CSCI 104
Hash Tables Intro

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Motivation

Suppose a company has a unique 3-digit ID for each of its 1000 employees.

- We want a data structure that, when given an employee ID, efficiently brings up that employee's record.

How should we implement this?

- An array gives O(1) access time!

Alright, how do we obtain this runtime when the keys are no longer so nicely ordered or non-integers??

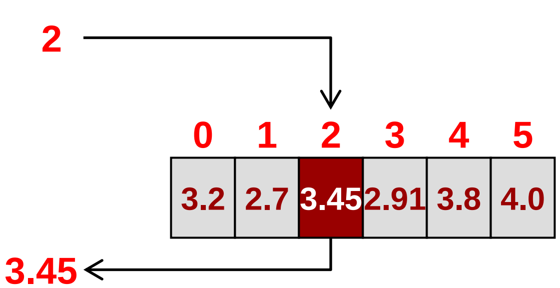
Maps/Dictionaries

Arrays

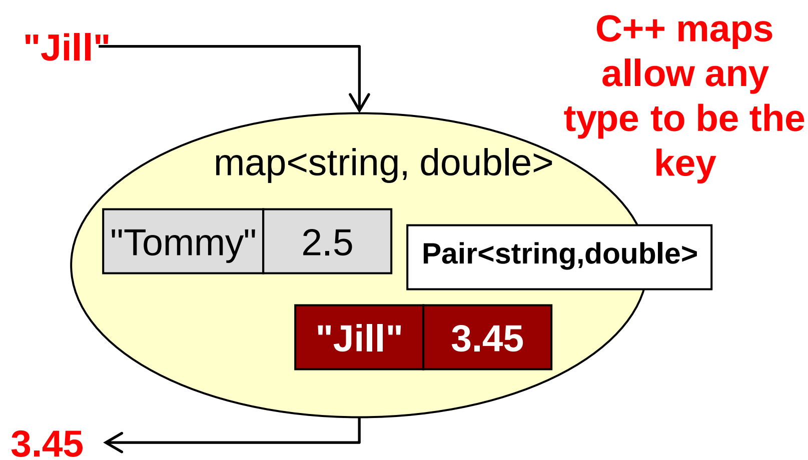
- An array maps **integers** to ***values***
  - Given i, array[i] returns the value in O(1)

Maps/Dictionaries

- Dictionaries map **keys** to ***values***
  - Given key, k, map[k] returns the associated value
  - Key can be anything provided…
    - It has a '<' operator defined for it (C++ map) or some other comparator functor (other languages require something similar)

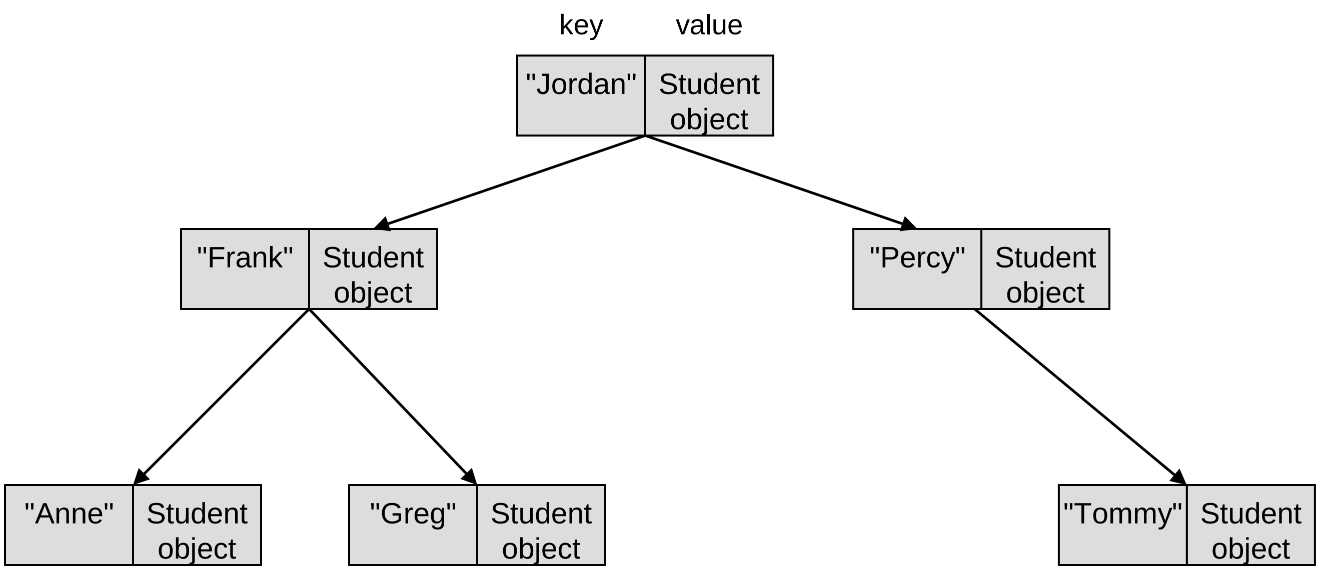


Arrays associate an integer with some arbitrary type as the value (i.e. the key is always an integer)



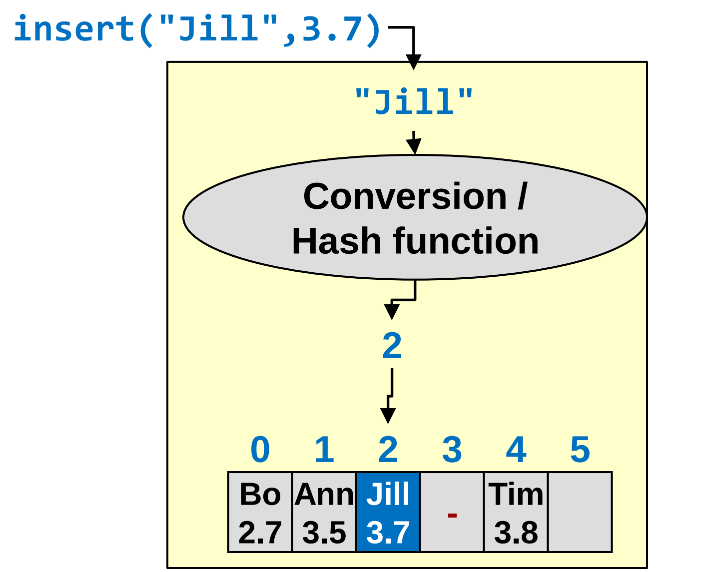
Dictionary Implementation

- A dictionary/map can be implemented with a balanced BST
  - Insert, Find, Remove = O(__________)
- Can we do better?
  - Hash tables (unordered maps) offer the promise of O(____) access time



