



**STATISTICS AND DISCRETE MATHEMATICS**

**(Course Code: 22MA3BSSDM)**

**UNIT-3: JOINT PROBABILITY AND MARKOV CHAINS**

**Joint Probability: Discrete Case**

1. The joint probability distribution of two random variables  $X$  and  $Y$  is given below:

$X \backslash Y$	2	3	4
1	0.06	0.15	0.09
2	0.14	0.35	0.21

Find the marginal distributions of  $X$  and  $Y$ . Also verify whether  $X$  and  $Y$  are stochastically independent.

2. The joint probability distribution of two random variables  $X$  and  $Y$  is given below:

$X \backslash Y$	-3	2	4
1	0.1	0.2	0.2
3	0.3	0.1	0.1

Find:

- (i) the marginal distributions of  $X$  and  $Y$
- (ii)  $\mu_X, \mu_Y, \sigma_X, \sigma_Y$  and  $Cov(X, Y)$

3. The joint probability distribution of two random variables  $X$  and  $Y$  is given below:

$X \backslash Y$	-4	2	7
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
5	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

Find: (i) the marginal distributions of  $X$  and  $Y$  (ii)  $E(X)$  and  $E(Y)$  (iii)  $\sigma_X, \sigma_Y$  (iv)  $Cov(X, Y)$  (v)  $\rho(X, Y)$

4. The joint probability distribution of two random variables  $X$  and  $Y$  is given below:

$X \backslash Y$	1	3	9
2	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$
4	$\frac{1}{4}$	$\frac{1}{4}$	0
6	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$

Find the marginal distributions of  $X$  and  $Y$ . Also evaluate  $Cov(X, Y)$ .

5. The joint probability function of two random variables  $X$  and  $Y$  is given by  $f(x, y) = c(2x + y)$  where  $x$  and  $y$  can assume all integral values such that  $0 \leq x \leq 2, 0 \leq y \leq 3$  and  $f(x, y) = 0$ , otherwise. Find:

- (i) the value of the constant  $c$
  - (ii)  $P(X \geq 1, Y < 2)$
  - (iii) Marginal Probability distribution of  $X$  and  $Y$ .
  - (iv) Check whether  $X$  and  $Y$  are independent.
6. A coin is tossed three times. Let  $X$  denote 0 or 1 according as tail or head occurs on the first toss. Let  $Y$  denote the total number of tails which occur. Determine:
    - (i) the marginal distributions of  $X$  and  $Y$
    - (ii) the joint distributions of  $X$  and  $Y$ . Also find the expected values of  $X + Y$  and  $XY$ .
  7. Two flashcards are selected at random from a box which contains five cards numbered 1, 1, 2, 2 and 3. Find the joint distributions of  $X$  and  $Y$  where  $X$  denotes the sum of two numbers and  $Y$  denote the maximum of two numbers drawn. Also determine  $Cov(X, Y)$ .
  8.  $X$  and  $Y$  are independent random variables.  $X$  takes values 2, 5, 7 with probabilities  $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$  respectively.  $Y$  takes values 3, 4, 5 with probabilities  $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ .
    - (i) Find the joint probability distribution of  $X$  and  $Y$
    - (ii) Show that the covariance of  $X$  and  $Y$  is equal to zero
    - (iii) Find the probability distribution of  $Z = X + Y$
  9. Two fruits are selected at random from a bag containing 3 Apples, 2 Oranges and 4 Mangoes. If  $X$  and  $Y$  are respectively, the number of Apples and the number of Oranges included among the two fruits drawn from the bag, find the probability associated with all possible pair of values  $(x, y)$ . Also find the correlation between the variables  $X$  and  $Y$ .
  10. Two marbles are selected at random from a box containing 3 Blue, 2 Red and 3 Green marbles. If  $X$  and  $Y$  are respectively, the number of Blue marbles and the number of Red marbles selected, find:
    - (i) the probability associated with all possible pair of values
    - (ii) Show that the random variables  $X$  and  $Y$  are not stochastically independent.

