \dfrac{1}{\sqrt{a^{2}-\ {}^{2}}}\ln\dfrac{a+\ +\sqrt{a^{2}-\ {}^{2}}\tanh(\ 2)}{a+\ -\sqrt{a^{2}-\ {}^{2}}\tanh(\ 2)} & \text{if } a^{2} > {}^{2}. \end{cases}$$

S.3.4-2. Integrals containing sinh .

17. $\displaystyle \int \sinh(a+\quad) \qquad = \frac{1}{}\cosh(a+\quad).$

18. $\displaystyle \int \quad \sinh \qquad = \quad \cosh \quad -\sinh \quad .$

19. $\displaystyle \int {}^{2}\sinh \qquad = ({}^{2}+2)\cosh \quad -2\quad \sinh \quad .$

20. $\displaystyle \int {}^{2}\quad \sinh \qquad = (2\quad)! \quad \underset{=0}{} \frac{2}{(2k)!}\cosh \quad - \quad \underset{=1}{} \frac{2\ -1}{(2k-1)!}\sinh \quad .$

21. $\displaystyle\int \ {}^{2\ \ +1}\sinh \qquad = (2\ \ +1)! \ {}_{=0}\ \frac{2\ \ +1}{(2k+1)!}\cosh \ \ - \frac{2}{(2k)!}\sinh \quad .$

22. $\displaystyle\int \quad \sinh \qquad = \quad \cosh \ \ - \quad {}^{-1}\sinh \ \ + \ (\ \ -1)\int \ \ {}^{-2}\sinh \qquad .$

23. $\displaystyle\int \sinh^2 \qquad = -\tfrac{1}{2} \ \ + \tfrac{1}{4}\sinh 2 \ \ .$

24. $\displaystyle\int \sinh^3 \qquad = -\cosh \ \ + \tfrac{1}{3}\cosh^3 \ \ .$

25. $\displaystyle\int \sinh^2 \qquad = (-1) \ \ {}_2 \ \frac{\ }{2^2} + \frac{1}{2^{2\ \ -1}} \ {}_{=0}^{-1} (-1) \ \ {}_2 \ \frac{\sinh[2(\ \ -k) \ ]}{2(\ \ -k)}, \qquad = 1,\ 2,$

26. $\displaystyle\int \sinh^{2\ \ +1} \qquad = \frac{1}{2^2} \ {}_{=0} \ (-1) \ \ {}_{2\ \ +1}\frac{\cosh[(2\ \ -2k+1) \ ]}{2\ \ -2k+1} = \ {}_{=0} \ (-1)^{\ +} \quad \frac{\cosh^{2\ \ +1}}{2k+1},$
$= 1,\ 2,$

27. $\displaystyle\int \sinh \qquad = \frac{1}{\ }\sinh^{\ -1} \ \ \cosh \ \ - \frac{\ -1}{\ }\int \sinh^{\ -2} \qquad .$

28. $\displaystyle\int \sinh a \ \ \sinh \qquad = \frac{1}{a^2 - \ {}^2}\big(a\cosh a \ \ \sinh \ \ - \ \cosh \ \ \sinh a \ \ .$

29. $\displaystyle\int \frac{\ }{\sinh a} = \frac{1}{a}\ln \tanh \frac{a}{2} \ .$

30. $\displaystyle\int \frac{\ }{\sinh^2} = \frac{\cosh}{2\ \ -1} \ -\frac{1}{\sinh^{2\ \ -1}}$
$\displaystyle + \ {}_{=1}^{-1}(-1)^{\ -1}\frac{2\ (\ \ -1)(\ \ -2) \quad (\ \ -k)}{(2\ \ -3)(2\ \ -5) \quad (2\ \ -2k-1)}\frac{1}{\sinh^{2\ \ -2\ \ -1}} \ , \qquad = 1,\ 2,$

31. $\displaystyle\int \frac{\ }{\sinh^{2\ \ +1}} = \frac{\cosh}{2} \ \frac{1}{\sinh^2} + \ {}_{=1}^{-1}(-1)^{\ -1}\frac{(2\ \ -1)(2\ \ -3) \quad (2\ \ -2k+1)}{2\ (\ \ -1)(\ \ -2) \quad (\ \ -k)}\frac{1}{\sinh^{2\ \ -2}}$
$\displaystyle + (-1) \ \ \frac{(2\ \ -1)!!}{(2\ \ )!!}\ln\tanh \frac{\ }{2}, \qquad = 1,\ 2,$

