

## 18.100B: Problem Set 9

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### Problem 1

Let  $f_n(x) = \frac{1}{nx+1}$  and  $g_n(x) = \frac{x}{nx+1}$  for  $x \in (0, 1)$  and  $n \in \mathbb{N}$ . Prove that  $f_n$  converges pointwise but not uniformly on  $(0, 1)$ , and that  $g_n$  converges uniformly on  $(0, 1)$ .

*Proof.* We can see that  $f_n(x) = \frac{1}{nx+1}$  converges pointwise, since for any  $x \in (0, 1)$ :

$$\lim_{n \rightarrow \infty} \frac{1}{nx+1} = 0$$

However,  $f$  is not uniformly convergent since for any  $N > 1$  we can find point  $x_N = \frac{1}{N}$  such that  $f_N(x_N) = \frac{1}{2}$  and

$$|f_N(x_N) - f(x_N)| = \frac{1}{2}$$

Clearly, for any  $\epsilon < \frac{1}{2}$  criterion of uniform convergence fails at  $a_N$  for any  $N > 1$ .

We can also see that  $g_n(x) = \frac{x}{nx+1}$  converges pointwise, since for any  $x \in (0, 1)$ :

$$\lim_{n \rightarrow \infty} \frac{x}{nx+1} = \lim_{n \rightarrow \infty} \frac{1}{n + \frac{1}{x}} = 0$$

Then we notice that

$$x < 1 \Rightarrow \frac{1}{n + \frac{1}{x}} < \frac{1}{n+1} \Rightarrow \frac{x}{nx+1} < \frac{1}{n+1}$$

Therefore, for any  $\epsilon > 0$  we can choose  $N$  such that  $\frac{1}{N+1} < \epsilon$ . Then, for any  $x \in (0, 1)$  and for any  $m \geq N$ :

$$|g_m(x) - 0| = g_m(x) < \frac{1}{m+1} \leq \frac{1}{N+1} < \epsilon.$$

Therefore,  $g(x)$  converges uniformly.

□

## Problem 2

Let  $f_n(x) = \frac{x}{1+nx^2}$  if  $x \in \mathbb{R}$  and  $n \in \mathbb{N}$ . Find the limit function  $f$  of the sequence  $(f_n)$  and the limit function  $g$  of the sequence  $(f'_n)$ .

*Proof.* We notice that

$$\frac{x}{1+nx^2} = \frac{1}{\frac{1}{x} + nx},$$

which for arbitrary  $x \neq 0$  goes to 0 as  $n$  goes to infinity. We check  $x = 0$  separately and confirm that

$$f = \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \frac{x}{1+nx^2} = 0$$

To find limit function of  $f'_n$  we consider derivative function of  $f_n$ :

$$f'_n = \frac{(1+nx^2) - x(2xn)}{(1+nx^2)^2} = \frac{1-nx^2}{1+2nx^2+n^2x^4}$$

We consider two cases ( $x = 0$  and  $x \neq 0$ ) and use L'Hospital's rule, which is valid since both numerator and denominator of  $f'_n$  are differentiable on  $\mathbb{R}$ .

$$g = \lim_{n \rightarrow \infty} f'_n = \lim_{n \rightarrow \infty} \frac{1-nx^2}{1+2nx^2+n^2x^4} = \begin{cases} 1, & \text{for } x = 0 \\ \lim_{n \rightarrow \infty} \frac{-x^2}{2x^2+2nx^4} = 0, & \text{otherwise} \end{cases}$$

□

Prove that  $f'(x)$  exists for every  $x$  but that  $f'(0) \neq g(0)$ . For what values of  $x$  is  $f'(x) = g(x)$ ?

*Proof.* Derivative of constant function  $f$  exists and is equal to 0 for any  $x \in \mathbb{R}$ , including  $x = 0$ . Thus,  $f'(x) = g(x)$  for any  $x \neq 0$ . However,  $g(0) = 1$ .

