# Hall-Effect and Thermo-Current

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**What is this recitation about and why is it interesting?**

- The recitation discusses two important experimental situations/observations for which a statistical description is important: the Hall-effect and Thermo-current.
- The Hall-effect manifests itself through two experimental observations: magnetoresistance and appearance of so called Hall-voltage (see below for introduction). We use a statistical description of the problem in terms of the Boltzmann equation. In order to deal with the problem we introduce the so called relaxation-time approximation and make use of general properties of the Fermi-Dirac distribution of electrons at low temperatures. These approximations are widely used and very important in order to understand transport properties in metals.
- The problem of solving the Boltzmann eq. for the Hall-system will be part of your homework, as well as calculating the conductivity tensor and the Hall coefficient.
- We investigate the interplay between thermal current and electrical current for a gas of charged particles. The interplay is manifested in the response to gradients in the temperature distribution and gradients of the electro-chemical potential.
- In order to describe this we use again the Boltzmann equation in the relaxation-time approximation with local equilibrium distribution functions.
- The problem of obtaining an expression for the thermal conductivity for a specific setup (constraint of no electric current) will be part of your HP.

# 1 Introduction - Hall effect

Consider a metallic slab connecting two electrodes, with applied perpendicular magnetic field, H. A voltage is applied over the slab inducing a electric field $E = V/L$. Experimentally one can then observe i) A decrease of the current flow between the two electrodes depending on the strength of the magnetic field. This property is known as *Magnetoresistance*. ii) A voltage across the two sides of the plate not connected to electrodes (see figure). This voltage is called the *Hall voltage* and the corresponding electric field is known as the *Hall field*, named after the man who discovered this effect in 1879, Edwin Hall.

## 1.1 Basic description

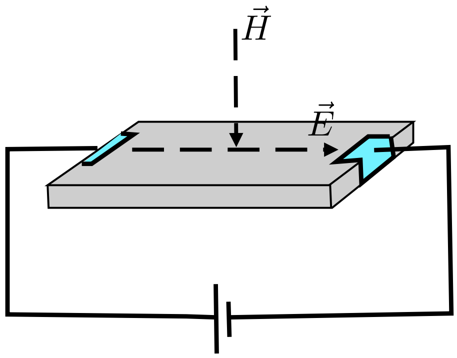
Electrons in the metal moving with a velocity $v$ will then feel the Lorentz force:

$$\vec{F}_L = e\left(\vec{E} + \vec{v} \times \vec{H}\right) \tag{1}$$

Let's choose our coordinate system so that the metallic plate lies in the $z = 0$ plane with the electric field in the $x$-direction, $\vec{E} = (E, 0, 0) = E\hat{x}$ and the magnetic field in the $z$-direction, $\vec{H} = (0, 0, H) = H\hat{z}$. Consider an electron injected from one of the electrodes. By Newtonian mechanics it will satisfy the following equation of motion:

$$m\frac{d\vec{v}}{dt} = \vec{F}_L \tag{2}$$

## 1.2 Drude model*(not compolsory reading)




Let's try to explain the experimental results for a simple description called the *Drude model*. Here every electron is assumed to have the same velocity $\vec{v} = \langle\vec{v}\rangle$. They are then accelerated by the electric field $E$ and the trajectory becomes bent towards the edges of the plate by the magnetic field (see figure). At the same time there is some inelastic scattering within the plate which we can account for by adding a friction term $\vec{F}_\gamma = \gamma\langle\vec{v}\rangle$ to Eq. (2). The equation of motion then becomes

$$m\frac{d\langle\vec{v}\rangle}{dt} = \vec{F}_L + F_\gamma \tag{3}$$

the steady state solution, $d\langle\vec{v}\rangle/dt = 0$, is determined by

$$0 = \vec{F}_L + \vec{F}_\gamma = eE\hat{x} + m\omega_c\langle\vec{v}\rangle \times \hat{z} + m\tau^{-1}\langle\vec{v}\rangle \tag{4}$$

