Differential Equations

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Spring 2018

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1 Introduction

Differential equations are equations which involve both a function (e.g. $y(t)$) and its derivative (e.g. $y'(t)$). For example,

$$\frac{dy}{dt} + 2ty = t.$$

To solve a differential equation is to determine a relation between the function and its independent variable(s), in a way that excludes any of the derivatives. One such way is to find an explicit formula:

$$y = \frac{1}{2} + ce^{t^2}, \quad c \in \mathbb{R}.$$

Sometimes this is difficult, and we instead write an implicit equation.

Because the differential equation involves derivatives, we lose some specific information about the solution function. Because we lose information, the general solution often takes the form of a *class* of functions which differ by some constant(s). Computationally, this arises because we usually perform indefinite integration to solve the differential equation, and this always produces a constant.

However, it is possible to specify some *initial conditions* for the solution function. These constraints allow us to resolve the constants and produce a single solution:

$$\frac{dy}{dt} + 2ty = t, \quad y(0) = 1$$

$$y = \frac{1}{2} + \frac{1}{2}e^{t^2}$$

Problems in which initial conditions are provided are known as initial-value problems. The number of initial conditions required to completely resolve the general solution of a differential equation is dependent the *order* of a differential equation (which will be discussed later).

There are several techniques to solve differential equations, and the choice of technique is strongly dependent on what type of differential equation you are dealing with.

2 First Order Ordinary Differential Equations

An *ordinary differential equation* (ODE) is a differential equation in which the function in question (i.e. $y(t)$) is a function of only one variable. In contrast, there are partial differential equations, in which the function may multivariable (i.e. $y(u, v)$). ODEs can be written in the form

$$F(t,\ y,\ y',\ y'',\ \ldots,\ y^{(n)}) = 0$$

where n is called the *order* of the differential equation. In other words, the order is the degree of the highest-degree derivative in the differential equation. This section deals with first-order ODEs.

2.1 Linear first-order ODEs

Recall that an ordinary differential equation can be written in the form

$$F(t,\ y,\ y',\ \ldots,\ y^{(n)}) = 0.$$

The equation is a *linear* equation iff F can be expressed as a linear combination of the derivatives of y. In other words, F is linear iff F can be written in the form

$$F = f(t) + a_0(t)y + a_1(t)y' + a_2(t)y'' + \cdots + a_n(t)y^{(n)}.$$

In this section, we will consider first-order linear ODEs. Note that we can rewrite the equation as

$$0 = F(t,\ y,\ y') = f(t) + a_0(t)y + a_1(t)y'.$$

By dividing both sides by $a_1(t)$ and rearranging, we can rewrite the differential equation into 'standard form':

$$\frac{dy}{dt} + a(t)y = b(t).$$

This is the equation we will solve. We will first consider the simple case of a *homogeneous* equation.

2.1.1 Homogeneous linear first-order ODEs

Any linear differential equation in which the constant term $f(t)$ is 0 is called a *homogeneous* equation. In the case of the standard form for a linear first-order ODE, we have

$$\frac{dy}{dt} + a(t)y = 0.$$

This can be rearranged into the form

$$\frac{dy}{dt} \cdot \frac{1}{y} = -a(t), \quad \text{if } y \neq 0.$$

From here, we can actually "cheat" and treat the derivative as a fraction of dy and dt. Then we can further manipulate the equation to get

$$\frac{1}{y}\, dy = -a(t)\, dt, \quad \text{if } y \neq 0.$$

Notice that the left side of the equation is expressed in terms of only y, while the right side is expressed in terms of only t. The process of manipulating the equation to this form is called *separation of variables*. This is important because it allows us to easily solve the differential equation by integrating both sides. If $y \neq 0$,

$$\begin{aligned}
\int \frac{1}{y}\, dy &= \int -a(t)\, dt \\
\ln|y| &= A(t) + c, \quad c \in \mathbb{R} \\
y &= \pm \exp(A(t) + c) = \pm e^c e^{A(t)} \\
&= c_2 e^{A(t)} \quad \text{where} \quad c_2 \neq 0
\end{aligned}$$

To handle the $y = 0$ case, we can just plug it into the differential equation and verify that it is indeed a solution. Then the full general solution is

$$y = ce^{A(t)} \quad \text{where} \quad c \in \mathbb{R} \quad \text{and} \quad A(t) = \int -a(t)\, dt.$$

Notice that we must be very careful about the signs of the integration constants, and we must be careful about restricting y if we divide by y.

