

Vector Calculus

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Fall 2017

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1 Lines and Planes

A vector function defines a set of vectors, rooted at the origin, whose end-points define a locus. This locus may be a path, a surface, or a higher-dimensional equivalent. We focus on three dimensions in these notes.

Vector definition of a line in 3-space:

$$\vec{r}(t) = \vec{a} + t\vec{v} = \langle a_x + tv_x, a_y + tv_y, a_z + tv_z \rangle,$$

where \vec{a} is a displacement constant, and \vec{v} is the direction vector. This is analagous to $y = mx + b$, where b is the displacement constant, and m is the direction. This can be abstracted to higher dimensions, such as to define a plane.

Vector definition of a plane:

$$\vec{r}(s, t) = \vec{a} + s\vec{u} + t\vec{v},$$

where the function has two independent variables to allow for movement in two dimensions (to define the plane). This plane is defined by the three points at \vec{a} , $\vec{a} + \vec{u}$, and $\vec{a} + \vec{v}$. If we let point $Q(a, b, c)$ be the point at \vec{a} , then we can define the plane as the set of points $P(x, y, z)$ that satisfy $\vec{n} \cdot \overrightarrow{QP} = 0$, where \vec{n} is the normal vector to the plane. Here, \overrightarrow{QP} defines all vectors parallel to the plane, and all of these must be orthogonal to the normal vector (by definition). If $\vec{n} = \langle A, B, C \rangle$, then we have

$$A(x - a) + B(y - b) + C(z - c) = 0 \quad \rightarrow \quad Ax + By + Cz = D$$

for some constant D . Thus, the normal vector can be used to find the explicit Cartesian equation of a plane in 3-space. Note that there is no way to express a line in 3-space using a single Cartesian equation; instead, we must use parametric equations for a line.

We can calculate the normal vector by taking the cross product of two arbitrary vectors on the plane.

2 Coordinate Systems

Polar (Cylindrical) coordinates: Given by the change-of-variables

$$(x, y, z) \mapsto (r, \theta, z), \text{ where } \begin{cases} x \mapsto r \cos(\theta) \\ y \mapsto r \sin(\theta) \\ z \mapsto z \end{cases}.$$

Here, r is the xy-distance from the origin, given by $r^2 = x^2 + y^2$, and θ is the angle from the +x-axis (ranging $[0, 2\pi]$).

Spherical coordinates: Given by the change-of-variables

$$(x, y, z) \mapsto (\rho, \phi, \theta), \text{ where } \begin{cases} x \mapsto \rho \sin(\phi) \cos(\theta) \\ y \mapsto \rho \sin(\phi) \sin(\theta) \\ z \mapsto \rho \cos(\phi) \end{cases}.$$

Here, ρ is the xyz-distance from the origin, given by $\rho^2 = x^2 + y^2 + z^2$, ϕ is the angle from +z-axis (ranging $[0, \pi]$), and θ is still the angle from the +x-axis (ranging $[0, 2\pi]$).

3 Surfaces and Level Curves

Level curves offer a way of visualizing surfaces in 3-space. These are horizontal slices of the surface at regular intervals of z . Each slice produces a curve, which can be thought of as a contour line; the projection of all of these level curves onto the xy-plane produces a contour map of the surface. Taking a level curve is important in that it “eliminates” a dimension; this property can be abstracted to higher levels.

The xy-projected level curves of a sphere are concentric circles, increasing in radius at first, and then decreasing. Likewise, for a plane, the xy-projected level curves are a series of equidistant parallel lines. Mathematically, this can be found by plugging in constant values for z in the equation for the surface and simplifying the equation:

$$25 = x^2 + y^2 + z^2 \quad \rightarrow \quad 16 = x^2 + y^2 \quad \text{when } z = 3.$$