where $\omega_c = eH/m$ is called the cyclotron frequency and $\tau = m/\gamma$ is called the scattering time. Since the electron is constrained to the $z = 0$ plane we have $\langle \vec{v} \rangle = (v_x, v_y, 0) = v_x\hat{x} + v_y\hat{y}$ and so, using $\hat{x} \times \hat{z} = -\hat{y}$, $\hat{y} \times \hat{z} = \hat{x}$, we get

$$\begin{aligned} 0 &= eE/m - \omega_c v_y + \tau^{-1} v_x \\ 0 &= \omega_c v_x + \tau^{-1} v_y \end{aligned} \tag{5}$$

solution of which yields

$$\langle \vec{v} \rangle = (v_x, v_y, 0) = \frac{e\tau E}{m} \frac{1}{1 + (\tau\omega_c)^2}(1, \tau\omega_c, 0) \tag{6}$$

The current is given by $\vec{j} = n\langle \vec{v} \rangle$ which means

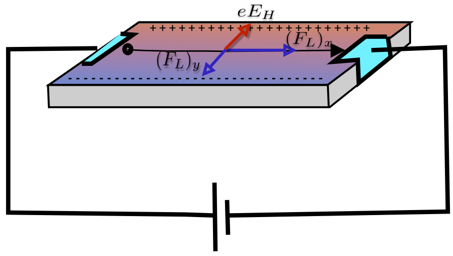
$$\begin{aligned} j_x &= \frac{ne\tau}{m} \frac{1}{1 + (\tau\omega_c)^2} E = \sigma_{xx} E \\ j_y &= \frac{ne\tau}{m} \frac{\tau\omega_c}{1 + (\tau\omega_c)^2} E = \sigma_{yx} E \end{aligned} \tag{7}$$

where the conductivity tensor $\sigma_{ij}$ is defined such that $j_i = \sigma_{ij} E_j$. In the limit $H \to 0$ we get the normal Drude conductivity $\sigma_{xx} = ne\tau/m \equiv \sigma_0$, $\sigma_{yx} = 0$.

What experimental observations can we verify with this model? From the expression Eq. (7) it seems the first observation that the current flow between the electrodes, $j_x$, decreases as a function of the magnetic field $H$, can be verified. This is due to the deflection of the electrons to the sides by the Lorentz force, producing a finite current in the $y$-direction. On the other hand the result that we have a finite current in the $y$-direction is unphysical for any finite-sized system. Taking into account the boundaries we realize that charges will build up at the edges which eventually will produce a field (the Hall field) which exactly compensates the $y$-component of the Lorentz force

$$eE_H = (F_L)_y$$

thus canceling the perpendicular current and also the magnetoresistance. To summarize, within the Drude model no magnetoresistance should be present in a finite system.




## 1.3 Shortcomings of Drude model

The reason why the Drude model makes false predictions is that it does not assume the statistical distribution of velocities and the Lorentz force depends on the velocity. To clarify: the Hall field exactly cancels the $y$-component of the Lorentz force corresponding to the **average** velocity $\langle \vec{v} \rangle$. However, for electrons moving with a velocity, $v_<$, less than the average the Hall field will be stronger than the Lorentz force and the electrons will be deflected towards one side of the plate, while for electrons with velocity, $v_>$, greater than the average the Lorentz force will be stronger than the Hall field and so will be deflected to the other side. As a result, no net current will flow between the sides but still the current between the electrodes, $j_x$, will be diminished due to the deflection of electrons with velocities different than the average velocity.

We have now seen that a statistical approach is necessary to correctly describe the problem. This will be the subject of the next section.

