18.100B: Problem Set 3

Dmitry Kaysin

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Problem 1

In vector spaces, metrics are usually defined in terms of norms which measure the length of a vector. If V is a vector space defined over $\mathbb{R}$, then a norm is a function from vectors to real numbers, denoted by $\|\cdot\|$ satisfying:

- $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$
- For any $\lambda \in \mathbb{R}, \|\lambda x\| = |\lambda|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$.

Prove that every norm defines a metric.

Proof. For given two vectors $p, q \in V$ and a norm $\|\cdot\|$, metric on V can be defined as $d(p, q) = \|p - q\|$. We will check the following properties of this metric:

- $d(p, q) > 0$ if $p \neq q$; d(p,p)=0
- $d(p, q) = d(q, p)$
- $d(p, q) \leq d(p, r) + d(r, q)$

for $p, q, r \in V$.

We first check that for $p \neq q : d(p, q) = \|p - q\| = \|x\| > 0$ for some $x \in V$. We also note that $d(p, p) = \|p - p\| = \|0\| = 0$.

We then check that $d(p, q) = \|p - q\| = \|-1 \cdot (q - p)\| = |-1|\|q - p\| = d(q, p)$. Finally, we check that

$$d(p, q) \leq d(p, r) + d(r, q)$$

Indeed

$$\|p - q\| \leq \|p - r\| + \|r - q\|$$
$$\|(p - r) + (r - q)\| \leq \|p - r\| + \|r - q\|$$

which is true by the triangle inequality for norms.

□

Problem 2

Let M be a metric space with metric d. Show that d_1 defined by

$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

is also a metric on M.

Proof. We will check whether d_1 satisfies properties of a metric. Let $p,q,r \in M$. Denote $a = d(p,q)$.

We first check that for $p \neq q : d_1(p,q) = 0$.

$$d_1(p,q) = \frac{d(x,y)}{1+d(x,y)} = \frac{a}{1+a} > 0$$

We also check that for $p = q : d_1(p,q) = 0$.

$$d_1(p,q) = \frac{0}{1+0} = 0$$

We then check that $d_1(p,q) = d_1(q,p)$.

$$d(p,q) = d(q,p) = a$$
$$d_1(p,q) = \frac{a}{1+a} = d_1(q,p)$$

We finally check the triangle inequality for d_1:

$$d_1(p,q) \leq d_1(p,r) + d_1(r,q)$$

Let $d(p,q) = a,\ d(p,r) = b,\ d(r,q) = c$. We now prove that

$$\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}$$
$$a(1+b)(1+c) \leq (1+a)(b+c+2bc)$$
$$a + ab + ac + abc \leq b + c + 2bc + ab + ac + 2abc$$
$$a \leq b + c + bc + abc$$

Since d is a metric: $a \leq b + c$:

$$a \leq a + bc + abc$$
$$0 \leq bc + abc$$

Which is always true because $a,b,c > 0$.

Since function d_1 satisfies all properties of a metric, it is a metric.

□

Observe that M itself is bounded in this metric.

Proof. We notice that set $D = \{d_1(p,q) : p,q \in \mathbb{R}_{\geq 0}\}$ is bounded. D is bounded from below by 0 since metric has a property of non-negativity. D is clearly bounded from above by 1.

□

Problem 3

Let A and B be two subsets of a metric space M. Recall that A°, the interior of A, is the set of interior points of A. Prove the following:
a) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

Proof. Consider $a \in A^\circ$ and $b \in B^\circ$. By definition of interior point we can find open neighbourhoods $N(a) \subset A$ and $N(b) \subset B$.

For any sets S, X, Y:

$$S \subset X \Rightarrow Y \subset (X \cup Y)$$

Therefore, $N(a) \subset A \cup B$ and $N(b) \subset A \cup B$. Thus, both points a and b are interior points of $A \cup B$. This should hold not only for metric spaces.

□

b) $A^\circ \cap B^\circ = (A \cap B)^\circ$

Proof. A point belongs to intersection of two sets if and only if it belongs to both sets. Consider point x that belongs to intersection of interiors of A and B ($x \in A^\circ, x \in B^\circ$). By definition of interior point we can find open neighbourhoods $N_a(x) \subset A$ and $N_b(x) \subset B$. Thus intersection of neighbourhoods $N_a(x) \cap N_b(x)$ lies within $A \cap B$. Thus, point x is an interior point of $A \cap B$, or:

$$A^\circ \cap B^\circ \subseteq (A \cap B)^\circ \tag{1}$$

Now consider y, interior point of intersection of A and B ($y \in (A \cap B)^\circ$). Since y is interior, we can find an open neighbourhood $N_{ab}(y) \in (A \cap B)$. Furthermore, $N_{ab}(y)$ must be a subset of both A and B and thus has open neighbourhoods in A and in B. Therefore y is an interior point of both A and B, or:

$$(A \cap B)^\circ \subseteq A^\circ \cap B^\circ \tag{2}$$

Considering expressions (1) and (2), we conclude that $A^\circ \cap B^\circ = (A \cap B)^\circ$. This should hold not only for metric spaces

□

Give an example of two subsets A and B of the real line such that $A^\circ \cup B^\circ \neq (A \cup B)^\circ$.

