1.5 Cross Product

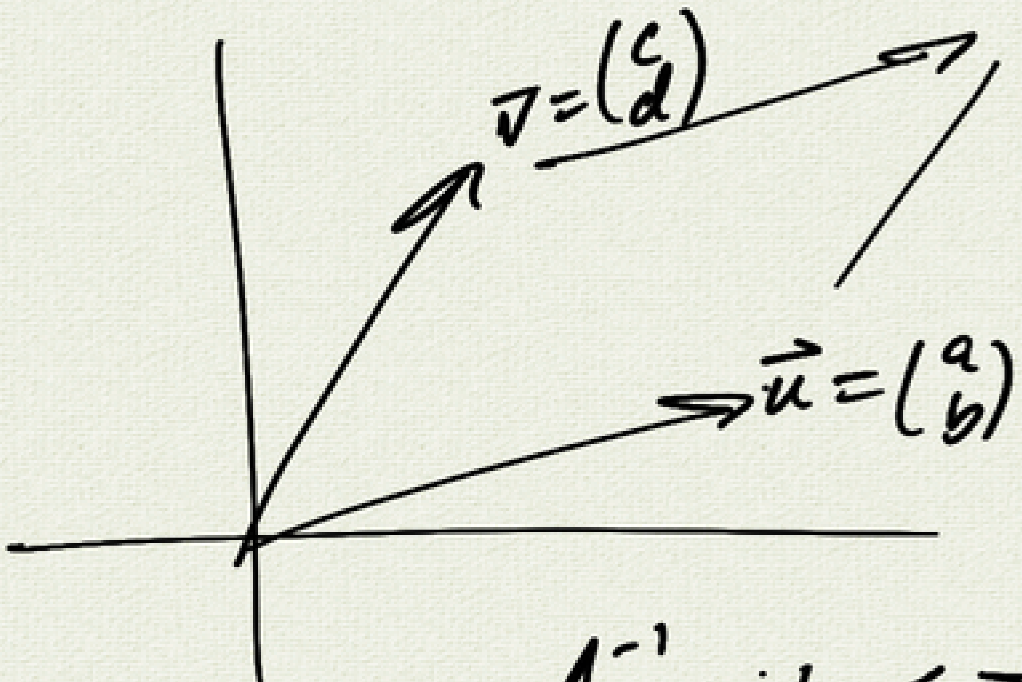
$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

example:

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 2\begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix} + 0\begin{vmatrix} 0 & 0 \\ 0 & 5 \end{vmatrix} + 0\begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix}$$

$$= 2 \cdot 3 \cdot 5$$



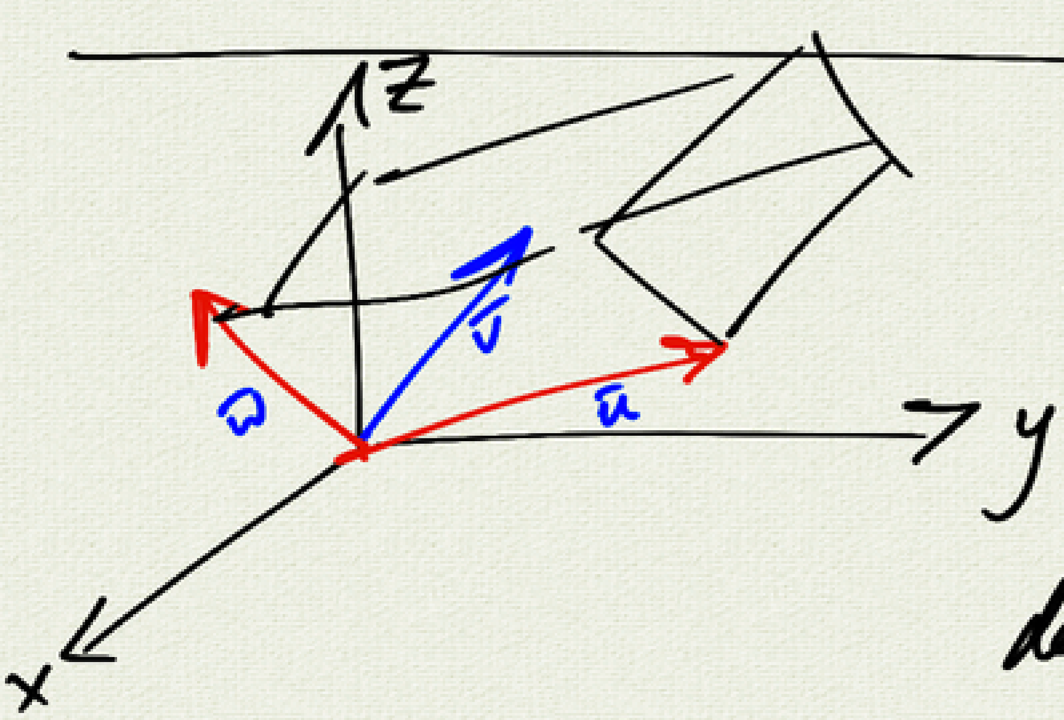
$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$|A| = ad - bc$

