18.100B: Problem Set 10

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Problem 1

Let (f_n) be the sequence of functions on $\mathbb{R}$ defined as follows

$$f_0(t) = \sin t \quad \text{and} \quad f_{n+1}(t) = \frac{2}{3} f_n(t) + 1 \quad \text{for } n \in \mathbb{N}$$

Show that $f_n \to 3$ uniformly on $\mathbb{R}$. What can you say if we choose $f_0(t) = t^2$?
Hint: Consider first the map $T(x) = \frac{2}{3}x + 1$ on $\mathbb{R}$.

Proof. We first note that T is a contraction mapping on $\mathbb{R}$, i.e. for any $a, b \in \mathbb{R}$:

$$T(a) - T(b) = \frac{2}{3}a + 1 - \frac{2}{3}b - 1 = \frac{2}{3}(a - b)$$

Therefore, there exists a fixed point of T, denote it L_t.

Consider sequence of functions that is identical to $f_n(t)$ but with arbitrary starting element x:

$$g_0(x) = x \quad \text{and} \quad g_{n+1}(x) = T(g_n(x)) \quad \text{for } n \in \mathbb{N}$$

Sequence $g_n(x)$ must converge to L_x:

$$\lim g_n(x) = L_x = T(L_x)$$

Furthermore:

$$\lim f_n(x) = \lim f_{n+1}(x) = \lim \left(\frac{2}{3} f_n(x) + 1 \right)$$
$$L_x = \frac{2}{3} L_x + 1$$
$$L_x = 3$$

Therefore, $g_n(x)$ converges to 3 pointwise.

We can see that g_n is monotone by considering the difference between subsequent elements of sequence g_n:

$$g_{n+1}(x) - g_n(x) = \frac{2}{3}g_n(x) + 1 - g_n(x) = \frac{3 - g_n(x)}{3}$$

Specifically, g_n is monotonically increasing in case $g_n(0) < 3$ and monotonically decreasing if $g_n(0) > 0$. Each g_n is a polynomial, thus continuous. Limit function $g(x) = 3$ is also continuous. From this we conclude that sequence g_n converges uniformly if domain of each g_n is compact.

If range of g_n is compact, then domain of g_{n+1} must be also compact since g_n is continuous. By induction, if range of g_0 is compact, domains of all g_n are also compact and thus g_n converges uniformly.

This is the case for $g_n(\sin t)$, therefore $g_n(\sin t) \to 3$ uniformly. However, this is not the case for $g_n(t^2)$ since range of $g_n(t^2) = t^2$ is $[0, \infty)$, thus not compact. $\square$

Problem 2

Suppose $\varphi : [0, \infty) \to \mathbb{R}$ is continuous and satisfies

$$0 \leq \varphi(t) \leq \frac{t}{2+t} \quad (t \geq 0)$$

Define the sequence (f_n) by setting $f_0(t) = \varphi(t)$ and $f_{n+1}(t) = \varphi(f_n(t))$ for $t \geq 0$ and $n \in \mathbb{N}$. Prove that the series $F(t) = \sum_{n=0}^{\infty} f_n(t)$ converges for every $t \geq 0$ and that F is continuous on $n = [0, \infty)$.

Proof. We first note that $f_n(t)$ is decreasing:

$$\frac{f_{n+1}(t)}{f_n(t)} = \frac{\varphi(f_n(t))}{f_n(t)} \leq \frac{f_n(t)}{f_n(t)(2 + f_n(t))} = \frac{1}{2 + f_n(t)} \leq \frac{1}{2}$$

and

$$f_{n+1}(t) \leq \frac{1}{2} f_n(t)$$

We also note that $f_1(t) \leq 1$, therefore:

$$f_{n+1}(t) \leq \frac{1}{2^n} f_1(t) \leq \frac{1}{2^n}$$

We apply Weierstrass M-test (Rudin 7.10) to examine convergence of the series $s(t) = \sum_{n=0}^{\infty} f_n(t)$:

$$f_{n+1}(t) \leq \frac{1}{2^n} \text{ and } \sum_{n=0}^{\infty} \frac{1}{2^n} \to 0 \text{ as } n \to \infty$$

We conclude that $s(t) \to F(t)$ uniformly.

Since $\varphi(t)$ is continuous each $f_n(t)$ must be continuous; thus partial sums of $s(t)$ must also be continuous. Sequence of continuous functions converges uniformly to a continuous function. Thus, $F(t)$ is continuous.

$\square$

Problem 3

> Does $f(t) = \sum_{k=1}^{\infty} \sin^2\left(\frac{t}{k}\right)$ define a differentiable function on $\mathbb{R}$?

