Negative Nominal Interest Rates and Monetary Policy*

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Abstract

A nonpar exchange rate between currency and reserves has been proposed as a potential policy tool to reduce the effective lower bound (ELB) on nominal interest rates. To study the implications of introducing this novel policy tool, I construct a model with multiple means of payment and frictions associated with the storage and transportation of currency. A nonpar exchange rate can reduce the ELB if there exist sufficient frictions that induce agents to exchange currency and reserves rather than avoiding the central bank. However, introducing a nonpar exchange rate can reduce welfare by increasing these frictions and distorting the allocation of means of payment. A negative nominal interest rate can be optimal for a nonstandard reason. But, if the negative interest rate is constrained by the ELB, the optimal policy is to set the interest rate at the ELB without introducing a nonpar exchange rate.

Keywords: Negative interest rate; Nonpar exchange rate; Money; Banking; Monetary policy

JEL Codes: E4; E5

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1 Introduction

Some macroeconomists have argued that the existence of the lower bound on nominal interest rates can be a major obstacle for operating monetary policy and have suggested several policy tools that could potentially remove or reduce the lower bound.[1] Understanding how and why a central bank could control the lower bound with these policy tools requires uncovering the determinants of the lower bound on nominal interest rates. What economic fundamentals or frictions determine the lower bound? If a central bank attempts to reduce the lower bound by manipulating the frictions, what would be the welfare implications? The main goal of this paper is to study the implications of reducing the lower bound on nominal interest rates.

In mainstream macroeconomic theory, monetary policy is constrained by the zero lower bound on nominal interest rates.[2] This constraint arises because, at a negative interest rate, borrowers can exploit an arbitrage opportunity by investing in zero-interest currency. However, in practice it is clear that negative nominal short-term interest rates are feasible, as the European Central Bank, the Swiss National Bank, the Swedish Riksbank, the Bank of Japan, and the National Bank of Denmark, among others, have demonstrated. The implication is that, in practice, there exist significant frictions that inhibit arbitrage, so that the effective lower bound (ELB) on the nominal interest rate is negative.

What are the frictions that make the ELB negative? It is generally recognized that these frictions arise principally because of the costs associated with storing, transporting, and exchanging large quantities of currency. For example, if a private bank is facing a negative nominal interest rate on reserves, it could hold currency instead of having reserve balances with the central bank. However, holding currency would entail costs of installing a large vault and hiring guards to watch it, and the currency would be of little or no use in making interbank transactions and online transactions.

Some macroeconomists have made the case that the ELB can be an important binding constraint on monetary policy in some circumstances. For some of these researchers, the frictions that determine the ELB are not welfare-reducing impediments, but frictions that should be enhanced. That is, greater friction means a lower ELB, which permits welfare-enhancing monetary policy, according to the argument (Goodfriend, 2016; Rogoff, 2017a,b). In this regard, several potential policy tools have been suggested to reduce the ELB. These policy tools range from abolishing paper currency to setting a quantitative limit on currency withdrawals.[3]

A market-based approach, and perhaps the most interesting policy tool presented in the liter-

[1] See, for example, Goodfriend (2016), Rogoff (2017a), and Rogoff (2017b).

[2] See, for example, Woodford (2003) and Curdia and Woodford (2010) for standard New Keynesian models.

[3] Complete elimination of paper currency with keeping coins outstanding, which would increase the storage cost of money, is discussed by Rogoff (2017a) and Rogoff (2017b). Goodfriend (2016) discusses introducing a quantitative limit to cash withdrawals at the central bank cash window. Interestingly, Correia et al. (2013) identify a tax policy that enables a negative interest rate, which requires a rising path of consumption taxes, a declining path of labor income taxes, and a temporary investment tax credit. Altering the one-to-one exchange rate between paper currency and reserves was first proposed by Eisler (1932) and later discussed by Buiter (2010), Agarwal and Kimball (2015), Goodfriend (2016), Rogoff (2017a), and Rogoff (2017b). Interested readers can refer to Agarwal and Kimball (2019) for a comprehensive survey of potential policy tools that enable substantially negative interest rates.

ature, involves altering the one-to-one exchange rate between currency and reserves (Eisler, 1932; Buiter, 2010; Agarwal and Kimball, 2015; Goodfriend, 2016; Rogoff, 2017a,b). This policy tool allows the central bank to set a different exchange rate for private banks' current currency withdrawals from the one for their future currency deposits, and potentially creates a negative nominal rate of return on currency. In particular, Agarwal and Kimball (2019) consider setting a nonpar exchange rate as "the first-best approach with the fewest undesirable side-effects" because the nominal rate of return on currency in units of reserves created by the central bank can be naturally transmitted to the rate of return on currency in units of other interest-bearing assets.

To fully understand how introducing this novel policy tool can reduce the ELB, it is crucial to explicitly consider the costs of handling currency and the imperfect role of currency as a means of payment. So, I develop a general equilibrium model with two different types of means of payment—currency and bank deposits—and include costs of holding currency, to answer the following questions. How does introducing a nonpar exchange rate between currency and reserves affect the frictions that determine the ELB? If the ELB falls as a result of this unconventional intervention, what implications does this have for the allocation of means of payment and welfare?

In my model, *private banks* issue deposit contracts to consumers and acquire an asset portfolio of currency, government bonds, and reserves. Deposit contracts allow consumers to choose one of two options. If a consumer needs currency in transactions, he or she can withdraw currency from the private bank. Otherwise, the consumer uses bank deposits to make payments in transactions. However, private banks are subject to limited commitment. To resolve the limited commitment problem, private banks post their assets as collateral to back their deposit liabilities. Although bank deposits are useful in transactions, there is inefficiency in the banking system due to binding collateral constraints for private banks.[4] So, consumers cannot obtain the first-best quantity of consumption goods in those transactions.

