Advanced Algorithms - Problem set 6

Whenever you give an algorithm, also argue for its running time and correctness. Always feel free to ask on MS Teams if you're stuck, need a clarification, or suspect a typo. We already discussed the first three exercises on November 5, now you can fill in the details.

1. Describe reductions from the following problems to SAT:

   (a) Vertex cover (input: graph G, number k).
   Hint: We discussed a reduction from Independent set in the lecture. This works similarly.

   (b) k-coloring (input: graph G, number k)

   (c) Hamiltonian cycle (input: graph G)

   Implement the reduction for Vertex Cover and use MiniSAT to verify that the cycle of length 24 has a vertex cover of size 12, but none of size 11. Also compute the minimum vertex covers for the attached graphs `vc_graph.txt` and `vc_graph_small.txt` (They are also on Teams.)

2. Fix $n = 100$. For a "reasonable" selection of $C \in \mathbb{N}$ and different $r \in \mathbb{Q}$ in the interval $[1, 20]$, pick C random 4-CNF formulas with n variables and rn clauses. (A random 4-CNF clause is obtained by choosing 4 of n variables, and then randomly choosing for each variable x whether it appears as x or $\bar{x}$ in the clause. A random 4-CNF formula with m clauses is obtained as the conjunction of m random 4-CNF clauses.) Using MiniSAT, compute for each selected r how many of the C random instances with rn clauses are satisfiable. It will make sense to specify a reasonable timeout value after which you the execution of MiniSAT is aborted.

   (a) Plot the ratio of satisfiable rn-clause formulas as a function of r. What do you observe?

   (b) Plot the average running time of MiniSAT as a function of r. What do you observe?

3. We approach the behaviour observed towards the right end of the plot in 2a theoretically.

   (a) Fix an assignment $a \in \{0,1\}^n$. Argue that the probability that a randomly drawn k-CNF clause is satisfied by a is
   $$(1 - 2^{-k}).$$

   (b) Fix an assignment $a \in \{0,1\}^n$. Using the above, argue that the probability that a randomly drawn k-CNF formula with rn clauses is satisfied by a is at least
   $$(1 - 2^{-k})^{rn}.$$

   (c) Using the above, argue that the expected number of satisfying assignments in a random k-CNF formula with n variables and rn clauses is at least
   $$2^n(1 - 2^{-k})^{rn}.$$

   (d) Let $r \geq 2^k \ln 2$, where ln denotes the natural logarithm. How does the expected number of satisfying assignments behave as n tends to infinity? Do you recognize this behaviour in your plot for 2a, where $k = 4$?

4. Implement the FPT-algorithm for Vertex Cover and run it on the test cases from Exercise 1 and two more test cases of your choice. (The algorithm should exceed reasonable running time for `vc_graph.txt`)