18.100B: Problem Set 5

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Problem 1

> Let $\mathcal{M}$ be a complete metric space, and let $X \subseteq \mathcal{M}$. Show that X is complete if and only if X is closed.

Proof. Suppose X is complete. Any limit point of X is the limit of some Cauchy sequence in X. Since any Cauchy sequence converges in X, its limit must be in X. Thus all limit points of X are in X and therefore X must be closed.

Now suppose X is closed. Any Cauchy sequence in $\mathcal{M}$ that is also in X must converge to some limit in $\mathcal{M}$. Limit of any Cauchy sequence in X must be either a point of X or a limit point of X. Since X is closed, it contains all its limit points. Therefore, all Cauchy sequences in X converge in X. Thus, X is a complete metric space.

□

Problem 2

> a) Show that a sequence in an arbitrary metric space $\{x_n\}$ converges if and only if the 'even' and 'odd' subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ both converge to the same limit.

Proof. Suppose sequence $\{x_n\}$ converges to x. Then for any ϵ we can find $N \in \mathbb{N}$ such that for all $n > N : d(x, x_n) < \epsilon$. Clearly, $d(x, x_{2n}) < \epsilon$ and $d(x, x_{2n-1}) < \epsilon$. Therefore subsequence $\{x_{2n}\}$ and subsequence $\{x_{2n-1}\}$ must converge to x.

Suppose both $\{x_{2n}\}$ and $\{x_{2n-1}\}$ converge to x. Then for any ϵ we can find $N, M \in \mathbb{N}$ such that for all $n > N : d(x, x_{2n}) < \epsilon$ and for all $n > M : d(x, x_{2n-1}) < \epsilon$. We notice that for all $n > 2N : d(x, x_n) < \epsilon$. Therefore, sequence $\{x_n\}$must converge to x.

□

b) Show that a sequence in an arbitrary metric space $\{x_n\}$ converges if and only if the subsequences $\{x_{2n}\}$, $\{x_{2n-1}\}$, and $\{x_{5n}\}$ all converge.

Proof. Consider $\{x_n\}$ and its subsequence $\{x_{f(n)}\}$ where n and $f(n)$ are natural numbers and $f(n) \geq n$. Suppose $\{x_n\}$ converges to x. Then for any $\epsilon > 0$ there exists N such that for all $n > N : d(x, x_n) < \epsilon$. For any given n we notice that $n > N \Rightarrow f(n) geqn > N$. Therefore $d(x, x_{f(n)}) < \epsilon$ and thus subsequence $\{x_{f(n)}\}$ converges to x.

Conversely, suppose that $\{x_{f(n)}\}$ converges to x. Then for any $\epsilon > 0$ there exists N such that for all $n : f(n) > N : d(x, x_{f(n)}) < \epsilon$. We notice that for any N there exists K such that $K > f(N)$ by the Archimedean property. Then $K > f(N) > N$. Therefore for all $n > K : d(x, x_n) < \epsilon$ and thus sequence $\{x_n\}$ converges to x.

We conclude that sequence converges to x if and only if all its subsequences converge to x. Case in hand is a special case of this general theorem for three chosen subsequences.

□

Problem 3

If $\{x_n\}$ and $\{y_n\}$ are two bounded sequences of real numbers, show that:

- $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$;
- $\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n$.

Proof. Let $x = \limsup x_n, \;\; y = \limsup y_n$. Since sequences x_n and y_n are bounded, $(x_n + y_n)$ must be also bounded, thus we can find a convergent subsequence $s_{n_k} = (x_{n_k} + y_{n_k});\;\; s_{n_k} \to s$. Since sequence x_{n_k} is bounded, there must exist its subsequence that is convergent; denote it $x_{n_a} \to a$. Now consider a bounded sequence y_{n_a}, which must contain a convergent subsequence $y_{n_b} \to b$. We note that x_{n_b} is a subsequence of convergent sequence, thus $x_{n_b} \to a$. Now consider sequence $(x_{n_b} + y_{n_b})$, which must converge to $a + b$. At the same time, $(x_{n_b} + y_{n_b})$ is a subsequence of convergent sequence s_{n_k}. Therefore s_{n_k} must also converge to $a + b$, which means $s = a + b$. We notice that $a \leq x$ and $b \leq y$. Therefore:

$$s \leq x + y$$

We can see that limit of any convergent subsequence of $(x_n + y_n)$ must be strictly less than $x + y$. In other words, $x + y$ is an upper bound of the set of subsequential limits of s_n. Supremum of the set of subsequential limits of s_n is its least upper bound, therefore:

