Serial Correlation and Heteroscedasticity in Time Series Regressions

Chapter 12

Wooldridge: Introductory Econometrics:
A Modern Approach, 5e

with some adjustments by me and others

Analyzing Time Series: Serial Correl. and Heterosced.

- Chapter 10 and 11, two crucial assumptions for OLS to be BLUE:
  - Homoskedastic errors (TS.4‵)
  - No serial correlation (TS.5‵)
- When a model is appropriately specified (dynamically complete), there is no serial correlation.
  - This Chapter: tests, consequences, and remedies in case of serial correlation
  - (and something more regarding heteroskedasticity)

Analyzing Time Series: Serial Correl. and Heterosced.

- *12.1* **Properties of OLS with serially correlated errors**

  - OLS still unbiased and consistent if errors are serially correlated
    - TS.1' – TS.3' do not require absence of serial correlation (but strict exogeneity is required)
    - TS.1' – TS.3' do not require absence of serial correlation (but weakly dependence is required)
    - However, OLS is not BLUE when there is serial correlation (TS.4', TS.5')

→ *see next slides*

Analyzing Time Series: Serial Correl. and Heterosced.

- **OLS standard errors, tests, and efficiency with serial correlation**
- Consider a first-order serially correlated error process:

$$u_t = \rho u_{t-1} + e_t, t = 1, 2, \ldots, n$$

with $|\rho| < 1$, and where the e_t are uncorrelated random variables with mean zero and constant variance (i.i.d.).

- What will be the variance of the OLS slope estimator in a simple y on x regression model, $y_t = \beta_0 + \beta_1 x_t + u_t$? (For simplicity, center the x series, so that $\bar{x} = 0$.)
- Then the OLS estimator will be:

$$\hat{\beta}_1 = \beta_1 + \mathrm{SST}_x^{-1} \sum_{t=1}^{n} x_t u_t$$

where SST_x is the sum of squares of the x series.

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- **OLS standard errors, tests, and efficiency with serial correlation**
- In computing the variance of $\hat{\beta}_1$, conditional on ***X***, we must account for the serial correlation in the u_t process:

$$\begin{aligned}\operatorname{Var}(\hat{\beta}_1) &= \mathrm{SST}_x^{-2}\operatorname{Var}\left(\sum_{t=1}^{n} x_t u_t\right) \\ &= \mathrm{SST}_x^{-2}\left(\sum_{t=1}^{n} x_t^2 \operatorname{Var}(u_t) + 2\sum_{t=1}^{n-1}\sum_{j=1}^{n-t} x_t x_{t+j}\,\mathrm{E}(u_t u_{t+j})\right) \\ &= \sigma^2/\mathrm{SST}_x + 2(\sigma^2/\mathrm{SST}_x^2)\sum_{t=1}^{n-1}\sum_{j=1}^{n-t}\rho^j x_t x_{t+j},\end{aligned}$$

where $\sigma^2 = \operatorname{Var}(u_t)$, and we have used the fact that

$$\mathrm{E}(u_t u_{t+j}) = \operatorname{Cov}(u_t, u_{t+j}) = \rho^j \sigma^2$$

- Notice that the first term in this expression is merely the OLS variance $\hat{\beta}_1$ in the absence of serial correlation (ρ=0).

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- **OLS standard errors, tests, and efficiency with serial correlation**
  - When will the second term be nonzero? When $\rho \neq 0$, and when also the x process itself is autocorrelated, this double summation will have a nonzero value.
  - Nothing prevents the explanatory variables from exhibiting autocorrelation (and in fact many explanatory variables take on similar values through time).
  - Hence, the only way in which this second term will vanish is if ρ is zero, that is, if u is not serially correlated.
  - In the presence of serial correlation, the second term will cause the standard OLS variances of our regression parameters to be biased and inconsistent.

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- **OLS standard errors, tests, and efficiency with serial correlation**
  - In the presence of serial correlation, the second term will cause the standard OLS variances of our regression parameters to be biased and inconsistent.
  - In most applications, when serial correlation arises, ρ is positive, so that successive errors are positively correlated.
  - In that case, the second term will be positive as well.
  - Recall that this expression is the true variance of the regression parameter, and realize that OLS will only consider the first term. Hence, in that case OLS will seriously underestimate the variance of the parameter, and the *t*-statistic will be much too high.

Analyzing Time Series: Serial Correl. and Heterosced.

- **Properties of OLS with serially correlated errors**

  Hence, summarizing:

  - OLS still unbiased and consistent if errors are serially correlated

  - OLS standard errors and tests will be invalid if there is serial correlation
  - OLS will not be efficient anymore if there is serial correlation

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- **<u>Serial correlation and the presence of lagged dependent variables</u>**
  - Is OLS inconsistent if there are serially correlated errors in the presence of lagged dependent variables?

