Nonequilibrium processes 2018
Homework 1

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1 Stochastic processes (6p)

a) (2p)

Consider the stochastic process $X_t = 2\sin(\omega t)$, where ω is a random variable with uniform distribution between $\omega = 2$ and $\omega = 6$. Calculate the

(a) ensemble average,

Uniform dist., “weight function” f should be equal to 1/(6-2)

(b) time average.

What is the definition of an ergodic process? Is this process ergodic?

b) (2p)

Do the same calculations for the stochastic process $X_t = 2\sin(\omega_0 t + \theta)$, where ω_0 is a fixed constant and θ is a random variable uniformly distributed in the interval $[0, 2\pi]$. Is this process ergodic?

c) (2p)

Confirm your calculation by doing computer simulations of the stochastic processes in a) and b). Numerically calculate the ensemble and time averages for the two processes. You can use Matlab, or another programming language of your choice. Include your code as an appendix.

2 Markov chains

2.1 Discrete time (5p)

Suppose we have three boxes and we divide a number of balls among the boxes. When an alarm sounds, 1/4 of the balls in the first box stay there (so the probability of remaining in box 1 is 1/4), 1/4 move to the third box and 1/2 of the balls move to the second box. Also, suppose that none of the balls in the second box move to the first, 1/2 stay in the second, and 1/2 move to the third box. For the third box, 1/8 of the balls move to the first box, 3/4 to the second, and 1/8 stay in the third.

(a) Construct the left-stochastic transition matrix W.

(b) We divide 1000 balls among the boxes. We initially place 500 balls in the first box, 300 in the second, and 200 in the third. How many balls are in each box after the first transition?

(c) Calculate the stationary distribution.

(d) Calculate the distribution of balls after s transitions.

2.2 Continuous time

2.2.1 Random walk (3p)

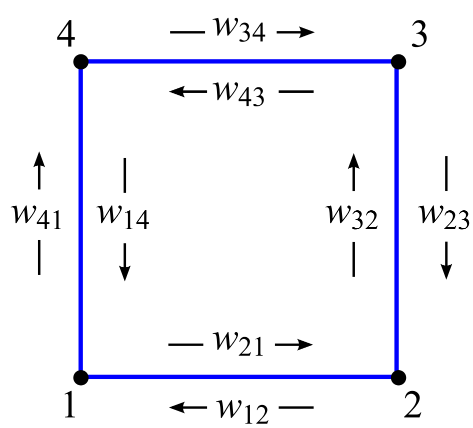

Figure 1: Illustration of a random walk

Consider the random walk shown in Fig. 1. The four possible states 1, 2, 3 and 4 correspond to the walker being in the respective corner of the square, and the transition rates to go from corner j to $i \neq j$ are $w_{ij} \geq 0$. For this task, we set

$$w_{21} = w_{34} = 1,\ w_{32} = w_{43} = \tfrac{3}{4},\ w_{12} = w_{23} = \tfrac{1}{4},$$

and 0 for all remaining transition rates.

(a) Write down the transition matrix G.

(b) Find the stationary state.

(c) Check whether detailed balance is fulfilled.

2.2.2 Time evolution (4p)

Consider the generator matrix

$$\mathrm{G} = \begin{pmatrix} -3 & 4 & 1 \\ 1 & -5 & 1 \\ 2 & 1 & -2 \end{pmatrix}, \tag{1}$$

for a continuous time Markov chain with three states.

(a) Calculate the stationary state.

(b) Calculate the time evolution from e^{tG}. Confirm that the system reaches the previously calculated stationary state for $t \to \infty$. What is the probability that the system is in state 3 at time t, given that it starts in state 1?

3 Master equation

3.1 Calculate moments (5p)

To better organize the ordering of coffee, the three employees working for the 24 hour hot-line of the Laphroaig customer service – Jim, Jack and John – have decided to analyze their daily coffee consumption. Jim is working during the less busy night shifts. He estimates that the probability for him to drink another cup of coffee rises at a constant rate of $\omega_{\mathrm{Ji}} = 0.8/\text{hour}$ until he drinks it. During the day shifts, Jack and John take over. As there is not as much time for coffee breaks, the probability for Jack to take a cup rises at a rate $\omega_{\mathrm{Ja}} = 0.4/\text{hour}$. John takes even less breaks, but since he needs his coffee, he also drinks more. He always consumes two cups at the same time, but at a probability that only rises at $\omega_{\mathrm{Jo}} = 0.2/\text{hour}$.

From these estimates, they intuitively expect that an average of 9.6 cups is consumed in both 12 hour shifts. However, since the availability of coffee is crucial, they also want to take possible fluctuations into account, and your job is now to help them. Model this problem separately for the day and night shift by considering Markovian time evolution for the probabilities $P_n(t)$ that $n \in \mathbb{N}^{\geq 0}$ cups have been consumed within a time interval $t \geq 0$ after the beginning of the shift. Obviously, no cup has been consumed at the beginning, meaning that $P_n(t = 0) = \delta_{n0}$ (with the Kronecker delta δ_{ab} equal to 1 if $a = b$ and equal to 0 if $a \neq b$).

(a) Write down a master equation for the probabilities $P_n(t)$ separately for the day and night shift. (2p)

(b) Confirm the above stated average consumption and state a reasonable amount of extra cups that need to be available for the day shift and the night shift in order to be prepared for the average fluctuations. Explain the difference! (3p)
Hint: Use the generating function.

3.2 Calculate the probability distribution (4p)

Consider traffic flow as a Markov process. We want to investigate the dissolution of a queue of cars standing in front of traffic lights. When the lights switch to green, the first car starts to move. After a certain time τ the next car moves, and so on. We look at the decay of traffic congestion. The stochastic variable $n(t)$ is the number of cars that are bounded in the jam at time t.

Define $p(n, t)$ as the probability of finding a jam of size n at time t. The initial jam size is n_0. The master equation with boundary conditions is

$$\begin{aligned}
\frac{\partial}{\partial t} p(n,t) =& \frac{1}{\tau} \left[p(n+1,t) - p(n,t) \right], \\
\frac{\partial}{\partial t} p(n_0,t) =& -\frac{1}{\tau} p(n_0,t), \\
\frac{\partial}{\partial t} p(0,t) =& \frac{1}{\tau} p(1,t).
\end{aligned}$$

Solve the master equation to find the general solution of the probability distribution $p(n,t)$ of observing a car cluster of size n at time t. What kind of distribution is it?