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n.$$

Similar logic can be used to show that

$$\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n.$$

$\square$

Moreover, show that if $\{x_n\}$ converges, then both inequalities are actually equalitites.

Proof. Suppose $\{x_n\}$ converges to p. Then its every subsequence converges to p. For $\limsup\{y_n\} = a$ there must exist a subsequence $\{y_{n_a}\}$ of $\{y_n\}$ that converges to a. Consider subsequence $\{x_{n_a} + y_{n_a}\}$ of the sequence $\{x_n + y_n\}$ for which:

$$\limsup\{x_{n_a} + y_{n_a}\} \leq \limsup\{x_n + y_n\}$$

We note that subsequence $\{x_{n_a} + y_{n_a}\}$ converges to $p + a$, therefore:

$$p + a \leq \limsup\{x_n + y_n\}$$

$$\limsup\{x_n\} + \limsup\{y_n\} \leq \limsup\{x_n + y_n\}$$

At the same time, using the result from the first part of the problem we can see that

$$\limsup\{x_n + y_n\} \leq \limsup\{x_n\} + \limsup\{y_n\}$$

From this follows the equality

$$\limsup\{x_n + y_n\} = \limsup\{x_n\} + \limsup\{y_n\}$$

Using the same logic we can prove that for convergent $\{x_n\}$ and bounded $\{y_n\}$ that

$$\liminf\{x_n\} + \liminf\{y_n\} = \liminf\{x_n + y_n\}$$

$\square$

Problem 4

The 'sequence of averages' of a sequence of real numbers $\{x_n\}$ is the sequence $\{a_n\}$ defined by

$$a_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

If $\{x_n\}$ is a bounded sequence of real numbers, then show that

$$\liminf x_n \leq \liminf a_n \leq \limsup a_n \leq \limsup x_n.$$

Proof. Denote $x^* = \limsup x_n$. For a given $\epsilon > 0$ consider $K = \{k \in \mathbb{N} : x_k \geq x^* + \epsilon\}$. Suppose K is infinite. Then bounded sequence $\{x_k\}_{k \in K}$ must contain a convergent subsequence $x_{k_c} \to c$. Limit of x_{k_c} must lie in the closed interval:

$$\liminf x_{k_c} \leq c \leq \limsup x_{k_c}$$

Since $x^* + \epsilon < \liminf x_{k_c}$ we've found a subsequence of x_n that converges to a point that is strictly greater than $\limsup x_n$. Contradiction. Therefore, K must be finite.

Define $\mathcal{S}_n = \{i \in \mathbb{N} : i \in K, i \leq n\}$ and $\mathcal{T}_n = \{i \in \mathbb{N} : i \notin K, i \leq n\}$ and define sequences s_n and t_n by:

$$s_n = \sum_{i \in \mathcal{S}_n} x_i, \quad t_n = \sum_{i \in \mathcal{T}_n} x_i$$

Since $\mathcal{S}_n \cup \mathcal{T}_n = \{i \in \mathbb{N} : i \leq n\}$, the set of the first n natural numbers, $s_n + t_n$ is equal to the sum of first n elements of the sequence x_n. Therefore:

$$a_n = \frac{s_n}{n} + \frac{t_n}{n}.$$

We notice that $\frac{s_n}{n} \to 0$ since $s_n \to s$, some finite number, and $n \to \infty$. Furthermore, we notice that every element of $\{x_k\}_{k \in \mathbb{N}, k \notin K}$ is less than $x^* + \epsilon$. Then the sum of any n elements from $\{x_k\}_{k \in \mathbb{N}, k \notin K}$ is less than $n(x^* + \epsilon)$ for any $n \in \mathbb{N}$. Therefore:

$$\frac{t_n}{n} < x^* + \epsilon$$

Using the result from problem 3 we find that:

$$\limsup\left(\frac{s_n}{n} + \frac{t_n}{n}\right) \leq \limsup \frac{s_n}{n} + \limsup \frac{t_n}{n}$$

