Further Issues Using OLS with Time Series Data

Chapter 11

Wooldridge: Introductory Econometrics:
A Modern Approach, 5e

with some adjustments by myself

Analyzing Time Series: Further Issues Using OLS

- *Where were we?*
  - A key requirement for large sample analysis of time series is that the time series in question are <u>**stationary and weakly dependent**</u>
  - Loosely speaking, a time series is stationary if its stochastic properties and its temporal dependence structure do not change over time
  - A stochastic process is <u>weakly dependent</u>, if x_t is „almost independent" of x_{t+h} if $h \rightarrow \infty$

→ MA(1) and AR(1) processes are nice examples of the latter (as long as, for the AR process, the stability condition $|\rho|<1$ holds)

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- **One more remark**
  - Note that a series may be nonstationary but weakly dependent:
  - In particular, a trending series, though certainly nonstationary, can be stationary about its time trend as well as weakly dependent: a **trend-stationary process.**

➔ Such processes can be used in time series regression models, provided that we include the appropriate time trend in the model (as in Ch. 10)

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- *11.2* **Asymptotic properties of OLS**
- **Assumption TS.1` (Linear in parameters)**
  - Same as assumption TS.1
  - … but now the dependent and independent variables are assumed to be stationary and weakly dependent, …
  - … and now there may be lagged dependent variables among the explanatory variables
- **Assumption TS.2` (No perfect collinearity)**
  - Same as assumption TS.2

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- **Assumption TS.3' (Zero conditional mean)**
  - Now the explanatory variables are assumed to be only contempo-raneously exogenous rather than strictly exogenous, i.e.

$$E(u_t|\mathbf{x}_t) = 0$$

The explanatory variables of the same period are uninformative about the mean of the error term

  - By stationarity, if contemporaneous exogeneity holds for one time period, it holds for them all.

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- **Theorem 11.1 (Consistency of OLS)**

$$TS.1'–TS.3' \quad \Rightarrow \quad plim\ \widehat{\beta}_j = \beta_j, \quad j = 0, 1, \ldots, k$$

Note: For consistency it would even suffice to assume that the explanatory variables are merely contemporaneously *uncorrelated* with the error term: E[u_t]=0, and Cov[x_{tj} ,u_j]=0, for all j=1, .., k.

- The theorem establishes consistency of the OLS estimator, but not necessarily unbiasedness (as we had in Ch. 10).

- In comparison with Ch. 10, we have weakened the exogeneity requirements, but weak dependence of the time series remains crucial

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- **Why is it important to relax the strict exogeneity assumption?**
  - Strict exogeneity was a serious restriction because it ruled out all kinds of dynamic relationships between explanatory variables and the error term
  - In particular, it ruled out feedback from the dependent variable on future values of the explanatory variables (which is very common in economic contexts)

➔ *Strict exogeneity precluded the use of lagged dependent variables as regressors*

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- **Why do lagged dependent variables violate strict exogeneity?**

$$y_t = \beta_0 + \beta_1 y_{t-1} + u_t$$

This is the simplest possible regression model with a lagged dependent variable

Contemporanous exogeneity: $E(u_t | y_{t-1}) = 0$

Strict exogeneity: $E(u_t | y_0, y_1, \ldots, y_{n-1}) = 0$

Strict exogeneity would imply that the error term is uncorrelated with all y_t, t=1, …, n-1

This leads to a contradiction because:

$$Cov(y_t, u_t) = \beta_1 Cov(y_{t-1}, u_t) + Var(u_t) > 0$$

Hence, strict exogeneity cannot be satisfied with a lagged dependent variable

- **OLS estimation in the presence of lagged dependent variables**
  - Under contemporaneous exogeneity, OLS is consistent but biased

