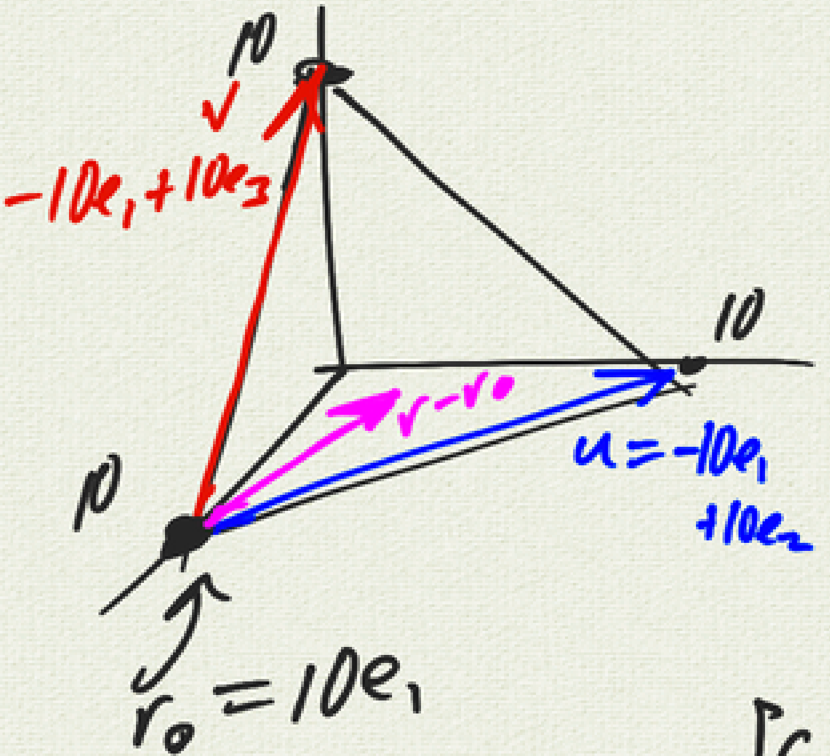


plane:

$$(r-r_0)\wedge u \wedge v = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xe_1+ye_2+ze_3$$

point on plane

$$\underbrace{\left[(x-10)e_1+ye_2+ze_3\right]}_{r-r_0}\wedge(-10e_1+10e_2)$$
$$\wedge(-10e_1+10e_3)$$
$$=0$$

$$(x-10)(10)(10)e_1e_2e_3 + y(-10)(10)\underbrace{e_2e_1e_3}_{-e_1e_2e_3}$$
$$+ z(10)(-10)\underbrace{e_3e_2e_1}_{-e_1e_2e_3} = 0$$