2.1.2 Inhomogeneous linear first-order ODEs

To solve the general form

$$\frac{dy}{dt} + a(t)y = b(t)$$

we must employ a few tricks. Our goal will be to use some function $\mu(t)$ to put the equation in the product form

$$\frac{d}{dt}(y \cdot \mu) = g(t)$$

so that we can integrate both sides by t and solve for y. The motivation for putting the equation in product form is that we can actually reverse-engineer the product rule. Notice that

$$\frac{d}{dt}(y \cdot \mu) = \mu\frac{dy}{dt} + \mu' y$$

which looks quite similar to the left side of the original differential equation. In fact, if we multiply the original differential equation by μ we have

$$\mu(t)\frac{dy}{dt} + \mu(t)a(t)y = \mu(t)b(t),$$

which is already almost the same as the product form. In fact, all we have to do is satisfy the constraint

$$\mu'(t) = \mu(t)a(t)$$

and then we will be able to rewrite it in the desired form. The challenge, then, is to find an 'integrating factor' μ which satisfies the constraint.

We can use separation of variables to conclude that

$$\mu = ce^{\int a(t)}, \quad c \neq 0.$$

But since we are multiplying the whole differential equation by μ, we can drop the constant factor. Solving, we get

$$y = \frac{1}{\mu(t)} \int g(t)\ dt$$

2.1.3 Solving initial-value problems

There are two general methods for solving an initial-value problem. One is to arrive at the general solution, and then use the initial value to restrict the general solution to a single function. The other method is to perform definite integrals with dummy variables, where the lower limits are the initial values, and the upper limits are the variables we actually want.

Suppose we have the initial value problem

$$\frac{dy}{dt} - 2ty = 0, \quad y(0) = 2.$$

The general solution is then

$$y = ce^{t^2}.$$

The initial condition $y(0) = 2$ forces $c = 2$. We have solved the initial value problem: $y = 2e^{t^2}$. That was easy.

The second method, which involves definite integrals, is certainly applicable to problems like the above, but it is usually more long-winded. However, when faced with difficult integration, the definite integral approach is appropriate. In fact, when the solution must be left in integral form, we have no choice but to use the second method.

Suppose we have the initial value problem

$$\frac{dy}{dt} - \sin\bigl(t^2\bigr)y = 0, \quad y(2) = 3.$$

Then

$$\frac{1}{y}\, dy = \sin\bigl(t^2\bigr)\, dt, \quad y \neq 0.$$

Notice that the right side is very tricky to integrate; it'd be better to leave it in integral form. However, because we have an initial value problem, we have to eliminate the constants. But this time, we have to do it *without integrating*. The only way to do this is to specify limits of integration:

$$\int_3^y \frac{1}{s}\, ds = \int_2^t \sin\bigl(s^2\bigr)\, ds$$

$$\ln|y| - \ln 3 = \int_2^t \sin\bigl(s^2\bigr)\, ds$$

$$y = 3\exp\left(\int_2^t \sin(s^2)\ ds\right).$$

Note that we can drop the condition $y \neq 0$ because the initial condition already satisfies that. We can also resolve the $\pm$ from the absolute value using the initial condition.

This technique makes intuitive sense. Suppose the differential equation represents a physical system which is undergoing some change. What we are doing here is equivalent to starting at the initial value, and then summing up all the change that has ocurred between the initial point and any future time; this will give us the state of the future system.

2.2 Exact Equations

Consider an equation of the form

$$M + N\frac{dy}{dx} = 0$$

where M and N are functions of both y and x. We can again "cheat" and treat the derivative as a fraction, and separate the variables. Now we have an equation in the form

$$Mdx + Ndy = 0$$

where $Mdx + Ndy$ is a differential 1-form. This differential form is called *exact* if there exists a potential function $\phi(x,\ y)$ whose total differential $d\phi$ is equal to $Mdx + Ndy$. Note that most differential forms are not exact; this is part of the reason why many differential equations don't have solutions.

If the differential form is exact, then we know that there is a potential function, and we can draw some conclusions about it. Notably, since its total differential is 0, we know that the potential function must be equal to a constant (for the same reason that the derivative of a constant is 0). The challenge, then is to find the potential function; once we have that, then we have an implicit general solution $\phi = c$.

