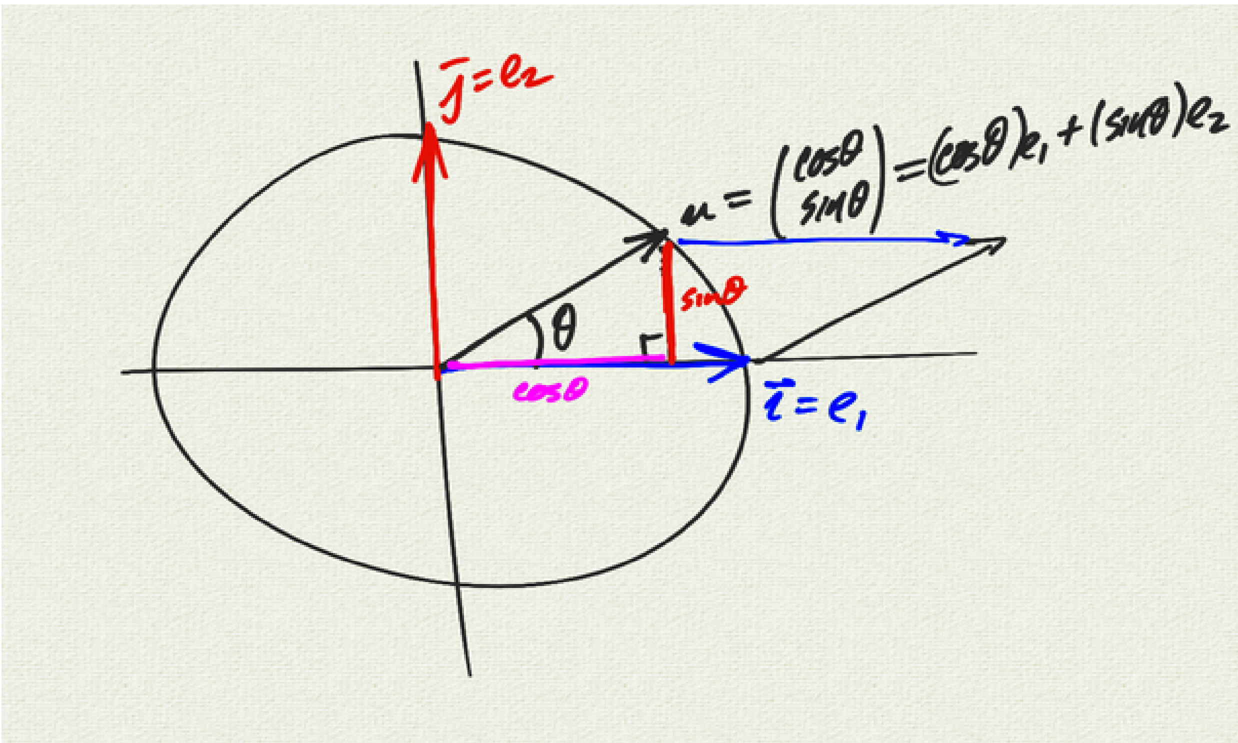
Geometric Algebra Notes 1 (Wedge Product)
MultiV 2021-22 / Dr. Kessner

First we're going to think about the unit circle in $\mathbb{R}^2$, and change our notation as well.

