Midterm 2 Suggested Answers

1. Consider the 1D Laplacian discretized on the grid $x \in \{0, 1/n, 2/n, \dots, 1\}$ with Dirichlet boundary conditions.

   (a) Show that for any positive integer $k < n$, $u(x) = \sin(k\pi x)$ is an eigenvector of the centered difference operator

   $$(Lu)(x) = n^2\Big(- u(x - 1/n) + 2u(x) - u(x + 1/n)\Big)$$

   with homogeneous Dirichlet boundary conditions. Hint: $\sin(a + b) + \sin(a - b) = 2 \sin a \cos b$.

   (b) As a function of n, what are the maximum and minimum eigenvalues of L?

   (c) The optimal relaxation parameter w for Richardson iteration

   $$u_{j+1} = u_j - w(Lu_j - b)$$

   is

   $$w_* = \operatorname{argmin}_w \{|1 - w\lambda_{\min}|, |1 - w\lambda_{\max}|\}.$$

   Write a closed form expression for w_*.

   (d) How many iterations would be required to reduce the error by one order of magnitude?

Answer: (a) Compute

$$\begin{aligned} L\sin(k\pi x) &= n^2\Big[- \sin\big(k\pi(x - 1/n)\big) + 2\sin(k\pi x) - \sin\big(k\pi(x + 1/n)\big)\Big] \\ &= n^2\left(2 - 2\cos\frac{k\pi}{n}\right)\sin k\pi x. \end{aligned}$$

(b) The min and max eigenvalues arise for $k = 1$ and $k = n - 1$ respectively,

$$\begin{aligned} \lambda_{\min} &= n^2\left(2 - 2\cos\frac{\pi}{n}\right) \approx \pi^2 \\ \lambda_{\max} &= n^2\left(2 - 2\cos\frac{(n-1)\pi}{n}\right) \approx 4n^2. \end{aligned}$$

(c) The first term is positive and decreasing with w while the second is negative and increasing in magnitude with w. The optimum thus occurs when they are equal in magnitude,

$$1 - w\lambda_{\min} = w\lambda_{\max} - 1$$

which yields

$$w_* = \frac{2}{\lambda_{\min} + \lambda_{\max}}.$$

(d) The optimal convergence factor is

$$\rho = 1 - w_*\lambda_{\min} = \frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}}$$

and $\rho^k < 0.1$ for

$$k = \lceil \log_\rho 0.1 \rceil = \left\lceil \frac{\log 0.1}{\log \rho} \right\rceil.$$

2. Consider the third order equation

$$u'''(x) = f(x)$$

on an infinite grid $x \in \mathbb{Z}$.

(a) Using the grid functions $\phi(x,\theta) = e^{i\theta x}$, show that the symbol of the exact third derivative operator is purely imaginary.

(b) An antisymmetric operator has the property that $A^T = -A$. Antisymmetry implies that the stencil has the form

$$\left[-s_k, \ldots, -s_1, 0, s_1, \ldots, s_k\right].$$

Show that all antisymmetric stencils have purely imaginary symbols.

(c) Write a consistent anti-symmetric stencil for evaluating the third derivative.

(d) Sketch the symbol of your stencil for $\theta \in [-\pi, \pi)$. Is your stencil stable?

Answer: (a) Applying $\phi'(x,\theta) = i\theta\phi(x,\theta)$ three times yields

$$\phi'''(x,\theta) = (i\theta)^3\phi(x,\theta) = \underbrace{-i\theta^3}_{\text{symbol}}\phi(x,\theta).$$

(b) Compute

$$\begin{aligned} A\phi(x,\theta) &= \sum_{j=1}^{k} s_j\big(\phi(x+j,\theta) - \phi(x-j,\theta)\big) \\ &= \sum_{j=1}^{k} s_j\big(e^{i\theta j} - e^{-i\theta j}\big)\phi(x,\theta) \\ &= i\sum_{j=1}^{k} 2s_j \sin(j\theta). \end{aligned}$$

(c) One choice is to compute the second derivative at ± 1 and difference those via centered difference, yielding the 5-point stencil

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -1 \end{bmatrix}.$$

From above, this has symbol

$$\hat{A}(\theta) = i\big(2\sin\theta - \sin 2\theta\big).$$

Using the Taylor series

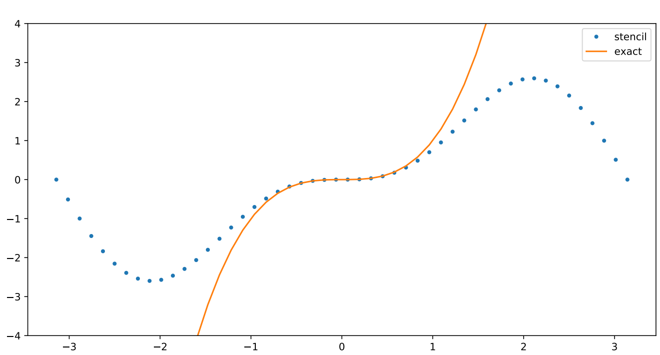
$$\sin\theta = \theta - \theta^3/6 + O(\theta^5)$$

we have

$$\hat{A}(\theta) = -i\big(\theta^3 + O(\theta^5)\big)$$

which demonstrates its consistency with the exact value for well-resolved functions (small θ).