How can we check if the equation is exact? Since $d\phi$ is a total differential, we know that M is ϕ_x and N is ϕ_y. Then by Clairaut's theorem, $M_y = N_x$. Checking this fact is how we check if the equation is exact; if it is not exact, then this equality will not hold.

If the equation is exact, we can integrate Mdx and Ndy separately. Notice that the constant of integration will actually be an arbitrary function of the

other variable (here, we treat y and x as independent of each other, but satisfying a constraint). If the resulting equations match up, term by term, and the constants of integration can be resolved, then the final equation is the potential function ϕ.

If the equation is not exact, there is still hope. We may be able to modify the equation and make it exact, without changing the original differential equation. Again, we use an integrating factor μ such that

$$\mu M dx + \mu N dy = 0$$

is exact. In other words, we want to find an integrating factor such that Clairaut's theorem will hold:

$$(\mu M)_y = (\mu N)_x.$$

If we make certain assumptions about μ, then this is actually not too hard of a problem. In particular, we assume that μ is either a function of only x or only y, but not a function of both x and y. Then we can use the product rule on the equation for Clairaut's theorem, simplify, and solve for μ. Once we have found μ, we can convert our non-exact equation into an exact equation, and proceed as normal.

2.3 Picard Iterations

For an arbitrary initial-value problem

$$y' = f(t,\, y) \quad \text{and} \quad y(t_0) = y_0$$

we can construct a solution using an iterative algorithm. In some cases, this iterative algorithm actually reproduces the Taylor series for the function.

First, we notice that any solution to an initial value problem can be taken by summing the initial value with the sum of all changes from the initial time to the present. In other words,

$$y(t) = y_0 + \int_{t_0}^{t} y' = y_0 + \int_{t_0}^{t} f(s,\, y)\, ds.$$

Then, the recursive algorithm is

$$y_{n+1} = y_0 + \int_{t_0}^{t} f(s,\, y_n)\, ds$$

starting at the given y_0. As n increases, the approximations converge towards $y(t)$; however, integration tends to become much messier.

3 Second Order ODEs

Now we will discuss equations which involve a second derivative of y. First, we should make some observations about the homogeneous equation. We'll first notice that the left side of the homogeneous equation

$$y'' + a(t)y' + b(t)y = 0$$

can be rewritten in terms of a *linear operator*: $L[y](t) = 0$. An operator can be thought of as a function that acts on a function; the t in parentheses indicates the independent variable of the argument function. I'll omit it from now on for clarity, since all our functions are functions of t.

In our case, our operator takes in y and returns $y'' + a(t)y' + b(t)y$. A linear operator has the additional properties

- $L[y_1 + y_2] = L[y_1] + L[y_2]$
- $L[cy] = cL[y]$ where c is a constant

and we can verify that these properties hold for the operator we're discussing.

We can exploit these properties to analyze solutions. In particular, we can combine the two properties to conclude that

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2].$$

Now, let's suppose that y_1 and y_2 are both solutions of the homogeneous differential equation; in other words, they both satisfy $L[y] = 0$. Then we can easily show that $L[c_1y_1 + c_2y_2] = 0$ as well, which implies that the sum $c_1y_1 + c_2y_2$ is also a solution. In other words, if we have two solutions, then any linear combination of those solutions is also a solution!

Does this linear combination represent the general solution to the differential equation? We might certainly think so; there are two unresolved constants, just as we'd expect from a second order equation. However, another condition must be satisfied: the two functions y_1 and y_2 must be linearly independent. If they are not, then we would be able to combine the constants, and that would leave us with less than two constants. In linear algebra terms, the dimension of the vector space of solutions must be equal to the order of the equation.

We can verify the linear independence of two solutions by either checking that

- Neither solution is a multiple of the other, or
- The Wronskian is nonzero (easier to generalize to higher order equations)

where the Wronskian is the determinant given by

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

If the solutions are independent, then they form a basis for the solution space, or a *fundamental solution set*. All their linear combinations $(c_1y_1 + c_2y_2)$ comprise the general solution.

What if the equation is not homogeneous? It can be shown that the general solution of a non-homogeneous equation is the sum of the general solution of the associated homogeneous equation and a *particular solution*, where a particular solution ψ is any solution that satisfies the non-homogeneous equation:

$$y = c_1y_1 + c_2y_2 + \psi.$$

The proof of this involves the following important lemma: The difference of any two particular solutions (of the non-homogeneous equation) yields a solution of the associated homogeneous equation.

