18.701: Problem Set 4

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Problem 1

a) Let $x(t)$ and $y(t)$ be quadratic polynomials with real coefficients. Prove that image of the path $(x(t), y(t))$ is contained in a conic, i.e., that there is a real quadratic polynomial $f(x, y)$ such that $f(x(t), y(t))$ is identically zero.

Proof. Let

$$x(t) = a_1t^2 + b_1t + c_1,$$

and

$$y(t) = a_2t^2 + b_2t + c_2.$$

Then

$$\begin{aligned} a_2x(t) - a_1y(t) &= a_2a_1t^2 + a_2b_1t + a_2c_1 - a_1a_2t^2 - a_1b_2t - a_1c_2 \\ &= a_2b_1t + a_2c_1 - a_1b_2t - a_1c_2 \\ &= t(a_2b_1 - a_1b_2) + a_2c_1 - a_1c_2. \end{aligned}$$

If $a_2b_1 - a_1b_2 = 0$, then x can be expressed as a multiple of y plus some constant, i.e. there exists a linear function l such that $x(t) = l(y(t))$. In such case, linear polynomial in two variables $f(x, y) = -x + l(y)$ is identically zero, as requested.

If $a_2b_1 - a_1b_2 \neq 0$, then:

$$t = \frac{a_2x(t) - a_1y(t)}{a_2c_1 - a_1c_2}.$$

Substituting t to the original expression for either $x(t)$ or $y(t)$ we will arrive at a quadratic polynomial equation in $x(t)$ and $y(t)$:

$$p(x(t), y(t)) = 0$$

This expression is true for all t, thus polynomial p is identically zero, as requested.

□

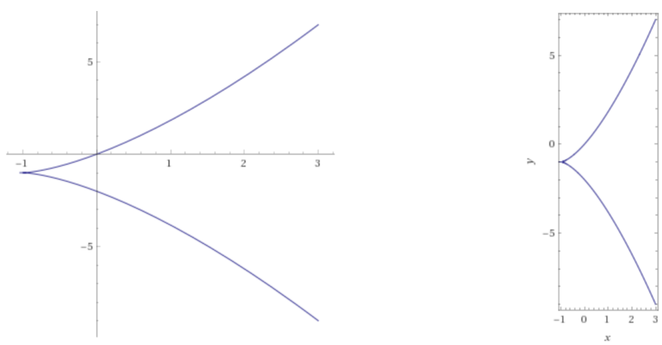
b) Let $x(t) = t^2 - 1$ and $y(t) = t^3 - 1$. Find a nonzero real polynomial $f(x, y)$ such that $f(x(t), y(t))$ is identically zero. Sketch the locus $\{f(x, y) = 0\}$ and the path $(x(t), y(t))$ in $\mathbb{R}^2$.

We notice that for $x(t) = t^2 - 1$ and $y(t) = t^3 - 1$, polynomial $p(x, y) = (x + 1)^3 - (y + 1)^2$ is identically zero. Expanding p we have:

$$\begin{aligned} p(x, y) &= (x + 1)^3 - (y + 1)^2 \\ &= x^3 + 3x^2 - y^2 + 3x - 2y. \end{aligned}$$

By plotting locus $\{f(x, y) = 0\}$ and path $(x(t), y(t))$, we confirm that they coincide.

(a) Locus $\{f(x, y) = 0\}$ (b) Path $(x(t), y(t))$



c) Prove that every pair $x(t), y(t)$ of real polynomials satisfies some real polynomial relation $f(x, y) = 0$.

Proof. Fix polynomials $x(t)$ and $y(t)$. Denote n the higher of the degrees of $x(t)$ and $y(t)$. Consider W_k, vector space of polynomials in two variables of degree at most k. Monomials of degrees $0, 1, \ldots, k$ form a basis for the vector space of polynomials of degree at most k. For a polynomial in two variables, there exist $p+1$ different monomials of degree p (for example, for $p = 3$ possible monomials are: x^3, x^2y, xy^2, y^3). Thus, the total number of possible monomials for a polynomial in two variables of degree k is

$$\frac{(k + 1)(k + 2)}{2}.$$

This is also the dimension of W_k.

For every polynomial $f(x, y) \in W_k$ we can substitute $x(t)$ and $y(t)$ into $f(x, y)$, simplify and arrive at a polynomial in one variable of degree at most

nk, which is an element of vector space V_{nk} of dimension $nk+1$. By doing so, for a given $x(t)$ and $y(t)$, we have defined a map $\varphi_{xy} : W_k \to V_{nk}$.