32. $\displaystyle\int \frac{\ }{a + \ \ \sinh} = \frac{1}{\sqrt{a^2 + \ {}^2}}\ln \frac{a\tanh(\ \ 2) - \ \ + \sqrt{a^2 + \ {}^2}}{a\tanh(\ \ 2) - \ \ - \sqrt{a^2 + \ {}^2}}.$

33. $\displaystyle\int \frac{A \ \ + B\sinh}{a + \ \ \sinh} \qquad = \frac{B}{\ } \ \ + \frac{A \ \ - Ba}{\sqrt{a^2 + \ {}^2}}\ln \frac{a\tanh(\ \ 2) - \ \ + \sqrt{a^2 + \ {}^2}}{a\tanh(\ \ 2) - \ \ - \sqrt{a^2 + \ {}^2}}.$

S.3.4-3. Integrals containing tanh or coth .

34. $\displaystyle\int \tanh \qquad = \ln\cosh \ \ .$

35. $\displaystyle\int \tanh^2 \qquad = \quad -\tanh \ \ .$

36. $\displaystyle\int \tanh^3 \qquad = -\tfrac{1}{2}\tanh^2 \ \ + \ln\cosh \ \ .$

37. $\displaystyle\int \tanh^2 \qquad = \quad - \ {}_{=1} \ \frac{\tanh^{2\ \ -2\ \ +1}}{2\ \ -2k+1}, \qquad = 1,\ 2,$

38. $\displaystyle\int \tanh^{2\ \ +1} \qquad = \ln\cosh \ \ - \ {}_{=1} \ \frac{(-1)}{2k\cosh^2} = \ln\cosh \ \ - \ {}_{=1} \ \frac{\tanh^{2\ \ -2\ \ +2}}{2\ \ -2k+2}, \qquad = 1,\ 2,$

39. $\int \tanh \qquad = -\frac{1}{-1} \tanh^{-1} \quad + \int \tanh^{-2} \qquad .$

40. $\int \coth \qquad = \ln|\sinh \ |.$

41. $\int \coth^2 \qquad = \quad -\coth \ .$

42. $\int \coth^3 \qquad = -\tfrac{1}{2}\coth^2 \quad + \ln|\sinh \ |.$

43. $\int \coth^2 \qquad = \quad - \ {}_{=1} \ \frac{\coth^{2 \ -2 \ +1}}{2 \ -2k+1}, \qquad = 1, 2,$

44. $\int \coth^{2 \ +1} \qquad = \ln|\sinh \ | - \ {}_{=1} \ \frac{}{2k \sinh^2} = \ln|\sinh \ | - \ {}_{=1} \ \frac{\coth^{2 \ -2 \ +2}}{2 \ -2k+2}, \qquad = 1, 2,$

45. $\int \coth \qquad = -\frac{1}{-1} \coth^{-1} \quad + \int \coth^{-2} \qquad .$

S.3.5. Integrals Containing Logarithmic Functions

1. $\int \ln a \qquad = \quad \ln a \ - \ .$

2. $\int \quad \ln \qquad = \tfrac{1}{2} \ ^2 \ln \ - \tfrac{1}{4} \ ^2.$

3. $\int \quad \ln a \qquad = \begin{cases} \dfrac{1}{+1} \ ^{+1} \ln a \ - \dfrac{1}{(\ +1)^2} \ ^{+1} & \text{if } \ \neq -1, \\ \tfrac{1}{2} \ln^2 a & \text{if } \ = -1. \end{cases}$

4. $\int (\ln \ )^2 \qquad = \ (\ln \ )^2 - 2 \ \ln \ + 2 \ .$

5. $\int \ (\ln \ )^2 \qquad = \tfrac{1}{2} \ ^2 (\ln \ )^2 - \tfrac{1}{2} \ ^2 \ln \ + \tfrac{1}{4} \ ^2.$