As an aside, the lemma has an important consequence. Using three appropriate particular solutions, we can find the two linearly independent solutions to the homogeneous equation, construct the general solution (of the homogeneous equation), and then add one of the particular solutions to arrive at the general solution for the non-homogeneous equation.

The trick, of course, is to be able to find particular solutions. Or, if we're just considering a homogeneous equation, we'd have to find two linearly independent solutions. From there, the rest is easy.

3.1 Homogeneous Linear with Constant Coefficients

Suppose our linear second-order equation is homogeneous, and the coefficients are constants (rather than functions of t):

$$y'' + by' + cy = 0.$$

The trick to solving such an equation is to find the roots of the associated *characteristic polynomial*:

$$r^2 + br + c = 0.$$

Let's suppose the roots are r_1 and r_2. Then we have three cases: two distinct, real roots; one real root with multiplicity 2; and two complex roots. The general solution takes different forms depending on the case.

One physical application of this differential equation is to describe a damped oscillator. The b term is the damping coefficient (where $b > 0$), and c is the square of the natural oscillation frequency. The three cases of solutions correspond to the cases of overdamped, critically damped, and underdamped oscillation.

1. Distinct, real roots (Overdamped):

$$y(t) = Ae^{r_1 t} + Be^{r_2 t}$$

2. Real root with multiplicity = 2 (Critically damped):

$$y(t) = (A + Bt)e^{rt}$$

3. Complex roots (Underdamped): $r_1,\ r_2 = \alpha \pm \beta i$

$$y(t) = Ae^{\alpha t}\cos(\beta t) + Be^{\alpha t}\sin(\beta t)$$

In all the above equations, A and B are the undetermined constants of integration.

Where do these equations come from? First, we notice that the original differential equation has derivatives which "cancel out" to 0. This suggests that the derivatives should take a similar form to the original function, so that when we add them, their coefficients make the total sum vanish. One possibility we can try is $y = e^{rt}$. Plugging in, we arrive at $(r^2+br+c)e^{rt} = 0$, which explains where the characteristic equation comes from. For two real roots, this gives us two linearly independent solutions, so case 1 makes sense.

Extending this to complex numbers for case 3 just involves the following realization: If $y(t) = u(t) + i \cdot v(t)$ is a complex solution to a differential equation, then $u(t)$ and $v(t)$ are two real solutions. We can use Euler's formula to expand each complex solution (e^{rt} with $r = \alpha \pm i\beta$) into sines and

cosines; then we separate the real part from the imaginary part, and treat each as a real solution.

However, if we have only one root (case 2), then we have arrive at only one solution. We still have to find another linearly independent solution. The other solution comes from a method called *reduction of order* which won't be discussed.

3.2 Variation of Parameters

Ok, so what about the non-homogeneous equation?

$$y'' + by' + cy = F(t).$$

As mentioned earlier, we just need to find a) the general solution of the associated homogeneous equation; and b) a particular solution of the non-homogeneous equation. We know how to do the first part, from the previous section. Now, we just need to find a particular solution.

The reasoning goes as follows. If $y = c_1y_1 + c_2y_2$ is a homogeneous solution, then let's allow c_1 and c_2 to be functions of t rather than constants, and maybe the resulting function will satisfy the homogeneous equation. This strategy of allow the parameters c_1 and c_2 to vary with t is known as variation of parameters.

For convenience, we will also impose the additional constraint that

$$u_1'y_1 + u_2'y_2 = 0$$

So we get

$$\begin{aligned} y &= u_1y_1 + u_2y_2 \\ y' &= u_1y_1' + u_2y_2' \\ y'' &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''. \end{aligned}$$

where several terms vanished because of our constraint. When we plug these into the original non-homogeneous equation and simplify, several terms vanish again since we chose y_1 and y_2 that satisfy the homogeneous equation. The resulting equation, along with our imposed constraint, are

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= F. \end{aligned}$$

This can be written as a matrix equation, which allows us to use Cramer's rule to solve for u_1' and u_2':

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ F \end{bmatrix}$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ F & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2 F}{W} \quad \text{and} \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & F \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 F}{W}$$

where W is the Wronskian. From there, we just integrate to find the functions u_1 and u_2, and the particular solution is $y = u_1 y_1 + u_2 y_2$.