□

In what subintervals of  $\mathbb{R}$  does  $f_n \rightarrow f$  uniformly? In what subintervals of  $\mathbb{R}$  does  $f'_n \rightarrow g$  uniformly?

*Proof.* For any  $n$  function  $f_n$  is continuous on  $\mathbb{R}$  and the following holds:

$$f_n(x) \rightarrow 0 \text{ as } x \rightarrow \pm\infty; \quad f(0) = 0$$

Derivative of  $f_n$  is also continuous on  $\mathbb{R}$ , thus we can find minimum and maximum of  $f_n$  by setting  $f'_n$  to 0:

$$\begin{aligned}\frac{1 - nx^2}{1 + 2nx^2 + n^2x^4} &= 0 \\ nx^2 &= 1 \\ x &= \pm \frac{1}{\sqrt{n}}\end{aligned}$$

More specifically,  $f_n$  at  $x = \frac{1}{\sqrt{n}}$  attains its global maximum. Maximum of  $f_n$  is:

$$f_n^{\max} = \frac{1}{\sqrt{n}(1 + \frac{n}{n})} = \frac{1}{2\sqrt{n}}$$

Denote  $s_n = \sup_{x \in \mathbb{R}} |f_n(x) - f(x)|$ . Based on the above,  $s_n = f_n\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{2\sqrt{n}}$ .

For any  $\epsilon > 0$  we can find  $n \in \mathbb{N}$  such that  $s_n < \epsilon$ , thus  $f_n$  converges uniformly to  $f$  on  $\mathbb{R}$ .

Since  $f'_n(0) = 1$  and  $f'_n$  is continuous, image of every open neighbourhood of  $x = 0$  contains points arbitrarily close to 1. For such points  $|f'_n(x) - f(x)|$  is arbitrarily close to 1 and thus  $s_n = 1$ . Clearly,  $f'_n$  cannot be uniformly convergent on any set that has  $x = 0$  as a limit point.

Otherwise we have:

$$|f'_n(x) - f(x)| = \left| \frac{1 - nx^2}{(1 + nx^2)^2} \right|,$$

which can be made arbitrarily small for any  $x$  by setting  $n$  sufficiently large (degree of the polynomial in the denominator is larger than the degree of the polynomial in the numerator). □

### Problem 3

Let  $\mathcal{M}$  be a metric space and  $(f_n)$  a sequence of functions defined on a subset  $E \subseteq \mathcal{M}$ . We say that  $(f_n)$  is uniformly bounded if there exists a constant  $M$  such that  $|f_n(x)| \leq M$  for every  $n \in \mathbb{N}$  and  $x \in E$ .

Prove that if  $(f_n)$  is a sequence of bounded real valued functions that converges uniformly to a function  $f$ , then  $(f_n)$  is uniformly bounded.

Prove that in this case  $f$  is also bounded.

*Proof.* Choose arbitrary  $\epsilon > 0$ . Since  $f_n$  converges to  $f$  uniformly, there exists some  $N \in \mathbb{N}$  such that for all  $m \geq N$  and  $x \in E$  the following holds:

$$|f_m(x) - f(x)| \leq \epsilon$$

Since  $f_m(x)$  is bounded,  $f(x)$  must also be bounded. We also have finitely many functions  $f_k(x)$  such that  $k < N$ . For each  $x \in E$  consider an upper bound of  $|f_n(x)|$ , which can be constructed as follows:

$$g(x) = \max ( \max |f_k(x)|, |f(x)| + \epsilon )$$

Since each  $f_n(x)$  is bounded and  $f(x)$  is bounded,  $g(x)$  must also be bounded. Any upper bound of  $g(x)$ , for example  $M = \sup_{x \in E} g(x)$  is an upper bound for any  $|f_n(x)|$  ( $x \in E, n \in \mathbb{N}$ ) by construction. Therefore,  $f_n$  is uniformly bounded.  $\square$

If  $(f_n)$  is a sequence of bounded functions converging pointwise to  $f$ , need  $f$  be bounded?