The central bank, which conventionally determines the nominal interest rate on reserves, can set the exchange rate between currency and reserves off par to create a negative nominal rate of return on currency. In particular, withdrawing one unit of currency from the central bank requires more than one unit of reserves (a nonpar exchange rate applied), while depositing one unit of currency with the central bank provides only one unit of reserves (a one-to-one exchange rate). However, private banks can acquire currency from other private individuals to avoid costly currency withdrawals from the central bank cash window. In other words, there is a possibility of side trading in the private sector, which the central bank cannot observe. Due to side trades of currency, the rate of return on currency set by the central bank can differ from the actual rate of return on currency determined by market participants.

To create a negative rate of return on currency, there must be both currency deposits and withdrawals through the central bank cash window. If private individuals can participate in side

[4] More precisely, the fiscal authority determines the total value of consolidated government debt outstanding. If the fiscal authority chooses the value that is sufficiently low, then there arises a shortage of collateralizable assets, leading to binding collateral constraints for private banks.

trades of currency with no frictions, currency might not come back to the central bank cash window when side trading is more profitable than depositing currency with the central bank. So, it is possible that a nonpar exchange rate fails to create a negative rate of return on currency. However, in practice it is costly to transport currency due to the risk of theft. For instance, private banks hire armed security guards and use armored vehicles to transport a large volume of currency. To reflect potential friction in side trades of currency, I introduce endogenous theft into the model, as another cost of holding currency. Theft can take place if private individuals attempt to side trade currency. In particular, some individuals can acquire a theft technology at a cost and steal currency from those participating in side trading.[5]

A key result is that a nonpar exchange rate can indeed reduce the ELB on nominal interest rates, in line with the idea presented by Eisler (1932), Buiter (2010), and Agarwal and Kimball (2015). In the model, a nonpar exchange rate policy is effective if it can reduce the nominal rate of return on currency faced by both private banks and private individuals. In particular, if the cost of theft is sufficiently low (or equivalently, if there exists theft in equilibrium), the actual rate of return on currency perceived by private individuals tracks the one faced by private banks.[6] This occurs because the presence of theft acts to make some individuals deposit their currency with their banks, in which case one unit of currency is effectively exchanged for one unit of reserves.[7] Since private individuals deposit and withdraw currency at different exchange rates, the actual rate of return on currency is determined by the target rate of return set by the central bank.

Introducing a nonpar exchange rate between currency and reserves, however, is not innocuous for the economy. With a decrease in the rate of return on currency, private banks issue a deposit contract that offers a smaller quantity of currency withdrawal. As a result, the quantity of goods traded in transactions using currency (*currency transactions*, hereafter) decreases and the quantity traded in transactions using bank deposits (*bank deposit transactions*, hereafter) increases in equilibrium. Also, a nonpar exchange rate acts to increase the market price of currency by increasing private banks' demand for currency. As an increase in the market price of currency encourages both side trading and theft, the fraction of individuals bearing the cost of theft increases. As a consequence, the total cost of theft increases, decreasing welfare.

In contrast, if the cost of theft is sufficiently high (or equivalently, if no theft takes place in equilibrium), introducing a nonpar exchange rate does not reduce the rate of return on currency perceived by private individuals. With no risk of theft, individuals strictly prefer side trading and do not deposit their currency with the central bank. As the one-to-one exchange rate (applied

[5]There could also be other sources of inefficiency in a currency system stemming from side trades of currency. The side trades of currency, especially if occurring in a large volume, could encourage opportunistic behaviors such as fraud and counterfeiting as well as theft. In the literature, the risk of theft is considered in He et al. (2005), He et al. (2008), and Sanches and Williamson (2010), and counterfeiting of currency is introduced in Williamson (2002), Nosal and Wallace (2007), Li and Rocheteau (2011), and Kang (2017).

[6]In the model, the central bank can observe the quantity of currency held by each private bank and ask private banks to deposit their currency with the central bank (one-to-one exchange rate).

[7]More precisely, private individuals deposit their currency with private banks and then the private banks are required to exchange the currency for reserves one-for-one with the central bank. Therefore, the private individuals indirectly exchange their currency for reserves at par.

to currency deposits) is no longer effective for them, the actual rate of return on currency for individuals does not fall in response to an increase in the nonpar exchange rate (applied to currency withdrawals). However, a nonpar exchange rate can reduce the ELB if the ELB is determined by the rate of return on currency for private banks. There is a range of nonpar exchange rates within which increasing the nonpar exchange rate leads to a fall in the ELB. In particular, an increase in the nonpar exchange rate decreases the ELB if the nonpar exchange rate is sufficiently low, but increases the ELB if the nonpar exchange rate is sufficiently high.[8]

If the cost of theft is sufficiently low, the optimal nominal interest rate on reserves can be negative for an unconventional reason. If currency is costly to store, consumers in currency transactions need to bring more currency to compensate for the cost that will be paid by their counterparts.[9] Then, a negative nominal interest rate can mitigate the inefficiency in currency transactions because it increases the rate of return on currency relative to reserves, which indirectly reduces the cost of holding currency. However, even if a negative nominal interest rate is constrained by the ELB, introducing a nonpar exchange rate between currency and reserves cannot increase welfare. This occurs because the optimal nominal interest rate falls in response to an increase in the nonpar exchange rate. So, implementing a deeper negative nominal interest rate with a nonpar exchange rate cannot attain the optimal equilibrium allocation while increasing costly theft. Therefore, optimal monetary policy is to set the nominal interest rate at the ELB and maintain the one-to-one exchange rate between currency and reserves.

To quantify the magnitude of the welfare cost incurred by a nonpar exchange rate, I calibrate my model to the U.S. economy. Three different scenarios are considered where the cost of theft is given by 2.5 percent, 5 percent, and 10 percent of consumption.[10] In each scenario, I measure the welfare cost of increasing the nonpar exchange rate by asking how much consumption private individuals would need to be compensated to endure the welfare loss. I find that the costs of introducing the nonpar exchange rate of 1.05 (or reducing the ELB by around 5 percentage points) are 0.11 percent, 0.22 percent, and 0.44 percent of consumption, respectively, in the three scenarios.

The frictions that create a negative ELB could also arise from potential disintermediation. Disintermediation happens when consumers opt to use currency rather than bank deposits to make transactions given a sufficiently low interest rate.[11] This is a practical concern because bank deposits serve as a primary and stable funding source for financing bank loans, and thus, disintermediation

[8]This result arises from a collateral role of currency. Private banks use currency as collateral while private individuals do not. As currency bears a liquidity premium due to its role as collateral, it is more costly for private individuals to invest in currency than private banks.