3.3 Non-homogeneous special cases

The method of variation of parameters can get messy quickly. However, if the non-homogeneous part $F(t)$ is a polynomial or an exponential,

$$ay'' + by' + cy = (a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) e^{\alpha t}$$

then we can make a good guess at a solution and save some steps. We can guess a polynomial and/or exponential solution, since when you differentiate such solutions and add them up, as in the left side of differential equation, you get another polynomial/exponential, as seen in the right side.

In particular, for a second-degree equation with constant coefficients (like the equation above), we first find the characteristic polynomial and its roots, and compare the exponent α to these roots. Of course, if there is no exponential term in $F(t)$, then $\alpha = 0$. Depending on the multiplicity of α, we guess different solutions:

$$\begin{aligned} &\psi = (A_0 + A_1 t + \cdots + A_n t^n) e^{\alpha t} && \text{if} \quad r_1, r_2 \neq \alpha \\ &\psi = t(A_0 + A_1 t + \cdots + A_n t^n) e^{\alpha t} && \text{if} \quad r_1 = \alpha,\ r_2 \neq \alpha \\ &\psi = t^2(A_0 + A_1 t + \cdots + A_n t^n) e^{\alpha t} && \text{if} \quad r_1 = r_2 = \alpha \end{aligned}$$

Then we plug in our guess for ψ into the differential equation and solve for the constants $A_0,\ A_1, \ldots,\ A_n$. This will give us our particular solution.

3.4 Series Solutions

Now we'll go back to considering homogeneous equations, but this time with function coefficients:

$$a(t)y'' + b(t)y' + c(t)y = 0.$$

It's hard for us to find explicit solutions for y, but by Taylor's theorem, we can express any continuous function in a given region as an infinite-degree polynomial. Once we substitute, differentiate, and simplify:

$$a(t)\left(\sum_{n=0}^{\infty}(n-1)(n)a_n t^{n-2}\right) + b(t)\left(\sum_{n=0}^{\infty} n a_n t^{n-1}\right) + c(t)\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0$$

This is still not that useful because we can't do much with the function coefficients – unless, of course, they happen to also be a simple polynomial[1]. In that case, we can pull in the coefficient and combine it directly with the series solution:

$$\sum_{n=0}^{\infty}(n-1)(n)a_n t^{n-2+a} + \sum_{n=0}^{\infty} n a_n t^{n-1+b} + \sum_{n=0}^{\infty} a_n t^{n+c} = 0$$

Now, we'd like to be able to combine all the sums into one big sum. To do this, we need all the sums to have the same power of t, while still maintaining the same lower index for the sums. Suppose we had

$$\sum_{n=0}^{\infty}(n-1)(n)a_n t^{n-2} - 2\sum_{n=0}^{\infty} n a_n t^n - 2\sum_{n=0}^{\infty} a_n t^n = 0.$$

We want the sums to all have matching powers of t; it will be helpful to shift lower-powered sums to the same power as the highest-power sum. In this example, we want the t^{n-2} term to be shifted to t^n. This can be done by rewriting the sum as

$$\sum_{n=0}^{\infty}(n-1)(n)a_n t^{n-2+a} = \sum_{n=-2}^{\infty}(n+1)(n+2)a_{n+2}t^n,$$

[1] If they aren't polynomials, then we're out of luck.

which is clearly equivalent. But now we have sums whose lower indices don't match, so we still can't combine them. However, we can change the lower index of our modified sum from -2 to 0 by pulling out the first two terms:

$$\sum_{n=-2}^{\infty}(n+1)(n+2)a_{n+2}t^n = (-1)(0)a_0t^{-2}+(0)(1)a_1t^{-1}+\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}t^n$$

So then the full equation is

$$0+0+\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}t^n-2\sum_{n=0}^{\infty}na_nt^n-2\sum_{n=0}^{\infty}a_nt^n=0.$$

Now we can finally combine the sums and pull out the t^n:

$$\sum_{n=0}^{\infty}((n+2)(n+1)a_{n+2}-2na_n-2a_n)t^n=0$$

which implies[2] that the coefficients must be 0 if the entire polynomial is identically 0:

$$(n+2)(n+1)a_{n+2}-2na_n-2a_n=0$$

which allows us to set up a recursion relation:

$$a_{n+2}=\frac{2a_n(n+1)}{(n+2)(n+1)}=\frac{2a_n}{n+2}.$$

Since the gap between a values in the recursion is 2, we need two initial conditions (a_0 and a_1) in order to figure out the rest of the values for a. We can use any values we want; for convenience, we can use $a_0=0$ and $a_1=1$. We examine the recursion relation and find an explicit solution for a_n, and use it to find the solution:

$$y_1=\sum_{n=0}^{\infty}a_nt^n.$$

This is to find a single solution for our differential equation; to find another (linearly independent) solution, we use a different pair of values for a_0 and a_1. The general solution, of course, is the set of all linear combinations of these two solutions.