Not necessarily.

*Proof.* Counterexample: Consider function

$$f_n(x) = \begin{cases} |x|, & \text{if } x \leq n \\ n, & \text{otherwise} \end{cases}$$

Each function  $f_n(x)$  is bounded (by  $n$ ). Limit function of  $f(x)$  is  $|x|$ , which is unbounded.  $\square$

#### Problem 4

Prove that if  $f_n \rightarrow f$  uniformly and  $g_n \rightarrow g$  uniformly on a set  $E$  then:  
a)  $f_n + g_n \rightarrow f + g$  uniformly on  $E$ .

*Proof.* Since for any fixed  $x$  sequences  $f_n(x) \rightarrow f(x)$  and  $g_n(x) \rightarrow g(x)$  as  $n \rightarrow \infty$ :

$$\lim f_n(x) + g_n(x) = \lim f_n(x) + \lim g_n(x)$$

Fix  $\epsilon > 0$ . By the uniformity of convergence, for  $\epsilon$  there exist  $K, M \in \mathbb{N}$  such that for all  $x \in E$ :

$$|f_k(x) - f(x)| \leq \frac{\epsilon}{2}$$

for  $k \geq K$  and

$$|g_m(x) - g(x)| \leq \frac{\epsilon}{2}$$

for  $m \geq M$ . Suppose, without loss of generality, that  $K \geq M$ , then by Triangle inequality:

$$|(f_k(x) + g_k(x)) - (f(x) + g(x))| \leq |f_k(x) - f(x)| + |g_k(x) - g(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

We conclude that  $f_n(x) + g_n(x) \rightarrow f(x) + g(x)$  uniformly on  $E$ .  $\square$

b) If each  $f_n$  and each  $g_n$  is bounded on  $E$ , prove that  $f_n g_n \rightarrow fg$  uniformly.

*Proof.* Since each  $f_n$  and  $g_n$  are bounded, and from uniformity of convergence, by Problem 3 we have that  $f_n$  and  $g_n$  are uniformly bounded, i.e. there exist some  $M_f$  and  $M_g$  such that  $|f_n| \leq M_f$ ,  $|f| \leq M_f$ ,  $|g_n| \leq M_g$ ,  $|g| \leq M_g$  for all  $n \in \mathbb{N}$ .

Fix  $\epsilon > 0$ . From the uniformity of convergence of  $f_n$  and  $g_n$  we have

$$|f_k - f| \leq \frac{\epsilon}{M_f + M_g}$$

$$|g_k - g| \leq \frac{\epsilon}{M_f + M_g}$$

for all  $k > N$  for some  $N \in \mathbb{N}$ .

We will now show that all elements of sequence  $f_n(x)g_n(x)$  after the  $N$ -th one are within  $\epsilon$  of  $f(x)g(x)$  for arbitrary  $x \in E$ :

$$\begin{aligned} |f_k g_k - f g| &= |f_k g_k - f_k g + f_k g - f g| = |f_k(g_k - g) + g(f_k - f)| \\ &\leq |f_k| |g_k - g| + |g| |f_k - f| \\ &\leq |f_k| \left( \frac{\epsilon}{M_f + M_g} \right) + |g| \left( \frac{\epsilon}{M_f + M_g} \right) \\ &= \frac{\epsilon}{M_f + M_g} (|f_k| + |g|) \leq \frac{\epsilon}{M_f + M_g} (M_f + M_g) = \epsilon \end{aligned}$$

Therefore,  $f_n g_n \rightarrow fg$  uniformly. □

## Problem 5

Define two sequences  $(f_n)$  and  $(g_n)$  as follows:

$$f_n(x) = x \left( 1 + \frac{1}{n} \right) \text{ if } x \in \mathbb{R}, n \geq 1$$

$$g_n(x) = \begin{cases} \frac{1}{n}, & \text{if } x = 0 \text{ or } x \text{ is irrational} \\ q + \frac{1}{n}, & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ in reduced form} \end{cases}$$

Show that, on any interval  $[a, b]$  both  $f_n$  and  $g_n$  converge uniformly, but  $f_n g_n$  does not converge uniformly.