[9]Retailers who accept currency usually hold currency by the end of the day or overnight. So, the cost of storing currency here can be interpreted as the cost of purchasing a safe, using a security system, or hiring a security guard.

[10]Since theft does not occur in the model given a one-to-one exchange rate, the cost of theft must be directly calibrated outside the model. However, to the best of my knowledge, there is no data that allows measuring the cost of theft.

[11]In fear of disintermediation, private banks might not be able to actively adjust their deposit rates in response to monetary policy in negative territory. See Eggertsson et al. (2019) and Ulate (2021), who study the implication of negative interest rates for the transmission mechanism of monetary policy.

might cause long-run inefficiency in the financial system.[12] To understand the implications of introducing a nonpar exchange rate for potential disintermediation, I modify the assumption about available means of payment in transactions. In the modified version of the model, some individuals accept only currency, and others accept both currency and bank deposits as a means of payment. In other words, currency is a universally accepted means of payment, and therefore, potential depositors can opt out of bank deposit contracts and use only currency in transactions.[13]

A key finding is that a nonpar exchange rate between currency and reserves can allow the central bank to set a negative nominal interest rate without harming the banking system, consistent with the conventional view. That is, disintermediation does not take place if the central bank implements a negative nominal interest rate and a nonpar exchange rate so that the nominal rate of return on currency relative to other assets remains constant. However, introducing a nonpar exchange rate encourages costly theft and reduces social welfare as in the baseline model.

Interestingly, in an economy subject to disintermediation, the nominal interest rate must be sufficiently high to prevent a complete flight to currency. In this economy, a complete flight-to-currency episode cannot be supported in equilibrium if there is a shortage of government bonds. If all the depositors decided to opt out of banking contracts, the central bank would have to purchase a sufficient quantity of government bonds to issue a required quantity of currency. But, this would be infeasible if there is a shortage of government bonds. Therefore, the nominal interest rate must be sufficiently high to encourage banking activities. Also, note that the set of nominal interest rates that support banking activities differs from the set of rates that prevent arbitrage. In an economy subject to disintermediation, the former can determine the ELB rather than the latter.

The rest of the paper is organized as follows. I construct the baseline model in Section 2 and define and characterize an equilibrium in Section 3. In Section 4, optimal monetary policy is analyzed, and quantitative analysis is presented in Section 5. In Section 6, a modified version of the model is constructed to analyze the policy implication for disintermediation, and Section 7 is a conclusion.

Related literature This paper is related to the existing literature on how to make negative nominal interest rates implementable by removing or reducing the ELB. Among others, Agarwal and Kimball (2019) provide a comprehensive survey and discuss the pros and cons of policy tools suggested in the literature. The idea of a nonpar exchange rate between currency and reserves was first proposed by Eisler (1932) in the form of a dual currency system where one currency (physical

[12]Replacing physical currency with central bank digital currency (CBDC) can also help reduce the ELB. This is because the central bank can directly set a negative nominal interest rate on CBDC, which is impossible with physical currency. However, it has raised more concern about disintermediation and financial instability as CBDC is a type of electronic means of payment that can completely substitute for bank deposits. Interested readers could refer to Williamson (2022a), Williamson (2022b), and Keister and Sanches (2022), for example.

[13]Recall that, in the baseline model, some transactions require currency while other transactions require bank deposits as a means of payment. In practice, some transactions, especially online transactions, cannot be made with currency. In contrast, some transactions can be made only with currency because either the consumer or the retailer does not have access to the banking system or because they value privacy. Also, there are transactions where both means of payment can be accepted. Therefore, reality may fit somewhere between the two versions of the model.

currency) is used as a means of payment, and the other (electronic money) plays a unit-of-account role. Eisler's proposal was revived recently by Buiter (2010), who developed a simple model of a dual currency system where the central bank can reduce the ELB by adjusting the exchange rate between two currencies. Although the simple model helps understand how a dual currency system would work, the main limitation is that the central bank is assumed to frictionlessly determine the ELB with a proper exchange rate between the two currencies.

The key contribution of this paper is to explicitly consider the frictions that determine the ELB, including the imperfect substitutability between currency and bank deposits and inefficiency stemming from storing or transporting currency. Therefore, my paper can provide the implications of a nonpar exchange rate policy itself for the ELB, the allocation of means of payment, and welfare. In contrast, a nonpar exchange rate policy is neutral to the allocation of means of payment and welfare in Buiter (2010), aside from reducing the ELB. More recently, a nonpar exchange rate between currency and reserves has been favorably discussed, but without a formal model by Agarwal and Kimball (2015), Goodfriend (2016), Rogoff (2017a), and Rogoff (2017b). My paper complements these papers by providing a full-fledged theoretical model that helps understand under what conditions the policy works and why it can go wrong.

This paper is also related to a few theoretical papers studying the implications of a negative nominal interest rate. The most closely related paper is the work of He et al. (2008), who construct a model of two competing means of payment, currency and bank deposits, where using currency is relatively less safe due to the risk of theft. Given endogenous theft and banking, they show that the ELB can be negative, and a negative nominal interest rate can be optimal for some parameters. My paper departs from theirs by differentiating private banks' deposits with the central bank (reserves) from private individuals' bank deposits. This structure allows me to study the transmission of a nonpar exchange rate between currency and reserves into the actual rate of return on currency and the terms of bank deposit contracts. Another paper related to the current one is Brunnermeier and Koby (2019), who study the ELB on nominal interest rates in a model with private banks. They define "reversal interest rate" as the interest rate at which lowering the rate further becomes contractionary and interpret it as the ELB. In contrast, I define the ELB as the lower bound on implementable nominal interest rates and interpret their "reversal interest rate" as the optimal monetary policy rate. Relative to this literature, the main contribution of the current paper is not only to study the welfare implications of implementing a negative nominal interest rate but also to introduce a policy tool that can reduce the ELB.[14]

Search-theoretic models with currency (outside money) and bank deposits (inside money) have been analyzed in Cavalcanti et al. (1999), Williamson (1999), He et al. (2005), Li (2006), He et al. (2008), Li (2011), and Williamson (2012) among others. However, these papers do not answer questions such as how the currency-related frictions determine the ELB on nominal interest rates and how the ELB can be lowered by a policy tool. Endogenous theft is also not new in the literature.

