Coulomb's Law

(Gravity) $\boxed{F = G \frac{m_1 m_2}{r^2} \hat{r}}$

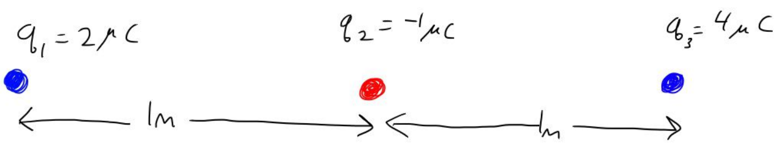
(E&M) $\boxed{F_{12} = k \frac{|q_1||q_2|}{r_{12}^2}}$ ← absolute value

$k = 9 \times 10^9 \frac{N \cdot m^2}{C^2}$, $G = 6.67 \times 10^{-11} \frac{N \cdot m^2}{kg}$, $\underbrace{m_e = 9.1 \times 10^{-31}\ kg}_{\text{electron mass}}$, $\underbrace{m_p = 1.6 \times 10^{-27}\ kg}_{\text{proton mass}}$

Superposition Principle

$$\boxed{\vec{F}_{net} = \sum_i \vec{F}_i}$$

Ex | 1-D)



What is the total force on q_3?

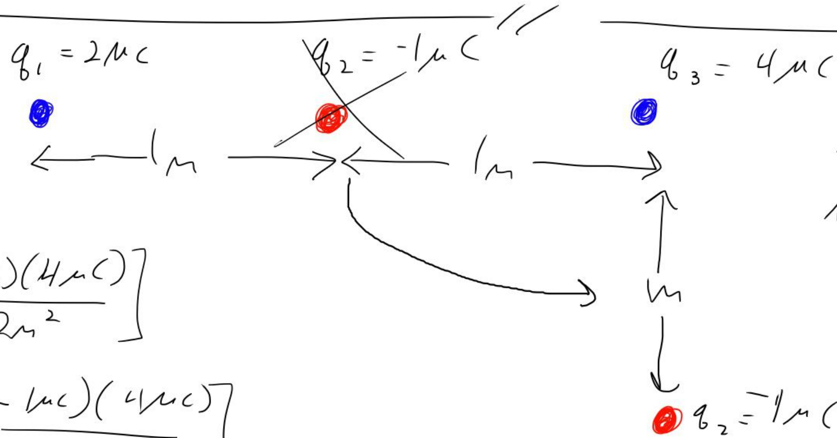
$*\ \vec{F}_{net} = \sum_i \vec{F}_i$, $F_{12} = k \frac{|q_1||q_2|}{r_{12}^2}$

negative because it is a force in the opposite direction.

$$\Rightarrow F_3 = F_{13} + F_{23} = k\left[\frac{(2\mu C)(4\mu C)}{(2m)^2} - \frac{|(-1\mu C)(4\mu C)|}{1m^2}\right] = k\left[2\frac{\mu C^2}{m^2} - \frac{4\mu C^2}{m^2}\right]$$

$$= k\left[-2\frac{(\mu C)^2}{m^2}\right] \approx \boxed{-0.018N}$$

Ex | 2-D



Now what is the Net force on q_3?

$$\vec{F}_{12} = \hat{x}\left[k \frac{(2\mu C)(4\mu C)}{2m^2}\right]$$

$$\vec{F}_{32} = \hat{y}\left[k \frac{(-1\mu C)(4\mu C)}{1m^2}\right]$$

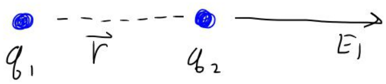
$\Rightarrow \vec{F}_{net_3} = 0.018N\,\hat{x} - 0.036N\,\hat{y}$, you can det. the angle from these values

$\Rightarrow \boxed{|\vec{F}_{net_3}| \approx (0.040N)}$ $\Rightarrow \theta = \tan^{-1}\left(\frac{-0.036}{0.018}\right) \approx \boxed{-63^\circ} \approx \theta$

Unit 2 – Electric Fields

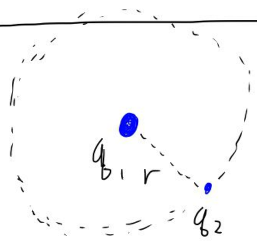
$$\vec{E}_1 = \frac{\vec{F}_{12}}{q_2} = k\frac{q_1}{r^2}\hat{r}$$

used for the situation:



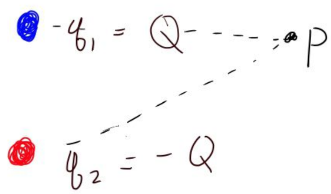
Generally $\boxed{\vec{E} = \frac{\vec{F}}{q}}$

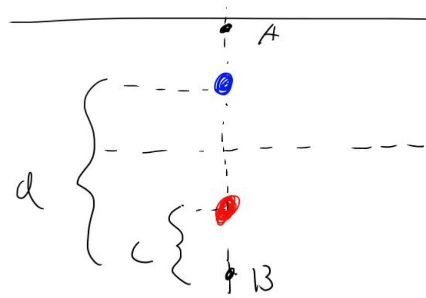
Ex



$$\boxed{\vec{E}_{q_1} = \frac{\vec{F}}{q} = k\frac{q_1}{r^2}\hat{r}}$$

Electric Dipole

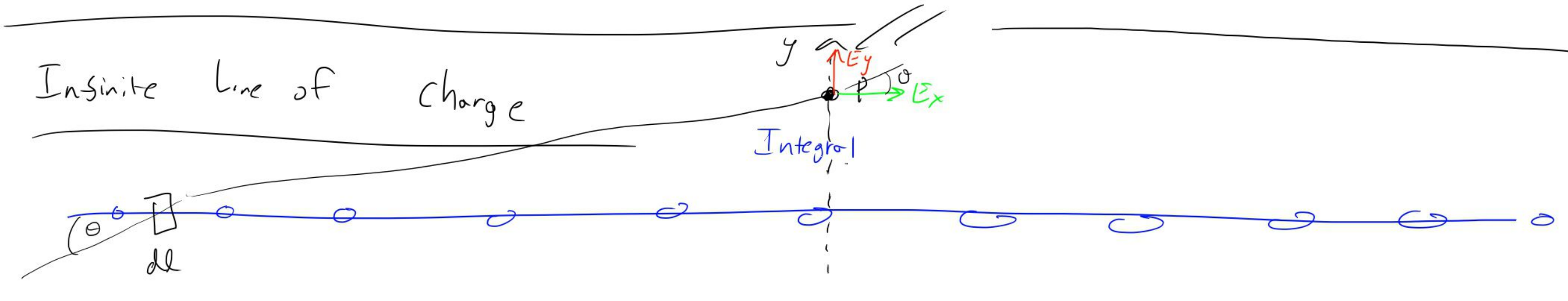




$$\vec{E}_A = k\frac{Q}{c^2} - k\frac{Q}{d^2}$$

$$\vec{E}_B = k\frac{Q}{c^2} - k\frac{Q}{d^2}$$

Infinite Line of Charge



$$E_x = \int_{x=-\infty}^{x=\infty} dE_x = 0 \quad , \quad E_y = \int_{x=-\infty}^{x=\infty} dE_y$$

$$dE_y = k\frac{dq}{s^2}\cos^2(\theta)$$

$$\Rightarrow E_y = \int_{-\infty}^{\infty} k\frac{dq}{s^2}\cos(\theta) \qquad dq = \lambda dx$$

$$= \int_{-\infty}^{\infty} \frac{\lambda dx}{s^2}\cos(\theta) \quad , \quad s = \frac{r}{\cos(\theta)}$$

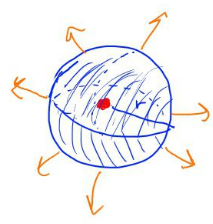
$$\Rightarrow E_y = \frac{k\lambda}{r}\int_{-\pi/2}^{\pi/2}\cos(\theta)d\theta$$

$$\boxed{E_y = \frac{2k\lambda}{r}}$$

An infinite line of charge symmetry has cylindrical symmetry & a point charge has spherical symmetry

Electric Flux & Field Lines

Recall $\boxed{\vec{E} = K\frac{q}{r^2}\hat{r}}$ for a point charge



Consider a spherical surface surrounding the point charge •

The surface area of the sphere is given by: $A = 4\pi r^2$

The density of field lines going through the sphere is given by:

$D = \frac{N}{4\pi r^2}$, N = # of Field Lines

Note the electric field is proportional to the Density of electric field lines

This is rewritten using several Equations

$$E = \frac{kq}{r^2} \Rightarrow E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

$$N \equiv \frac{q}{\varepsilon_0}$$

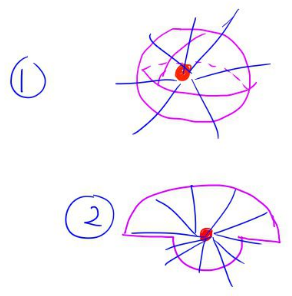
$$D = \frac{N}{4\pi r^2} \Leftrightarrow E = \frac{N}{4\pi r^2}$$

Electric Flux Φ_E

$$\boxed{\Phi_E = \int_{\text{surface}} \vec{E}\cdot d\vec{A}}$$

dA is the area element of the surface you are integrating over. The direction of which is orthogonal to the surface

Ex ① Flux



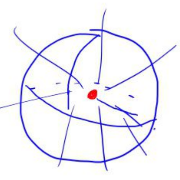
$\Phi = \int \vec{E}\cdot d\vec{A}$, $\vec{E}$ is constant

$= \vec{E}\int d\vec{A}$

$\Phi = EA$

② The flux through each hemisphere is the same & the flux through the straight lines in the middle on each side is zero.

Gauss's Law



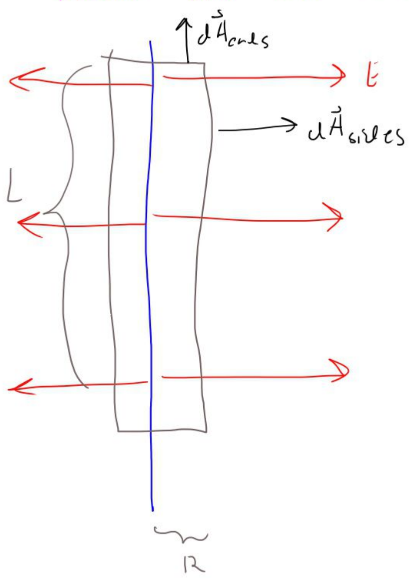
$\Phi = \frac{q}{\varepsilon_0}$

$$\Phi_E = \frac{q_{enclosed}}{\varepsilon_0}$$

$$\Rightarrow \boxed{\Phi = \oint_{\text{surface}} \vec{E}\cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_0}}$$

Ex Infinite Line of charge

Recall from unit 2. The direction of the field is perpendicular to the line of charge.



- Recall from the hemisphere example that surfaces parallel to the direction of the electric field have zero flux through them
- $d\vec{A}_{sides}$ is radially outward in the $\hat{r}$ direction from the line of charge

$\Phi_E = \oint_s \vec{E}\cdot d\vec{A}$, $\vec{E}$ is constant in $\hat{r}$ the same as $d\vec{A}$

$\Rightarrow \Phi_E = E\oint_s dA$

$= EA_{Barrel}$

$= E(2\pi R)L$, the charge we are working with for a line of charge is, λ,

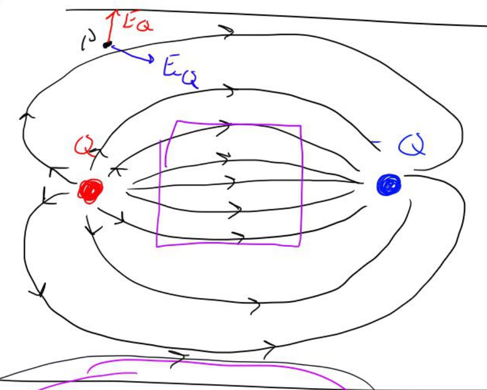
$= \left[\frac{\lambda}{2\pi\varepsilon_0 R}\right](2\pi R)L$, the $\left[\frac{\lambda}{2\pi\varepsilon_0 R}\right]$ comes from the derivation of the E-field of an infinite line of charge

$= \frac{\lambda L}{\varepsilon_0}$, where $\lambda L = q_{enclosed}$

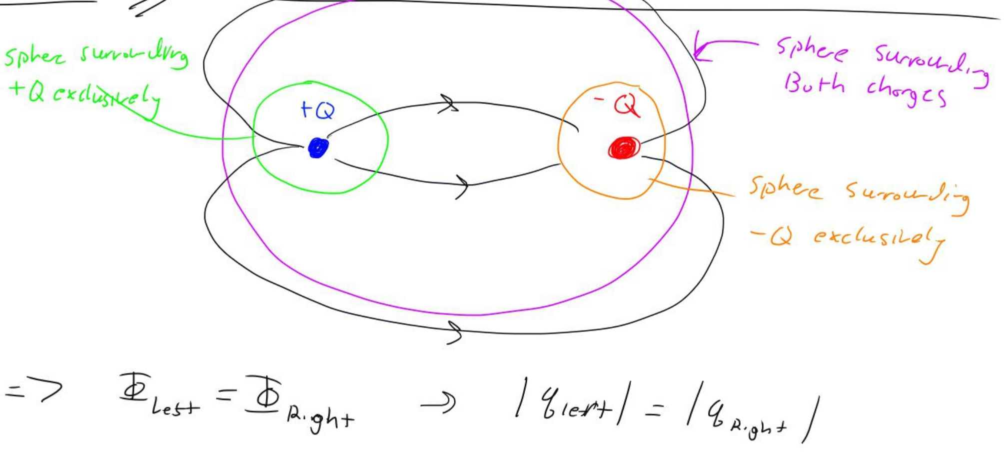
$$\boxed{\Phi_E = \frac{q_{enc}}{\varepsilon_0}}$$ Gauss's Law.

Electric Dipole Field Lines

The selected area is where the electric field is strongest as the density of field lines is greatest in between the two charges.



Gauss's Law & dipoles



Flux is $\frac{Q_{encl}}{\varepsilon_0} = \Phi_E$

(Therefore) $\therefore$ $\Phi = 0$

Flux is $\frac{+Q}{\varepsilon_0} = \Phi_E$

Flux is $\frac{-Q}{\varepsilon_0} = \Phi_E$

$\Rightarrow \Phi_{Left} = \Phi_{Right} \rightarrow |q_{Left}| = |q_{Right}|$

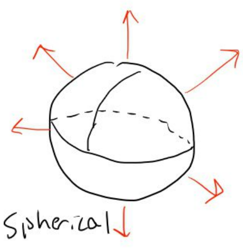
Gauss's Law

$$\boxed{\oint_S \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}}$$

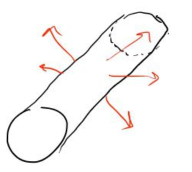
Symmetry

In general you want to choose Gaussian surface that results in a constant E field at all points along the surface of the shape.