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- **Serial correlation and the presence of lagged dependent variables**
  - OLS can be consistent even when lagged dependent variables and serially correlated errors are at hand.
  - To illustrate, suppose a linear expected value:

    $$\mathrm{E}(y_t|y_{t-1}) = \beta_0 + \beta_1 y_{t-1}$$

    and assume stability, $|\beta_1| < 1$.
  - We can always write this with an error term as:

    $$y_t = \beta_0 + \beta_1 y_{t-1} + u_t,$$

    $$\mathrm{E}(u_t|y_{t-1}) = 0.$$
  - By construction, this model satisfies the key zero conditional mean Assumption TS.3' for consistency of OLS – even when, without further assumptions, the errors { u_t } can be serially correlated

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- **Serial correlation and the presence of lagged dependent variables**
  - Nevertheless, when are OLS estimators inconsistent with lagged dep.?
  - Consider

$$y_t = \beta_0 + \beta_1 y_{t-1} + u_t,$$

$$u_t = \rho u_{t-1} + e_t, t = 1, 2, \ldots, n$$

$$|\rho| < 1,$$

  - Hence, the same model as before but assuming a stable AR(1) error process, where $\mathrm{E}(e_t|u_{t-1}, u_{t-2}, \ldots) = \mathrm{E}(e_t|y_{t-1}, y_{t-2}, \ldots) = 0$
  - Because, by assumption, e_t is uncorrelated with y_{t-1} , we get

$$\mathrm{Cov}(y_{t-1}, u_t) = \rho \mathrm{Cov}(y_{t-1}, u_{t-1})$$

which is nonzero unless ρ=0, and renders OLS estimates inconsistent

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- **Serial correlation and the presence of lagged dependent variables**
  - Note that, combining the above equations, we see that y_t effectively follows an AR(2) process:

$$\begin{aligned} y_t &= \beta_0 + \beta_1 y_{t-1} + \rho(y_{t-1} - \beta_0 - \beta_1 y_{t-2}) + e_t \\ &= \beta_0(1 - \rho) + (\beta_1 + \rho)y_{t-1} - \rho\beta_1 y_{t-2} + e_t \\ &= \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + e_t, \end{aligned}$$

    and thus:

$$\mathrm{E}(y_t|y_{t-1}, y_{t-2}, \ldots) = \mathrm{E}(y_t|y_{t-1}, y_{t-2}) = \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 y_{t-2}.$$

  - Hence, the expected value of y_t, given all past y, depends on two lags of y. For any practical purpose, that is the model we are interested in.
  - Under the appropriate stability conditions for an AR(2) model, OLS estimation produces consistent and asymptotically normal estimators of the α_j – but the AR(1) model is a dynamic misspecification.

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- **Serial correlation and lagged dependent variables**

  Hence, summarizing:

  - „Is OLS inconsistent if there are serially correlated errors in the presence of lagged dependent variables?"

  - No, not necessarily: Including enough lags so that TS.3' holds guarantees consistency
  - Including too few lags will cause an omitted variable problem and serial correlation because some lagged dep. var. end up in the error term

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- *12.2* **Testing for serial correlation**
- **Testing for AR(1) serial correlation with strictly exog. regressors**

$$y_t = \beta_0 + \beta_1 x_{t1} + \cdots + \beta_k x_{tk} + u_t$$

  - The error terms can be serially correlated in numerous ways, but the most popular model, and the simplest to work with, is the AR(1) model:

$$u_t = \rho u_{t-1} + e_t$$

AR(1) model for serial correlation, with an i.i.d. series e_t

  - The null hypothesis of the test is that the appropriate Gauss-Markov assumption is true, that is: $H_0 : \rho = 0$
  - An asymptotic *t*-test, replacing the true but unobserved errors by estimated residuals, $\hat{u}_t = \rho \hat{u}_{t-1} + error$, can now be performed

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- **<u>Testing for serial correlation</u>**

Example: Static Phillips curve

- Remember that:

$$\widehat{inf}_t = \underset{(1.72)}{1.42} + \underset{(.289)}{.468}\ unem_t$$

$$n = 49, R^2 = .053, \bar{R}^2 = .033$$

- Then, an AR(1) regression using the estimated residuals gives:

$$\hat{\rho} = .573, t = 4.93, p - value = .000$$

and thus, reject null hypothesis of no serial correlation

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- **Testing for AR(1) serial correlation with general regressors**
  - The t test for autocorrelation can be easily generalized to allow for the possibility that the explanatory variables are not strictly exogenous (such as, lagged dependent variables):

$$\widehat{u}_t = \alpha_0 + \alpha_1 x_{t1} + \cdots + \alpha_k x_{tk} + \boxed{\rho}\,\widehat{u}_{t-1} + error$$

The test now allows for the possibility that the strict exogeneity assumption is violated.

Test for $H_0: \rho = 0$

  - The test may be carried out in a heteroscedasticity robust way

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- **General Breusch-Godfrey test for AR(q) serial correlation**
  - One of the disadvantages of tests for AR(1) errors is that they consider precisely that alternative hypothesis.
  - In many cases, serial correlation may manifest itself in a more complex relationship, involving higher-order autocorrelations; e.g. AR(*q*).
  - A logical extension to the *t* test described before and the Durbin "h" test is the Breusch–Godfrey test, which considers the null of nonautocorrelated errors against an alternative that they are AR(*q*):

$$\widehat{u}_t = \alpha_0 + \alpha_1 x_{t1} + \cdots + \alpha_k x_{tk} + \boxed{\rho_1} \widehat{u}_{t-1} + \cdots + \boxed{\rho_q} \widehat{u}_{t-q} + \ldots$$

