18.701: Problem Set 5

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Preliminary Problem 1

Let A be $m \times m$ and B be $n \times n$ complex matrices, and consider the linear operator T on the space $\mathbb{C}^{m\times n}$ of all complex $m \times n$ matrices defined by $T(M) = AMB$.
a) Show how to construct an eigenvector for T out of a pair of column vectors X, Y, where X is an eigenvector for A and Y is an eigenvector for B^t.

Denote α the eigenvalue of the eigenvector X, i.e. $AX = \alpha X$. Denote β the eigenvalue of the eigenvector Y, i.e. $B^tY = \beta Y$. Eigenvector $P \in \mathbb{C}^{m\times n}$ of T must satisfy:

$$T(P) = APB = tP,$$

which means that for any vector $V \in \mathbb{C}^n$:

$$T(P)V = APBV = tPV,$$

where t is the same for all V.

Consider linear map $P : \mathbb{C}^n \to \operatorname{span} X$, which can be represented as $P = f(V) \cdot X$, where f is linear functional $f : \mathbb{C}^n \to \mathbb{C}$.

For such P and for any $V \in \mathbb{C}^n$ we have:

$$PV = Xf(V),$$
$$APBV = AXf(BV) = \alpha Xf(BV),$$

and

$$T(P)V = \alpha\frac{f(BV)}{f(V)}PV.$$

If $\frac{f(BV)}{f(V)}$ is equal to constant λ for all V, then P is an eigenvector of T with eigenvalue $\alpha\lambda$. We can represent f as left-multiplication by some row vector R^t:

$$f(V) = R^tV.$$

Therefore we must have

$$\begin{aligned}\frac{R^t BV}{R^t V} &= \lambda,\\ R^t BV &= \lambda R^t V,\\ (B^t R)^t V &= \lambda R^t V,\end{aligned}$$

which is the case if and only if R is an eigenvector of B^t, i.e. $R = Y$. In this case, $\lambda = \beta$. We conclude that map XY^t is an eigenvector of T.

We also note that nonzero vectors X_0 and Y_0 in the kernel of A and B^t (if any exist) are eigenvectors of A and B^t, respectively, with eigenvalue 0. Any map $X_0 Y_0^t$ will be a nonzero eigenvector of T with eigenvalue 0.

> b) Determine the eigenvalues of T in terms of those of A and B.

Choose some eigenvector P of T, which has the form XY^t with X being an eigenvector of A, and Y being an eigenvector of B^t. Eigenvalue of P is $\alpha\beta \neq 0$, where $\alpha \neq 0$ is the eigenvalue of X and $\beta \neq 0$ is the eigenvalue of Y.

Therefore, the set of eigenvalues of T is the set of all pairwise products between eigenvalues of A and eigenvalues of B^t. Since B^t and B have the same characteristic polynomial, they must have the same sets of eigenvalues. We conclude that the set of eigenvalues of T is

$$\{\alpha\beta : \alpha \text{ is an eigenvalue of } A,\ \beta \text{ is an eigenvalue of } B\}.$$

> c) Determine the trace of this operator.

Trace of the operator is equal to the sum of its eigenvalues:

$$\operatorname{tr} T = \sum_i \sum_j \alpha_i \beta_j,$$

where α_i and β_j is the enumeration of nonzero eigenvalues of A and B, respectively. We also notice that

$$\begin{aligned}\operatorname{tr} T &= \sum_i \sum_j \alpha_i \beta_j = a_1 \sum_j \beta_j + a_2 \sum_j \beta_j + \cdots = \sum_i \alpha_i \cdot \sum_j \beta_j\\ &= \operatorname{tr} A \cdot \operatorname{tr} B.\end{aligned}$$

Problem 1

> Let T be a linear operator on a vector space V. Let K_r and W_r denote the kernel and image, respectively, of T^r.
> a) Show that $K_1 \subseteq K_2 \subseteq \cdots$ and that $W_1 \supseteq W_2 \supseteq \cdots$.

