18.100B: Problem Set 9

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Problem 1

Let $f_n(x) = \frac{1}{nx+1}$ and $g_n(x) = \frac{x}{nx+1}$ for $x \in (0,1)$ and $n \in \mathbb{N}$. Prove that f_n converges pointwise but not uniformly on $(0,1)$, and that g_n converges uniformly on $(0,1)$.

Proof. We can see that $f_n(x) = \frac{1}{nx+1}$ converges pointwise, since for any $x \in (0,1)$:

$$\lim_{n\to\infty} \frac{1}{nx+1} = 0$$

However, f is not uniformly convergent since for any $N > 1$ we can find point $x_N = \frac{1}{N}$ such that $f_N(x_N) = \frac{1}{2}$ and

$$|f_N(x_N) - f(x_N)| = \frac{1}{2}$$

Clearly, for any $\epsilon < \frac{1}{2}$ criterion of uniform convergence fails at a_N for any $N > 1$.

We can also see that $g_n(x) = \frac{x}{nx+1}$ converges pointwise, since for any $x \in (0,1)$:

$$\lim_{n\to\infty} \frac{x}{nx+1} = \lim_{n\to\infty} \frac{1}{n+\frac{1}{x}} = 0$$

Then we notice that

$$x < 1 \;\Rightarrow\; \frac{1}{n+\frac{1}{x}} < \frac{1}{n+1} \;\Rightarrow\; \frac{x}{nx+1} < \frac{1}{n+1}$$

Therefore, for any $\epsilon > 0$ we can choose N such that $\frac{1}{N+1} < \epsilon$. Then, for any $x \in (0,1)$ and for any $m \geq N$:

$$|g_m(x) - 0| = g_m(x) < \frac{1}{m+1} \leq \frac{1}{N+1} < \epsilon.$$

Therefore, $g(x)$ converges uniformly.

□

Problem 2

Let $f_n(x) = \frac{x}{1+nx^2}$ if $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Find the limit function f of the sequence (f_n) and the limit function g of the sequence (f'_n).

Proof. We notice that

$$\frac{x}{1+nx^2} = \frac{1}{\frac{1}{x}+nx},$$

which for arbitrary $x \neq 0$ goes to 0 as n goes to infinity. We check $x = 0$ separately and confirm that

$$f = \lim_{n\to\infty} f_n = \lim_{n\to\infty} \frac{x}{1+nx^2} = 0$$

To find limit function of f'_n we consider derivative function of f_n:

$$f'_n = \frac{(1+nx^2) - x(2xn)}{(1+nx^2)^2} = \frac{1-nx^2}{1+2nx^2+n^2x^4}$$

We consider two cases ($x = 0$ and $x \neq 0$) and use L'Hospital's rule, which is valid since both numerator and denominator of f'_n are differentiable on $\mathbb{R}$.

$$g = \lim_{n\to\infty} f'_n = \lim_{n\to\infty} \frac{1-nx^2}{1+2nx^2+n^2x^4} = \begin{cases} 1, & \text{for } x = 0 \\ \lim_{n\to\infty} \frac{-x^2}{2x^2+2nx^4} = 0, & \text{otherwise} \end{cases}$$

□

Prove that $f'(x)$ exists for every x but that $f'(0) \neq g(0)$. For what values of x is $f'(x) = g(x)$?

Proof. Derivative of constant function f exists and is equal to 0 for any $x \in \mathbb{R}$, including $x = 0$. Thus, $f'(x) = g(x)$ for any $x \neq 0$. However, $g(0) = 1$.

□

In what subintervals of $\mathbb{R}$ does $f_n \to f$ uniformly? In what subintervals of $\mathbb{R}$ does $f'_n \to g$ uniformly?

Proof. For any n function f_n is continuous on $\mathbb{R}$ and the following holds:

$$f_n(x) \to 0 \text{ as } x \to \pm\infty; \quad f(0) = 0$$

Derivative of f_n is also continuous on $\mathbb{R}$, thus we can find minimum and maximum of f_n by setting f_n' to 0:

$$\frac{1-nx^2}{1+2nx^2+n^2x^4} = 0$$
$$nx^2 = 1$$
$$x = \pm\frac{1}{\sqrt{n}}$$

More specifically, f_n at $x = \frac{1}{\sqrt{n}}$ attains its global maximum. Maximum of f_n is:

$$f_n^{\max} = \frac{1}{\sqrt{n}(1+\frac{n}{n})} = \frac{1}{2\sqrt{n}}$$

Denote $s_n = \sup_{x\in\mathbb{R}} |f_n(x) - f(x)|$. Based on the above, $s_n = f_n\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{2\sqrt{n}}$.