$$(x-10+y+z)e_1e_2e_3=0$$

$$x-10+y+z=0$$

$$x+y+z=10$$

$r_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

line: $(r - r_0) \wedge v = 0$

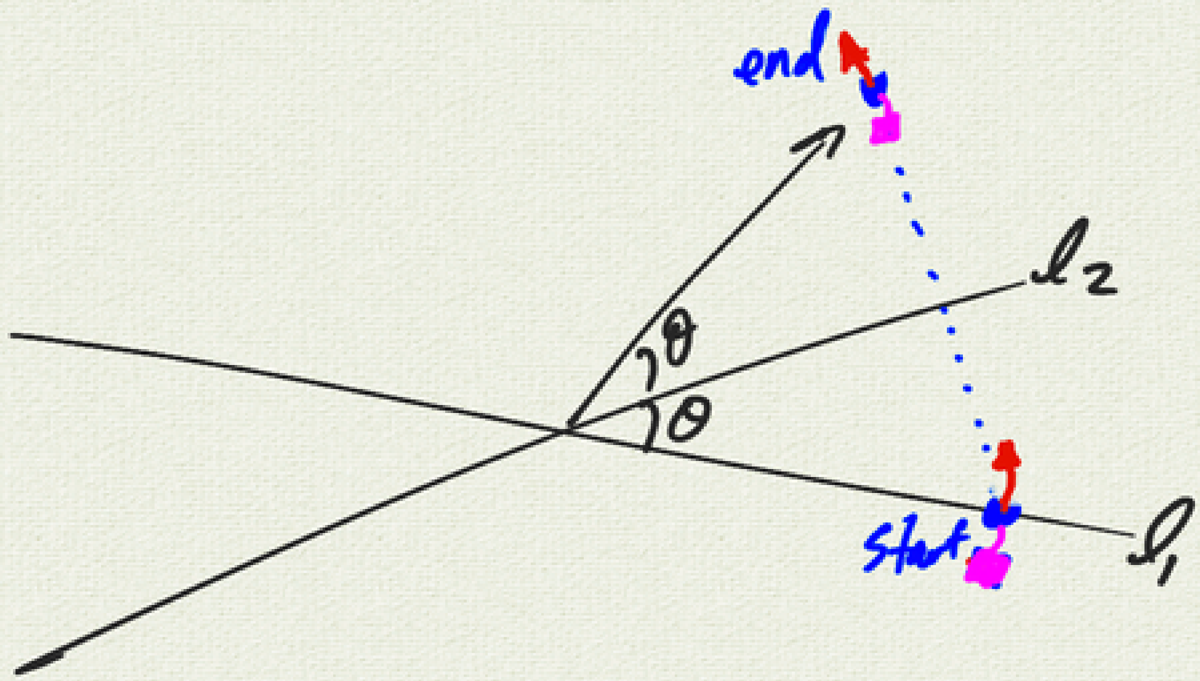
$\begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad$ $\begin{pmatrix} v_x \\ v_y \end{pmatrix}$

$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \wedge \begin{pmatrix} v_x \\ v_y \end{pmatrix} = 0$$