Proof. Suppose $v \in V$ is in K_r. Since $T^r v = 0$, then $TT^r v = 0$ and thus v is in K_{r+1}. By induction we have $K_1 \subset K_2 \subset \cdots$.

Suppose $v \in V$ is in W_r. There must exist some $u \in V$ such that $v = T^r u = T^{r-1}(Tu)$. Therefore v is the image of some $Tu \in V$ under T^{r-1}, therefore v is in W_{r-1}. By induction we have $W_1 \supseteq W_2 \supseteq \cdots$. $\square$

b) The following conditions might or might not hold for a particular value of r.

$$(1)\ K_r = K_{r+1}, \quad (2)\ W_r = W_{r+1},$$
$$(3)\ W_r \cap K_1 = \{0\}, \quad (4)\ W_1 + K_r = V.$$

Find all implications among the conditions (1)−(4) when V is finite-dimensional.

All four conditions are equivalent for finite-dimensional vector spaces.

We first prove (1) $\iff$ (2).

Proof. By the rank-nullity theorem, $\dim K_r + \dim W_r = \dim V$. By the result of part a), conditions (1) and (2) are equivalent. $\square$

We then prove (2) $\iff$ (3).

Proof. Consider $T\big|_{W_r}$, restriction of T to W_r. By the rank-nullity theorem,

$$\dim W_r = \dim \ker T\big|_{W_r} + \dim W_{r+1}.$$

We can write $\ker T\big|_{W_r} = W_r \cap \ker T = W_r \cap K_1$, therefore:

$$\dim W_r = \dim(W_r \cap K_1) + \dim W_{r+1}.$$

We note that $\dim(W_r \cap K_1) = 0$ if and only if $W_r \cap K_1 = \{0\}$, which concludes the proof. $\square$

We then prove (2) $\implies$ (4).

Proof. Consider arbitrary $v \in V$. Since $W_r = W_{r+1}$, then

$$T^r v = T^r T v, \quad T^r(v - Tv) = 0.$$

Therefore there exists some k in the kernel of K_r such that $k = v - Tv$. Rearranging, we have $v = Tv + k$. We notice that Tv is in W_1. Therefore every v is equal to the sum of some element of K_r and some element of W_1, which implies $W_1 + K_r = V$, as requested. $\square$

Finally, we prove (4) $\implies$ (2).

Proof. Consider arbitrary $v \in V$. Since $V = W_1 + K_r$, v can be represented as $v = w + k$ for some $w \in W_1$ and $k \in K_r$. Choose some $x \in V$ such that $w = Tx$. Image of v under T^r is as follows:

$$T^r v = T^r Tx + T^r k = T^{r+1} x.$$

Since $T^{r+1}x$ is in W_{r+1}, we conclude that $T^r v$ is in W_{r+1}, i.e. $W_r \subseteq W_{r+1}$. Combining this with the result of part a) yields $W_r = W_{r+1}$, as requested. □

Problem 2

Let A and B be $m \times n$ and $n \times m$ real matrices.
a) Prove that if λ is a nonzero eigenvalue of the $m \times m$ matrix AB then it is also an eigenvalue of the $n \times n$ matrix BA.
Show by example that this need not be true if $\lambda = 0$.

Proof. Since λ is a nonzero eigenvalue of AB, there must exist a nonzero eigenvector v such that $ABv = \lambda v$. Consider vector $w = Bv$. Left multiply w by BA:

$$BAw = BABv = B\lambda v = \lambda Bv = \lambda w.$$

We can prove that $w \neq 0$. Since $\lambda \neq 0$ and $v \neq 0$, then $0 \neq \lambda v = ABv = Aw$. From this, w must be nonzero. Thus, w is an eigenvector of BA with eigenvalue λ.

This is not necessarily true for $\lambda = 0$, as the following example suggests:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is invertible, thus has no zero eigenvalues, while

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

which is singular, thus has a zero eigenvalue. □

b) Prove that $I_m - AB$ is invertible if and only if $I_n - BA$ is invertible.

