(57) $\bar{r}(t) = \begin{pmatrix} 3\cos 4t \\ 3\sin 4t \\ 5t \end{pmatrix}$ $\quad 1 \le t \le 2$

find $T(t) = \dfrac{\bar{r}'(t)}{|\bar{r}'(t)|}$

$\bar{r}'(t) = \begin{pmatrix} -12\sin 4t \\ 12\cos 4t \\ 5 \end{pmatrix}$

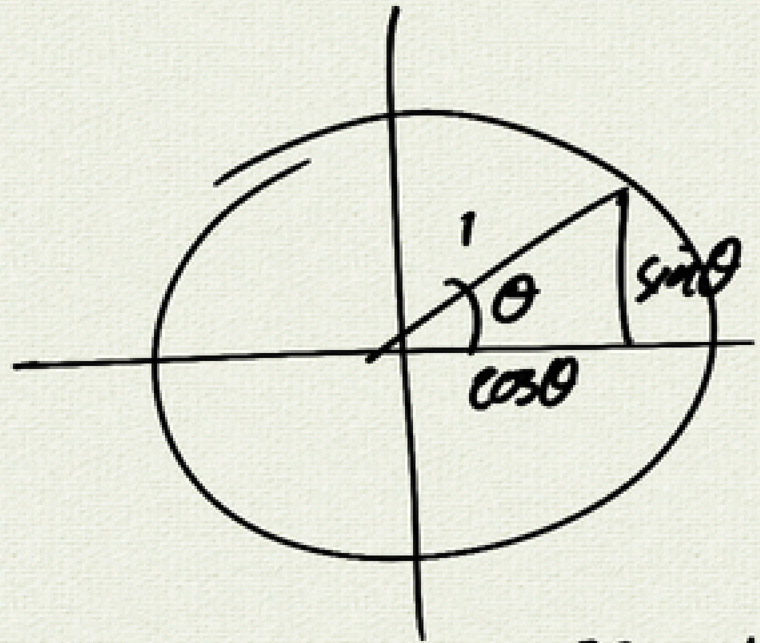
$\rightarrow |\bar{r}'(t)|^2 = 12^2 \underline{\sin^2 4t} + 12^2 \underline{\cos^2 4t} + 5^2$

$= 12^2 + 5^2$

$= 13^2$

$|\bar{r}'(t)| = 13$

$\Rightarrow T(t) = \frac{1}{13} \begin{pmatrix} -12\sin 4t \\ 12\cos 4t \\ 5 \end{pmatrix}$



$\cos^2\theta + \sin^2\theta = 1$

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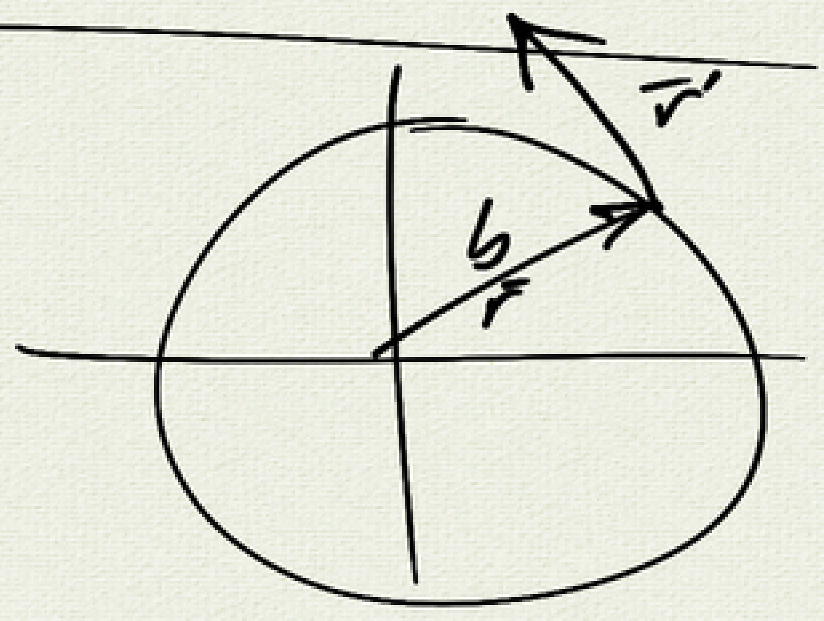
(67) $r(t) = \begin{pmatrix} b\cos(\omega t) \\ b\sin(\omega t) \end{pmatrix}$

period $\frac{2\pi}{\omega}$



$\bar{r}'(t) = \begin{pmatrix} -b\omega\sin(\omega t) \\ b\omega\cos(\omega t) \end{pmatrix}$