$$x + y + z = 14 \quad \rightarrow \quad y = 20 - x \quad \text{when } z = -6.$$

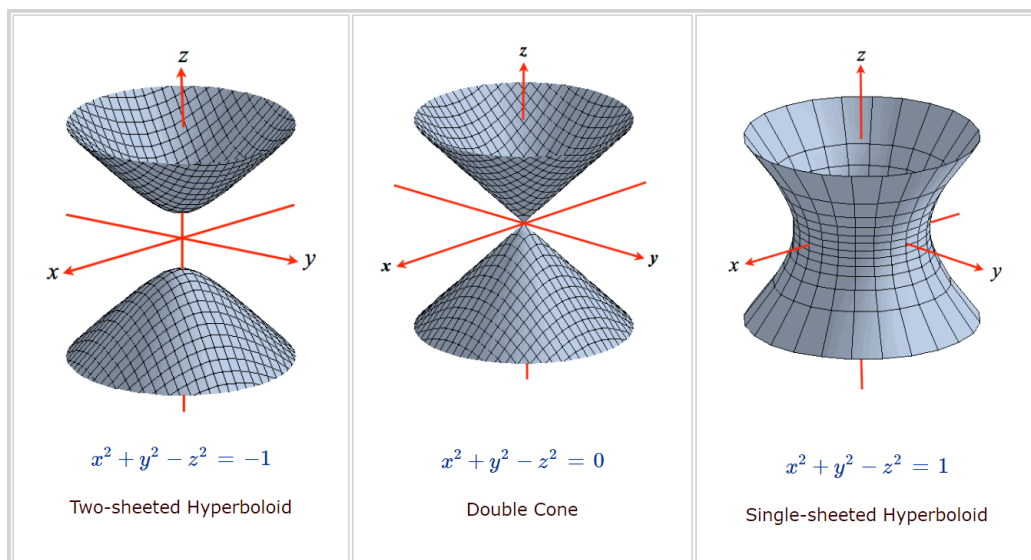
Likewise, we can take level surfaces of solids defined by functions of 3 variables: $w = f(x, y, z)$. “Slicing” these solids along a value of w will produce surfaces. The level surfaces of $w = x^2 + y^2 + z^2$, for example, produce concentric spheres of increasing radii. For the plane, we get parallel planes, each translated in the normal direction. For other functions, we get more interesting behavior.

This brings us to a discussion of surface classification in 3-space. We can group surfaces using the solids described above: $w = f(x, y, z)$. A sphere,

therefore, is any surface satisfying $w = x^2 + y^2 + z^2$. Of course, each of the independent variables can also undergo any arbitrary linear transformation (i.e. $w = (2x - 5)^2 + (1 - y)^2 + z^2$), but we will ignore these for brevity.

Surface	Equation	z -isolated
Plane	$w = x + y + z$	
Sphere	$w = x^2 + y^2 + z^2$	
Paraboloid	$w = x^2 + y^2 - z$	$z = x^2 + y^2 + w$
Hyperboloid	$w = x^2 + y^2 - z^2$	$z^2 = x^2 + y^2 + w$
Hyperbolic Paraboloid	$w = x^2 - y^2 - z$	$z = x^2 - y^2 + w$
General Cylinder	$w = f(x, y) + 0z$	

The hyperboloid has different properties depending on which level surface we take:



4 Differentiation

On a path $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t) \rangle$, we may be interested in finding a vector which is tangent to the path. We can do this by taking the derivative of the path (or more precisely, the vector function which defines the path):

$$\vec{\mathbf{r}}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

Similarly, we can perform differentiation on a surface, if we provide a direction to differentiate in. This is equivalent to first taking a trace of the surface in one direction, and then differentiating along that one-dimensional trace. This is easily defined in the x , y , or z direction as the *partial derivative* of the curve with respect to x , y , or z :

$$f_x = \frac{\partial}{\partial x} f(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

which is equivalent to differentiating the equation with respect to only one variable, while treating the others as constants. Then, at point $(a, b, f(a, b))$, the slope m in the y -direction is given by $\partial f / \partial y$, and the tangent vector is given by $\langle 0, 1, m \rangle$.

Now we consider surfaces of the form $z = f(x, y)$. The *gradient vector* is the vector given by

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

at every point (x, y) in the domain. The gradient vector at any point points in the direction of greatest increase (greatest slope), with its magnitude being equal to this greatest slope. As a corollary, this means that a gradient vector will always be perpendicular to the level curve that it's on.