Proof. We will use proof by contrapositive. Matrix is invertible if and only if it has no zero eigenvalues, so it is singular if it has at least one zero eigenvalue.

Suppose $I_m - AB$ is singular and has zero eigenvalue, i.e. there exists nonzero v such that

$$(I_m - AB)v = 0.$$

Then

$$v = ABv$$

and, noting that $v \neq 0$, matrix AB must have eigenvalue 1. By part a) of the problem, BA must also have eigenvalue 1, i.e.:

$$BAw = w$$

for some nonzero w. From this we have

$$w - BAw = 0,$$

$$(I_n - BA)w = 0,$$

which implies that $I_n - BA$ is singular.

The proof in the other direction is symmetric.

□

Problem 3

Let A be an 3×3 orthogonal matrix with $\det A = 1$, whose angle of rotation is different from 0 or π, and let $M = A - A^t$.

a) Show that M has rank 2, and that a nonzero vector X in the nullspace of M is an eigenvector of A with eigenevalue 1.

Proof. Consider spin $\rho_{(U,\theta)}$ of the rotation A. Rotation A fixes U, therefore U is an eigenvector of A with eigenvalue 1:

$$AU = U.$$

We know that A^{-1} is rotation with spin $\rho(U, -\theta)$, which also fixes U:

$$A^{-1}U = U.$$

From this we have:

$$AU = A^{-1}U, \quad (A - A^{-1})U = 0.$$

Since A is an orthogonal matrix, $A^t = A^{-1}$, therefore

$$0 = (A - A^t)U = MU.$$

Therefore, nullspace of M is nontrivial and includes, at least, a subspace spanned by U.

We will now prove that $\operatorname{null} M = \operatorname{span} U$. Consider nonzero X, an arbitrary element of the nullspace of M. Following the line of reasoning above in the reverse order we have

$$AX = A^{-1}X,$$

left-multiplying by A:

$$A^2X = X,$$

and X is an eigenvector of A^2. Rotation A^2 is equivalent to the rotation with spin $\rho(U, 2\theta)$. Since θ is neither 0 nor π, X can be an eigenvector of A^2 only if it lies in the axis of rotation of A^2:

$$X \in \operatorname{span} U.$$

Therefore, $\operatorname{null} M = \operatorname{span} U$.

Since $\dim \operatorname{null} M = 1$, we conclude, by the rank-nullity theorem, that the rank of M is 2, as required.

We have that X lies in the subspace spanned by vector U, let $X = cU$. Since U is an eigenvector of A with eigenvalue 1, we have:

$$AU = U, \quad cAU = cU, \quad AX = X,$$

so X is also an eigenvector of A with eigenvalue 1, as required.

□

b) Find such an eigenvector explicitly in terms of the entries of the matrix A.

Proof. We can write M in terms of entries of A explicitly:

$$M = \begin{pmatrix} 0 & a_{12} - a_{21} & a_{13} - a_{31} \\ a_{21} - a_{12} & 0 & a_{23} - a_{32} \\ a_{31} - a_{13} & a_{32} - a_{23} & 0 \end{pmatrix}$$

It can be shown that vector

$$X = \left(\frac{a_{23} - a_{32}}{a_{12} - a_{21}}, \frac{a_{31} - a_{13}}{a_{12} - a_{21}}, 1 \right)$$

spans the nullspace of M. Any nonzero multiple of X is therefore an eigenvector of A.

□

Problem 4

The space $\mathcal{C}$ of continuous functions $f(u)$ on the interval $[0,1]$ is one of many infinite-dimensional analogues of $\mathbb{R}^n$, and continuous functions $A(u,v)$ on the square $0 \le u, v \le 1$ are infinite-dimensional analogues of matrices. The integral

$$A \cdot f = \int_0^1 A(u,v) f(v) dv$$

is analogous to multiplication of a matrix and a vector. This problem treats the integral as a linear operator.
For the function $A = u+v$, determine the image of the operator explicitly. Determine its nonzero eigenvalues, and describe its kernel in terms of the vanishing of some integral.