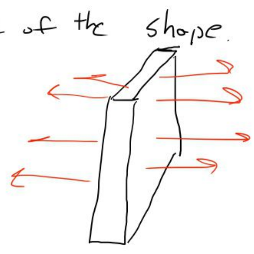
Ex:



Spherical
(3D)



Cylindrical
(2D)



Planar
(1D)

Solid spherical Insulator

Charge, Q
Radius, a

(Solid sphere)

Gauss's Law

Let's choose a sphere of radius r around the insulator

$\Rightarrow \Phi_E = \oint_S \vec{E} \cdot d\vec{A}$, $d\vec{A}$ & $\vec{E}$ have the same direction

$\Rightarrow \frac{Q_{encl}}{\varepsilon_0} = E \oint_S dA$

$\Rightarrow \frac{Q_{enc}}{\varepsilon_0} = E A_{surface}$

$\Rightarrow \frac{Q_{enc}}{\varepsilon_0} = E(4\pi r^2)$ $\Rightarrow$ $\boxed{E = \frac{Q_{encl}}{4\pi\varepsilon_0 r^2}}$

This is for $r > a$

Inside the insulator

Now a spherical gaussian surface inside the sphere

$\Phi_E = \oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$, Now $Q_{enclosed}$ is no longer Q but a portion of it.

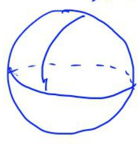
- The charge of the insulator is uniform
  Consider the volume charged density $\rho = \frac{Q}{V} = \frac{Q}{\frac{4}{3}\pi a^3}$

$\Rightarrow E A_s = \frac{\rho \cdot \frac{4}{3}\pi r^3}{\varepsilon_0}$

$\Rightarrow E(4\pi r^2) = \frac{\rho}{\varepsilon_0} \frac{4}{3}\pi r^3 \Rightarrow \boxed{E = \frac{\rho}{3\varepsilon_0} \cdot r \quad (r < a)}$

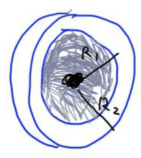
- Notice, this is Linear & not squared

Conductor, Charges on the Surface



Solid Conductor
charges are free to move
$\Rightarrow$ distributed evenly across the surface.

$\boxed{E_{inside} = 0 \quad \boxed{r<a} \quad q_{enclosed} = 0}$ $\Rightarrow \Phi_{E\,inside} = 0 = \frac{q_{encl}}{\varepsilon_0} \Rightarrow q_{enclosed} = 0$ Inside



Conducting shell

Q on outside of the shell

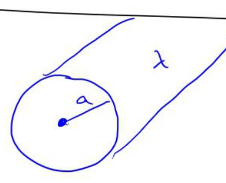
Now Imagine the black dot is a charge q_0 centered in the sphere.
It will induce a charge on the inner surface to cancel out the field it produces additionally adding charge to the exterior surface.

$Q_{inner} = -q_0$

$\sigma_i = \frac{-q_0}{4\pi R_i^2}$ Inner charge density

$\sigma_o = \frac{Q + q_0}{4\pi R_o^2}$ Outer charge density

Solid Infinite Cylindrical Conductor



placing a gaussian surface in the shape of a cylinder around it results in the expected

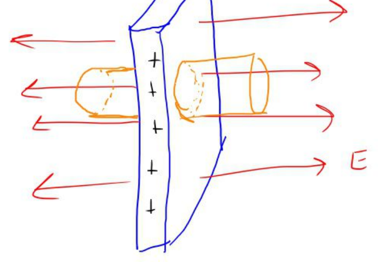
$E(2\pi r L) = \frac{q_{enc}}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0}$

$\Rightarrow \boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r} \quad (r > a)}$ Identical to infinite line of charge

---

Infinite plane of charge

Surface charge density σ

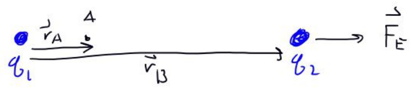


Cylinder Gaussian surface
only flux going through the ends

$\Rightarrow \oint_S \vec{E} \cdot d\vec{A} = \underbrace{2EA_{ends}}_{\text{2 due to the two surfaces}} = \frac{q_{enc}}{\varepsilon_0} = \frac{\sigma(\pi r^2)}{\varepsilon_0} = 2E(\pi r^2)$

$\Rightarrow \boxed{E = \frac{\sigma}{2\varepsilon_0}}$ (No dependence on the distance from the sheet)

Electric Potential Energy



Coulomb force is conservative => path does not matter, only the begining & endpoints.

$$\boxed{W_{A\to B} = \int_{r_A}^{r_B} \vec{F}_E \cdot d\vec{r} = \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)}$$

Electric Potential Energy

$$\Delta U = -W$$

$$\Rightarrow \Delta U_{AB} = \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

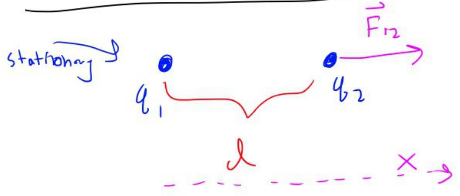
It is natural to select the 1st distance to be where the potential is zero. This is at ∞ for a charge giving a value of $\frac{1}{r_A} = \frac{1}{\infty} = 0$

$$\Rightarrow \boxed{U_r = \Delta U_{AB} = \frac{q_1 q_2}{4\pi\varepsilon_0 r}}$$

, r being the distance between the charges

- If the charges are the same, the potential is positive
- If the charges are opposites the potential is negative

Calculate speed ex



Since V(x)

· Note

=> The both start at rest the kinetic energy gained will be the same as the change in potential energy due to energy conservation.

$$\Rightarrow U_i - U_f = K_{final}$$

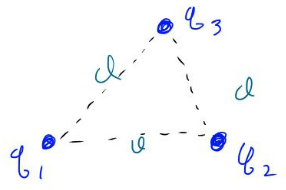
$$\Rightarrow U_i - U_f = \tfrac{1}{2} m_2 v^2$$

$$\Rightarrow \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{d} - \frac{1}{x}\right) = \frac{1}{2} m_2 v^2$$

$$\Rightarrow \boxed{v = \sqrt{\frac{q_1 q_2}{2\pi\varepsilon_0 m_2}\left(\frac{1}{d} - \frac{1}{x}\right)}}$$

$$\boxed{v_{max} = \sqrt{\frac{q_1 q_2}{2\pi\varepsilon_0 m_2 d}}}$$

System of 3 - charges fixed



- Determining the electric potential energy of this static system is simply calculating the energy to bring the charges from ∞ to their current positions.

=> q_1, takes no work as there is nothing there to interact

=> q_2, Interacts with q_1

$$\Delta U_2 = \frac{k q_1 q_2}{d}$$

=> q_3, interacts with both q_1 & q_2

$$\Delta U_3 = \frac{k q_3 q_2}{d} + \frac{k q_3 q_1}{d}$$

$$\Rightarrow \boxed{U_{system} = \frac{k}{d}\left[q_1 q_2 + q_1 q_3 + q_2 q_3\right]}$$

∴ For N-charged Particles

$$\boxed{U_{system} = k \sum_{pairs} \frac{q_i q_j}{r_{ij}}}$$

Electric Potential

From electric potential energy

$\vec{F} = \frac{kq_1q_2}{r^2}\hat{r}$

$\vec{E} = \frac{kq}{r^2}\hat{r} = \frac{\vec{F}}{q}$

$\Delta V \equiv \frac{\Delta U}{q}$

$W_{A\to B} = \int_A^B \vec{F}\cdot d\vec{l}$

$= \int_A^B q\vec{E}\cdot d\vec{l}$

$\Rightarrow \Delta U_{A\to B} = -\int_A^B q\vec{E}\cdot d\vec{l}$

$\Rightarrow \boxed{\frac{\Delta U_{A\to B}}{q} = -\int_A^B \vec{E}\cdot d\vec{l} = \Delta V_{AB}}$

$V(r) \equiv \Delta V_{r_0\to r} = kQ\left(\frac{1}{r} - \frac{1}{r_0}\right)$

r_0 where the potential is zero which is usually @ $r_0 = \infty$

$\Rightarrow \boxed{V(r) = \frac{kQ}{r}}$

Going from $E \to V$ & $V \to E$

1-D $\quad V(x) = -\int_{x_0}^{x} \vec{E}\cdot d\vec{x}$

3-D $\quad V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l}$

Field to potential

Euclidean Geometry

$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

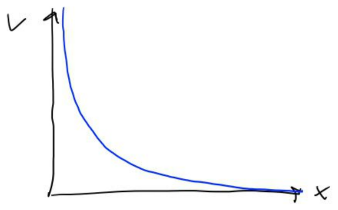
1-D $\quad \vec{E}(x) = -\frac{dV}{dx}\hat{i}$

3-D $\quad \vec{E}(r) = -\vec{\nabla}V$

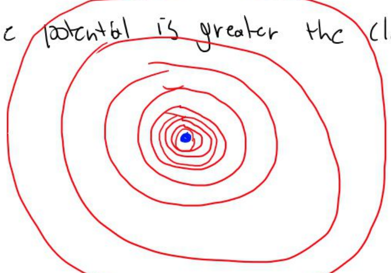
Potential to Field

Equipotentials

- The potential is greater the closer you get to a charge

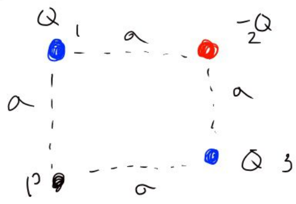


$\Rightarrow$



Equipotential lines
(use your imagination that the sketch looks good)
And that the writing is legible

Ex



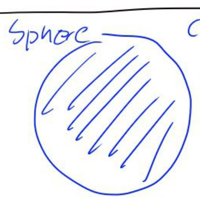
$V_p = k\frac{Q}{r_1} + k\frac{-Q}{r_2} + k\frac{Q}{r_3}$

$= k\frac{Q}{a} + k\frac{-Q}{a\sqrt{2}} + k\frac{Q}{a}$

$\boxed{V_p = k\frac{Q}{a}\left[2 - \frac{\sqrt{2}}{2}\right]}$

Charged solid insulator

Ex



charge Q
radius a

Recall from previous units:

$E_{in} = \frac{kQ}{a^3}r \quad (r<a)$

$E_{out} = \frac{kQ}{r^2} \quad (r>a)$

$\Rightarrow \Delta V_{A\to B} = -\int_A^B \vec{E}\cdot d\vec{l}$

0 at ∞

$\Rightarrow V(r) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{l}$

$\Rightarrow (r<a) \quad V(r) = -\int_{-\infty}^{r}\vec{E}\cdot d\vec{l} = -\int_{-\infty}^{a} E_{out}\,dr - \int_a^r E_{in}\,dr$

$V(r) = k\frac{Q}{2a^3}(3a^2 - r^2) \quad (r<a)$

$\Rightarrow (r>a) \quad V(r) = k\frac{Q}{r} \quad (r>a)$

Electric Potential (scalar)

$\boxed{V_Q \equiv \frac{U}{Q} \text{ (scalar)} \qquad \vec{E}_Q \equiv \frac{\vec{F}}{Q} \text{ (vector)}}$

Conductors & Capacitance

Conductors & Equipotentials

A conductor is an equipotential, since the field inside of a conductor is zero that means that a potential will be zero everywhere in the conductor => it is an equipotential.

- Field lines are always perpendicular to a conductors surface

---

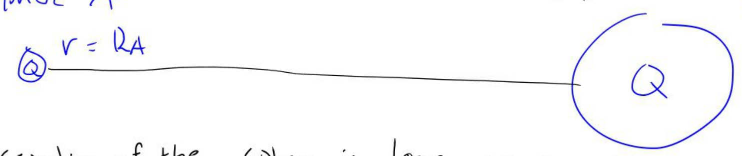
Ex

Sphere A

$r = R_A$

Sphere B

$r = 4R_A$



If the separation of the spheres is large we can calculate the pot. of A at its surface

$V_A = k\frac{Q}{R_A}$, $V_B = k\frac{Q}{4R_A}$

Now connect the two spheres by a wire.

This causes them to become a single conductor & the charge will be spread so that their potentials will be equal

$$k\frac{Q_A}{R_A} = k\frac{Q_B}{R_B}$$

$$k\frac{Q_A}{R_A} = k\frac{Q_B}{4R_A}$$

The total charge must remain the same $Q_A + Q_B = 2Q$ =>

=> $4Q_A = Q_B$

Plug that in to $Q_A + Q_B = 2Q$ => $Q_A + 4Q_A = 2Q$

=> $5Q_A = 2Q$ => $Q_A = \frac{2}{5}Q$

$4Q_A = Q_B$

=> $Q_B = \frac{8}{5}Q$

Note the charge density is still 4 times larger on A

$$\boxed{\sigma_A = 4\sigma_B}$$

This tells you that the charge density of a conductor is largest at the points of greatest curvature

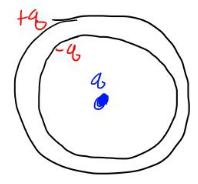
---

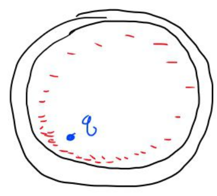
Charge distribution on Conductors

Recall the ex. of the conducting shell & the point charge inside.

Induced charges.

- Equal charge distribution

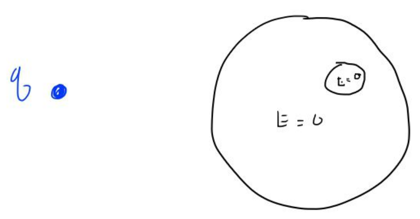




The density of negative charge induced will be greater nearest to the charge inside

- The exterior positive charge distribution does not change

Shielding in a conductor w/ a cavity



The same thing as above except the charge is outside & the charges on the inner surface stay the same & you find using Gauss's law that the field inside the cavity is zero.