*Proof.* We can see that  $f_n(x)$  converges pointwise to  $f(x) = x$ . For  $x \in [a, b]$  we have

$$|f_n(x) - f(x)| = \left| x + \left( 1 + \frac{1}{n} \right) - x \right| = \left| \frac{x}{n} \right| \leq \frac{\max(|a|, |b|)}{n},$$

which can be made arbitrarily small by setting  $n$  sufficiently large. Therefore,  $f_n \rightarrow f$  uniformly. We also note that  $f_n$  and  $f$  are bounded on  $[a, b]$ .

We examine pointwise convergence of  $g_n(x)$ .

For  $x = 0$  and irrational  $x$ :

$$g_x^a = \lim g_n(x) = \lim \frac{1}{n} = 0.$$

For rational  $x \neq 0$  (in reduced form:  $x = \frac{p}{q}$ ):

$$g_x^b = \lim g_n(x) = \lim q + \frac{1}{n} = q.$$

For both rational and irrational  $x \in [a, b]$  we have:

$$|g_n(x) - g(x)| = \frac{1}{n},$$

which can be made arbitrarily small by setting  $n$  sufficiently large. Therefore,  $g_n \rightarrow g$  uniformly. However, we note that neither  $f_n$  nor  $f$  are bounded on  $[a, b]$  since neighbourhood of any irrational number contains infinitely many rationals with arbitrarily large denominator in reduced form.

$$f_n(x)g_n(x) = \begin{cases} x \left(1 + \frac{1}{n}\right) \frac{1}{n}, & \text{if } x = 0 \text{ or } x \text{ is irrational} \\ \frac{p}{q} \left(1 + \frac{1}{n}\right) \left(q + \frac{1}{n}\right), & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{x}{n} \left(1 + \frac{1}{n}\right), & \text{if } x = 0 \text{ or } x \text{ is irrational} \\ \frac{p}{qn} \left(1 + \frac{1}{n} + \frac{1}{q}\right), & \text{otherwise} \end{cases}$$

For irrational  $x$  and  $x = 0$ :

$$|f_n(x)g_n(x) - f(x)g(x)| = \left| \frac{x}{n} \left(1 + \frac{1}{n}\right) - x \cdot 0 \right| \leq \left| \max(a, b) \left(\frac{1}{n} + \frac{1}{n^2}\right) \right|,$$

which can be made arbitrarily small by setting  $n$  sufficiently large.

For rational  $x \neq 0$ :

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &= \left| \frac{p}{q} \left(1 + \frac{1}{n}\right) \left(q + \frac{1}{n}\right) - p \right| \\ &= \left| \frac{p}{qn} + \frac{p}{qn^2} + \frac{p}{n} \right| = \frac{1}{n} \left| x + \frac{x}{n} + p \right| \end{aligned}$$

For any interval  $[a, b]$  and sufficiently large  $n \in \mathbb{N}$  we can always find  $x = \frac{p}{q} \in [a, b]$  such that  $p \geq n$ . Thus, above expression cannot be less than 1 for some values of  $x$ . Therefore,  $f_n g_n$  does not converge uniformly to  $f g$  on interval  $[a, b]$ .  $\square$

## Problem 6

Assume that  $(f_n)$  is a uniformly bounded sequence of functions converging uniformly to  $f$  on a set  $E$ , define  $M$  as in Problem 3. Let  $g$  be continuous on  $[-M, M]$ , prove that  $g \circ f_n \rightarrow g \circ f$  uniformly on  $E$ .