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- **Assumption TS.4' (Homoscedasticity)**

$$Var(u_t|\mathbf{x}_t) = Var(u_t) = \sigma^2$$

The errors are *contemporaneously* homoscedastic

- **Assumption TS.5' (No serial correlation)**

$$Corr(u_t, u_s|\mathbf{x}_t, \mathbf{x}_s) = 0, \ \ t \neq s$$

Conditional on the explanatory variables in periods *t* and *s*, the errors are uncorrelated

  - Notice: in contrast with TS.4 and TS.5 (Ch. 10), for the large sample properties we are not conditioning on the full matrix **X.**
  - For TS.5', the key idea remains the same: errors from different time periods are uncorrelated

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- **Theorem 11.2 (Asymptotic normality of OLS)**
  - Under assumptions TS.1' – TS.5', the OLS estimators are asymptotically normally distributed.
  - Further, the usual OLS standard errors, *t*-statistics and *F*-statistics are asymptotically valid.

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- **Example: Efficient Markets Hypothesis (EMH)**

The EMH in a strict form states that information observable to the market prior to week t should not help to predict the return during week t. A simplification assumes that only past returns are considered as relevant information to predict the return in week t. This implies that

$$E(return_t | return_{t-1}, return_{t-2}, \ldots) = E(return_t)$$

A simple way to test the EMH is to specify an AR(1) model. Under the EMH assumption,TS.3' holds, so that an OLS regression can be used to test whether this week's returns depend on last week's.

$$\widehat{return}_t = \underset{(.081)}{.180} + \boxed{\underset{(.038)}{.059}}\, return_{t-1}$$

$$n = 689, R^2 = .0035, \bar{R}^2 = .0020$$

There is no evidence against the EMH. Including more lagged returns yields similar results.

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- *11.3* **Using highly persistent time series in regression analysis**

  - Unfortunately many economic time series violate weak dependence because they are *highly persistent* (= *strongly dependent*)
  - In this case OLS methods are generally invalid (unless the CLM hold)
  - In some cases transformations to weak dependence are possible

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- **Random walks**

$$\boxed{y_t = y_{t-1} + e_t}$$

The random walk is called random walk because it wanders from the previous position y_{t-1} by an i.i.d. random amount e_t

$$\Rightarrow \; y_t = (y_{t-2} + e_{t-1}) + e_t = \ldots = e_t + e_{t-1} + \ldots + e_1 + y_0$$

The value today is the accumulation of all past shocks plus an initial value.
This is the reason why the random walk is highly persistent: the effect of a shock will be contained in the series forever.

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- **Random walks vs. Stable AR(1)**
  - Consider a general AR(1) process: $y_t = \rho_1 y_{t-1} + e_t$
  - For ρ_1=1, we have the random walk process, and derive that

    $$y_{t+h} = e_{t+h} + e_{t+h-1} + \ldots + e_{t+1} + y_t$$

    and thus $\mathrm{E}(y_{t+h}|y_t) = y_t$, for all $h \geq 1$

    Hence, for ρ_1=1, the importance of y_t remains persistent.
  - For ρ_1≠1, we have $\mathrm{E}(y_{t+h}|y_t) = \rho_1^h y_t$, for all $h \geq 1$
  - Under stability, $|\rho_1| < 1$, and thus $\mathrm{E}(y_{t+h}|y_t)$ approaches zero when h approaches infinity, hence the value of y_t becomes less and less important and eventually $\mathrm{E}(y_t) = 0$. the conditional expectation gets closer and closer to the unconditional expectation

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- **Random walks**

$$y_t = y_{t-1} + e_t$$

The effect of a shock will be contained in the series forever.

$$E(y_t) = E(y_0)$$

Moreover,

$$y_t = (y_{t-2} + e_{t-1}) + e_t = \ldots = e_t + e_{t-1} + \ldots + e_1 + y_0$$

implies that:

$$Var(y_t) = \sigma_e^2 t$$

$$Corr(y_t, y_{t+h}) = \sqrt{t/(t+h)}$$

The random walk is <u>not covariance stationary</u> because its variance and its covariance depend on time.

