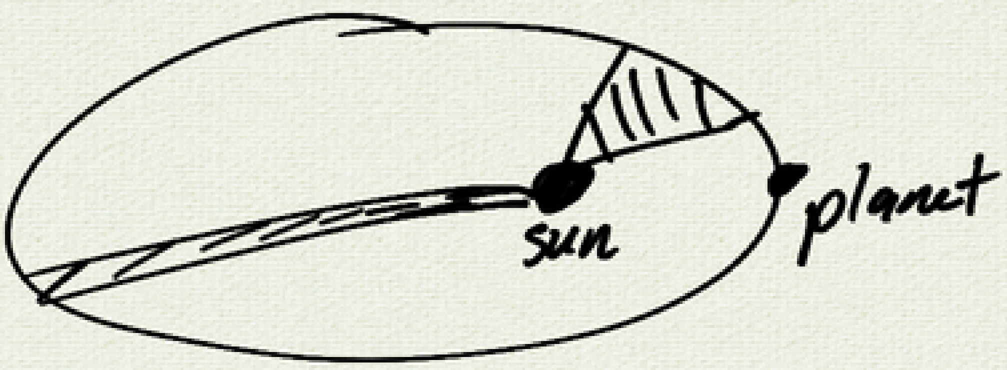
2.7 Kepler's Laws



① ellipses

② equal areas in equal times

③ $T^2 \sim r^3$

period   radius (average)

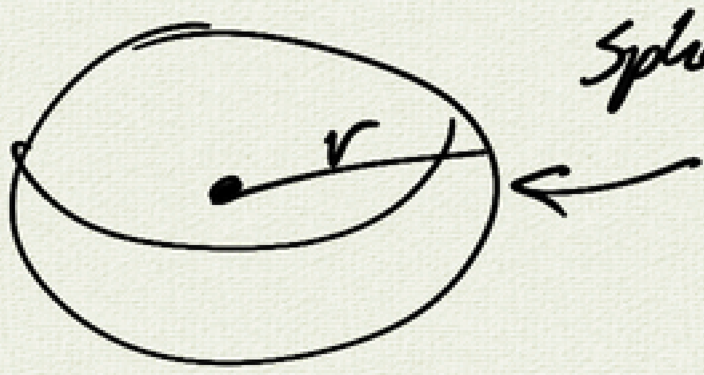
Newton

calculus

vectors

$$\left.\begin{array}{l}\bar{F} = m\bar{a} \\ \text{gravity} \sim \dfrac{1}{r^2}\end{array}\right\} \Longrightarrow \text{Kepler's Laws}$$