[2]If the pulled-out terms didn't vanish, then we'd have to constrain their values to zero as well to satisfy the equation.

4 Systems of Differential Equations

Sometimes a differential equation can be better approached by first decomposing it into a system of coupled differential equations. Such a system is of the form

$$\begin{aligned}
\dot{x}_1 &= f_1(t,\ x_1,\ x_2,\ \ldots,\ x_n) \\
\dot{x}_2 &= f_2(t,\ x_1,\ x_2,\ \ldots,\ x_n) \\
&\vdots \\
\dot{x}_n &= f_n(t,\ x_1,\ x_2,\ \ldots,\ x_n).
\end{aligned}$$

For an arbitrary differential equation, we can set $x_1 = y$, $x_2 = y'$, etc. Then we can set up the following system, whose solutions give solutions to the original equation:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_n &= f(t,\ x_1,\ x_2,\ \ldots,\ x_n)
\end{aligned}$$

where the final expression comes from the original equation.

4.1 Linear Systems with Constant Coefficients

We can analyze the behavior of any system of linear homogeneous coupled differential equations with constant coefficients. Such a system can be written in matrix form:

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where $\mathbf{x}$ is a column vector whose entries are x_1, x_2, etc. and A is the scalar matrix which contains the coefficients. The challenge, then is to find all solutions of this system.

The existence-uniqueness theorem tells us that if we have an initial value problem, there is a unique solution. Linear algebra tells us that the dimension of the solution space is n, and the general solution is a linear combination of n particular solutions. We can try to convert the system into a single equation, but the process can be very tricky, and we may not even know how to solve the resulting equation.

However, we can use linear algebra here. If the matrix has n eigenvalues, then an independent solution to the system is given by

$$\mathbf{x} = e^{\lambda t}\mathbf{v}$$

for every eigenvalue/eigenvector pair λ and $\mathbf{v}$. An important theorem states that if a set of eigenvectors each correspond to a different eigenvalue, then they are linearly independent; therefore, if we have n eigenvalues, we have necessarily found n linearly independent solutions.

Since A is real-valued, any complex eigenvalue comes in pairs: let's say $\lambda = a \pm bi$. Each eigenvalue has a corresponding eigenvector, and you can use the formula above to find the complex-valued solutions. But we're only interested in the real part of these, and it turns out that their real parts are the same. This is a problem: we have two eigenvalues, but only one real solution. Fortunately, another theorem states that if our complex valued solution is $\mathbf{x}(t) = \mathbf{y}(t) + i\mathbf{z}(t)$, then both $\mathbf{y}(t)$ and $\mathbf{z}(t)$ are real-valued solutions.

However, we still have made a crucial assumption – that we have n distinct eigenvalues. If we have eigenvalues with multiplicity > 1, then the dimension of their eigenspace may not be equal to their multiplicity. This means that the number of distinct, independent eigenvectors will be $< n$, and we'll have to find others.

The general method for finding the general solution to $\dot{\mathbf{x}} = A\mathbf{x}$ hinges on the observation that the firs-order equation $\dot{x} = ax$ has the general solution $x = ce^{at}$. Then, we would expect the general solution to our matrix equation to have the solution

$$\mathbf{x} = \mathbf{v}e^{At}$$

where we define e^{At} using the familiar series expansion:

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!}\ldots$$

We can rewrite our solution as $\mathbf{x} = \mathbf{v}e^{\lambda t}e^{(A-\lambda I)t}$. This has the crucial property that if we satisfy the condition $(A - \lambda I)\mathbf{v} = \mathbf{0}$, then $e^{(A-\lambda I)t}\mathbf{v}$ collapses in the first-order and higher terms:

$$\begin{aligned} e^{(A-\lambda I)t}\mathbf{v} &= I\mathbf{v} + (A - \lambda I)\mathbf{v}t + \frac{((A - \lambda I)t)^2\mathbf{v}}{2!}\ldots \\ &= \mathbf{v} + 0 + 0 + \ldots \end{aligned}$$