6. $\int \quad (\ln \ )^2 \qquad = \begin{cases} \dfrac{^{+1}}{+1} (\ln \ )^2 - \dfrac{2 \ ^{+1}}{(\ +1)^2} \ln \ + \dfrac{2 \ ^{+1}}{(\ +1)^3} & \text{if } \ \neq -1, \\ \tfrac{1}{3} \ln^3 & \text{if } \ = -1. \end{cases}$

7. $\int (\ln \ ) \qquad = \frac{}{+1} \ {}_{=0} \ (-1) \ (\ +1) \qquad (\ -k+1)(\ln \ )^{-} , \qquad = 1, 2,$

8. $\int (\ln \ ) \qquad = \ (\ln \ ) \ - \ \int (\ln \ )^{-1} \quad , \quad \neq -1.$

9. $\int \quad (\ln \ ) \qquad = \frac{^{+1}}{+1} \ {}_{=0} \ \frac{(-1)}{(\ +1)^{\ +1}} (\ +1) \qquad (\ -k+1)(\ln \ )^{-} , \qquad , \quad = 1, 2,$

10. $\int \quad (\ln \ ) \qquad = \frac{1}{+1} \ ^{+1} (\ln \ ) \ - \frac{}{+1} \int \quad (\ln \ )^{-1} \qquad , \qquad , \quad \neq -1.$

11. $\int \ln(a + \ ) \qquad = \frac{1}{}(a \ + \ ) \ln(a \ + \ ) - \ .$

12. $\int \quad \ln(a + \ ) \qquad = \frac{1}{2} \ \ ^2 - \frac{a^2}{^2}\Big) \ln(a + \ ) - \frac{1}{2} \ \frac{^2}{2} - \frac{a}{} \ \Big).$

13. $\int \ ^2 \ln(a + \ ) \qquad = \frac{1}{3} \ \ ^3 - \frac{a^3}{^3}\Big) \ln(a + \ ) - \frac{1}{3} \ \frac{^3}{3} - \frac{a \ ^2}{2} + \frac{a^2}{^2}\Big).$

14. $\displaystyle\int \frac{\ln}{(a+\quad)^2} = -\frac{\ln}{(a+\quad)} + \frac{1}{a}\ln\frac{}{a+}.$

15. $\displaystyle\int \frac{\ln}{(a+\quad)^3} = -\frac{\ln}{2\ (a+\quad)^2} + \frac{1}{2a\ (a+\quad)} + \frac{1}{2a^2}\ln\frac{}{a+}.$

16. $\displaystyle\int \frac{\ln}{\overline{a+}} = \begin{cases} \frac{2}{}\,(\ln\quad -2)\ \overline{a+}\quad + \ \overline{a}\,\ln\dfrac{\overline{a+}\ \ +\ \overline{a}}{\overline{a+}\ \ -\ \overline{a}} & \text{if } a>0, \\ \frac{2}{}\,(\ln\quad -2)\ \overline{a+}\quad +2\ \overline{-a}\,\arctan\dfrac{\overline{a+}}{\overline{-a}} & \text{if } a<0. \end{cases}$

17. $\displaystyle\int \ln(\ ^2+a^2)\quad = \ \ln(\ ^2+a^2)-2\quad +2a\arctan(\quad a).$

18. $\displaystyle\int \ \ln(\ ^2+a^2)\quad = \tfrac{1}{2}\big[(\ ^2+a^2)\ln(\ ^2+a^2)-\ \ ^2\big].$

19. $\displaystyle\int \ ^2\ln(\ ^2+a^2)\quad = \tfrac{1}{3}\big[\ ^3\ln(\ ^2+a^2)-\tfrac{2}{3}\ \ ^3+2a^2\quad -2a^3\arctan(\quad a)\big].$

S.3.6. Integrals Containing Trigonometric Functions

S.3.6-1. Integrals containing cos (= 1, 2,).