Hash Tables - Insert

- Can we use non-integer keys to index an array?
- Yes. Let us convert (i.e. "hash") the non-integer key to an integer
- To **insert** a key, we hash it and place the key (and value) at that index in the array
  - For now, make the unrealistic assumption that each unique key hashes to a unique integer
- The conversion function is known as a ***hash function, h(k)***
- A hash table implements a set/map ADT
  - `insert(key)` / `insert(key,value)`
  - `remove(key)`
  - `lookup/find(key) => value`
- ***Question to address***: What should we do if two keys ("Jill" and "Erin") hash to the same location (aka a **COLLISION**)?



A map implemented as a hash table (key=name, value = GPA)

Hash table parameter definitions:

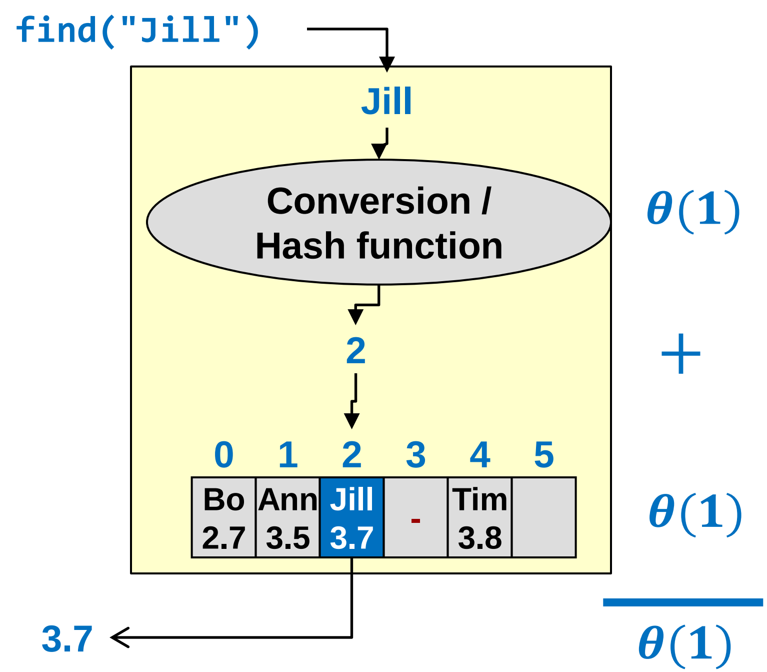
n = # of keys entered (=4 above)

m = tableSize (=6 above)

$\alpha = \frac{n}{m}$ **= Loading factor = (4/6 above)**

Hash Tables - Find

- To **find** a key, we simply hash it again to find the index where it was inserted and access it in the array
- How might we hash a string to an integer?
  - Use ASCII codes for each character and **add, multiply, or shift/mix** them
  - We then can use simple a **modulo m** operation to convert the sum to a value between **0** to m-1 where **m** is the table size
  - Note: All data in a computer is already bits (1s and 0s). Any object can be viewed as a long binary number and hashed



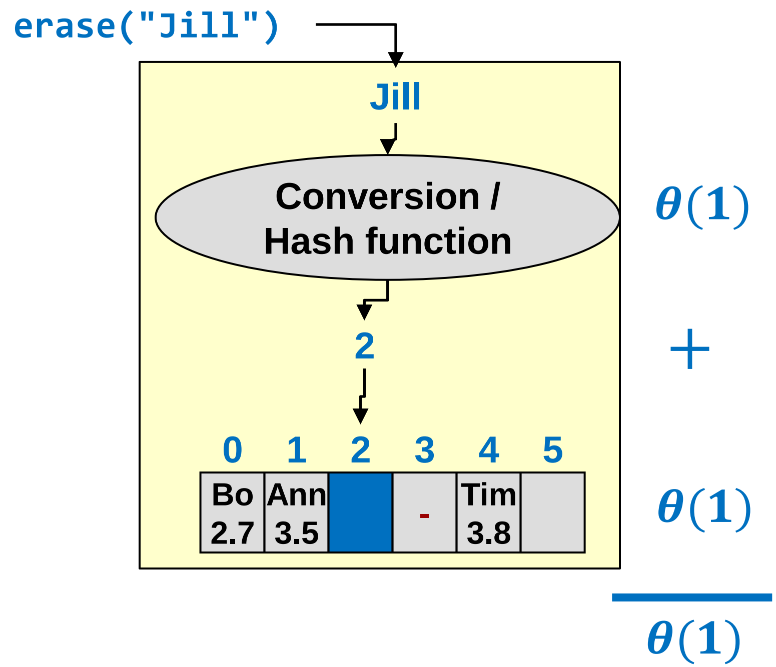
We could sum the ASCII values.

```
'h' = 104   'e' = 101   'l' = 108
'l' = 108   'o' = 111
_______________________________
h("hello") = 532 % m
```

Is this a good way to hash a string?