When we generalize this to the 3-dimensional case $w = f(x, y, z)$, we get a gradient vector equal to

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

This vector does not tell us about slopes on the surface; rather, it tells us about how quickly the surface morphs with respect to w . Since a gradient vector is perpendicular to level curves in the 2-dimensional case, here we get a vector which is perpendicular to the level surface. We can use this technique to find normal vectors for surfaces other than planes.

Proofs for the 3 properties of gradient vectors mentioned above involve the chain rule. Consider the composition of two functions $z = f(x, y)$ and $\vec{g} = \langle x(s, t), y(s, t) \rangle$, where z is a surface and \vec{g} is a relation between variables x, y to s, t . The chain rule shows

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}.$$

In the case where $\vec{r} = \langle x(t), y(t) \rangle$, signifying a path, then we have $\vec{g}(t) = f \circ r = f(x(t), y(t))$. The chain rule for paths is then

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} = \nabla f \cdot \vec{r}'(t).$$

Using the chain rule for paths, we can show that the derivative of some surface in a given direction \vec{v} is given by

$$Df_{\vec{v}}(a, b) = (\nabla f)(a, b) \cdot \vec{v}.$$

Trivially, the directional derivative in the direction of \hat{i} is equal to f_x etc.

This allows us to show that the maximal value of the directional derivative occurs when the direction is that of the gradient. The other properties of the gradient (magnitude equals slope and perpendicular to level curve) can also be shown.

Higher-order partial derivatives can also be taken: f_{xx} , f_{xy} , f_{yx} , and f_{yy} . Clairaut's Theorem shows that if f_{xy} and f_{yx} are continuous, then they are equal. These derivatives can be used to make Taylor approximations of functions near a point. The linear approximation of a function $f(x, y)$ at point (a, b) is given by

$$L(x, y) = f(a, b) + f_x x' + f_y y',$$

where all partial derivatives are evaluated at (a, b) , $x' = (x - a)$, and $y' = (y - b)$. Similarly, a quadratic approximation can be built off the linear approximation:

$$Q(x, y) = L(x, y) + \frac{1}{2}f_{xx}(x')^2 + f_{xy}(x')(y') + \frac{1}{2}f_{yy}(y')^2$$

5 Extrema

A critical point on a surface $z = f(x, y)$ is defined as a point where $\nabla f = 0$. Tangent planes at critical points are always horizontal. A critical point is an *extrema* if all local points are either higher or lower than the critical point; if some points are higher and others are lower, then it is a *saddle point*.

Using the equation for critical points, we can solve for all critical points. To determine whether a critical point is an extrema, we calculate the *discriminant* at that point, defined as

$$D = f_{xx}f_{yy} - (f_{xy})^2.$$

The second derivative test for surfaces classifies critical points:

Local maximum	$D > 0$ and $f_{xx} < 0$
Local minimum	$D > 0$ and $f_{xx} > 0$
Saddle point	$D < 0$
Inconclusive	$D = 0$

Finding extrema has a variety of applications, including minimizing an error function in regression. The Least Squares Error is a method of finding a regression line for a set of data, and is found by minimizing the function

$$E(m, b) = (p_1 - y_1)^2 + (p_2 - y_2)^2 + \cdots + (p_n - y_n)^2,$$

where p_n is the predicted y -value of data point n (i.e. $p_n = mx_n + b$), and y_n is the actual y -value of data point n .

6 Lagrange Multipliers

Using Lagrange multipliers is a technique for finding extrema on a constraint. These problems are generally called “constrained optimization problems.” We typically constrain a surface $z = f(x, y)$ to a given path on the xy -plane, $g(x, y) = 0$. We try to find the minimum/maximum value of z along that path.

We try to find points along the path such that the path is tangent to a contour line. Since we are “walking along” the contour line, that point must be a min or max. Since the gradient vectors of f and g are perpendicular to a common tangent, they are parallel to each other (i.e. scalar multiples). This is formalized by the Lagrange multiplier:

$$\begin{aligned}(\nabla f)(a, b) &= \lambda(\nabla g)(a, b) \\ g(a, b) &= 0, \quad (\nabla g)(a, b) \neq 0\end{aligned}$$

This method generalizes easily to higher dimensions.

