Further Issues Using OLS with Time Series Data

Chapter 11

Wooldridge: Introductory Econometrics:
A Modern Approach, 5e

with some adjustments by myself

Analyzing Time Series: Further Issues Using OLS

- *11.1* **The assumptions used so far seem to be very restricitive**
  - Strict exogeneity, homoscedasticity, and no serial correlation are very demanding requirements, especially in the time series context
  - Statistical inference rests on the validity of the normality assumption
  - Much weaker assumptions are needed if the sample size is large
  - A key requirement for large sample analysis of time series is that the time series in question are <u>**stationary and weakly dependent**</u>

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- **Stationary time series**

  - Loosely speaking, a time series is stationary if its stochastic properties and its temporal dependence structure do not change over time

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- **Stationary stochastic processes**

  A stochastic process $\{x_t : t = 1, 2, \ldots\}$ is <u>stationary</u>, if for every collection of indices $1 \leq t_1 \leq t_2 \leq \cdots \leq t_m$ the joint distribution of $(x_{t_1}, x_{t_2}, \ldots, x_{t_m})$ is the same as that of $(x_{t_1+h}, x_{t_2+h}, \ldots, x_{t_m+h})$ for all integers $h \geq 1$.

- **Implications**
  - All x_t , t =1,2, …, are identically distributed (implied by setting m=1 and t_1=1)
  - All pairs (x_t , x_{t+1}) have identical joint distribution
  - Correlation between x_t and x_{t+1} is not ruled out, but the nature of the correlation must remain the same over time

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- **Non-stationary stochastic processes**
  - A stochastic process that is not stationary is said to be a **nonstationary process**.
- **Discussion of stationary stochastic processes**
  - Stationarity is an aspect of the underlying stochastic process, not of the available single realization, hence it can be difficult to determine whether actual data were generated by a stationary process.
  - However, it is easy to spot certain sequences that are not stationary.
    - A process with a time trend is clearly nonstationary: at a minimum, its mean changes over time.

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- **Covariance stationary stochastic processes**
  - Sometimes, a weaker form focused on first two moments suffices: a **covariance stationarity stochastic process**:

A stochastic process $\{x_t : t = 1, 2, \ldots\}$ is <u>covariance stationary</u>, if its expected value, its variance, and its covariances are constant over time:

(a) $E(x_t) = \mu$, (b) $Var(x_t) = \sigma^2$, and (c) $Cov(x_t, x_{t+h}) = f(h)$

- **Implications**
  - The first two moments are constant across time
  - The covariance and correlation between x_t and x_{t+h} depends only on the distance h between the two terms, and not on the location of the initial time t

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- **Stationarity vs. Covariance stationarity**
  - If a stationary process has a finite second moment, then it is also covariance stationary.
  - The converse is not true: a finite second moment does not guarantee stationarity
  - Hence, stationarity is a stronger requirement than covariance stationarity, and therefore sometimes referred to as *strict stationarity*

➔ In practice, because it simplifies many assumptions, we will always refer to *strict stationarity* when talking about stationarity

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- **Weakly dependent time series**

  - (Strict) stationarity has to do with the joint distributions of a process as it moves through time

  - A different concept is that of *weak dependence*, which places restrictions on how strongly related the random variables x_t and x_{t+h} can be as the time distance between them, h, gets large

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- **Weakly dependent time series**

  A stochastic process $\{x_t : t = 1, 2, \ldots\}$ is weakly dependent, if x_t is „almost independent" of x_{t+h} if h grows to infinity (for all t).

- **Discussion of the weak dependence property**
  - An implication of weak dependence is that the correlation between x_t and x_{t+h} must converge to zero "sufficiently quickly" if h grows to infinity.
  - Covariance stationary sequences where Corr(x_t , x_{t+h}) tends to 0 as h tends to infinity are said to be *asymptotically uncorrelated*.