Test $H_0 : \rho_1 = \cdots = \rho_q = 0$

Toward a Lagrange Multiplier (*LM*) Test

- Equation of interest:

$$y_t = \beta_1 + \beta_2 x_t + e_t$$

- If e_t and e_{t-1} are correlated, then one way to model the relationship between them is to write:

$$e_t = \rho e_{t-1} + v_t$$

  - We can substitute this into a simple regression equation:

$$y_t = \beta_1 + \beta_2 x_t + \rho e_{t-1} + v_t$$

  - Assume that v_t and e_{t-1} are independent

- To derive the relevant auxiliary regression for the autocorrelation *LM* test, we write the test equation as:

$$y_t = \beta_1 + \beta_2 x_t + \rho \hat{e}_{t-1} + v_t$$

  - But since we know that $y_t = b_1 + b_2 x_t + \hat{e}_t$, we get:

$$b_1 + b_2 x_t + \hat{e}_t = \beta_1 + \beta_2 x_t + \rho \hat{e}_{t-1} + v_t$$

- Rearranging, we get:

$$\begin{aligned}\hat{e}_t &= (\beta_1 - b_1) + (\beta_2 - b_2)x_t + \rho\hat{e}_{t-1} + v_t \\ &= \gamma_1 + \gamma_2 x_t + \rho\hat{e}_{t-1} + v_t\end{aligned}$$

  - If H_0: $\rho = 0$ is true, then $LM = T\,R^2$ has an approximate $\chi^2_{(1)}$ distribution
    - T and R^2 are the sample size and goodness-of-fit statistic, respectively, from least squares estimation of Eq. 9.26

- An advantage of this Lagrange Multiplier test is that it readily generalizes to a joint test of correlations at more than one lag

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- **Durbin-Watson test under classical assumptions**
  - The test cannot be used on a model without a constant term
  - It is not appropriate if there are any lagged dependent variables.

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- **Durbin-Watson test under classical assumptions**
  - A very common test for AR(1) serial correlation is the Durbin-Watson test.
  - Also the Durbin-Watson test is based on OLS residuals from the regression of interest.
  - Under assumptions TS.1 – TS.6, the Durbin-Watson test is an exact test (whereas the previous *t*-test is only valid asymptotically).

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- **Durbin-Watson test under classical assumptions**
  - It is defined as:

$$DW = \sum_{t=2}^{n} (\hat{u}_t - \hat{u}_{t-1})^2 / \sum_{t=2}^{n} \hat{u}_t^2 \approx 2(1-\hat{\rho})$$

    with $H_0 : \rho = 0$ generally tested against $H_1 : \rho > 0$.
  - Note that $\hat{\rho} \approx 0$ implies *DW*≈2, and ρ>0 implies *DW*<2, hence we are looking for a value of *DW* that is significantly less than two
  - Unfortunately, the Durbin-Watson test works with a lower and an upper bound for the critical value. In the area between the bounds the test result is inconclusive.

Reject if $DW < d_L$, „Accept" if $DW > d_U$

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- **Example: Static Phillips curve (see above)**
  - Critical values d_L and d_U of the exact sampling distribution depend on the sample size, the number of explanatory variables, and the desired significance level, and are listed in extensive tables.
  - For k=1 explanatory variable and n=50 observations, and 1% signifcance level, Savin and White (1977) state that d_L =1.324 and d_U =1.403.
  - In the example, it can be calculated that DW=0.80 < d_L =1.32, and thus we reject null hypothesis of no serial correlation (against the alternative of positive serial correlation) at 1%.

Table A-1

Models with an intercept (from Savin and White)