1. $\displaystyle\int \cos(a+\quad)\quad = \frac{1}{}\sin(a+\quad).$

2. $\displaystyle\int \quad\cos\quad = \cos\quad + \quad\sin\quad .$

3. $\displaystyle\int \ ^2\cos\quad = 2\quad\cos\quad + (\ ^2-2)\sin\quad .$

4. $\displaystyle\int \ ^2\quad\cos\quad = (2\quad)!\ \sum_{=0} (-1)\ \frac{\ ^{2\ \ -2}}{(2\quad -2k)!}\sin\quad + \sum_{=0}^{-1}(-1)\ \frac{\ ^{2\ \ -2\ \ -1}}{(2\quad -2k-1)!}\cos\quad .$

5. $\displaystyle\int \ ^{2\ \ +1}\cos\quad = (2\quad+1)!\ \sum_{=0}(-1)\ \frac{\ ^{2\ \ -2\ \ +1}}{(2\quad -2k+1)!}\sin\quad + \frac{\ ^{2\ \ -2}}{(2\quad -2k)!}\cos\quad .$

6. $\displaystyle\int \quad\cos\quad = \quad\sin\quad + \quad\ ^{-1}\cos\quad - \ (\quad -1)\int \ ^{-2}\cos\quad .$

7. $\displaystyle\int \cos^2\quad = \tfrac{1}{2}\quad + \tfrac{1}{4}\sin 2\quad .$

8. $\displaystyle\int \cos^3\quad = \sin\quad - \tfrac{1}{3}\sin^3\quad .$

9. $\displaystyle\int \cos^2\quad = \frac{1}{2^2}\ \ _2\quad + \frac{1}{2^{2\ \ -1}}\sum_{=0}^{-1}\ \ _2\ \frac{\sin[(2\quad -2k)\quad]}{2\quad -2k}.$

10. $\displaystyle\int \cos^{2\ \ +1}\quad = \frac{1}{2^2}\sum_{=0}\ \ _{2\ \ +1}\frac{\sin[(2\quad -2k+1)\quad]}{2\quad -2k+1}.$

11. $\displaystyle\int \frac{}{\cos} = \ln\ \tan\ \frac{}{2}+\frac{}{4}\quad .$

12. $\displaystyle\int \frac{}{\cos^2} = \tan\quad .$

13. $\displaystyle\int \frac{}{\cos^3} = \frac{\sin}{2\cos^2} + \frac{1}{2}\ln\ \tan\ \frac{}{2}+\frac{}{4}\quad .$

14. $\int \frac{}{\cos} = \frac{\sin}{(\ \ -1)\cos^{\ -1}} + \frac{-2}{-1}\int \frac{}{\cos^{\ -2}}, \qquad >1.$

15. $\int \frac{}{\cos^2} = \ {}^{-1}_{=0}\ \frac{(2\ \ -2)(2\ \ -4)\quad (2\ \ -2k+2)}{(2\ \ -1)(2\ \ -3)\quad (2\ \ -2k+3)}\ \frac{(2\ \ -2k)\ \ \sin\ \ -\cos}{(2\ \ -2k+1)(2\ \ -2k)\cos^{2\ -2\ +1}}$
$+\frac{2^{\ -1}(\ \ -1)!}{(2\ \ -1)!!}(\ \ \tan\ \ +\ln|\cos\ \ |\ ).$

16. $\int \cos a\ \ \cos\ \ = \frac{\sin[(\ \ -a)\ \ ]}{2(\ \ -a)} + \frac{\sin[(\ \ +a)\ \ ]}{2(\ \ +a)}, \qquad a \neq\ \ .$

17. $\int \frac{}{a+\ \ \cos} = \begin{cases} \frac{2}{\sqrt{a^2-\ ^2}}\arctan\frac{(a-\ \ )\tan(\ \ 2)}{\sqrt{a^2-\ ^2}} & \text{if } a^2 >\ ^2, \\ \frac{1}{\sqrt{\ ^2-a^2}}\ln\frac{\sqrt{\ ^2-a^2}+(\ \ -a)\tan(\ \ 2)}{\sqrt{\ ^2-a^2}-(\ \ -a)\tan(\ \ 2)} & \text{if }\ ^2 > a^2. \end{cases}$

18. $\int \frac{}{(a+\ \ \cos\ \ )^2} = \frac{\sin}{(\ ^2-a^2)(a+\ \ \cos\ \ )} - \frac{a}{\ ^2-a^2}\int \frac{}{a+\ \ \cos}.$

19. $\int \frac{}{a^2+\ ^2\cos^2} = \frac{1}{a\sqrt{a^2+\ ^2}}\arctan\frac{a\tan}{\sqrt{a^2+\ ^2}}.$