Inverse square law



expanding spherical shell

← surface area

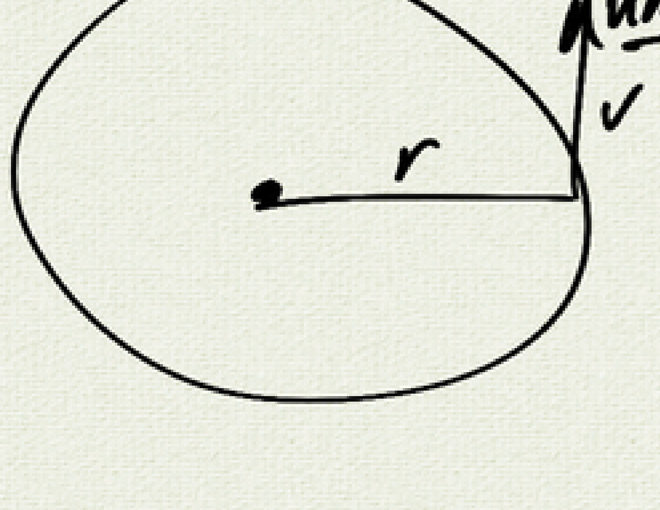
$V = \frac{4}{3}\pi r^3$

$SA = 4\pi r^2$

expands, but mass is constant

$$\Longrightarrow \text{density} \sim \frac{m}{SA} \sim \frac{1}{r^2}$$

$\boxed{T^2 \sim r^3}$ from data/observation



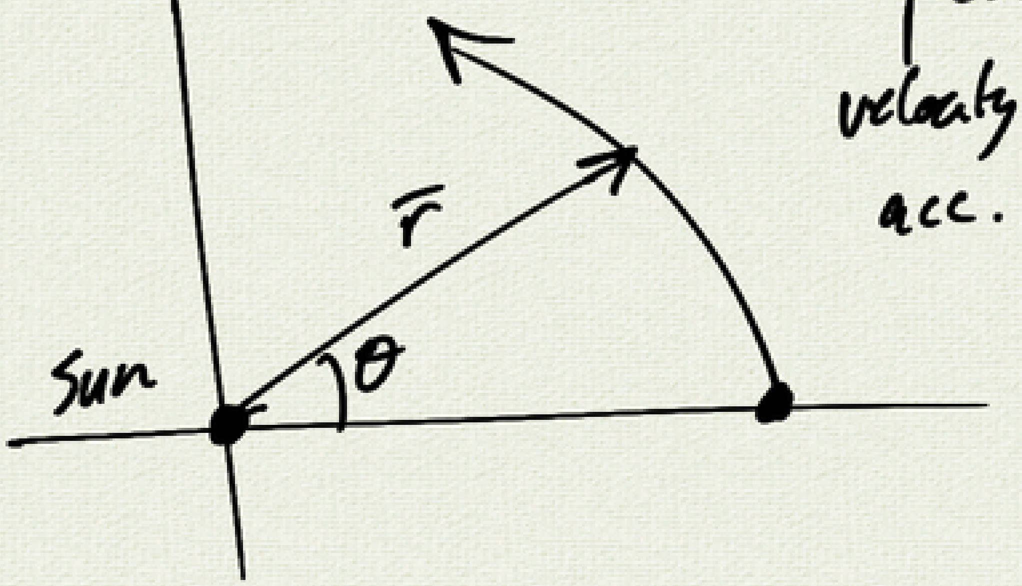
uniform circular motion

T = period

$$\Longrightarrow v = \frac{2\pi r}{T} \sim \frac{r}{T}$$

$$\Longrightarrow a \sim \frac{v}{T}$$

$$\Longrightarrow a \sim \frac{v}{T} \sim \frac{r}{T^2} \sim \frac{r}{r^3} \sim \frac{1}{r^2}$$