Durbin-Watson Statistic: 1 Per Cent Significance Points of dL and dU

	k'*=1		k'=2		k'=3		k'=4		k'=5		k'=6		k'=7		k'=8		k'=9		k'=10	
n	dL	dU	dL	dU	dL	dU	dL	dU	dL	dU	dL	dU	dL	dU	dL	dU	dL	dU	dL	dU
6	0.390	1.142	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
7	0.435	1.036	0.294	1.676	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
8	0.497	1.003	0.345	1.489	0.229	2.102	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
9	0.554	0.998	0.408	1.389	0.279	1.875	0.183	2.433	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
10	0.604	1.001	0.466	1.333	0.340	1.733	0.230	2.193	0.150	2.690	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
11	0.653	1.010	0.519	1.297	0.396	1.640	0.286	2.030	0.193	2.453	0.124	2.892	-----	-----	-----	-----	-----	-----	-----	-----
12	0.697	1.023	0.569	1.274	0.449	1.575	0.339	1.913	0.244	2.280	0.164	2.665	0.105	3.053	-----	-----	-----	-----	-----	-----
13	0.738	1.038	0.616	1.261	0.499	1.526	0.391	1.826	0.294	2.150	0.211	2.490	0.140	2.838	0.090	3.182	-----	-----	-----	-----
14	0.776	1.054	0.660	1.254	0.547	1.490	0.441	1.757	0.343	2.049	0.257	2.354	0.183	2.667	0.122	2.981	0.078	3.287	-----	-----
15	0.811	1.070	0.700	1.252	0.591	1.465	0.487	1.705	0.390	1.967	0.303	2.244	0.226	2.530	0.161	2.817	0.107	3.101	0.068	3.374
16	0.844	1.086	0.738	1.253	0.633	1.447	0.532	1.664	0.437	1.901	0.349	2.153	0.269	2.416	0.200	2.681	0.142	2.944	0.094	3.201
17	0.873	1.102	0.773	1.255	0.672	1.432	0.574	1.631	0.481	1.847	0.393	2.078	0.313	2.319	0.241	2.566	0.179	2.811	0.127	3.053
18	0.902	1.118	0.805	1.259	0.708	1.422	0.614	1.604	0.522	1.803	0.435	2.015	0.355	2.238	0.282	2.467	0.216	2.697	0.160	2.925
19	0.928	1.133	0.835	1.264	0.742	1.416	0.650	1.583	0.561	1.767	0.476	1.963	0.396	2.169	0.322	2.381	0.255	2.597	0.196	2.813
20	0.952	1.147	0.862	1.270	0.774	1.410	0.684	1.567	0.598	1.736	0.515	1.918	0.436	2.110	0.362	2.308	0.294	2.510	0.232	2.174
21	0.975	1.161	0.889	1.276	0.803	1.408	0.718	1.554	0.634	1.712	0.552	1.881	0.474	2.059	0.400	2.244	0.331	2.434	0.268	2.625
22	0.997	1.174	0.915	1.284	0.832	1.407	0.748	1.543	0.666	1.691	0.587	1.849	0.510	2.015	0.437	2.188	0.368	2.367	0.304	2.548
23	1.017	1.186	0.938	1.290	0.858	1.407	0.777	1.535	0.699	1.674	0.620	1.821	0.545	1.977	0.473	2.140	0.404	2.308	0.340	2.479
24	1.037	1.199	0.959	1.298	0.881	1.407	0.805	1.527	0.728	1.659	0.652	1.797	0.578	1.944	0.507	2.097	0.439	2.255	0.375	2.417
25	1.055	1.210	0.981	1.305	0.906	1.408	0.832	1.521	0.756	1.645	0.682	1.776	0.610	1.915	0.540	2.059	0.473	2.209	0.409	2.362
26	1.072	1.222	1.000	1.311	0.928	1.410	0.855	1.517	0.782	1.635	0.711	1.759	0.640	1.889	0.572	2.026	0.505	2.168	0.441	2.313
27	1.088	1.232	1.019	1.318	0.948	1.413	0.878	1.514	0.808	1.625	0.738	1.743	0.669	1.867	0.602	1.997	0.536	2.131	0.473	2.269
28	1.104	1.244	1.036	1.325	0.969	1.414	0.901	1.512	0.832	1.618	0.764	1.729	0.696	1.847	0.630	1.970	0.566	2.098	0.504	2.229
29	1.119	1.254	1.053	1.332	0.988	1.418	0.921	1.511	0.855	1.611	0.788	1.718	0.723	1.830	0.658	1.947	0.595	2.068	0.533	2.193
30	1.134	1.264	1.070	1.339	1.006	1.421	0.941	1.510	0.877	1.606	0.812	1.707	0.748	1.814	0.684	1.925	0.622	2.041	0.562	2.160
31	1.147	1.274	1.085	1.345	1.022	1.425	0.960	1.509	0.897	1.601	0.834	1.698	0.772	1.800	0.710	1.906	0.649	2.017	0.589	2.131
32	1.160	1.283	1.100	1.351	1.039	1.428	0.978	1.509	0.917	1.597	0.856	1.690	0.794	1.788	0.734	1.889	0.674	1.995	0.615	2.104
33	1.171	1.291	1.114	1.358	1.055	1.432	0.995	1.510	0.935	1.594	0.876	1.683	0.816	1.776	0.757	1.874	0.698	1.975	0.641	2.080
34	1.184	1.298	1.128	1.364	1.070	1.436	1.012	1.511	0.954	1.591	0.896	1.677	0.837	1.766	0.779	1.860	0.722	1.957	0.665	2.057
35	1.195	1.307	1.141	1.370	1.085	1.439	1.028	1.512	0.971	1.589	0.914	1.671	0.857	1.757	0.800	1.847	0.744	1.940	0.689	2.037
36	1.205	1.315	1.153	1.376	1.098	1.442	1.043	1.513	0.987	1.587	0.932	1.666	0.877	1.749	0.821	1.836	0.766	1.925	0.711	2.018
37	1.217	1.322	1.164	1.383	1.112	1.446	1.058	1.514	1.004	1.585	0.950	1.662	0.895	1.742	0.841	1.825	0.787	1.911	0.733	2.001
38	1.227	1.330	1.176	1.388	1.124	1.449	1.072	1.515	1.019	1.584	0.966	1.658	0.913	1.735	0.860	1.816	0.807	1.899	0.754	1.985
39	1.237	1.337	1.187	1.392	1.137	1.452	1.085	1.517	1.033	1.583	0.982	1.655	0.930	1.729	0.878	1.807	0.826	1.887	0.774	1.970
40	1.246	1.344	1.197	1.398	1.149	1.456	1.098	1.518	1.047	1.583	0.997	1.652	0.946	1.724	0.895	1.799	0.844	1.876	0.749	1.956
45	1.288	1.376	1.245	1.424	1.201	1.474	1.156	1.528	1.111	1.583	1.065	1.643	1.019	1.704	0.974	1.768	0.927	1.834	0.881	1.902
50	1.324	1.403	1.285	1.445	1.245	1.491	1.206	1.537	1.164	1.587	1.123	1.639	1.081	1.692	1.039	1.748	0.997	1.805	0.955	1.864
55	1.356	1.428	1.320	1.466	1.284	1.505	1.246	1.548	1.209	1.592	1.172	1.638	1.134	1.685	1.095	1.734	1.057	1.785	1.018	1.837
60	1.382	1.449	1.351	1.484	1.317	1.520	1.283	1.559	1.248	1.598	1.214	1.639	1.179	1.682	1.144	1.726	1.108	1.771	1.072	1.817