Capacitance

E-field In between $\boxed{E = \frac{Q}{\varepsilon_0 A}}$

Parallel Plate Capacitor

$$\boxed{C = \frac{\varepsilon_0 A}{d}}$$

, A = Area of the plates

, d = distance between the plates

Energy Storage

* $C = QV$ => $V = \frac{Q}{C}$

$$\boxed{U = \frac{1}{2}QV \quad \text{or } U = \frac{1}{2}\frac{Q^2}{C} \quad \text{or } U = \frac{1}{2}CV^2}$$

Energy in Electric Fields

The potential between the plates is $\boxed{V = Ed}$

(Energy density) $u = \frac{1}{2}\varepsilon_0 E^2$

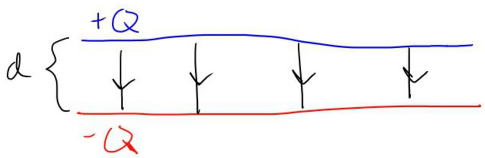
$U = \frac{1}{2}\overset{C = \frac{\varepsilon_0 A}{d}}{C}V^2$, $V = Ed$

=> $U = \frac{1}{2}\frac{\varepsilon_0 A}{d}\left[E^2 d^2\right] = \frac{1}{2}\varepsilon_0 A E^2 d$

$$\boxed{u = \frac{U}{Ad} = \frac{1}{2}\varepsilon_0 E^2}$$

Capacitors

Parallel-Plate Capacitors



If D increases

- Q remains constant
- E remains constant
- ΔV increases
- C decreases
- U increases

$$C = \frac{Q}{\Delta V} = \frac{Q}{Ed} = \frac{Q}{\frac{\sigma}{\varepsilon_0} d} = \frac{Q}{\frac{Q}{\varepsilon_0 A} d} = \frac{\varepsilon_0 A}{d} = C$$

$$E = \frac{\sigma}{\varepsilon_0} \qquad \sigma = \frac{Q}{A}$$

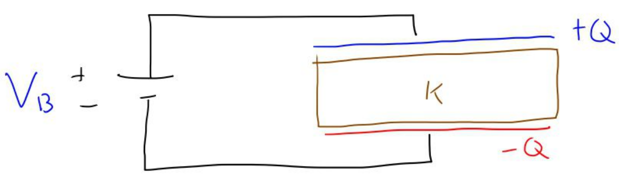
Dielectrics

- Dielectrics change the capacitance by a factor of K

$\Rightarrow$ $\boxed{C_{new} = K\, C_{original}}$

Dielectric added to an isolated capacitor

- E decreases
- V decreases
- C increases

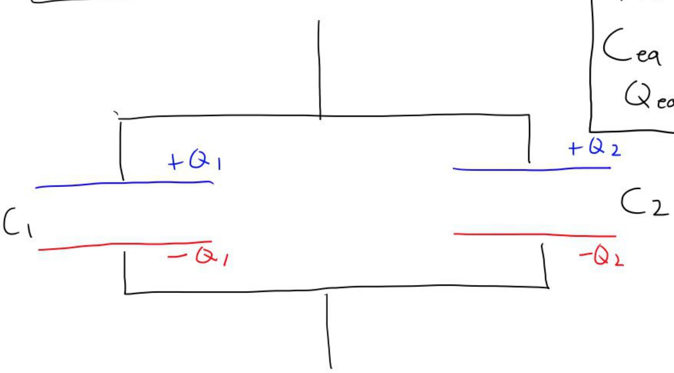


Dielectric Added

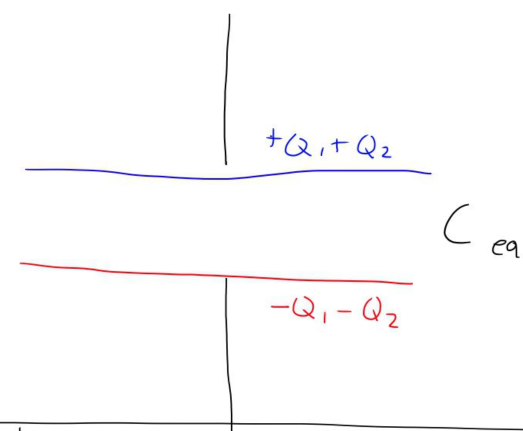
- C increases
- V_c remains constant
- Q increases
- U increases

Capacitors in Parallel

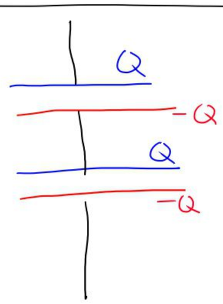
$$\boxed{\begin{aligned} V_{eq} &= V_1 = V_2 \\ C_{eq} &= C_1 + C_2 \\ Q_{eq} &= Q_1 + Q_2 \end{aligned}}$$



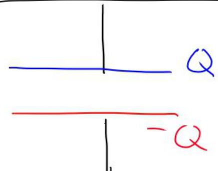
=



Capacitors in Series



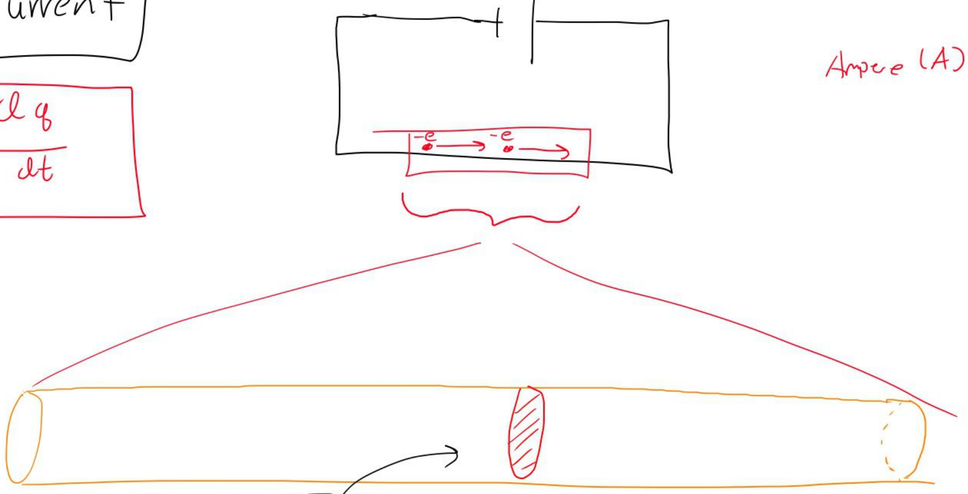
=



$$\boxed{\begin{aligned} Q_{eq.} &= Q_1 = Q_2 \\ V_{eq} &= V_1 + V_2 \\ \frac{1}{C_{eq}} &= \frac{1}{C_1} + \frac{1}{C_2} \end{aligned}}$$

Electric Current

$$\boxed{I = \frac{dq}{dt}}$$



Ampere (A) = $\frac{\text{Coulomb (C)}}{\text{Seconds (s)}}$

Current density $\quad J \equiv \frac{I}{A} = n_e e\, v_{drift}$, n_e = # of electrons (charge density carriers), e = fundamental charge, v_{drift} = Velocity of electrons

$n_e = N_A \frac{\rho_{mass}}{M}$ — Molar mass — This is specific to materials

Ohm's Law

$$\boxed{\begin{aligned} J &= \sigma E \\ J &= n_e e\, v_{drift} \\ \Rightarrow \quad v_{drift} &= \frac{\sigma}{n_e e} E \end{aligned}}$$

, σ = conductivity, specific to materials

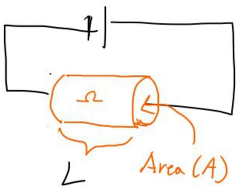
Resistance

$$R = \frac{1}{\sigma}\frac{L}{A} \quad \text{or} \quad R = \rho \frac{L}{A}$$

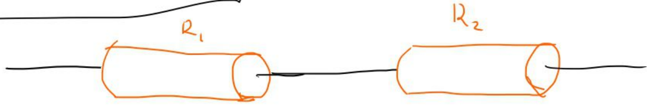
Ohm (Ω) = $\frac{\text{Volt (V)}}{\text{Ampere (A)}}$

Ohm's Law

$$J = \sigma E \quad \text{or} \quad V = IR$$

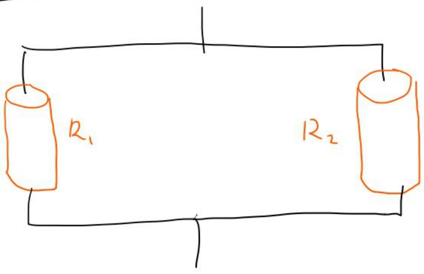


Resistors in Series



$$\boxed{\begin{aligned} I_{eq.} &= I_1 = I_2 \\ V_{eq} &= V_1 + V_2 \\ R_{eq} &= R_1 + R_2 \end{aligned}}$$

Resistors in Parallel



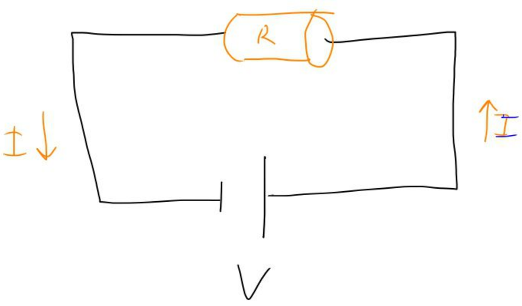
$$\boxed{\begin{aligned} I_{eq} &= I_1 + I_2 \\ V_{eq} &= V_1 = V_2 \\ \frac{1}{R_{eq}} &= \frac{1}{R_1} + \frac{1}{R_2} \end{aligned}}$$

Power

$$\boxed{P = IV} \quad \text{or} \quad \boxed{P = I^2 R}$$

Power (W)$_{watt}$ = Volt · Ampere (V·A)

$$P = \frac{dU}{dt} \quad , \quad I = \frac{dq}{dt} \quad \Rightarrow \quad dq = I\,dt$$



$$dU = dq\, V$$

$$\Rightarrow dU = I\,dt\, V$$

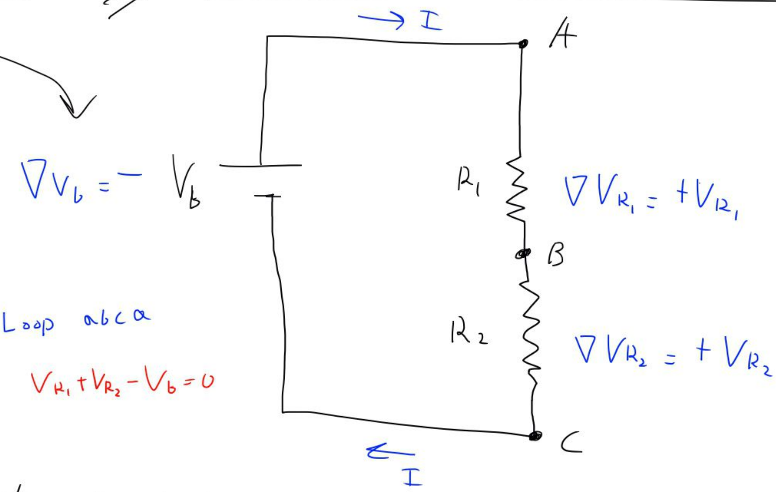
$$\Rightarrow \frac{dU}{dt} = IV \Rightarrow \boxed{P = IV}$$

Kirkhoff's Rules

Voltage Rule	Current Rule
(Any closed Loop) $\sum \Delta V_n = 0$	$\sum I_{in} = \sum I_{out}$ (at any node)

Convention

(I disagree with this)



- This is the given Convetion where you consider each resistor and the voltage across each as an "additive" component & the battery as a "subtractive" component. which doesn't make much sese when you think about how Resistors Impede the movement of charge in a circuit & how Batteries add current & Voltage to a circuit,

Feel Free to ignore this however, this is how I interpret the above Circuit.

You start at the Bottom left & go through the Battery

(1) $+V_b$

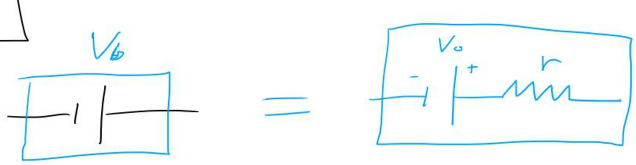
Next you travel through R_1

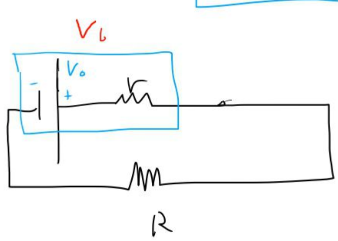
(2) $+V_b - V_{R_1}$ (A voltage drop across the Resistor)

Next you trave through R_2 & finish the circuit

(3) $\boxed{+V_b - V_{R_1} - V_{R_2} = 0}$

Real Batteries





$$I = \frac{V_0}{R+r}$$

$$\Rightarrow V_b = V_0 - Ir$$

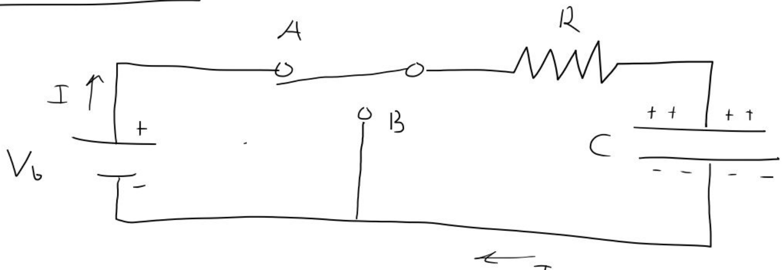
$$= V_0 - \frac{V_0}{R+r} r$$

$$V_b = V_0 \left[1 - \frac{r}{R+r} \right]$$

$$= V_0 \left[\frac{R+r}{R+r} - \frac{r}{R+r} \right]$$

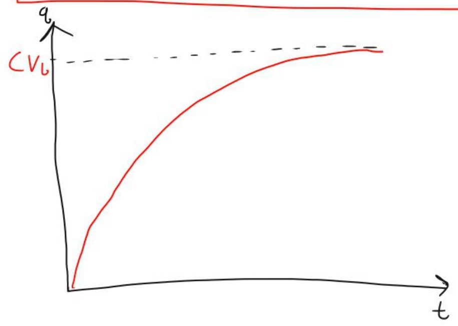
$$\boxed{V_b = V_0 \left[\frac{R}{R+r} \right] = V_0 \left[\frac{R/r}{\frac{R}{r}+1} \right]}$$

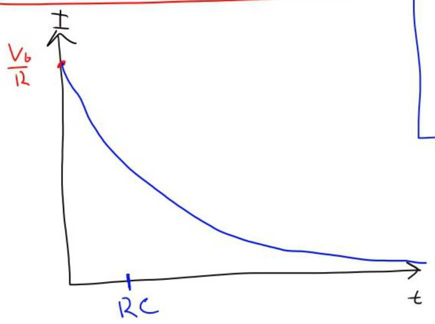
RC - Circuit



$$\boxed{q(t) = CV_b(1-e^{-t/RC})}$$

$$\boxed{I(t) = \frac{V_b}{R}e^{-t/RC}}$$





$t = 0$
$V_c(0) = 0$
$I(0) = \frac{V_b}{R}$

t increases
q & V_c increases
I, V_R decreases

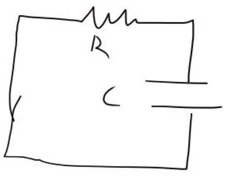
$t \to \infty$
$q \to CV_b$
$I \to 0$

Charging

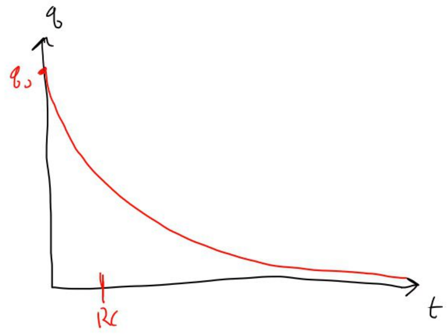
Breaking the Circuit By switching to "B"

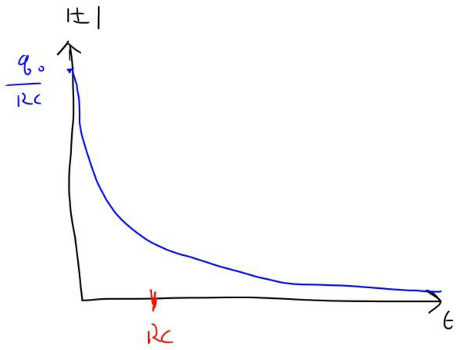
$$\boxed{q(t) = q_0 e^{-t/RC}}$$

$$\boxed{I(t) = \frac{-q_0}{RC}e^{-t/RC}}$$



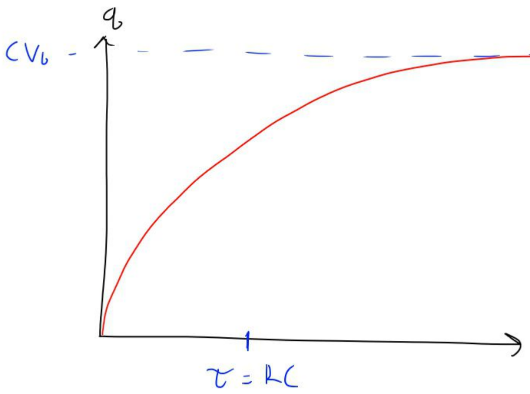
Boundary Conditions	
$t=0$	$t=\infty$
$q(0) = q_0$	$q(\infty) = 0$
$\lvert I(0)\rvert = \frac{q_0}{RC}$	$I(\infty) = 0$