## 2 Boltzmann Equation

Starting from a statistical description with a $N$-particle distribution function we can obtain a kinetic equation for the 1-particle distribution function $f$, called the Boltzmann equation:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \vec{F}_{\text{ext}} \cdot \frac{\partial f}{\partial \vec{p}} = I\{f\} \tag{8}$$

where $\vec{F}_{\text{ext}} = \vec{F}_L$ is the external force and $I\{f\}$ is the collision integral due to the interaction between particles. Solving this equation with the exact scattering integral is equivalent to solving the kinetic equation for the many-particle distribution function, which is impossibly difficult. Naturally we need to make considerable simplifications on this term. The most common simplification is the so-called *relaxation-time approximation* where one assumes that inelastic collisions between the particles drives the distribution function towards the equilibrium distribution function $f^0$ on a time-scale $\tau$. To illustrate how this enters the Boltzmann equation we can start out by ignoring the interaction with other particles, $I\{f\} = 0$. This means we have a system of only one particle. If the external force is conservative (originates from some potential: $\vec{F}_{\text{ext}} = -\nabla U_{\text{ext.}}$), then Liouvilles theorem tells us that we have $df/dt = 0$:

$$\left(\frac{df}{dt}\right)_{1\text{p}} \equiv \frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \vec{F}_{\text{ext}} \cdot \frac{\partial f}{\partial \vec{p}} = 0, \qquad I\{f\} = 0 \tag{9}$$

where I used $(\ldots)_{1\text{p}}$ to indicate that this corresponds to the time derivative for a noninteracting system. Now we are taking into account the interaction with other particles by thinking of them as instantaneous, random, inelastic collisions. The right hand side of Eq. (8) then describes the change in the distribution function due to these collisions:

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll.}} \equiv I\{f\} \tag{10}$$

which "breaks" the Liouville theorem $(df/dt)_{1\text{p}} = (\partial f/\partial t)_{\text{coll.}} \neq 0$. The collisions drive the single-particle distribution function into its equilibrium form $f^0$, for which detailed balance ensures that $I\{f^0\} = 0$. The relaxation-time approximation suggests that this happens on a time scale[1] $\tau$, i.e. we can write

$$I\{f\} \equiv \left(\frac{\partial f}{\partial t}\right)_{\text{coll.}} = -\frac{1}{\tau}\left(f - f^0\right) \tag{11}$$

[1] In general there may be a hierarchy of time-scales $\tau_i$ corresponding to different processes responsible for the decay into equilibrium but one usually only considers the longest such time-scale.

describing a decay $\sim e^{-t/\tau}$. The Boltzmann equation in this approximation is thus given by

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \vec{F}_{\text{ext}} \cdot \frac{\partial f}{\partial \vec{p}} = -\frac{1}{\tau}(f - f^0) \tag{12}$$

## 2.1 Hall effect

The problem with the Hall effect will now be treated with the Boltzmann equation, where now the external force $\vec{F}_{\text{ext}} = \vec{F}_L$.

Even with the relaxation-time approximation the Boltzmann equation can be very difficult to solve. We shall in our problem use some symmetry-properties in order to simplify the equation. First, note that it is difficult to introduce spatial boundary conditions describing the effect of a finite plate. Instead we shall consider a infinite metallic plate and simulate the effect of finite size by eventually requiring that the net current in the $y$-direction be zero. In this description the plate becomes translationally invariant, i.e.

Clever cookies alright

$$f(\vec{r} + \vec{\Delta r}) = f(\vec{r}) \Rightarrow \frac{\partial f}{\partial \vec{r}} = 0 \tag{13}$$

Furthermore, we are looking for a stationary solution. I.e. the distribution function does not change in time

$$\frac{\partial f}{\partial t} = 0 \tag{14}$$

The Boltzmann equation within the relaxation-time appr. Eq (12) then reduces to solving

$$\vec{F}_L \cdot \frac{\partial f}{\partial \vec{p}} = -\frac{1}{\tau}(f - f^0) \tag{15}$$

The equilibrium distribution function we are dealing with must be the Fermi-Dirac distribution since we are dealing with electrons:

$$f^0(\vec{p}) = \underbrace{2}_{\text{spin. deg.}} f_{FD}(\vec{p}) = 2\left[1 + e^{\beta[\epsilon(\vec{p})-\mu]}\right]^{-1} \tag{16}$$

where $\epsilon(\vec{p}) \equiv \vec{p}^2/2m$, $\mu \simeq \epsilon_F$ and $\beta \equiv (k_B T)^{-1}$. The derivative wrt the momentum can then be written:

$$\frac{\partial f^0}{\partial \vec{p}} = \frac{\partial f^0}{\partial \epsilon}\frac{\partial \epsilon}{\partial \vec{p}} = \frac{\partial f^0}{\partial \epsilon}\vec{v} \tag{17}$$