Hash Tables - Remove

- To **remove** a key, we simply hash the key and mark the location as "free" again
  - Could use a `bool` in the struct for each array entry (more later) to indicate it is free
- The ***hash function, h(k), should***
  - Be fast/easy to compute
    - O(|k|) – where |k| is the length of the key
    - But in terms of n [# of keys in the set/map] this runtime is constant since |k| << n [e.g. O(1)]
  - Be consistent and output the same result any time it is given the same input
  - Distribute keys well
    - We'd like every unique key to map to a different index, but that turns out to be almost impossible.
    - We'll settle for a "**good**" hash function where the probability of a key mapping to any location x is 1/m (i.e. uniform)



Hash table parameter definitions:

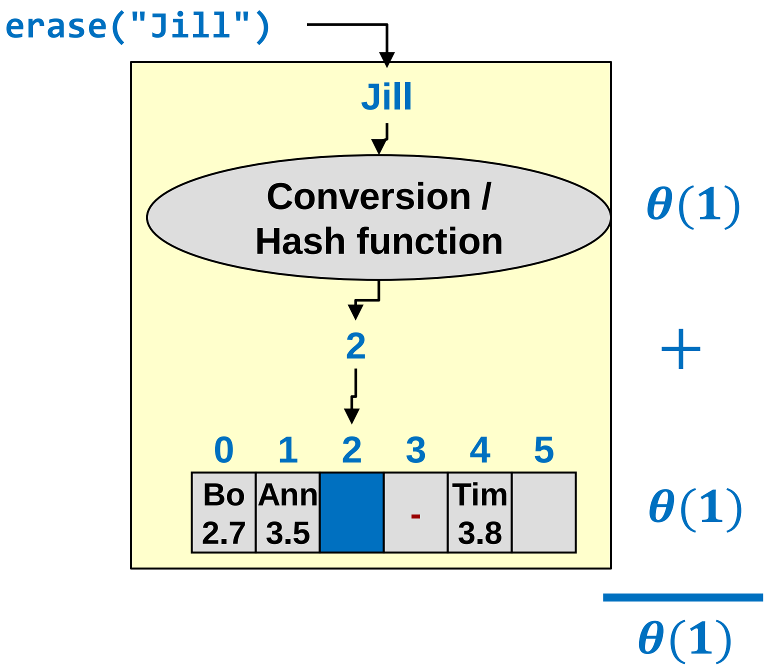
n = # of keys entered
m = tableSize
$\alpha = \frac{n}{m}$ = Loading factor

Possible Hash Functions

- Define **n** = # of keys stored, **m** = table size and suppose k is non-negative integer key
- Evaluate the following possible hash functions
  - h(k) = 0 ?
  - h(k) = rand() mod **m** ?
  - h(k) = k mod **m** ?
- Rules of thumb
  - The hash function should examine the entire search key (i.e. all bits/characters), not just a few digits or a portion of the key
  - When modulo hashing is used, the base should be prime

Hashing Efficiency

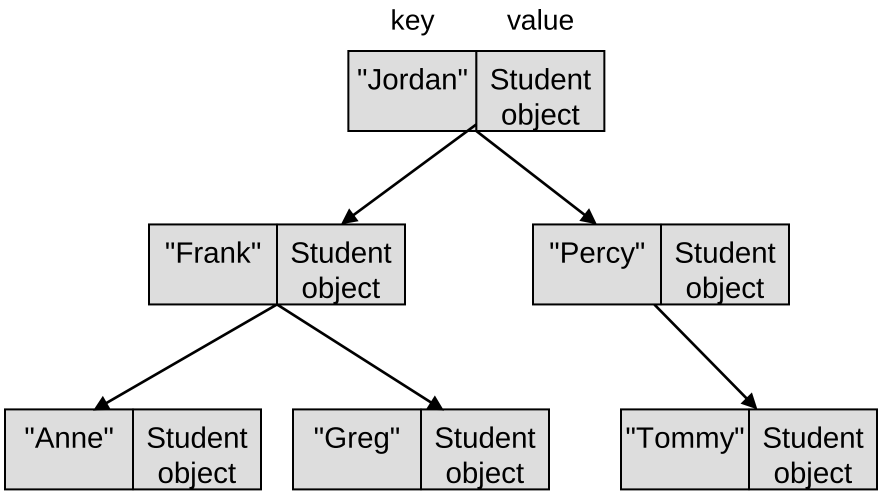
- If computing the hash function, **h(k)**, is O(1) and the array access is O(1),
- Then the runtime of the operations is O(1)
- What might prevent us from achieving this O(1)?
  - Collisions



Ordered vs. Unordered

Ordered Map/Set

- `map/set`
  (implemented as balanced BST)
- Log(n) runtime for insert/find/remove
- If we print each key via an in-order traversal of the tree, in what order will the keys be printed?



Unordered Map/Set

- `unordered_map/unordered_set`
  (implemented as hash table)
- Each uses a hash table for O(1) average runtime to insert, find, and remove
- New to C++11 and requires compilation with the `-std=c++11` option in g++
- Iteration will print the keys in an undefined order (unordered)
- Provides hash functions for basic types: int, string, etc. but for any other type you must provide your own hash function (like the operator< for BSTs)

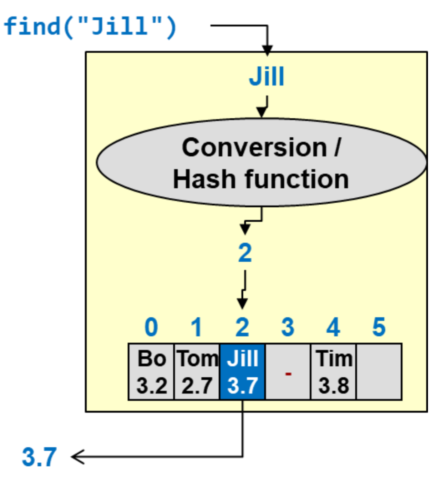
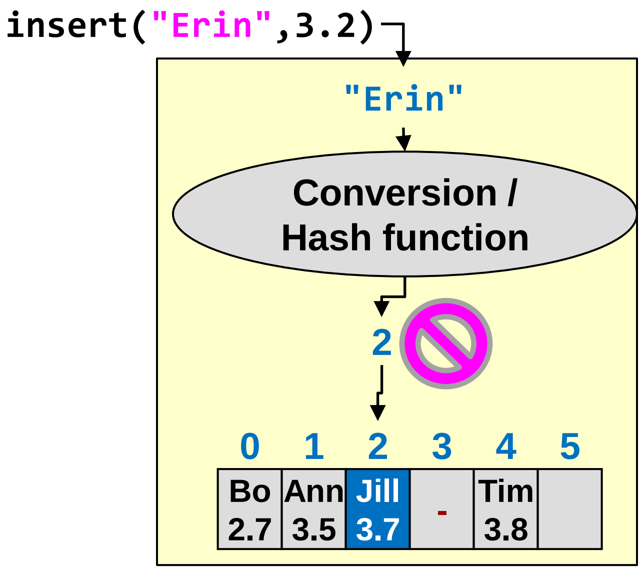


Table Size and Collisions

- Suppose we want to store USC student info using their 10-digit USC ID as the key
  - The set of all POSSIBLE keys, S, has size $|S| = 10^{10}$
  - But the number of keys we'd actually store, **n**, is likely much less (i.e. **n** << |S|)
- So how large should the table size (m) be?