It is also <u>not weakly dependent</u> because the correlation between observations vanishes very slowly and this depends on how large *t* is.

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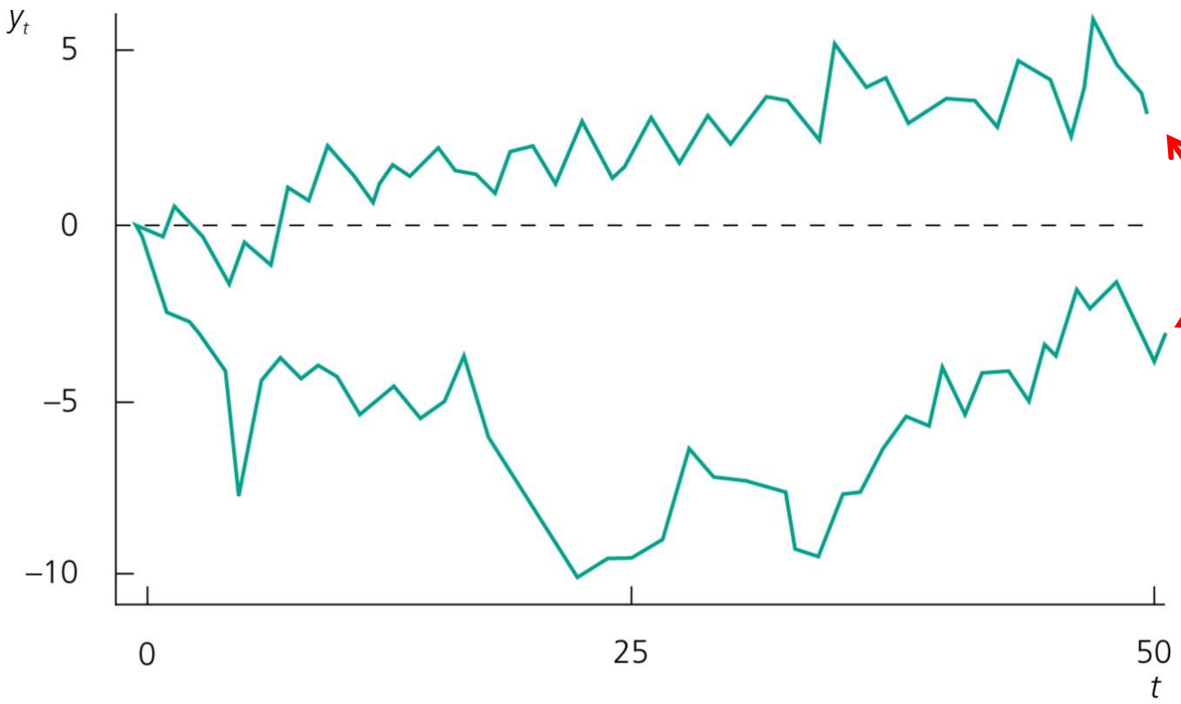
- **Examples for random walk realizations**



The random walks wander around with no clear direction

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- **Unit root process**
  - A random walk is a special case of what is known as a **unit root process**, a name that comes from the fact that ρ=1 in the AR(1) model.
  - A more general class of unit root processes is generated as in $y_t = \rho_1 y_{t-1} + e_t$ but allowing $\{e_t\}$ to be a general, weakly dependent series.
  - For example, $\{e_t\}$ could be an MA(1) process, or a stable AR(1) process, instead of an i.i.d. sequence.
  - The properties of the random walk no longer hold, but the key feature of $\{y_t\}$ is preserved: the value of y today is highly correlated with y even in the distant future.

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- **Unit root process**

  - From an economic point of view it is important to know whether a time series is highly persistent. In highly persistent time series, shocks or policy changes have lasting/permanent effects, in weakly dependent processes their effects are transitory.

