

18.100B: Problem Set 3

Dmitry Kaysin

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Problem 1

In vector spaces, metrics are usually defined in terms of norms which measure the length of a vector. If V is a vector space defined over \mathbb{R} , then a norm is a function from vectors to real numbers, denoted by $\|\cdot\|$ satisfying:

- $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$
- For any $\lambda \in \mathbb{R}$, $\|\lambda x\| = |\lambda| \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$.

Prove that every norm defines a metric.

Proof. For given two vectors $p, q \in V$ and a norm $\|\cdot\|$, metric on V can be defined as $d(p, q) = \|p - q\|$. We will check the following properties of this metric:

- $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$
- $d(p, q) = d(q, p)$
- $d(p, q) \leq d(p, r) + d(r, q)$

for $p, q, r \in V$.

We first check that for $p \neq q$: $d(p, q) = \|p - q\| = \|x\| > 0$ for some $x \in V$. We also note that $d(p, p) = \|p - p\| = \|0\| = 0$.

We then check that $d(p, q) = \|p - q\| = \| -1 \cdot (q - p) \| = | -1 | \|q - p\| = d(q, p)$. Finally, we check that

$$d(p, q) \leq d(p, r) + d(r, q)$$

Indeed

$$\begin{aligned} \|p - q\| &\leq \|p - r\| + \|r - q\| \\ \|(p - r) + (r - q)\| &\leq \|p - r\| + \|r - q\| \end{aligned}$$

which is true by the triangle inequality for norms.

□

Problem 2

Let M be a metric space with metric d . Show that d_1 defined by

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on M .

Proof. We will check whether d_1 satisfies properties of a metric. Let $p, q, r \in M$. Denote $a = d(p, q)$.

We first check that for $p \neq q : d_1(p, q) = 0$.

$$d_1(p, q) = \frac{d(x, y)}{1 + d(x, y)} = \frac{a}{1 + a} > 0$$

We also check that for $p = q : d_1(p, q) = 0$.

$$d_1(p, q) = \frac{0}{1 + 0} = 0$$

We then check that $d_1(p, q) = d_1(q, p)$.

$$d(p, q) = d(q, p) = a$$

$$d_1(p, q) = \frac{a}{1 + a} = d_1(q, p)$$

We finally check the triangle inequality for d_1 :

$$d_1(p, q) \leq d_1(p, r) + d_1(r, q)$$

Let $d(p, q) = a$, $d(p, r) = b$, $d(r, q) = c$. We now prove that

$$\frac{a}{1 + a} \leq \frac{b}{1 + b} + \frac{c}{1 + c}$$

$$a(1 + b)(1 + c) \leq (1 + a)(b + c + 2bc)$$

$$a + ab + ac + abc \leq b + c + 2bc + ab + ac + 2abc$$

$$a \leq b + c + bc + abc$$

Since d is a metric: $a \leq b + c$:

$$a \leq a + bc + abc$$

$$0 \leq bc + abc$$

Which is always true because $a, b, c > 0$.

Since function d_1 satisfies all properties of a metric, it is a metric.

□

Observe that M itself is bounded in this metric.

Proof. We notice that set $D = \{d_1(p, q) : p, q \in \mathbb{R}_{\geq 0}\}$ is bounded. D is bounded from below by 0 since metric has a property of non-negativity. D is clearly bounded from above by 1. \square

Problem 3

Let A and B be two subsets of a metric space M . Recall that A° , the interior of A , is the set of interior points of A . Prove the following:
a) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

Proof. Consider $a \in A^\circ$ and $b \in B^\circ$. By definition of interior point we can find open neighbourhoods $N(a) \subset A$ and $N(b) \subset B$.

For any sets S, X, Y :

$$S \subset X \Rightarrow Y \subset (X \cup Y)$$

Therefore, $N(a) \subset A \cup B$ and $N(b) \subset A \cup B$. Thus, both points a and b are interior points of $A \cup B$. This should hold not only for metric spaces. \square

b) $A^\circ \cap B^\circ = (A \cap B)^\circ$

Proof. A point belongs to intersection of two sets if and only if it belongs to both sets. Consider point x that belongs to intersection of interiors of A and B ($x \in A^\circ, x \in B^\circ$). By definition of interior point we can find open neighbourhoods $N_a(x) \subset A$ and $N_b(x) \subset B$. Thus intersection of neighbourhoods $N_a(x) \cap N_b(x)$ lies within $A \cap B$. Thus, point x is an interior point of $A \cap B$, or:

$$A^\circ \cap B^\circ \subseteq (A \cap B)^\circ \quad (1)$$

Now consider y , interior point of intersection of A and B ($y \in (A \cap B)^\circ$). Since y is interior, we can find an open neighbourhood $N_{ab}(y) \subseteq (A \cap B)$. Furthermore, $N_{ab}(y)$ must be a subset of both A and B and thus has open neighbourhoods in A and in B . Therefore y is an interior point of both A and B , or:

$$(A \cap B)^\circ \subseteq A^\circ \cap B^\circ \quad (2)$$

Considering expressions 1 and 2, we conclude that $A^\circ \cap B^\circ = (A \cap B)^\circ$. This should hold not only for metric spaces. \square

Give an example of two subsets A and B of the real line such that $A^\circ \cup B^\circ \neq (A \cup B)^\circ$.