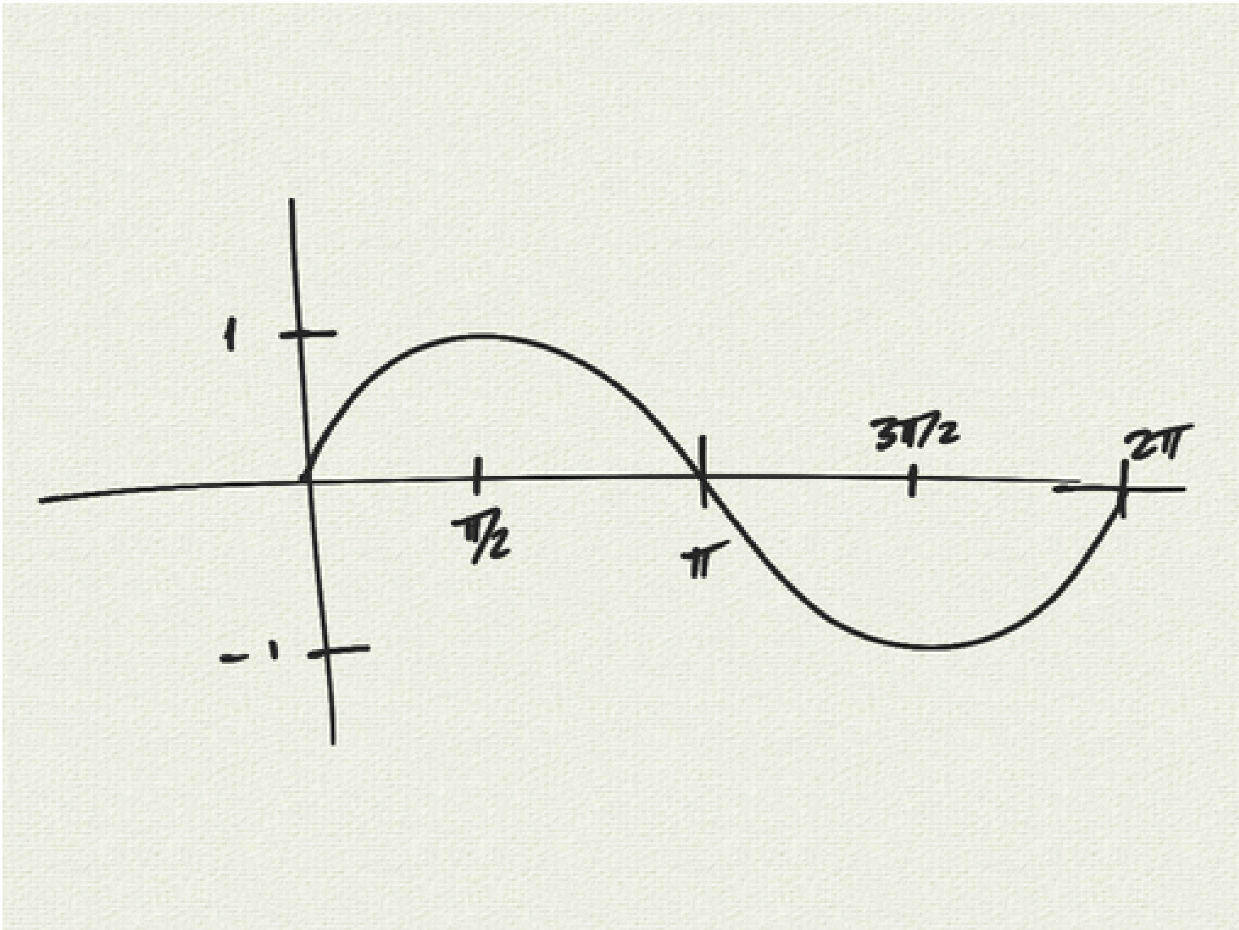
We're going to call our unit vectors $e_1 = \mathbf{i} = \begin{pmatrix}1\\0\end{pmatrix}$ and $e_2 = \mathbf{j} = \begin{pmatrix}0\\1\end{pmatrix}$.

Let $u = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ be a vector on the unit circle.