Since $\limsup \frac{s_n}{n} = 0$ and $\limsup \frac{t_n}{n} \leq x^* + \epsilon$:

$$\limsup a_n \leq x^* + \epsilon$$

We notice that $\limsup x_n$ is a lower bound for the set $\{x^* + \epsilon : \epsilon > 0\}$, therefore it cannot be greater than x^*. Thus $\limsup x_n \leq x^*$. We conclude that:

$$\limsup a_n \leq \limsup x_n$$

Similar logic can be employed to show that $\liminf x_n \leq \liminf a_n$.

□

In particular, if $x_n \to x$ then show that $a_n \to x$.

Proof. If $x_n \to x$ then $\liminf x_n = \limsup x_n = x$. Therefore $\liminf a_n = \limsup a_n = x$. From this $a_n \to x$.

□

Does the convergence of $\{a_n\}$ imply the convergence of $\{x_n\}$?

Answer: Convergence of a_n does not necessarily imply the convergence of x_n.

Proof. Counterexample: Consider sequence $\{x_n\} = \{(-1)^n : n \in \mathbb{N}\}$. Sequence a_n for such x_n converges to 0. Indeed a_n can be rewritten as:

$$a_n = \frac{0 \cdot k + (-1)^{n-2k}}{n} = \frac{0}{n} + \frac{(-1)^{n-2k}}{n}$$

where $k = \left\lfloor \frac{n}{2} \right\rfloor$. From this it is clear that $\frac{(-1)^{n-2k}}{n} \to 0$ as $n \to 0$. At the same time, x_n clearly does not converge.

□

Problem 5

Consider any sequence (x_n) defined by choosing $0 < x_1 < 1$ and then defining $x_{n+1} = 1 - \sqrt{1 - x_n}$ for $n \geq 0$. Show that x_n is a decreasing sequence converging to zero.

Proof. Suppose that $0 < x_n < 1$. Then

$$0 < 1 - x_n < 1$$
$$0 < \sqrt{1 - x_n} < 1$$
$$0 < 1 - \sqrt{1 - x_n} < 1$$
$$0 < x_{n+1} < 1$$

By induction with $0 < x_1 < 1$ we conclude that all elements of sequence x_n are between 0 and 1.

Furthermore, x_n is decreasing. To prove this, consider the difference:

$$d = x_{n+1} - x_n = 1 - \sqrt{1 - x_n} - x_n = 1 - x_n - \sqrt{1 - x_n} = \sqrt{1 - x_n}\left(\sqrt{1 - x_n} - 1\right)$$

We notice that $0 < \sqrt{1 - x_n} < 1$, then $d < 0$, thus for any $n \in \mathbb{N} : x_{n+1} < x_n$. Therefore, limit of x_n must lie in $[0, 1)$.

Suppose that $\lim x_n = L > 0$. Then for all x_n:

$$x_n > L$$
$$x_{n+1} > L$$
$$1 - \sqrt{1 - x_n} > L$$
$$\sqrt{1 - x_n} < 1 - L$$
$$1 - x_n < (1 - L)^2$$
$$x_n - 1 > -(1 - L)^2$$
$$x_n > 1 - (1 - L)^2$$

We then examine the difference

$$1-(1-L)^2-L=(1-L)-(1-L)^2=(1-L)(1-1+L)=L(1-L),$$

which is greater than 0 for any $0<L<1$. Therefore

$$x_n > 1-(1-L)^2 > L$$

However, we then can find ϵ-neighbourhood of L that contains no points of x_n. Therefore $L>0$ cannot be a limit of x_n. Contradiction. We conclude that $\lim x_n = 0$.

