18.701: Problem Set 10

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Problem 1

> a) Let SL_2 be the special linear group of real matrices with determinant 1. Determine the possible eigenvalues λ (real or complex) of the elements of SL_2, and make a drawing showing the points λ in the complex plane.

We start with 2×2 matrix of the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $\det A = ad - bc = 1$.

Characteristic polynomial of A is $t^2 - tr + 1$ where $r = \operatorname{trace} A = a + d$. Eigenvalues of A are thus

$$\lambda = \frac{r \pm \sqrt{r^2 - 4}}{2}.$$

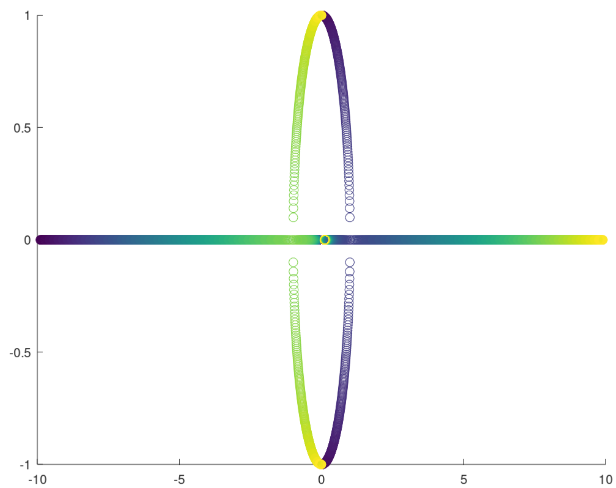
As $r \to \infty : \lambda \to \pm\infty$ and as $r \to -\infty : \lambda \to \mp 0$. For r in the interval $(-\infty, -2] \cup [2, \infty)$ eigenvalues of A are real. For r in the interval $(-2, 2)$ eigenvalues of A are complex, occur in conjugate pairs and their locus is upper and lower half of the unit circle on the complex plane.

> b) For each λ, decompose the set of matrices $P \in \mathrm{SL}_2$ with eigenvalue λ into SL_2-conjugacy classes.

Proof. We claim that for every matrix A with complex eigenvalues λ and $\overline{\lambda}$ there exists a matrix $Q \in \mathrm{SL}_2$ such that $Q^t A Q = \Lambda = \begin{pmatrix} \lambda & \\ & \overline{\lambda} \end{pmatrix}$.

Let $X = (u, v)^t$ be a unit eigenvector of A with eigenvalue λ. Then $Y = (-v, u)^t$ is also a unit eigenvector of A with eigenvalues $\overline{\lambda}$. [PROVE] Let $Q =$

Figure 1: Possible eigenvalues of $A \in \mathrm{SL}_2$ in the complex plane.



$\begin{pmatrix} u & -v \\ v & u \end{pmatrix}$, then

$$Q^t A Q = \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u & -v \\ v & u \end{pmatrix} = \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \begin{pmatrix} \lambda u & -\overline{\lambda} v \\ \lambda v & \overline{\lambda} u \end{pmatrix}$$

$$= \begin{pmatrix} \lambda(u^2+v^2) & -\overline{\lambda} uv + \overline{\lambda} uv \\ -\lambda uv + \lambda uv & \overline{\lambda}(v^2+u^2) \end{pmatrix} = \begin{pmatrix} \lambda & \\ & \overline{\lambda} \end{pmatrix}.$$

□

c) Determine the matrices $P \in \mathrm{SL}_2$ that can be obtained as $P = e^A$ for some real matrix A.

Proof. □

Problem 2

According to Sylvester's Law, every 2×2 real symmetric matrix is congruent to exactly one of six standard types. List them. If we consider the operation of GL_2 on 2×2 matrices by $P * A = PAP^t$, then Sylvester's Law asserts that the symmetric matrices form six orbits. We may view the symmetric matrices as points in $\mathbb{R}^3$, letting (x, y, z) correspond to the matrix $\begin{pmatrix} x & y \\ y & z \end{pmatrix}$. Describe the decomposition of $\mathbb{R}^3$ into orbits geometrically, and make a clear drawing depicting it.

Proof. □

Problem 3

This problem is about the space V of real polynomials in the variables x and y. If f is a polynomial, ∂_f will denote the operator $f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\right)$, and $\partial_f(g)$ will denote the result of applying this operator to a polynomial g.
a) The rule $\langle f, b\rangle = \partial_f(g)_0$ defines a bilinear form on V, the subscript denoting evaluation of a polynomial at the origin. Prove that this form is symmetric and positive definite, and that the monomials $x^i y^j$ form an orthogonal basis of V (not an orthonormal basis).
b) We also have the operator of multiplication by f, which we write as m_f. So $m_f(g) = fg$. Prove that ∂_f and m_f are adjoint operators.
c) When $f = x^2 + y^2$, the operator ∂_f is the Laplacian, which is often written as Δ. A polynomial h is harmonic if $\Delta h = 0$. Let H denote the space of harmonic polynomials. Identify the space $H^\perp$ orthogonal to H with respect to the given form.

Proof. □

Problem 4

Show that the vector cross product makes $\mathbb{R}^3$ into a Lie algebra L_1.

Proof. □