  - It is important not to confuse trending and highly persistent behaviors; sequences may be highly persistent, but without obvious upward or downward trend.
  - However, often highly persistent series also contain a clear trend

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- **Random walks with drift**

$$y_t = \boxed{\alpha_0} + y_{t-1} + e_t$$

In addition to the usual random walk mechanism, there is a deterministic increase/decrease (= drift) in each period

$$\Rightarrow \quad y_t = \boxed{\alpha_0 t} + e_t + e_{t-1} + \ldots + e_1 + y_0$$

This leads to a linear time trend around which the series follows its random walk behaviour. As there is no clear direction in which the random walk develops, it may also wander away from the trend.

$$E(y_t) = \boxed{\alpha_0 t} + E(y_0)$$

Otherwise, the random walk with drift has similar properties as the random walk without drift.

$$Var(y_t) = \sigma_e^2 t$$

$$Corr(y_t, y_{t+h}) = \sqrt{t/(t+h)}$$

Random walks with drift are <u>not covariance stationary</u> and <u>not weakly dependent</u>.

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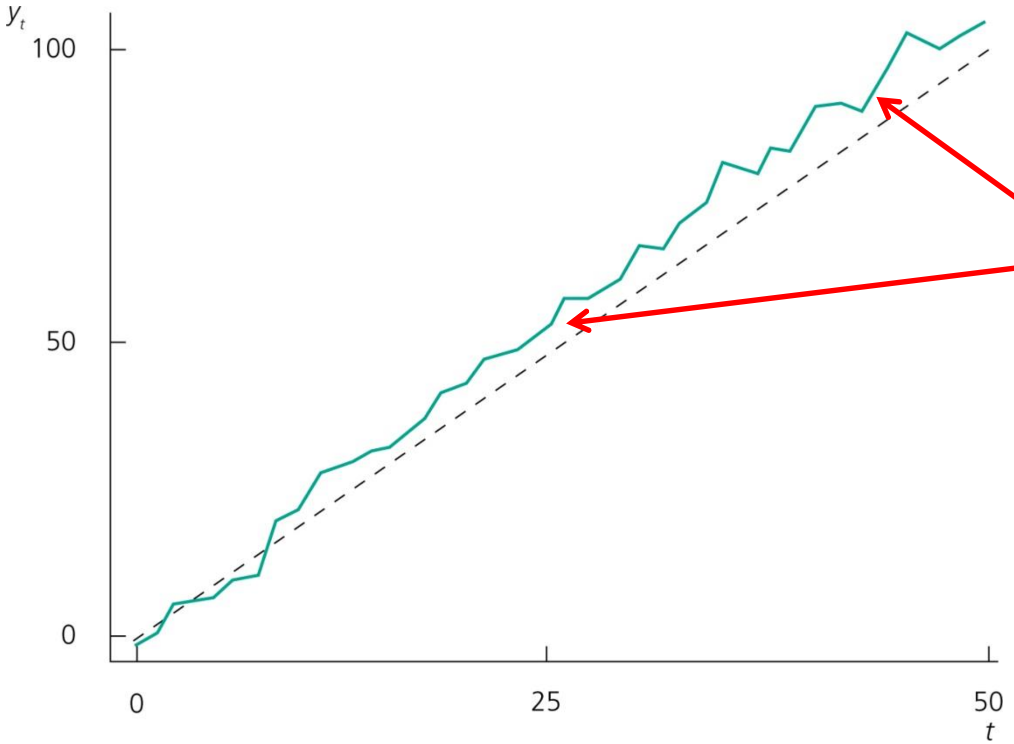
- **Sample path of a random walk with drift**



Note that the series does not regularly return to the trend line.

Random walks with drift may be good models for time series that have an obvious trend but are not weakly dependent.