If we introduce $f^1 = f - f^0$ Eq. (15) can be written in the nicer form

$$\boxed{\left[\vec{F}_L \cdot \frac{\partial}{\partial \vec{p}} + \frac{1}{\tau}\right] f^1 = -(\vec{F}_L \cdot \vec{v})\frac{\partial f^0}{\partial \epsilon} \qquad (18)}$$

It now remains to calculate $f^1$ and substitute into $f = f^0 + f^1$ to obtain the one-particle distribution function. With this distribution function it is then possible to calculate physical averages such as the current density[2]:

$$\vec{j} = \langle e\vec{v}\rangle = (e/m)\int \frac{d^2p}{(2\pi\hbar)^2}\ \vec{p}\ f(\vec{p}) \tag{19}$$

where $d^2p = dp_x dp_y$ since we are looking at a 2-dimensional plate and $(2\pi\hbar)^{-2}$ is the density of states[3].

It will be part of your home problem to solve the differential equation (18) and calculate the conductivity tensor **for arbitrary magnetic field**. In order for you to do this you will need to perform some approximations. In the next section I will show how to do such a calculation in the absence of a magnetic field, which should give you hints on how to solve your problem.

### 2.2 Conductivity tensor for $H = 0$

For zero magnetic field we have $\vec{F}_L = e\vec{E}$. We are interested in the response of the current to a small electrical field and therefore we only keep first order terms. Since the deviation $f^1$ from the equilibrium distribution must be a function of the external force which vanishes for $E = 0$ we know that $f^1$ is at least of first order in $E$. We thus have

$$e\vec{E}\cdot\frac{\partial f^1}{\partial \vec{p}} \sim \mathscr{O}(\vec{E}^2) \approx 0, \quad \text{Êsince } f^1 \sim \mathscr{O}(\vec{E}^1). \tag{20}$$

so that, within the *linear response regime*, i.e. neglecting second order terms, this part is zero and we have only

$$f^1 = -\tau\frac{\partial f^0}{\partial \epsilon}(e\vec{v}\cdot\vec{E}) \tag{21}$$

Now turning our attention to the current, we can calculate the $i$th component of the equilibrium current

$$\begin{aligned} j_i^0 = \langle ev_i\rangle_0 &= e\int \frac{d^2p}{(2\pi\hbar)^2}\ v_i\ f^0(p), \qquad i = x, y \\ &= e\int_{-\infty}^{\infty}\frac{dp_k}{2\pi\hbar}\underbrace{\left(\int_{-\infty}^{\infty}\frac{dp_i}{2\pi\hbar}v_i f^0(v_i, v_k)\right)}_{=0}, \quad k \neq i \\ &= 0 \end{aligned} \tag{22}$$

[2]If we had an inhomogeneous system the distribution function would depend on the position and so would the current : $\vec{j}(\vec{r}) = \int d^2p(e\vec{p}/m)f(\vec{p},\vec{r})$

[3]In quantum mechanics a particle can not have **any** value of $p$, but only a discrete set of states. The density of these states for a d-dimensional system per unit volume is given by $1/(2\pi\hbar)^d$. For more detail see the QM notes that will be published on my homepage

where in the last equality we used that the integrand is odd under $v_i \to -v_i$ since $f^0(-v_i, v_k) = f^0(v_i, v_k)$. This simply tells us that the equilibrium current is zero. Now the finite response comes from the deviation $f^1$:

$$
\begin{aligned}
j_i = e \int \frac{d^2p}{(2\pi\hbar)^2}\, v_i\, f^1(p) &= -e^2\tau \int \frac{d^2p}{(2\pi\hbar)^2} v_i \frac{\partial f^0}{\partial \epsilon} \sum_k (v_k E_k), \qquad i = x, y \\
&= \sum_k \underbrace{\left(-e^2\tau \int \frac{d^2p}{(2\pi\hbar)^2} v_i v_k \frac{\partial f^0}{\partial \varepsilon}\right)}_{\sigma_{ik}} E_k
\end{aligned}
\tag{23}
$$