Answer: $A = \mathbb{Q}$, $B = \mathbb{R} \setminus \mathbb{Q}$.

Proof. Since \mathbb{Q} is dense in \mathbb{R} , every neighbourhood of every point of both sets \mathbb{Q} and $\mathbb{R} \setminus \mathbb{Q}$ contains infinitely many points of both \mathbb{Q} and $\mathbb{R} \setminus \mathbb{Q}$. This means that no point of these sets is interior:

$$\mathbb{Q}^\circ = \emptyset, (\mathbb{R} \setminus \mathbb{Q})^\circ = \emptyset$$

However, interior of the union of these sets is \mathbb{R} :

$$(\mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}))^\circ = \mathbb{R}^\circ = \mathbb{R}$$

Therefore, $\mathbb{Q}^\circ + (\mathbb{R} \setminus \mathbb{Q})^\circ \neq (\mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}))^\circ$. □

Problem 4

Let A be a subset of a metric space M . Recall that \overline{A} , the closure of A , is the union of A and its limit points. Recall that a point x belongs to the boundary of A , ∂A , if every open ball centered at x contains points of A and points of A^c , the complement of A . Prove that:
a) $\partial A = \overline{A} \cap \overline{A^c}$

Proof. Note: We shall prove validity of claims a)-d) for general topological spaces that do not necessarily have a metric.

Consider point $p \in \partial A$. Every neighbourhood of p must contain points of both A and A^c . Thus every neighbourhood of p must contain points of A , so $p \in \overline{A}$. Every neighbourhood of p must also contain points of A^c , so $p \in \overline{A^c}$. Since p is both an element of set \overline{A} and of set $\overline{A^c}$, p must be an element of $\overline{A} \cap \overline{A^c}$, so:

$$\partial A \subseteq \overline{A} \cap \overline{A^c} \tag{3}$$

Now consider point $p \in \overline{A} \cap \overline{A^c}$. Point p must belong to the set \overline{A} , thus every open neighbourhood of p must contain elements of A . Point p must also belong to the set $\overline{A^c}$, thus every open neighbourhood of p must contain elements of A^c . We can see that every open neighbourhood of p must contain elements of both A and A^c so p must lie in the boundary of A , or:

$$\overline{A} \cap \overline{A^c} \subseteq \partial A \tag{4}$$

Considering expressions 3 and 4 we can see that $\partial A = \overline{A} \cap \overline{A^c}$. □

b) $p \in \partial A \iff p$ is in \bar{A} but not in A° (symbolically, $\partial A = \bar{A} \setminus A^\circ$)

Proof. Consider $p \in \partial A$. Every open neighbourhood of p must contain both points of A and points of A^c . Since every open neighbourhood of p must contain points of A , p must be an element of \bar{A} , so:

$$A \subseteq \bar{A}$$

Furthermore, since every open neighbourhood of p must contain points of A^c , such neighbourhood cannot be a subset of A , so p cannot be an interior point. Therefore

$$A \not\subseteq A^\circ$$

We conclude that

$$\partial A \subseteq \bar{A} \setminus A^\circ \tag{5}$$

Now consider $p \in \bar{A} \setminus A^\circ$. Since p is an element of \bar{A} , every neighbourhood of p must contain points of A . Since p cannot be an element of A° , every neighbourhood of p includes points that belong to A^c . Therefore every neighbourhood of p includes both points of A and points of A^c and we conclude that

$$\bar{A} \setminus A^\circ \subseteq \partial A \tag{6}$$

Considering expressions 5 and 6 we can see that $\partial A = \bar{A} \setminus A^\circ$. □

c) ∂A is a closed set

Proof. Set ∂A is closed if it contains all its limit points. If the set of limit points of ∂A is not \emptyset , consider an arbitrary limit point p of ∂A . Now we will prove that p is a boundary point of A . Consider $N(p)$, an arbitrary open neighbourhood of p . Since p is a limit point, $N(p)$ must contain points of ∂A other than p . Denote $q : q \in \partial A, q \in N(p)$ one of these points. Every open neighbourhood of q must contain points that lie in A and in A^c by definition of boundary. Notice that $N(p)$ is an open neighbourhood of q . Thus $N(p)$ must include points that lie in A and in A^c . Therefore, p is a boundary point of A . We conclude that all limit points of ∂A lie in ∂A , or set ∂A is closed.