[14]See also, Eggertsson et al. (2019), Jung (2019), and Ulate (2021), who study the implication of a negative nominal interest rate.

In the models of He et al. (2005) and He et al. (2008), theft can happen in bilateral meetings if one individual holds currency and the other is a thief. Due to the risk of theft, using currency in transactions is riskier than making a payment with bank deposits. Sanches and Williamson (2010) assume that theft can happen in a subset of bilateral meetings, and acquiring a theft technology incurs resource costs. A key difference is that, in my model, theft can happen in a meeting where one individual carries currency for side-trading purposes, not transaction purposes, and the other has invested in the theft technology.

2 Baseline Model

The basic structure of the model is similar to Lagos and Wright (2005) and Rocheteau and Wright (2005) with additional details incorporated to address the particular issues related to this problem. Time is indexed by $t = 0, 1, 2, ...$, and there are three subperiods in each period. The theft market (TM) opens in the first subperiod, the centralized market (CM) opens in the following subperiod, and the decentralized market (DM) opens in the last subperiod. There is a continuum of buyers with unit mass, each of whom maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[-\tilde{H}_t - H_t + u(x_t) \right], \tag{1}$$

where $0 < \beta < 1$, $\tilde{H}_t$ and H_t denote the buyer's labor supply in the TM and the CM respectively and x_t denotes his or her consumption in the DM. Assume that $u(\cdot)$ is strictly increasing, strictly concave, and twice continuously differentiable with $u'(0) = \infty$, $u'(\infty) = 0$, and $-\frac{xu''(x)}{u'(x)} < 1$. There is also a continuum of sellers with unit mass, each of whom maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[-\tilde{H}_t^s + X_t^s - h_t^s \right], \tag{2}$$

where X_t^s denotes the seller's consumption in the CM, and $\tilde{H}_t^s$ and h_t^s denote his or her labor supply in the TM and the DM. Finally, there is a continuum of private banks each of which maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[X_t^b - H_t^b \right], \tag{3}$$

where X_t^b is the bank's consumption in the CM and H_t^b is its labor supply in the CM. Private banks are agents who are active only in the CM. In the CM or the DM, one unit of perishable consumption good can be produced with one unit of labor supply while no production takes place in the TM. Buyers cannot produce goods in the DM, while sellers cannot produce in the CM.

At the beginning of the TM, sellers holding currency can deposit the currency with the central bank in exchange for reserve balances.[15] At this stage, currency trades one-to-one for reserve

[15] In practice, private individuals cannot have a reserve account with a central bank. It could be interpreted that

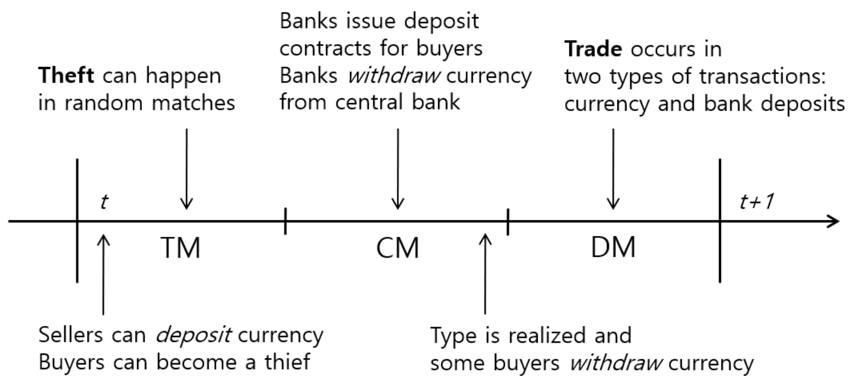

Figure 1: Timing of events

balances, and then sellers can exchange reserve balances for goods in the following CM. After sellers make currency deposits with the central bank, buyers can incur κ units of labor to acquire a theft technology (e.g., producing a weapon). Then, buyers and sellers are randomly matched. If a seller with currency meets a buyer with the theft technology, the buyer steals all of the seller's currency.

At the beginning of the CM, debts are paid off, then production, consumption, and exchange take place in a perfectly competitive market. Private banks can obtain currency in three different ways—from a seller, from a buyer (stolen currency), or by acquiring reserves with the central bank and exchanging the reserves for currency. Also, private banks write deposit contracts with buyers before buyers learn their types. A type for a buyer is the type of seller he or she will meet in the following DM, as specified in what follows. Bank deposit contracts provide insurance, by allowing buyers to withdraw currency at the end of the CM when they learn their types. Buyers' types are publicly observable.

In the DM, each buyer is randomly matched with a seller and makes a take-it-or-leave-it offer to the seller. In any DM matches, a matched buyer and seller do not know each others' histories (no memory or record keeping) and are subject to limited commitment. This implies that no buyers' IOUs can be traded in the DM. There are two types of sellers. Fraction ρ of sellers accepts only currency, and fraction $1-\rho$ accepts only claims on banks. The timing of events is summarized in Figure 1.

Some sellers who acquire currency in the DM may want to exchange the currency for goods in the following CM, instead of safely depositing it with the central bank. Let α_t^s denote the fraction of sellers who carry currency into the CM conditional on having acquired the currency in the previous DM and let α_t^b denote the fraction of buyers who acquire the technology to steal currency. Then, the probability that each seller meets a buyer with the theft technology is α_t^b and the probability that each buyer meets a seller with currency is $\rho\alpha_t^s$.

an individual deposits the currency with a private bank and then the private bank deposits the currency in exchange for reserve balances. Suppose the central bank can ask private banks to deposit their currency at the beginning of the TM. Alternatively, I could assume that, when an individual deposits the currency with the central bank, the central bank credits the payment to the corresponding private bank which in turn credits it to the individual's bank account.

