18.100B: Problem Set 8

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Problem 1

> Let $f : [0, \infty) \to \mathbb{R}$ be continuous, and suppose
> $$f(x^2) = f(x)$$
> holds for every $x \geq 0$. Prove that f has to be a constant function.

Proof. Consider sequence $a_n^x = \{x, x^2, x^4, \dots\}$ for $x \in [0, 1)$. We notice that $a_n^x \to 0$. Image of any element of a_n^x under f must be the same value $f(x)$, thus $\lim f(a_n^x) = f(x)$. Since f is continuous at 0, $\lim f(a_n^x)$ must be equal to $f(0)$. Therefore
$$f(x) = f(0) \text{ for all } x \in [0, 1)$$

Consider sequence $b_n^x = \{x, x^{\frac{1}{2}}, x^{\frac{1}{4}}, \dots\}$ for any $x \in (0, \infty) \setminus \{1\}$. We notice that $b_n^x \to 1$. Image of any element of b_n^x under f must be the same value $f(x)$, thus $\lim f(b_n^x) = f(x)$. Since f is continuous at 1, $\lim f(b_n^x)$ must be equal to $f(1)$. Therefore
$$f(x) = f(1) \text{ for all } x \in (0, \infty) \setminus \{1\}$$

We conclude that for any $x \in [0, \infty) : f(x) = f(0) = f(1)$, thus f must be constant.

□

Problem 2

> Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

Proof. Consider arbitrary $\epsilon > 0$. Since $f'(x) \to 0$ as $x \to +\infty$, there exists a such that for all $x > a : |f'(x)| < \epsilon$.

Now consider arbitrary point $b > a$. We note that $f(x)$ is continuous and differentiable on $(b, b+1)$, thus applying Mean Value Theorem (Rudin 5.10):
$$g(b) = f(b+1) - f(b) = (b + 1 - b)f'(z) = f'(z)$$

for some $z \in (b, b+1)$. Since $z > b > a$, $f'(z) < \epsilon$, therefore

$$|g(b)| < \epsilon$$

We conclude that there exists a such that for all points $b > a$ absolute value of function g at b is less than ϵ, which is exactly the definition of $g(x) \to 0$ as $x \to +\infty$. □

Problem 3

If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

where $C_0, \ldots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

Proof. Consider polynomial $C(x)$ such that

$$C'(x) = C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n,$$

specifically:

$$C(x) = C_0 x + C_1 \frac{x^2}{2} + \cdots + C_{n-1} \frac{x^n}{n} + C_n \frac{x^{n+1}}{n+1}$$

Evaluate $C(x)$ at $x = 0$ and $x = 1$:

$$C(0) = 0$$

$$C(1) = C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

By Mean Value Theorem

$$0 = C(1) - C(0) = C'(z)$$

for some $z \in (0, 1)$, which is equivalent to the original equation having a root between 0 and 1. □

Problem 4

Suppose f is a real function defined on $\mathbb{R}$. We call $x \in \mathbb{R}$ a fixed point of f if $f(x) = x$.
a) If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

Proof. Suppose there exist two fixed points of f, namely x_1 and x_2 and without loss of generality $x_2 > x_1$. Applying Mean Value Theorem:

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(z)$$

for some $z \in (x_1, x_2)$. Using definition of fixed point:

$$(x_2 - x_1) = (x_2 - x_1)f'(z)$$

$$1 = f'(z)$$

for some z, which contradicts the original assumption. $\square$

b) Show that the function f defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t.

Proof. We notice that $f(t) > t$ for any $t \in \mathbb{R}$:

$$t + (1 + e^t)^{-1} - t = (1 + e^t)^{-1} > 0$$

for any $t \in \mathbb{R}$. Therefore, no point of $\mathbb{R}$ can be a fixed point of f. $\square$

However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, \ldots$

Proof. First we notice from MVT that

$$f(b) - f(a) = (b - a)f'(z)$$

for some $z \in (a, b)$. Since $|f'(z)| \leq A < 1$, f is a contraction mapping, i.e. $|f(b) - f(a)| \leq A|b - a|$.