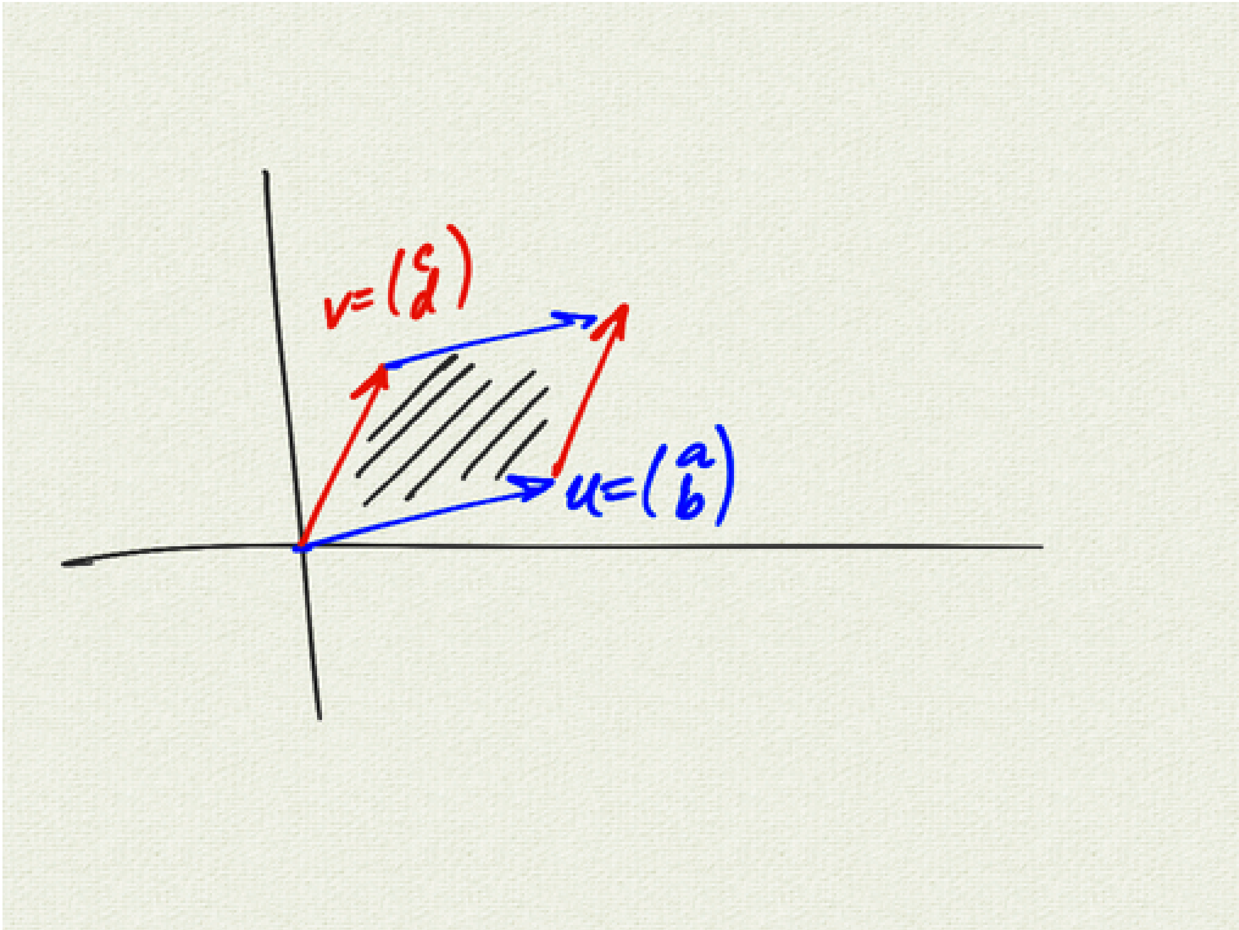


Observe that the projection of u on the x-axis is given by $\cos\theta$, and the area of the parallelogram determined by e_1 and u is $\sin\theta$.

We can graph the area of the parallelogram as u moves around the unit circle. Notice that for $\theta \in [\pi, 2\pi]$, the area is negative.

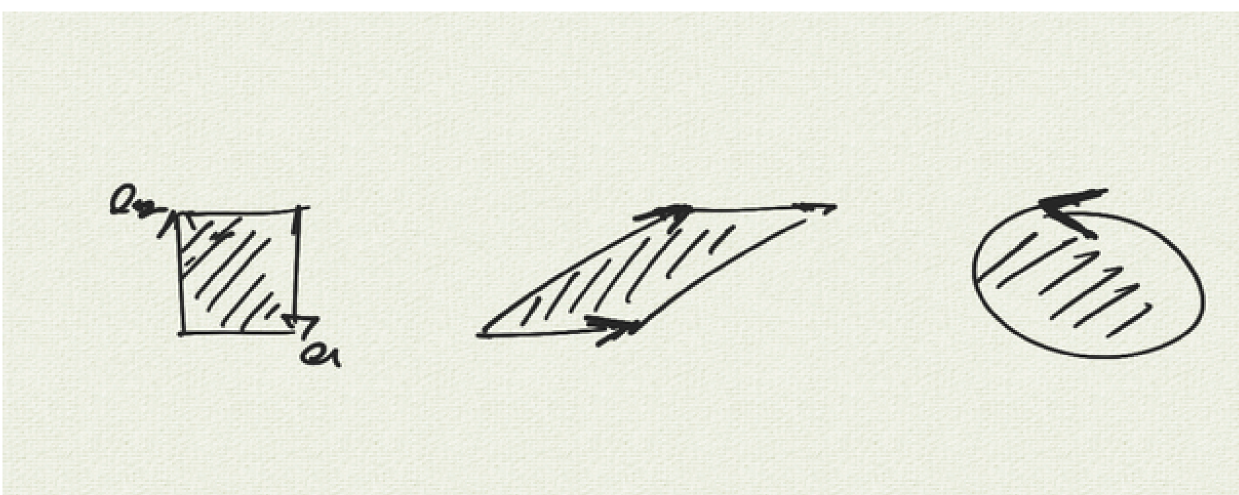


We have seen previously that $u = \begin{pmatrix} a \\ b \end{pmatrix}$ and $v = \begin{pmatrix} c \\ d \end{pmatrix}$, the area of the parallelogram is given by the determinant $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.$