This gives us a explicit expression for the conductivity tensor $\sigma_{ij}$. It can immediately be seen that this conductivity must be diagonal $\sigma_{ik} = \sigma_{ii}\delta_{ik}$ since the integrand is odd under $v_i \to -v_i$ and vanishes on integration over symmetric domain, unless $i = k$ in which case $v_i v_k = v_i^2$ and the integrand is even.

The integral in Eq. (23) is evaluated in the appendix:

$$
\int \frac{d^2p}{(2\pi\hbar)^2} v_i^2 \frac{\partial f^0}{\partial \varepsilon} = -\frac{n}{m} \tag{24}
$$

where $n = N/V$ is the particle density. The fact that this number appears can be anticipated from the fact that the distribution function is defined to have the normalization $\int \frac{d^2p}{(2\pi\hbar)^2} f^0(p) = n$. The conductivity is then

$$
\sigma \equiv \sigma_{ii} = \frac{ne^2\tau}{m} \tag{25}
$$

coinciding with the Drude model.

## 3 Thermo-current

In this section we consider a dilute gas of particles with charge q, where we have artificially introduced an uneven distribution of temperature and chemical- and electrical potential[4]. We may consider a local equilibrium distribution function:

$$
f^0(\vec{r}, \vec{p}) \sim e^{-\frac{[\epsilon(\vec{p}) - \mu(\vec{r})]}{k_B T(\vec{r})}} \tag{26}
$$

with the normalization

$$
\int d^3p\, f^0(\vec{r}, \vec{p}) = n(\vec{r}) \tag{27}
$$

Of course convection/drift will try to equalize these gradients in temperature etc. This causes the distribution function to deviate from its local equilibrium form:

$$
f = f^0 + f^1, \qquad f^1 \ll f^0 \tag{28}
$$

[4] What this means is that we can define local thermodynamic variables $T(\vec{r})$, $\mu(\vec{r})$ which enter the local...

The Boltzmann equation within the relaxation-time approximation again has the form

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \vec{F}_{ext} \cdot \frac{\partial f}{\partial \vec{p}} = -\frac{1}{\tau}(f - f^0) \tag{29}$$

where $\vec{F}_{\text{ext}} = -q\nabla\phi$, and $\phi$ is the electric potential. We are again looking for a stationary solution so that

$$\frac{\partial f}{\partial t} = 0 \tag{30}$$

Another important simplification follows from the **separation of length scales**. Namely that the characteristic scale of variations of the macroscopic variables $T$, $\mu$, $\phi$, which we denote $L$, is much longer than the characteristic length-scale of microscopic processes, $\ell$.

$$\ell \ll L \tag{31}$$

To be a little more specific we can compare, for example, the relative difference in temperature at two points $\vec{r}$ and $\vec{r} + \delta\vec{r}$:

$$\frac{T(\vec{r} + \delta\vec{r}) - T(\vec{r})}{T(\vec{r})} \approx \frac{\nabla T \cdot \delta r}{T} \tag{32}$$

The characteristic scale of variation of the macroscopic variables ($\delta r \sim L$) is one for which the relative change of temperature is significant, i.e. of order 1:

$$\frac{T(\vec{r} + L) - T(\vec{r})}{T(\vec{r})} \approx \frac{\nabla T \cdot L}{T} \sim 1 \tag{33}$$

A requirement for establishing local equilibrium is that the microscopic length scale is small enough so that on this scale the temperature can be treated as constant, i.e. that the relative change of the temperature on these length scales is negligble:

$$\frac{T(\vec{r} + \ell) - T(\vec{r})}{T(\vec{r})} \approx \frac{\nabla T \cdot \ell}{T} \ll 1 \tag{34}$$