$\bar{r}(t) \cdot \bar{r}'(t) = \begin{pmatrix} b\cos\omega t \\ b\sin\omega t \end{pmatrix} \cdot \begin{pmatrix} -b\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix}$

$= -b^2\omega\sin\omega t\cos\omega t + b^2\omega\sin\omega t\cos\omega t$

$= 0$

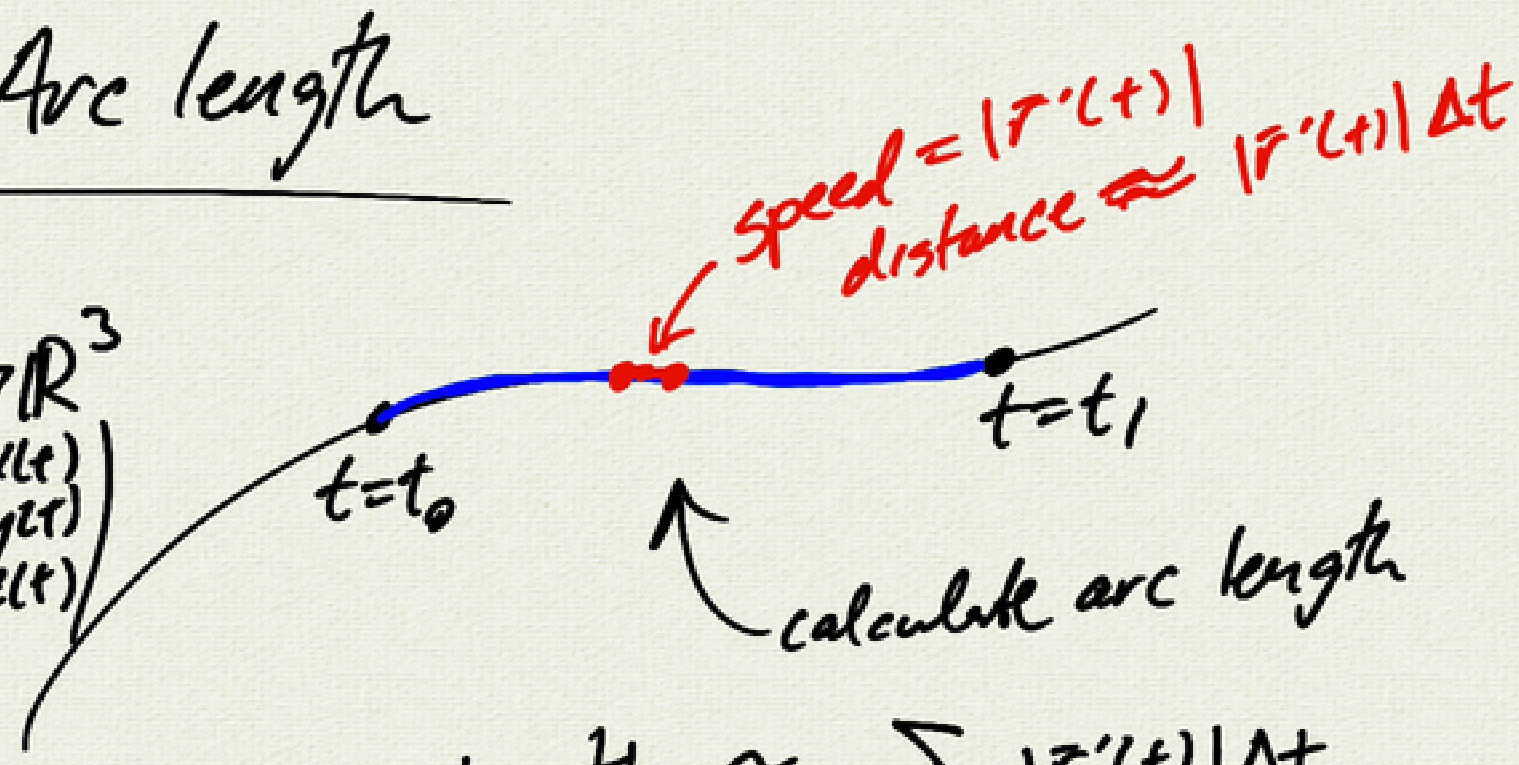
$\Rightarrow$ orthogonal (perpendicular)