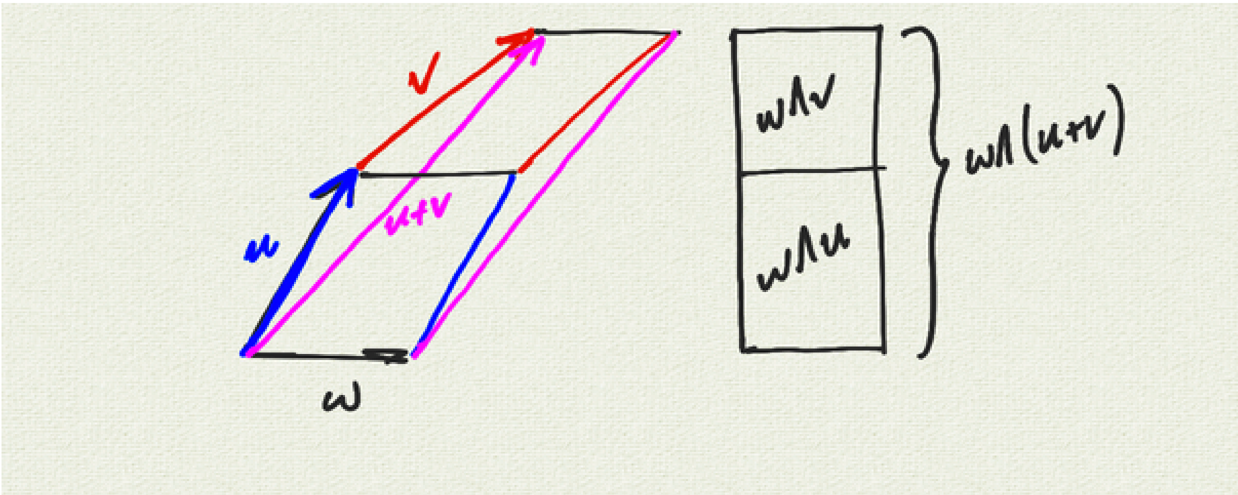
We define the wedge product $u \wedge v$ to be the directed (signed) area of the parallelogram determined by the two vectors, but with "units" (like meters^2). We call this directed area a *bivector*.

We define $e1 \wedge e2$ to be the "unit bivector". It represents the directed area of the square determined by e_1 and e_2. A general bivector will be a scalar multiple of $e_1 \wedge e_2$. However, the actual shape of of the bivector is not specified: we can think of it as a square, or reshape it to a parallelogram, or an amorphous shape in the plane.



From the definition of the wedge product, we observe that:
$e_1 \wedge e_1 = 0 = e_2 \wedge e_2$
$e_2 \wedge e_1 = -e_1 \wedge e_2$.

The distributive property is not so obvious:
$w \wedge (u + v) = w \wedge u + w \wedge v$