### Joint Probability: Continuous Case

1. Find the constant  $k$  so that

$$f(x, y) = \begin{cases} k(x+1)e^{-y}, & 0 < x < 1, y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

is a joint probability density function. Are  $x$  and  $y$  independent?

2. Suppose the joint p.d.f of  $(x, y)$  is given by

$$p(x, y) = \begin{cases} e^{-y}, & x > 0, y > x \\ 0 & \text{elsewhere} \end{cases}$$

Find: (i) the marginal density function of  $x$  (ii) the marginal density function of  $y$   
Check whether  $x$  and  $y$  are independent.

3. Let  $x$  and  $y$  be random variables having the joint density function

$$f(x, y) = \begin{cases} 4xy, & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal density functions of  $x$  and  $y$ . Verify that  $E(x + y) = E(x) + E(y)$  and  $E(xy) = E(x)E(y)$

4. The joint probability density function of two random variables  $x$  and  $y$  is given by

$$p(x, y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the marginal density functions of  $x$  and  $y$ . Also find the covariance between  $x$  and  $y$ .

5. The joint density function of two random variables  $x$  and  $y$  is given by

$$f(x, y) = \begin{cases} kxy, & 0 < x < 4, 1 < y < 5 \\ 0 & \text{elsewhere} \end{cases}$$

Find: (i) the value of  $k$  (ii)  $E(x), E(y), E(xy), E(2x + 3y)$

6. If the joint probability function for  $(x, y)$  is

$$f(x, y) = \begin{cases} c(x^2 + y^2), & 0 \leq x \leq 1, 0 \leq y \leq 1, c \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Determine the following: (i) the value of the constant  $c$  (ii) marginal density functions of  $x$  and  $y$  (iii)  $P\left(x < \frac{1}{2}, y > \frac{1}{2}\right)$  (iv)  $P\left(\frac{1}{4} < x < \frac{3}{4}\right)$  (v)  $P\left(y < \frac{1}{2}\right)$

7. The joint probability function for  $(x, y)$  is given by

$$f(x, y) = \begin{cases} cxy, & 0 \leq x \leq 2, 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

Determine  $c$  and hence evaluate  $P\left(\frac{1}{2} < x < 1\right)$

8. Verify that  $f(x, y) = \begin{cases} e^{-(x+y)}, & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

is a density function of a joint probability distribution. Hence evaluate:

- (i)  $P\left(\frac{1}{2} < x < 2, 0 < y < 4\right)$  (ii)  $P(x < 1)$  (iii)  $P(x \leq y)$  (iv)  $P(x > y)$   
(v)  $P(x + y \leq 1)$

9. Suppose that the joint density function of two random variables  $x$  and  $y$  is

$$f(x, y) = \begin{cases} x^2 + \frac{1}{3}xy, & 0 < x < 1, 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Evaluate the following:

- (i)  $P\left(x \geq \frac{1}{2}\right)$  (ii)  $P(x + y \geq 1)$  (iii)  $P(Y \leq x)$

10. For the distribution given by the density function

$$f(x, y) = \begin{cases} \frac{1}{96}xy, & 0 < x < 4, 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}$$

Evaluate: (i)  $P(1 < x < 2, 2 < y < 3)$  (ii)  $P(x \geq 3, y \leq 2)$  (iii)  $P(x + y \leq 3)$  (iv)  $P(x + y \leq 3)$

1. Alice and Bob meet for lunch every day at a random time between noon and 1 P.M. Bob always arrives after Alice, but otherwise, they are equally likely to arrive at any time. The joint p.d.f. of  $X$ , the time Alice arrives (in minutes after 12 P.M.), and  $Y$ , the time Bob arrives (in minutes after 12 P.M.), is given

$$f(x, y) = \begin{cases} c & 0 \leq x < y \leq 60 \\ 0 & \text{otherwise} \end{cases}$$

where  $c$  is a constant.

- Sketch a birds-eye view of the joint p.d.f. (Because the p.d.f. is constant, you should be able to calculate all probabilities using geometry, rather than integration).
- Determine the value of  $c$  that makes this a valid joint p.d.f.
- Calculate the probability that Bob arrives more than 25% later than Alice. That is, what is  $P(Y > 1.25X)$ ? (Sketch a picture of this region.)
- Suppose  $X$  and  $Y$  are continuous random variables with joint p.d.f.  $f(x, y)$ . What is  $P(X=Y)$  and why?