(d) The symbol goes to zero at the Nyquist frequency $\theta = \pm\pi$, thus is not strictly stable.



3. Suppose the operator A is split as $A = A_+ + A_-$.

   (a) Write a compact expression for the error iteration matrix to solve $Ax = b$ using Richardson iteration preconditioned by A_+^{-1}.

   (b) Suppose A_+ and A_- have symbols
   $$\hat{A}_+(\theta) = 2 - e^{-i\theta}$$
   $$\hat{A}_-(\theta) = e^{i\theta}$$
   and compute the maximum absolute value of the symbol over high frequencies $|\theta| \in [\pi/2, \pi]$. This is called the "smoothing factor".

   (c) Recall that a square matrix B is a projector if $B^2 = B$. Show that the "Galerkin projection" $G = P(P^T AP)^{-1} P^T A$ is indeed a projector.

   (d) If B is a projector, show that $I - B$ is also a projector.

   (e) Show that all eigenvalues of a projector are either 1 or 0.

Answer: (a) Begin with the error iteration matrix and compute
$$\begin{aligned} I - A_+^{-1} A &= I - A_+^{-1}(A_+ + A_-) \\ &= I - I - A_+^{-1} A_- = -A_+^{-1} A_- \end{aligned}$$

(b) We can compute
$$\left| -\frac{\hat{A}_-(\theta)}{\hat{A}_+(\theta)} \right| = \frac{|e^{i\theta}|}{|2 - e^{-i\theta}|} = \frac{1}{|2 - e^{-i\theta}|}.$$
For high frequency θ, this expression ranges from $1/3$ at $\theta = \pm\pi$ to $\sqrt{1/5}$ at $\theta = \pm\pi/2$.

(c) Compute
$$G^2 = P \underbrace{(P^T AP)^{-1} P^T AP}_{\text{identity}} (P^T AP)^{-1} P^T A = \underbrace{P(P^T AP)^{-1} P^T A}_{G}.$$

(d) Compute
$$(I - B)^2 = I - 2B + \underbrace{B^2}_{B} = I - B.$$

(e) If $Bx = \lambda x$ then
$$B(Bx) = B(\lambda x) = \lambda^2 x \neq Bx$$
unless $\lambda^2 = \lambda$, i.e., $\lambda \in \{0, 1\}$.

4. The equilibrium diffusion equation

$$-\nabla \cdot \big(\kappa \nabla u\big) = f(x,y,z) \text{ on } \Omega \subset \mathbb{R}^3 \qquad\qquad u|_{\partial\Omega} = 0$$

is solved in three dimensions using a conservative finite difference method (approximating κ at staggered points).

(a) If κ is independent of u, this equation is linear and can be discretized to yield the matrix equation $Au = b$. How many nonzeros per row are present in the matrix A?

(b) If κ depends on ∇u, as in the p-Laplacian

$$\kappa(\nabla u) = \left(\frac{\epsilon^2}{2} + \frac{\nabla u \cdot \nabla u}{2}\right)^{(p-2)/2},$$

our discrete system will have the form $F(u) = 0$. To compute $F(u)$, we need to compute the full gradient ∇u at staggered points such as $(x - h/2, y, z)$. The aligned component

$$u_x(x - h/2, y, z) \approx \frac{u(x,y,z) - u(x-h,y,z)}{h}$$

is simple, but the transverse components are trickier. In 2D, we might approximate transverse derivatives using a scheme such as

$$u_y(x - h/2, y) \approx \frac{u(x-h/2, y+h) - u(x-h/2, y-h)}{2h}$$

where $u(x - h/2, y + h) \approx \frac{1}{2}\big[u(x-h, y+h) + u(x-h, y)\big]$. For the 3D problem, how many nonzeros per row are present in the Jacobian matrix

$$J = \frac{\partial F}{\partial u}?$$

(c) If κ is independent of u, but discontinuous, what order of convergence can we expect from the discretization above under grid refinement $h \to 0$?

Answer: (a) In 3D, we need fluxes at the six faces of the dual cube $[-h/2, h/2]^3$. Each of these fluxes is computed by directional derivatives of the form

$$\kappa(-h/2, 0, 0)\frac{u(0,0,0) - u(-h,0,0)}{h},$$

each of which contains the center point and one neighbor. Summing over the 6 faces, our stencil depends on a total of 7 grid values of u. Consequently, the matrix has 7 nonzeros per row, except possibly for boundary conditions.

(b) With the transverse derivatives included, our gradients ∇u at staggered points depend on all points

$$\Big\{(x,y,z) : x,y,z \in \{-h,0,h\} \text{ and } xyz = 0\Big\}$$

where the $xyz = 0$ condition excludes the 8 "corners" from the $3^3 = 27$ possible points, leaving $27 - 8 = 19$ nonzeros in the Jacobian.

(c) If κ is discontinuous, the true gradient ∇u will have a jump at the discontinuity so that $\kappa \nabla u \cdot \hat{n}$ is continuous across the interface with normal $\hat{n}$. The error in our pointwise formulas for gradient can thus be $O(1)$ and after integrating over the surface of an element of size h, $O(h)$.