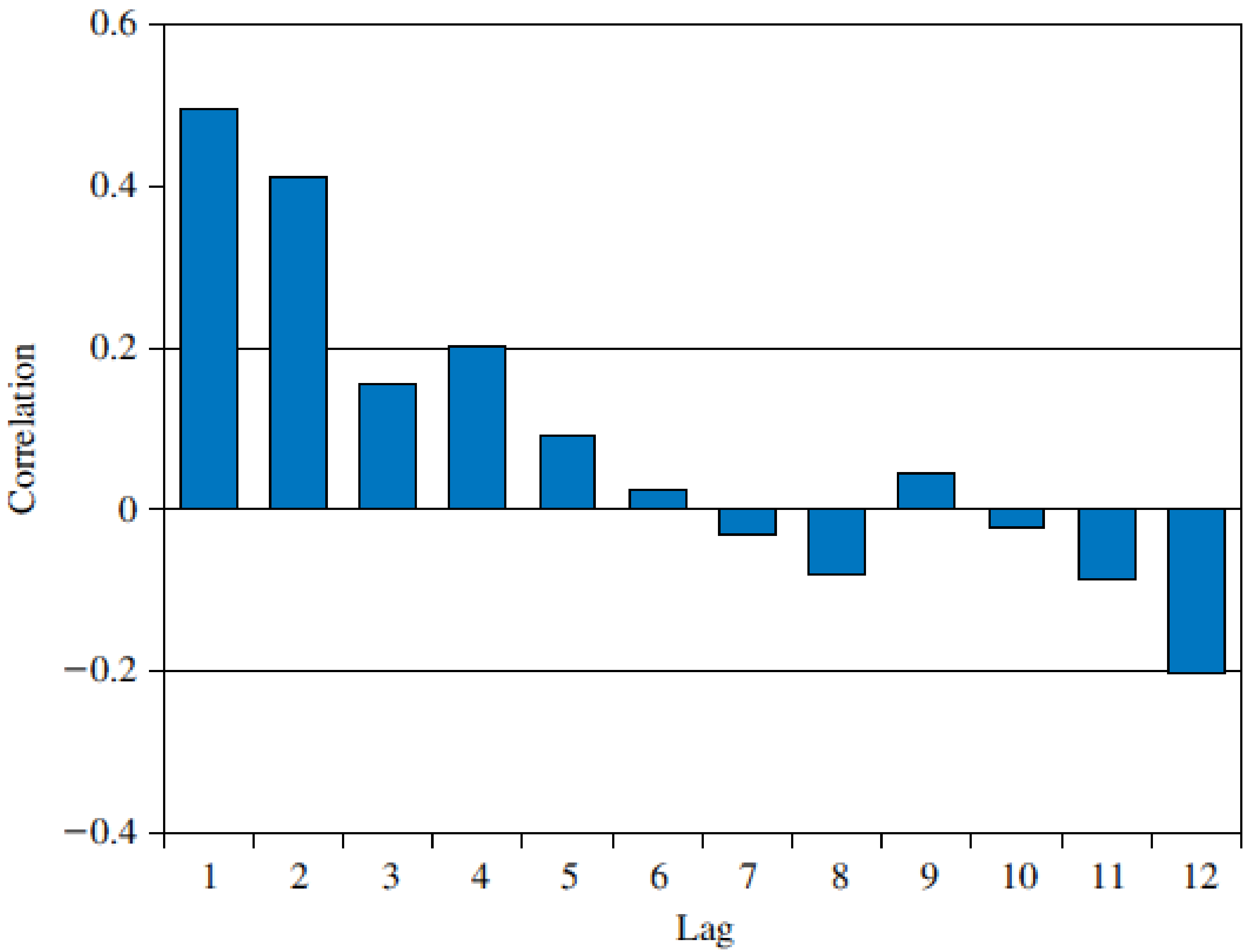
- The ***k*-th order sample autocorrelation** for a series *y* that gives the correlation between observations that are *k* periods apart is:

$$r_k = \frac{\sum_{t=k+1}^{T} (y_t - \bar{y})(y_{t-k} - \bar{y})}{\sum_{t=1}^{T} (y_t - \bar{y})^2}$$

where estimates of both covariance and variance were obtained dividing the sums by $T-k$

- The **correlogram**, also called the **sample autocorrelation function**, is the sequence of autocorrelations r_1, r_2, r_3, …
  - It shows the correlation between observations that are one period apart, two periods apart, three periods apart, and so on

FIGURE 9.6 Correlogram for *G*



- The correlogram can also be used to check whether the multiple regression assumption $\text{cov}(e_t, e_s) = 0$ for $t \neq s$ is violated

Analyzing Time Series: Serial Correl. and Heterosced.