which yields the solutions $\mathbf{x} = e^{\lambda t}\mathbf{v}$. This is the same solutions which we had seen before. Notice that our arbitrary condition is equivalent to finding the eigenvectors of A, just as we were doing before. But now, there's more to the story – if we need to find more solutions, then we just satisfy the second-order condition $(A - \lambda I)^2\mathbf{v} = \mathbf{0}$ for the same λ. Then $e^{(A-\lambda I)t}\mathbf{v}$ collapses in second-order and higher terms:

$$\begin{aligned} e^{(A-\lambda I)t}\mathbf{v} &= I\mathbf{v} + (A - \lambda I)\mathbf{v}t + \frac{((A - \lambda I)t)^2\mathbf{v}}{2!} \dots \\ &= \mathbf{v} + (A - \lambda I)\mathbf{v}t + 0 + 0 + \dots \end{aligned}$$

And so on for higher order terms until we have n linearly independent solutions.

If we apply this method to our 2nd-order linear equation with constant coefficients, we can easily derive the three cases which we saw. The seemingly peculiar case where we had a solution of the form $y = (A + Bt)e^{rt}$, which we earlier ascribed to a technique called reduction of order, can now be explained as satisfying the second-order condition.

If we know the linearly independent solutions of $\dot{\mathbf{x}} = A\mathbf{x}$, then we can arrange the solutions as columns of an $n \times n$ matrix $X(t)$. Then we have an explicit formula $e^{At} = X(t)X^{-1}(0)$.

4.2 Analyzing Systems

What can we deduce about the solutions of a system of equations (not necessarily linear) without solving the system? Suppose we have $\dot{\mathbf{x}} = f(\mathbf{x}, t)$. Then we can find *equilibrium points* where the solution does not change with time by solving $\mathbf{0} = f(\mathbf{x}, t)$.

Furthermore, we can determine the stability of solutions of $\dot{\mathbf{x}} = A\mathbf{x}$ without solving fully. Stability roughly means that solutions which "start together" at some t will "stay together" for all future t.

We can determine whether solutions are stable by looking at the eigenvalues $\lambda = a + bi$ of A:

- If there exists an eigenvalue with $a > 0$: every solution unstable. This is because $e^{\lambda t}$ will diverge.
- If all eigenvalues have $a < 0$: every solution is stable. This is because $e^{\lambda t}$ will converge.

- If most eigenvalues have $a < 0$ and the rest have $a = 0$: we can factor the characteristic polynomial as

$$0 = (\lambda_1 - b_1 i)^{k_1} \dots (\lambda_m - b_m i)^{k_m} q(\lambda)$$

  where $q(\lambda)$ represents all the factors with λ for which $a < 0$. Clearly, the first factors represent the distinct λ for which $a = 0$, with multiplicity k. For each factor $(\lambda_j - b_j i)^{k_j}$, if λ_j has k_j linearly independent eigenvectors, then all solutions are stable. Other, all solutions are unstable.

We can also analyze systems by analyzing their *phase portraits*. If we imagine a system with $n = 2$, then we can represent solutions as curves in 3-space, where the coordinates are x_1, x_2, and t. Stable solutions are curves which stay together or converge as $t \to \infty$, while unstable solutions are curves which diverge as $t \to \infty$. Equilibrium solutions are lines which are parallel to the t axis.

Now, if project the curves onto the $x_1 - x_2$ plane, we can get a 2-dimensional view at the behavior of solutions. The projection of a solution curve is called an orbit; we can think of each orbit as a trajectory, following the location of a point on the plane as t progresses. It's important to note that an orbit which contains an equilibrium point is actually a union of multiple orbits.

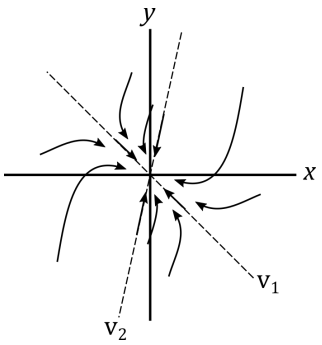
If we rename $x = x_1$ and $y = x_2$, then we can write $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$ from our original system. We can explicitly solve for the orbits by noting that

$$\frac{g(x, y)}{f(x, y)} = \frac{\dot{y}}{\dot{x}} = \frac{dy}{dx}$$

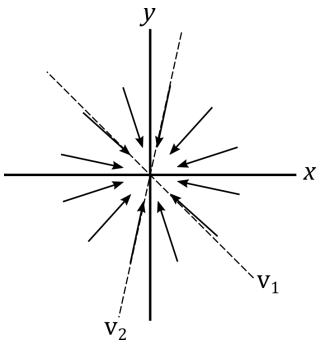
and solving. This applies for any 2-variable system of differential equations.