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- **Practical relevance**
  - If we want to understand the relationship between two or more variables, we need to assume some sort of stability over time.
    - If we would allow the relationship between two variables to change arbitrarily in each time period, then we cannot hope to learn much about how a change in one variable affects the other variable
    - We are assuming a certain form of stationarity in that the β_j do not change over time.
  - Assumptions TS.4 and TS.5 imply that the variance of the error process is constant over time and that the correlation between errors in two adjacent periods is equal to zero

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- **Practical relevance**
  - Stationarity simplifies statements of the law of large numbers (LLN) and the Central Limit Theorem (CLT)
  - For the LLN and the CLT to hold, the individual observations must not be too strongly related to each other; in particular their relation must become weaker (and this fast enough) the farther they are apart
    - Weak dependence essentially replaces the assumption of random sampling

➔ Thus, stationary, weakly dependent time series are ideal for use in multiple regression analysis

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- **<u>Examples for weakly dependent time series</u>**

0) An i.i.d. sequence

An independent, identically distributed sequence $\{e_t\}$ is trivially weakly dependent: observations from different time period aparts are uncorrelated and thus have nothing in common and are therefore uncorrelated.

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- **<u>Examples for weakly dependent time series</u>**

1) Moving average process of order one (MA(1))

$$x_t = e_t + \alpha_1 e_{t-1}$$

The process x_t is a short moving average of an i.i.d. series e_t with zero mean and variance σ_e^2

The process $\{x_t\}$ is called a moving average process because each x_t is a weighted average of e_t and e_{t-1}.

The process $\{x_t\}$ is weakly dependent because observations that are more than one time period apart have nothing in common and are therefore uncorrelated.

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- **Why is an MA(1) process weakly dependent?**
  - Adjacent terms in the sequence are correlated:

    $x_t = e_t + \alpha_1 e_{t-1}$ implies that $\mathrm{Cov}(x_t, x_{t+1}) = \alpha_1 \mathrm{Var}(e_t) = \alpha_1 \sigma_e^2$.

    and because $\mathrm{Var}(x_t) = (1 + \alpha_1^2)\sigma_e^2$, we obtain that $\mathrm{Corr}(x_t, x_{t+1}) = \alpha_1/(1 + \alpha_1^2$
  - However, variables that are two or more periods apart, are uncorrelated:

    For example, $x_{t+2} = e_{t+2} + \alpha_1 e_{t+1}$ is independent of x_t, because the sequence $\{e_t\}$ is independent across time.
    - It is easy to see that Cov(x_t, x_{t-1}) = 0)
  - Notice that the process $\{x_t\}$ is also stationary, because of the identical distribution assumption of the e_t.

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- **Examples for weakly dependent time series**

2) Autoregressive process of order one (AR(1))

$$y_t = \rho_1 y_{t-1} + e_t$$

The process carries over to a certain extent the value of the previous period (plus random shocks from an i.i.d. series e_t)

The process $\{y_t\}$ is called an autoregressive process because each y_t is a function of its value in the previous period, y_{t-1}, and an i.i.d. sequence $\{e_t\}$.

Further, assume the starting point y_0 is independent of e_t, with $E[y_0]=0$

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- **Why is an AR(1) process weakly dependent?**
  - Assume that the process $\{y_t\}$ is covariance stationary – hence, with constant mean, *i.e.* $\mathrm{E}(y_t) = \mathrm{E}(y_{t-1})$, what, for general $\rho_1 \neq 1$, implies that $\mathrm{E}(y_t) = 0$
  - Taking the variance of the AR(1) equation, and using that e_t and y_{t-1} are independent (uncorrelated), gives that $\mathrm{Var}(y_t) = \rho_1^2 \mathrm{Var}(y_{t-1}) + \mathrm{Var}(e_t)$, which, under covariance stationarity – hence, with constant variance, implies that $\sigma_y^2 = \rho_1^2 \sigma_y^2 + \sigma_e^2$, and thus $\sigma_y^2 = \sigma_e^2/(1 - \rho_1^2)$