Answer: $A = \mathbb{Q},\ B = \mathbb{R} \setminus \mathbb{Q}$.

Proof. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, every neighbourhood of every point of both sets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ contains infinitely many points of both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$. This means that no point of these sets is interior:

$$\mathbb{Q}^\circ = \emptyset,\ \ (\mathbb{R} \setminus \mathbb{Q})^\circ = \emptyset$$

However, interior of the union of these sets is $\mathbb{R}$:

$$(\mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}))^\circ = \mathbb{R}^\circ = \mathbb{R}.$$

Therefore:

$$\mathbb{Q}^\circ + (\mathbb{R} \setminus \mathbb{Q})^\circ \neq (\mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}))^\circ.$$

□

Problem 4

Let A be a subset of a metric space M. Recall that $\overline{A}$, the closure of A, is the union of A and its limit points. Recall that a point x belongs to the boundary of A, ∂A, if every open ball centered at x contains points of A and points of A^c, the complement of A. Prove that:
a) $\partial A = \overline{A} \cap \overline{A^c}$

Proof. Note: We shall prove validity of claims a)-d) for general topological spaces that do not necessarily have a metric.

Consider point $p \in \partial A$. Every neighbourhood of p must contain points of both A and A^c. Thus every neighbourhood of p must contain points of A, so $p \in \overline{A}$. Every neighbourhood of p must also contain points of A^c, so $p \in \overline{A^c}$. Since p is both an element of set $\overline{A}$ and of set $\overline{A^c}$, p must be an element of $\overline{A} \cap \overline{A^c}$, so:

$$\partial A \subseteq \overline{A} \cap \overline{A^c} \tag{3}$$

Now consider point $p \in \overline{A} \cap \overline{A^c}$. Point p must belong to the set $\overline{A}$, thus every open neighbourhood of p must contain elements of A. Point p must also belong to the set $\overline{A^c}$, thus every open neighbourhood of p must contain elements of A^c. We can see that every open neighbourhood of p must contain elements of both A and A^cm so p must lie in the boundary of A, or:

$$\overline{A} \cap \overline{A^c} \subseteq \partial A \tag{4}$$

Considering expressions (3) and (4) we can see that $\partial A = \overline{A} \cap \overline{A^c}$.

□

b) $p \in \partial A \iff p$ is in $\overline{A}$ but not in A° (symbolically, $\partial A = \overline{A} \setminus A^\circ$)

Proof. Consider $p \in \partial A$. Every open neighbourhood of p must contain both points of A and points of A^c. Since every open neighbourhood of p must contain points of A, p must be an element of $\overline{A}$, so:

$$A \subseteq \overline{A}$$

Furthermore, since every open neighbourhood of p must contain points of A^c, such neighbourhood cannot be a subset of A, so p cannot be a interior point. Therefore

$$A \nsubseteq A^\circ$$

We conclude that

$$\partial A \subseteq \overline{A} \setminus A^\circ \tag{5}$$

Now consider $p \in \overline{A} \setminus A^\circ$. Since p is an element of $\overline{A}$, every neighbourhood of p must contain points of A. Since p cannot be an element of A°, every neighbourhood of p includes points that belong to A^c. Therefore every neighbourhood of p includes both points of A and points of A^c and we conclude that

$$\overline{A} \setminus A^\circ \subseteq \partial A \tag{6}$$

Considering expressions 5 and 6 we can see that $\partial A = \overline{A} \setminus A^\circ$.

$\square$

c) ∂A is a closed set

Proof. Set ∂A is closed if it contains all its limit points. If the set of limit points of ∂A is not $\emptyset$, consider an arbitrary limit point p of ∂A. Now we will prove that p is a boundary point of A. Consider $N(p)$, an arbitrary open neighbourhood of p. Since p is a limit point, $N(p)$ must contain points of ∂A other than p. Denote $q : q \in \partial A, q \in N(p)$ one of these points. Every open neighbourhood of q must contain points that lie in A and in A^c by definition of boundary. Notice that $N(p)$ is an open neighbourhood of q. Thus $N(p)$ must include points that lie in A and in A^c. Therefore, p is a boundary point of A. We conclude that all limit points of ∂A lie in ∂A, or set ∂A is closed.