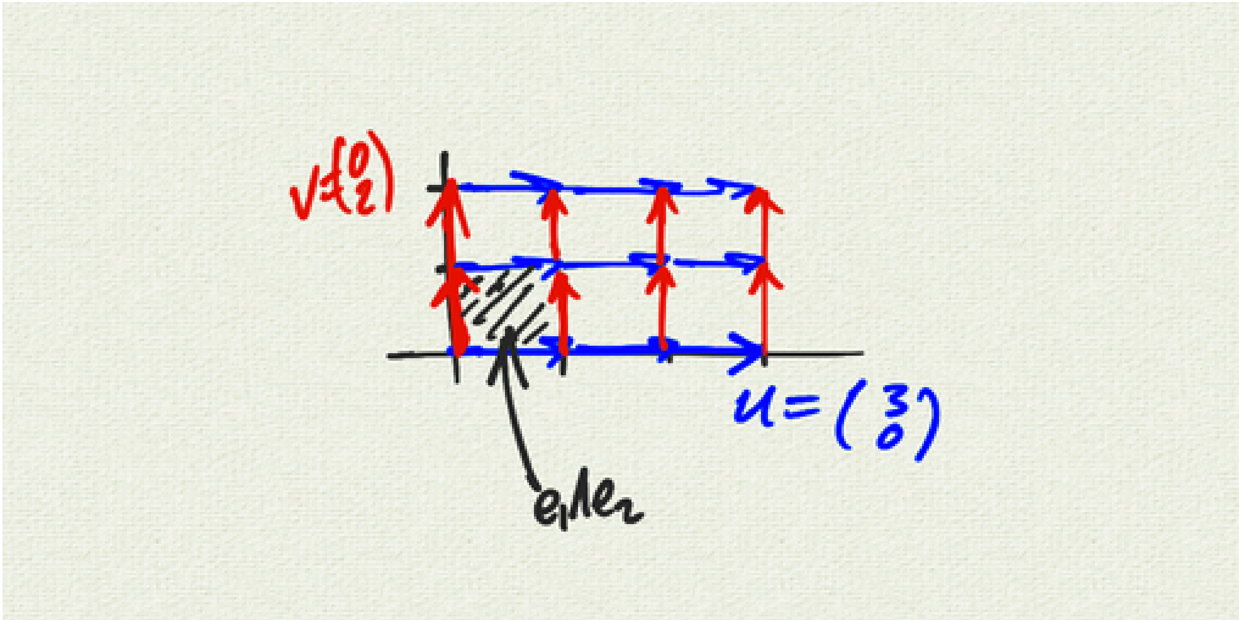


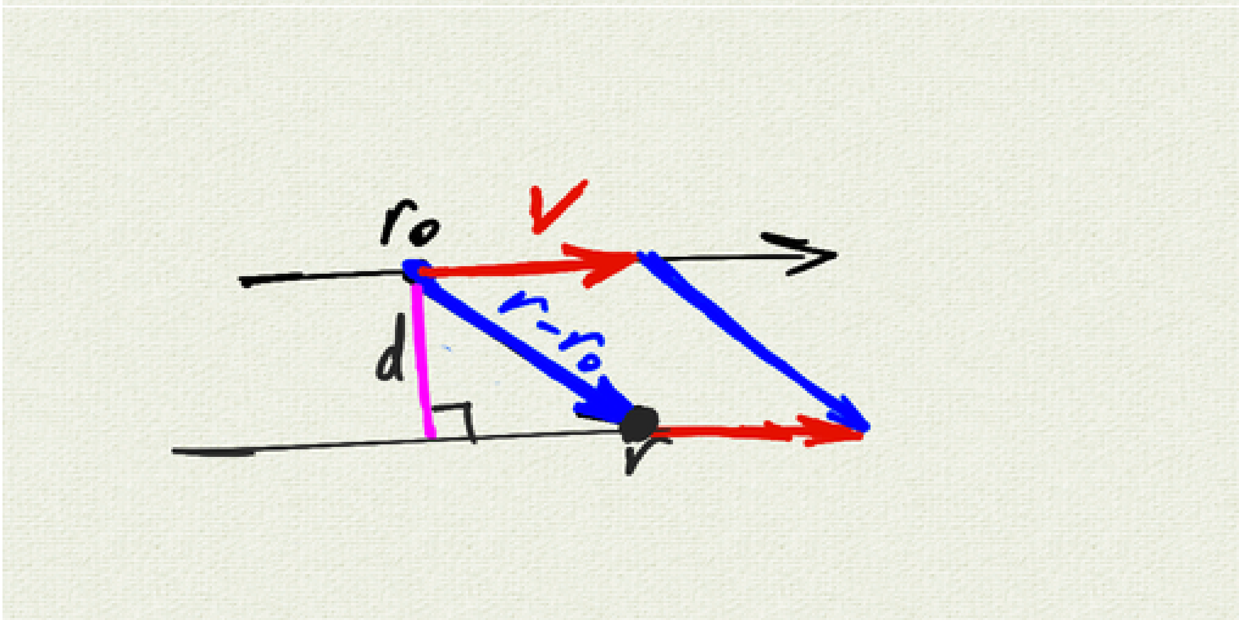
Once we believe the distributive property, we can do FOIL.
Let $u = \begin{pmatrix} a \\ b \end{pmatrix} = ae_1 + be_2$
and $v = \begin{pmatrix} c \\ d \end{pmatrix} = ce_1 + de_2$.

Then

$$\begin{aligned} u \wedge v &= (ae_1 + be_2) \wedge (ce_1 + de_2) \\ &= (ae_1 \wedge ce_1) + (ae_1 \wedge de_2) + (be_2 \wedge ce_1) + (be_2 \wedge de_2) \\ &= ac(e_1 \wedge e_1) + +ad(e_1 \wedge e_2) + bc(e_2 \wedge e_1) + bd(e_2 \wedge e_2) \\ &= (ad - bc)(e_1 \wedge e_2) \end{aligned}$$

Notice that the determinant $ad - bc$ emerges as a consequence of the elementary properties of the wedge product.


$v=\binom{0}{2}$
$u=\binom{3}{0}$
$e_1 \wedge e_2$


r_0
v
$r-r_0$
d
r