We notice that φ is a linear map. Indeed, consider two polynomials $f, g \in W_k$:

$$\varphi(f+g) = f(x,y) + g(x,y) = \varphi(f) + \varphi(g),$$
$$\varphi(cf) = cf(x,y) = c\varphi(f).$$

We notice that for sufficiently large values of k:

$$\frac{(k+1)(k+2)}{2} > nk,$$
$$\dim W_k > \dim V_{nk}.$$

Therefore, by rank-nullity theorem, $\dim \ker \varphi_{xy} > 0$ and $\ker \varphi_{xy} \neq \{0\}$. Thus, there exists non-zero polynomial $f \in W_k$ such that $\varphi_{xy}(f) = f(x(t), y(t)) = 0$.

□

Problem 2

Prove that every $m \times n$ matrix A of rank 1 has the form $A = XY^t$, where X, Y, are m- and n-dimensional column vectors. How uniquely determined are these vectors.

Proof. Dimension of the column-space of matrix A of rank 1 is 1, i.e. it is a line. Therefore, any one of column vectors of A is the basis of column-space of A. Choose one column-vector B_i, then all other column vectors of A are multiples of B_i:

$$A = \begin{bmatrix} | & | & & | & & | \\ c_1 B_i & c_2 B_i & \cdots & B_i & \cdots & c_n B_i \\ | & | & & | & & | \end{bmatrix}$$

This is equivalent to:

$$A = B_i C(i)^t, \tag{1}$$

where

$$C(i) = \begin{pmatrix} c_1 \\ \vdots \\ c_{i-1} \\ 1 \\ c_{i+1} \\ \vdots \\ c_n \end{pmatrix}.$$

B_i is a m-dimensional column vector and $C(i)$ is a n-dimensional column vector, as required.

□

Since we could choose any one of column vectors of A as a basis B_i, there exist n possible combinations of B_i and $C(i)$ that satisfy (1), specifically:

$$B_i = c_i B_1, \qquad C(i) = \frac{1}{c_i} C(1),$$

where c_i is one of entries of $C(1)$.

Problem 3

a) Let U and W be vector spaces over a field F. Show that the two operations $(u, w) + (u', w') = (u + u', w + w')$ and $c(u, w) = (cu, cw)$ on pairs of vectors make the product set $U \times W$ into a vector space. It is called a product space.

Proof. We will prove that V is a vector space directly from the definition of a vector space.

We first check that $+$ makes $U \times W$ into abelian group V^+. Indeed, V^+ is the product group $U^+ \times W^+$ of abelian groups, thus it is itself an abelian group. We also note that V^+ includes identity $(0_U, 0_W)$.

We then check that $1v = v$, for all $v \in V$. Indeed:

$$1v = 1(u, w) = (1u, 1w) = (u, w) = v.$$

We check the associativity of scalar multiplication. Indeed:

$$(ab)(u, w) = (abu, abw) = a(bu, bw) = a(bv).$$

Finally, we check distributive laws. For scalar multiplication:

$$\begin{aligned}(a + b)(u, w) &= ((a + b)u, (a + b)w) = (au + bu, aw + bw) \\ &= (au, aw) + (bu, bw) \\ &= a(u, w) + b(u, w).\end{aligned}$$

For scalar addition:

$$\begin{aligned}a((u, w) + (u' + w')) &= a(u + u', w + w') = (au + au', aw + aw') \\ &= (au, aw) + (au', aw') \\ &= a(u, w) + a(u', w')\end{aligned}$$

Therefore, V is a vector space.

$\square$

b) Let U and W be subspaces of a vector space V. Show that the map $T : U \times W \to V$ defined by $T(u, w) = u + w$ is a linear transformation.

Proof. We check linearity of T directly from the definition of a linear transformation:

$$T((u,w)+(u',w')) = T(u+u', w+w') = u+u'+w+w' = T(u,w)+T(u',w').$$
$$T(c(u,w)) = T(cu,cw) = cu+cw = c(u+w) = cT(u,w).$$

Therefore, T is a linear transformation.

$\square$

> c) Express the dimension formula for T in terms of the dimension of subspaces of V.