There are three underlying assets in this economy — currency, reserves, and nominal government bonds. Currency and reserves are issued by the central bank. Currency is perfectly divisible, portable, and storable, and bears a nominal interest rate of zero. Reserves are private banks' account balances with the central bank, and one unit of reserves acquired in the CM of period t pays off R_{t+1}^m units of reserves at the beginning of the CM in period $t+1$. Private banks visit the central bank if they want to withdraw currency from their reserve accounts. Following the ideas presented in Eisler (1932), Buiter (2010) and Agarwal and Kimball (2015) among others, the central bank can set an exchange rate between currency and reserves off par to create a negative nominal rate of return on currency. The exchange rate, denoted by $\eta_t \geq 1$, measures the units of reserves exchanged to withdraw one unit of currency in period t while deposited currency is always exchanged one-to-one for reserves. Nominal government bonds, issued by the fiscal authority, are one-period bonds with a gross nominal interest rate of R_t^b.

In addition to the underlying assets issued by the consolidated government, there are bank deposit claims that private banks create endogenously. I assume that private banks have a collateral technology that allows creditors to seize at least part of the asset if the bank defaults. This implies that private banks can issue bank claims that can be accepted in transactions by some sellers as the claims will be backed by collateral. Private banks hold government bonds and reserves as collateral. A bank could potentially hold currency in its portfolio from one CM until the next (also using it as collateral). But, in doing so it bears a cost γc_t, where $0 < \gamma < 1$ and c_t is the real value of currency held by the bank when it is acquired, and the cost is born in the CM of period $t+1$. Sellers have the same storage technology as banks, but buyers cannot hold currency across periods.[16] In addition to the proportional cost of storing currency across periods, there is a fixed cost μ of holding currency incurred by individuals at the beginning of the TM.

2.1 Government

Confine attention to stationary equilibria where all real variables and government policies are constant across periods. Assume that the consolidated government starts issuing its liabilities with no unsettled debt outstanding in period 0. Then, the consolidated government budget constraint at $t = 0$ can be written as

$$\eta \bar{c} + \bar{m} + \bar{b} = \tau_0, \tag{4}$$

where $\bar{c}$, $\bar{m}$, and $\bar{b}$ denote the real quantities of currency, reserves, and nominal government bonds outstanding at the end of period 0 (and in every following period). Also, τ_0 is the real quantity of the lump-sum transfer to each buyer. Assume that the fiscal authority can levy lump-sum taxes on buyers in equal amounts. So, the consolidated government budget constraint at $t = 1, 2, ...$ can be

[16] This implies that an equilibrium nominal interest rate must satisfy no arbitrage conditions for both private banks and sellers from holding currency across periods.

written as

$$\eta\bar{c} + \bar{m} + \bar{b} = \frac{\bar{c} + R^m\bar{m} + R^b\bar{b}}{\pi} + \tau, \tag{5}$$

where π is the gross inflation rate and τ is the real quantity of the lump-sum transfer (or the lump-sum tax if $\tau < 0$) to each buyer. The left-hand side of (5) represents the revenue of the consolidated government from issuing new liabilities, and the right-hand side represents the expenditure on repayments of government debt issued in the previous period and the transfer to buyers.

As in Andolfatto and Williamson (2015) and Williamson (2016), I assume that the fiscal authority determines the real value of the consolidated government debt outstanding, v, where

$$v = \eta\bar{c} + \bar{m} + \bar{b}, \tag{6}$$

for all periods.[17] So, the fiscal authority adjusts lump-sum transfers, in response to a change in monetary policy, to achieve the fiscal policy goal. Given the fiscal policy that targets the total value of the consolidated government debt, the central bank's monetary policy changes its composition. A key feature I can capture through this fiscal policy rule is the scarcity of collateral and resulting low real interest rates. That is, a low supply of consolidated government debt, or a low v, will lead to low real interest rates, and I will eventually analyze the effects of monetary policy given a sufficiently low v.

3 Equilibrium

In this section, I will describe how buyers and sellers make their decisions in the TM and how private banks choose their deposit contract and asset portfolio in the CM. Then, I will define and characterize an equilibrium and show why the effective lower bound on the target policy rate can be negative.

3.1 Side Trading and Theft

In the CM, private banks can withdraw one unit of currency from the central bank by paying η units of their reserve balances. If there are would-be sellers of currency in the CM, private banks would participate in side trades of currency to reduce the cost of acquiring currency. In equilibrium,

[17] The central bank purchases government bonds by issuing currency and reserves in period 0 and transfers its profits to the fiscal authority in every following period. This implies that the real value of the central bank's assets must be equal to that of its liabilities in every period, that is,

$$\eta\bar{c} + \bar{m} = \hat{b},$$

where $\hat{b}$ denotes the real quantity of government bonds held by the central bank. Therefore, the fiscal authority determines the real quantity of total government bonds issued by the fiscal authority because

$$v = \hat{b} + \bar{b}.$$

private banks must be indifferent between withdrawing currency from the central bank and obtaining currency from any would-be sellers of currency. This implies that the price of currency in terms of reserves in the CM must be η in equilibrium.[18]

Side trading in currency between private banks and currency holders can take place only when there are sellers who choose not to deposit their currency with the central bank in the TM. However, in the TM, some buyers may incur κ units of labor supply to acquire the theft technology. Suppose that the representative currency-holding seller carries c^s units of currency in real terms into the TM. As acquiring the theft technology is costly, each buyer's decision on stealing currency must be incentive compatible in equilibrium, that is,

$$\text{if} \quad \kappa > \rho\alpha^s\eta c^s, \quad \text{then} \quad \alpha^b = 0, \tag{7}$$

$$\text{if} \quad \kappa = \rho\alpha^s\eta c^s, \quad \text{then} \quad 0 \leq \alpha^b \leq 1, \tag{8}$$

$$\text{if} \quad \kappa < \rho\alpha^s\eta c^s, \quad \text{then} \quad \alpha^b = 1, \tag{9}$$

where $\rho\alpha^s$ is the probability of meeting a currency-holding seller and ηc^s is the real value of currency held by the seller. Conditions (7)-(9) state that no theft takes place if the buyer strictly prefers not to steal currency, the buyer sometimes steals if he or she is indifferent between two options, and the buyer always steals if theft is strictly preferred. Also, each seller's decision on carrying currency into the TM must be incentive compatible, that is,

$$\text{if} \quad (1-\alpha^b)\eta < 1, \quad \text{then} \quad \alpha^s = 0, \tag{10}$$

$$\text{if} \quad (1-\alpha^b)\eta = 1, \quad \text{then} \quad 0 \leq \alpha^s \leq 1, \tag{11}$$

$$\text{if} \quad (1-\alpha^b)\eta > 1, \quad \text{then} \quad \alpha^s = 1. \tag{12}$$