Proof. Choose an arbitrary closed interval $[a, b] \subseteq \mathbb{R}$ and denote $s = \max(|a|, |b|)$. Denote $g_k(t) = \sin^2\left(\frac{t}{k}\right)$. First we notice that for each $g_k(t)$ on $[a, b]$:

$$|g_k(t)| = \left|\sin\left(\frac{t}{k}\right)^2\right| \leq \left(\frac{t}{k}\right)^2 \leq \frac{s^2}{k^2}$$

Here we use the fact that $|\sin x| \leq |x|$. Series $\sum_{k=1}^{\infty} \frac{s^2}{k^2} = s^2 \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, therefore $f_k(t)$ converges pointwise on $[a, b]$.

Function g_k is differentiable on $[a, b]$ for any k:

$$g_k'(t) = \frac{2}{k} \sin\left(\frac{t}{k}\right) \cos\left(\frac{t}{k}\right)$$

Thus any partial sum of f_k is also differentiable on $[a, b]$.

We notice that for each $g_k'(t)$ on $[a, b]$:

$$\left|\frac{2}{k} \sin\left(\frac{t}{k}\right) \cos\left(\frac{t}{k}\right)\right| \leq \left|\frac{2t}{k^2}\right| \leq \frac{2s}{k^2}$$

Series $\sum_{k=1}^{\infty} \frac{s}{k^2} = s \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, therefore $\sum g_k'(t)$ converges uniformly on $[a, b]$.

By Rudin 7.17 we have that f is differentiable on $[a, b]$. Therefore, f is differentiable on $\mathbb{R}$.

$\square$

Problem 4

> Suppose (f_n) is a sequence of continuous functions such that $f_n \to f$ uniformly on a set E. Prove that
>
> $$\lim_{n\to\infty} f_n(x_n) = f(x)$$
>
> for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. Is the converse of this true?

Proof. Fix $\epsilon > 0$. Since $f_n \to f$ uniformly, there must exist M such that for all $m > M$ for any $z \in E$:

$$\|f_m(z) - f(z)\| \leq \frac{\epsilon}{2}$$

By Rudin 7.12 f must be continuous. Since f is continuous, and $x_n \to x$ there exists K such that for all $k \geq K$:

$$\|f(x_k) - f(x)\| \leq \frac{\epsilon}{2}$$

Combining the above:

$$\|f_m(x_k) - f(x)\| \leq \|f_m(x_k) - f(x_k)\| + \|f(x_k) - f(x)\| \leq \epsilon$$

Taking $N = \max(M, K)$ we have that for all $n \geq N$:

$$\|f_n(x_n) - f(x)\| \leq \epsilon,$$

which proves that $\lim_{n\to\infty} f_n(x_n) = f(x)$.

$\square$

Now consider the converse. As we will see, the fact that for every convergent sequence $x_n \to x$ the following holds:

$$\lim_{n\to\infty} f_n(x_n) = f(x),$$

it does not necessarily imply that convergent sequence of continuous functions f_n converges uniformly, even if the limit function of f_n is continuous.

Proof. Counterexample: Consider sequence of functions $\frac{x^2}{n}$, which converges to 0.

Take arbitrary convergent sequence $x_n \to x$ with $x_n, x \in \mathbb{R}$. Fix $\epsilon > 0$. Since $x_n \to x$ there exists M such that for all $m \geq M$:

$$\|x_m - x\| \leq \epsilon$$
$$\|x_m\| \leq \|x\|\|+\epsilon\|$$

We will now prove that $f_n(x_n) \to f(x)$:

$$\|f_m(x_m) - f(x)\| = \left\| \frac{x_m^2}{m} - 0 \right\| \leq \frac{(\|x\| + \epsilon)^2}{m}.$$

The last expression can be set arbitrarily close to 0 by choosing sufficiently high m.

However, it is easy to see that convergence of $\frac{x^2}{n}$ is not uniform on $\mathbb{R}$.

$\square$

Problem 5

Suppose (f_n) is a sequence of real-valued functions that are Riemann-integrable on all compact subintervals of $[0,\infty)$. Assume further that:
a) $f_n \to 0$ uniformly on every compact subset of $[0,\infty)$;
b) $0 \le f_n(t) \le e^{-t}$ for all $t \ge 0$ and $n \in \mathbb{N}$.
Prove that

$$\lim_{n\to\infty} \int_0^\infty f_n(t)dt = 0,$$

where the improper integral $\int_0^\infty f_n(t)dt$ is defined as $\lim_{b\to\infty} \int_0^b f_n(t)dt$.