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- **Why is an AR(1) process weakly dependent?**
  - For $h \geq 1$, we can find the covariance between y_t and y_{t+h}. By repeated substitution: $$y_{t+h} = \rho_1 y_{t+h-1} + e_{t+h} = \rho_1(\rho_1 y_{t+h-2} + e_{t+h-1}) + e_{t+h}$$ $$= \rho_1^2 y_{t+h-2} + \rho_1 e_{t+h-1} + e_{t+h} = \cdots$$ $$= \rho_1^2 y_t + \rho_1^{h-1} e_{t+1} + \cdots + \rho_1 e_{t+h-1} + e_{t+h}$$ *Note the mistake in this last line*
  - Because $\mathrm{E}(y_t) = 0$ for all t, multiplication of the last equation with y_t and taking expectations gives: $$\mathrm{Cov}(y_t, y_{t+h}) = \mathrm{E}(y_t y_{t+h}) = \rho_1^h \mathrm{E}(y_t^2) + \rho_1^{h-1}\mathrm{E}(y_t e_{t+1}) + \cdots + \mathrm{E}(y_t e_{t+h})$$ $$= \rho_1^h \mathrm{E}(y_t^2) = \rho_1^h \sigma_y^2.$$ and thus $$\mathrm{Corr}(y_t, y_{t+h}) = \mathrm{Cov}(y_t, y_{t+h})/(\sigma_y \sigma_y) = \rho_1^h.$$
  - Hence, it shows that y_t and y_{t+h} are always correlated for $h \geq 1$, but that for $h \to \infty$, $\rho_1^h \to 0$ as long as $|\rho_1| < 1$, which is a crucial assumption for the AR(1) to be weakly dependent: the *stability condition*.

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- **One more remark**
  - Note that a series may be nonstationary but weakly dependent:
  - In particular, a trending series, though certainly nonstationary, can be stationary about its time trend as well as weakly dependent: a **trend-stationary process.**
  - Such processes can be used in time series regression models, provided that we include the appropriate time trend in the model

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- *Summarizing …*
  - A key requirement for large sample analysis of time series is that the time series in question are **stationary and weakly dependent**
  - Loosely speaking, a time series is stationary if its stochastic properties and its temporal dependence structure do not change over time
  - A stationary variable is one that is not explosive, nor is it trending, and nor does it wander aimlessly without returning to its mean

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- *Summarizing ...*
  - A stochastic process is <u>weakly dependent</u>, if x_t is „almost independent" of x_{t+h} if $h \rightarrow \infty$

→ MA(1) and AR(1) processes are nice examples of the latter (as long as, for the AR process, the stability condition $|\rho|<1$ holds)