2.3 Arc length

curve

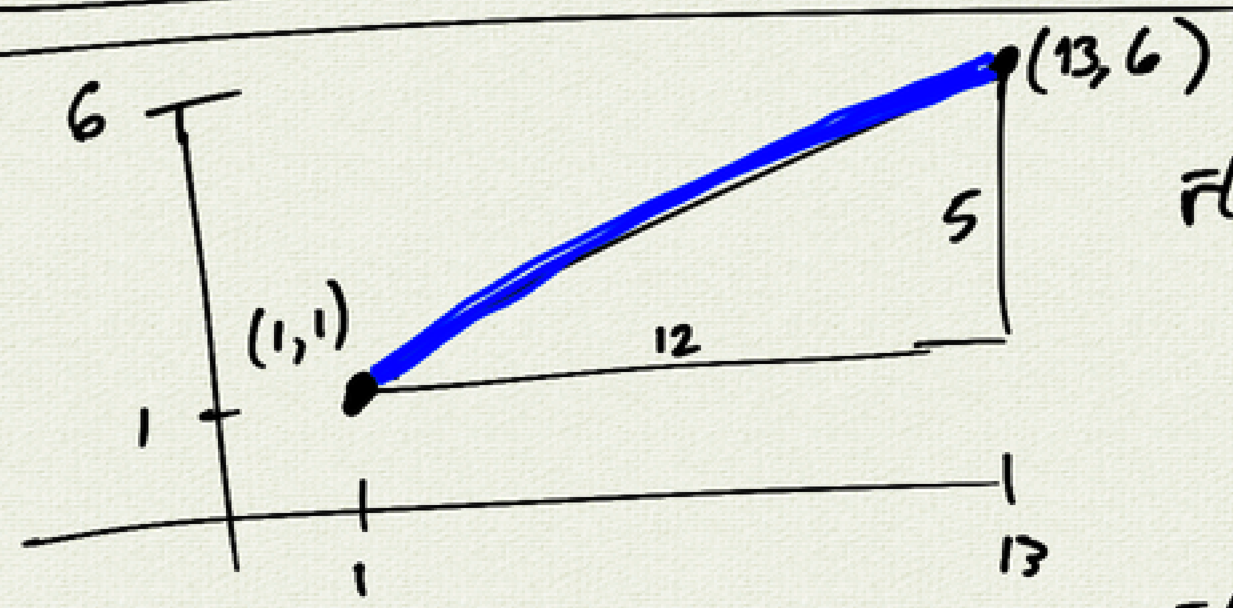
$\bar{r}: \mathbb{R} \to \mathbb{R}^3$

$t \mapsto \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$



arc length $\approx \sum |\bar{r}'(t)| \Delta t$

$$s = \int_{t_0}^{t_1} |\bar{r}'(t)| \, dt$$



$$\bar{r}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 12 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 + 12t \\ 1 + 5t \end{pmatrix}$$

$\bar{r}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\bar{r}(1) = \begin{pmatrix} 13 \\ 6 \end{pmatrix}$ ✓

$t=0$ $t=1$

$\bar{r}'(t) = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$ $|\bar{r}'(t)| = 13$

arc length
$$s = \int_0^1 |\bar{r}'(t)| \, dt = \int_0^1 13 \, dt = 13$$

general arc length:

$$s(t) = \int_0^t |\bar{r}'(t')| \, dt' = \int_0^t 13 \, dt = 13t \Big|_0^t$$

$$\boxed{s(t) = 13t}$$

drop

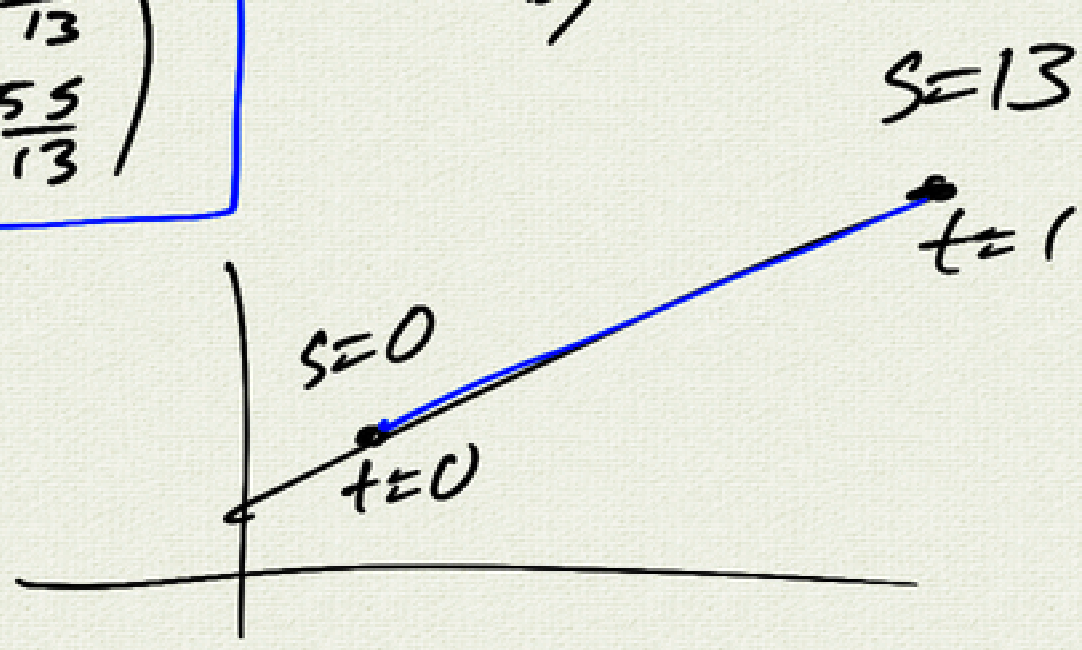
$s = 13t$

$\rightarrow t = s/13$

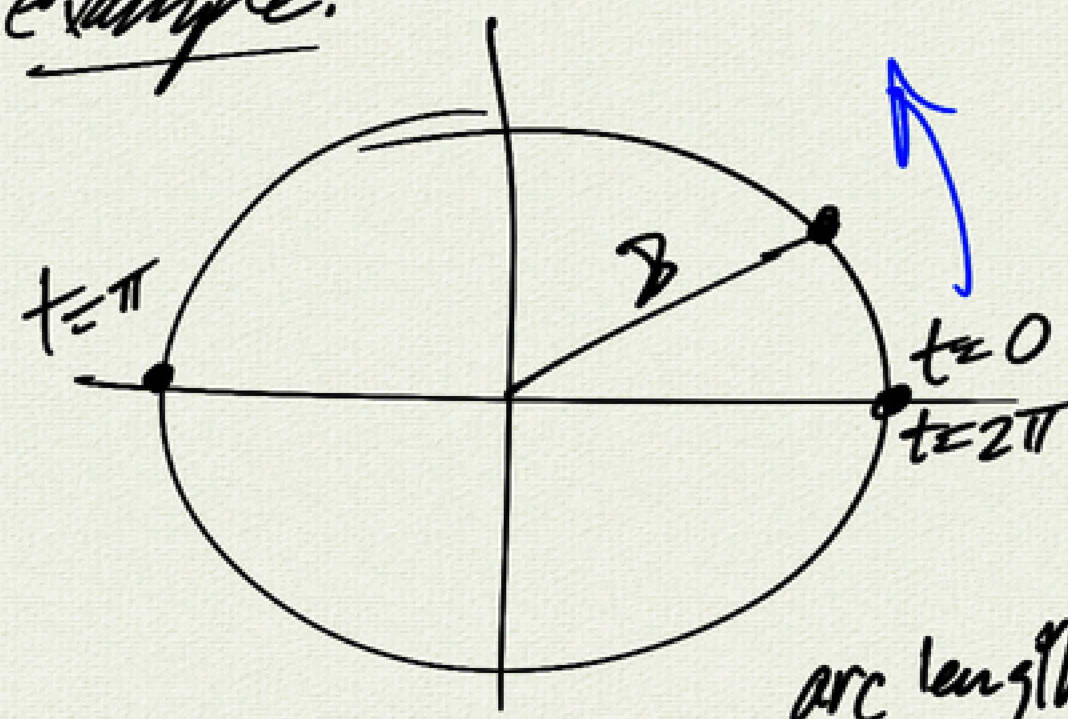
$\bar{r}(s) = $ "$\begin{pmatrix} 1 + 12t \\ 1 + 5t \end{pmatrix}$"