20. $\int \frac{}{a^2-\ ^2\cos^2} = \begin{cases} \frac{1}{a\sqrt{a^2-\ ^2}}\arctan\frac{a\tan}{\sqrt{a^2-\ ^2}} & \text{if } a^2 >\ ^2, \\ \frac{1}{2a\sqrt{\ ^2-a^2}}\ln\frac{\sqrt{\ ^2-a^2}-a\tan}{\sqrt{\ ^2-a^2}+a\tan} & \text{if }\ ^2 > a^2. \end{cases}$

21. $\int e\ \ \cos\ \ = e\ \ \Big(\frac{}{a^2+\ ^2}\sin\ \ + \frac{a}{a^2+\ ^2}\cos\ \ \Big).$

22. $\int e\ \ \cos^2\ \ = \frac{e}{a^2+4}\Big(a\cos^2\ \ +2\sin\ \ \cos\ \ +\frac{2}{a}\Big).$

23. $\int e\ \ \cos\ \ = \frac{e\ \ \cos^{\ -1}}{a^2+\ ^2}(a\cos\ \ +\ \ \sin\ \ ) + \frac{(\ \ -1)}{a^2+\ ^2}\int e\ \ \cos^{\ -2}\ \ .$

S.3.6-2. Integrals containing sin (= 1, 2,).

24. $\int \sin(a+\ \ )\ \ = -\frac{1}{}\cos(a+\ \ ).$

25. $\int\ \ \sin\ \ = \sin\ \ -\ \ \cos\ \ .$

26. $\int\ ^2\sin\ \ = 2\ \ \sin\ \ -(\ ^2-2)\cos\ \ .$

27. $\int\ ^3\sin\ \ = (3\ ^2-6)\sin\ \ -(\ ^3-6\ \ )\cos\ \ .$

28. $\int\ ^2\ \ \sin\ \ = (2\ \ )!\ \ {}_{=0}\ (-1)^{\ +1}\frac{^{2\ -2}}{(2\ \ -2k)!}\cos\ \ + \ {}^{-1}_{=0}(-1)\ \ \frac{^{2\ -2\ -1}}{(2\ \ -2k-1)!}\sin\ \ .$

29. $\int\ ^{2\ +1}\sin\ \ = (2\ \ +1)!\ \ {}_{=0}\ (-1)^{\ +1}\frac{^{2\ -2\ +1}}{(2\ \ -2k+1)!}\cos\ \ +(-1)\ \ \frac{^{2\ -2}}{(2\ \ -2k)!}\sin\ \ .$

30. $\int\ \ \sin\ \ = -\ \ \cos\ \ +\ \ ^{-1}\sin\ \ -\ \ (\ \ -1)\int\ ^{-2}\sin\ \ .$

31. $\int \sin^2\ \ = \tfrac{1}{2}\ \ -\tfrac{1}{4}\sin 2\ \ .$

32. $\int \sin^2 \quad = \frac{1}{4}\ {}^2 - \frac{1}{4}\ \sin 2\ - \frac{1}{8}\cos 2\ .$

33. $\int \sin^3 \quad = -\cos\ + \frac{1}{3}\cos^3\ .$

34. $\int \sin^2 \quad = \frac{1}{2^2}\ {}_2\ + \frac{(-1)}{2^{2\ -1}}\ {}^{-1}_{=0}(-1)\ {}_2\ \frac{\sin[(2\ -2k)\ ]}{2\ -2k},$

where $= \frac{!}{k!(\ -k)!}$ are binomial coefficients $(0! = 1)$.

35. $\int \sin^{2\ +1} \quad = \frac{1}{2^2}\ {}_{=0}(-1)^{\ +\ +1}\ {}_{2\ +1}\frac{\cos[(2\ -2k+1)\ ]}{2\ -2k+1}.$

36. $\int \frac{}{\sin} = \ln\ \tan\frac{}{2}\ .$

37. $\int \frac{}{\sin^2} = -\cot\ .$

38. $\int \frac{}{\sin^3} = -\frac{\cos}{2\sin^2} + \frac{1}{2}\ln\ \tan\frac{}{2}\ .$