The seller does not carry the currency into the TM if the expected payoff from carrying one unit of currency $(1-\alpha^b)\eta$ is less than the payoff from depositing it with the central bank and obtaining one unit of reserves. If the seller is indifferent between two choices, he or she sometimes carries the currency into the TM. Otherwise, the seller always carries the currency into the TM.

3.2 Deposit Contracts

Private banks write deposit contracts for buyers in the CM before buyers learn their types. Deposit contracts provide insurance to buyers as in Williamson (2012, 2016, 2022b) by giving them an option to withdraw currency when they learn their types. Those buyers who do not exercise the option will use bank claims as a means of payment in DM meetings. Suppose a bank proposes a deposit contract (k, c', d), where k is the quantity of CM goods deposited by each buyer at the beginning of the CM, c' is the real quantity of currency that the buyer can withdraw at the end of the CM,

[18] This happens if the real quantity of currency traded in the CM does not exceed the real quantity demanded by private banks. Here, I focus on stationary equilibria with sufficiently low v and γ so that there is always inflation $\pi > 1$ and the real value of currency decreases over time. See Appendix A.1 for the details on stationary equilibria with deflation.

and d is the quantity of claims to CM goods in the following period that the buyer can exchange in the DM if currency has not been withdrawn. Also, the bank acquires an asset portfolio (b, m, c), where b is the quantity of government bonds, m is the quantity of reserves, and c is the quantity of currency in real terms. In equilibrium, the bank's problem can be written as

$$\max_{k,c',d,b,m,c} \left\{ -k + \rho u \left([1 - \alpha^s + \alpha^s(1-\alpha^b)\eta] \frac{\beta c'}{\pi} - \beta\mu \right) + (1-\rho) u(\beta d) \right\} \tag{13}$$

subject to

$$k - b - m - \eta c + \beta \left[-(1-\rho)d + \frac{R^m m + R^b b + c - \rho c'}{\pi} - \gamma(c - \rho c') \right] \geq 0, \tag{14}$$

$$-(1-\rho)d + \frac{R^m m + R^b b + c - \rho c'}{\pi} \geq \frac{\delta\left(R^m m + R^b b + c\right)}{\pi}, \tag{15}$$

$$k, c', d, b, m, c, c - \rho c' \geq 0. \tag{16}$$

The objective function (13) is the representative buyer's expected utility, implying that the bank chooses a contract that maximizes the buyer's expected utility in equilibrium. With probability ρ, the buyer realizes that, in the following DM, he or she will be matched with a seller who accepts only currency. In this case, the buyer visits the bank to withdraw c' units of currency at the end of the CM. In the following DM, the buyer makes a take-it-or-leave-it offer to the matched seller and acquires $[1 - \alpha^s + \alpha^s(1-\alpha^b)\eta]\frac{\beta c'}{\pi} - \beta\mu$ units of goods.[19] With probability $1 - \rho$, the buyer learns that he or she will meet a seller who accepts a claim on the bank. As the buyer does not withdraw currency in this case, he or she receives a claim to d units of goods in the next CM. So, the buyer's take-it-or-leave-it offer implies that the buyer trades d deposit claims for βd units of goods.

Constraint (14) states that the bank earns a nonnegative discounted net payoff in equilibrium. In the CM, the bank receives k deposits from the buyer and acquires a portfolio of government bonds b, reserves m, and currency c. At the end of the CM, the bank pays off currency to the fraction ρ of buyers, each of whom withdraws c' currency. The remaining fraction $1 - \rho$ of buyers exchange their deposit claims in the DM. So, in the following CM, the bank pays off d units of goods to each holder of the deposit claims. Notice that the bank stores the remaining $c - \rho c'$ units of currency until the next CM which incurs $\gamma(c - \rho c')$ units of labor supply. At the beginning of the next CM, the bank must deposit the remaining currency with the central bank at a one-to-one exchange rate.

As for any agents in the economy, the bank is subject to limited commitment. So, the bank's

[19]As the buyer makes a take-it-or-leave-it offer to the seller, the buyer can extract all the surplus from trade. By accepting the buyer's offer, the seller receives c' units of currency (in real terms) from the buyer in the DM. Then, in the next period, the seller will be holding $\frac{c'}{\pi}$ units of currency and bear μ units of fixed cost (in terms of labor supply) at the beginning of the TM. The ex-ante expected payoff per unit of currency is $1 - \alpha^s + \alpha^s(1-\alpha^b)\eta$ because with probablity $1 - \alpha^s$ the seller deposits the currency to receive one unit of reserves, and with probability $\alpha^s(1-\alpha^b)$ the seller successfully sells the currency in the CM at price η. Since the seller's surplus from trade is zero, the quantity of goods produced by the seller and transferred to the buyer (or equivalently, the disutility from producing goods) is equal to the seller's discounted expected net payoff from acquiring c' units of currency in the DM.

deposit liabilities must be backed by collateral and (15) is a collateral constraint. Assume that the bank can abscond with a fraction δ of its assets, pledged as collateral, when it defaults. Then, the collateral constraint tells us that the bank must weakly prefer to repay its deposit liabilities in the CM and in the next CM rather than absconding with collateral. If the bank were to default, it would not let buyers withdraw currency as in Williamson (2022b). Finally, constraint (16) demonstrates that all real quantities must be nonnegative.

