18.701: Problem Set 10

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Problem 1

a) Let SL_2 be the special linear group of real matrices with determinant 1. Determine the possible eigenvalues λ (real or complex) of the elements of SL_2, and make a drawing showing the points λ in the complex plane.

We start with 2×2 matrix of the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $\det A = ad - bc = 1$.

Characteristic polynomial of A is $t^2 - tr + 1$ where $r = \operatorname{trace} A = a + d$. Eigenvalues of A are thus

$$\lambda = \frac{r \pm \sqrt{r^2 - 4}}{2}.$$

As $r \to \infty : \lambda \to \pm\infty$ and as $r \to -\infty : \lambda \to \pm 0$. For r in the interval $(-\infty, -2] \cup [2, \infty)$ eigenvalues of A are real. For r in the interval $(-2, 2)$ eigenvalues of A are complex, occur in conjugate pairs and their locus is upper and lower half of the unit circle of the complex plane.

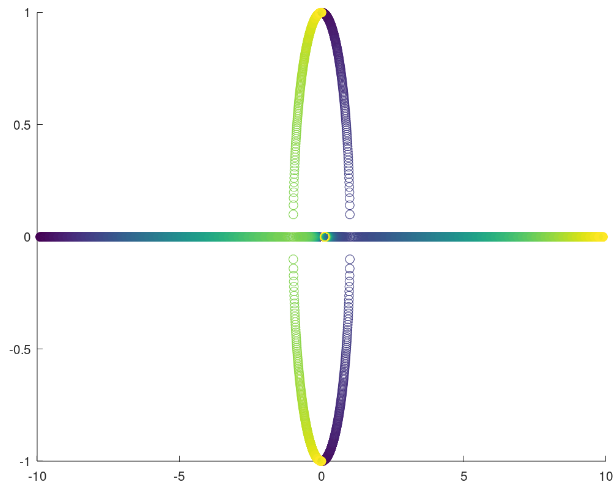
b) For each λ, decompose the set of matrices $P \in \mathrm{SL}_2$ with eigenvalue λ into SL_2-conjugacy classes.

Proof. Conjugate matrices have the same characteristic polynomial and the same eigenvalues. □

c) Determine the matrices $P \in \mathrm{SL}_2$ that can be obtained as $P = e^A$ for some real matrix A.

Proof. □

Figure 1: Possible eigenvalues of $A \in \mathrm{SL}_2$ in the complex plane.



Problem 2

Proof. $\square$