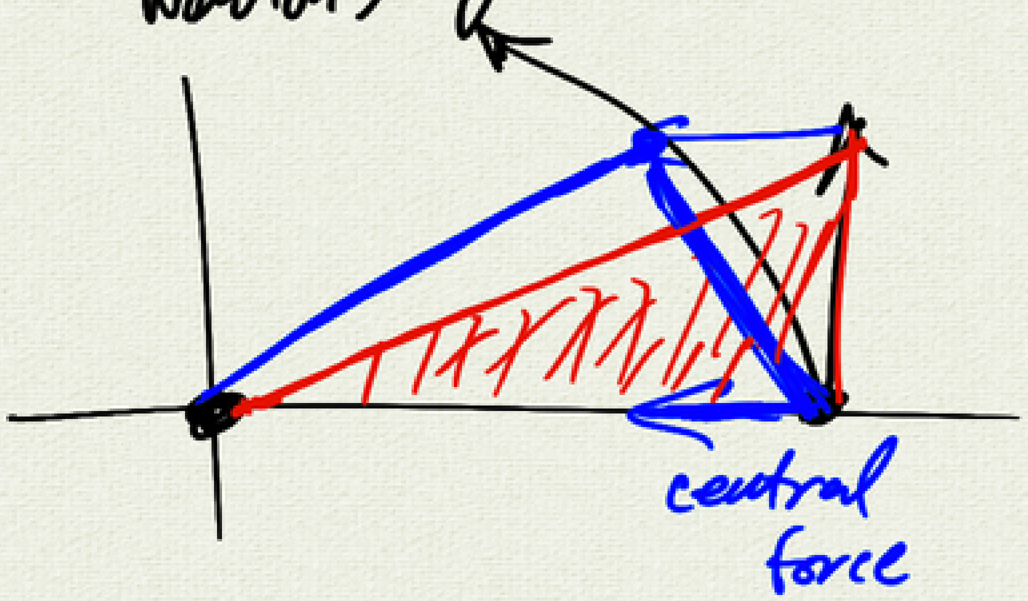


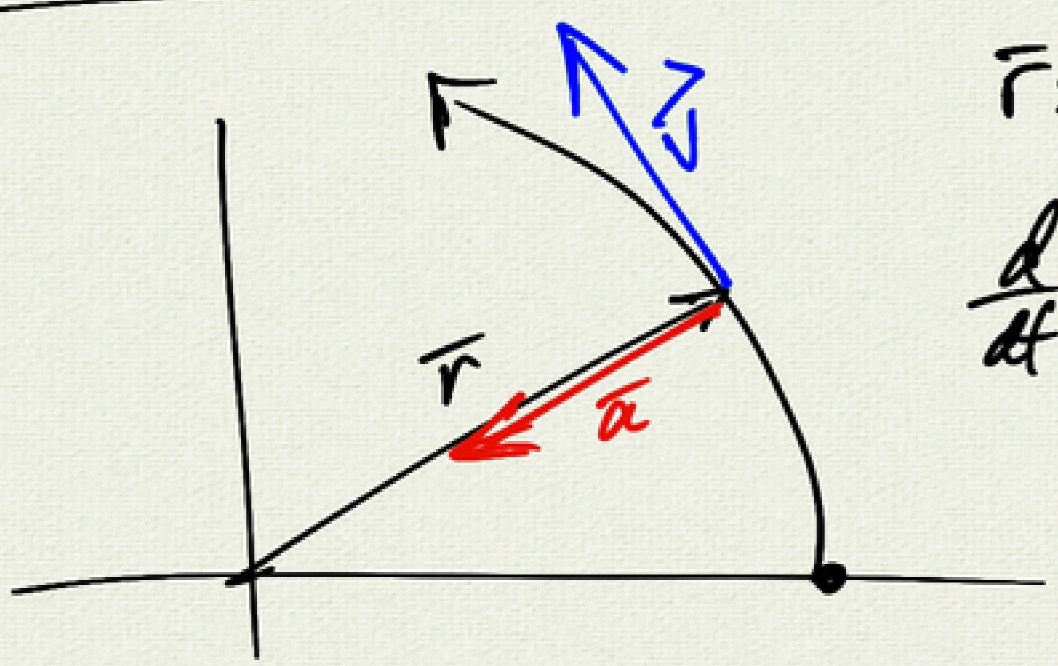
position $\bar{r}(t)$

velocity $\bar{v} = \bar{r}'$

acc. $\bar{a} = \bar{v}' = \bar{r}''$

Newton's geometric argument: central force $\Rightarrow$ equal areas in equal times





$\bar{r} \times \bar{v}$ (angular momentum)

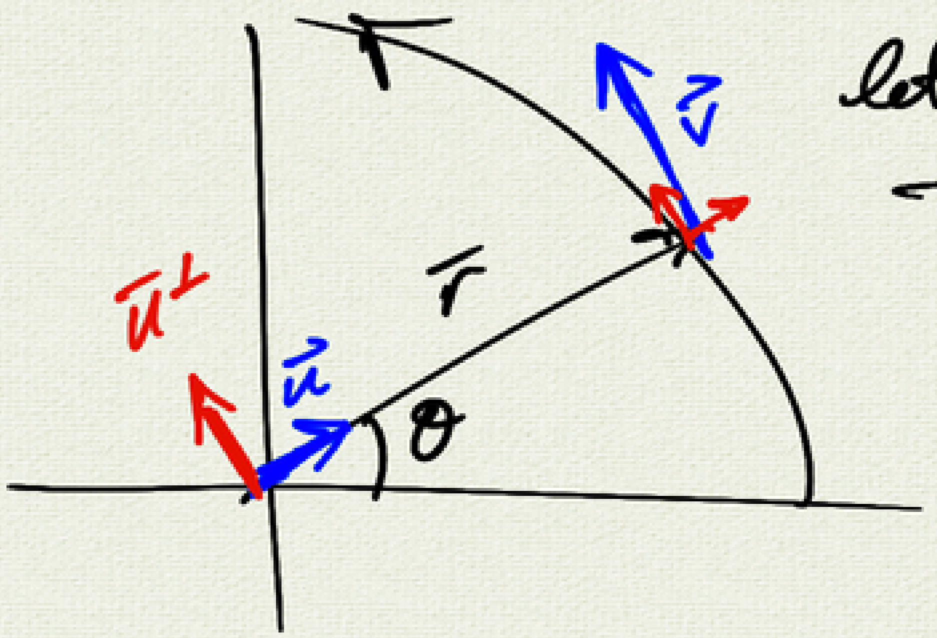
$$\frac{d}{dt}(\bar{r} \times \bar{v}) = \bar{r}' \times \bar{v} + \bar{r} \times \bar{v}'$$

$$= \underbrace{\bar{v} \times \bar{v}}_{0} + \underbrace{\bar{r} \times \bar{a}}_{\substack{0 \\ \text{for central force}}}$$