2. An ecologist selects a point inside a circular sampling region according to a uniform distribution. Let  $X$  be the  $x$ -coordinate of the point selected and  $Y$  be the  $y$ -coordinate of the point selected. If the circle is centered at  $(0,0)$  and has radius  $r$ , then the joint pdf

of  $X$  and  $Y$  is 
$$f(x, y) = \begin{cases} c^2 & x^2 + y^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases}$$

- Determine the value of  $c$  that makes this a valid joint p.d.f.
- What is the probability that the selected point is within  $r/2$  of the center of the circular region? (*Hint: Use geometry.*)
- What is the probability that *both*  $X$  and  $Y$  differ from 0 by at most  $r/2$ ?

3. A company produces cans of mixed nuts containing almonds, cashews, and peanuts. Each can is exactly 1 lb, but the amount of each type of nut is random. The joint p.d.f. of  $X$ , the amount of almonds, and  $Y$ , the amount of cashews, is

$$f(x, y) = \begin{cases} 24xy & 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

Show that the probability there are more almonds than cashews is 0.50.

**Markov Chain**

1. Define (i) Stochastic matrix (ii) Absorbing State (iii) Fixed Probability Vector.
2. Verify that the matrix  $A = \begin{pmatrix} 0 & 0 & 1 \\ 0.5 & 0.25 & 0.25 \\ 0 & 1 & 0 \end{pmatrix}$  is a regular stochastic matrix.
3. Verify that the matrix  $A = \begin{pmatrix} 0 & 1 \\ 0.3 & 0.7 \end{pmatrix}$  is a regular stochastic matrix.
4. Prove that the Markov Chain whose transition probability matrix is  $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$  is irreducible.
5. Find the fixed probability vector for the transition matrix  $P = \begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0 & 0.4 & 0.6 \\ 0 & 0.2 & 0.8 \end{bmatrix}$ .
6. Find the unique fixed probability vector for the regular stochastic matrix  $A = \begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix}$ .
7. Find the unique fixed probability vector for the regular stochastic matrix  $A = \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{pmatrix}$ .
8. Find the unique fixed probability vector for the regular stochastic matrix  $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$ .
9. A student's study habits are as follows. If he studies one night, he is 70% sure not to study the next night. On the other hand, if he does not study one night, he is 60% sure not to study the next night. In a long run how often does he study?
10. Consider a game of "ladder climbing". There are 5 levels in the game, level 1 is the lowest (bottom) and level 5 is the highest (top). A player starts at the bottom. Each time, a fair coin is tossed. If it turns up heads, the player moves up one rung. If tails, the player moves down to the very bottom. Once at the top level, the player moves to the very bottom if a tail turns up, and stays at the top if head turns up.
  - a) Find the transition probability matrix.
  - b) Find the two-step transition probability matrix.
  - c) Find the steady-state distribution of the Markov chain.