_________ < _______________ < __________

- But anything smaller than the size of all possible keys admits the chance of COLLISION
  - A collision is when two keys map to the same location [i.e. h(k1) == h(k2)]
  - The **probability** of this should be low
  - How we handle collisions is the major remaining question to answer
- You will see that table size (**m**) should usually be a **prime** number



COLLISION!!
h("Jill") = h("Erin")

Resolving Collisions

- Collisions occur when two keys, k1 and k2, are not equal, but h(k1) = h(k2).
- Collisions are inevitable if the number of entries, **n**, is greater than table size, **m** (***by pigeonhole principle***) and are likely even if **n** < **m** ***(by the birthday paradox…more in our probability unit)***
- Methods
  - **Closed Addressing** (e.g. buckets or **chaining**): Keys MUST live in the location they hash to (thus requiring multiple locations at each hash table index)
    - Methods: 1.) Buckets, 2.) Chaining
  - **Open Addressing (aka probing):** Keys MAY NOT live in the location they hash to (only requiring a single 1D array as the hash table)
    - Methods: 1.) Linear Probing, 2.) Quadratic Probing, 3.) Double-hashing

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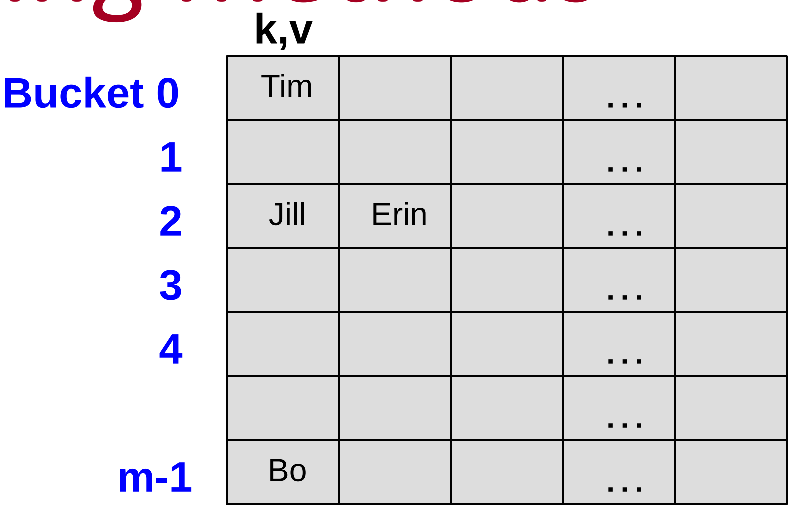
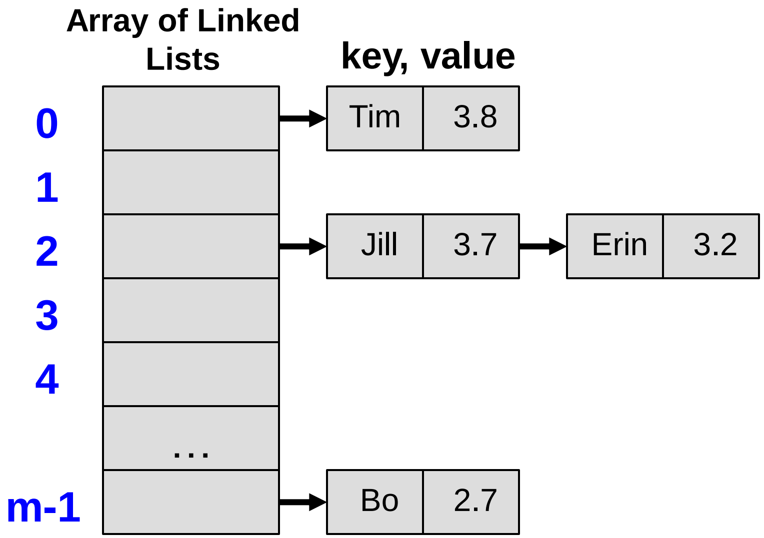
Closed Addressing Methods

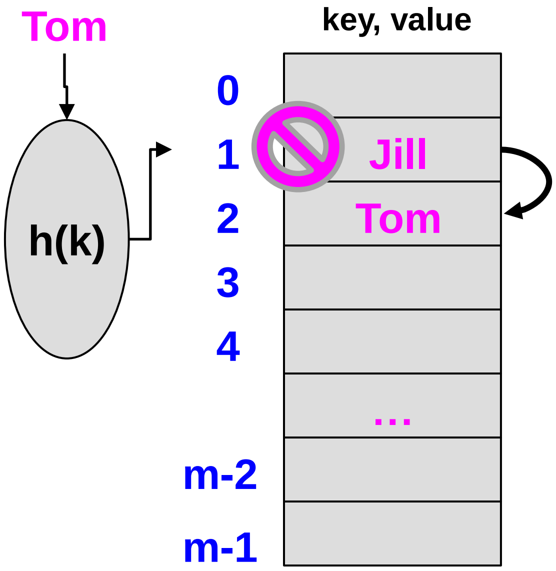
- Make each entry in the table a fixed-size ARRAY (bucket) or LINKED LIST (chain) of items/entries so all keys that hash to a location can reside at that index
  - **Close Addressing** => A key **will reside in the location it hashes to** (it's just that there may be many keys (and values) stored at that location
- **Buckets**
  - How big should you make each array?
  - Too much wasted space
- **Chaining**
  - Each entry is a linked list (or, potentially, vector)