7 Jacobians and Transformations

Sometimes we want to transform from a certain basis to another. This can be viewed as a mapping from $\mathbb{R}^2 \mapsto \mathbb{R}^2$ for the 2-space case. A *linear transformation* is a mapping that always maps straight lines to straight lines,

parallel lines to parallel lines, and keeps the origin fixed. This restricts the “movement of the gridlines” to stretches and rotations only (no translations, warping, etc). Linear transformations can be codified as $n \times n$ matrices in \mathbb{R}^n . Multiplying this matrix to any vector in \mathbb{R}^n results in the mapped vector. An example of a nonlinear transformation is the transformation between cartesian coordinates and polar coordinates.

Coordinate transformations are useful for finding the areas of irregular regions. Instead of integrating of an irregular area, we can construct a mapping which maps the region to a rectangle; integration then becomes much easier. The issue is that these mappings often distort the area; we have to add a distortion factor to any integrals in order to find the right area of the original region. This distortion factor is the *Jacobian* of the mapping.

Consider a region D we want to integrate in (x, y) . If we find a mapping Φ which maps $\Phi(u, v) \mapsto (x, y)$ and $\Phi(D^*) \mapsto D$, where D^* is a rectangle in (u, v) . Then

$$\int \int_D f(x, y) \, dydx = \int \int_{D^*} f(\Phi(u, v)) \, |\mathbf{J}| \, dudv$$

where $|\mathbf{J}|$ is the absolute value of the Jacobian. The Jacobian can be found by taking the determinant of a “derivative matrix”:

$$\mathbf{J} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

For the case of the polar transformation, the Jacobian simplifies to a factor of r , hence the extra r factor when integrating.

In 3-space, this can be generalized to find distortion factors for volumes. The Jacobian is the determinant of a 3×3 matrix of derivatives. For spherical coordinates, the Jacobian is $\rho^2 \sin \phi$.

8 Describing Motion

For a vector path $\vec{\mathbf{r}}(t)$, we can imagine a point travelling through 3-space along the path, with its location at time t given by

$$\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t) \rangle.$$

Then, we can calculate the velocity vector and acceleration vector:

$$\vec{v}(t) = \vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

$$\vec{a}(t) = \vec{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle$$

These are related to the tangent vector $\vec{T}(t)$ and the normal vector $\vec{N}(t)$, which are unit vector that are tangent and normal to the path of travel, respectively:

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \quad \vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

Note that while the tangent vector points in the same direction as velocity, the normal vector does not necessarily have the same direction as acceleration. The cross product of \vec{T} and \vec{N} is the binormal vector, $\vec{B}(t)$. These three unit vectors form a set of mutually perpendicular vectors in 3-space.

The curvature of a path at any point is inversely proportional to the size of the largest circle that fits in the curve of the path. Trivially, the curvature at all points on a circular path of radius R is $\kappa = 1/R$. The tighter the curve, the smaller the circle and the larger the curvature. Curvature is formally defined as the rate of change of the tangent vector with respect to arc length:

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$$

In terms of t :

$$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

A useful result of this is showing that the acceleration vector can be broken down into tangent and normal components:

$$\vec{a}(t) = v'(t)\vec{T}(t) + \kappa(t)v(t)^2\vec{N}(t)$$

where $v(t)$ is the scalar speed of the particle.

9 Vector Fields

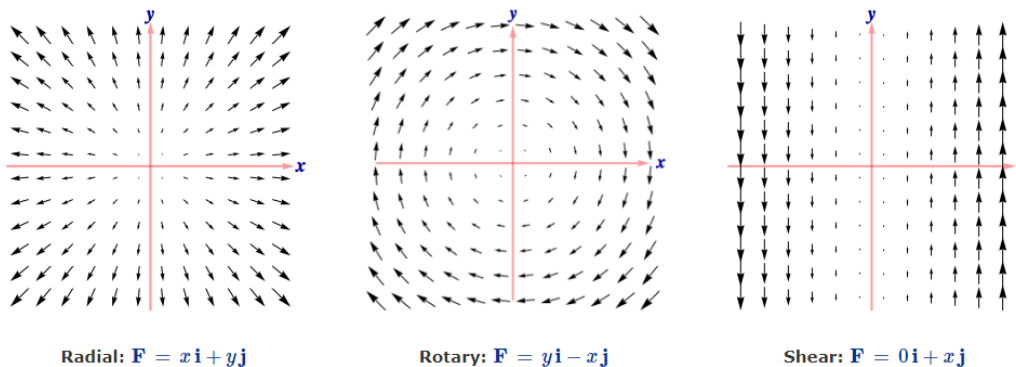
A vector field $\vec{F}(x_1, \dots, x_n)$ is a function that assigns a vector to each point in \mathbb{R}^n . An example of a vector field is a function which plots wind flow over