Proof. Substituting $A = u + v$ we have

$$A \cdot f = \int_0^1 (u+v) f(v) dv = u \int_0^1 f(v) dv + \int_0^1 v f(v) dv,$$

which is a linear function in u. Therefore, every function in the image of A is linear.

Since A is a linear operator, its image must be a subspace of $\mathcal{C}$. We will prove that A has at least rank 2. Consider image of function $h_1 : u \mapsto 1$ under A:

$$A \cdot h_1 = \int_0^1 1 \cdot (u+1) dv = u + 1.$$

Consider image of function $h_2 : u \mapsto u$ under A:

$$A \cdot h_2 = \int_0^1 v(u+v) dv = \int_0^1 (vu + v^2) dv = \frac{u}{2} + \frac{1}{3}.$$

We can see that $A \cdot h_1$ and $A \cdot h_2$ are linearly independent, thus rank of A is, at least, 2. Subspace of linear functions on $[0,1]$ has dimension 2. Therefore, image of A is the subspace of linear functions on $[0,1]$.

Consider w, a nonzero eigenvector of A with eigenvalue λ.

$$A \cdot w = \lambda w.$$

Denote $a = \int_0^1 w(v) dv$, $b = \int_0^1 v w(v) dv$, and $w(u) = au + b$. Substituting:

$$(\lambda a) u + \lambda b = u \int_0^1 (av + b) dv + \int_0^1 v(av + b) dv.$$

Thus:

$$\lambda a = \int_0^1 (av+b)dv, \qquad\qquad \lambda b = \int_0^1 v(av+b)dv,$$

$$\lambda a = \frac{a}{2} + b, \qquad\qquad \lambda b = \frac{a}{3} + \frac{b}{2},$$

$$2\lambda a = a + 2b, \qquad\qquad 6\lambda b = 2a + 3b,$$

$$(1-2\lambda)a + 2b = 0, \qquad\qquad 2a + (3-6\lambda)b = 0,$$

We can rewrite this system of equations in the matrix form:

$$PX = 0.$$

where

$$P = \begin{pmatrix} 1-2\lambda & 2 \\ 2 & 3-6\lambda \end{pmatrix}, \quad X = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Since $X \neq 0$, matrix P must be singular. We solve $\det P = 0$ for λ with

$$\det P = 12\lambda^2 - 12\lambda - 1,$$

and get eigenvalues of A:

$$\lambda = \frac{1}{2} \pm \frac{1}{\sqrt{3}}.$$

Kernel of A is a subspace of functions $g(u)$ in $\mathcal{C}$ such that:

$$A \cdot g = au + b = 0,$$

which is only the case as long as $a = 0$ and $b = 0$, i.e.:

$$\int_0^1 g(v)dv = 0 \quad \text{and} \quad \int_0^1 vg(v)dv = 0.$$

Since g is continuous, the second expression, with integral bounds $[0,1]$, is zero only if $g(v) = 0$ for all $v \in [0,1]$. Therefore, kernel of A is trivial.

$\square$

Do the same for the function $A = u^2 + v^2$.

Proof. Substituting $A = u^2 + v^2$ we have

$$A \cdot f = \int_0^1 (u^2+v^2)f(v)dv = u^2 \int_0^1 f(v)dv + \int_0^1 v^2 f(v)dv,$$

which is a quadratic polynomial. Therefore, image of A lies in the subspace P_2 of polynomials of degree at most 2 (dimension 3). Consider image of function $h_1 : u \to 1$ under A:

$$A \cdot h_1 = u^2 + \frac{1}{3}.$$

Consider image of function $h_2 : u \to u$ under A:

$$A \cdot h_2 = \frac{u^2}{2} + \frac{1}{4}.$$

We can see that $A \cdot h_1$ and $A \cdot h_2$ are linearly independent, thus rank of A is, at least, 2.