We can rewrite the two above equations into the from

$$\nabla T \sim \frac{T}{L}, \qquad \nabla T \ll \frac{T}{\ell} \tag{35}$$

Similar equations hold for $\nabla\mu$ and $\nabla\phi$, and we shall assume that they share a common scale. Furthermore, since the correction, $f^1$ to the distribution function is induced by these gradients, it is natural to assume that this correction shares the same characteristic length scale.

$$\frac{\partial f^1}{\partial \vec{r}} = \nabla f^1 \sim \frac{1}{L} f^1 \tag{36}$$

So far I have not quantified the microscopic length scale in terms of the parameters we have available. Such a scale can be obtained by combining to other characteristic scales, namely the relaxations time $\tau$, which is associated with microscopic scattering events, with the mean- (or perhaps better: mode-) velocity $\bar{v}$ of the microscopic particles:

$$\ell = \tau \bar{v} \tag{37}$$

The requirement that $\ell \ll L$ then reads

$$\tau \bar{v} \ll L, \quad \text{or} \quad \frac{\bar{v}}{L} \ll \frac{1}{\tau} \tag{38}$$

From this analysis we can immediately see that the term:

$$\vec{v} \cdot \frac{\partial f^1}{\partial \vec{r}} \sim \frac{\bar{v}}{L} f^1 \ll \frac{1}{\tau} f^1 \tag{39}$$

can be neglected compared to the collision integral term. Furthermore, the term

$$q \nabla \phi \cdot \frac{\partial f^1}{\partial \vec{p}} \tag{40}$$

is a second order correction and can thus also be neglected. This leaves us with a simple algebraic equation for $f^1$ the solution of which is:

$$f^1 = -\tau \left( \vec{v} \cdot \frac{\partial f^0}{\partial \vec{r}} - q \nabla \phi \cdot \frac{\partial f^0}{\partial \vec{p}} \right) \tag{41}$$

To get a better expression lets write down the derivatives:

$$\begin{aligned} \frac{\partial f^0}{\partial \vec{r}} &= \frac{\partial f^0}{\partial T} \nabla T + \frac{\partial f^0}{\partial \mu} \nabla \mu = \left[ \frac{(\epsilon - \mu)}{k_B T^2} \nabla T + \frac{1}{k_B T} \nabla \mu \right] f^0 \\ \frac{\partial f^0}{\partial \vec{p}} &= \frac{\partial f^0}{\partial \epsilon} \frac{\partial \epsilon}{\partial \vec{p}} = -\vec{v} \frac{1}{k_B T} f^0 \end{aligned} \tag{42}$$

the expression for the correction $f^1$ is then given by

$$f^1 = -\tau \left( \frac{(\epsilon - \mu)}{k_B T^2} \vec{v} \cdot \nabla T + \frac{q}{k_B T} \vec{v} \cdot \nabla \Phi \right) f^0 \tag{43}$$

where $\Phi = \mu / q + \phi$ is the electro-chemical potential.

## 3.1 Thermo-current

Now that we have solved the kinetic equation for the distribution function we can try to evaluate the electric current:

$$(j_e)_i = \langle q v_i \rangle = q \int d^3 p v_i f = q \underbrace{\int d^3 p v_i f^0}_{=0} + q \int d^3 p v_i f^1 \tag{44}$$

The equilibrium term cancels due to the even nature of $f^0$ and what remains can be written:

$$(j_e)_i = \sum_k \underbrace{\left(-\frac{q\tau}{k_B T^2}\int d^3p\, v_i v_k (\epsilon-\mu) f^0\right)}_{\tilde{L}_{ik}} (\nabla T)_k + \sum_k \underbrace{\left(-\frac{q^2\tau}{k_B T}\int d^3p\, v_i v_k f^0\right)}_{\sigma_{ik}} (\nabla\Phi)_k \tag{45}$$