For linear systems given by $\dot{\mathbf{x}} = A\mathbf{x}$, there are several cases for phase portraits, which (again) depend on the eigenvalues of A. For eigenvalues λ_1 and λ_2 corresponding to eigenvectors v_1 and v_2 respectively, we have the following cases:

- $\lambda_2 < \lambda_1 < 0$: The arrows seem to approach v_1 because it converges slower.

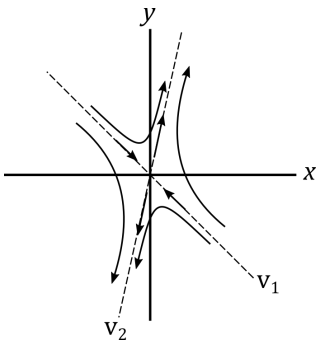


- $0 < \lambda_1 < \lambda_2$: Same as above but with the arrows reversed.
- $\lambda_2 = \lambda_1 < 0$:

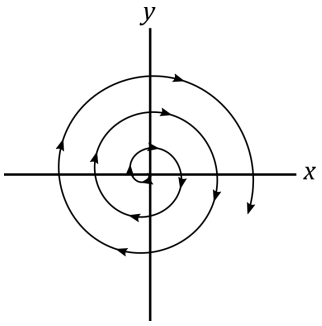


- $0 < \lambda_1 = \lambda_2$: Same as above but with the arrows reversed.

- $\lambda_1 < 0 < \lambda_2$:



- $\lambda = a \pm bi$: If $a > 0$ then solutions spiral outward; otherwise, inward. The handedness can be determined by the sign of $\dot{y}$ when $x > 0$ which can be found by examining the system.



5 The Heat Equation

The heat equation is a partial differential equation, which means that it deals with the partial derivatives of a multivariate function. In this case, it governs the time evolution of a temperature distribution across a one-dimensional interval (for example, a metal bar). The heat equation is given by

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial u}{\partial x}$$

where $u(x, t)$ is the temperature function and α is a conductivity constant related to the material. We can find many solutions (although not necessarily the general solution) using a technique known as *separation of variables*: We

assume $u = X(x)T(t)$ is a product of two single-variable functions. Then we can simplify and manipulate our heat equation to get

$$\frac{X''}{X} = \frac{T'}{\alpha T}.$$

The key insight is that this equation can only be valid for all x and t if the two sides of the equations are constants. Setting everything equal to λ, we can separate the equations and rearrange to get

$$X'' + \lambda X = 0 \quad \text{and} \quad T' + \alpha\lambda T = 0.$$

If we place some initial restrictions, we can eliminate some possible values of λ. First, we can set *boundary conditions*, making this a *boundary value problem*: for example, $X(0) = X(L) = 0$ if we're considering the interval $[0, L]$. Then, we can set the initial temperature distribution $u(x, 0) = f(x)$. This will help later.

Examining $X(x)$, the given boundary conditions force $\lambda > 0$. The general solution is then $X = a\sin\Big(\sqrt{\lambda}x\Big) + b\cos\Big(\sqrt{\lambda}x\Big)$. But the boundary conditions then also force $b = 0$ and $\lambda = n^2\pi^2/L^2$ for $n \in \mathbb{N}$. Solving fr T and plugging in our value for λ, we get

$$\begin{aligned} X_n(x) &= c_{xn}\sin\Big(\frac{n\pi x}{L}\Big) \\ T_n(t) &= c_{tn}e^{-\alpha n^2\pi^2 t/L^2} \\ u_n(x,t) &= X_nT_n = c_n\sin\Big(\frac{n\pi x}{L}\Big)e^{-\alpha n^2\pi^2 t/L^2} \end{aligned}$$

and any linear combinations of u_n for various values of n also yields a solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n\sin\Big(\frac{n\pi x}{L}\Big)e^{-\alpha n^2\pi^2 t/L^2}.$$

In fact, the exact values of c_n are determined by the initial value $u(x, 0) = f(x)$. By using Fourier analysis, we can determine the coefficients:

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi x}{L}\Big)dx.$$

Thus the heat equation is fully solved given appropriate boundary conditions and an initial temperature distribution.