11. The pattern of sunny and rainy days on the planet Rainbow is a homogeneous Markov chain with two states. Every sunny day is followed by another sunny day with probability 0.8. Every rainy day is followed by another rainy day with probability 0.6.
- If today is sunny on Rainbow, what is the chance of rain the day after tomorrow?
  - Compute the probability that April 1 next year is rainy on Rainbow.
12. A computer device can be either in a busy mode (state 1) processing a task, or in an idle mode (state 2), when there are no tasks to process. Being in a busy mode, it can finish a task and enter an idle mode any minute with the probability 0.2. Thus, with the probability 0.8 it stays another minute in a busy mode. Being in an idle mode, it receives a new task any minute with the probability 0.1 and enters a busy mode. Thus, it stays another minute in an idle mode with the probability 0.9. The initial state is idle. Let  $X_n$  be the state of the device after  $n$  minutes.
- Find the distribution of  $X_2$ .
  - Find the steady-state distribution of  $X_n$ .
13. In a certain city, the weather on a day is reported as sunny, cloudy or rainy. If a day is sunny, the probability that the next day is sunny is 70%, cloudy is 20% and rainy is 10%. If a day is cloudy, the probability that the next day is sunny is 30%, cloudy is 20% and rainy is 50%. If a day is rainy, the probability that the next day is sunny is 30%, cloudy is 30% and rainy is 40%. If a Sunday is sunny, find the probability that the Wednesday is rainy.
14. A housewife buys 3 kinds of cereals A, B, and C. She never buys the same cereal in successive weeks. If she buys cereal A, the next week she buys cereal B. However, if she buys cereal B or C, the next week she is three times as likely to buy cereal A as the other cereal. In the long run how often she buys each of the three cereals?
15. A salesman's territory consists of 3 cities A, B and C. He never sells in the same city on successive days. If he sells in city A, then the next day he sells in city B. However, if he sells in either B or C, then the next day he is twice as likely to sell in city A as in other city. In the long run, how often does he sell in each of the cities?
16. A housewife buys 3 brands of soaps: A, B, C. She never buys the same brand on successive weeks. If she buys brand A in a week, she buys brand B in the next week. If she buys the brand other than A in a week, then in the next week she is three times as likely to buy the brand A as the other brand. Supposing that she has brought brand B in the first week, find the probability of her buying each of the three brands in the fourth week.
17. Each year a man trades his car for a new car in 3 brands of the popular company Maruti Udyog Limited. If he has "Standard", he trades it for "Zen". If he has a "Zen", he trades it for "Esteem". If he has "Esteem", he is just as likely to trade it for a new "Esteem" or for a "Zen" or a "Standard". In 2010 he bought his first car which was "Esteem". Find the probability that he has on 2012 "Esteem" and on 2013 "Zen".
18. Three boys A, B, C are throwing ball to each other. A always throws the ball to B and B always throws the ball to C. C is just as likely to throw the ball to B as to A. If C was the first person to throw the ball. Find the probability after three throws that
- A has the ball
  - B has the ball
  - C has the ball
19. A habitual gambler is a member of two clubs A and B. He visits either of the clubs everyday for playing cards. He never visits club A on two consecutive days. But, if he visits club B on a particular day, then the next day he is as likely to visit club B or club A.

- (i) Find the transition matrix of this Markov chain.
  - (ii) Show that the matrix is a regular stochastic matrix.
  - (iii) Find the unique fixed probability vector.
  - (iv) If the person had visited club B on Monday, find the probability that he visits club A on Thursday.
20. A gambler's luck follows a pattern. If he wins a game, the probability of winning the next game is 0.6. However, if he loses a game, the probability of losing the next game is 0.7. There is an even chance of gambler winning the first game. If so
- i. What is the probability of he winning the second game?
  - ii. What is the probability of he winning the third game?
  - iii. In the long run, how often he will win?
21. An auto insurance company classifies its customers in three categories: poor, satisfactory and preferred. No one moves from poor to preferred or from preferred to poor in one year. 40% of the customers in the poor category become satisfactory, 30% of those in the satisfactory category moves to preferred, while 10% become poor; 20% of those in the preferred category are downgraded to satisfactory. (i) Write the transition matrix for the model (ii) Define Irreducible Markov Chain (iii) Is the Markov chain irreducible?
22. Assume that a man's profession can be classified as professional, skilled labourer, or unskilled labourer. Assume that, of the sons of professional men, 80 percent are professional, 10 percent are skilled labourers, and 10 percent are unskilled labourers. In the case of sons of skilled labourers, 60 percent are skilled labourers, 20 percent are professional, and 20 percent are unskilled. Finally, in the case of unskilled labourers, 50 percent of the sons are unskilled labourers, and 25 percent each are in the other two categories. Assume that every man has at least one son, and form a Markov chain by following the profession of a randomly chosen son of a given family through several generations. Set up the matrix of transition probabilities.
- (i) Find the probability that a randomly chosen grandson of an unskilled labourer is a professional man.
  - (ii) In the long run, what is the probability that a great grandson of a skilled labourer is a professional man.