We notice that in this case we have a finite electric current even in the absence of a gradient in the electro-chemical potential (electric field*). The temperature gradient also naturally produces a current due to the flow of charged particles from hotter to colder regions. It turns out that the electrical- and heat-currents are intimately connected. Calculating the heat/energy current we get:

$$\begin{aligned}(j_Q)_i &= \langle v_i(\epsilon-\mu)\rangle = \int d^3p\, v_i(\epsilon-\mu) f = \underbrace{\int d^3p\, v_i(\epsilon-\mu) f^0}_{=0} + q\int d^3p\, v_i(\epsilon-\mu) f^1 \\ &= \sum_k \underbrace{\left(-\frac{\tau}{k_B T^2}\int d^3p\, v_i v_k (\epsilon-\mu)^2 f^0\right)}_{\tilde{M}_{ik}} (\nabla T)_k + \sum_k \underbrace{\left(-\frac{q\tau}{k_B T}\int d^3p\, v_i v_k (\epsilon-\mu) f^0\right)}_{\tilde{L}_{ik}} (\nabla\Phi)_k\end{aligned} \tag{46}$$

and we see the response to gradients in the electro-chemical potential and temperature can be related in the matrix form

$$\begin{pmatrix}(j_e)_i \\ (j_Q)_i\end{pmatrix} = \begin{pmatrix}\sigma_{ik} & \tilde{L}_{ik} \\ T\tilde{L}_{ik} & \tilde{M}_{ik}\end{pmatrix}\begin{pmatrix}(\nabla\Phi)_k \\ (\nabla T)_k\end{pmatrix} \tag{47}$$

since all the integrals defining the different tensors are zero for $i \neq k$ (check!) the tensors are diagonal. Furthermore, the response is isotropic so that for instance $\sigma_{ii} = \sigma$. Thus we can write directly

$$\boxed{\begin{pmatrix}j_e \\ j_Q\end{pmatrix} = \begin{pmatrix}\sigma & \tilde{L} \\ T\tilde{L} & \tilde{M}\end{pmatrix}\begin{pmatrix}\nabla\Phi \\ \nabla T\end{pmatrix}} \tag{48}$$

**NOTE:** I use $\tilde{M}$ and $\tilde{L}$ to emphasize that I use a slightly different notation than Vitaly, which I find more convenient for the current problem. Basically the difference arises because I prefer to relate transport coefficients directly to the temperature gradient as opposed to the gradient of $1/k_B T$. The conversion between the two notations is (check yourselves):

$$\tilde{L} = -\frac{L}{k_B T^2}, \qquad \tilde{M} = -\frac{M}{k_B T^2} \tag{49}$$

The homework problem consists of finding the thermal conductivity $\kappa$, i.e. the coefficient describing the response of the heat current to a direct temperature gradient.

$$j_Q = -\kappa \nabla T \tag{50}$$

Form Eq. (48) we see that in general the response of a temperature gradient is both a heat current *and* an electric current. Also a finite gradient of the thermoelectric potential will also induce a heat current. In an actual experiment it can be very difficult to obtain this coefficient, one way is to invent a setup such as to make sure there is no electric current in which case the temperature gradient only produces a heat current. In your home-problem you will be asked to calculate the thermal conductivity with the constraint that the electric current should be zero $j_e = 0$, and gradients in temperature and electrochemical potential be constant but non-zero $\nabla T, \nabla\Phi = \text{const}$.

In the following I will evaluate the thermal conductivity for the case of zero electrical current and zero gradient of the electrochemical potential. In this case $\kappa = -\tilde{M} = M/k_B T$ which is given by

$$\kappa = \frac{\tau}{k_B T^2} \int d^3 p v_i^2 (\epsilon - \mu)^2 f^0 \tag{51}$$

This may seem like a potentially difficult integral. One way to perform the integral is to notice that since $f^0 \sim e^{-\beta(\epsilon-\mu)}$, $\beta = (k_B T)^{-1}$ we can write

$$(\epsilon - \mu)^n f^0 = (-1)^n \frac{\partial^n}{\partial \beta^n} f^0 \tag{52}$$

and thus

$$\kappa = \frac{\tau}{k_B T^2} \frac{\partial^2}{\partial \beta^2} \int d^3 p v_i^2 f^0 \tag{53}$$

we then only need to evaluate the integral $\int d^3 p v_i^2 f^0$ which can be evaluated by noting that the distribution function is normalized according to Eq. (27) such that

$$\int d^3 p v_i^2 f^0 = n \frac{\int d^3 p v_i^2 e^{-\beta(\epsilon-\mu)}}{\int d^3 p e^{-\beta(\epsilon-\mu)}} = n \frac{\int d p_i v_i^2 e^{-\beta m v_i^2/2}}{\int d p_i e^{-\beta m v_i^2/2}} = \frac{n}{m} \beta^{-1} \tag{54}$$

?