By the Rank-nullity theorem:

$$\dim V = \dim\ker T + \dim \operatorname{im} T.$$

Claim. $\dim\ker T = \dim(U\cap W)$

Proof. We notice that for any $u\in U, w\in W$ the value of $T(u,w)$ is zero if and only if $u=-w$. Indeed:

$$T(u,w) = u+w = 0 \iff u=-w.$$

This can be the case only when u and w are in the same subspace, i.e. $\ker T = U\cap W$. Therefore, $\dim\ker T = \dim(U\cap W)$ $\square$

Claim. $\dim\operatorname{im} T = \dim U + \dim W - \dim(U\cap W)$

Proof. We notice that $\operatorname{im} T$ is the sum of all possible vectors from U and W, i.e. $\operatorname{im} T = U+W$. We know that $\dim(U+W) = \dim U + \dim W - \dim(U\cap W)$, as requested. $\square$

We conclude that

$$\begin{aligned}\dim V &= \dim(U\cap W) + \dim U + \dim W - \dim(U\cap W)\\ &= \dim U + \dim W.\end{aligned}$$

We can double-check this result directly. Let $\{u_1,\cdots u_n\}$ be a basis of U ($\dim U = n$) and let $\{w_1,\cdots w_m\}$ be a basis of W ($\dim W = m$). Set of vectors $B = \{(u_1,0),\ldots,(u_n,0),(0,w_1),\ldots,(0,w_m)\}$ is linearly independent and spans $V = U\times W$. Therefore, B is a basis of V and $\dim V = n+m = \dim U + \dim W$.

Problem 4

> Let $A = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$ be a 2×2 matrix with eigenvalue λ.
> a) Show that unless it is zero, the vector $v = (b, \lambda - a)^t$ is an eigenvector.

Proof.

$$Av = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} ab + \lambda b - ab \\ bc + \lambda d - ad \end{bmatrix} = \begin{bmatrix} \lambda b \\ \lambda d - (ad - bc). \end{bmatrix} \tag{2}$$

We know that the characteristic polynomial for a 2×2 matrix can be expressed as follows:

$$p(t) = t^2 - (\operatorname{tr} A)t + (\det A),$$

and λ is the root of the equation $p(t) = 0$, thus:

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0,$$
$$\lambda d - (ad - bc) = \lambda(\lambda - a).$$

Substituting into the expression (2):

$$Av = \begin{bmatrix} \lambda b \\ \lambda(\lambda - a) \end{bmatrix} = \lambda \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \lambda v.$$

Therefore, v must be an eigenvector of A with eigenvalue λ.

□

b) Find a matrix P such that $P^{-1}AP$ is diagonal, assuming that $b \neq 0$ and that A has distinct eigenvalues.

Proof. Since characteristic polynomial $p(t)$ of A is degree 2 (see part b), A has at most two eigenvalues. One of them, λ, is provided to us, and we denote another one λ' (assume $\lambda \neq \lambda'$).

It is known that trace of a matrix is the sum of its eigenvalues. For matrix A we have:

$$\lambda + \lambda' = a + d. \tag{3}$$

By Artin 4.6.10, matrix $\Lambda = P^{-1}AP$ is a diagonal matrix for

$$P = \begin{bmatrix} | & | \\ P_1 & P_2 \\ | & | \end{bmatrix},$$

where P_1 and P_2 are the coordinates (column vectors) of eigenvectors of A in the standard basis.

As we have proved, in the Part a), eigenvectors of A have the form $v = (b, \lambda - a)^t$, thus:

$$v_1 = (b, \lambda - a)^t,$$
$$v_2 = (b, \lambda' - a)^t.$$

Substituting (3):

$$v_2 = (b, a + d - \lambda - a)^t = (b, d - \lambda)^t.$$

Therefore:

$$P = \begin{bmatrix} b & b \\ \lambda - a & d - \lambda \end{bmatrix}.$$

We need to make sure that P is invertible:

$$\det P = b(a + d - 2\lambda) \neq 0.$$

Since $b \neq 0$, we just need to prove that $a + d - 2\lambda \neq 0$. This is indeed the case for $\lambda \neq \lambda'$, since otherwise we would have:

$$a + d = 2\lambda$$

using (3):

$$\begin{aligned} \lambda + \lambda' &= 2\lambda, \\ \lambda' &= \lambda, \end{aligned}$$

which produces a contradiction. Thus, P is invertible. Therefore, matrix $\Lambda = P^{-1}AP$ is diagonal.