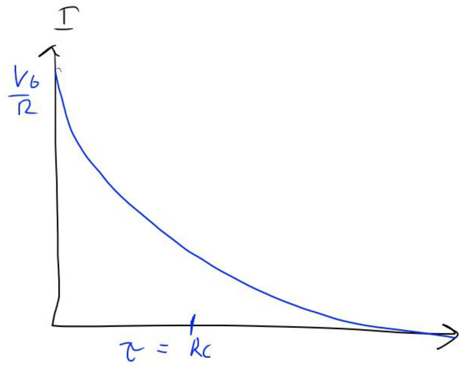




time constant

$$\tau = RC$$





Power In the RC Circuit

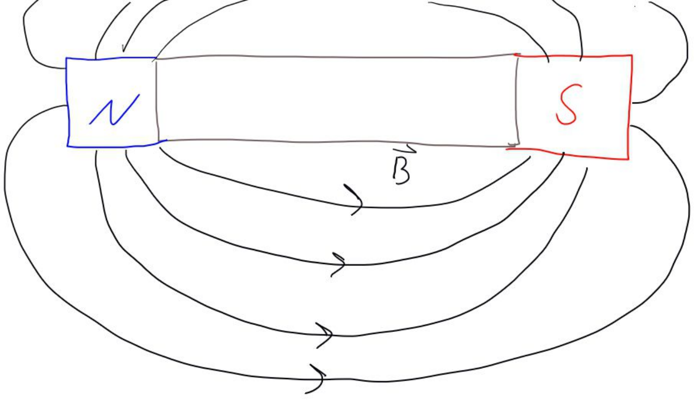
$P_{Battery}(t) = V_b I_0 e^{-t/RC}$

$P_R(t) = RI_0^2 e^{-2t/RC}$

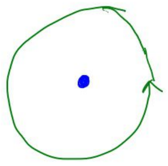
$P_c(t) = \left[\frac{q_0}{C}(1-e^{-t/RC})\right]\left[I_0 e^{-t/RC}\right]$

$$P_{Battery}(t) = P_R(t) + P_c(t)$$

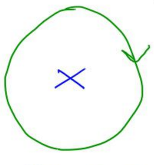
Magnetism



Line of Current

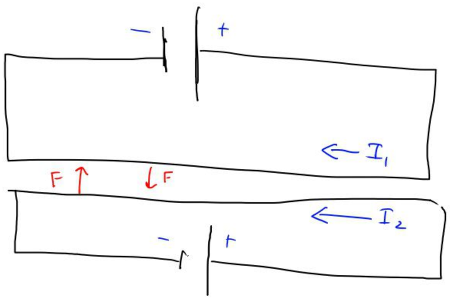


out of page



Into page

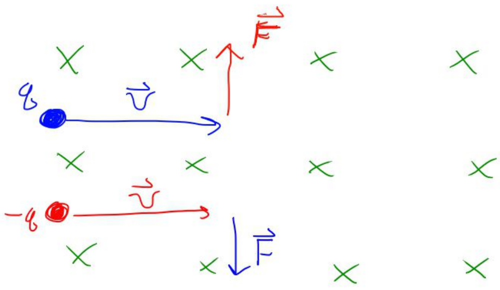
[RHR] point your thumb in the direction of the current & the way your fingers naturally curl is the direction of the $\vec{B}$-field



Magnetic Force

Cross product

$$\boxed{\vec{F} = q\vec{v}\times\vec{B}}$$



Right hand Rule (RHR)

- Point your fingers in the direction of the field & your thumb in the direction of the velocity.

The direction your palm points is the direction of the force "felt" by the positive charge.

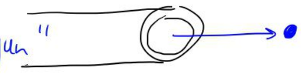
Flip this for negative charges

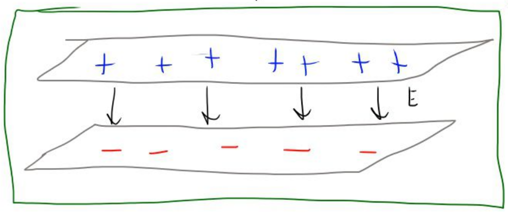
Recall: $|\vec{A}\times\vec{B}| = AB\sin(\theta)$

Velocity Selector

× $\vec{B}$ Into page in Green zone

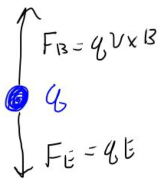
Charge "gun"





There are two forces acting on particles passing through

- Electric
- Magnetic



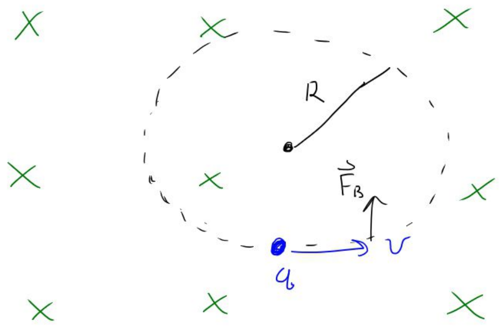
For a particle to pass through:

$$\vec{F}_e = -\vec{F}_{mag}$$

$$\Rightarrow q\vec{E} = -q\vec{v}\times\vec{B}$$

$$\Rightarrow E = vB \quad \Rightarrow \boxed{v = \frac{E}{B}}$$

Motion in a constant $\vec{B}$-field



Uniform circular motion

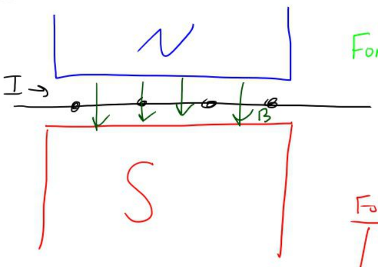
$$F = qvB = ma_c$$

$$a_c = \frac{v^2}{R}$$

$$\Rightarrow \frac{qvB}{m} = \frac{v^2}{R}$$

$$\Rightarrow \boxed{R = \frac{mv}{qB}}$$

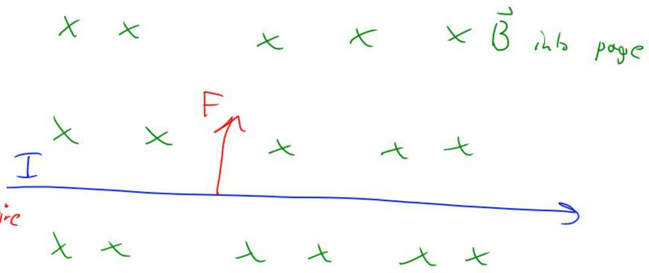
Forces & Torques on Currents



Force into page

Force on a current carrying wire

$$\boxed{\vec{F} = I\vec{L}\times\vec{B}}$$



$\vec{F} = q\sum \vec{v}_i \times \vec{B}$

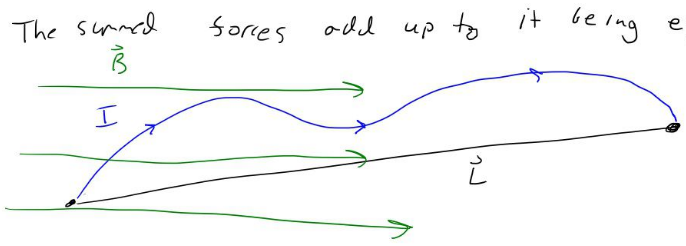
$= q(N\vec{v}_{avg})\times\vec{B}$, $N = nAL$ ← # of charges; AL — Volume of wire in B-field

$= qnAL\,\vec{v}_{avg}\times\vec{B}$, $I = nAqV_{avg}$

$\vec{F} = I\vec{L}\times\vec{B}$ //

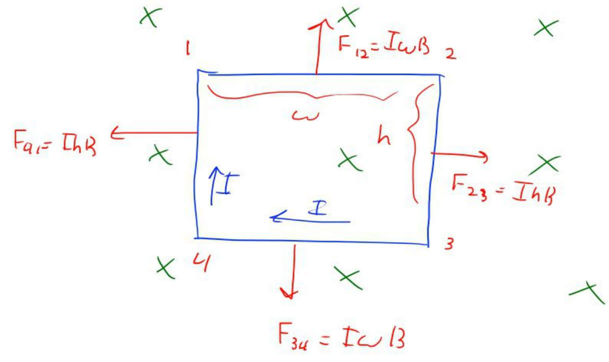
Curved wire

The summed forces add up to it being equivalent to a straight wire from point A to point B



Force on a Current Loop

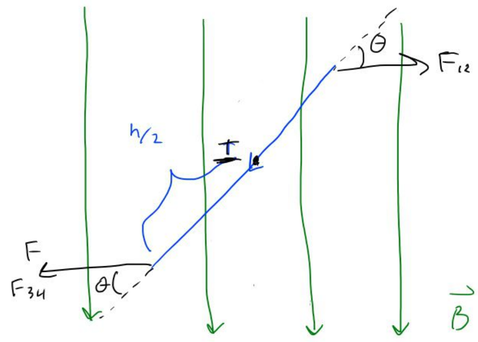
$$\sum F_{closed\ loop} = 0$$



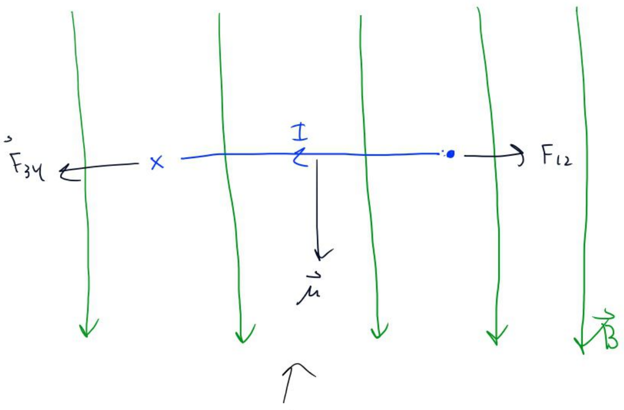
τorque on a Current Loop

$$\boxed{\begin{aligned}&\tau_{loop} = IAB\sin(\theta)\\ &\Rightarrow \theta = 0^\circ \rightarrow \tau = 0\\ &\Rightarrow \theta = 90^\circ \rightarrow \tau = IAB\end{aligned}}$$

A = Area of loop



Dipole Moment of a Current Loop



$\tau = 0$

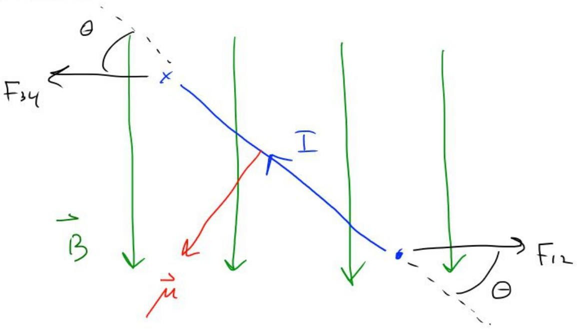
Magnetic Dipole Moment	Torque on a Loop
$\vec{\mu} = NI\vec{A}$	$\vec{\tau} = \vec{\mu}\times\vec{B}$

N = # of Loops

$\vec{A}$ is the area vector, the area vector direction is determined by curling your fingers in the direction of the current & your thumb points in the direction of the Area element.

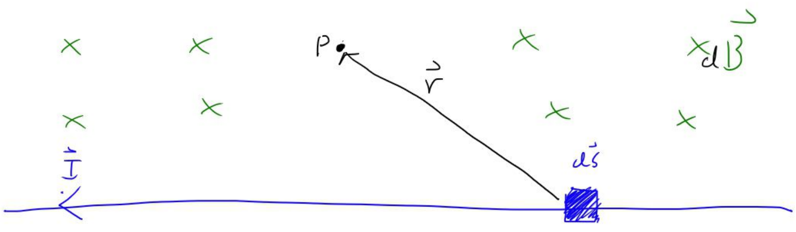
Magnetic Potential Energy

$$\boxed{U(\theta) = -\vec{\mu}\cdot\vec{B} = -\mu B\cos(\theta)}$$

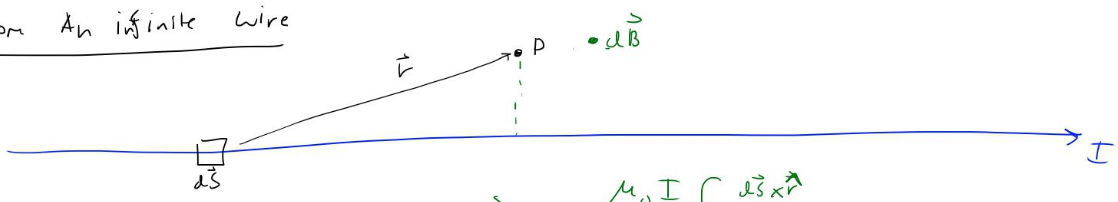


Biot-Savart Law

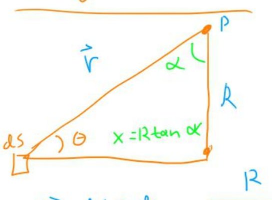
$$\vec{dB} = \frac{\mu_0 I}{4\pi} \frac{\vec{ds} \times \hat{r}}{r^2}$$



From An infinite wire



triangle from above



$\Rightarrow ds = dx = \frac{R}{\cos^2(\alpha)} d\alpha$

$\Rightarrow r = \frac{R}{\cos(\alpha)}$

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{\vec{ds} \times \hat{r}}{r^2}$$

$$= \frac{\mu_0 I}{4\pi} \int \frac{\sin(\theta)\, ds}{r^2} \hat{z}$$

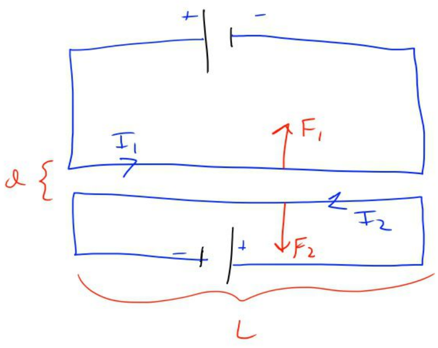
$$= \frac{\mu_0 I}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{\cos(\alpha)}{r^2} \frac{R}{\cos^2(\alpha)} d\alpha\, \hat{z}$$

$$= \frac{\mu_0 I}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{\cos(\alpha)}{\left(\frac{R}{\cos(\alpha)}\right)^2} \frac{R}{\cos^2(\alpha)} d\alpha\, \hat{z}$$

$$= \frac{\mu_0 I}{4\pi R} \underbrace{\int_{-\pi/2}^{\pi/2} \cos(\alpha)\, d\alpha}_{2}\, \hat{z}$$

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi R} \hat{z}}$$

Force Between two current Carrying Wires



$F_1 = -F_2$

$$\boxed{|F_2| = |F_1| = \frac{\mu_0}{2\pi d} I_1 I_2 L}$$

B @ I_1 caused by I_2

$$B_2 = \frac{\mu_0 I_2}{2\pi d}$$

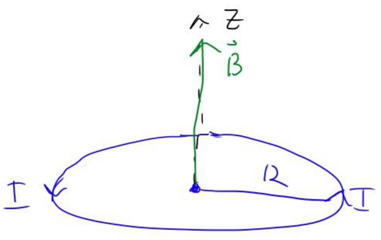
$\Rightarrow$ F on I_1 due to I_2's B_2

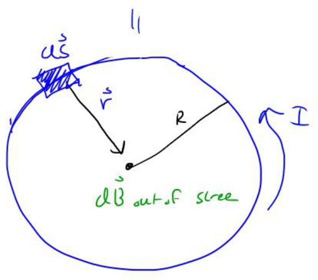
$$\vec{F}_1 = I_1 \vec{L} \times \vec{B}_2$$

$$\Rightarrow \boxed{\vec{F}_1 = I_1 L \frac{\mu_0 I_2}{2\pi d} = \frac{\mu_0}{2\pi d} I_1 I_2 L}$$