*Proof.* Since  $g$  is continuous on a compact set, it must be uniformly continuous (Rudin 4.19). Fix  $\epsilon > 0$ . There must exist  $\delta > 0$  such that for any  $|f_n(x) - f(x)| \leq \delta$  it must hold that  $|g \circ f_n(x) - g \circ f(x)| \leq \epsilon$ . Since  $f_n \rightarrow f$  uniformly, there exists  $n \in \mathbb{N}$  such that for all  $x \in E$  and all  $m \geq N$ :

$$|f_m(x) - f(x)| \leq \delta$$

Therefore:

$$|g \circ f_m(x) - g \circ f(x)| \leq \epsilon$$

We conclude that  $g \circ f_n \rightarrow g \circ f$  uniformly. □

## Problem 7

a) Show that the sequence of polynomials defined inductively by

$$\begin{aligned} P_0(x) &= 0, \\ P_{n+1}(x) &= P_n(x) + \frac{1}{2}(x - P_n^2(x)) \end{aligned}$$

converges uniformly on the interval  $[0, 1]$  to the function  $f(x) = \sqrt{x}$ .

*Proof.* We first notice that  $P_n$  is increasing:

$$\sqrt{x} \geq P_n(x) \Rightarrow x \geq P_n^2(x) \Rightarrow \frac{1}{2}(x - P_n^2(x)) = P_{n+1}(x) - P_n(x) \geq 0$$

We then notice that if  $P_n \leq \sqrt{x}$  then  $P_{n+1} \leq \sqrt{x}$ . This is true for  $n = 0$ . We proceed by induction on  $n$ . Suppose  $0 \leq P_n \leq \sqrt{x} \leq 1$ , then

$$\begin{aligned} P_{n+1}(x) &= P_n(x) + \frac{1}{2}(x - P_n^2(x)) \\ P_{n+1}(x) &\leq \sqrt{x} + \frac{1}{2}(x - P_n^2(x)) \\ 0 &\leq \frac{1}{2}(x - P_n^2(x)) \leq \sqrt{x} - P_{n+1}(x) \\ P_{n+1}(x) &\leq \sqrt{x} \end{aligned}$$

Therefore  $\sqrt{x}$  is an upper bound for  $P_n(x)$ .

Monotonic and bounded sequence  $P_n(x)$  must converge for each  $x$ . Suppose  $P_n(x) \rightarrow L_x$ . We have:

$$\begin{aligned} L_x &= \lim P_{n+1}(x) = \lim \left( P_n(x) + \frac{1}{2}(x - P_n(x)^2) \right) = L_x + \frac{1}{2}(x - L_x^2) \\ x - L_x^2 &= 0 \\ L_x &= \pm\sqrt{x} \end{aligned}$$

Limit of  $P_n$  cannot be negative, so we conclude that  $P_n$  converges to  $f(x) = +\sqrt{x}$  pointwise. Moreover, each  $P_n(x)$  must be continuous since it is a polynomial and  $f(x) = +\sqrt{x}$  is also continuous on  $[0, 1]$ . Monotonically increasing sequence of continuous functions over a compact set  $([0, 1])$  converging pointwise to a continuous function converges uniformly. Therefore,  $P_n(x) \rightarrow +\sqrt{x}$  uniformly.  $\square$

b) Deduce that there exists a sequence of polynomials converging uniformly on  $[-1, 1]$  to the function  $f(x) = |x|$ .

*Proof.* Sequence of polynomials defined on  $[0, 1]$  as recurrence relation:

$$\begin{aligned} P_0(x) &= 0, \\ P_{n+1}(x) &= P_n(x) + \frac{1}{2}(x^2 - P_n^2(x)) \end{aligned}$$

converges to  $|x|$ . This is easy to see since  $P_n(x)$  as defined in Problem 7a) converges to the positive root of  $\sqrt{x}$ . Thus  $P_n(x^2)$  converges to  $+\sqrt{x^2} = |x|$  on  $[0, 1]$ .  $\square$