For any $\epsilon > 0$ we can find $n \in \mathbb{N}$ such that $s_n < \epsilon$, thus f_n converges uniformly to f on $\mathbb{R}$.

Since $f_n'(0) = 1$ and f_n' is continuous, image of every open neighbourhood of $x = 0$ contains points arbitrarily close to 1. For such points $|f_n'(x) - f(x)|$ is arbitrarily close to 1 and thus $s_n = 1$. Clearly, f_n' cannot be uniformly convergent on any set that has $x = 0$ as a limit point.

Otherwise we have:

$$|f_n'(x) - f(x)| = \left|\frac{1-nx^2}{(1+nx^2)^2}\right|,$$

which can be made arbitrarily small for any x by setting n sufficiently large (degree of the polynomial in the denominator is larger than the degree of the polynomial in the numerator).

□

Problem 3

Let $\mathcal{M}$ be a metric space and (f_n) a sequence of functions defined on a subset $E \subseteq \mathcal{M}$. We say that (f_n) is uniformly bounded if there exists a constant M such that $|f_n(x)| \leq M$ for every $n \in N$ and $x \in E$.
Prove that if (f_n) is a sequence of bounded real valued functions that converges uniformly to a function f, then (f_n) is uniformly bounded. Prove that in this case f is also bounded.

Proof. Choose arbitrary $\epsilon > 0$. Since f_n converges to f uniformly, there exists some $N \in \mathbb{N}$ such that for all $m \geq N$ and $x \in E$ the following holds:

$$|f_m(x) - f(x)| \leq \epsilon$$

Since $f_m(x)$ is bounded, $f(x)$ must also be bounded. We also have finitely many functions $f_k(x)$ such that $k < N$. For each $x \in E$ consider an upper bound of $|f_n(x)|$, which can be constructed as follows:

$$g(x) = \max\left(\max|f_k(x)|, \ |f(x)| + \epsilon \right)$$

Since each $f_n(x)$ is bounded and $f(x)$ is bounded, $g(x)$ must also be bounded. Any upper bound of $g(x)$, for example $M = \sup_{x \in E} g(x)$ is an upper bound for any $|f_n(x)|$ $(x \in E, n \in \mathbb{N})$ by construction. Therefore, f_n is uniformly bounded.

□

If (f_n) is a sequence of bounded functions converging pointwise to f, need f be bounded?

Not necessarily.

Proof. Counterexample: Consider function

$$f_n(x) = \begin{cases} |x|, & \text{if } x \le n \\ n, & \text{otherwise} \end{cases}$$

Each function $f_n(x)$ is bounded (by n). Limit function of $f(x)$ is $|x|$, which is unbounded.

□

Problem 4

Prove that if $f_n \to f$ uniformly and $g_n \to g$ uniformly on a set E then:
a) $f_n + g_n \to f + g$ uniformly on E.

Proof. Since for any fixed x sequences $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ as $n \to \infty$:

$$\lim f_n(x) + g_n(x) = \lim f_n(x) + \lim g_n(x)$$

Fix $\epsilon > 0$. By the uniformity of convergence, for ϵ there exist $K, M \in \mathbb{N}$ such that for all $x \in E$:

$$|f_k(x) - f(x)| \le \frac{\epsilon}{2}$$

for $k \ge K$ and

$$|g_m(x) - g(x)| \le \frac{\epsilon}{2}$$

for $m \ge M$. Suppose, without loss of generality, that $K \ge M$, then by Triangle inequality:

$$|(f_k(x) + g_k(x)) - (f(x) + g(x))| \le |f_k(x) - f(x)| + |g_k(x) - g(x)| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

We conclude that $f_n(x) + g_n(x) \to f(x) + g(x)$ uniformly on E.

□

b) If each f_n and each g_n is bounded on E, prove that $f_n g_n \to fg$ uniformly.

Proof. Since each f_n and g_n are bounded, and from uniformity of convergence, by Problem 3 we have that f_n and g_n are uniformly bounded, i.e. there exist some M_f and M_g such that $|f_n| \leq M_f$, $|f| \leq M_f$, $|g_n| \leq M_g$, $|g| \leq M_g$ for all $n \in \mathbb{N}$.