B - along axis of Current Loop

$$\boxed{\vec{B} = \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z^2)^{3/2}}}$$





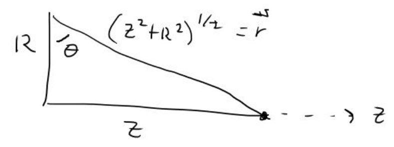
$$* \vec{dB} = \frac{\mu_0 I}{4\pi} \frac{\vec{ds} \times \hat{r}}{r^2}$$

$$\int \vec{dB} = \frac{\mu_0 I}{4\pi R^2} \underbrace{\int \vec{ds}}_{2\pi R}$$

$$\boxed{B_{center} = \frac{\mu_0 I}{2R}}$$

* Moving in the z-direction

$\Rightarrow$



$$* B = \int dB_z$$

$$= \int \left(\frac{\mu_0 I}{4\pi} \frac{ds}{r^2}\right) \cos(\theta)$$

$$= \frac{\mu_0 I}{4\pi} \frac{R}{(R^2+z^2)^{3/2}} \underbrace{\int ds}_{2\pi R}$$

$$\Rightarrow \boxed{B = \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z^2)^{3/2}}}$$

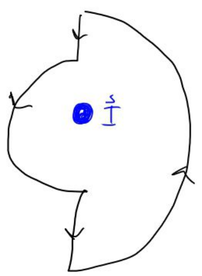
Ampere's Law

Recall Gauss's Law $\boxed{\Phi_E = \oint_S \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}}$

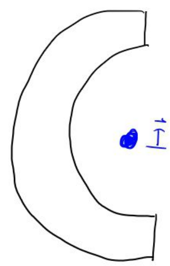
Ampere's Law

$$\boxed{\oint_{Loop} \vec{B}\cdot d\vec{l} = \mu_0 I}$$

Ex's

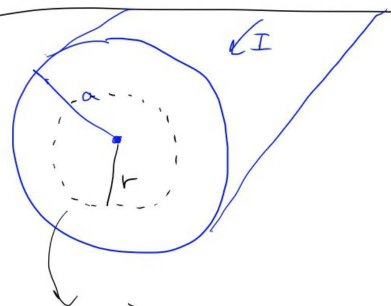


$\boxed{\oint_{Loop} \vec{B}\cdot d\vec{l} = \mu_0 I}$



$\oint_{Loop} \vec{B}\cdot d\vec{l} = 0$

Mag field Inside a current Carrying wire



$$\oint_{Loop} \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$$

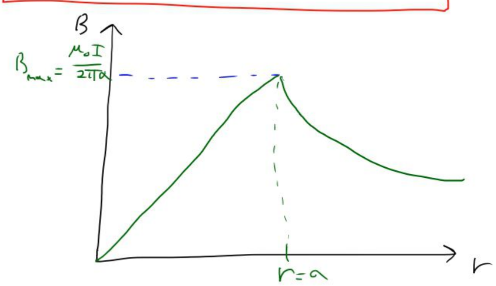
$$\Rightarrow B \oint_{Loop} dl = \mu_0 I_{encl}$$

$$\Rightarrow B(2\pi r) = \mu_0 I_{encl}$$

$$\Rightarrow B(2\pi r) = \mu_0 I\left(\frac{\pi r^2}{\pi a^2}\right)$$

$$\Rightarrow \boxed{B = \frac{\mu_0}{2\pi a^2} r \quad (r<a)}$$

$$\boxed{\begin{aligned} B &= \frac{\mu_0 I}{2\pi a^2} r \quad (r<a) \\ B &= \frac{\mu_0 I}{2\pi r} \quad (r>a) \end{aligned}}$$

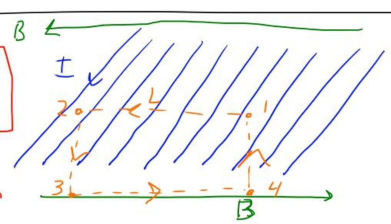


Infinite Sheet of Current

- Think of it as densely packed current carrying wires

$\boxed{B = \frac{1}{2}\mu_0 n I}$

$n \equiv \frac{\text{\# of wires}}{\text{unit length}}$



$$\oint_{Loop} \vec{B}\cdot d\vec{l} = \int_1^2 \vec{B}\cdot d\vec{l} + \underbrace{\cancel{\int_2^3 \vec{B}\cdot d\vec{l}}}_{0} + \int_3^4 \vec{B}\cdot d\vec{l} + \underbrace{\cancel{\int_4^1 \vec{B}\cdot d\vec{l}}}_{0}$$

$$= \int_1^2 \vec{B}\cdot d\vec{l} + \int_3^4 \vec{B}\cdot d\vec{l}$$

$$= BL + BL$$

$$= 2BL = \mu_0 I_{encl}, \quad I_{encl} = nLI$$

$$= 2BL = \mu_0 nLI \rightarrow \boxed{B = \frac{1}{2}\mu_0 n I}$$

Motional EMF

Electrodynamics

Maxwell's Equations

Gauss's Law for $\vec{E}$-fields

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

Gauss's Law for $\vec{B}$-fields

$$\oint \vec{B}\cdot d\vec{A} = 0$$

Faraday's Law

$$\oint \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$$

Ampere-Maxwell Law

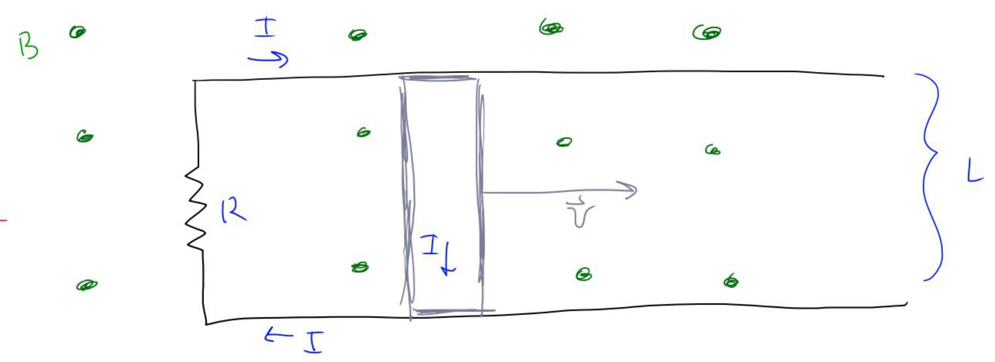
$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I + \mu_0\varepsilon_0 \frac{d}{dt}\int \vec{E}\cdot d\vec{A}$$

At equilibrium

$$\boxed{E = vB}$$

Potential Difference

$$\boxed{\mathcal{E} = vBL}$$



Equilibrium

$F_E = F_B$

$qE = qvB$

$\Rightarrow \boxed{E = vB}$

$\Rightarrow$

$\Delta V = EL$

$\Delta V = vBL$

$$\boxed{\mathcal{E} = vBL}$$

Power Considerations

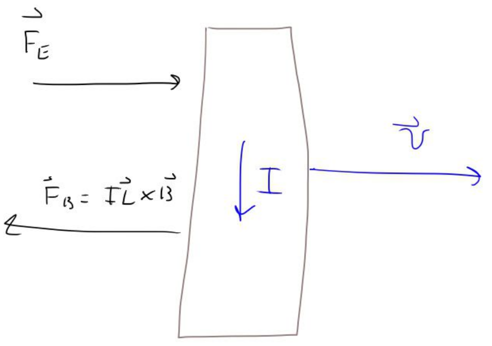
Current

$$\boxed{I = \frac{vBL}{R}}$$

$V = IR$

$\mathcal{E} = vBL = IR$

$\Rightarrow \boxed{I = \frac{vBL}{R}}$



Power Dissipated

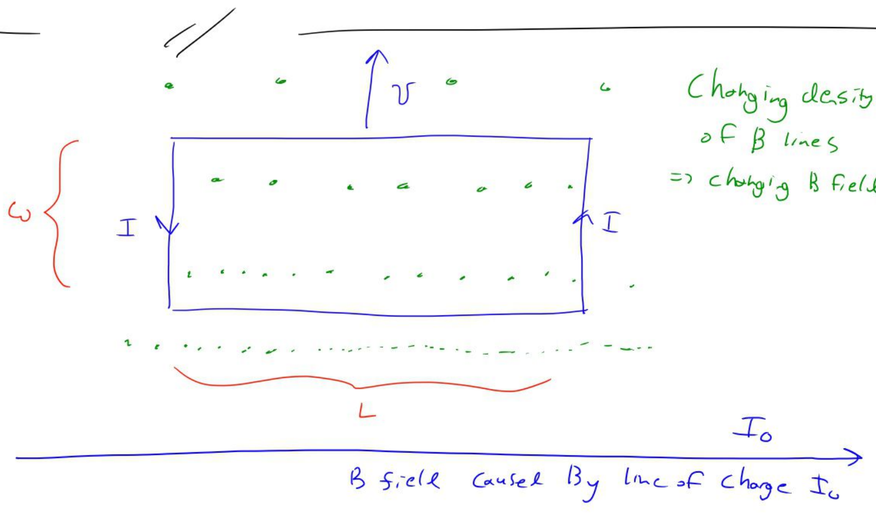
$$\boxed{P_R = \frac{(vBL)^2}{R} = I^2R}$$

Wire Loop & Straight Current

EMF

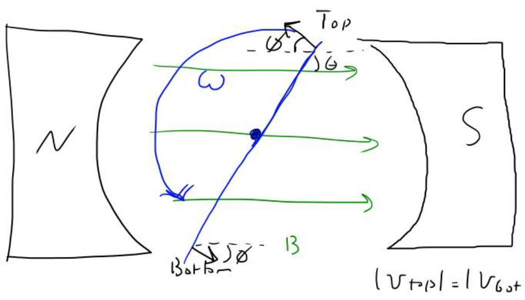
$$\boxed{\mathcal{E}_{loop} = vL(B_{bottom} - B_{top})}$$

$$\boxed{B_{top} = \frac{\mu_0 I_0}{2\pi(R+\omega)} \qquad B_{bottom} = \frac{\mu_0 I_0}{2\pi R}}$$



Changing density of B lines
$\Rightarrow$ changing B field

Generator



Area of Loop $A = WL$

Motional EMF

$$\boxed{\mathcal{E} = \omega AB\cos(\omega t)}$$

$\vec{F}_{charge} = q\vec{v}\times\vec{B}$, $F_{B\,top} = F_{B\,bottom} = qvB\sin\phi$
$= qvB\cos(\theta)$

$F_E = qE$

At Equilibrium $F_E = F_B \Rightarrow qE = qvB\cos(\theta)$

$\Rightarrow E = vB\cos(\theta)$

$|v_{top}| = |v_{bottom}| = \omega\left(\frac{W}{2}\right)$

* Motional Emf $\mathcal{E} = E(L_{top} + L_{bottom})$

$\mathcal{E} = 2EL$

$\Rightarrow \mathcal{E} = 2vLB\cos(\theta)$

* Note $\theta = \omega t$

$\Rightarrow \mathcal{E} = 2vLB\cos(\omega t)$

$= 2\omega\frac{W}{2}LB\cos(\omega t)$

$\Rightarrow \mathcal{E} = \omega \underbrace{WL}_{A} B\cos(\omega t)$

$\Rightarrow \boxed{\mathcal{E} = \omega AB\cos(\omega t)}$

Connections

Recall $\Phi_B = \int \vec{B}\cdot d\vec{A}$

Faraday's Law

$* \ \Phi_B = \vec{B} \cdot \vec{A}$

Magnetic Flux

- motional EMF $\quad |\mathcal{E}| = \frac{\Delta \Phi_B}{\Delta t}$

- Flux through a rotating Loop

$$|\varepsilon| = \frac{d\Phi_B}{dt} = \omega BA\cos(\omega t)$$

Faraday's Law

$$\boxed{\mathcal{E}_{induced} = -\frac{d\Phi_B}{dt}}$$

$$\boxed{\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}}$$

Lenz's Law

$\mathcal{E}_{induced}$ opposes $\frac{d\Phi_B}{dt}$

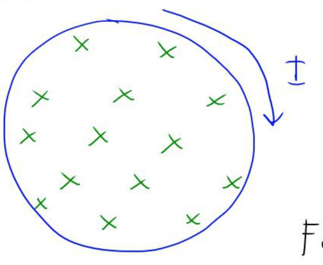
Generator

$$\boxed{\mathcal{E} = -\omega NBA\cos(\omega t)}$$

N = # of turns

A = Area of loop

Induction & RL-Circuits



Farday's Law

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

$$= -\frac{d(LI)}{dt}$$

$$\boxed{\mathcal{E} = -L\frac{dI}{dt}}$$

$\Phi_B = LI$
$\Rightarrow d\Phi_B = d(LI)$

Self-Inductance

$$\boxed{L = \frac{\Phi_B}{I}}$$

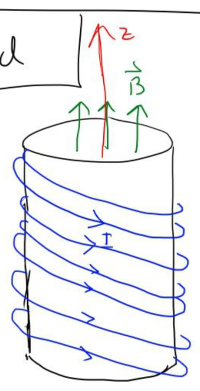
SI unit

$H = T \cdot m^2/A$

Inductor Voltage

$$\boxed{\mathcal{E} = -L\frac{dI}{dt}}$$

Solenoid



Magnetic Field

$$\boxed{B = \mu_0 n I}$$

$n = \frac{\#\text{ of turns}}{\text{Length}}$

use Ampere's Law to get $\vec{B}$

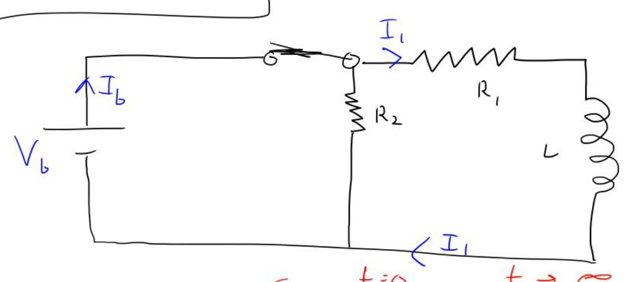
Magnetic flux

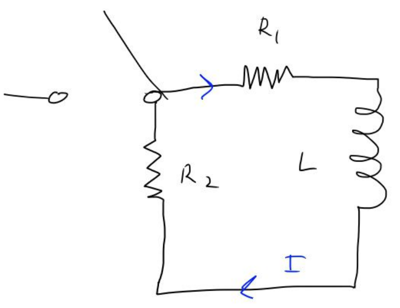
$$\boxed{\Phi_B = \mu_0 n^2 z \pi r^2 I}$$

Self Inductance

$$\boxed{L = \mu_0 n^2 z \pi r^2}$$

RL-Circuit





closing switch

$t = 0$	$t \to \infty$
$V_R(0) = 0$	$V_R(\infty) = V_b$
$V_L(0) = V_b$	$V_L(\infty) = 0$
$I_1(0) = 0$	$I_1(\infty) = \frac{V_b}{R}$

now acts like a wire

opening switch

$t = 0$	$t \to \infty$
$V_R(0) = V_b$	$V_R(\infty) = 0$
$V_L(0) = -2V_b$	$V_L(\infty) = 0$
$I(0) = \frac{V_b}{R}$	$I(\infty) = 0$

