

# Application of Eigenvalues in Predator–Prey Relationship Models

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# Our Objectives

- Use linear algebra to analyze real-world dynamical systems.
- See how eigenvalues/eigenvectors describe long-run behavior of a linear (or linearized) system.
- Compute the state  $\mathbf{r}_t$  of a discrete dynamical system at time  $t$ .

# Discrete Dynamical Systems

**Dynamical system:** shows how a state changes over time, represented by state variables.

Example variables:

$r_t$  = number of rabbits in year  $t$ .

If we measure once per year and start with  $r_0 = 1000$  (initial condition), we need a rule taking year  $t$  to year  $t + 1$ .

**Rule set:**  $\{(y_0 \rightarrow y_1), (y_1 \rightarrow y_2), \dots, (y_n \rightarrow y_{n+1})\}$ .

## Example: Constant Growth

Suppose the rabbit population increases by 8% each year.

- Year-to-year change:  $r_{t+1} - r_t = 0.08 r_t$ .
- Equivalent update:  $r_{t+1} = 1.08 r_t$ , with  $r_0 = 1000$ .
- Then  $r_t = 1000 \cdot 1.08^t$ .

## Predator–Prey State

Let

$$\mathbf{x}_t = \begin{bmatrix} W_t \\ R_t \end{bmatrix}$$

where  $W_t$  is wolves,  $R_t$  is rabbits (in thousands) at time  $t$ .

Heuristics:

- Wolves need rabbits to survive; without rabbits, many wolves die.
- Rabbits reproduce but are eaten by wolves.

# Difference Equations (Model)

Consider the *difference* (not differential) equations

$$W_{t+1} = 0.4 W_t + 0.3 R_t,$$

$$R_{t+1} = -0.5 W_t + 1.2 R_t.$$

Interpretation:

- $0.4 W_t$ : with no rabbits, only 40% of wolves survive a period.
- $0.3 R_t$ : abundant rabbits support additional surviving wolves.
- $1.2 R_t$ : with no wolves, rabbits grow by 20%.
- $-0.5 W_t$ : predation reduces rabbits; two wolves consume about one thousand rabbits per year.

# Matrix Form and Eigen Analysis

Write  $\mathbf{x}_{t+1} = A\mathbf{x}_t$  with

$$A = \begin{bmatrix} 0.4 & 0.3 \\ -0.5 & 1.2 \end{bmatrix}.$$

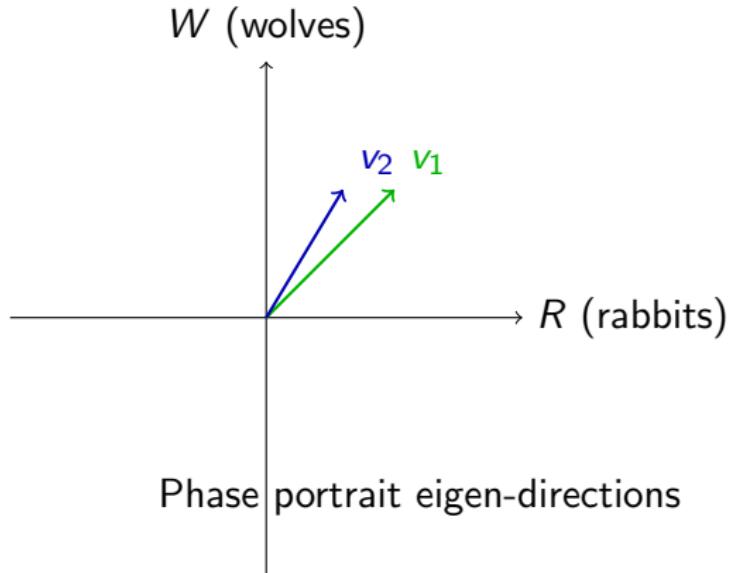
Eigenvalues solve  $\det(\lambda I - A) = 0$ :

$$\lambda^2 - 1.6\lambda + 0.63 = 0 \quad \Rightarrow \quad \lambda_1 = 0.7, \lambda_2 = 0.9.$$