a region; at every location, there is a wind vector which specifies the speed and direction of the wind. All gradient vector functions are vector fields. In fact, many vector fields are gradients of another function f , which we call the *potential function*, from physics terminology.

$$\vec{\mathbf{F}} = \nabla f$$

The wind flow vector field, for example, can be viewed as the gradient of an air pressure potential. Level curves along the potential function are called *equipotentials*.

In \mathbb{R}^2 , there are three main types of vector fields, which are shown below. All of these can be encoded as matrices representing linear transformations from $\mathbb{R}^2 \mapsto \mathbb{R}^2$.



The *divergence* of a vector field at some location in the field is a measure of the “compressibility” of the vector field: a positive divergence indicates that the vector field is expansive, while a negative divergence indicates compression. For a vector field $\vec{\mathbf{F}} = \langle F_1, F_2, F_3 \rangle$ in \mathbb{R}^3 , the divergence is defined as

$$\operatorname{div} \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

A vector field is said to be incompressible if $\operatorname{div} \vec{\mathbf{F}} = 0$ everywhere.

The *curl* of a vector field at some location in the field is a measure of the “vorticity” of the vector field. It results in a vector which is normal to the

plane of rotation in the fluid. Curl is defined as

$$\text{curl } \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

A vector field is said to be *conservative* if $\text{curl } \vec{\mathbf{F}} = 0$ everywhere. All gradient vector fields are conservative.

10 Line Integrals

Line integrals are integrals over some path in 3-space. When we integrate over the path, we can apply either a scalar value or a vector value. When we apply a scalar value, it can be thought of as an integral over a density function, with the end goal of finding the mass of the wire. Conversely, when we apply a vector value, it can be thought of as an integral over a force field, with the end goal of finding the work done by the vector field on a particle constrained to the path.

10.1 Scalar Line Integrals (aka Path Integrals)

The general form for these integrals over a path C is

$$\int_C f \, ds$$

where f is the density function in terms of $\vec{\mathbf{r}}(t)$, and ds is the scalar line element. Since the scalar line element is given by the magnitude of $\vec{\mathbf{r}}'(t)$, the integral becomes

$$\int_C f \, ds = \int_a^b f \|\vec{\mathbf{r}}'(t)\| \, dt.$$

Note that when $f = 1$, we have the arclength formula for the path.

The scalar line element for a polar graph simplifies to

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta.$$

10.2 Vector Line Integrals

The general form for these integrals over a path C is

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \int_a^b \vec{\mathbf{F}} \cdot \vec{\mathbf{r}}'(t) dt.$$

where $\vec{\mathbf{F}}$ is the vector field in terms of $\vec{\mathbf{r}}(t)$, and $d\vec{\mathbf{s}} = \vec{\mathbf{r}}'(t)dt$ is the vector line element. If $\vec{\mathbf{F}} = \langle M, N, P \rangle$, then an alternate form is

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \int_a^b M dx + N dy + P dz$$

When using a gradient vector $\vec{\mathbf{F}} = \nabla f$, we can use the fundamental theorem:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)).$$

Geometrically, this is equivalent to saying that the work done by a conservative force over a path is equal to the change in potential between the start and end location. Notably, in a closed path, the change in potential must be 0, meaning that the work done is always 0.

Observe that the path integral in a circle around a point P gives the “rotational force” around that point; as the size of the circle approaches 0, the path integral approaches the component of $\text{curl } \vec{\mathbf{F}}$ in the normal direction.

$$\lim_{r \rightarrow 0} \frac{1}{\text{Area } C_r} \int_{C_r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = (\text{curl } \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}.$$

11 Parametric Surfaces

A parametric surface is defined in 3-space as the set of all points generated by the function

$$\Phi(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle.$$

Common parametrizations include $\Phi(x, y)$, $\Phi(r, \theta)$, and $\Phi(\theta, \phi)$.