If the set of limit points of ∂A is \emptyset we note that $\emptyset \subseteq \partial A$. In this case, vacuously, ∂A is a closed set. □

d) A is closed $\iff \partial A \subseteq A$

Proof. Consider b , a boundary point of a closed set A . Consider an arbitrary open neighbourhood of b : $N(b)$. Since b is a boundary point, $N(b)$ contains at least one point of A and at least one point of A^c . Consider an arbitrary point $p \in A$ that lies in $N(b)$. Either $p = b$ or $p \neq b$. If $p = b$, then $p = b \in A$, so b , a boundary point of A , is an element of A . If $p \neq b$, then b must be a limit point. But then we know that a closed set contains all its limit points, thus b , a boundary point of A , is an element of A . We conclude that if A is closed then $\partial A \subseteq A$.

Now consider a set A such that $\partial A \subseteq A$. Suppose p is a limit point of A such that $p \notin A$. Then $p \in A^c$. Therefore each open neighbourhood of p must include points of A (since it is a limit point of A) and at least one point of A^c , specifically p . But then p must be a boundary point of A and since $\partial A \subseteq A$, p must be an element of A , which presents a contradiction. Therefore, p , a limit point of A must be a point of A . We conclude that if $\partial A \subseteq A$ then A is closed.

Thus, A is closed $\iff \partial A \subseteq A$. \square

Problem 5

Show that, in \mathbb{R}^n with the usual (Euclidean) metric, the closure of the open ball $B_R(p)$, $R > 0$, is the closed ball

$$\{q \in \mathbb{R}^n : d(p, q) \leq R\}.$$

Proof. We claim that for every metric space M with metric that is induced by a norm, closure of an open ball is a closed ball:

$$\overline{B_R(p)} = B_R[p]$$

where

$$B_R(p) = \{r : d(p, r) < R\}$$

$$B_R[p] = \{r : d(p, r) \leq R\}$$

and $d(p, r) = \|p - r\|$ for some norm $\|\cdot\|$; $R > 0$.

Set $B_R[p]$ is closed since it contains all its limit points. Suppose, there exists x , a limit point of $B_R[p]$, that belongs to $B_R[p]^c = \{r : d(p, r) > R\}$. Set $B_R[p]^c$ is open, so all its points internal to itself. This means that for such x we can find an open ball that is a subset of $B_R[p]^c$. But then, it cannot contain points of $B_R[p]$, which is a contradiction with x being a limit point of $B_R[p]$. Thus, $B_R[p]$ contains all its limit points and is closed. We also note that set $B_R(p) \subseteq B_R[p]$. Closure of $B_R(p)$ is the smallest closed subset that includes $B_R(p)$, therefore it is a subset of any closed set that includes $B_R(p)$, including $\subseteq B_R[p]$:

$$\overline{B_R(p)} \subseteq B_R[p]$$

Consider $r \in B_R[p]$. We will prove that $r \in \overline{B_R(p)}$. We need to prove that r is either an element or a limit point of $B_R(p)$. All r such that $d(p, r) < R$

are clearly elements of $B_R(p)$. The set of remaining points is $B_R[p] \setminus B_R(p)$. All points of this set are given by $\{r : d(p, r) = R\}$. Point r is a limit point of $B_R(p)$ if every open ball $B_\epsilon(r)$ contains at least one point x such that $x \in B_R(p) \iff d(p, x) < R$. We can find such x explicitly, provided M is a normed vector space, which follows from existence of norm $\|\cdot\|$ for M .

We claim that

$$x = r + \frac{\epsilon}{2} \frac{1}{d(p, r)}(p - r)$$

is such a point. To prove this we show that $d(r, x) < \epsilon$ and $d(p, x) < R$.

$$\begin{aligned} d(r, x) = \|r - x\| &= \left\| r - r + \frac{\epsilon}{2} \frac{1}{d(p, r)}(p - r) \right\| = \left\| \frac{\epsilon}{2} \frac{1}{d(p, r)}(p - r) \right\| = \\ &= \frac{\epsilon}{2} \frac{\|p - r\|}{d(p, r)} = \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Thus x lies in $B_\epsilon(r)$.

$$\begin{aligned} d(p, x) = \|p - x\| &= \left\| p - r - \frac{\epsilon}{2} \frac{1}{d(p, r)}(p - r) \right\| = \\ &= \left\| (p - r) \left(1 - \frac{\epsilon}{2} \frac{1}{d(p, r)} \right) \right\| = \left(1 - \frac{\epsilon}{2} \frac{1}{d(p, r)} \right) d(p, r) = \\ &= d(p, r) - \frac{\epsilon}{2} \leq R - \frac{\epsilon}{2} < R. \end{aligned}$$

Thus x lies in $B_R(p)$.