Immediately after disconnecting $I(0) = \frac{V_b}{R}$

$\Rightarrow$ Kirkhoff's Voltage Rule/Law, $R_1 = R_2$

$\Rightarrow -I(0)R - I(0)R + V_L(0) = 0$

$\Rightarrow -2I(0)R + V_L(0) = 0$

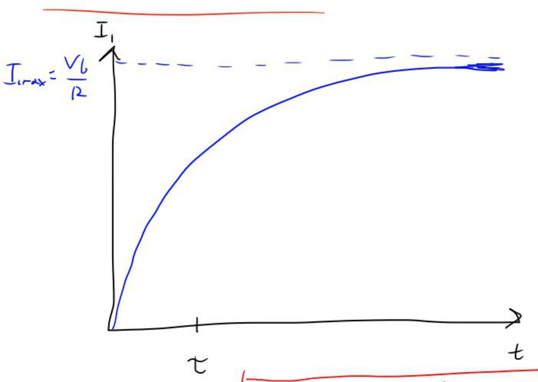
$\Rightarrow V_L(0) = 2I(0)R$

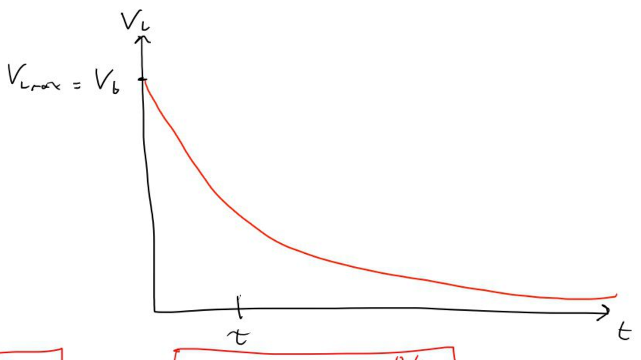
$$\boxed{V_L(0) = 2\frac{V_b}{R}R = 2V_b}$$

Quantitative approach

$\tau = \frac{L}{R}$

closing switch



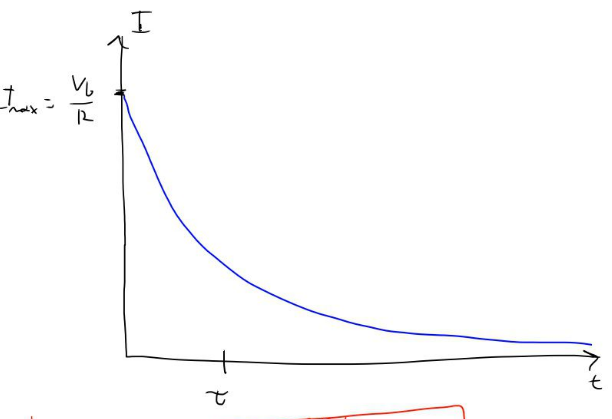


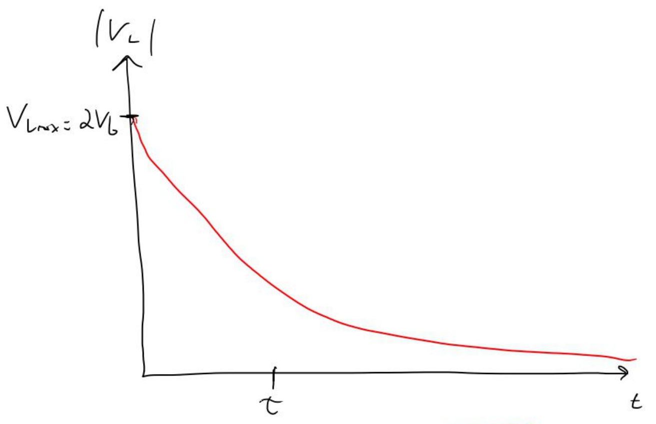
$$\boxed{I(t) = \frac{V_b}{R}\left(1 - e^{-\frac{Rt}{L}}\right)}$$

$$\boxed{V_L = V_b e^{-\frac{Rt}{L}}}$$

Opening Switch

$\tau = \frac{L}{2R}$





$$\boxed{I(t) = \frac{V_b}{R}e^{-\frac{2Rt}{L}}}$$

$$\boxed{V_L(t) = -2V_b e^{-\frac{2Rt}{L}}}$$

Energy in an Inductor

Mag. Energy Density (solenoid)

$$\boxed{u_B = \frac{B^2}{2\mu_0}}$$

Parallel plate capacitor

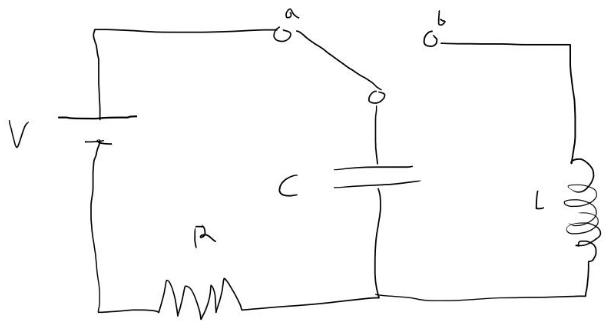
$$\boxed{u_E = \frac{1}{2}\varepsilon_0 E^2}$$

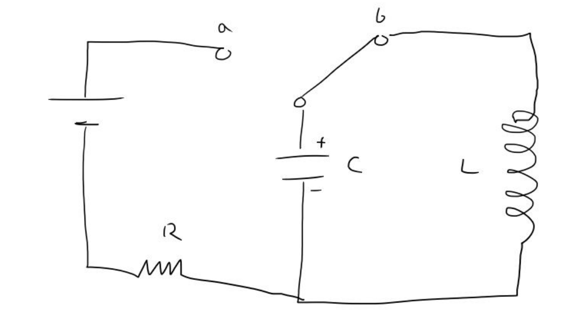
semi-Reminiscent of

Inductor Energy

$$\boxed{U_L = \frac{1}{2}LI^2}$$

LC & RLC Circuits





$C_{max} \Rightarrow \quad Q = CV, \ \tau = RC$

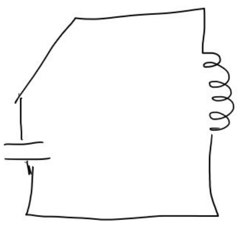
$Q = CV$.

Capacitor Discharges increasingly & then is fully discharged, the inductor opposes the change and Imposes an $\mathcal{E}$ in the same direction charging the capacitor in the opposite direction as before to repeat causing an oscillation

Quantitative

Current & charge over time on the capacitor

$$\boxed{L\frac{d^2Q}{dt^2} = -\frac{Q}{C}} \quad \text{KVR}$$



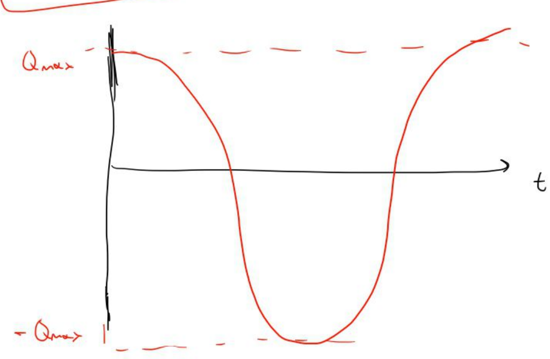
Solution

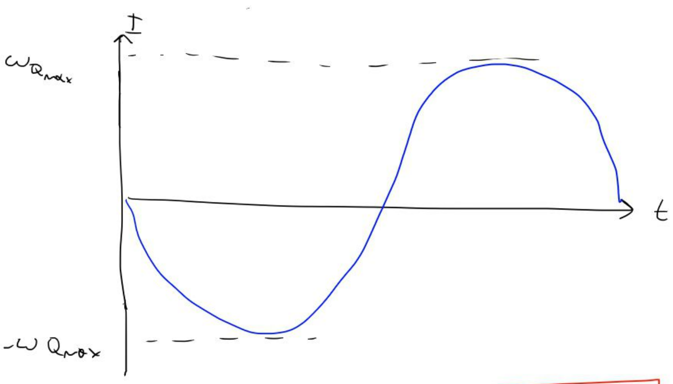
$$\boxed{Q(t) = Q_{max}\cos(\omega t + \phi)}$$

ϕ = phase angle

Q_{max} = max charge = CV

$$\boxed{\omega = \frac{1}{\sqrt{LC}}}$$

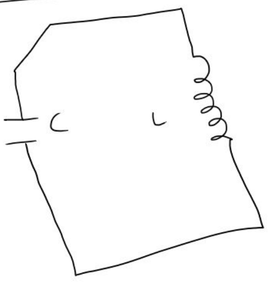


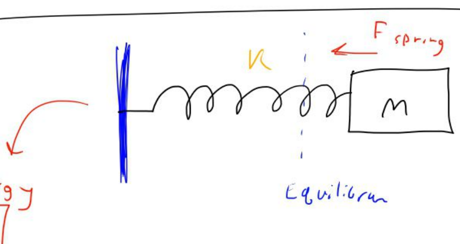


$$\boxed{Q(t) = Q_{max}\cos(\omega t + \phi)}$$

$$\boxed{I(t) = \frac{dQ}{dt} = -\omega Q_{max}\sin(\omega t + \phi)}$$

LC circuits & Energy





Inductor Energy

$$\boxed{U_L = \frac{1}{2}LI^2} \rightarrow$$

Kinetic Energy

$$\boxed{K = \frac{1}{2}mv^2}$$

Capacitor Energy

$$\boxed{U_C = \frac{1}{2}\frac{Q^2}{C}} \rightarrow$$

Spring Pot. Energy

$$\boxed{U_{spring} = \frac{1}{2}kx^2}$$

$$\boxed{U_C = \frac{1}{2C}Q_{max}^2\cos^2(\omega t + \phi)}$$

$$\boxed{U_L = \frac{1}{2}L\omega^2 Q_{max}^2\sin^2(\omega t + \phi)}$$

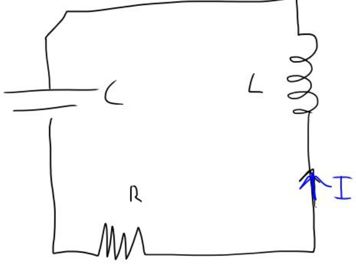
$$\boxed{U_{C_{max}} = \frac{Q_{max}^2}{2C}}$$

$$\boxed{U_{L_{max}} = \frac{1}{2}L\omega^2 Q_{max}^2 = \frac{Q_{max}^2}{2C}}$$

$* \ \omega^2 = \frac{1}{LC}$

$$\boxed{U_{total} = U_C + U_L = \frac{Q_{max}^2}{2C}}$$

RLC - circuit



(Damped Oscillations due to R)

If R is small

Large R, No oscillations

Rate of Energy Loss

$P = I^2R$

KVR

$$\frac{Q}{C} + R\frac{dQ}{dt} + L\frac{d^2Q}{dt^2} = 0$$

• Solution = combo of exponential & cos

$$\boxed{Q(t) = Ae^{-\beta t}\cos(\omega' t + \phi)}$$

$\beta = \frac{R}{2L}$ (Damping factor)

$\omega'^2 = \omega_0^2 - \beta^2$ (oscillation frequency)

$\omega_0 = \frac{1}{\sqrt{LC}}$ (natural frequency)

Critical Damping

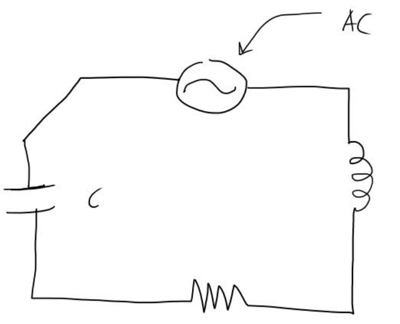
$R = 2\sqrt{\frac{L}{C}}$

$\beta = \omega_0$

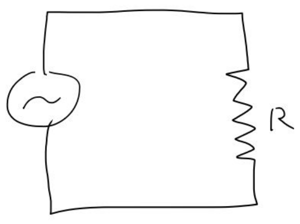
$\omega' = 0$

AC circuits

Driven LCR circuits



Generator & Resistor



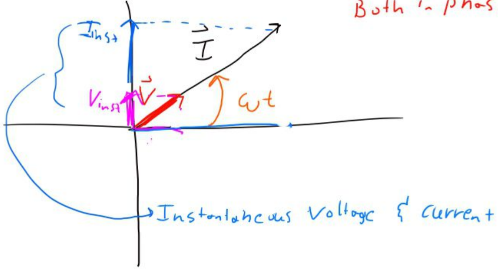
$\mathcal{E}(t) = \mathcal{E}_{in}\sin(\omega t)$

$$\boxed{I_R = \frac{\mathcal{E}_{in}}{R}\sin(\omega t)}$$

Current & voltage are "in phase"

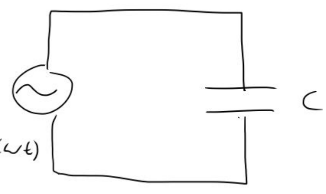
$$\boxed{V_R(t) = I_R R = \mathcal{E}_{in}\sin(\omega t)}$$

Phasor diagram



- As these both have $\sin(\omega t)$, when $\omega t = \pi/2$ they will be at a maximum & when $\omega t = 0, \pi$ they will be at 0.