$=$ area of

A^{-1} exists $\Longleftrightarrow$ $\det A \neq 0$

area $\neq 0$



parallelopiped

$$\begin{array}{cccc} & \bar u & \bar v & \bar w \\ A = & \Bigg(& & \Bigg) \end{array}$$

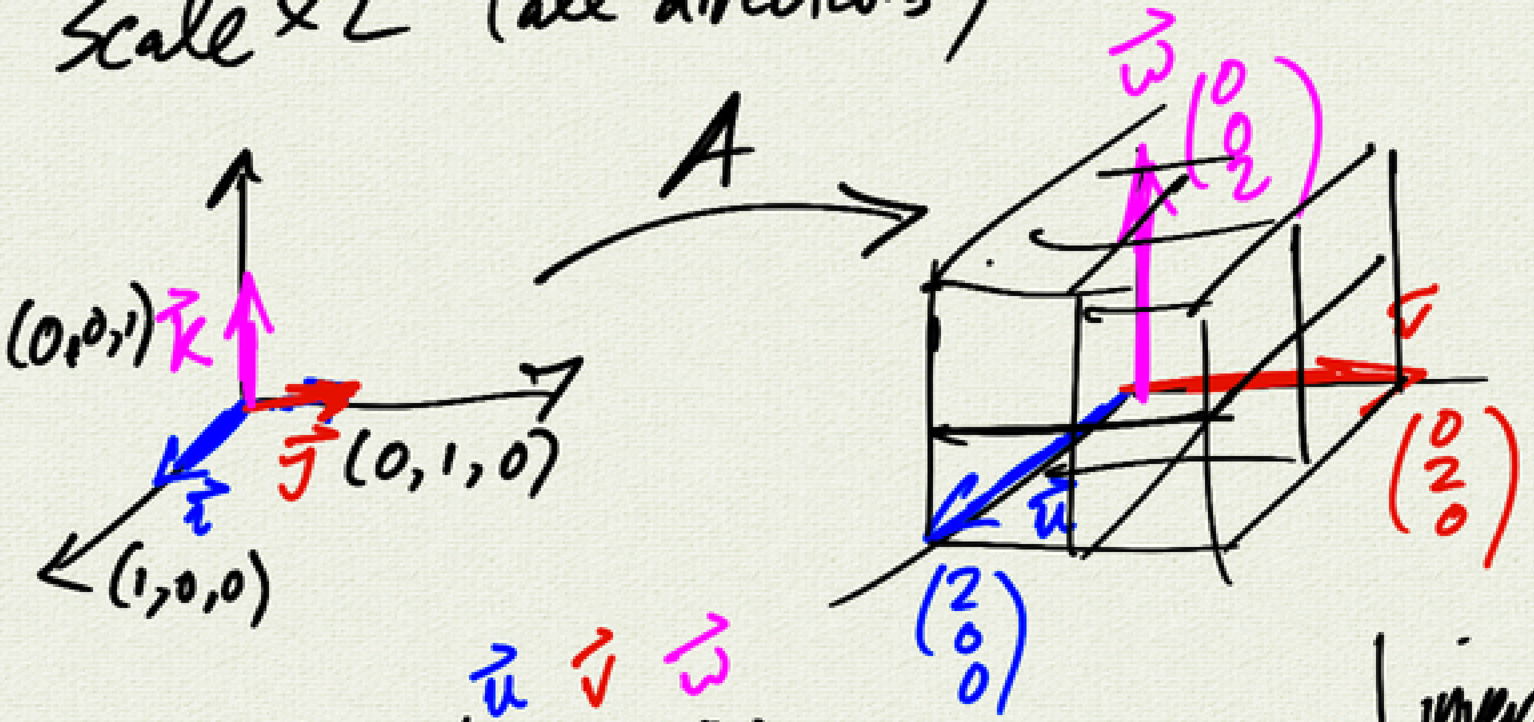
$\det A =$ volume

A^{-1} exists $\Longleftrightarrow$ $\det A \neq 0$

volume $\neq 0$

3D linear transformations

Scale $\times 2$ (all directions)



$$A = \begin{array}{c} \begin{array}{ccc} \vec{u} & \vec{v} & \vec{w} \end{array} \\ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{array} = 2I$$