□

Problem 5

Let $v = (a_1, \ldots, a_n)$ be a real row vector. We may form the $n! \times n$ matrix M whose rows are obtained by permuting the entries of v in all possible ways. The rows can be listed in an arbitrary order. Thus if $n = 3$, M might be

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_1 & a_3 & a_2 \\ a_2 & a_3 & a_1 \\ a_2 & a_1 & a_3 \\ a_3 & a_1 & a_2 \\ a_3 & a_2 & a_1 \end{bmatrix}.$$

Determine the possible ranks that such matrix could have.

Rank of a matrix is the dimension of its column space and also the dimension of its row space. Thus, rank of A is at most n.

We explore some concrete examples. We notice that for $v = (0, \ldots, 0)$, all entries of matrix M are zero, thus M has rank 0.

We then notice that for $v = (a, 0, \ldots, 0)$, $a \in \mathbb{R}$, the set of permutations of the entries of v contains linearly independent set of vectors

$$\{\, (a, 0, \ldots, 0),\ (0, a, 0, \ldots, 0),\ \ldots,\ (0, \ldots, 0, a) \,\}.$$

In this case, rank of A is n.

We notice that for $v = a(1, 1, \dots, 1)$, $a \in \mathbb{R}$, all entries of matrix M are equal to a, thus M has rank 1.

Now we consider a general case with at least two entries of v not being equal to one another: let $v_m \neq v_k$.

Let the first row of A (vector v_e) correspond to the original vector v. Consider the row vector $v_{(mk)}$ that corresponds to the permutation (mk) of v (swap m-th and k-th entries of v).

$$v_e = (a_1, a_2, \dots, a_m, \dots, a_k, \dots, a_n),$$
$$v_{(mk)} = (a_1, a_2, \dots, a_k, \dots, a_m, \dots, a_n).$$

Consider the following vector p:

$$p = \frac{1}{a_m - a_k}(v_e - v_{(mk)}), \tag{4}$$

which results in

$$p = (0, \dots, 1, 0, \dots, -1, 0, \dots, 0).$$

Denote R the row space of M. Since p is a linear combination of the rows of M, p must be in R.

We notice that we could have chosen different permutations of v instead of v_1 and $v_{(mk)}$. Then, after (4), we would arrive at a vector that is a permutation of the entries of p. In fact, we can achieve any permutation of the entries of p by using the appropriate permutations of the entries of v. Therefore, denoting P the set of all permutations of the entries of p, we conclude that $P \subseteq R$. Furthermore, since P is in R, a vector space spanned by the vectors of P must be a subspace of R.

Consider $P_m \subseteq P$, the set of vectors that correspond to permutations of the entries of p that fix the m-th element of (i.e., 1).

Claim. P_m is a basis of the span of P.

Proof. To prove this, we first notice that set P_m is linearly independent. We also claim that every $p' \in P$ is a linear combination of vectors in P_M. Let the x-th entry of p' be 1 and let its y-th entry be -1, then:

$$p' = p_{(ky)} - p_{(kx)}.$$

Since both $p_{(ky)}, p_{(kx)}$ are elements of P_m, then p' is a linear combination of elements of P_m. Since every element of P can be represented as a linear combination of elements of P_m, P_m is a basis of the span of P.

□

Basis P_m has $n-1$ vectors, thus dimension of the span of P is $n-1$. Since $\operatorname{span} P \subseteq R$, dimension of R must be at least $n-1$ and, thus, M has rank of at least $n-1$.

Therefore, the possible ranks of matrix M are: $0, 1, n-1$ and n.

Problem 6

Determine the finite-dimensional spaces W of differentiable functions $f(x)$ with this property: If f is in W, then $\frac{df}{dx}$ is in W.

We assume vector space W is over field of complex numbers $\mathbb{C}$. Let n be dimension of W.