AC & Capacitor

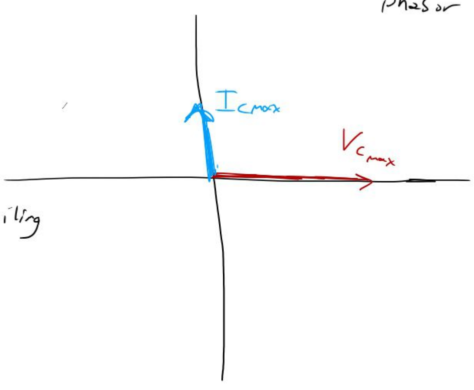


$Q = C\mathcal{E}_{in}\sin(\omega t)$

$$\boxed{\begin{aligned} I_C &= \frac{\mathcal{E}_{in}}{X_C}\cos(\omega t) \\ V_C &= \frac{Q}{C} = \mathcal{E}_{in}\sin(\omega t) \end{aligned}}$$

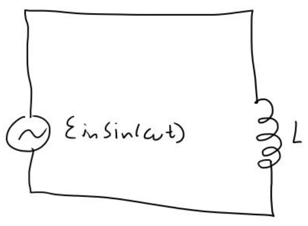
$$\boxed{X_C = \frac{1}{\omega C} \quad \text{(Reactance)}}$$

phasor diagram



- 90° out of phase w/ V_{cmax} trailing behind the current
  - I_C maxed at $0, \pi$
  - V_C maxed at $\pi/2, -\pi/2$

AC & Inductor

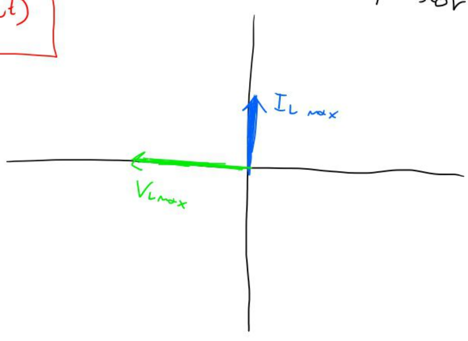


$$\boxed{\begin{aligned} I_L &= -\frac{\mathcal{E}_{in}}{X_L}\cos(\omega t) \\ V_L &= L\frac{dI_L}{dt} = \mathcal{E}_{in}\sin(\omega t) \end{aligned}}$$

Reactance of Inductor

$$\boxed{X_L = \omega L}$$

Phasor Diagram

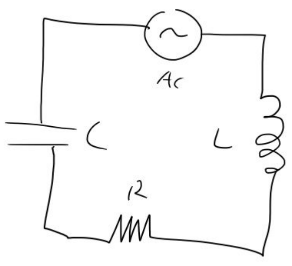


- 90° out of phase w/ $I_{L\,max}$ trailing $V_{L\,max}$
  - $|I_L|$ maxed at $0, \pi$
  - $|V_L|$ maxed at $\pi/2, -\pi/2$

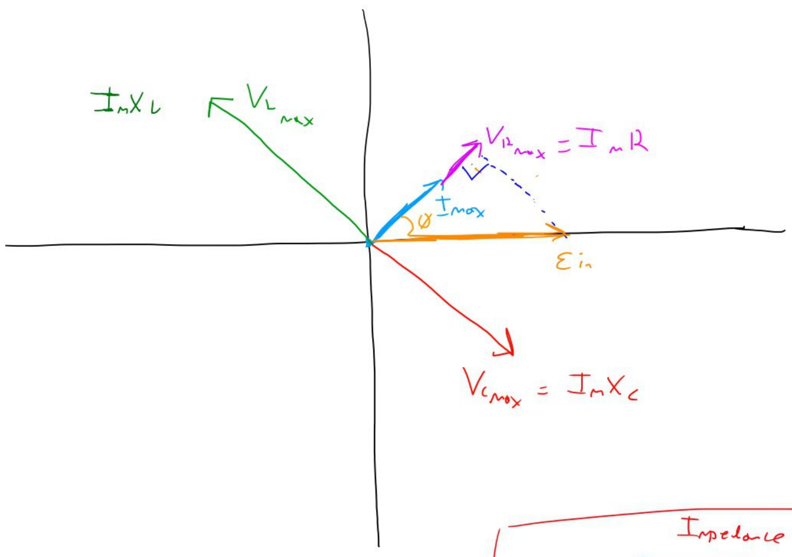
Driven LCR

Very Complicated

Instead use phasors



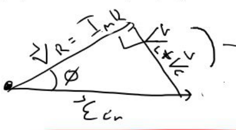
Phasors



Kirchhoff

KVR

$\vec{V}_R + \vec{V}_L + \vec{V}_C = \vec{\mathcal{E}}_{in}$



$\vec{V}_L + \vec{V}_C = I_m(X_L - X_C)$

$$\Rightarrow \boxed{\tan(\phi) = \frac{X_L - X_C}{R}}$$

$$\boxed{I_m = \frac{\mathcal{E}_{in}}{\sqrt{R^2 + (X_L - X_C)^2}}}$$

Impedance

$$\boxed{Z = \sqrt{R^2 + (X_L - X_C)^2} \qquad Z = \frac{\mathcal{E}_{in}}{I_m}}$$

Resonance & Power

* Recall;

Phase Relation

$$\boxed{\tan\phi = \frac{X_L - X_C}{R}}$$

Impedance

$$\boxed{Z = \sqrt{R^2 + (X_L - X_C)^2}}$$

Max Current

$$\boxed{I_m = \frac{\mathcal{E}_{in}}{Z}}$$

Resonance

- Occurs when:
  - I_m is maximized $\quad I_m = \frac{\mathcal{E}_{in}}{R}$
  - $\omega = \omega_0$
  - Z is minimized $\quad X_L = X_C$
  - $\phi = 0°$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

Frequency Dependence of Max Current

$$x \equiv \frac{\omega}{\omega_0}, \quad Q^2 \equiv \frac{L}{R^2 C}$$

$$\boxed{I_m = \frac{\mathcal{E}_m}{R} \frac{1}{\sqrt{1 + Q^2 \frac{(x^2-1)^2}{x^2}}}}$$

Power

Average power per cycle

$\langle P_{generator} \rangle = \langle P_{resistor} \rangle$

* avg. Power in a Capacitor & Inductor is zero

$$\langle P_R \rangle = \langle I^2 R \rangle = I_m^2 \langle \underbrace{\sin^2(\omega t - \phi)}_{1/2} \rangle R = \frac{1}{2} I_m^2 R = \frac{1}{2} I_m \mathcal{E}_{in} \cos(\phi)$$

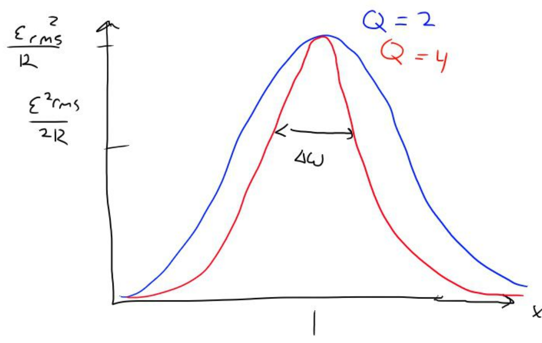
$$\boxed{\langle P_{gen} \rangle = \langle P_R \rangle = \frac{1}{2} I_m \mathcal{E}_{in} \cos(\phi)}$$

⋮ Lot's o' math later

$$\boxed{\langle P_{gen} \rangle = \frac{\mathcal{E}_{rms}^2}{R} \cdot \frac{x^2}{x^2 + Q^2 (x^2-1)^2}}$$

$$\mathcal{E}_{rms} = \frac{\mathcal{E}_{in}}{\sqrt{2}}$$

$$I_{rms} = \frac{I_m}{\sqrt{2}}$$



$$FWHM \; \Delta\omega \xrightarrow{Q \gg 1} \frac{1}{Q}$$

Q Factor

Quality factor

$$\boxed{Q \equiv 2\pi \left[\frac{U_{max}}{\Delta U}\right]_{cycle}} \longrightarrow \text{evaluate at } \omega = \omega_0 \text{ (Resonance)}$$

Drive LCR Circuit

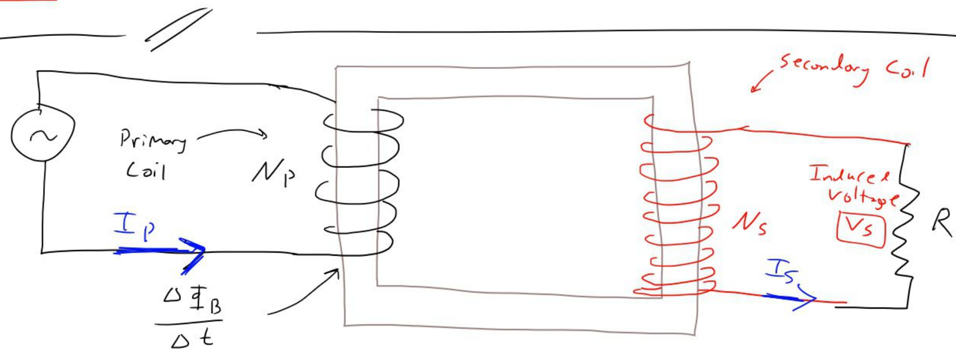
$$Q^2 = \frac{L}{R^2 C} \longrightarrow$$

Q Large when:
- $U_{max} \gg \Delta U$
- $R \ll X_L, X_C$

$$V_{L\,max}\big|_{\omega=\omega_0} = V_{C\,max}\big|_{\omega=\omega_0} = Q\mathcal{E}_{in}$$

Ideal Transformer

$$\boxed{V_s = \frac{N_s}{N_p} V_p}$$



Voltage Relation

$$V_s = \frac{N_s}{N_p} V_p$$

Current Relation

$$\boxed{I_p = \frac{N_s}{N_p} I_s}$$

Summary of Equations up to this point:

- **Gauss's Law for $\vec{E}$-fields**

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_0}$$

Applies to spheres, cylinders & planes

- **Gauss's Law for $\vec{B}$-fields**

$$\oint \vec{B} \cdot d\vec{A} = 0$$

- **Ampere's Law**

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$$

- **Faraday's Law**

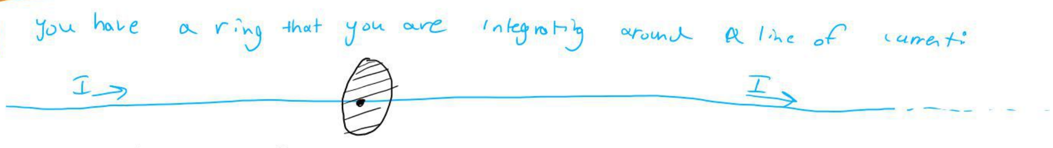
$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{A} = -\frac{d}{dt}\Phi_B$$

These are Maxwell's Equations

The equations as we have them above are inconsistent to charge & Maxwell offers a correction to Ampere's Law to fix that:

The problem –

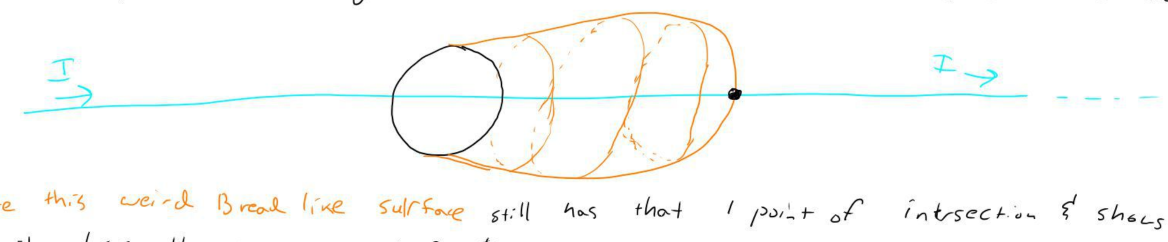
Say you have a ring that you are integrating around a line of current:



This shows the point of intersection where the loop & its area meet the wire & this small spot is where the loop "encloses" the wire.

⇒ Taken to the extreme you could change this internal shape (The black lined area)

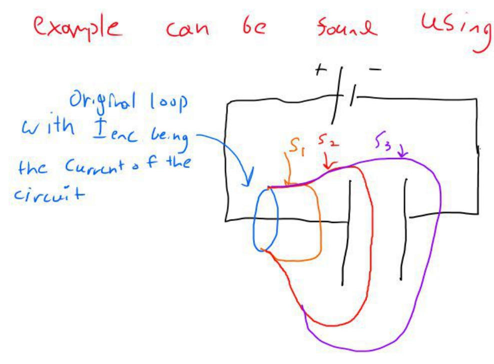
⇒



Where this weird Bread like surface still has that 1 point of intersection & shows that it encloses the same amount of ~~the~~ current as before

---

Another example can be found using Capacitors.



S_1 = The same situation as above

S_2 = A new shape that Doesn't enclose Any Current!!!

S_3 = the same situation as above

Solution to this problem? $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl} + ?$

Since the above capacitor setup also has changing $\vec{E}$-fields Maxwell considered displacement currents as that "?" factor. Note, changing $\vec{E}$-fields induce changing Magnetic flux

$$\Rightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 I_{displ.}$$

Let's analyze this by going back to some Def^{n}s.

$I = \frac{dQ}{dt}$ { In the wire, this is the charge flowing past some point per unit time; Capacitor, this is the charge flowing onto the plate & off the other plate.

$\Phi_E = EA$, Electric flux between the parallel plates of the capacitor

$= \frac{\sigma}{\varepsilon_0}A$, E is proportional to the charge density on each plate, σ, over ε_0

$= \frac{Q}{\varepsilon_0 A}A$, $\sigma = \frac{Q}{A}$

$$\Phi_E = \frac{Q}{\varepsilon_0}$$

$\therefore Q = \varepsilon_0 \Phi_E$, The charge on the plates is proportional to the Flux in between the plates.

$$\Rightarrow \frac{dQ}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$\therefore \boxed{I_{displacement} = \frac{dQ}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt}}$$

$\therefore$ Ampere's Law (Modified)

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl.} = \mu_0 (I + I_D) \boxed{= \mu_0\left[I + \varepsilon_0 \frac{d\Phi_E}{dt}\right]}$$

What are the consequences of modifying A fundamental Law?

Here's where things get interesting:

Amp's Law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$$

Means: "A changing electric field can create a magnetic field."

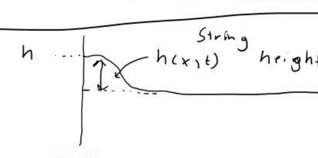
Faraday's Law

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{A}$$

Means: "A changing magnetic field can create an electric field."

Now there can be self sustaining E & M waves in Space.

Analyzing waves:



1-D wave equation:

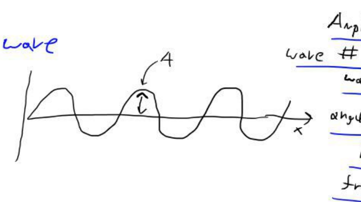
$$\frac{d^2 h}{dx^2} = \frac{1}{v^2}\frac{d^2 h}{dt^2}, \quad h = \text{disturbance of the wave}$$

Gen. Solution. A linear combo of a wave going to the left & right

$$h(x,t) = h_1(x - vt) + h_2(x + vt)$$

We will use the example of a harmonic plane wave

$$\boxed{h(x,t) = A\cos(kx - \omega t)}$$



- Amplitude: A
- Wave # k: $k = \frac{2\pi}{\lambda}$
- wavelength: λ
- angular frequency: $\omega = \frac{2\pi}{T}$
- Period: T
- frequency: $\frac{1}{T} = f$
- Wave velocity: $v = \lambda f = \frac{\omega}{k}$

If you would like to see why I have the following equations I would just recommend you watch Prelecture Unit 22 "Plane waves in Maxwell's equations"

Faraday's Law

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{A}$$

Modified Ampere's Law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0\varepsilon_0 \frac{d}{dt}\int \vec{E} \cdot dA$$

* A plane wave in the z direction { $\vec{E} = \vec{E}(z,t)$, $\vec{B} = \vec{B}(z,t)$

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$

$$\Rightarrow \frac{\partial^2 E_x}{\partial z^2} = -\frac{\partial}{\partial z}\frac{\partial B_y}{\partial t}$$

$$\frac{\partial B_y}{\partial z} = -\mu_0\varepsilon_0 \frac{\partial E_x}{\partial t}$$

$$\frac{\partial}{\partial t}\frac{\partial B_y}{\partial z} = -\mu_0\varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 E_x}{\partial z^2} = \mu_0\varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}}$$ the 1-D wave eq. for the electric field.

Comparing this to the 1-D wave equation:

$$\Rightarrow \mu_0\varepsilon_0 = \frac{1}{v^2}$$

$$\therefore \boxed{v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c = 3.00 \times 10^8 \tfrac{m}{s} \text{ speed of light!!}}$$

Light is an electromagnetic wave.