By the general form of $A \cdot f$ we can see that no elements f of $\mathcal{C}$ is mapped to a polynomial with nonzero linear monomial. Therefore, image of A is quadratic polynomials of the form $p(u) = au^2 + b$.

Consider w, nonzero eigenvalue of A. Denote $a = \int_0^1 w(v)dv$, $b = \int_0^1 v^2 w(v)dv$, and $w(u) = au^2 + b$. We have:

$$A \cdot w = \lambda w,$$

$$(\lambda a)u^2 + \lambda b = u^2 \int_0^1 (av^2 + b)dv + \int_0^1 (av^4 + bv^2)dv$$

Thus:

$$\begin{aligned} \lambda a &= \int_0^1 (av^2 + b)dv, & \lambda b &= \int_0^1 (av^4 + bv^2)dv, \\ \lambda a &= \frac{a}{3} + \frac{b}{2}, & \lambda b &= \frac{a}{5} + \frac{b}{3}, \\ 6\lambda a &= 2a + 3b, & 15\lambda b &= 3a + 5b, \\ (2 - 6\lambda)a + 3b &= 0, & 3a + (5 - 15\lambda)b &= 0. \end{aligned}$$

Equivalently:

$$PX = 0.$$

where

$$P = \begin{pmatrix} 2 - 6\lambda & 3 \\ 3 & 5 - 15\lambda \end{pmatrix}, \quad X = \begin{pmatrix} a \\ b \end{pmatrix}.$$

We solve $\det P$ for λ with:

$$\det P = 90\lambda^2 - 60\lambda + 1,$$

and get eigenvalues of A:

$$\lambda = \frac{1}{3} \pm \frac{1}{\sqrt{10}}.$$

To find the kernel of A we write

$$A \cdot g = au^2 + b = 0,$$

which is only the case as long as $a = 0$ and $b = 0$, i.e.:

$$\int_0^1 g(v)dv = 0 \quad \text{and} \quad \int_0^1 v^2 g(v)dv = 0.$$

Since g is continuous, the second expression, with integral bounds $[0, 1]$, is zero only if $g(v) = 0$ for all $v \in [0, 1]$. Therefore, kernel of A is trivial.

$\square$

Problem 5

> Let f and g be rotations of the plane about distinct points, with arbitrary nonzero angles of rotation θ and ϕ. Prove that the group generated by f and g contains a translation.

Proof. We can represent isometry f as $t_v\rho_\theta$; and g as $t_w\rho_\phi$. Let G be a group generated by isometries f and g.

We consider homomorphism $\pi\big|_G : G \to O_2$. Consider element $h = g^{-1}f^{-1}gf$ of G.

$$\pi(h) = \pi(g^{-1}f^{-1}gf) = \pi(g^{-1})\pi(f^{-1})\pi(g)\pi(f) = \rho_\phi^{-1}\rho_\theta^{-1}\rho_\phi\rho_\theta$$

We know that $\rho_\alpha^{-1} = \rho_{(-\alpha)}$ and $\rho_\alpha\rho_\beta = \rho_{(\alpha+\beta)}$, hence:

$$\pi(h) = \rho_{(-\phi-\theta+\phi+\theta)} = \rho_0 = 1.$$

Kernel of $\pi\big|_G$ is the group of translations of G. Since h is in the kernel of π, it is a translation.

We will check that h is not the identity of G. Choose coordinates such that f is rotation around origin, i.e. $f = \rho_\theta$. Suppose $h = g^{-1}f^{-1}gf = 1$, then:

$$\begin{aligned} gf &= fg, \\ t_w\rho_\phi\rho_\theta &= \rho_\theta t_w\rho_\phi, \\ t_w &= t_{w'}, \\ w &= w', \end{aligned}$$

where $w' = \rho_\theta(w)$. Expression $w = \rho_\theta(w)$ is true if and only if w is the origin. However, then both v and w must be the origin and $v = w$, which contradicts the problem statement. Therefore, $h \neq 1$.

We conclude that element $g^{-1}f^{-1}gf$ of G is a translation by nonzero vector. $\square$