Therefore all points in $B_R[p]$ are either elements of $B_R(p)$ or its limit points, thus are elements of the closure of $B_R(p)$:

$$B_R[p] \subseteq \overline{B_R(p)}$$

We conclude that:

$$\overline{B_R(p)} = B_R[p]$$

Give an example of a metric space for which the corresponding statement is false.

This may not be the case for metrics that are not induced by a norm. For example, for discrete metric on \mathbb{R} , which is not induced by a norm,

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y \end{cases}$$

open ball $B_1(0) = \{0\}$, closed ball $B_1[0] = \mathbb{R}$ and closure of $\overline{B_1(0)} = \{0\}$. Clearly

$$\overline{B_1(0)} \subseteq B_1[0]$$

but

$$\overline{B_1(0)} \neq B_1[0].$$

□

Problem 6

Prove directly from the definition that the set $K \subseteq \mathbb{R}$ given by

$$K = \left\{ 0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots \right\}$$

is compact.

Proof. Let C be an open cover of K , i.e. a collection of open sets such that:

$$K \subseteq \bigcup_{S \in C} S$$

For each $k \in K$ we can find a set within the collection C that contains k (not necessarily unique). Denote such open set $C(k)$.

Specifically, for $0 \in K$ there must exist open set $C(0)$. Since $C(0)$ is open, it must contain some open ball $B_\epsilon(0)$. We notice that $B_\epsilon(0)$ contains all points of K that are less than ϵ :

$$x \in K : x < \epsilon \Rightarrow x \in B_\epsilon(0)$$

We can see that there are only finitely many points of K that are greater than or equal to ϵ . Such points can be covered by union of finitely many open sets $C(k)$. The rest of the points of K can be covered by $B_\epsilon(0)$.

Therefore the following finite collection covers K :

$$\{C(k) : k \in K, k \geq \epsilon\} \cup \{B_\epsilon(0)\}$$

We conclude that K is compact. □

Problem 7

Let K be a compact subset of a metric space M , and let $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ be an open cover of K . Show that there is a positive real number δ with the property that for every $x \in K$ there is some $\alpha \in A$ with

$$B_\delta(x) \subseteq \mathcal{U}_\alpha$$

Proof. Since all sets in collection \mathcal{U} are open, for each $x \in K$ there must exist an open ball $B_{\epsilon(x)}(x)$ where $\epsilon(x) > 0$ that is a subset of some $\mathcal{U}_\alpha, \alpha \in A$. Open ball with half the radius, $B_{\epsilon(x)/2}(x)$, is also a subset of the same \mathcal{U}_α . Collection of such open balls with half the radius, \mathcal{H} , is an open cover of K since each $x \in K$ belongs to at least one of the open sets in collection \mathcal{H} .

Since K is compact, there must exist a finite subcover $\mathcal{F} \in \mathcal{H}$ that covers K . Since collection \mathcal{F} is finite, we can enumerate all its sets:

$$B_{\epsilon(x_1)/2}(x_1), B_{\epsilon(x_2)/2}(x_2), \dots, B_{\epsilon(x_N)/2}(x_N)$$

where $N \in \mathbb{N}$. Radius of each of these open balls must be strictly greater than zero. Find the minimum of these radii:

$$\delta = \min\{\epsilon(x_n)/2 : n \in \mathbb{N}, 1 \leq n \leq N\},$$

which will be also strictly greater than zero. We claim that δ satisfies the desired property for K and \mathcal{U} .

We prove that for any $x \in K$ open ball $B_\delta(x)$ is a subset of at least one of the sets of the collection \mathcal{U} . Indeed, every $x \in K$ must belong to at least one of the sets of its open cover \mathcal{F} ; denote such set $B_{\epsilon(x_k)/2}(x_k)$. We notice that

$$B_{\epsilon(x_k)/2}(x_k) \subset B_{\epsilon(x_k)}(x_k)$$

and since $\delta \leq \epsilon(x_k)/2$:

$$B_\delta(x) \subset B_{\epsilon(x_k)}(x_k) \subseteq \mathcal{U}_\alpha, \text{ for some } \alpha \in A$$

Therefore, there exists $\delta > 0$ such that for any point in K there exists an open ball with radius δ that is a subset of at least one set of an open cover of K . \square