Fix $\epsilon > 0$. From the uniformity of convergence of f_n and g_n we have

$$|f_k - f| \leq \frac{\epsilon}{M_f + M_g}$$

$$|g_k - g| \leq \frac{\epsilon}{M_f + M_g}$$

for all $k > N$ for some $N \in \mathbb{N}$.

We will now show that all elements of sequence $f_n(x)g_n(x)$ after the N-th one are within ϵ of $f(x)g(x)$ for arbitrary $x \in E$:

$$\begin{aligned}
|f_k g_k - fg| &= |f_k g_k - f_k g + f_k g - fg| = |f_k(g_k - g) + g(f_k - f)| \\
&\leq |f_k||g_k - g| + |g||f_k - f| \\
&\leq |f_k| \left(\frac{\epsilon}{M_f + M_g} \right) + |g| \left(\frac{\epsilon}{M_f + M_g} \right) \\
&= \frac{\epsilon}{M_f + M_g}(|f_k| + |g|) \leq \frac{\epsilon}{M_f + M_g}(M_f + M_g) = \epsilon
\end{aligned}$$

Therefore, $f_n g_n \to fg$ uniformly.

□

Problem 5

Define two sequences (f_n) and (g_n) as follows:

$$f_n(x) = x\left(1 + \frac{1}{n}\right) \text{ if } x \in \mathbb{R}, n \geq 1$$

$$g_n(x) = \begin{cases} \frac{1}{n}, & \text{if } x = 0 \text{ or } x \text{ is irrational} \\ q + \frac{1}{n}, & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ in reduced form} \end{cases}$$

Show that, on any interval $[a, b]$ both f_n and g_n converge uniformly, but $f_n g_n$ does not converge uniformly.

Proof. We can see that $f_n(x)$ converges pointwise to $f(x) = x$. For $x \in [a, b]$ we have

$$|f_n(x) - f(x)| = \left| x + \left(1 + \frac{1}{n}\right) - x \right| = \left| \frac{x}{n} \right| \leq \frac{\max(|a|, |b|)}{n},$$

which can be made arbitrarily small by setting n sufficiently large. Therefore, $f_n \to f$ uniformly. We also note that f_n and f are bounded on $[a, b]$.

We examine pointwise convergence of $g_n(x)$.

For $x = 0$ and irrational x:

$$g_x^a = \lim g_n(x) = \lim \frac{1}{n} = 0.$$

For rational $x \neq 0$ (in reduced form: $x = \frac{p}{q}$):

$$g_x^b = \lim g_n(x) = \lim q + \frac{1}{n} = q.$$

For both rational and irrational $x \in [a, b]$ we have:

$$|g_n(x) - g(x)| = \frac{1}{n},$$

which can be made arbitrarily small by setting n sufficiently large. Therefore, $g_n \to g$ uniformly. However, we note that neither f_n nor f are bounded on $[a, b]$ since neighbourhood of any irrational number contains infinitely many rationals with arbitrarily large denominator in reduced form.

$$f_n(x)g_n(x) = \begin{cases} x\left(1 + \frac{1}{n}\right)\frac{1}{n}, & \text{if } x = 0 \text{ or } x \text{ is irrational} \\ \frac{p}{q}\left(1 + \frac{1}{n}\right)\left(q + \frac{1}{n}\right), & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{x}{n}\left(1 + \frac{1}{n}\right), & \text{if } x = 0 \text{ or } x \text{ is irrational} \\ \frac{p}{qn}\left(1 + \frac{1}{n} + \frac{1}{q}\right), & \text{otherwise} \end{cases}$$

For irrational x and $x = 0$:

$$|f_n(x)g_n(x) - f(x)g(x)| = \left|\frac{x}{n}\left(1 + \frac{1}{n}\right) - x \cdot 0\right| \leq \left|\max(a, b)\left(\frac{1}{n} + \frac{1}{n^2}\right)\right|,$$

which can be made arbitrarily small by setting n sufficiently large.