Instead of eliminating B in the derivation of the wave equation for E, we can eliminate E

$$\Rightarrow \boxed{\frac{\partial^2 B_y}{\partial z^2} = \mu_0\varepsilon_0 \frac{\partial^2 B_y}{\partial t^2}}$$

Noting that once again the wave velocity will be $c = 3.00 \times 10^8 \tfrac{m}{s}$

Let's look at the relationship between E & B noting from above $\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$

Harmonic Solution

$$E_x = E_0\cos(kz - \omega t) \Rightarrow \frac{\partial B_y}{\partial t} = kE_0\sin(kz - \omega t)$$

$$\Rightarrow B_y = \int kE_0\sin(kz - \omega t)\,dt$$

$$\Rightarrow B_y = \frac{k}{\omega}E_0\cos(kz - \omega t)$$

Note: - E_x & B_y are in phase w/ one another

- $B_0 = \frac{E_0}{c} = \frac{k}{\omega}$

Properties of E&M waves

Velocity

$$C = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = \frac{E_o}{B_o} = 3\times10^{8}\ \frac{m}{s}$$

E_x

$$E_x = E_o \sin(kz-\omega t)$$

B_y

$$B_y = B_o \sin(kz-\omega t)$$

Doppler shift

$$\boxed{f' = f\sqrt{\frac{1\pm\beta}{1\mp\beta}} \qquad \beta \equiv \frac{v}{c}} \xrightarrow{\beta \ll 1} f' \approx f(1\pm\beta)$$

Decreasing Separation

$$f' = f\sqrt{\frac{1+\beta}{1-\beta}} \qquad (f' > f)$$

Incresing Separation

$$f' = f\sqrt{\frac{1-\beta}{1+\beta}} \qquad (f' < f)$$

Energy in E&M waves

Total Energy Density

$$\boxed{u = \varepsilon_o E^2}$$

Intensity

$$\boxed{I = \frac{1}{2} c\,\varepsilon_o E_o^2}$$

Average Energy Density

$$\boxed{\langle u \rangle = \frac{1}{2}\varepsilon_o E_o^2}$$

Poynting Vector

$$\boxed{\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_o} \xrightarrow{\text{magnitude}} S = c\,\varepsilon_o E^2 \qquad \langle S \rangle = I}$$

Ex found In Pre-lecture

Photons

h = planck's Constant
$= 1.24\times10^{-6}$ eV·m

Energy

$$\boxed{E = hf}$$

Momentum

$$\boxed{p = \frac{E}{c} = \frac{h}{\lambda}}$$

Polarization

$\omega = kc$

$E_0 = cB_0$

$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

Linear polarization

• plane harmonic wave $E_x = E_0 \sin(kz - \omega t)$
$B_y = B_0 \sin(kz - \omega t)$

This wave is polarized in the x-direction

Polarized in the z-direction

$\vec{E} = \hat{e}\, E_0 \sin(kz - \omega t + \phi)$, ϕ is the e field strength @ $z = t = 0$

$\Rightarrow \left.\begin{array}{l} E_x = E_0 \cos\theta \sin(kz - \omega t + \phi) \\ E_y = E_0 \sin(\theta) \sin(kz - \omega t + \phi) \end{array}\right\}$ θ = angle between $\hat{e}$-axis & the x-axis

These resulting in expanding around θ

Polarizers

• Most light is not polarized

• Intensity $I_{final} \leq I_{initial}$

θ = angle with respect to the transmission axis & the original transmission angle

Law of Malus:

$$I_{final} = I_0 \cos^2(\theta) \quad \text{(Intensity of Light)}$$

Ex) Unpolarized light goes through 3 polarizers

	1	2	3
transmission axis	$\theta = 90°$	$\theta = 30°$	$\theta = 60°$

• Unpolarized light going through a polarizer obeys this relation:

$$I_1 = \frac{1}{2} I_0$$

$\Rightarrow$ After the 1st there is $\frac{1}{2} I_0$ of intensity

Using Malus's Law

$$I_2 = \frac{1}{2} I_0 \cos^2(30°) = \frac{1}{2} I_0 \left[\frac{3}{4}\right] = \boxed{\frac{3}{8} I_0} = I_2$$

$$\Rightarrow I_3 = \frac{3}{8} I_0 \cos^2(60°) = \frac{3}{8} I_0 \left[\frac{1}{4}\right] = \boxed{\frac{3}{32} I_0} = I_3$$

Other polarizations

Dropping same ϕ relation

$$\left.\begin{array}{l} E_x = E_0 \cos(\theta) \sin(kz - \omega t + \phi_x) \\ E_y = E_0 \sin(\theta) \sin(kz - \omega t + \phi_y) \end{array}\right\} \text{satisfying} \left\{\begin{array}{l} \frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} \\ \frac{\partial^2 E_y}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} \end{array}\right.$$

Linear polarization

Relative phase: $\phi \equiv \phi_x - \phi_y = 0$

Circular polarization

Relative phase: $\phi \equiv \phi_x - \phi_y = \pm \frac{\pi}{2}$

Note: $\phi_x - \phi_y = \oplus \frac{\pi}{2} \longrightarrow \left\{\begin{array}{l} E_x = E_0 \cos(kz - \omega t + \phi_y) \\ E_y = E_0 \sin(kz - \omega t + \phi_y) \end{array}\right.$

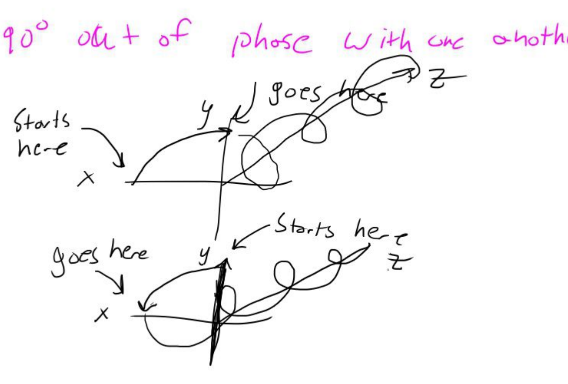
$\phi_x - \phi_y = \ominus \frac{\pi}{2} \longrightarrow \left\{\begin{array}{l} E_x = E_0 \sin(kz - \omega t + \phi_x) \\ E_y = E_0 \cos(kz - \omega t + \phi_x) \end{array}\right.$

$\therefore$ Amplitudes are the same but they are 90° out of phase with one another

Circular polarization

Right handed polarization: $\phi_x - \phi_y = \frac{\pi}{2}$

Left handed polarization: $\phi_x - \phi_y = -\frac{\pi}{2}$



Birefringence Material

Creates a phase difference between the orthogonal waves, speed of light is different in different directions

$$\Delta\phi = \phi_y - \phi_x = \omega d \left(\frac{1}{v_{fast}} - \frac{1}{v_{slow}} \right)$$

$\phi_y = \phi_x - \pi/2$

d = thickness of material

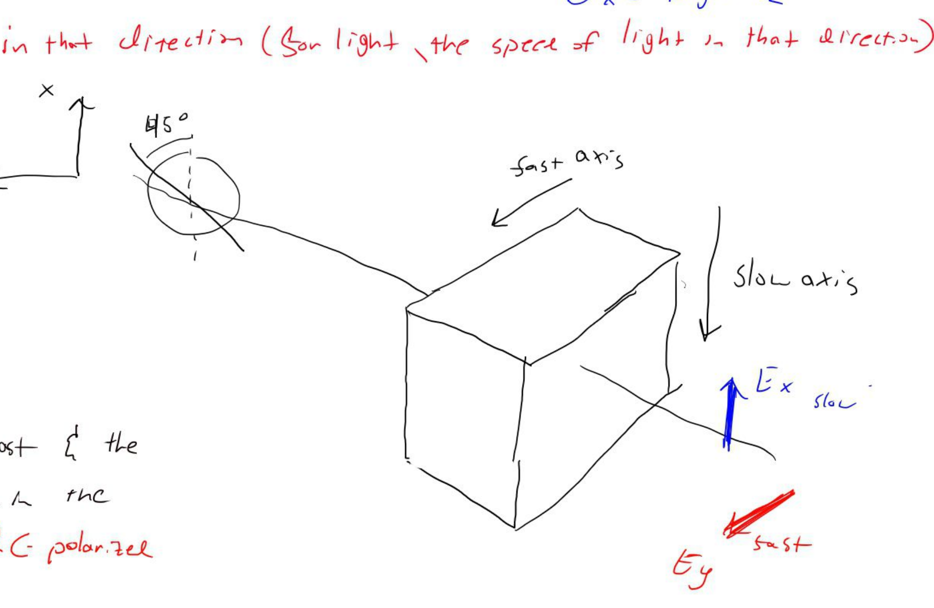
ω = angular frequency

$\phi_x = \phi_y - \pi/2$

v_i = velocity of the wave in that direction (for light, the speed of light in that direction)

Circularly polarized waves

• Using a Quarter wave plate

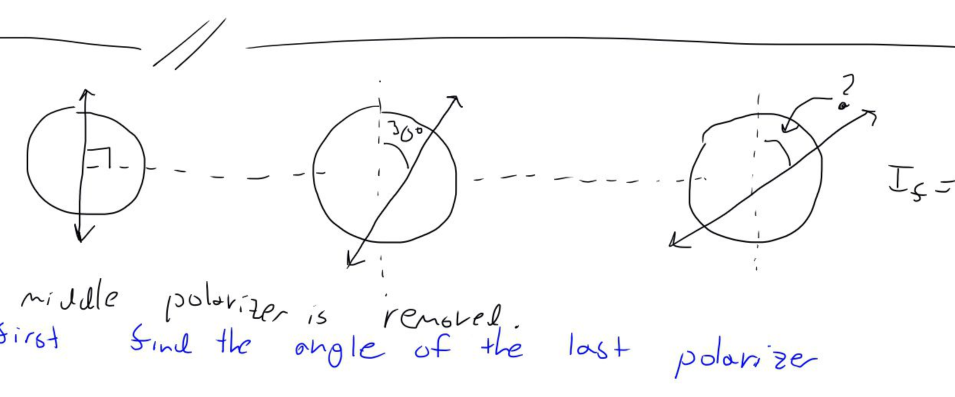


Handedness:

Curl your Right hand fingers from the slow component to the fast & the direction of your thumb should point in the same direction as the wave if it is RC polarized & in the opposite direction if it is LC

C stands for circularly polarized

Worked Example



$I_f = \frac{1}{3} I_0$

What is the I_f when the middle polarizer is removed.

[Method 1]

• We want to first find the angle of the last polarizer

Step 1

• Incident light upon Polarizer 1 (P1).

$I_1 = \frac{1}{2} I_0$ ("Circularly polarized light on a linear polarizer")

Step 2

• Light, now linearly polarized on P_2, using the Law of Malus

$* \quad I_f = I_0 \cos^2(\theta)$

$\Rightarrow I_2 = \frac{1}{2} I_0 \cos^2(\pi/6) = \frac{1}{2} I_0 \left[\frac{3}{4}\right] = \frac{3}{8} I_0$

$\Rightarrow I_2 = \frac{3}{8} I_0$

Step 3

• Light on P_3, we know the end result but we want to know ϕ_3

$\Rightarrow \frac{1}{3} I_0 = \frac{3}{8} I_0 \cos^2(\theta_3)$

$\Rightarrow \frac{8}{9} = \cos^2(\theta_3)$

$\Rightarrow \frac{\sqrt{8}}{3} = \cos(\theta_3) \Rightarrow \cos^{-1}\left(\frac{\sqrt{8}}{3}\right) = \boxed{\theta_3 \approx 19.5°}$

Now that we have this value we know that θ_3 or rather P_3's transmission axis is $\approx 30° + 19.5°$ off of the initial value so $\approx 49.5°$

$\therefore$ If the second polarizer was not there the steps would result in:

$I_1 = \frac{1}{2} I_0$

$\rightarrow I_2 = \frac{1}{2} I_0 \cos^2(49.5°) \approx \boxed{0.21\, I_0 = I_f}$ Notice that this is less than when we had the second polarizer

[Method 2]

Using Malus's Law & simply keeping things in relations

$I_0 \rightarrow I_1 = \frac{1}{2} I_0 \rightarrow I_2 = \frac{1}{2} I_0 \cos^2(\pi/6) \rightarrow I_f = \frac{1}{3} I_0 = \frac{1}{2} I_0 \cos^2(\pi/6) \cos^2(\theta_3)$

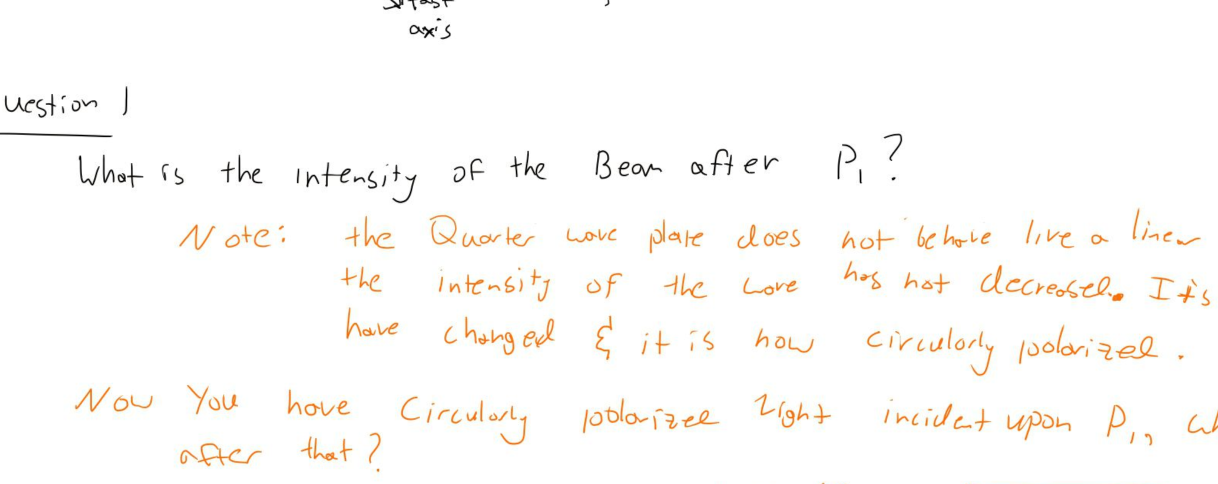
$\Rightarrow \sqrt{\frac{2}{3\cos^2(\pi/6)}} = \cos(\theta_3) \Rightarrow \theta_3 = \cos^{-1}\left(\sqrt{\frac{2}{3\cos^2(\pi/6)}}\right)$

$\Rightarrow$ w/out the second polarizer

$I_0 \rightarrow I_1 = \frac{1}{2} I_0 \rightarrow \boxed{I_f = \frac{1}{2} I_0 \cos^2\left(\cos^{-1}\left(\sqrt{\frac{2}{3\cos^2(\pi/6)}}\right)\right)}$

$\boxed{\approx 0.21\, I_0}$

Polarizers & a Quarter-Wave Plate



$I_0 = 131 \frac{W}{m^2}$

$\theta_1 = 69°$

Question 1

What is the intensity of the Beam after P_1?

Note: the Quarter wave plate does not behave like a linear polarizer so the intensity of the wave has not decreased. It's relative phases have changed & it is now circularly polarized.

Now you have circularly polarized light incident upon P_1, what is the intensity after that?

$$\rightarrow I_f = \frac{1}{2} I_0 = \frac{1}{2}\, 131 \frac{W}{m^2} = \boxed{65.5 \frac{W}{m^2} = I_f}$$

Question 2

What is I_f after P_2?

After P_1 $I_2 = \frac{1}{2} I_0$

$$\Rightarrow I_f = \frac{1}{2} I_0 \cos^2(69°) \approx 0.064\, I_0 = 0.64 \left(131 \frac{W}{m^2}\right) \approx \boxed{8.412 \frac{W}{m^2} = I_f}$$

.
.
.