$$\begin{vmatrix} x - x_0 & v_x \\ y - y_0 & v_y \end{vmatrix} = 0$$

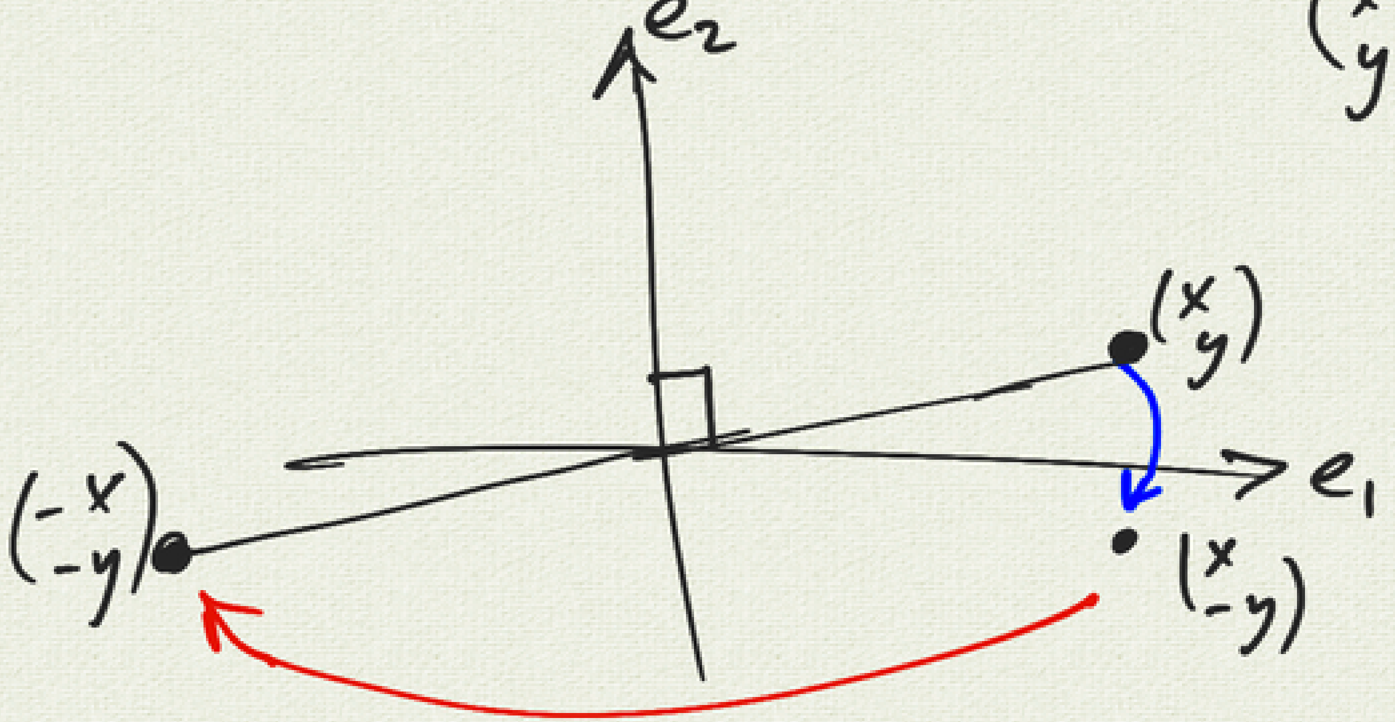
$$v_y (x - x_0) - v_x (y - y_0) = 0$$

$$y - y_0 = \underbrace{\frac{v_y}{v_x}}_{\text{slope}} (x - x_0)$$



reflect in ℓ_1, then in ℓ_2 $\Longleftrightarrow$ rotate by 2θ

$$\begin{pmatrix}x\\y\end{pmatrix} \xrightarrow{e_1} \begin{pmatrix}x\\-y\end{pmatrix} \xrightarrow{e_2} \begin{pmatrix}-x\\-y\end{pmatrix}$$

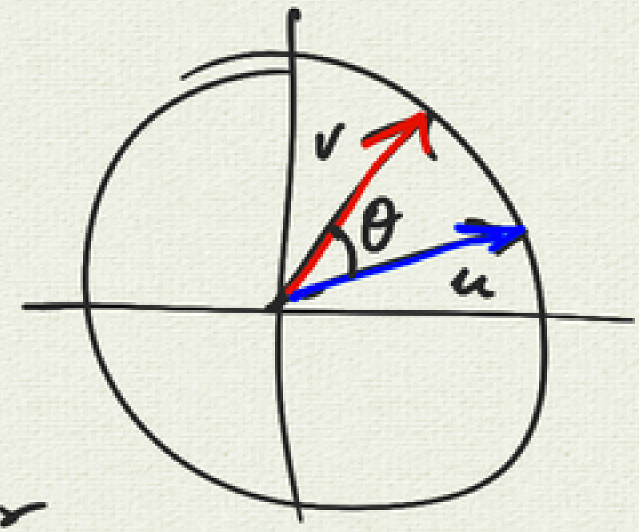


reflection in e_1, then e_2

$\Updownarrow$

rotation by π

recall: rotation in $\mathbb{R}^2$



u, v unit vectors

$$\Rightarrow \quad uv = u \cdot v + u \wedge v$$

$$= \underset{\text{scalar}}{\cos\theta} + \underset{\text{bivector}}{(\sin\theta) e_1 e_2}$$

rotor

rotation by θ (in $\mathbb{R}^2$)