If the set of limit points of ∂A is $\emptyset$ we note that $\emptyset \subseteq \partial A$. In this case, vacuously, ∂A is a closed set.

$\square$

d) A is closed $\iff \partial A \subseteq A$

Proof. Consider b, a boundary point of a closed set A. Consider an arbitrary open neighbourhood of b: $N(b)$. Since b is a boundary point, $N(b)$ contains at least one point of A and at least one point of A^c. Consider an arbitrary point $p \in A$ that lies in $N(b)$. Either $p = b$ or $p \neq b$. If $p = b$, then $p = b \in A$, so b, a boundary point of A, is an element of A. If $p \neq b$, then b must be a limit point. But then we know that a closed set contains all its limit points, thus b, a boundary point of A, is an element of A. We conclude that if A is closed then $\partial A \subseteq A$.

Now consider a set A such that $\partial A \subseteq A$. Suppose p is a limit point of A such that $p \notin A$. Then $p \in A^c$. Therefore each open neighbourhood of p must include points of A (since it is a limit point of A) and at least one point of A^c, specifically p. But then p must be a boundary point of A and since $\partial A \subseteq A$, p must be an element of A, which presents a contradiction. Therefore, p, a limit point of A must be a point of A. We conclude that if $\partial A \subseteq A$ then A is closed.

Thus, A is closed $\iff$ $\partial A \subseteq A$.

$\square$

Problem 5

Show that, in $\mathbb{R}^n$ with the usual (Euclidean) metric, the closure of the open ball $B_R(p), R > 0$, is the closed ball

$$\{q \in \mathbb{R}^n : d(p,q) \leq R\}.$$

Proof. We claim that for every metric space M with metric that is induced by a norm, closure of an open ball is a closed ball:

$$\overline{B_R(p)} = B_R[p]$$

where

$$B_R(p) = \{r : d(p,r) < R\}$$
$$B_R[p] = \{r : d(p,r) \leq R\}$$

and $d(p,r) = \|p - r\|$ for some norm $\|\cdot\|$; $R > 0$.

Set $B_R[p]$ is closed since it contains all its limit points. Suppose, there exists x, a limit point of $B_R[p]$, that belongs to $B_R[p]^c = \{r : d(p,r) > R\}$. Set $B_R[p]^c$ is open, so all its points internal to itself. This means that for such x we can find an open ball that is a subset of $B_R[p]^c$. But then, it cannot contain points of $B_R[p]$, which is a contradiction with x being a limit point of $B_R[p]$. Thus, $B_R[p]$ contains all its limit points and is closed. We also note that set $B_R(p)m \subseteq B_R[p]$. Closure of $B_R(p)$ is the smallest closed subset that includes $B_R(p)$, therefore it is a subset of any closed set that includes $B_R(p)$, including $\subseteq B_R[p]$:

$$\overline{B_R(p)} \subseteq B_R[p]$$

Consider $r \in B_R[p]$. We will prove that $r \in \overline{B_R(p)}$. We need to prove that r is either an element or a limit point of $B_R(p)$. All r such that $d(p,r) < R$ are clearly elements of $B_R(p)$. The set of remaining points is $B_R[p] \setminus B_R(p)$. All points of this set are given by $\{r : d(p,r) = R\}$. Point r is a limit point of $B_R(p)$ if every open ball $B_\epsilon(r)$ contains at least one point x such that $x \in B_R(p) \iff d(p,x) < R$. We can find such x explicitly, provided M is a normed vector space, which follows from existence of norm $\|\cdot\|$ for M.

We claim that

$$x = r + \frac{\epsilon}{2}\frac{1}{d(p,r)}(p-r)$$

is such a point. To prove this we show that $d(r,x) < \epsilon$ and $d(p,x) < R$.

$$\begin{aligned} d(r,x) = \|r-x\| &= \left\| r - r + \frac{\epsilon}{2}\frac{1}{d(p,r)}(p-r) \right\| \\ &= \left\| \frac{\epsilon}{2}\frac{1}{d(p,r)}(p-r) \right\| = \frac{\epsilon}{2}\frac{\|p-r\|}{d(p,r)} = \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Thus x lies in $B_\epsilon(r)$.

$$\begin{aligned} d(p,x) = \|p-x\| &= \left\| p - r - \frac{\epsilon}{2}\frac{1}{d(p,r)}(p-r) \right\| \\ &= \left\| (p-r)\left(1 - \frac{\epsilon}{2}\frac{1}{d(p,r)}\right) \right\| = \left(1 - \frac{\epsilon}{2}\frac{1}{d(p,r)}\right) d(p,r) \\ &= d(p,r) - \frac{\epsilon}{2} \leq R - \frac{\epsilon}{2} < R. \end{aligned}$$

Thus x lies in $B_R(p)$.