$$\bar{b} = \bar{r} \times \bar{v} = \text{constant}$$

$\bar{b} \perp \bar{r}$ and $\bar{b} \perp \bar{v}$

$\Rightarrow \bar{r}, \bar{v}$ always in the same plane



let $r = |\bar{r}|$

$$\bar{u}(t) = \begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{pmatrix} \Longrightarrow \bar{u}'(t) = \begin{pmatrix} (-\sin\theta)\frac{d\theta}{dt} \\ \underbrace{(\cos\theta)}_{\bar{u}^{\perp}}\frac{d\theta}{dt} \end{pmatrix}$$

$$\bar{r}(t) = r\bar{u}$$

$$(= r(t)\bar{u}(t))$$

$$= \frac{d\theta}{dt}\bar{u}^{\perp}$$

$$\bar{r}'(t) = (r(t)\bar{u}(t))'$$

$$= r'(t)\bar{u}(t) + r(t)\underbrace{\bar{u}'(t)}$$

$$\boxed{\bar{v} = r'\bar{u} + r\frac{d\theta}{dt}\bar{u}^{\perp}}$$

$$\bar{r}\times\bar{v} = \bar{r}\times\left(r'\bar{u} + r\frac{d\theta}{dt}\bar{u}^{\perp}\right)$$

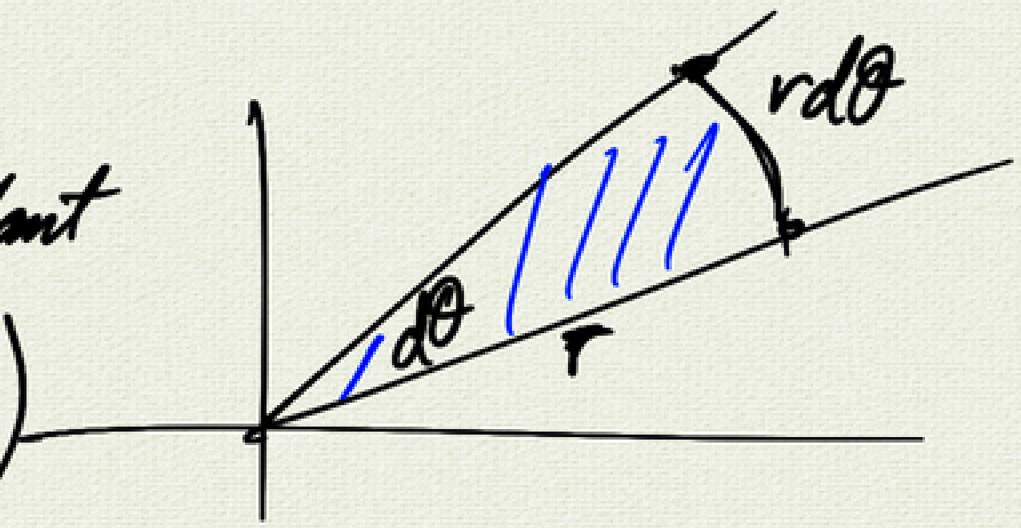
$$= r'\underbrace{(\bar{r}\times\bar{u})}_{0} + r\frac{d\theta}{dt}\underbrace{(\bar{r}\times\bar{u}^{\perp})}_{r\bar{k}}$$

$$\bar{b} = \boxed{\bar{r}\times\bar{v} = \boxed{\left(r^2\frac{d\theta}{dt}\right)}\bar{k}}$$ constant

$$dA \approx \frac{1}{2}r^2 d\theta$$

$\frac{dA}{dt}$ constant

(equal areas)



$$\bar{a} = -\frac{GM}{r^2}\bar{u} \qquad \text{gravity} \sim \frac{1}{r^2}$$

$$\bar{b} = \bar{r} \times \bar{v}$$

$$\Longrightarrow \underbrace{\bar{a}}_{\bar{v}'} \times \bar{b} = GM\,\bar{u}'$$

integrate

$$\bar{v} \times \bar{b} = GM\bar{u} + \bar{c}$$

const

$$\Longrightarrow \underbrace{|b|^2}_{\text{const}} = \underline{GM}\,r + \underline{c}\,r\cos\theta$$

$$r = \frac{|b|^2}{GM + c\cos\theta}$$

polar form of conic $\quad r = \dfrac{ed}{1 + e\cos\theta}$

$$e = \frac{c}{GM}$$

< 1 ellipse

$= 1$ parabola

> 1 hyperbola