$$w(uv) = w \text{ rotated by } \theta$$

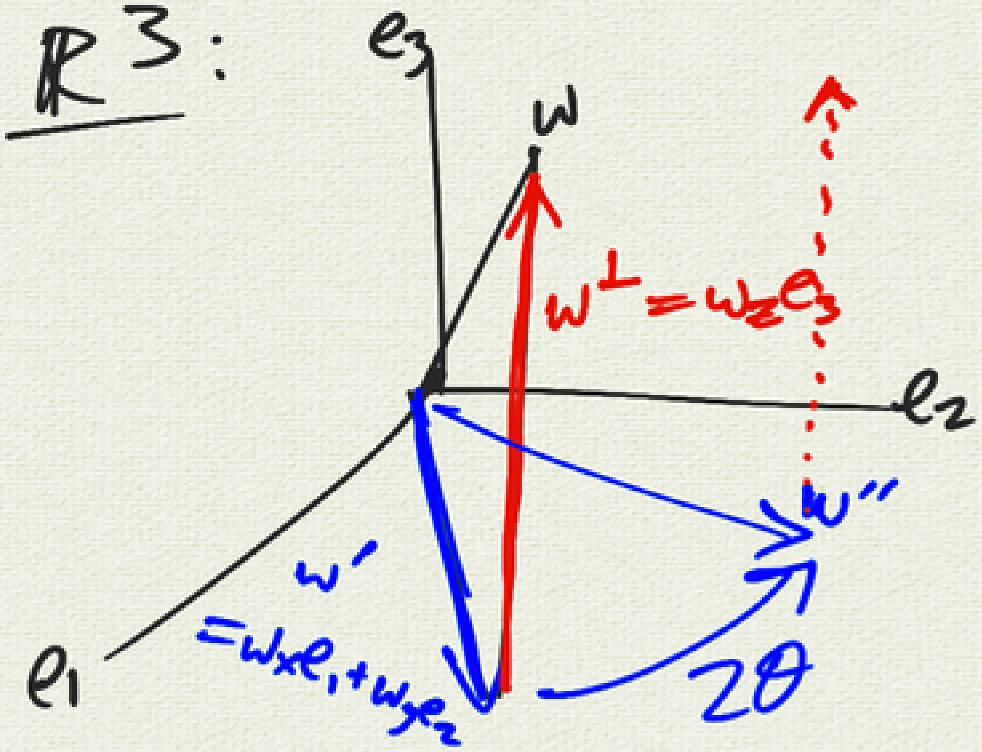
$$(w_x e_1 + w_y e_2)(\cos\theta + \sin\theta\, e_1 e_2)$$

$\Rightarrow$ in $\mathbb{R}^2$:

$$(vu)w = w \text{ rotated by } \theta$$

$$(vu)w(uv) = w \text{ rotated by } 2\theta$$

$\mathbb{R}^3$:



$$w = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} = w_x e_1 + w_y e_2 + w_z e_3$$

let uv = rotation by θ in $e_1 e_2$ plane

$$uv = \cos\theta + (\sin\theta) e_1 e_2$$
$$= c + s e_1 e_2 \qquad \begin{array}{|l} c = \cos\theta \\ s = \sin\theta \end{array}$$

rotor

$$vu = c - s e_1 e_2$$

$$(vu)w(uv) = (c - s e_1 e_2)(w' + w^\perp)(c + s e_1 e_2)$$
$$= \underbrace{(vu)w'(uv)}_{w'' = w' \text{ rotated by } 2\theta} + \underbrace{(c - s e_1 e_2) w^\perp (c + s e_1 e_2)}$$

$$(c - s e_1 e_2) \underbrace{w_z e_3 (c + s e_1 e_2)}$$
$$= (c - s e_1 e_2)\left(c w_z e_3 + s w_z \underbrace{e_3 e_1 e_2}_{e_1 e_2 e_3}\right)$$
$$= c^2 w_z e_3 + \underline{c s w_z e_1 e_2 e_3} - \underline{s c w_z e_1 e_2 e_3} - s^2 w_z \underbrace{e_1 e_2 e_1 e_2 e_3}_{-e_3}$$
$$= \underbrace{(c^2 + s^2)}_{1} w_z e_3$$
$$= w_z e_3$$