- *12.3* **Correcting for serial correlation with strictly exog. regressors**

====== Certainly interesting, but I won't go into the details ======

====== OLS on transformed variables – GLS, FGLS – to remove serial correlation in errore ======

  - Under the assumption of AR(1) errors, one can transform the model so that it satisfies all Gauss-Markov assumptions:

$$y_t = \beta_0 + \beta_1 x_t + u_t$$

Simple case of regression with only one explanatory variable. The general case works analogously.

$$\rho y_{t-1} = \rho\beta_0 + \rho\beta_1 x_{t-1} + \rho u_{t-1}$$

Lag and multiply by ρ

$$\Rightarrow \boxed{y_t - \rho y_{t-1}} = \beta_0 \boxed{(1-\rho)} + \beta_1 \boxed{(x_t - \rho x_{t-1})} + \boxed{u_t - \rho u_{t-1}}$$

$$u_t = \rho u_{t-1} + e_t \Leftrightarrow u_t - \rho u_{t-1} = \boxed{e_t}$$

The transformed error satisfies the Gauss-Markov assumptions.

Analyzing Time Series: Serial Correl. and Heterosced.

- **Correcting for serial correlation (cont.)**
  - Hence, we can write this as

    $$\tilde{y}_1 = (1-\rho)\beta_0 + \beta_1\tilde{x}_t + e_t,$$

    for t>1, where $\tilde{y}_1 = y_t - \rho y_{t-1}$, and $\tilde{x}_t = x_t - \rho x_{t-1}$,

    i.e., a model with *quasi-differenced data* and i.i.d. errors e_t.
  - We can estimate β_0 and β_1, provided we divide the estimated intercept by (1–ρ).
  - For this model, OLS is BLUE, as long as we rescue the first observatoin by writing the equation for t=1 as $y_1 = \beta_0 + \beta_1 x_1 + u_1$

Analyzing Time Series: Serial Correl. and Heterosced.

- **Correcting for serial correlation (cont.)**
  - But, note that, using the AR(1) error structure, that

$$\mathrm{Var}(u_1) = \sigma_e^2/(1-\rho^2) > \sigma_e^2 = \mathrm{Var}(e_t)$$

  - And thus, to have the same variance for the first observation, we have to multiply it by $(1-\rho^2)^{1/2}$ to get errors of the same variance:

$$(1-\rho^2)^{1/2}y_1 = (1-\rho^2)^{1/2}\beta_0 + \beta_1(1-\rho^2)^{1/2}x_1 + (1-\rho^2)^{1/2}u_1$$

    or, by using the applicable transformations, as

$$\tilde{y}_1 = (1-\rho^2)^{1/2}\beta_0 + \beta_1\tilde{x}_1 + \tilde{u}_1,$$

  - Adding more regressors changes very little.

→ *Example of a Generalized Least Squares estimator (OLS on transformed data)*

Analyzing Time Series: Serial Correl. and Heterosced.

- **Correcting for serial correlation (cont.)**
  - Problem: The AR(1)-coefficient ρ is not known and has to be estimated
  - Replacing the unknown ρ by $\hat{\rho}$, obtained from a regression of the OLS residuals on their lagged counterparts, leads to a FGLS-estimator.

Feasible GLS Estimation of the AR(1) Model:

(i) Run the OLS regression of y_t on $x_{t1}, \ldots, x_{tk}$ and obtain the OLS residuals, $\hat{u}_t$, $t = 1, 2, \ldots, n$.
(ii) Run the regression in equation (12.14) and obtain $\hat{\rho}$.
(iii) Apply OLS to equation (12.33) to estimate $\beta_0, \beta_1, \ldots, \beta_k$. The usual standard errors, t statistics, and F statistics are asymptotically valid.

Analyzing Time Series: Serial Correl. and Heterosced.

- **Correcting for serial correlation (cont.)**
  - There are two variants:
    - Cochrane-Orcutt estimation omits the first observation
    - Prais-Winsten estimation adds a transformed first observation
  - Asymptotically it makes no difference; in smaller samples, Prais-Winsten estimation should be more efficient
  - In practice, an iterative scheme is often used: repeat the whole process many times, until the estimate of ρ changes by very little from the previous iteration.

Analyzing Time Series: Serial Correl. and Heterosced.

- **Comparing OLS and FGLS**
  - Sometimes FGLS estimates differ in practically important ways from the OLS estimates, which has been interpreted as a verification of FGLS's superiority over OLS. Of course, things are not so simple.
  - For consistency of FGLS, more than TS.3�much is needed (e.g. TS.3) because the transformed regressors include variables from different periods.
  - It can be shown that at least $\mathrm{Cov}[(x_{t-1} + x_{t+1}), u_t] = 0$ is required, or in practical terms, u_t should be uncorrelated with x_{t-1}, x_t, and x_{t+1}.
  - The reason is that we use transformed data, $x_t - \rho x_{t-1}$ and $u_t - \rho u_{t-1}$, and thus we neeed that $\mathrm{E}[(x_t - \rho x_{t-1})(u_t - \rho u_{t-1})] = 0$. Expanding the expectation, and using stationarity, gives the requirement.