39. $\int \frac{}{\sin} = -\frac{\cos}{(\ -1)\sin^{\ -1}} + \frac{\ -2}{\ -1}\int\frac{}{\sin^{\ -2}}, \qquad > 1.$

40. $\int \frac{}{\sin^2} = -\ {}^{-1}_{=0}\frac{(2\ -2)(2\ -4)\quad(2\ -2k+2)}{(2\ -1)(2\ -3)\quad(2\ -2k+3)}\frac{\sin\ +(2\ -2k)\ \cos}{(2\ -2k+1)(2\ -2k)\sin^{2\ -2\ +1}}$

$+\frac{2^{\ -1}(\ -1)!}{(2\ -1)!!}\big(\ln|\sin\ | -\ \cot\ \ .$

41. $\int \sin a\ \sin \quad = \frac{\sin[(\ -a)\ ]}{2(\ -a)} - \frac{\sin[(\ +a)\ ]}{2(\ +a)}, \quad a \neq\ .$

42. $\int \frac{}{a+\ \sin} = \begin{cases} \frac{2}{\sqrt{a^2-\ ^2}}\arctan\frac{\ +a\tan\ 2}{\sqrt{a^2-\ ^2}} & \text{if } a^2 >\ ^2, \\ \frac{1}{\sqrt{\ ^2-a^2}}\ln\frac{\ -\sqrt{\ ^2-a^2}+a\tan\ 2}{\ +\sqrt{\ ^2-a^2}+a\tan\ 2} & \text{if }\ ^2 > a^2. \end{cases}$

43. $\int \frac{}{(a+\ \sin\ )^2} = \frac{\cos}{(a^2-\ ^2)(a+\ \sin\ )} + \frac{a}{a^2-\ ^2}\int\frac{}{a+\ \sin}.$

44. $\int \frac{}{a^2+\ ^2\sin^2} = \frac{1}{a\sqrt{a^2+\ ^2}}\arctan\frac{\sqrt{a^2+\ ^2}\ \tan}{a}.$

45. $\int \frac{}{a^2-\ ^2\sin^2} = \begin{cases} \frac{1}{a\sqrt{a^2-\ ^2}}\arctan\frac{\sqrt{a^2-\ ^2}\ \tan}{a} & \text{if } a^2 >\ ^2, \\ \frac{1}{2a\sqrt{\ ^2-a^2}}\ln\frac{\sqrt{\ ^2-a^2}\ \tan\ +a}{\sqrt{\ ^2-a^2}\ \tan\ -a} & \text{if }\ ^2 > a^2. \end{cases}$

46. $\int \frac{\sin}{\sqrt{1+k^2\sin^2}} = -\frac{1}{k}\arcsin\frac{k\cos}{\sqrt{1+k^2}}.$

47. $\int \frac{\sin}{\sqrt{1-k^2\sin^2}} = -\frac{1}{k}\ln\ k\cos\ +\sqrt{1-k^2\sin^2}\ .$

48. $\int \sin\ \sqrt{1+k^2\sin^2}\quad = -\frac{\cos}{2}\sqrt{1+k^2\sin^2} - \frac{1+k^2}{2k}\arcsin\frac{k\cos}{\sqrt{1+k^2}}.$

49. $\int \sin\ \sqrt{1-k^2\sin^2}\quad = -\frac{\cos}{2}\sqrt{1-k^2\sin^2} - \frac{1-k^2}{2k}\ln\ k\cos\ +\sqrt{1-k^2\sin^2}\ .$

50. $\displaystyle\int e \quad \sin \quad\quad = e \quad \frac{a}{a^2 + \ ^2} \sin \quad - \frac{}{a^2 + \ ^2} \cos \quad .$

51. $\displaystyle\int e \quad \sin^2 \quad\quad = \frac{e}{a^2+4} \quad a \sin^2 \quad - 2 \sin \quad \cos \quad + \frac{2}{a} \quad .$

52. $\displaystyle\int e \quad \sin \quad\quad = \frac{e \quad \sin^{\ -1}}{a^2 + \ ^2} (a \sin \quad - \quad \cos \quad) + \frac{(\quad -1)}{a^2 + \ ^2} \int e \quad \sin^{\ -2} \quad .$

S.3.6-3. Integrals containing sin and cos .