3.3 Definition of Equilibrium

Any contract that provides a positive discounted net payoff for the bank cannot be supported in equilibrium. If private banks were to earn a positive discounted net payoff, a bank would design an alternative contract that provides a slightly lower payoff per contract, but a higher total payoff by attracting all buyers. So, constraint (14) must hold with equility in equilibrium.

Let λ denote the Lagrange multiplier associated with the collateral constraint (15). Then, I can derive the first-order conditions for the bank's maximization problem, (13) subject to (14)-(16), as follows.

$$\frac{\beta[1-\alpha^s+\alpha^s(1-\alpha^b)\eta]}{\pi}u'\left([1-\alpha^s+\alpha^s(1-\alpha^b)\eta]\frac{\beta c'}{\pi}-\beta\mu\right)-\eta-\frac{\lambda\delta}{\pi}=0, \tag{17}$$

$$\beta u'(\beta d)-\beta-\lambda=0, \tag{18}$$

$$-1+\frac{\beta R^m}{\pi}+\frac{\lambda R^m(1-\delta)}{\pi}=0, \tag{19}$$

$$-1+\frac{\beta R^b}{\pi}+\frac{\lambda R^b(1-\delta)}{\pi}=0, \tag{20}$$

$$-\eta+\frac{\beta}{\pi}-\beta\gamma+\frac{\lambda(1-\delta)}{\pi}\leq 0, \tag{21}$$

$$\lambda\left[-(1-\rho)d+\frac{(1-\delta)\left(R^m m+R^b b+c\right)}{\pi}-\frac{\rho c'}{\pi}\right]=0. \tag{22}$$

A necessary condition for an equilibrium to exist is that sellers do not hold currency from the CM to the next CM. So, the expected payoff from holding currency across periods must be nonpositive at the margin. That is,

$$-\eta+\beta\left[\frac{1-\alpha^s+\alpha^s(1-\alpha^b)\eta}{\pi}-\gamma\right]\leq 0. \tag{23}$$

Also, in equilibrium, asset markets clear in that the demand for each asset is equal to the supply. That is,

$$c=\bar{c};\quad m=\bar{m};\quad b=\bar{b}. \tag{24}$$

For convenience, let x^c and x^d denote the consumption quantities in DM meetings, respectively, with currency and deposit claims being traded, i.e.,

$$x^c = [1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta]\frac{\beta c'}{\pi} - \beta\mu, \tag{25}$$

$$x^d = \beta d. \tag{26}$$

I assume that the central bank conducts monetary policy under a floor system where a sufficiently large quantity of reserve balances are held by private banks and the central bank sets the nominal interest rate on reserves R^m. Under this system, private banks treat reserves and government bonds as identical assets at the margin, so the nominal interest rate on reserves pegs the nominal interest rate on government bonds in equilibrium, i.e., $R^m = R^b$ from (19) and (20).[20] Also, the central bank can expand its balance sheet through swaps of reserves for government bonds. Let $\omega = \eta\bar{c} + \bar{m}$ denote the size of the balance sheet, which is equivalent to the real value of the central bank's liabilities.[21]

Then, I can define an equilibrium as follows.

Definition *Given exogenous fiscal policy v and monetary policy (R^m, ω, η), a stationary equilibrium consists of DM consumption quantities (x^c, x^d), asset quantities (k, c', d, b, m, c), the fraction of buyers who choose to steal currency in the TM α^b, the fraction of sellers who choose to carry currency into the TM conditional on having acquired the currency in the previous DM α^s, transfers (τ_0, τ), gross inflation rate π, and gross nominal interest rate on government bonds R^b, satisfying the consolidated government budget constraints (4) and (5), the fiscal policy rule (6), the first-order conditions for the bank's problem (17)-(22), no arbitrage condition for sellers (23), the incentive compatibility conditions for buyers and sellers (7)-(12), and market clearing conditions (24).*

Notice that, according to the definition, the fiscal and monetary policies are given exogenously. The fiscal authority determines the total value of consolidated government debt, in real terms, while the central bank has three policy targets: (i) the nominal interest rate on reserves, (ii) the size of the central bank's balance sheet, and (iii) the exchange rate between currency and reserves. As the total value of consolidated government debt is exogenously set by the fiscal policy rule, the central bank's monetary policy effectively determines the composition of the debt. Also, note that the fiscal authority manipulates transfers, in response to monetary policy, so as to satisfy its fiscal policy target and the government budget constraints.

[20] In practice, reserves are considered as a useful means of payment in intraday trading in the banking system. However, a key property of the U.S. financial system in the post-financial crisis period is that a large volume of reserves has been held by private banks without being used in intraday financial transactions. This observation allows us to simply assume that reserves and government bonds share the same properties by abstracting from the transaction role of reserves. Also, note that only reserves can be turned into currency through the central bank cash window. But, this property does not play an important role in equilibrium.

[21] The real value of liabilities is equal to that of assets because the central bank is assumed to transfer any profits/losses to the fiscal authority and the central bank's net worth is zero.

3.4 Characterization of Equilibrium

I first characterize the effective lower bound (ELB) on the gross nominal interest rate R^m. Note that inequality (21) represents no arbitrage for private banks from acquiring currency in the CM, holding it across periods, and redepositing it in the next CM. Substituting (19) into (21) gives

$$R^m \geq \frac{1}{\eta + \beta\gamma}.$$

Also, using (19), the no arbitrage condition for sellers from holding currency across periods (23) can be rewritten as

$$R^m \geq \frac{1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta}{(\eta + \beta\gamma)[(1 - \delta)u'(x^d) + \delta]}.$$

So, the ELB on the gross nominal interest rate in equilibrium is given by

$$R^m \geq \max\left\{\frac{1}{\eta + \beta\gamma}, \frac{1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta}{(\eta + \beta\gamma)[(1 - \delta)u'(x^d) + \delta]}\right\}. \tag{27}$$

Suppose the ELB is determined by the first argument in the above maximization problem. Then, the gross nominal interest rate on reserves R^m can be less than one, or the net nominal interest rate $R^m - 1$ can be negative, for two reasons. If the exchange rate for currency withdrawals is not one-to-one, $\eta > 1$, then a negative net nominal interest rate on reserves can be supported in equilibrium, as proposed by Eisler (1932), Buiter (2010), and Agarwal and Kimball (2015). Another reason why the net nominal interest rate can be negative comes from the proportional cost of storing currency. If holding currency across periods is costly and the cost is proportional to the quantity of currency, i.e., $\gamma > 0$, negative nominal interest rates can be supported in equilibrium. This result can explain why some central banks could implement negative nominal interest rates without causing a flight to currency.