Analyzing Time Series: Serial Correl. and Heterosced.

- **Comparing OLS and FGLS with autocorrelation (cont.)**
  - *Bottom line*: We do not expect OLS and FGLS to be the same, but if they differ dramatically this might indicate violation of TS.3 …
  - … and then, OLS may then be preferred over FGLS, because OLS is consistent under the (single) assumption of $\mathrm{Cov}(x_t, u_t) = 0$.
  - But, remember large sample properties for OLS require stationarity and weak dependence: e.g., I(1) series will violate this assumption

Analyzing Time Series: Serial Correl. and Heterosced.

- **Correcting for higher-order serial correlation**
  - Assume AR(2) serially correlated errors

$$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + e_t$$

  - The stability conditions are more complicated now, can be shown to be

$$\rho_2 > -1, \rho_2 - \rho_1 < 1, \text{ and } \rho_1 + \rho_2 < 1$$

  - In a regression with one explanatory var., the transformation now is

$$y_t - \rho_1 y_{t-1} - \rho_2 y_{t-2} = \beta_0(1 - \rho_1 - \rho_2) + \beta_1(x_t - \rho_1 x_{t-1} - \rho_2 x_{t-2}) + e_t$$

    which gives rise to a model in quasi-differenced data

$$\tilde{y}_t = \beta_0(1 - \rho_1 - \rho_2) + \beta_1 \tilde{x}_t + e_t, t = 3, 4, \ldots, n.$$

  - Again, ρ_1 and ρ_2 are typically unknown, but can be estimated using an OLS of $\hat{u}_t$ on $\hat{u}_{t-1}, \hat{u}_{t-2}, t = 3, \ldots, n$ and use $\hat{\rho}_1$ and $\hat{\rho}_2$ to obtain transformed variables.

Analyzing Time Series: Serial Correl. and Heterosced.

- *12.4* **Differencing and serial correlation**
  - We have seen before that differencing makes I(1) series weakly dependent
  - In general, suppose highly persistent data and a simple regression model with AR(1) errors u_t:

$$y_t = \beta_0 + \beta_1 x_t + u_t, t = 1, 2, \ldots$$

  - Then OLS can be very misleading; in case of random walks, the variance of u_t will even grow with t.

Analyzing Time Series: Serial Correl. and Heterosced.

- **Differencing and serial correlation (cont.)**
  - An equation in differences makes more sense:

$$\Delta y_t = \beta_1 \Delta x_t + \Delta u_t, t = 2, \ldots, n$$

  - If u_t follows a random walk, $e_t \equiv \Delta u_t$ has mean zero, constant variance, and is serially uncorrelated.
  - Thus, assuming e_t and Δx_t are uncorrelated enables use of OLS.
  - If not a random walk but ρ is large, first differencing may still be a good option as it will eliminate most of the serial correlation.
  - We can have more faith in the OLS standard errors and t statistics in the model in first differences.

Analyzing Time Series: Serial Correl. and Heterosced.

- *12.5* **Serial correlation-robust inference after OLS**
  - In the presence of serial correlation, OLS standard errors overstate statistical significance because there is less independent variation
  - It has become popular to compute serial correlation-robust standard errors after OLS
  - This is useful because FGLS (section 12.3) requires strict exogeneity and assumes a very specific form of serial correlation (AR(1), or, more general, AR(*q*))

Analyzing Time Series: Serial Correl. and Heterosced.

- **Serial correlation-robust inference after OLS (cont.)**

  ====== Certainly interesting, but I won't go into the details ======

  - Consider a standard multiple linear regression model

    $$y_t = \beta_0 + \beta_1 x_{t1} + \ldots + \beta_k x_{tk} + u_t,\ t = 1, 2, \ldots, n$$

    estimated by OLS.

  - Remember the variance of the parameter estimates,

    $$\begin{aligned}\mathrm{Var}(\hat{\beta}_1) &= \mathrm{SST}_x^{-2}\mathrm{Var}\left(\sum_{t=1}^{n} x_t u_t\right) \\ &= \mathrm{SST}_x^{-2}\left(\sum_{t=1}^{n} x_t^2 \mathrm{Var}(u_t) + 2\sum_{t=1}^{n-1}\sum_{j=1}^{n-t} x_t x_{t+j}\,\mathrm{E}(u_t u_{t+j})\right) \\ &= \sigma^2/\mathrm{SST}_x + 2(\sigma^2/\mathrm{SST}_x^2)\sum_{t=1}^{n-1}\sum_{j=1}^{n-t}\rho^j x_t x_{t+j},\end{aligned}$$

  - Serial correlation-robust *F*- and *t*-tests are also available

Analyzing Time Series: Serial Correl. and Heterosced.

- **Serial correlation-robust inference after OLS (cont.)**
  - Serial correlation-robust standard errors have to approximate that variance with an adjustment factor:

$$se(\hat{\beta}_j) = \left[\text{“}se(\hat{\beta}_j)\text{”}/\hat{\sigma}\right]^2 \sqrt{\hat{v}}$$

The usual OLS standard errors are normalized and then „inflated“ by a correction factor.