reflections $u^2 = uu = u \cdot u + \underbrace{u \wedge u}_{0}$

$$u^2 = |u|^2$$

$$\Longrightarrow \quad u^{-1} = \frac{u}{|u|^2}$$

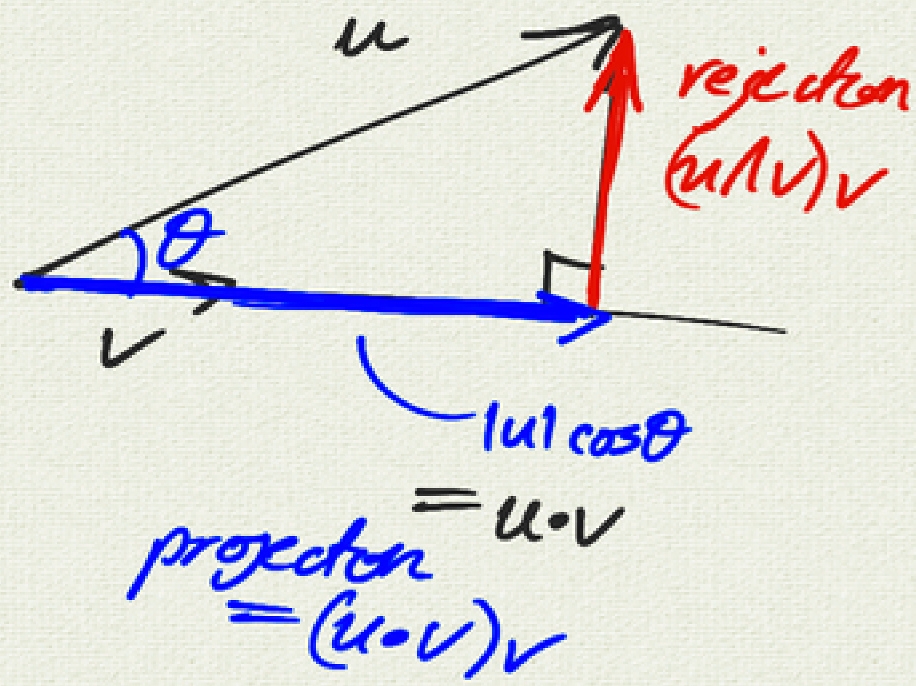
u unit vector $\Longrightarrow u^2 = 1$

$$u^{-1} = u$$

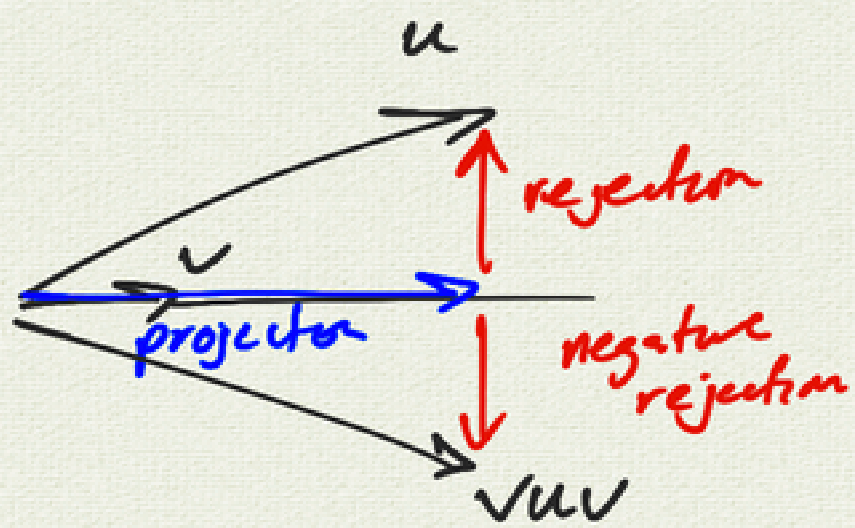
---

v unit vector

$$u = \underline{uvv}$$
$$= (uv)v$$
$$= (u \cdot v + u \wedge v)v$$
$$= \underbrace{(u \cdot v)v}_{\text{projection}} + \underbrace{(u \wedge v)v}_{\text{rejection}}$$



$$vuv = (vu)v$$
$$= (u \cdot v + v \wedge u)v$$
$$= (u \cdot v - u \wedge v)v$$
$$= \underbrace{(u \cdot v)v}_{\text{projection}} - \underbrace{(u \wedge v)v}_{\text{rejection (but negative)}}$$