$$\boxed{\bar{r}(s) = \begin{pmatrix} 1 + \frac{12s}{13} \\ 1 + \frac{5s}{13} \end{pmatrix}}$$

different parametrization by arc length



example:



$\vec{r}(t) = \begin{pmatrix} 8\cos t \\ 8\sin t \end{pmatrix}$

$\vec{r}\,'(t) = \begin{pmatrix} -8\sin t \\ 8\cos t \end{pmatrix}$

$|\vec{r}\,'(t)| = 8$

arc length (circumference)

$$s = \int_0^{2\pi} |\vec{r}\,'(t)|\,dt$$

$$= \int_0^{2\pi} 8\,dt$$

$$= 16\pi$$

---

parametrize by arc length

$$s(t) = \int_0^t |\vec{r}\,'(u)|\,du$$

$$= \int_0^t 8\,du$$

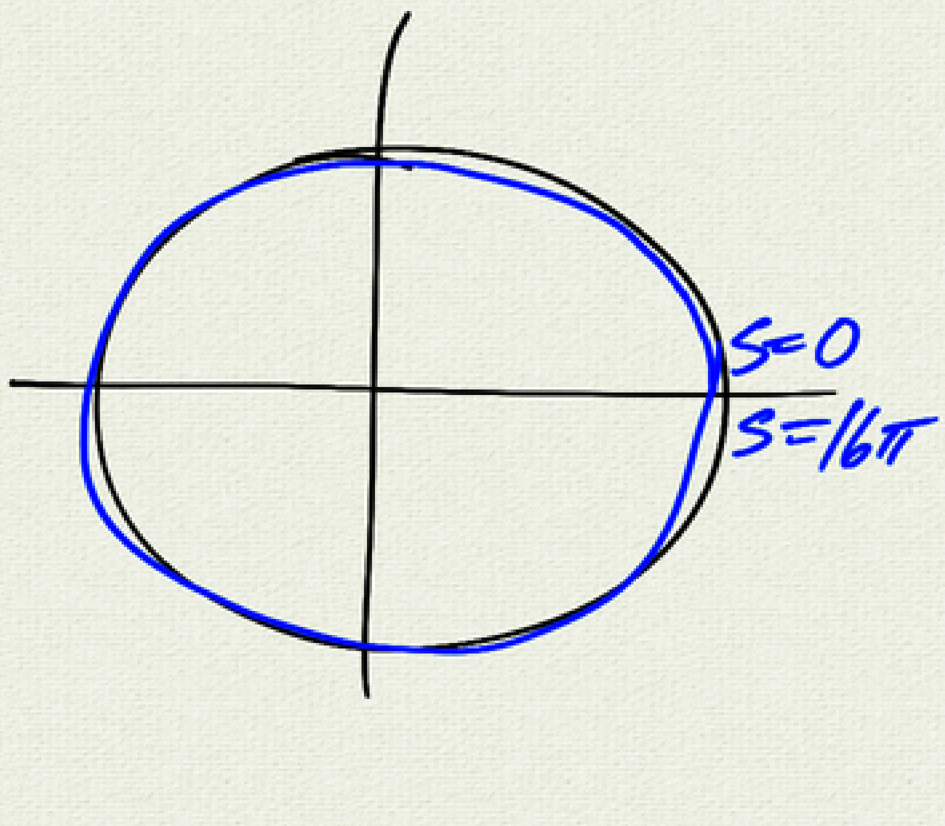
$$= 8u\Big|_0^t$$

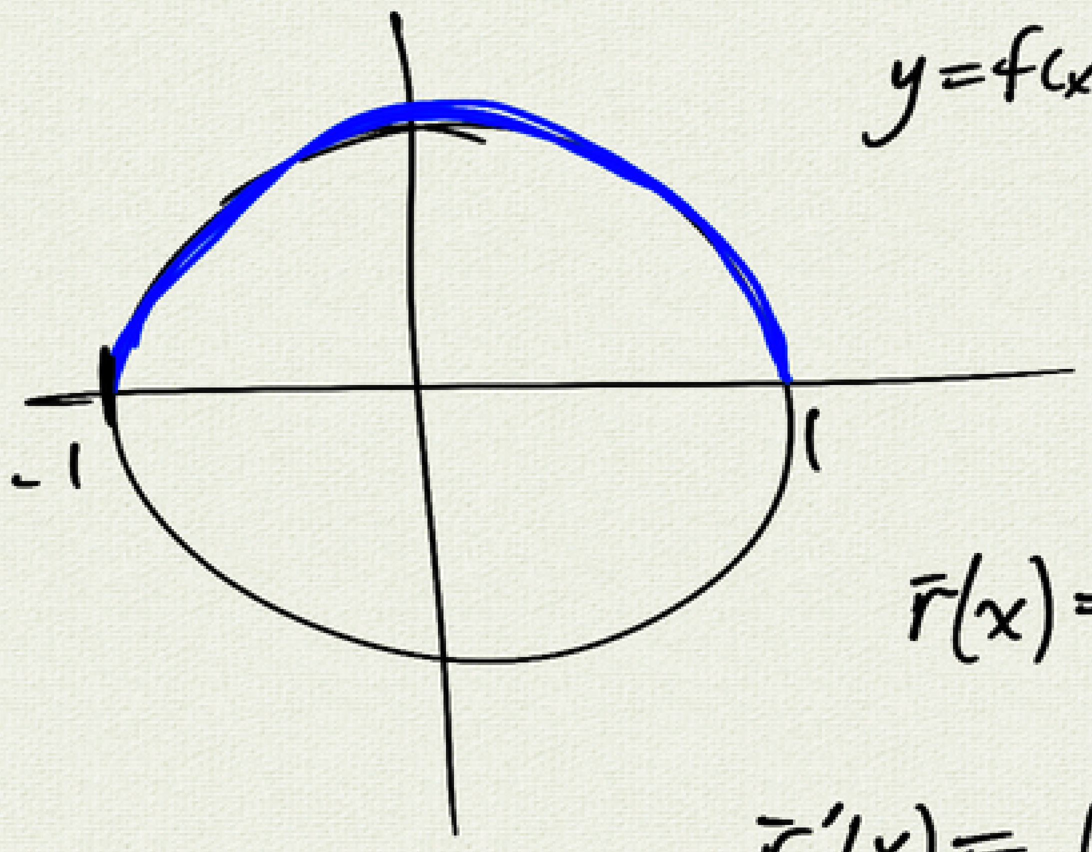
$$\boxed{s(t) = 8t}$$ arc length function

reparametrize: $s = 8t$

$t = s/8$

$$\vec{r}(s) = \begin{pmatrix} 8\cos(s/8) \\ 8\sin(s/8) \end{pmatrix}$$