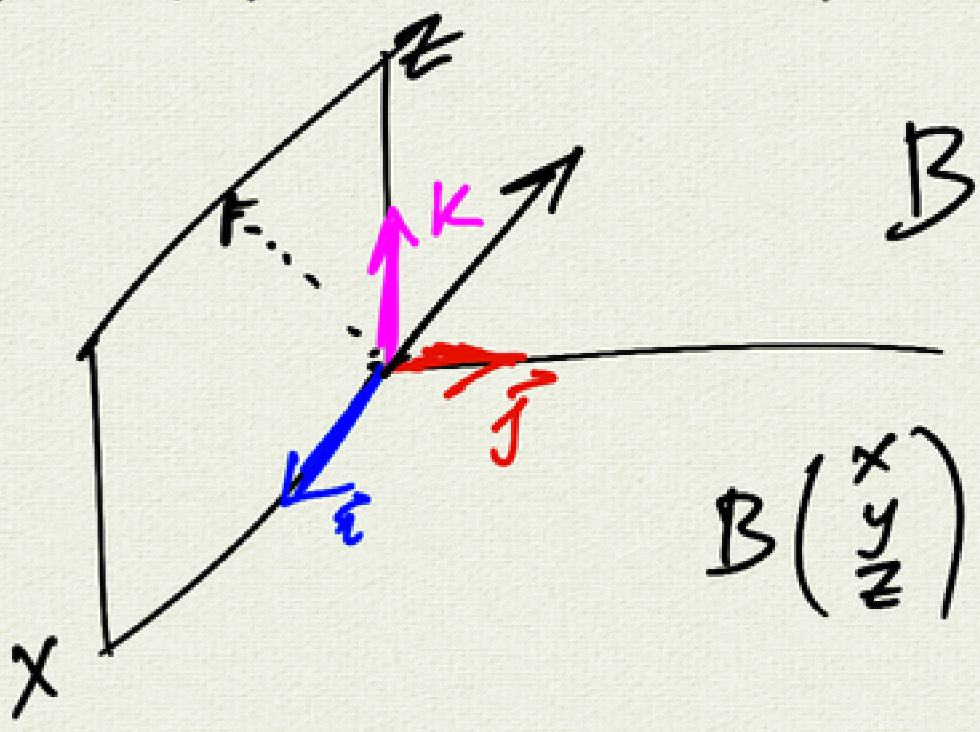
$\det A = 8 \leftarrow$ volume scaling

inverse

$$A^{-1} = \tfrac{1}{2} I = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$$

$$\det A^{-1} = \tfrac{1}{8}$$

B = reflection in xz plane



$$B = \begin{array}{c} \begin{array}{ccc} & -\vec{j} & \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

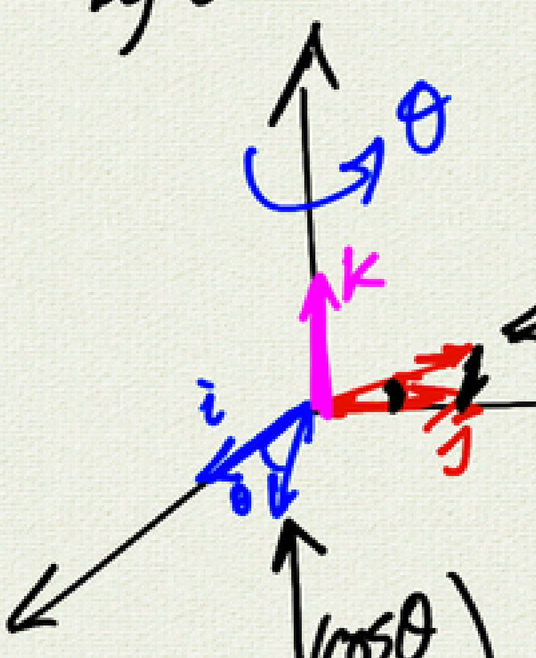
$$B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ z \end{pmatrix}$$