Another way to see that the limit of sequence x_n is 0 is to evaluate its first few elements:

$$x_2 = 1-\sqrt{1-x_1} = 1-(1-x_1)^{\left(\frac{1}{2}\right)^1}$$
$$x_3 = 1-\sqrt{1-x_2} = 1-\sqrt{1-\left(1-\sqrt{1-x_1}\right)} = 1-(1-x_1)^{\left(\frac{1}{2}\right)^2}$$

It is easy to see that:

$$x_n = 1-(1-x_1)^{\left(\frac{1}{2}\right)^n}$$

For $0<x_1<1$ we notice that $(1-x_1)^p \to 1$ as $p\to 0$. Therefore $x_n \to 0$ as $n\to\infty$.

□

> Also, show that $\frac{x_{n+1}}{x_n} \to \frac{1}{2}$.

Proof.

$$\frac{x_{n+1}}{x_n} = \frac{1-\sqrt{1-x_n}}{x_n} = \frac{x_n}{x_n\left(1+\sqrt{1-x_n}\right)} = \frac{1}{1+\sqrt{1-x_n}}$$

As $x_n \to 0$ we can see that limit of the denominator of the above expression is 2. Thus, $\frac{x_{n+1}}{x_n} = \frac{1}{2}$.

□

Problem 6

> The Greeks thought that the number Φ, known as the Golden Mean, was the ratio of the sides of the most aesthetically pleasing rectangles. Imagine a line segment A divided into two smaller line segments B and C, with lengths a, b, and c respectively and $b>c$. If the proportion between a and b is the same as the proportion between b and c, then we call this proportion Φ.
> a) Show that with a,b,c as above, $\Phi = \frac{b}{c}$ satisfies $\Phi^2 = \Phi + 1$. Conclude that $\Phi = \frac{1+\sqrt{5}}{2}$

Proof.

$$\Phi = \frac{a}{b} = \frac{b}{c}$$
$$ac = b^2$$
$$ac^2 = b^2c$$
$$\frac{b^2}{c^2} = \frac{a}{c}$$
$$\frac{b^2}{c^2} = \frac{b+c}{c}$$
$$\frac{b^2}{c^2} = \frac{b}{c} + 1$$
$$\Phi^2 = \Phi + 1$$

Solving for Φ we get:

$$\Phi = \frac{1 \pm \sqrt{5}}{2}$$

Since Φ is a ratio between lengths of line segments, it cannot be negative. Thus, $\Phi = \frac{1+\sqrt{5}}{2}$.

□

b) Show that:

$$\Phi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$$

Proof. Consider sequence x_n, which is defined by a recurrence relation:

$$x_1 = 1, \quad x_{n+1} = 1 + \frac{1}{x_n}$$

Sequence x_n is positive and bounded. Indeed, for any $n \in \mathbb{N}$: $1 \le x_n \le 2$.

Now consider two subsequences of x_n: odd x_n^o and even x_n^e:

$$x_n^o = x_n, \qquad x_{n+1}^o = 1 + \frac{1}{1 + \frac{1}{x_n}}$$
$$x_n^e = x_{n+1}, \quad x_{n+1}^e = 1 + \frac{1}{1 + \frac{1}{x_{n+1}}}$$

We examine difference between two subsequent elements of subsequence x_n^o:

$$\begin{aligned} x_{n+1}^o - x_n^o = 1 + \frac{1}{1 + \frac{1}{x_n}} - x_n = -\frac{x_n^2 - x_n - 1}{x_n + 1} \\ = -\frac{(x_n - \Phi)(x_n - \frac{1-\sqrt{5}}{2})}{x_n + 1} = \frac{(\Phi - x_n)(x_n + \frac{\sqrt{5}}{2} - 1)}{x_n + 1} \end{aligned} \tag{1}$$