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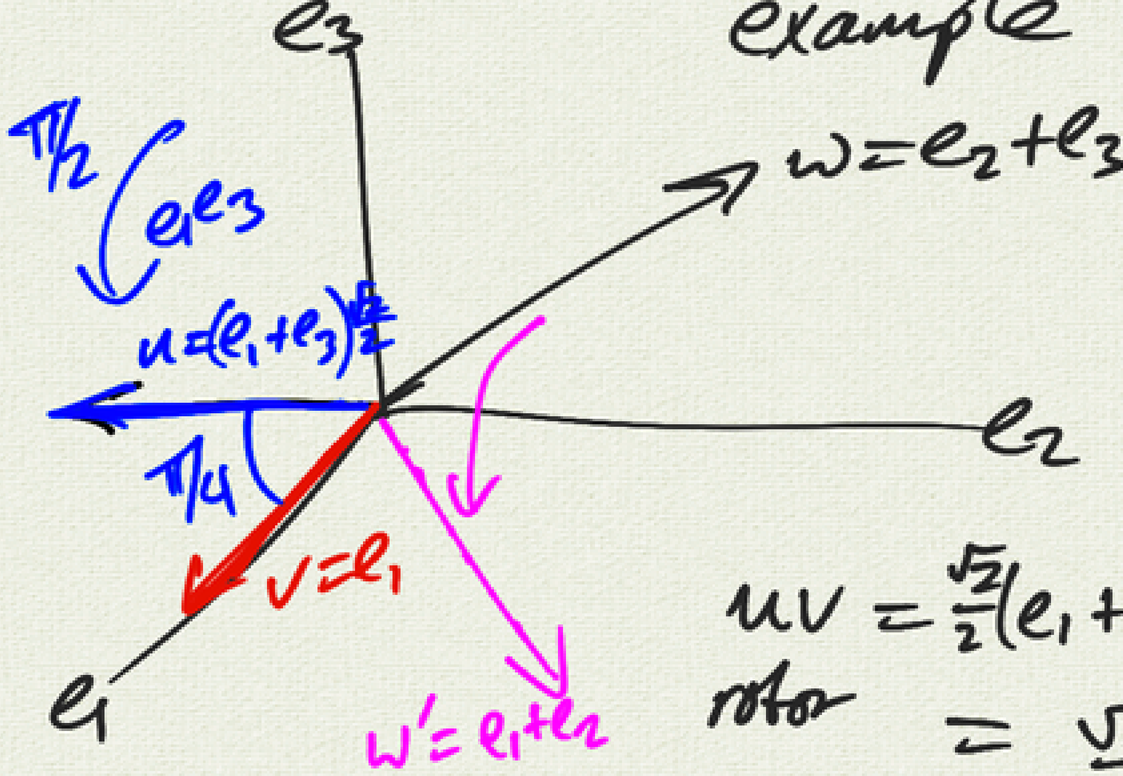
vuv = u reflected across v

$(vu)w(uv)$ rotation by 2θ (in uv plane)

$v(uwu)v$ reflect in u, then v

example



$$uv = \frac{\sqrt{2}}{2}(e_1 + e_3)(e_1)$$

rotor

$$= \frac{\sqrt{2}}{2}(1 + e_3 e_1)$$

$$vu = \frac{\sqrt{2}}{2}(1 - e_3 e_1)$$

rotate w by $\pi/2$ in $e_1 e_3$:

$$w' = (vu)\, w\, (uv)$$

$$= \frac{\sqrt{2}}{2}(1 - e_3 e_1)(e_2 + e_3)\left(\frac{\sqrt{2}}{2}\right)(1 + e_3 e_1)$$

$$= \frac{1}{2}(1 - e_3 e_1)\big(e_2 + \underbrace{e_2 e_3 e_1}_{e_1 e_2 e_3} + e_3 + \underbrace{e_3 e_3 e_1}_{e_1}\big)$$

$$= \frac{1}{2}\Big[e_2 + e_1 e_2 e_3 + e_3 + e_1$$

$$\underbrace{-e_3 e_1 e_2}_{-e_1 e_2 e_3} \underbrace{-e_3 e_1 e_1 e_2 e_3}_{+e_2} \underbrace{-e_3 e_1 e_3}_{+e_1} \underbrace{-e_3 e_1 e_1}_{-e_3}\Big]$$

$$= \frac{1}{2}(2e_1 + 2e_2)$$

$$= e_1 + e_2$$