reflection

$$\det B = -1$$

$$B^{-1} = B$$

rotation around z-axis by θ



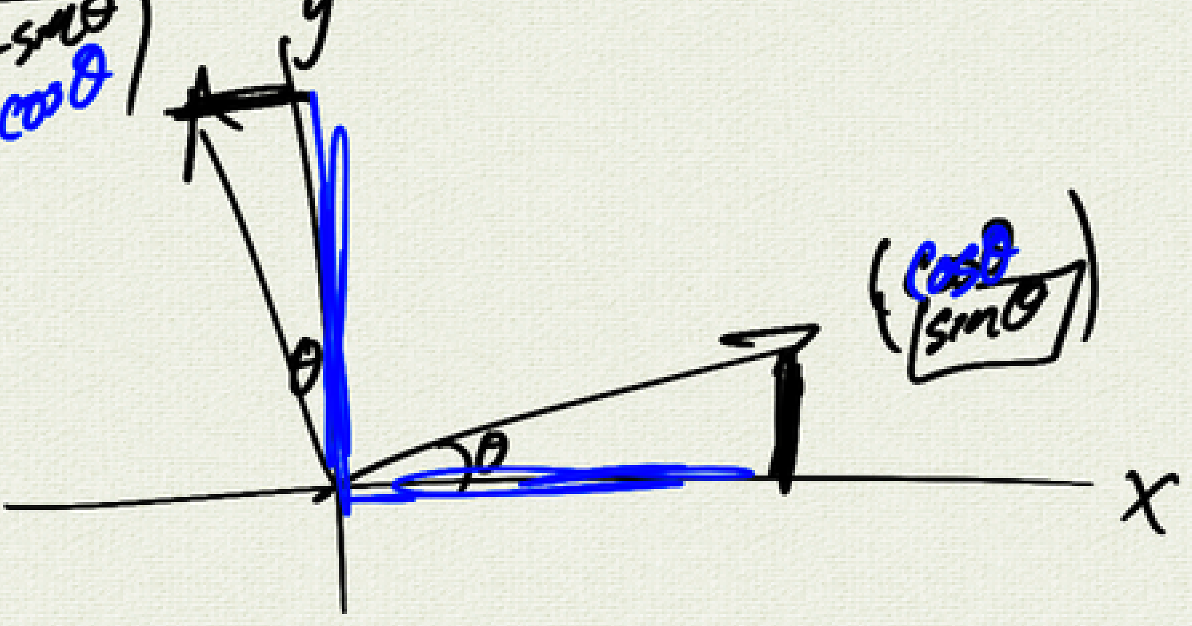
$$\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det C = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = 1$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} -y \\ x \end{pmatrix} = -xy + xy = 0$$

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = 0$$



$$\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \qquad \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

cross product

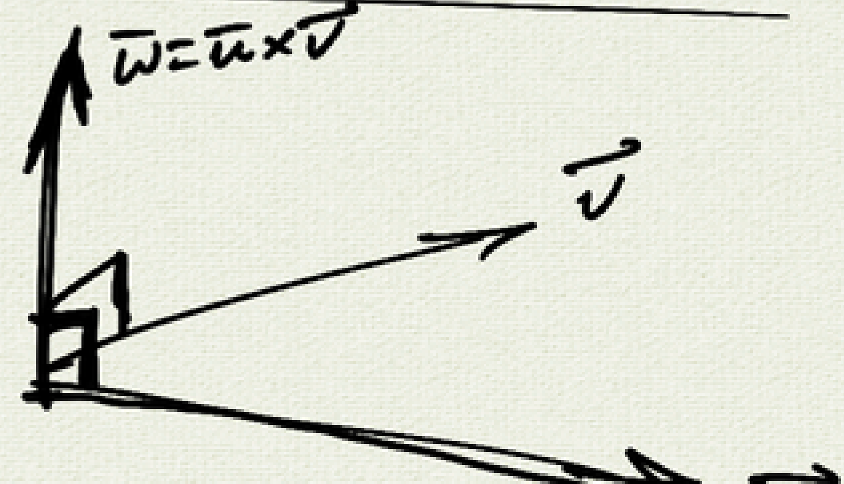
$\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

define $\vec{u} \times \vec{v} = $ " $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ "

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}$$

$\bar{w} = \bar{u} \times \bar{v}$



① $\bar{w} \perp \bar{u}$

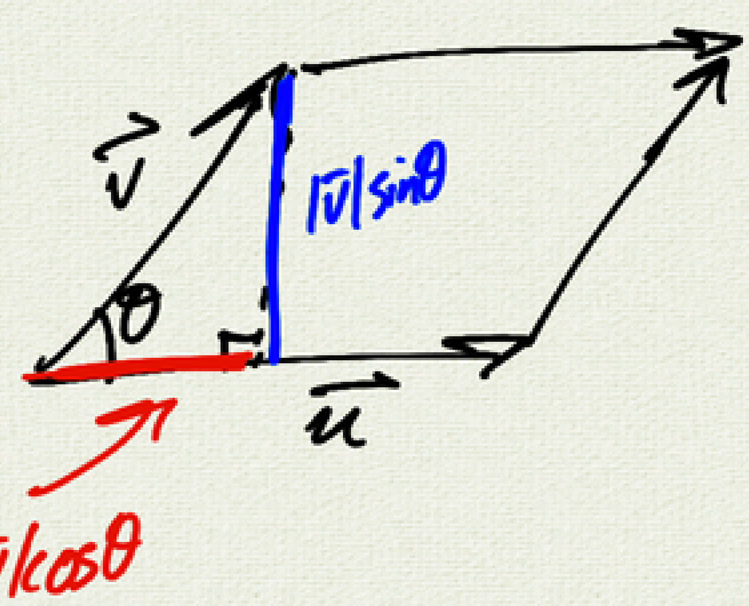
$\bar{w} \perp \bar{v}$ ⟵ check $\bar{w} \cdot \bar{u} = 0$

$\bar{w} \cdot \bar{v} = 0$

② $|\bar{w}| = $ area of ▱ determined by $\bar{u}, \bar{v}$

$|\bar{w}| = |\bar{u} \times \bar{v}|$

$= |\bar{u}||\bar{v}| \sin\theta$

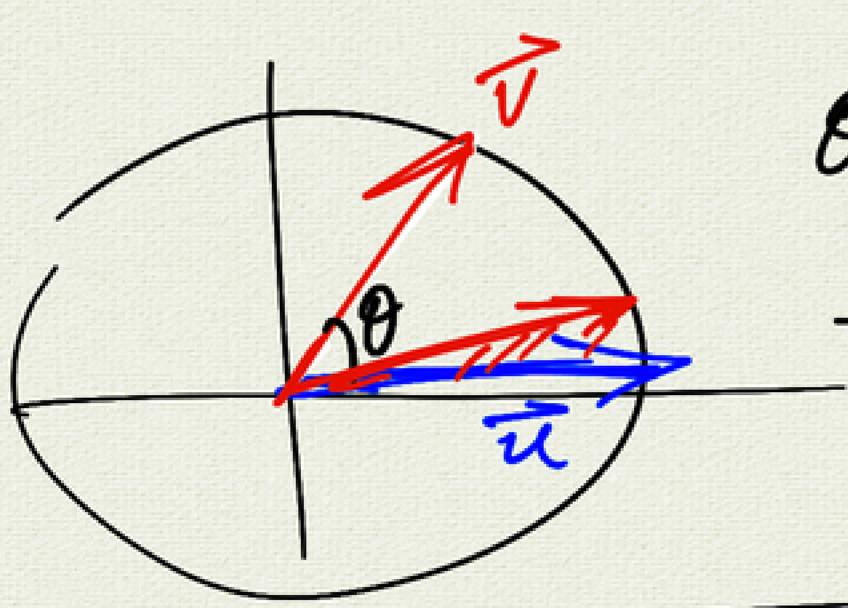


$\bar{u} \cdot \bar{v} = |\bar{u}||\bar{v}|\cos\theta$

$\sin^2\theta + \cos^2\theta = 1$

$(|\bar{u}||\bar{v}|\sin\theta)^2 + (|\bar{u}||\bar{v}|\cos\theta)^2 = (|\bar{u}||\bar{v}|)^2$

$|\bar{u} \times \bar{v}|^2 + |\bar{u} \cdot \bar{v}|^2 = (|\bar{u}||\bar{v}|)^2$



θ small : large projection, ⟷ $\cos\theta$ | dot product

small area ⟷ $\sin\theta$ | cross product

$\theta \approx \frac{\pi}{2}$: small projection

large area

$$\vec{i} \times \vec{j} = ? = \text{“}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}\text{”}$$

$$= \vec{i}\underbrace{\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}}_{0} - \vec{j}\underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}}_{0} + \vec{k}\underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{1}$$

$$\vec{i} \times \vec{j} = \vec{k} = 0\vec{i} - 0\vec{j} + 1 \cdot \vec{k}$$

$$\vec{j} \times \vec{k} = \begin{array}{c} \\ \vec{j} \\ \vec{k} \end{array} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= i\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - j\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Scalar triple product: $\bar{u} \cdot (\bar{v} \times \bar{w})$

order does not matter!

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

challenge: convince yourself