18.701: Problem Set 8

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Problem 1

Let G be a group of order 55.
a) Prove that G is generated by two elements x and y, with the relations $x^{11} = 1,\ y^5 = 1,\ yxy^{-1} = x^r,$ for some $r:\ 1 \leq r < 11$.
b) Decide which values of r are possible.
c) Prove that there are two isomorphism classes of groups of order 55.

Proof. By the First Sylow theorem, group G of order 55 contains at least one Sylow 11-subgroup H_{11} and at least one Sylow 5-subgroup H_5.

By the Third Sylow theorem, the number of Sylow 11-subgroups in G, must divide 5 and also must be congruent to 1 modulo 11. Therefore, there is only one 11-subgroup in G, and it must be normal, denote it H_{11}.

Since H_{11} is normal, by the First Isomorphism theorem, G/H_{11} is isomorphic to a subgroup of order $55/11 = 5$, one of the Sylow 5-subgroups, denote it H_5.

Since H_{11} and H_5 have prime order, they are both cyclic, abelian, and they are generated by any of their respective elements other than identity:

$$H_{11} = \langle x \rangle,\ x \neq 1,\ x^{11} = 1,$$
$$H_5 = \langle y \rangle,\ y \neq 1,\ y^5 = 1.$$

Since cosets of H_{11} partition G and G/H_{11} is isomorphic to H_5, any element of G can be represented as a product of $x^p y^q$ for some $0 \leq p < 11,\ 0 \leq q < 5$. Therefore, x and y generate G. and $H_{11}H_5 = G$.

We note that since H_{11} is normal, conjugate of $x \in H_{11}$ must be in H_{11}, i.e. for $x \neq 1$:

$$yxy^{-1} = x^r,\ 1 \leq r < 11.$$

By the Third Sylow theorem, the number of 5-subgroup in G, s, must divide 11 and must be congruent to 1 modulo 5. There are two such options: $s = 1$ and $s = 11$, which correspond to two possible isomorphism classes of groups of order 55.

Case 1. There is only one 5-subgroup in G, namely H_5. Since both H_{11} and H_5 are abelian, $yx = xy$ and $yxy^{-1} = x$. Therefore, $r = 1$ for the case $s = 1$.

Since there is only one 5-subgroup of G, it must be normal. We can also see that $H_{11} \cap H_5 = 1$. Thus, multiplication map $f : H_{11} \times H_5 \to G$, defined as $f(h,k) = hk$, is an isomorphism. We conclude that G is isomorphic to $H_{11} \times H_5$ for the case $s = 1$.

Case 2. There are 11 5-subgroups in G. Since $xy^5 = y^5x = 1$, we have:

$$x = y^5xy^{-5} = y^4(yxy^{-1})y^{-4} =$$

since $yxy^{-1} = x^r$:

$$= y^4x^ry^4 = y^3(yx^ry^{-1})y^{-3} =$$

since $(yxy^{-1})^r = yx^ry^{-1}$:

$$= y^3(yxy^{-1})^ry^{-3} = y^3(x^r)^ry^{-3} =$$

continuing:

$$= y^2x^{(r^3)}y^2 = yx^{(r^4)}y = x^{(r^5)}.$$

Therefore, r^5 must be congruent to 1 modulo order of x:

$$r^5 = 1 \mod 11.$$

We test possible integer r, such that $1 < r < 11$:

$$\begin{aligned}
2^5 = 32 = 10 &\mod 11,\\
3^5 = 243 = 1 &\mod 11,\\
4^5 = 1024 = 1 &\mod 11,\\
5^5 = 3125 = 1 &\mod 11,\\
6^5 = 7776 = 10 &\mod 11,\\
7^5 = 16807 = 10 &\mod 11,\\
8^5 = 32768 = 10 &\mod 11,\\
9^5 = 59049 = 1 &\mod 11,\\
10^5 = 100000 = 10 &\mod 11.
\end{aligned}$$

Therefore, possible values of r for case of $s = 11$ are $3, 4, 5$ and 9.

We will prove that groups G_r generated by

$$\langle x, y; x^{11} = 1, y^5 = 1, yxy^{-1} = x^r \rangle$$

are isomorphic for $r \in \{3, 4, 5, 9\}$.