53. $\displaystyle\int \sin a \quad \cos \quad\quad = -\frac{\cos[(a+\ )\ ]}{2(a+\ )} - \frac{\cos\left[(a-\ )\ \right]}{2(a-\ )}, \quad a \neq \quad .$

54. $\displaystyle\int \frac{}{\ ^2\cos^2 a \quad + \ ^2\sin^2 a} = \frac{1}{a} \arctan \quad \tan a \quad .$

55. $\displaystyle\int \frac{}{\ ^2\cos^2 a \quad - \ ^2\sin^2 a} = \frac{1}{2a} \ln \frac{\tan a \quad +}{\tan a \quad -} .$

56. $\displaystyle\int \frac{}{\cos^2 \quad \sin^2} = \sum_{=0}^{+\ -1} \quad \frac{}{+\ -1} \frac{\tan^{2\ -2\ +1}}{2k-2 \quad +1}, \quad , \quad = 1, 2,$

57. $\displaystyle\int \frac{}{\cos^{2\ +1} \quad \sin^{2\ +1}} = \quad _{+} \ln|\tan \quad | + \sum_{=0}^{+} \quad _{+} \frac{\tan^{2\ -2}}{2k-2}, \quad , \quad = 1, 2,$

S.3.6-4. Reduction formulas.

The parameters and below can assume any values, except for those at which the denominators on the right-hand side vanish.

58. $\displaystyle\int \sin \quad \cos \quad\quad = -\frac{\sin^{\ -1} \quad \cos^{\ +1}}{+} + \frac{-1}{+} \int \sin^{\ -2} \quad \cos \quad .$

59. $\displaystyle\int \sin \quad \cos \quad\quad = \frac{\sin^{\ +1} \quad \cos^{\ -1}}{+} + \frac{-1}{+} \int \sin \quad \cos^{\ -2} \quad .$

60. $\displaystyle\int \sin \quad \cos \quad\quad = \frac{\sin^{\ -1} \quad \cos^{\ -1}}{+} \quad \sin^2 \quad - \frac{-1}{+\quad -2}$
$\displaystyle\quad + \frac{(\quad -1)(\quad -1)}{(\quad +\quad)(\quad +\quad -2)} \int \sin^{\ -2} \quad \cos^{\ -2} \quad .$

61. $\displaystyle\int \sin \quad \cos \quad\quad = \frac{\sin^{\ +1} \quad \cos^{\ +1}}{+1} + \frac{+\quad +2}{+1} \int \sin^{\ +2} \quad \cos \quad .$

62. $\displaystyle\int \sin \quad \cos \quad\quad = -\frac{\sin^{\ +1} \quad \cos^{\ +1}}{+1} + \frac{+\quad +2}{+1} \int \sin \quad \cos^{\ +2} \quad .$

63. $\displaystyle\int \sin \quad \cos \quad\quad = -\frac{\sin^{\ -1} \quad \cos^{\ +1}}{+1} + \frac{-1}{+1} \int \sin^{\ -2} \quad \cos^{\ +2} \quad .$

64. $\displaystyle\int \sin \quad \cos \quad\quad = \frac{\sin^{\ +1} \quad \cos^{\ -1}}{+1} + \frac{-1}{+1} \int \sin^{\ +2} \quad \cos^{\ -2} \quad .$

S.3.6-5. Integrals containing tan and cot .

65. $\displaystyle\int \tan \quad\quad = -\ln|\cos \quad |.$

66. $\int \tan^2 \quad = \tan \quad - \quad .$

67. $\int \tan^3 \quad = \frac{1}{2}\tan^2 \quad + \ln|\cos \quad|.$

68. $\int \tan^2 \quad = (-1) \quad - \underset{=1}{} \frac{(-1) \ (\tan \ )^{2 \ -2 \ +1}}{2 \ -2k+1}, \qquad = 1,\ 2,$

69. $\int \tan^{2 \ +1} \quad = (-1)^{\ +1} \ln|\cos \ | - \underset{=1}{} \frac{(-1) \ (\tan \ )^{2 \ -2 \ +2}}{2 \ -2k+2}, \qquad = 1,\ 2,$

70. $\int \frac{}{a + \ \tan} = \frac{1}{a^2 + \ ^2}\big(a \quad + \quad \ln|a\cos \quad + \quad \sin \ | \ .$

