18.100B: Problem Set 6

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Problem 1

Prove that if $\sum |a_n|$ converges, then $\sum |a_n|^2$ also converges.

Proof. Since $\sum |a_n|$ converges $\lim_{n\to\infty} |a_n|$ must be equal to 0, which implies that for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n| - 0 < \epsilon$ for all $n > N$. Consider such N for $\epsilon = 1$.

Since series $\sum |a_n|$ converges, its subseries $\sum_{n=N+1}^{\infty} |a_n|$ must also converge. We notice that $|a_n|^2 < |a_n|$ for all $n > N$ since $|a_n| < 1$. Therefore $\sum_{n=N+1}^{\infty} |a_n|^2$ must also converge.

We then notice that series $\sum |a_n|^2$ can be written as a sum of two expressions:

$$\sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{N} |a_n|^2 + \sum_{n=N+1}^{\infty} |a_n|^2$$

.

The second expression is a convergent series, as we have just proved. The first expression must be finite since N is finite. We conclude that $\sum |a_n|^2$ converges. $\square$

Problem 2

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$$

Proof. We can write

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2n(n+1)} - \frac{1}{2(n+1)(n+2)}$$

Denoting

$$b_n = \frac{1}{2n(n+1)}$$

we can see that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \sum_{n=1}^{\infty} b_n - b_{n+1} = b_1 = \frac{1}{2 \cdot 1 \cdot 2} = \frac{1}{4}$$

□

Problem 3

Investigate the behavior (convergence and divergence) of $\sum_{n=1}^{\infty} a_n$ if
a) $a_n = \sqrt{n+1} - \sqrt{n}$

Answer: Series $\sum a_n$ diverges.

Proof. Using telescoping argument, we can rewrite

$$\sum_{n=1}^{k} a_n = \sqrt{1} + \sqrt{k+1}$$

which clearly diverges as $k \to \infty$.

□

b) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$

Answer: Series $\sum a_n$ converges.

Proof.

$$a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{n+1-n}{n(\sqrt{n+1}+\sqrt{n})} < \frac{1}{n\sqrt{n}} = \frac{1}{n^{\frac{3}{2}}}$$

Series $\sum \frac{1}{n^{\frac{3}{2}}}$ is convergent by Rudin 3.28, therefore series $\sum a_n$ is also convergent by comparison test.

□

c) $a_n = \frac{1}{1+\alpha^n}$ where $\alpha \geq 0$ is some fixed number

Answer: Series $\sum a_n$ diverges for $0 \leq \alpha \leq 1$ and converges for $\alpha > 1$.

Proof. If $\alpha = 0$ then $a_n = 1$, therefore $\sum a_n$ diverges. If $0 < \alpha < 1$ then $\alpha^n \to 0$ and $a_n \to 1$, therefore $\sum a_n$ diverges. If $\alpha = 1$ then $\alpha^n = 1$ and $a_n = \frac{1}{2}$, therefore $\sum a_n$ diverges. If $\alpha > 1$ then $\alpha^n \to \infty$ and $a_n \to 0$. Furthermore

$$\frac{1}{1+\alpha^n} < \frac{1}{\alpha^n} = \left(\frac{1}{\alpha}\right)^n$$

We notice that $\sum(\frac{1}{\alpha})^n$ is a convergent geometric series for $\alpha > 1$ by Rudin 3.26. In this case a_n is also convergent by comparison test.

□

Problem 4

Show that convergence of $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$ implies convergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$$

Proof. Following Cauchy–Schwarz inequality we notice that

$$\left(\sum \sqrt{a_n} \cdot \frac{1}{n}\right)^2 \leq \sum a_n \cdot \sum \frac{1}{n^2}$$

Since both $\sum a_n$ and $\sum \frac{1}{n^2}$ converge, $\sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$ must be a finite number. Therefore, $\left(\sum_{n=1}^{\infty} \sqrt{a_n} \cdot \frac{1}{n}\right)^2$ is also finite. We conclude that the series $\sum \frac{\sqrt{a_n}}{n}$ must converge. □

Problem 5

Assume $a_0 \geq a_1 \geq a_2 \geq \ldots$ and suppose that $\sum a_n$ converges. Prove that

$$\lim_{n \to \infty} (na_n) = 0$$

Proof. We first notice that a_n is monotonically non-increasing. Since $\sum a_n$ converges, $a_n \to 0$ as $n \to \infty$. Therefore all a_n are non-negative.

We then notice that $a_k \leq a_n$ for any $n < k$. Therefore

$$na_{2n} \leq \sum_{k=n+1}^{2n} a_k$$

and

$$na_{2n-1} \leq \sum_{k=n}^{2n-1} a_k$$

Since $\sum a_n$ converges, for arbitrary $\epsilon > 0$ we can always find such integer N that $\sum_{k=N+1}^{2N} a_k < \frac{\epsilon}{2}$ (by Rudin 3.22). We find that

$$Na_{(2N)} \leq \sum_{k=N+1}^{2N} a_k < \frac{\epsilon}{2}$$
$$(2N)a_{(2N)} < \epsilon$$

This is equivalent to subsequence $(2n)a_{(2n)}$ converging to a limit of 0.