The thermal conductivity is then given by

$$\kappa = \frac{n\tau}{m k_B T^2} \frac{\partial^2}{\partial \beta^2} \beta^{-1} = 2 \frac{n \tau k_B^2 T}{m} \tag{55}$$

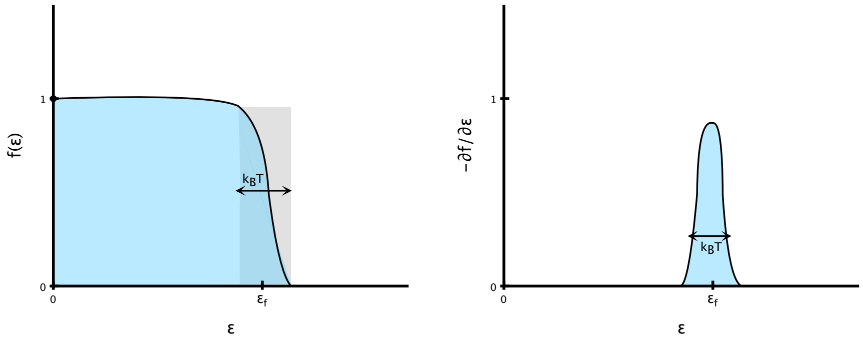
## A Evaluation of integral in Eq. (23)

In this appendix I shall show how to evaluate the integral

$$\int \frac{d^2 p}{(2\pi\hbar)^2} v_i^2 \frac{\partial f^0}{\partial \varepsilon} \tag{56}$$

First note that the distribution function is defined such that $\int \frac{d^2 p}{(2\pi\hbar)^2} f^0 = n = N/V$, the density of particles per unit volume. The value of $n$ can be obtained by performing the integral at zero temperature where the equilibrium distribution function becomes a Heaviside function

$$f^0(\epsilon) \equiv 2 f_{FD}(\epsilon) \to 2\Theta(\epsilon - \epsilon_F), \quad T \to 0 \tag{57}$$




The integral then reduces to the volume of a sphere of radius $p_F$ in momentum space which for 2 dimensions gives

$$n = \frac{2\pi p_F^2}{(2\pi\hbar)^2} = \frac{k_F^2}{2\pi} \tag{58}$$

As for the integral in Eq. (23), it can be evaluated by also looking at the zero temperature limit where the derivative of the distribution function then becomes a delta function:

$$\frac{\partial f^0}{\partial \epsilon} = -2\delta(\epsilon - \epsilon_F) \tag{59}$$

This approximation is in fact quite good even at higher temperatures since the deviation is proportional to $k_B T$. Since the Fermi-energy is usually much higher than the thermal energy $\epsilon_F \gg k_B T$, this deviation is negligible.

The integral can then be evaluated by going over to polar-coordinates with $p_i = p\cos\varphi$:

$$\begin{aligned}\int \frac{d^2p}{(2\pi\hbar)^2} v_i^2 \frac{\partial f^0}{\partial \epsilon} &= -\frac{2}{m^2}\int_0^\infty \frac{dp}{(2\pi\hbar)^2} p^3 \underbrace{\delta(\epsilon-\epsilon_F)}_{\frac{m}{p}\delta(p-p_F)} \int_0^{2\pi} \underbrace{d\varphi\cos^2\varphi}_{\pi} \\ &= -\frac{2}{m}\frac{k_F^2}{4\pi} = -\frac{n}{m}\end{aligned} \tag{60}$$