$y=f(x)$ unit circle

$x^2+y^2=1$

$y=\sqrt{1-x^2}$

$\bar{r}(x)=\begin{pmatrix} x \\ f(x) \end{pmatrix}$

$\bar{r}'(x)=\begin{pmatrix} 1 \\ f'(x) \end{pmatrix}$ $\quad |\bar{r}'(x)|=\sqrt{1+(f'(x))^2}$

arc length

$$s=\int_{-1}^{1} |\bar{r}'(x)|\, dx$$

$$=\int_{-1}^{1} \sqrt{1+(f'(x))^2}\, dx$$

$$=\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx$$

$$=\int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta\, d\theta$$

$$=\int_{-\pi/2}^{\pi/2} d\theta$$

$$=\pi$$

$f(x)=\sqrt{1-x^2}=(1-x^2)^{1/2}$

$f'(x)=\frac{1}{2}(1-x^2)^{-1/2}\cdot(-2x)$

$=\dfrac{-x}{\sqrt{1-x^2}}$

$1+f'(x)^2=1+\left(\dfrac{-x}{\sqrt{1-x^2}}\right)^2$

$=1+\dfrac{x^2}{1-x^2}$

$=\dfrac{1}{1-x^2}$

$x=\sin\theta$

$dx=\cos\theta\, d\theta$

trig substitution