Proof. We first note that every element of W must be infinitely differentiable. Furthermore, for arbitrary function f in W infinite set of vectors

$$W_f = \left\{ f, \frac{df}{dx}, \frac{d^2f}{dx^2}, \frac{d^3f}{dx^3}, \cdots \right\}.$$

is a subset of W. Since n is the dimension of W, every finite subset of W_f with more than n elements must be linearly dependent. Consider set of vectors

$$\left\{ f, \frac{df}{dx}, \frac{d^2f}{dx^2}, \cdots, \frac{d^nf}{dx^n} \right\},$$

which must be linearly dependent. Then there must exist some coefficients $a = \{a_0, a_1, \dots a_n\}$ such that

$$a_0 f + a_1 \frac{df}{dx} + a_2 \frac{d^2f}{dx^2} + \cdots + a_n \frac{d^nf}{dx^n} = 0. \tag{5}$$

Every $f \in W$ must satisfy this linear homogeneous differential equation for some coefficients a. Solution space of the general equation of the form (5) has basis of

$$B = \left\{ x^k e^{\alpha x} : k \in \mathbb{N},\ 0 \le k \le n,\ \alpha \in \mathbb{C} \right\}.$$

Therefore, any subset B' of B with n elements, such that for every element $g \in B'$, its derivative $\frac{dg}{dx}$ is also in B', is a basis for some space W.

For arbitrary $g = x^k e^{\alpha x}$ in B':

$$\frac{dg}{dx} = \frac{d(x^k e^{\alpha x})}{dx} = k x^{k-1} e^{\alpha x} + \alpha x^k e^{\alpha x},$$

therefore functions

$$B_{\alpha,k} = \left\{ x^k e^{\alpha x},\ x^{k-1} e^{\alpha x},\ \dots,\ e^{\alpha x} \right\}$$

are in B'.

Every W with dimension n can be represented as a direct sum of subspaces, each with a basis of the form B_{α_i,k_i}.

$$W = \bigoplus B_{\alpha_i,k_i},$$

such that every α_i is different and $\sum k_i = n$.

□

Additional Problem 1

> Let V be a finite-dimensional vector space. A linear operator $T: V \to V$ is called a projection if $T^2 = T$ (not necessarily an "orthogonal projection"). Let K and W be the kernel and image of a linear operator T. Prove
> a) T is a projection onto W if and only if the restriction of T to W is the identity map.

We first proof the following claim.

Claim. Any surjective operator $P: U \to U$ is invertible.

Proof. By the rank-nullity theorem:

$$\begin{gathered}\dim U = \dim \ker P + \dim \operatorname{im} P, \\ \dim U = \dim \ker P + \dim U, \\ \dim \ker P = 0,\end{gathered}$$

therefore, any surjective operator is invertible. $\square$

We proceed with the proof of the problem.

Proof. ($\Longrightarrow$): Suppose $T^2 = T$ on V, then $T^2\big|_W = T\big|_W$. Map $T\big|_W : W \to W$ is a surjective operator, thus invertible. Left multiplying $T^2\big|_W = T\big|_W$ by $\left(T\big|_W\right)^{-1}$ we have $T\big|_W = I$, as requested.

($\Longleftarrow$): Suppose $T\big|_W = I$. Since image of T is W, we have $T^2 = T\big|_W T = T$, as requested.

$\square$

> b) If T is a projection, then V is the direct sum $W \oplus K$.

Proof. Consider v, an arbitrary element of $W \cap K$. Then v is in the image of T, we have $Tv = v$. Since v is in the kernel of V, we have $Tv = 0$. We conclude that $v = 0$ and $W \cap K = \{0\}$.

Since intersection of W and K is zero and by the rank-nullity theorem, we have

$$\dim(W + K) = \dim W + \dim K = \dim V.$$

Since $W + K$ are subsets of V and dimension of their sum $W + K$ is equal to the dimension of the space V itself, we conclude that $W + K$ spans V. Since $W \cap K = \{0\}$, we have that V is a direct sum of W and K, as requested.

$\square$

c) The trace of a projection T is equal to its rank.

Proof. Suppose V has dimension n and suppose projection T has rank m. Choose arbitrary basis of W, denote it $\{w_1, \dots, w_m\}$. We can extend this basis to the basis of V:

$$B = \{w_1, \dots, w_m, k_1, \dots, k_{n-m}\}.$$

By part b) of the problem, $W \oplus K = V$, therefore each k_i must be an element of K. Then, each k_i is in the kernel of T, therefore $Tk_i = 0$. Since each vector w_i is in the image of T, by part a) of the problem, we have that $Tv_i = v_i$.

Matrix representation of T with regards to basis B will have the following form:

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

with m ones and $n - m$ zeroes on the diagonal. Such matrix has trace m. Therefore T has trace m equal to the rank of T, as required.

□