Therefore all points in $B_R[p]$ are either elements of $B_R(p)$ or its limit points, thus are elements of the closure of $B_R(p)$:

$$B_R[p] \subseteq \overline{B_R(p)}$$

We conclude that:

$$\overline{B_R(p)} = B_R[p]$$

> Give an example of a metric space for which the corresponding statement is false.

This may not be the case for metrics that are not induced by a norm. For example, for discreet metric on $\mathbb{R}$, which is not induced by a norm,

$$d(x,y) = \begin{cases} 0, & \text{if} x = y, \\ 1, & \text{if} x \neq y \end{cases}$$

open ball $B_1(0) = 0$, closed ball $B_1[0] = \mathbb{R}$ and closure of $\overline{B_1(0)} = 0$. Clearly

$$\overline{B_1(0)} \subseteq B_1[0]$$

but

$$\overline{B_1(0)} \neq B_1[0].$$

$\square$

Problem 6

Prove directly from the definition that the set $K \subseteq \mathbb{R}$ given by

$$K = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$$

is compact.

Proof. Let C be an open cover of K, i.e. a collection of open sets such that:

$$K \subseteq \bigcup_{S \in C} S$$

For each $k \in K$ we can find a set within the collection C that contains k (not necessarily unique). Denote such open set $C(k)$.

Specifically, for $0 \in K$ there must exist open set $C(0)$. Since $C(0)$ is open, it must contain some open ball $B_\epsilon(0)$. We notice that $B_\epsilon(0)$ contains all points of K that are less than ϵ:

$$x \in K : x < \epsilon \Rightarrow x \in B_\epsilon(0)$$

We can see that there are only finitely many points of K that are greater than or equal to ϵ. Such points can be covered by union of finitely many open sets $C(k)$. The rest of the points of K can be covered by $B_\epsilon(0)$.

Therefore the following finite collection covers K:

$$\{C(k) : k \in K, k \geq \epsilon\} \cup \{B_\epsilon(0)\}$$

We conclude that K is compact.

$\square$

Problem 7

Let K be a compact subset of a metric space M, and let $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ be an open cover of K. Show that there is a positive real number δ with the property that for every $x \in K$ there is some $\alpha \in A$ with

$$B_\delta(x) \subseteq \mathcal{U}_\alpha$$

Proof. Since all sets in collection $\mathcal{U}$ are open, for each $x \in K$ there must exist an open ball $B_{\epsilon(x)}(x)$ where $\epsilon(x) > 0$ that is a subset of some $\mathcal{U}_\alpha, \alpha \in A$. Open ball with half the radius, $B_{\epsilon(x)/2}(x)$, is also a subset of the same $\mathcal{U}_\alpha$. Collection of such open balls with half the radius, $\mathcal{H}$, is an open cover of K since each $x \in K$ belongs to at least one of the open sets in collection $\mathcal{H}$.

Since K is compact, there must exist a finite subcover $\mathcal{F} \in \mathcal{H}$ that covers K. Since collection $\mathcal{F}$ is finite, we can enumerate all its sets:

$$B_{\epsilon(x_1)/2}(x_1),\ B_{\epsilon(x_2)/2}(x_2),\ \ldots,\ B_{\epsilon(x_N)/2}(x_N)$$

where $N \in \mathbb{N}$. Radius of each of these open balls must be strictly greater than zero. Find the minimum of these radii:

$$\delta = \min\{\epsilon(x_n)/2 : n \in \mathbb{N}, 1 \leq n \leq N\},$$

which will be also strictly greater than zero. We claim that δ satisfies the desired property for K and $\mathcal{U}$.

We prove that for any $x \in K$ open ball $B_\delta(x)$ is a subset of at least one of the sets of the collection $\mathcal{U}$. Indeed, every $x \in K$ must belong to at least one of the sets of its open cover $\mathcal{F}$; denote such set $B_{\epsilon(x_k)/2}(x_k)$. We notice that

$$B_{\epsilon(x_k)/2}(x_k) \subset B_{\epsilon(x_k)}(x_k)$$

and since $\delta \leq \epsilon(x_k)/2$:

$$B_\delta(x) \subset B_{\epsilon(x_k)}(x_k) \subseteq \mathcal{U}_\alpha, \text{ for some } \alpha \in A$$

Therefore, there exists $\delta > 0$ such that for any point in K there exists an open ball with radius δ that is a subset of at least one set of an open cover of K.

$\square$