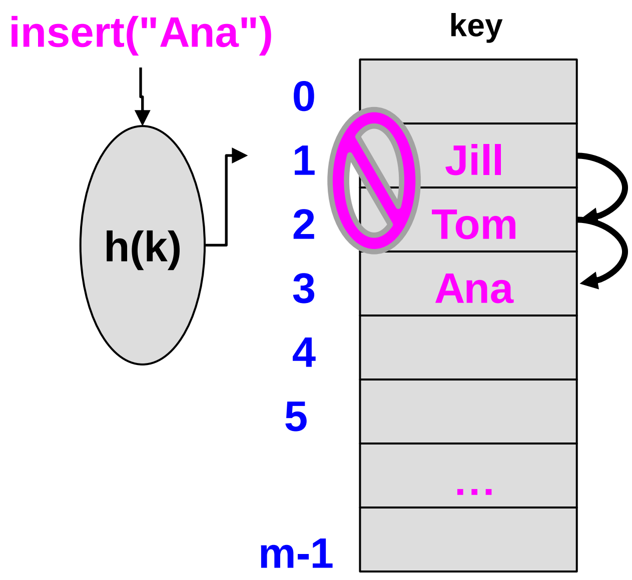
Open Addressing and Linear Probing

- With open addressing, we keep the hash table a 1D array (only one location per index) but when collisions occur we allow keys to reside in a location other than h(k)
  - **Open Addressing** => A key **may NOT reside in the location it hashes to** requiring extra searching in a process called **probing**
- For insertion: always start by checking location h(k)
  - If it is open, write the key (and value) there
  - Else "**probe"** for an empty location
- **Linear Probing (other techniques in a minute)**
  - Let **i** be number of **failed** checks to find a blank location (for insertion) or the key we are looking (for find/remove)
  - $h(k,i) = (h(k)+i)$ **mod m**
  - Example: If h(k) occupied (i.e. collision) then check h(k)+1, h(k)+2, h(k)+3, …



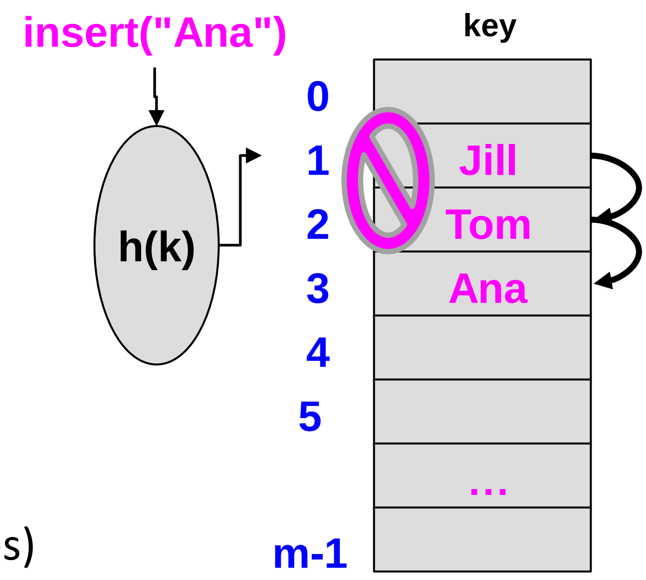
Probing Impact on Find

- If h(k) is occupied with another key, then probe
- **Insert**: probe until we find a blank location
- **Find/Remove**: probe until we…
  - Find the key we are looking for **..OR..**
  - ______________________________ **..OR..**
  - ______________________________



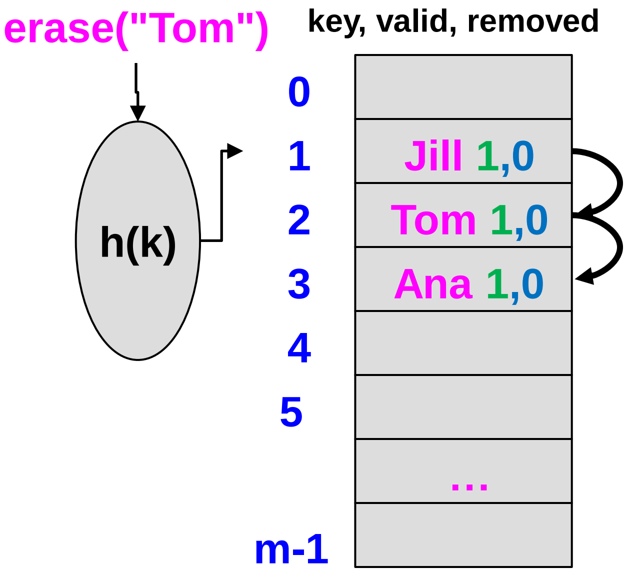
Probing Impact on Find

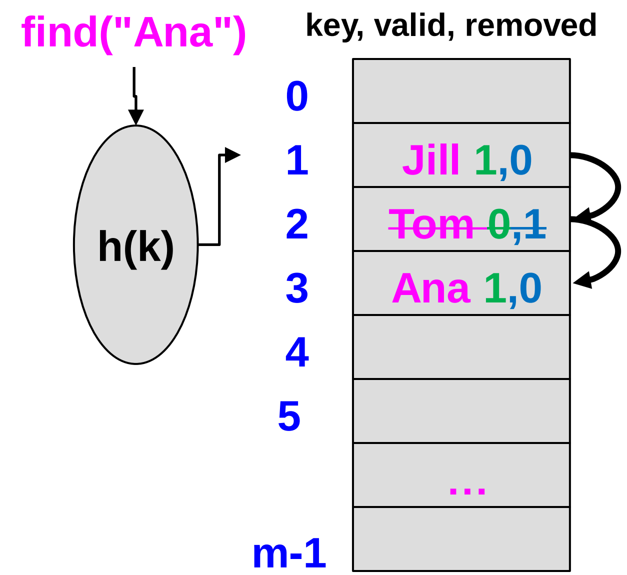
- If h(k) is occupied with another key, then probe
- **Insert**: probe until we find a blank location
- **Find/Remove**: probe until we…
  - Find the key we are looking for **..OR..**
  - We reach a free location **..OR..**
  - We have looked in all possible locations (i.e. wrapped back to h(k) or alternatively we've performed m probes)