Next we consider difference between any two successive elements of sequence x_n:

$$d_n = |x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq A|x_n - x_{n-1}| < |x_n - x_{n-1}|$$

Moreover:

$$d_n \leq A^{n-1}|x_2 - x_1|$$

Since $0 < A < 1$ and $|x_2 - x_1|$ is finite, for any $\epsilon > 0$ we can find n such that $d_n < \epsilon$. Therefore, $d_n \to 0$ and x_n is Cauchy sequence. Since $\mathbb{R}$ is complete, x_n must converge, denote $p = \lim x_n$. Then we notice:

$$p = \lim x_n = \lim f(x_{n-1}) = f(\lim x_{n-1}) = f(p),$$

where the third equality holds because f is differentiable and, thus, continuous. We conclude that p is a fixed point of f. $\square$

Problem 5

> Let f be a continuous real function on $\mathbb{R}$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?

Proof. By definition:

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x}$$

Numerator is differentiable in the punctured neighbourhood of 0. Denominator is differentiable and is not equal to 0 in the punctured neighbourhood of 0. Applying L'Hospital's rule:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{f'(x) - 0}{1} = \lim_{x\to 0} f'(x) = 3$$

Thus, $f'(0)$ exists and is equal to 3. $\square$

Problem 6

> Let f be a real function on $[a, b]$ and suppose $n \geq 2$ is an integer, $f^{(n-1)}$ is continuous on $[a, b]$, and $f^{(n)}(x)$ exists for all $x \in (a, b)$. Moreover, assume there exists $x_0 \in (a, b)$ such that
>
> $$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0, \;\; f^{(n)}(x_0) = A \neq 0$$
>
> Prove the following criteria: If n is even, then f has a local minimum at x_0 when $A > 0$, and f has a local maximum at x_0 when $A < 0$. If n is odd, then f does not have a local minimum or maximum at x_0.

Proof. By Taylor's theorem:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots$$

$$\cdots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x - x_0)^{n-1} + \frac{f^{(n)}(z)}{n!}(x - x_0)^n =$$

$$= f(x_0) + \frac{f^{(n)}(z)}{n!}(x - x_0)^n$$

for some z between x_0 and x. If n is even, the sign of the second term of the above expression is determined solely by the sign of $f^{(n)}(z)$. Since $f^{(n)}$ is differentiable at x_0, it must be also continuous at x_0. If $f^{(n)}(x_0) = A > 0$, there exists some neighbourhood of x_0 where all $f^{(n)}(x) > 0$. For any x in such neighbourhood, $f(x) > f(x_0)$; therefore x_0 is a local minimum. Conversely, if $f^{(n)}(x_0) = A < 0$, there exists some neighbourhood of x_0 where all $f^{(n)}(x) < 0$. For any x in such neighbourhood, $f(x - x_0) < f(x_0)$; therefore x_0 is a local maximum.

However, if n is odd, expression $\frac{f^{(n)}(z)}{n!}(x - x_0)^n$ changes sign when going from $x < x_0$ to $x > x_0$. Therefore, in any neighbourhood of x_0 there are points x such that $f(x) > f(x_0)$ and there are points x such that $f(x) < f(x_0)$. Therefore, x_0 can be neither local minimum nor local maximum. $\square$

Problem 7

For $f(x) = |x|^3$, compute $f'(x)$, $f''(x)$ for all real x, and show that $f^{(3)}(0)$ does not exist.

$$f(x) = |x|^3 = |x^3| = \begin{cases} x^3, & \text{if } x \geq 0 \\ -x^3, & \text{if } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 3x^2, & \text{if } x \geq 0 \\ -3x^2, & \text{if } x < 0 \end{cases}$$

$$f''(x) = \begin{cases} 6x, & \text{if } x \geq 0 \\ -6x, & \text{if } x < 0 \end{cases}$$

$$f^{(3)}(x) = \begin{cases} 6, & \text{if } x > 0 \\ -6, & \text{if } x < 0 \end{cases}$$

We notice that $f^{(3)}(0)$ does not exist since right-hand and left-hand derivatives of $f''(x)$ as $x \to 0$ differ:

$$f^{(3)}_{x \to 0^+}(x) = \lim_{x \to 0^+} \frac{f''(x) - f''(0)}{x - 0} = \lim_{x \to 0^+} \frac{6x}{x} = 6$$

$$f^{(3)}_{x \to 0^-}(x) = \lim_{x \to 0^-} \frac{f''(x) - f''(0)}{x - 0} = \lim_{x \to 0^-} \frac{-6x}{x} = -6$$