Consider group G_3 that has is generated by the following relation:

$$yxy^{-1} = x^3.$$

Also consider element $a = y^2$ of the subgroup H_5 of G_3:

$$axa^{-1} = y^2xy^{-2} = y(yxy^{-1})y^{-1} = yx^3y^{-1} = (x^3)^3 = x^9.$$

We note that a has order 5 and generates H_5. Thus, substituting a for y and keeping other relations unchanged generates G_9. Therefore, G_3 is isomorphic to G_9.

By the same logic, for $r = 4$ we substitute $b = y^3$ for y and we have:

$$bxb^{-1} = y^3xy^{-3} = (x^4)^3 = (x^{11})^5x^9 = x^9.$$

For $r = 5$ we substitute $c = y^4$ for y and we have:

$$cxc^{-1} = y^4xy^{-4} = (x^5)^4 = (x^{11})^{56}x^9 = x^9.$$

We conclude that $G_3 \simeq G_4 \simeq G_5 \simeq G_9$, which constitutes an isomorphism class for the case $s = 11$.

□

Problem 2

Use the Todd-Coxeter Algorithm to determine the order of the group generated by two elements x, y.
a) with relations $x^3 = 1,\ y^2 = 1,\ yxyxy = 1$.

We select $H = \langle x \rangle$ and denote it as 1. First steps of the Todd-Coxeter Algorithm are as follows:

	x	x	x
1	1	1	1
2	3	4	2
3	4	2	3

	y	y
1	2	1
3	4	3

	y	x	y	x	y
1	2	3	4	2	1
2	1	1	2	3	4

At this point we can see that coset 2 is the same as coset 4, which implies that cosets 2 and 3 are the same. This immediately "collapses" the group to a trivial group.

b) with relations $x^3 = 1,\ y^4 = 1,\ xyxy = 1$.

We select $H = \langle x \rangle$ and denote it as 1. Full table after applying the Todd-Coxeter Algorithm is as follows:

	x	x	x
1	1	1	1
2	3	4	2
5	6	7	5
8	8	8	8

	y	y	y	y
1	2	6	3	1
4	5	8	7	4

	x	y	x	y
1	1	2	3	1
2	3	1	1	2
3	4	5	6	3
4	2	6	7	4
5	6	3	4	5
6	7	4	2	1
7	5	8	8	7

Therefore, permutation representations are as follows:

$$x = (2\,3\,4)(5\,6\,7), \quad y = (1\,2\,6\,3)(4\,5\,8\,7).$$

Order of $\langle x \rangle$ is 3 and the number of cosets of $\langle x \rangle$ is 8. Therefore, the order of the group is $24 = 3 \cdot 8$.

Problem 3

Classify groups that are generated by two elements x and y of order 2.
Hint: It will be convenient to make use of the element $z = xy$.

Consider group G, which is generated by x and y such that $x^2 = y^2 = 1$. Denote element $z = xy$.

We first notice that G is generated by $\{z, y\}$ since $zy = xyy = x$. We also notice that:

$$zyzy = xyyxyy = x^2 = 1.$$

Presentation for G can be written as:

$$G = \langle z, y \mid y^2 = zyzy = 1 \rangle. \tag{1}$$

It is easy to see that subgroup $\langle z \rangle$ of G has infinite order, thus G is infinite. Presentation 1 is the usual presentation for infinite dihedral group.

In case G is finite, z must have finite order; denote it n. Presentation of G is then:

$$G = \langle z, y \mid z^n = y^2 = zyzy = 1 \rangle. \tag{2}$$

We claim that G is isomorphic to dihedral group D_{2n}.

Proof. Dihedral group D_{2n} is generated by two elements. Thus, by Artin, Corollary 7.10.14, there exists a surjective homomorphism $\varphi : \mathcal{G} \to G$. We need to prove that φ is injective. To do that we will show that the order of G is $2n$ using the Todd-Coxeter algorithm. We compute operations of elements of G on the cosets of $\langle z \rangle$ where $\langle z \rangle$ is represented as 1 in the table below:

	z	z	$\cdots$	z	
1	1	$\cdots$	1	1	
2	2	$\cdots$	2	2	

	y	y
1	2	1
2	1	2

	z	y	z	y
1	1	2	2	1
2	2	1	1	2

We conclude that the index of $\langle z \rangle$ is 2. Since the order of $\langle z \rangle$ is n, the order of G must be $2n$, which implies that φ is injective.

Therefore, G is isomorphic to D_{2n}.

$\square$

Introducing additional relations between z and y will result in more elements of G collapsing to 1.