Removal

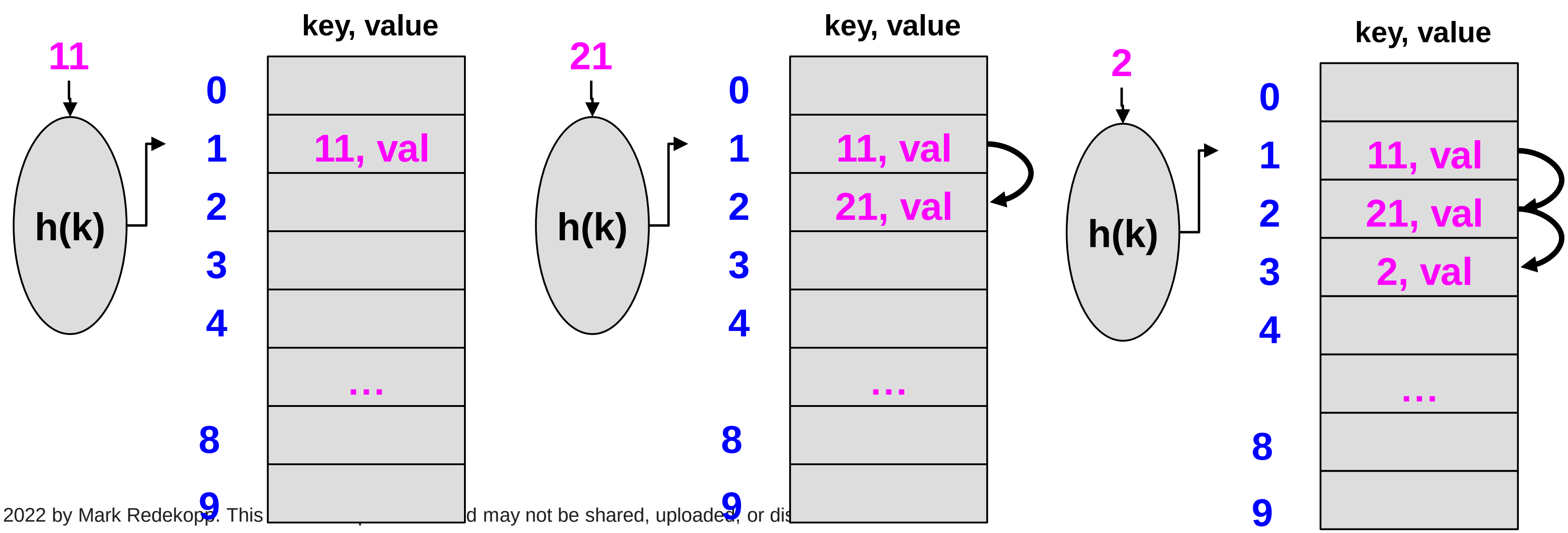
- Many implementations exist but we will show one simple way for illustration
- Each location stores two bools
  - **Valid**: a stored key exists in this location (or else is free)
  - **Removed**: a key was erased at this location (so it is free for insertion, but probing must continue for find/remove)
- Progression:
  - Initially: V=0,R=0 (Free/Never used),
  - On insert: V=1,R=0,
  - On erasure: V=0,R=1 (can return to V=1,R=0 on insert)
- For performance, we can periodically rebuild/rehash the hash table after some number of erasures to effectively return locations to free/never used





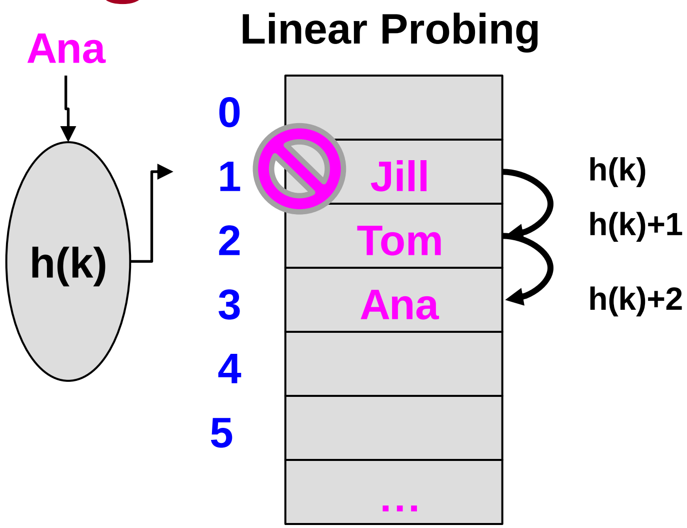
Linear Probing & Primary Clustering

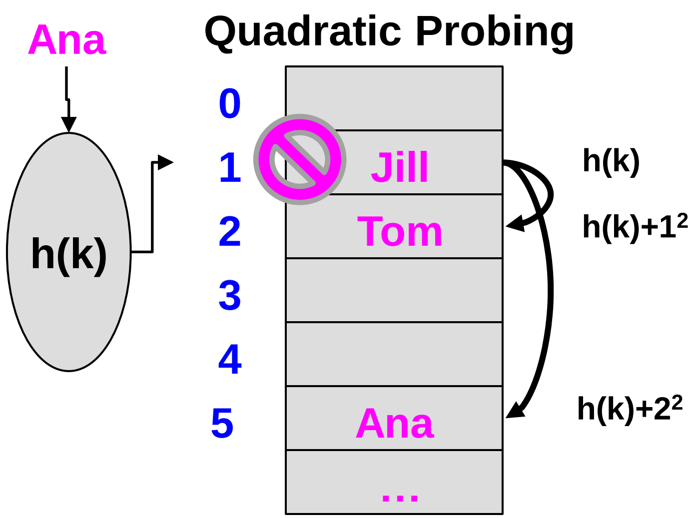
- Suppose a hash table (**m=10**) with integer keys and $h(k) = k\%m$
- Insert: 11, 21, 2, 31, 3
  - Notice, that the collisions of 11, 21, and 31 cause collisions for 2 and 3 which then may cause collisions for other nearby hash locations
- This is known as **primary clustering** (a few collisions to one location and the resulting probing cause collisions for other keys that would not have collided)