→ Typically, stationary variables have weak dependence

→ Formal tests known as **unit root tests** can detect non-stationarity

... but a preliminary graphical inspection is common

FIGURE 9.2 (a) Time series of a stationary variable

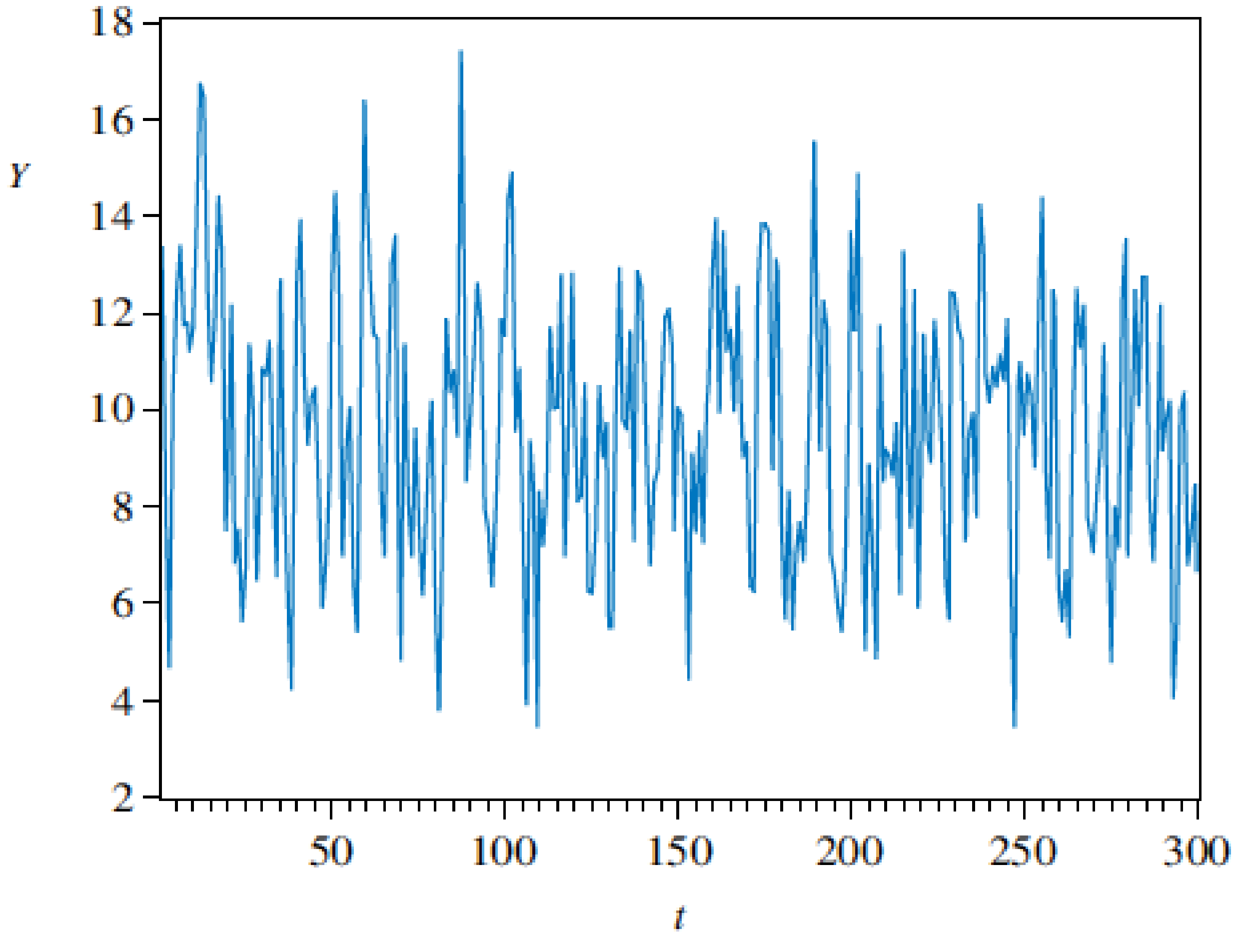


FIGURE 9.2 (b) time series of a nonstationary variable that is "slow-turning" or "wandering"