Furthermore, for arbitrary $\epsilon > 0$ we can always find such integer N that $\sum_{k=N}^{2N-1} a_k < \frac{\epsilon}{2}$. We find that

$$Na_{(2N-1)} \leq \sum_{k=N}^{2N-1} a_k < \frac{\epsilon}{2}$$
$$(2N-1)a_{(2N-1)} < (2N)a_{(2N-1)} < \epsilon$$

This is equivalent to subsequence $(2n-1)a_{(2n-1)}$ converging to a limit of 0.

Since both even and odd subsequences of sequence (na_n) converge to 0, a_n itself must converge to 0.

$\square$

Problem 6

If X and Y are metric spaces and $f : X \to Y$ is a mapping between them, show that the following statements are equivalent:
a) $f^{-1}(B)$ is open in X whenever B is open in Y.
b) $f^{-1}(B)$ is closed in X whenever B is closed in Y.
c) $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X.

Claim 1. For any set $B \subseteq Y$ and map f:

$$f^{-1}(B^c) = (f^{-1}(B))^c$$

Proof. Consider arbitrary $x \in f^{-1}(B^c)$:

$$f(x) \in B^c$$
$$f(x) \notin B$$
$$x \notin f^{-1}(B)$$
$$x \in (f^{-1}(B))^c$$

Since all above steps are bidirectional ("if and only if"), we conclude that

$$f^{-1}(B^c) = (f^{-1}(B))^c$$

$\square$

We then prove equivalence of properties a) and b).

Proof. Suppose a) holds. Consider a closed set $B \subseteq Y$. Its complement in Y must be open. Pre-image of B^c must also be open by property a). Its complement, $(f^{-1}(B^c))^c$, is closed in X. Using Claim 1 we find that $(f^{-1}(B^c))^c = f^{-1}(B)$. Therefore $f^{-1}(B)$ must be closed. Proof in the other direction follows the same argument (via considering open set B instead of closed set B).

$\square$

Claim 2. For arbitrary $B \in Y$ and map f:

$$f(f^{-1}(B)) \subseteq B$$

Proof. Consider arbitrary $y \in f(f^{-1}(B))$. Choose $x \in f^{-1}(B)$ such that $f(x) = y$.

$$x \in f^{-1}(B)$$
$$f(x) \in B$$
$$y \in B$$

□

Claim 3. For arbitrary $A \in X$ and map f:

$$A \subseteq f^{-1}(f(A))$$

Proof. Consider arbitrary $x \in A$:

$$f(x) \in f(A)$$
$$x \in f^{-1}(f(A))$$

□

We will now prove equivalence of properties b) and c).

Proof. Suppose b) holds. Consider arbitrary subset $A \subseteq X$.

$$A \subseteq f^{-1}(f(A))$$

Furthermore we notice that

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)})$$

Since $\overline{f(A)}$ is closed, its pre-image $f^{-1}(\overline{f(A)})$ must also be closed by property b). Closure of A is the smallest closed set containing A, in other words, any closed set that contains A contains $\overline{A}$. Therefore:

$$\overline{A} \subseteq f^{-1}(\overline{f(A)})$$
$$f(\overline{A}) \subseteq f(f^{-1}(\overline{f(A)}))$$

Using Claim 2 we conclude that

$$f(\overline{A}) \subseteq \overline{f(A)}$$

Now suppose c) holds. For arbitrary closed subset $B \subseteq Y$ consider $f^{-1}(B) \subseteq X$. Considering property c):

$$f(\overline{f^{-1}(B)}) \subseteq \overline{f(f^{-1}(B))}$$

Using Claim 2:

$$f(\overline{f^{-1}(B)}) \subseteq \overline{B}$$

Since B is closed $B = \overline{B}$, thus:

$$f(\overline{f^{-1}(B)}) \subseteq B f^{-1}(f(\overline{f^{-1}(B)})) \subseteq f^{-1}(B)$$

Using Claim 3:

$$\overline{f^{-1}(B)} \subseteq f^{-1}(B)$$

We conclude that $f^{-1}(B)$ is closed since it contains its closure.

□

Problem 7

Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{h \to 0} (f(x+h) - f(x-h)) = 0$$

for every $x \in \mathbb{R}$. Does this imply that f is continuous?

Answer: No.

Proof. Counterexample: consider function

$$f(x) = \frac{1}{x^2}.$$

It is clearly discontinuous at $x = 0$. However, it satisfies the stated condition. Specifically, if we evaluate $\lim_{h \to 0} f(x+h) - f(x-h)$ at $x = 0$ we find

$$\lim_{h \to 0} \frac{1}{(0+h)^2} - \frac{1}{(0-h)^2} = 0$$

as required.

□