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Transformations on highly persistent time series

- **Order of integration**
  - Weakly dependent time series are *integrated of order zero* (= I(0))
  - If a time series has to be differenced one time in order to obtain a weakly dependent series, it is called *integrated of order one* (= I(1))
- **Examples for I(1) processes**

$$y_t = y_{t-1} + e_t \;\Rightarrow\; \Delta y_t = y_t - y_{t-1} = e_t$$

$$\log(y_t) = \log(y_{t-1}) + e_t \;\Rightarrow\; \Delta \log(y_t) = e_t$$

After differencing, the resulting series are weakly dependent (because e_t is weakly dependent).

➔ **Differencing is often a way to achieve weak dependence**

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- **Deciding whether a time series is I(1)**
  - There are statistical tests for testing whether a time series is I(1) (= unit root tests, $H_0: \rho=1$ vs. $H_1: \rho<1$); covered in *Chapter 18*
  - Alternatively, look at the sample first order autocorrelation:

$$\widehat{\rho}_1 = \widehat{Corr}(y_t, y_{t-1})$$

Measures how strongly adjacent times series observations are related to each other.

  - If the sample first order autocorrelation is close to one, this suggests that the time series may be highly persistent (= contains a unit root) – but no hard-and-fast rule exists.
  - Alternatively, the series may have a deterministic trend
  - Both unit root and trend may be eliminated by differencing

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- **Example: Wages and productivity**

Include trend because both series display clear trends.

$$\widehat{\log}(hrwage) = \underset{(.37)}{-\ 5.33} + \underset{(.09)}{1.64}\ \log(outphr) - \underset{(.002)}{.018}\ t$$

$$n = 41, R^2 = .971, \bar{R}^2 = .970$$

The elasticity of hourly wage with respect to output per hour (=productivity) seems implausibly large.

It turns out that even after detrending, both series display sample autocorrelations close to one so that estimating the equation in first differences seems more adequate:

$$\Delta\widehat{\log}(hrwage) = \underset{(.0042)}{-\ .0036} + \underset{(.173)}{.809}\ \Delta \log(outphr)$$

$$n = 40, R^2 = .364, \bar{R}^2 = .348$$

This estimate of the elasticity of hourly wage with respect to productivity makes much more sense.

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- *11.4* **Dynamically complete models: serial correlation assumption**
  - A model is said to be dynamically complete if enough lagged variables have been included as explanatory variables so that further lags do not help to explain the dependent variable:

$$E(y_t|\mathbf{x}_t, y_{t-1}, \mathbf{x}_{t-1}, y_{t-2}, \ldots) = E(y_t|\mathbf{x}_t)$$

- **Dynamic completeness implies absence of serial correlation**
  - If further lags actually belong in the regression, their omission will cause serial correlation (if the variables are serially correlated)

➔ *In Chapter 12 we will see models that detect and correct serial correlation*

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- *11.5* **Homoskedasticity Assumption in Time Series Models**
  - The homoskedasticity assumption for time series regressions, particularly TS.4′, looks very similar to that for cross-sectional regressions
  - However, since $\mathbf{x}_t$ can contain lagged explanatory variables as well as lagged dependent variables y, the meaning of the assumption deserves some discussion.

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- **<u>Homoskedasticity in Time Series Models</u>**
  - For example, in a simple static model, $y_t = \beta_0 + \beta_1 z_t + u_t$, assumption TS.4′ requires that $\mathrm{Var}(u_t|z_t) = \sigma^2$ is constant, even though $\mathrm{E}(y_t|z_t)$ is a linear function of z_t .
  - For example, in an AR(1) model, the homoskedasticity assumption requires that $\mathrm{Var}(u_t|y_{t-1}) = \mathrm{Var}(y_t|y_{t-1}) = \sigma^2$, hence, constant variance, even though $\mathrm{E}(y_t|y_{t-1})$ clearly depends on y_{t-1}.

→ *In Chapter 12 we will see models that permit dynamic forms of heteroskedasticity.*