Analyzing Time Series: Serial Correl. and Heterosced.

- **Correction factor for serial correlation (Newey-West formula)**

======For illustration only; don´t worry about the details – I won´t ask them in the exam ======

$$\hat{v} = \sum_{t=1}^{n} \hat{a}_t^2 + 2 \sum_{h=1}^{g} [1 - h/(g+1)] \left(\sum_{t=h+1}^{n} \hat{a}_t \hat{a}_{t-h} \right)$$

$$\hat{a}_t = \hat{r}_t \hat{u}_t$$

This term is the product of the OLS residuals $\hat{u}_t$ and the residuals $\hat{r}_t$ of a regression of x_{tj} on all other explanatory variables, x_{t1} on $x_{t2}, x_{t3}, \ldots, x_{tk}$

The integer *g* controls how much serial correlation is allowed:

g=1: $\hat{v} = \sum_{t=1}^{n} \hat{a}_t^2 + \sum_{t=2}^{n} \hat{a}_t \hat{a}_{t-1}$

The weight of higher order autocorrelations is declining

g=2: $\hat{v} = \sum_{t=1}^{n} \hat{a}_t^2 + (4/3) \sum_{t=2}^{n} \hat{a}_t \hat{a}_{t-1} + (2/3) \sum_{t=3}^{n} \hat{a}_t \hat{a}_{t-2}$

Analyzing Time Series: Serial Correl. and Heterosced.

- **Discussion of serial correlation-robust standard errors**
  - The formulas are also robust to heteroscedasticity; they are therefore called „heteroscedasticity and autocorrelation consistent" (=HAC)
  - For the integer g, values such as g=1 or g=2 are normally sufficient (there are more involved rules of thumb for how to choose g)
  - Serial correlation-robust standard errors are only valid asymptotically; they may be severely biased if the sample size is not large enough
  - The bias is the higher the more autocorrelation there is; if the series are highly correlated, it might be a good idea to difference them first
  - Serial correlation-robust errors should be used if there is serial corr. and strict exogeneity fails (e.g. in the presence of lagged dep. var.)
  - Serial correlation-robust F- and t-tests are also available

Analyzing Time Series: Serial Correl. and Heterosced.

- *12.6* **Heteroscedasticity in time series regressions**
  - Heteroskedasticity, while not causing bias or inconsistency, invalidates the usual standard errors, t and F statistics
  - In time series, heteroscedasticity usually receives less attention than serial correlation
  - Heteroscedasticity-robust standard errors also work for time series (*Chapter 8; cross-section data*)
  - Heteroscedasticity is automatically corrected for if one uses the serial correlation-robust formulas for standard errors and test statistics (*previous section*)

Analyzing Time Series: Serial Correl. and Heterosced.

- **Testing for heteroscedasticity**
  - The usual heteroscedasticity tests (*Chapter 8: Breusch-Pagan test, White test*) assume absence of serial correlation
  - Before testing for heteroscedasticity one should therefore test for serial correlation first, using a heteroscedasticity-robust test if necessary
  - After serial correlation has been corrected for, test for heteroscedasticity

Analyzing Time Series: Serial Correl. and Heterosced.

- **Example: Serial correlation and homoscedasticity in the EMH**

$$return_t = \beta_0 + \beta_1 return_{t-1} + u_t$$

Test equation for the EMH

$$\underset{(.147)}{\hat{u}_t = .122} - \underset{(.647)}{.645}\, return_{t-1} + \boxed{\underset{(.648)}{.646}}\, \hat{u}_{t-1}$$

$$n = 688, R^2 = .0015, \bar{R}^2 = -.0015$$

Test for serial correlation: No evidence for serial correlation

$$\hat{u}_t^2 = \underset{(0.43)}{4.66} - \boxed{\underset{(0.201)}{1.104}}\, return_{t-1}$$

$$n = 688, R^2 = .0419, \bar{R}^2 = -.0405$$

Test for heteroscedasticity (Breusch-Pagan): Strong evidence for heteroscedasticity

Note: Volatility is higher if returns are low

Analyzing Time Series: Serial Correl. and Heterosced.

- *12.6c* **Autoregressive Conditional Heteroscedasticity (ARCH)**

====== Certainly interesting, especially for those with interests in Financial Econometrics ======
====== But I won't go into the details now – perhaps later in the course ======

Even if there is no heteroscedasticity in the usual sense (the error variance depends on the explanatory variables), there may be heteroscedasticity in the sense that the variance depends on how volatile the time series was in previous periods:

$$Var(u_t|\mathbf{X}, u_{t-1}, u_{t-2}, \ldots) = \alpha_0 + \alpha_1 u_{t-1}^2$$

ARCH(1) model

Analyzing Time Series: Serial Correl. and Heterosced.

- **What remains: Advanced time series topics**
  - Chapter 18 of Wooldridge, Chapters 9, 12, and 13 of Hill et al.
    - Testing for unit roots: Dickey-Fuller tests
    - Vector Autoregressive (VAR) models
    - Cointegration
    - Vector Error Correction (VEC) models
    - Forecasting
    - Impulse-response functions
    - ARCH, GARCH: time-varying volatility (Hill et al. Chapter 14)