Corresponding eigenvectors can be taken as

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3/5 \\ 1 \end{bmatrix}.$$

# Graph: Eigenvector Directions



# What the Graph Means

## Explanation for the audience:

- The green and blue arrows show the main directions that the populations move along over time.
- Each arrow represents an **eigenvector**, or a special balance between wolves and rabbits.
- The slope of each arrow tells us how the two species change together — if rabbits increase, how wolves respond, and vice versa.
- As years go by, no matter where we start, the populations move closer to one of these directions.
- Because both eigenvalues are less than 1, all movements shrink toward the **origin (0,0)**, meaning both populations eventually decline to a stable point.

## General Solution and Long-Run Behavior

If  $A$  is diagonalizable (it is), any initial state decomposes as

$$\mathbf{x}_0 = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 \text{ and}$$

$$\mathbf{x}_t = c_1 \lambda_1^t \mathbf{v}_1 + c_2 \lambda_2^t \mathbf{v}_2 = c_1(0.7)^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2(0.9)^t \begin{bmatrix} 3/5 \\ 1 \end{bmatrix}.$$

Ratio of coordinates (wolves/rabbits):

$$\frac{W_t}{R_t} = \frac{c_1(0.7)^t + \frac{3}{5}c_2(0.9)^t}{c_1(0.7)^t + c_2(0.9)^t} \xrightarrow[t \rightarrow \infty]{} \frac{3}{5}, \quad \text{since } 0.9 > 0.7.$$

**Therefore:** trajectories decay to the origin (both eigenvalues  $< 1$ ), and their direction approaches the dominant eigenvector  $\mathbf{v}_2 = \begin{bmatrix} 3/5 \\ 1 \end{bmatrix}$ .

## Attractors, Repellors, Saddle Points

For a linear discrete system  $\mathbf{x}_{t+1} = A\mathbf{x}_t$ :

- If the spectral radius  $\rho(A) < 1$  (all eigenvalues have  $|\lambda| < 1$ ), the origin is a **(globally) asymptotically stable attractor**.
- If all eigenvalues have  $|\lambda| > 1$ , the origin is a **repellor**.
- If some  $|\lambda| < 1$  and some  $|\lambda| > 1$ , the origin is a **saddle point**.
- Complex eigenvalues yield **spirals/oscillations**; real eigenvalues yield **straight-line** approach along eigendirections.

In our model,  $0.7, 0.9 < 1 \Rightarrow$  origin is an attractor; approach is along  $\mathbf{v}_2$ .

## Lotka–Volterra (Continuous-Time)

The classical Lotka–Volterra system (differential equations) is

$$\frac{dx}{dt} = ax - cxy, \quad \frac{dy}{dt} = -by + dxy, \quad a, b, c, d \geq 0.$$

Equilibria:  $(0, 0)$  and  $(\frac{b}{d}, \frac{a}{c})$ . Jacobian:

$$J(x, y) = \begin{bmatrix} a - cy & -cx \\ dy & dx - b \end{bmatrix}, \quad J\left(\frac{b}{d}, \frac{a}{c}\right) = \begin{bmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{c} & 0 \end{bmatrix}.$$

The linearization has purely imaginary eigenvalues  $\lambda = \pm i\sqrt{ab}$  (neutral center), so trajectories are closed orbits in the linear model (true nonlinear behavior depends on parameters and invariants).

# Conclusion

- Eigenvalues/eigenvectors characterize long-run behavior of linear (and linearized) dynamical systems.
- In the discrete predator-prey example, both  $|\lambda| < 1 \Rightarrow$  populations decay to the origin; direction aligns with the dominant eigenvector.
- Linear tools also clarify stability of nonlinear models via linearization (e.g., Lotka–Volterra).
- Applications span signal processing, structural stability, control, and resource exploration.

# References

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