Proof. Fix sufficiently small $\epsilon \in (0,1)$. Denote $\epsilon_s = \sqrt{\epsilon}$. Since $f_n \to 0$ uniformly, there exists M such that for all $m \ge M$:

$$|f_m(t) - 0| \le \epsilon_s$$

Consider point x such that $e^{-x} = \epsilon_s$:

$$x = -\ln(\epsilon_s)$$

We notice that

$$\left|\int_0^\infty f_m\right| = \left|\int_0^x f_m + \int_x^\infty f_m\right| \le \left|\int_0^x f_m\right| + \left|\int_x^\infty f_m\right|$$

We evaluate the first summand:

$$\left|\int_0^x f_m(t)dt\right| \le |x\epsilon_s| = |-\ln(\epsilon_s)\epsilon_s| = -\ln(\epsilon_s)\epsilon_s$$

We evaluate the second summand. Since $f_n(t) \le e^{-t}$:

$$\left|\int_x^\infty f_m\right| \le \left|\int_x^\infty e^{-t}dt\right| = \left|\int_{-\ln(\epsilon_s)}^\infty e^{-t}dt\right| = \left|0 - e^{-\ln \epsilon_s}\right| = \epsilon_s$$

We conclude that

$$\left|\int_0^\infty f_m\right| \le -\ln(\epsilon_s)\epsilon_s + \epsilon_s = \epsilon_s(1 - \ln(\epsilon_s))$$

We notice that for $0 < \epsilon < 1$:

$$\epsilon_s(1 - \ln(\epsilon_s)) < \epsilon_s^2 < \epsilon$$

Therefore $\int_0^\infty f_n \to 0$ as $n \to \infty$.

$\square$

Moreover, give an explicit example for a sequence (f_n), so that condition b) does not hold and the conclusion above fails.

Answer: $g_n(t) = \frac{1}{n(x+1)}$.

Proof. Sequence of functions g_n is Riemann-integrable on all compact subintervals of $[0, \infty)$ and converges uniformly to 0. However each improper integral $\int_0^\infty g_n$ diverges, therefore the conclusion of $\int_0^\infty g_n$ having a limit as $n \to \infty$ fails.

□

Problem 6

Suppose (f_n) is an equicontinuous sequence of functions on a compact set K, and $f_n \to f$ pointwise on K. Prove that $f_n \to f$ uniformly on K.

Proof. Fix $\epsilon > 0$. Since f_n is equicontinuous, there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ and $a, b \in K$ with $d(a, b) < \delta$ we have:

$$\|f_n(a) - f_n(b)\| < \frac{\epsilon}{3} \tag{1}$$

We also note that:

$$\|f(a) - f(b)\| = \left\|\lim_{n\to\infty} f_n(a) - \lim_{n\to\infty} f_n(b)\right\| = \lim_{n\to\infty} \|f_n(a) - f_n(b)\| \leq \frac{\epsilon}{3} \tag{2}$$

Consider open cover of K by open balls $B_\delta(x)$ with center t and radius δ. Denote such cover $\mathcal{U}_t$ (indexed over center points of δ-balls). Since K is compact, there exists finite subcover of $\mathcal{U}_t$; denote it $\mathcal{U}_{t_k}$. Since $f_n \to f$ pointwise, for each t_k we can find N_k such that for all $n_k \geq N_k$ we have:

$$\|f_{n_k}(t_k) - f(t_k)\| < \frac{\epsilon}{3}$$

Since there are finitely many t_k, we can select the largest N_k, denote it M. For each $m \geq M$ we have:

$$\|f_m(t_k) - f(t_k)\| < \frac{\epsilon}{3} \tag{3}$$

Consider arbitrary point $x \in K$. It must belong to at least one of the sets from $\mathcal{U}_k$. Choose one such open δ-ball with center t, then we have:

$$\|x - t\| < \delta$$

By (1) we have:

$$\|f_m(x) - f_m(t)\| < \frac{\epsilon}{3}$$

By (2) we have:

$$\|f(t) - f(x)\| < \frac{\epsilon}{3}$$

By (3) we have:

$$\|f_m(t) - f(t)\| < \frac{\epsilon}{3}$$

Finally, we notice that:

$$\|f_m(x) - f(x)\| \leq \|f_m(x) - f_m(t)\| + \|f(t) - f(x)\| + \|f_m(t) - f(t)\| < \epsilon$$

Therefore, $f_n \to f$ uniformly.

□

Problem 7

> Show that any uniformly bounded sequence of differentiable functions on a compact interval with uniformly bounded derivatives has a convergent subsequence.

Denote sequence of functions (f_n) over compact interval K. Suppose $|f_n'(x)| \leq M$.

Proof. We have the result by the Arzela-Ascoli theorem (Rudin 7.25) if the following conditions hold:
1) Each f_n is continuous;
2) Each f_n is pointwise bounded;
3) Sequence (f_n) is equicontinuous.

Condition 1 holds since each f_n is differentiable, thus continuous. Condition 2 holds since sequence (f_n) is uniformly bounded.

Next we will prove condition 3. Fix $\epsilon > 0$. Consider two arbitrary points $a, b \in K, b > a$ such that $d(a,b) < \frac{\epsilon}{M}$. By Mean Value Theorem there exists $c \in K$ such that:

$$f_n'(c) = \frac{f_n(b) - f_n(a)}{b - a},$$

from which we have:

$$f_n(b) - f_n(a) = (b-a)f_n'(c) < \frac{\epsilon}{M}M = \epsilon$$

We conclude that sequence (f_n) is equicontinuous. Thus, (f_n) has a convergent subsequence.

□