September 18, 2019

18.701 Problem Set 3

due Wednesday, September 25

1. Chapter 3, Exercise 4.4 (*order of* $GL_2(\mathbb{F}_p)$)

2. (*a homomorphism from* $GL_2(\mathbb{F}_3)$ *to* S_4) Let GL denote the group $GL_2(\mathbb{F}_3)$ of invertible matrices with entries modulo 3. This group operates on 2-dimensional vectors with entries mod 3 by matrix multiplication, as usual:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

There are 9 vectors modulo 3, and four pairs $\pm v$ of nonzero vectors, namely

$$s_1 = \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \; s_2 = \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \; s_3 = \pm \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \; s_4 = \pm \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The elements of GL permute the nonzero vectors, and they also permute the pairs of nonzero vectors. Sending a matrix to the permutation it defines gives us a homomorphism φ from GL to the symmetric group S_4 of permutations of $\{s_1, s_2, s_3, s_4\}$. For example, if $E = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$, then $\varphi(E)$ is the 3-cycle $(s_2\, s_3\, s_4)$.
(a) Show that φ is a surjective map, and determine its kernel.
(b) Determine the subgroup of GL that corresponds, by the Correspondence Theorem, to the alternating subgroup A_4 of S_4.
(c) Determine the subgroup of S_4 that corresponds to the subgroup of GL of upper triangular matrices.

3. Chapter 2, Exercise M.6a,b (*paths in* $\mathbb{R}^k$)

4. Chapter 2, Exercise M.7 (*paths in* GL_n)

5. Chapter 2, Exercise M.8a (SL_n *is connected*)