We notice that expression 1 is positive only if $x_n < \Phi$. Furthermore we claim that if $x_n^o < \Phi$ then $x_{n+1}^o < \Phi$. To prove that we consider the difference

$$\begin{aligned} x_{n+1}^o - \Phi &= \frac{2x_n + 1}{x_n + 1} - \frac{1+\sqrt{5}}{2} \\ &= \frac{4x_n + 2 - (x + \sqrt{5}x_n + 1 + \sqrt{5})}{2(x_n + 1)} = \frac{x_n(3-\sqrt{5}) + (1-\sqrt{5})}{2(x_n+1)} \\ &= \frac{4x_n + 3 + \sqrt{5} - 3\sqrt{5} - 5}{2(x_n+1)(3+\sqrt{5})} = \frac{2x_n - 1 - \sqrt{5}}{(x_n+1)(3+\sqrt{5})}, \end{aligned}$$

which is positive if and only if $x_n < \Phi$. Therefore $x_n^o < \Phi \Rightarrow x_{n+1}^o < \Phi$. We notice that $x_1^o = 1 < \Phi$, therefore the whole sequence x_n^o is increasing.

Moreover, subsequence x_n^o is bounded by boundedness of x_n; therefore, it must converge. Any convergent sequence in complete metric space is a Cauchy sequence. Therefore expression 1 must be equal to zero as $n \to \infty$.

$$\lim_{n\to\infty} \frac{(\Phi - x_n)(x_n + \frac{\sqrt{5}}{2} - 1)}{x_n + 1} = \frac{(\Phi - \lim x_n^o)(\lim x_n^o + \frac{\sqrt{5}}{2} - 1)}{\lim x_n^o + 1} = 0,$$

which is true if and only if $\lim x_n^o = \Phi$.

We can prove in a similar way that subsequence x_n^e also converges to Φ.

Since odd and even subsequences of sequence x_n converge to Φ, x_n itself must converge to Φ.

$\square$

c) Show that:

$$\Phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}}}$$

Proof. Consider sequence y_n, which is defined by a recurrence relation:

$$y_1 = 1, \qquad y_{n+1} = \sqrt{1 + y_n}$$

$\square$

We consider the ratio of two subsequent elements of y_n:

$$\frac{y_{n+1}}{y_n} = \frac{\sqrt{1+y_n}}{y_n}$$

$$\begin{aligned} y_{n+1} > y_n \iff \frac{\sqrt{1+y_n}}{y_n} > 1 \iff \frac{1+y_n}{y_n^2} - 1 > 0 \iff \\ \iff -\frac{y_n^2 - y_n - 1}{y_n^2} > 0 \iff \frac{(\Phi - y_n)(y_n + \frac{\sqrt{5}}{2} - 1)}{y_n^2} > 0. \end{aligned}$$

This expression is true if and only if $y_n < \Phi$. Furthermore, we can prove that $y_n < \Phi \Rightarrow y_{n+1} < \Phi$ (the same argument as in part (c) of the problem). We also note that $y_1 = 1 < \Phi$. We conclude that y_n is increasing. Furthermore, y_n is bounded, therefore it must converge and, furthermore, it must be a Cauchy sequence. Therefore, denoting $\lim y_n = L$:

$$\lim_{n\to\infty} \frac{\sqrt{1+y_n}}{y_n} = 1 \iff \frac{\sqrt{1+L}}{L} = 1$$
$$L^2 - L - 1 = 0$$

From this we find that $L = \dfrac{1 \pm \sqrt{5}}{2}$. Since all y_n are positive:

$$\lim y_n = \frac{1+\sqrt{5}}{2} = \Phi.$$

d) The Fibonacci sequence is defined by $z_1 = 1, z_2 = 1, z_{n+2} = z_{n+1} + z_n$. Show that the sequence of ratios of successive elements, $\frac{z_{n+1}}{z_n}$, converges to Φ.

Proof. Consider sequence $x_n = \frac{z_{n+1}}{z_n}$. We can rewrite this as

$$x_n = \frac{z_{n+1}}{z_n} = \frac{z_n + z_{n-1}}{z_n} = 1 + \frac{z_{n-1}}{z_n} = 1 + \frac{1}{\frac{z_n}{z_{n-1}}} = 1 + \frac{1}{x_{n-1}}$$

We also notice that $x_1 = 1$. Thus, by 6b), sequence x_n must converge to Φ.

$\square$