Quadratic Probing

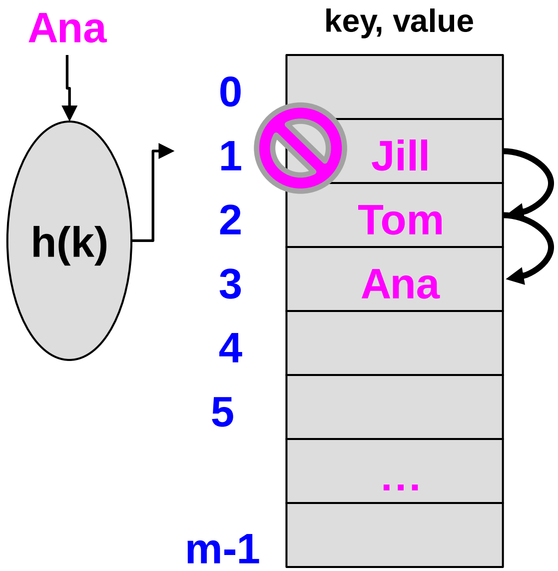
- If certain data patterns lead to many collisions, linear probing leads to clusters of occupied areas in the table called ***primary clustering***
- **Quadratic probing** tends to spread out data across the table by taking larger and larger steps until it finds an empty location
- **Quadratic Probing**
  - (Again, let **i** be number of **failed** probes)
  - $h(k,i) = (h(k)+i^2)$ **mod m**
  - If h(k) occupied, then check $h(k)+1^2$, $h(k)+2^2$, $h(k)+3^2$, …

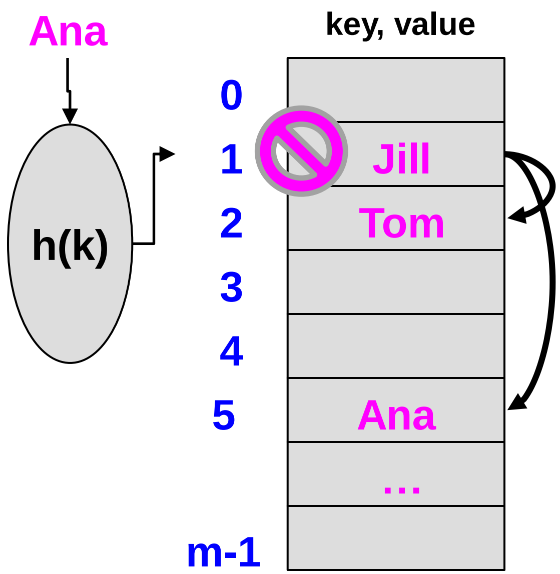




Linear vs. Quadratic Probing

- If certain data patterns lead to many collisions, linear probing leads to clusters of occupied areas in the table called ***primary clustering***
- How would quadratic probing help fight primary clustering?
  - Quadratic probing tends to spread out data across the table by taking larger and larger steps until it finds an empty location





Quadratic Probing Practice

- Use the hash function h(k)=k%9 to find the contents of a hash table (m=9) after inserting keys 36, 27, 18, 9, 0 using quadratic probing
- If your **loading factor** rises above 0.5, bad things can happen!

0	1	2	3	4	5	6	7	8

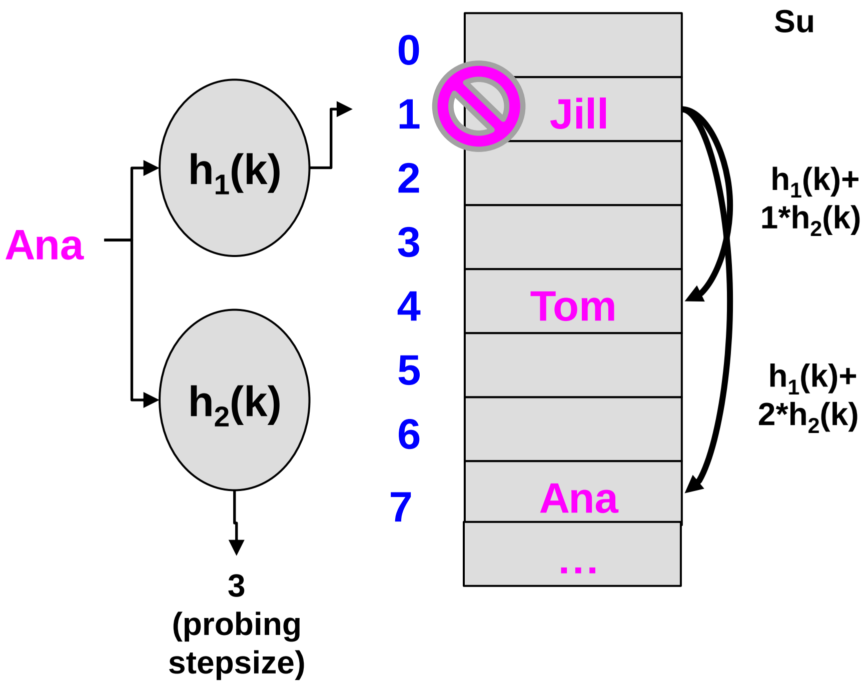
- Use the hash function h(k)=k%7 to find the contents of a hash table (m=10) after inserting keys 14, 8, 21, 2, 7 using quadratic probing

0	1	2	3	4	5	6

- Quadratic probing only works well for prime table sizes, and keeping the load factor < 0.5

Double Hashing

- Note: In linear and quadratic probing, if two keys hash to the same place ($h_1(k1) == h_1(k2)$) we will probe the ***same*** sequence
- Could we probe a ***different*** sequence even if two keys have collided?
  - Let's use ANOTHER hash function, $h_2(k)$ to choose the **step size** of our probing sequence
- **Double Hashing**
  - (Again, let i be number of failed probes)
  - Pick a second hash function $h_2(k)$ in addition to the primary hash function, $h_1(k)$
  - $h(k,i) = [\ h_1(k) + i*h_2(k)\ ]$ **mod m**



Sequence:
- Start at h1(k),
- If needed, probe h1(k) + h2(k)
- If needed, probe h1(k) + 2*h2(k)
- If needed, probe h1(k) + 3*h2(k)