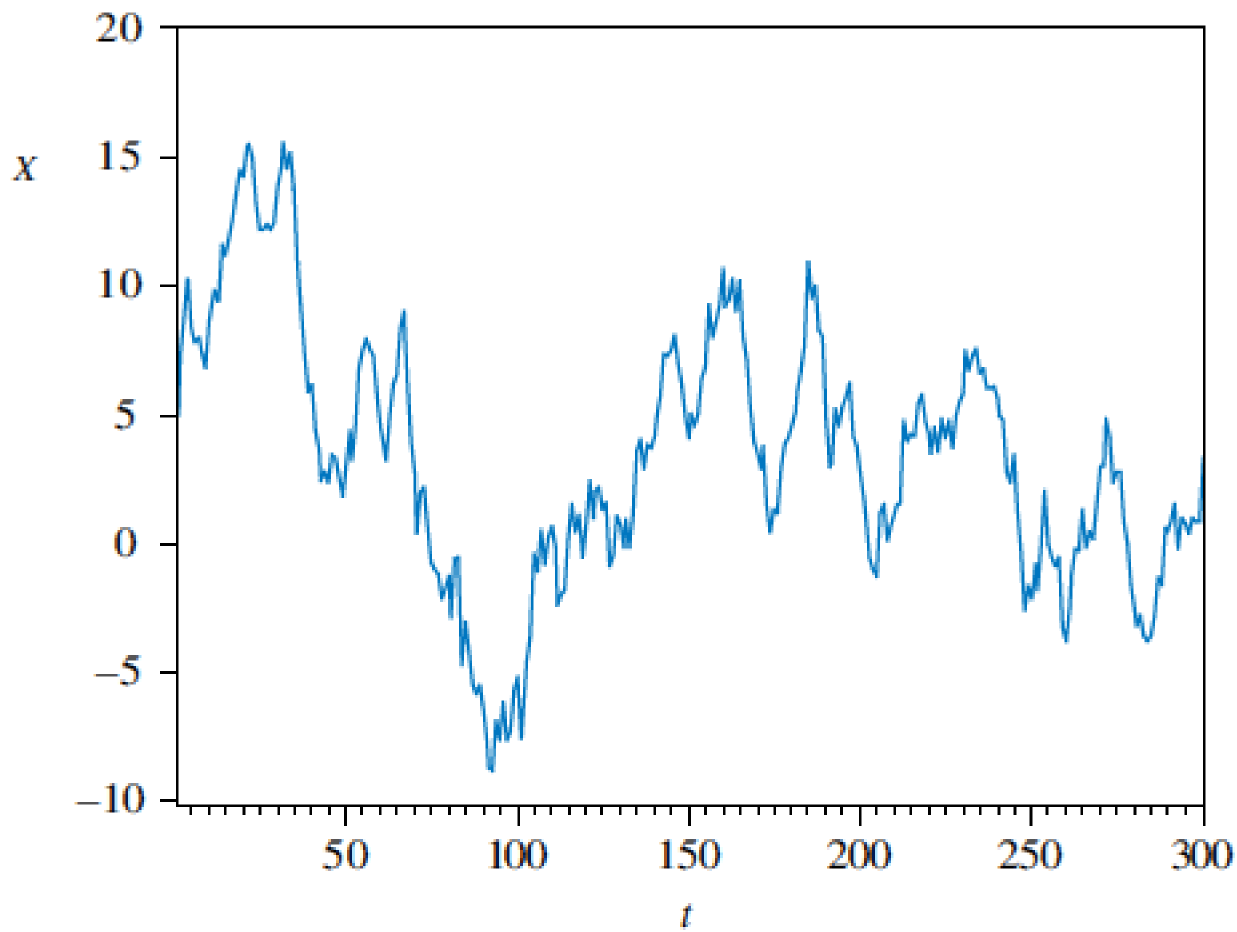
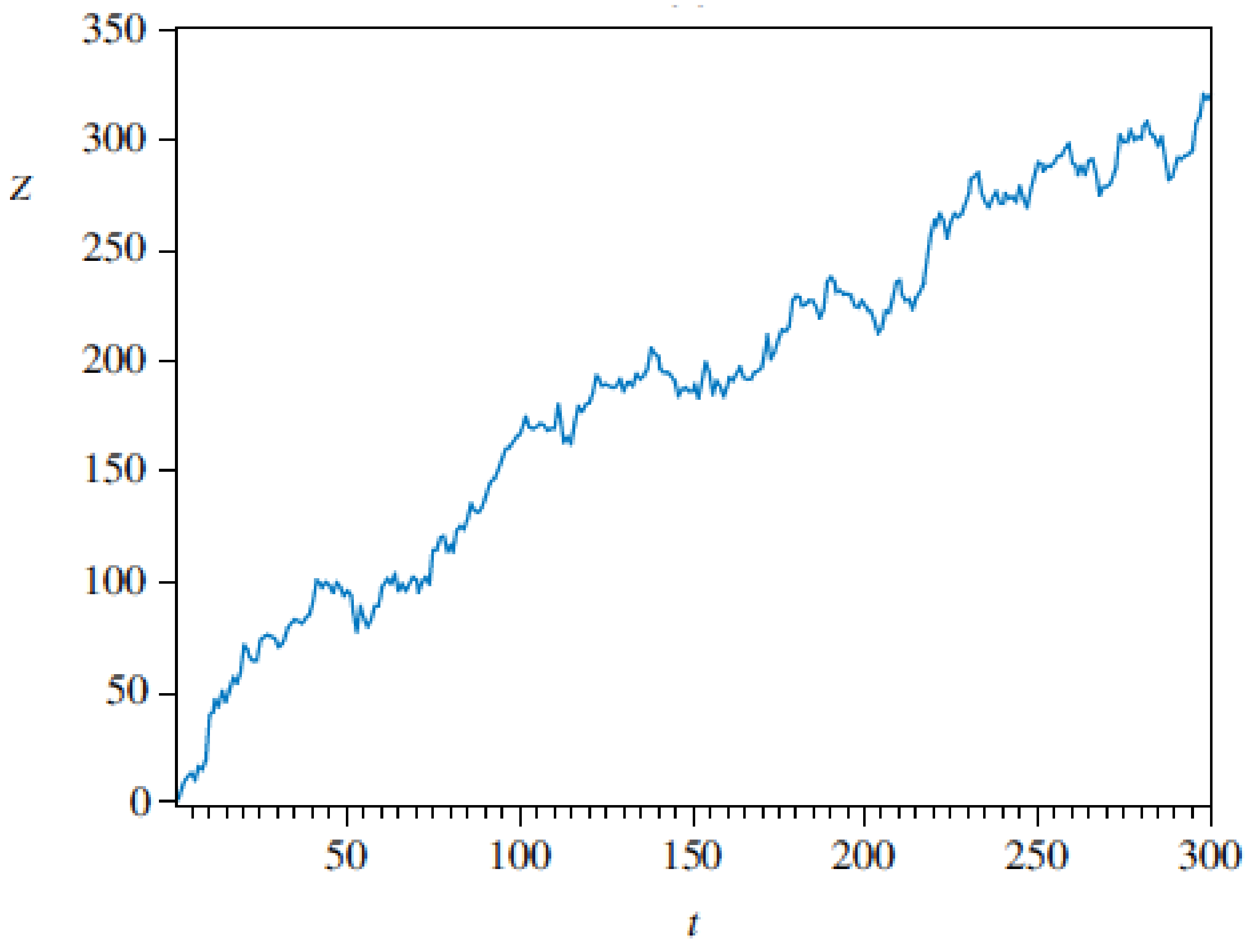