71. $\int \frac{\tan}{\overline{a + \ \tan^2}} = \frac{1}{\overline{\quad - a}} \arccos \ \sqrt{1 - \frac{a}{\ }\cos} \ \Big), \qquad > a, \quad > 0.$

72. $\int \cot \quad = \ln|\sin \ |.$

73. $\int \cot^2 \quad = -\cot \quad - \quad .$

74. $\int \cot^3 \quad = -\frac{1}{2}\cot^2 \quad - \ln|\sin \ |.$

75. $\int \cot^2 \quad = (-1) \quad + \underset{=1}{} \frac{(-1) \ (\cot \ )^{2 \ -2 \ +1}}{2 \ -2k+1}, \qquad = 1,\ 2,$

76. $\int \cot^{2 \ +1} \quad = (-1) \ \ln|\sin \ | + \underset{=1}{} \frac{(-1) \ (\cot \ )^{2 \ -2 \ +2}}{2 \ -2k+2}, \qquad = 1,\ 2,$

77. $\int \frac{}{a + \ \cot} = \frac{1}{a^2 + \ ^2}\big(a \quad - \quad \ln|a\sin \quad + \quad \cos \ | \ .$

S.3.7. Integrals Containing Inverse Trigonometric Functions

1. $\int \arcsin \frac{}{a} \quad = \quad \arcsin \frac{}{a} + \quad \overline{a^2 - \ ^2}.$

2. $\int \quad \arcsin \frac{}{a} \quad ^2 \quad = \quad \arcsin \frac{}{a} \quad ^2 - 2 \quad + 2 \quad \overline{a^2 - \ ^2} \ \arcsin \frac{}{a}.$

3. $\int \quad \arcsin \frac{}{a} \quad = \frac{1}{4}(2 \ ^2 - a^2) \arcsin \frac{}{a} + \frac{}{4} \quad \overline{a^2 - \ ^2}.$

4. $\int \ ^2 \arcsin \frac{}{a} \quad = \frac{^3}{3} \arcsin \frac{}{a} + \frac{1}{9}(\ ^2 + 2a^2) \quad \overline{a^2 - \ ^2}.$

5. $\int \arccos \frac{}{a} \quad = \quad \arccos \frac{}{a} - \quad \overline{a^2 - \ ^2}.$

6. $\int \quad \arccos \frac{}{a} \quad ^2 \quad = \quad \arccos \frac{}{a} \quad ^2 - 2 \quad - 2 \quad \overline{a^2 - \ ^2} \ \arccos \frac{}{a}.$

7. $\int \quad \arccos \frac{}{a} \quad = \frac{1}{4}(2 \ ^2 - a^2) \arccos \frac{}{a} - \frac{}{4} \quad \overline{a^2 - \ ^2}.$

8. $\int \ ^2 \arccos \frac{}{a} \quad = \frac{^3}{3} \arccos \frac{}{a} - \frac{1}{9}(\ ^2 + 2a^2) \quad \overline{a^2 - \ ^2}.$

9. $\int \arctan \frac{}{a} \quad = \quad \arctan \frac{}{a} - \frac{a}{2} \ln(a^2 + \ ^2).$

10. $\int \quad \arctan \frac{}{a} \quad = \frac{1}{2}(\ {}^2 + a^2) \arctan \frac{}{a} - \frac{a}{2}.$

11. $\int \ {}^2 \arctan \frac{}{a} \quad = \frac{{}^3}{3} \arctan \frac{}{a} - \frac{a\ {}^2}{6} + \frac{a^3}{6} \ln(a^2 + \ {}^2).$

12. $\int \arccot \frac{}{a} \quad = \quad \arccot \frac{}{a} + \frac{a}{2} \ln(a^2 + \ {}^2).$

13. $\int \quad \arccot \frac{}{a} \quad = \frac{1}{2}(\ {}^2 + a^2) \arccot \frac{}{a} + \frac{a}{2}.$

14. $\int \ {}^2 \arccot \frac{}{a} \quad = \frac{{}^3}{3} \arccot \frac{}{a} + \frac{a\ {}^2}{6} - \frac{a^3}{6} \ln(a^2 + \ {}^2).$

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