Tangent vectors to a surface can be found by taking partial derivatives of the parametrization. From these, we can find the normal vector by crossing them:

$$\vec{\mathbf{n}} = \Phi_u \times \Phi_v$$

and the norm of the cross product is the scalar surface area element, since it describes the area of the parallelogram between the tangent vectors:

$$dS = \|\vec{n}\|$$

Now we can discuss the scalar surface integral over some surface S with density function f . This can be thought of as the mass of a sheet with density function f , and is given by

$$\int \int_S f dS = \int \int_R f(\Phi(u, v)) \|\Phi_u \times \Phi_v\| du dv$$

where R is a rectangular region in u, v space. Observe that when $f = 1$ we have an integral for the surface area.

For a function $z = f(x, y)$, the scalar surface area element is given by

$$dS = \sqrt{(f_x)^2 + (f_y)^2 + 1} dA.$$

Similarly, the vector surface integral over some surface S with vector field \vec{F} can be thought of as the flux of \vec{F} through the surface. It is given by

$$\int \int_S \vec{F} \cdot d\vec{S} = \int \int_R (\vec{F} \cdot \vec{n}) \|\Phi_u \times \Phi_v\| du dv = \int \int_R \vec{F} \cdot (\Phi_u \times \Phi_v) du dv$$

where \vec{n} is the oriented unit normal vector.

12 Fundamental Theorems

We will discuss three theorems which are essentially multi-dimensional generalizations of the one-dimensional Fundamental Theorem of Calculus. All of these theorems relate an integral over a region to an evaluation over the region's boundary. For example, the Fundamental Theorem of Calculus relates the integral of a function to an evaluation over its endpoints:

$$\int_a^b f'(x) dx = f(b) - f(a).$$

In particular, all of these theorems are specific instances of the Generalized Stokes Theorem, which states:

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega.$$

In other words, the integral of an exterior derivative $d\omega$ over an orientable manifold Ω is equal to the integral of the differential form ω over the boundary of Ω .

12.1 Green's Theorem

This theorem relates the integral over a plane region D to an integral over its positively-oriented boundary ∂D . In this case, positively oriented means that if one were to walk along the boundary in the positively-oriented direction, standing upright in the positive- z direction, the region D would always lie to the left.

The theorem states that, for any differentiable functions P and Q ,

$$\int \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} P dx + Q dy.$$

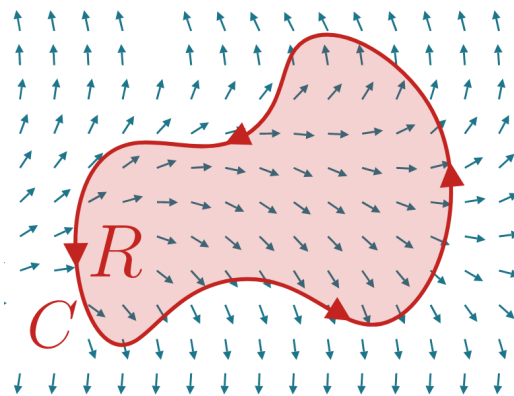
This can be reinterpreted with a vector field $\vec{\mathbf{F}} = \langle P, Q \rangle$ as

$$\int_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \int \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

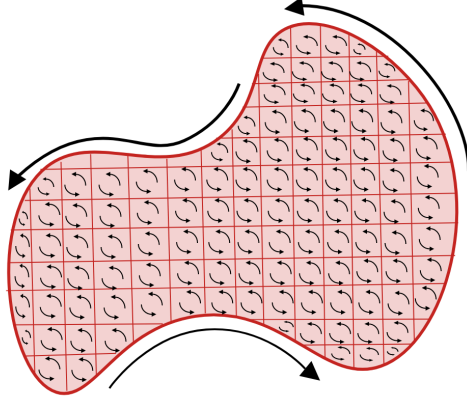
Note that for certain $\vec{\mathbf{F}}$, such as $\vec{\mathbf{F}} = \langle 0, x \rangle$, we get

$$\int_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \int \int_D dA = A,$$

which allows us to solve for the area of irregular figures by just taking a vector line integral.