FIGURE 9.2 (c) time series of a nonstationary variable that "trends"



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- *11.2* **Asymptotic properties of OLS**
- **Assumption TS.1` (Linear in parameters)**
  - Same as assumption TS.1
  - … but now the dependent and independent variables are assumed to be stationary and weakly dependent, …
  - … and now there may be lagged dependent variables among the explanatory variables

- **Assumption TS.2` (No perfect collinearity)**
  - Same as assumption TS.2

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- **Assumption TS.3' (Zero conditional mean)**
  - Now the explanatory variables are assumed to be only contemporaneously exogenous rather than strictly exogenous, i.e.

$$E(u_t|\mathbf{x}_t) = 0$$

The explanatory variables of the same period are uninformative about the mean of the error term

  - By stationarity, if contemporaneous exogeneity holds for one time period, it holds for them all.

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- <u>**Theorem 11.1 (Consistency of OLS)**</u>

$$TS.1'\text{–}TS.3' \quad \Rightarrow \quad plim\ \hat{\beta}_j = \beta_j, \quad j = 0, 1, \ldots, k$$

<u>Note</u>: For consistency it would even suffice to assume that the explanatory variables are merely contemporaneously *uncorrelated* with the error term: $E[u_t]=0$, and $Cov[x_{tj}, u_j]=0$, for all $j=1, .., k$.

- The theorem establishes <u>consistency</u> of the OLS estimator, but not necessarily unbiasedness (as we had in Ch. 10).
- In comparison with Ch. 10, we have weakened the exogeneity requirements, but <u>weak dependence</u> of the time series remains crucial