Double Hashing

- Assume
  - m=13,
  - h1(k) = k % 13
  - h2(k) = 5 – (k % 5)
- What sequence would I probe if k = 31
  - h1(31) = ___, h2(31) = ________________
  - Seq: ____
  - Notice we ________________________ in the table. Why? **A _____ table size!**

Double Hashing

- Assume
  - m=13,
  - $h_1(k) = k \% 13$
  - $h_2(k) = 5 - (k \% 5)$
- What sequence would I probe if k = 31
  - $h_1(31) = 5$
  - $h_2(31) = 5-(31 \% 5) = 4$ (which is the step size)
  - 5 + 0*4 = 5 % 13 = 5
  - 5 + 1*4 = 9 % 13 = 9
  - 5 + 2*4 = 13 % 13 = 0
  - 5 + 3*4 = 17 % 13 = 4
  - And then onto 8, 12, 3, 7, 11, 2, 6, 10, 1
  - Notice we visited each index in the table. Why? **A prime table size!**

Rehashing

- For probing (open-addressing), as α approaches 1 the expected number of probes/comparisons will get very large
  - Capped at the tableSize, **m** (i.e. O(**m**))
- Similar to resizing a vector, we can allocate a larger prime size table/array
  - Must **rehash** items to location in new table size and **cannot just copy items to corresponding location in the new array**
  - Example: h(k) = k % 7 != h(k) = k % 11 (e.g. k=9)
  - For quadratic probing if table size m is prime, then first m/2 probes will go to unique locations
- General guideline for probing: keep α < ____

0	1	2	3	4	5	6
	1	9	38	18		

h(k) = k % 7

0	1	2	3	4	5	6	7	8	9	10

h(k) = k % 11

Rehashing

- For probing (open-addressing), as α approaches 1 the expected number of probes/comparisons will get very large
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→

0	1	2	3	4	5	6	7	8	9	10
	1				38		18		9	

h(k) = k % 11

Probing Technique Summary

- If h(k) is occupied with another key, then probe
- Let i be number of **failed** probes
- **Linear Probing**
  - $h(k,i) = (h(k)+i) \bmod m$
- **Quadratic Probing**
  - $h(k,i) = (h(k)+i^2) \bmod m$
  - If h(k) occupied, then check $h(k)+1^2$, $h(k)+2^2$, $h(k)+3^2$, …
- **Double Hashing**
  - Pick a second hash function $h_2(k)$ in addition to the primary hash function, $h_1(k)$
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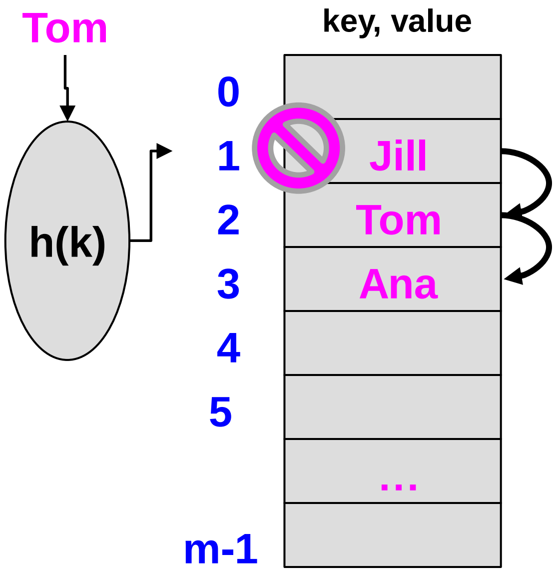

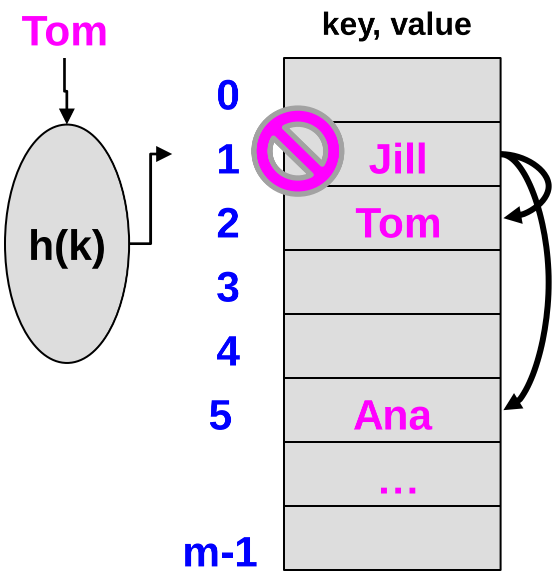

Hash Function Goals

- A "perfect hash function" should map each of the **n** keys to a unique location in the table
  - Recall that we will size our table to be larger than the expected number of keys…i.e. **n < m**
  - Perfect hash functions are not practically attainable
- A "good" hash function
  - Is easy and fast to compute
  - Scatters data uniformly throughout the hash table
    - $P(h(k) = x) = 1/\mathbf{m}$ (i.e. **pseudorandom**)

Hashing Efficiency

- Loading factor, α, defined as:
  - $\alpha = n / m$ (Really it is just the fraction of locations currently occupied)
  - n=number of items in the table, m=tableSize
- For open addressing, $\alpha \leq 1$
  - Good rule of thumb: resize and rehash after $\alpha > 0.5$
- For closed addressing (chaining), α, can be greater than 1
  - This is because $n > m$
  - What is the average length of a chain in the table (e.g. 10 total items in a hash table with table size of 5)?
  - Need to keep α constant (usually $\alpha \leq 1$)

Hashing Efficiency

- Loading factor, α, defined as:
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  - This is because $n > m$
  - What is the average length of a chain in the table (e.g. 10 total items in a hash table with table size of 5)?
    - Average length of chain will be $\alpha = n / m$
  - Need to keep α constant (usually $\alpha \le 1$)

Hash Tables are Awesome!

Hash tables provide a very lucrative potential runtime. However, they are **probabilistic**.

- There was a similar problem with Splay Trees: they had a good **average** runtime, but a poor **worst**-case runtime.

As of this moment, we do not have the necessary mathematical framework to analyze either of these structures.

- We're going to start remedying that… now.