The conceptual proof starts by showing that Green's Theorem holds for an infinitesimal region. Then, by induction, we can say that the theorem still holds after stitching together many infinitesimal rectangles. This is because the overlapping edges will cancel out, since two regions that share an edge will invariably integrate over that edge in opposite directions.



To prove that Green's Theorem holds for an infinitesimal region, consider what it means in terms of a vector field: the net rotational force around an infinitesimal region is equal to some quantity over its area. In section 10.2 we've seen that as a region gets smaller, the net rotational force approaches $\text{curl } \vec{\mathbf{F}}$ around that point. Then we can say that

$$\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = (\text{curl } \vec{\mathbf{F}}) \cdot \hat{\mathbf{k}}$$

Since $\vec{\mathbf{F}}$ is a plane vector field, this curl expression actually simplifies to Green's Theorem. However, this relationship to curl allows us to generalize Green's theorem to any 2-dimensional surface, not just plane regions. This brings us to Stokes' Theorem.

12.2 Stokes' Theorem

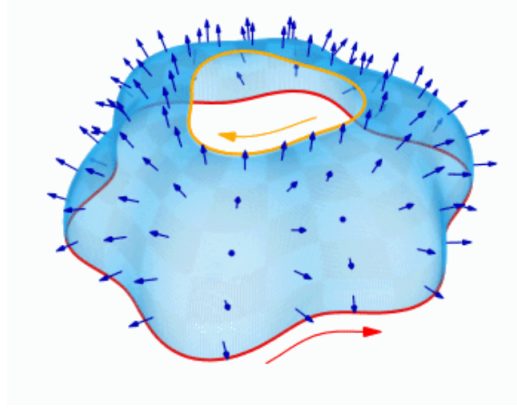
The Kelvin-Stokes Theorem, often referred to as just Stokes' Theorem, states: for an oriented surface S and a vector field $\vec{\mathbf{F}}$, the flux of $\text{curl } \vec{\mathbf{F}}$ over the surface is equal to the vector line integral of $\vec{\mathbf{F}}$ over its boundary:

$$\int \int_S \text{curl } \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_{\partial S} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}.$$

Note that for a closed surface, there is no boundary, and so the flux of curl is 0.

The boundary must be positive oriented, which follows a similar definition to the Green's Theorem case. If one were to walk along the boundary in the positively-oriented direction, standing upright in the \vec{n} direction, the surface S would always lie to the left. Note that this definition requires the direction of \vec{n} to be well-defined; in other words, S must be an oriented surface.

The proof for Stokes' Theorem follows a very similar argument to that of Green's Theorem. Indeed, Green's Theorem is just a specific instance of the Kelvin-Stokes Theorem.



12.3 Divergence Theorem

Also known as Gauss' Theorem, this theorem relates divergence in a solid to flux through its enclosing surface. If W is a solid in 3-space, and it is enclosed by a surface ∂W oriented with normal vectors pointing out from w , then the integral of $\text{div } \vec{F}$ through the volume of W is equal to the flux of \vec{F} through ∂W :

$$\int \int \int_W \text{div } \vec{F} \, dV = \int \int_{\partial W} \vec{F} \cdot d\vec{S}.$$

If we choose \vec{F} carefully, such as $\vec{F} = \langle x, 0, 0 \rangle$, then we can use this to calculate the volume of an irregular solid:

$$\int \int \int_W dV = V = \int \int_{\partial W} \vec{F} \cdot d\vec{S}.$$

Conceptually, divergence represents the compression and expansion within a vector field. We can talk about this in terms of sinks and sources. The divergence theorem then states that the flux of a field through a closed surface depends **ONLY** on the sinks and sources enclosed within the surface. The proof of the divergence theorem follows a similar argument to the previous proofs we've seen.

A physical application of the divergence theorem is Gauss' law, which states that electric flux through a closed surface is proportional to the charge enclosed within the surface:

$$\int \int_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\epsilon_0}.$$

Here, a positive charge represents a source of electric field lines, and a negative charge represents a sink. A net outwards flux indicates the presence of a net positive charge enclosed within the surface.