For rational $x \neq 0$:

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &= \left|\frac{p}{q}\left(1 + \frac{1}{n}\right)\left(q + \frac{1}{n}\right) - p\right| \\ &= \left|\frac{p}{qn} + \frac{p}{qn^2} + \frac{p}{n}\right| = \frac{1}{n}\left|x + \frac{x}{n} + p\right| \end{aligned}$$

For any interval $[a, b]$ and sufficiently large $n \in \mathbb{N}$ we can always find $x = \frac{p}{q} \in [a, b]$ such that $p \geq n$. Thus, above expression cannot be less than 1 for some values of x. Therefore, $f_n g_n$ does not converge uniformly to fg on interval $[a, b]$. $\square$

Problem 6

Assume that (f_n) is a uniformly bounded sequence of functions converging uniformly to f on a set E, define M as in Problem 3. Let g be continuous on $[-M, M]$, prove that $g \circ f_n \to g \circ f$ uniformly on E.

Proof. Since g is continuous on a compact set, it must be uniformly continuous (Rudin 4.19). Fix $\epsilon > 0$. There must exist $\delta > 0$ such that for any $|f_n(x) - f(x)| \leq \delta$ it must hold that $|g \circ f_n(x) - g \circ f(x)| \leq \epsilon$. Since $f_n \to f$ uniformly, there exists $n \in \mathbb{N}$ such that for all $x \in E$ and all $m \geq N$:

$$|f_m(x) - f(x)| \leq \delta$$

Therefore:

$$|g \circ f_m(x) - g \circ f(x)| \leq \epsilon$$

We conclude that $g \circ f_n \to fg$ uniformly. $\square$

Problem 7

a) Show that the sequence of polynomials defined inductively by

$$\begin{gathered} P_0(x) = 0, \\ P_{n+1}(x) = P_n(x) + \frac{1}{2}\left(x - P_n^2(x)\right) \end{gathered}$$

converges uniformly on the interval $[0, 1]$ to the function $f(x) = \sqrt{x}$.

Proof. We first notice that P_n is increasing:

$$\sqrt{x} \geq P_n(x) \;\; \Rightarrow \;\; x \geq P_n^2(x) \;\; \Rightarrow \;\; \frac{1}{2}\left(x - P_n^2(x)\right) = P_{n+1}(x) - P_n(x) \geq 0$$

We then notice that if $P_n \leq \sqrt{x}$ then $P_{n+1} \leq \sqrt{x}$. This is true for $n = 0$. We proceed by induction on n. Suppose $0 \leq P_n \leq \sqrt{x} \leq 1$, then

$$\begin{gathered} P_{n+1}(x) = P_n(x) + \frac{1}{2}\left(x - P_n^2(x)\right) \\ P_{n+1}(x) \leq \sqrt{x} + \frac{1}{2}\left(x - P_n^2(x)\right) \\ 0 \leq \frac{1}{2}\left(x - P_n^2(x)\right) \leq \sqrt{x} - P_{n+1}(x) \\ P_{n+1}(x) \leq \sqrt{x} \end{gathered}$$

Therefore $\sqrt{x}$ is an upper bound for $P_n(x)$.

Monotonic and bounded sequence $P_n(x)$ must converge for each x. Suppose $P_n(x) \to L_x$. We have:

$$L_x = \lim P_{n+1}(x) = \lim \left(P_n(x) + \frac{1}{2}(x - P_n(x)^2) \right) = L_x + \frac{1}{2}(x - L_x^2)$$
$$x - L_x^2 = 0$$
$$L_x = \pm\sqrt{x}$$

Limit of P_n cannot be negative, so we conclude that P_n converges to $f(x) = +\sqrt{x}$ pointwise. Moreover, each $P_n(x)$ must be continuous since it is a polynomial and $f(x) = +\sqrt{x}$ is also continuous on $[0, 1]$. Monotonically increasing sequence of continious functions over a compact set ($[0, 1]$) converging pointwise to a continous function converges uniformly. Therefore, $P_n(x) \to +\sqrt{x}$ uniformly.

□

b) Deduce that there exists a sequence of polynomials converging uniformly on $[-1, 1]$ to the function $f(x) = |x|$.

Proof. Sequence of polynomials defined on $[0, 1]$ as recurrence relation:

$$P_0(x) = 0,$$
$$P_{n+1}(x) = P_n(x) + \frac{1}{2}\left(x^2 - P_n^2(x)\right)$$

converges to $|x|$. This is easy to see since $P_n(x)$ as defined in Problem 7a) converges to the positive root of $\sqrt{x}$. Thus $P_n(x^2)$ converges to $+\sqrt{x^2} = |x|$ on $[0, 1]$.

□