Now, suppose the second argument in the above maximization problem determines the ELB on $R^m - 1$. Then, there is a nonstandard reason for a negative ELB. As I will show later, the term $(1 - \delta)u'(x^d) + \delta$ is higher than one due to low real interest rates on interest-bearing assets. Real interest rates are low because those assets are useful as collateral and therefore bear a liquidity premium. However, sellers do not use currency or any interest-bearing assets as collateral. As low real interest rates are accompanied by a high inflation rate, currency yields a low real return to sellers, which makes a negative nominal interest rate feasible. For analytical convenience, I will focus on cases where (27) holds with strict inequality. That is, the nominal interest rate on reserves is not constrained by the ELB.

Suppose the fraction ρ of sellers, who accept currency in the DM, holds c^s currency each in real terms at the beginning of the TM. Then, the following lemma shows the values of α^b and α^s that are consistent with optimal decisions of buyers and sellers in equilibrium.

Lemma 1 *Suppose that $\eta = 1$. Then, no theft occurs in equilibrium, i.e., $\alpha^b = 0$. Furthermore, $\alpha^s \in [0,1]$ for all $\kappa \geq \rho\eta c^s$ and $\alpha^s \in [0, \bar{\alpha}^s]$ for all $\kappa < \rho\eta c^s$ where $\bar{\alpha}^s = \frac{\kappa}{\rho\eta c^s}$. Alternatively, suppose that $\eta > 1$. Then, no theft occurs in equilibrium with $\alpha^b = 0$ and $\alpha^s = 1$ for all $\kappa \geq \rho\eta c^s$ while theft exists in equilibrium with $\alpha^b = \frac{\eta-1}{\eta}$ and $\alpha^s = \frac{\kappa}{\rho\eta c^s}$ for all $\kappa < \rho\eta c^s$.*

Proof See Appendix □

According to Lemma 1, there is no theft in equilibrium if private banks can withdraw currency from their reserve accounts at par. In this case, the market price of currency is identical to the price of reserves, so sellers are indifferent between depositing the currency with the central bank and trading the currency with a private bank. If the central bank sets a nonpar exchange rate between currency and reserves, i.e., $\eta > 1$, then the policy tends to encourage sellers to trade currency with private banks. However, sellers become indifferent to depositing the currency at the central bank in equilibrium because there are some buyers trying to steal the currency, given that the cost of theft is sufficiently low.

From (25) and Lemma 1, I can show that

$$x^c = \frac{\beta c' \eta}{\pi} - \beta\mu, \qquad \forall \kappa \geq \frac{\rho\eta c'}{\pi} \tag{28}$$

$$x^c = \frac{\beta c'}{\pi} - \beta\mu. \qquad \forall \kappa < \frac{\rho\eta c'}{\pi} \tag{29}$$

In what follows, I will consider the case where the real value of the consolidated government debt outstanding v is sufficiently low, so as to confine attention to an equilibrium with binding collateral constraints.

3.5 Equilibrium with No Theft

In this section, I characterize an equilibrium where all currency-holding sellers trade currency with private banks in the CM with no threat of theft in the TM. Suppose that the cost of acquiring the theft technology is sufficiently high, so that condition (28) holds, $\alpha^b = 0$, and $\alpha^s = 1$ in equilibrium. Then, from (17)-(20) and (28), the inflation rate π and the nominal interest rates on reserves and government bonds R^m and R^b are given by

$$\pi = \frac{\beta}{\eta}\left[\eta u'(x^c) - \delta u'(x^d) + \delta\right], \tag{30}$$

$$R^m = R^b = \frac{\eta u'(x^c) - \delta u'(x^d) + \delta}{\eta\left[u'(x^d) - \delta u'(x^d) + \delta\right]}, \tag{31}$$

and the real interest rate is given by

$$r^m = r^b = \frac{1}{\beta\left[u'(x^d) - \delta u'(x^d) + \delta\right]}. \tag{32}$$

In equilibrium, the quantity of consumption in DM trades that involve using bank claims is inefficiently low due to a binding collateral constraint, leading to a low real interest rate.

From (6), (24)-(26), and (30)-(31), the binding collateral constraint (15) can be rewritten as

$$\left[u'(x^c) + \frac{\delta}{(1-\delta)\eta}\right]\rho(x^c + \beta\mu) + \left[u'\left(x^d\right) + \frac{\delta}{1-\delta}\right](1-\rho)x^d = v. \tag{33}$$

Equation (33) implies that the aggregate demand for collateral (the left-hand side) must be equal to the aggregate supply (the right-hand side) in equilibrium. From (28), a necessary condition for buyers to not invest in the theft technology is given by

$$\kappa \geq \frac{\rho(x^c + \beta\mu)}{\beta}. \tag{34}$$

Finally, the ELB on the gross nominal interest rate in equilibrium is given by

$$R^m \geq \max\left\{\frac{1}{\eta + \beta\gamma}, \frac{\eta}{(\eta + \beta\gamma)[(1-\delta)u'(x^d) + \delta]}\right\}. \tag{35}$$

If the nonpar exchange rate between currency and reserves η is sufficiently close to one and the real interest rate $r^m = r^b$ is sufficiently low so that the term $(1-\delta)u'(x^d) + \delta$ is sufficiently high, then the ELB is determined by the first argument. In this case, a higher η implies a lower ELB, consistent with the claims made by Eisler (1932), Buiter (2010), and Agarwal and Kimball (2015). However, if η is sufficiently larger than one and $r^m = r^b$ is sufficiently high, then the second argument governs the ELB on the nominal interest rate. In this case, an increase in η does not necessarily lower the ELB, and it can even increase the ELB. This happens because there are insufficient frictions to prevent sellers from carrying currency across periods. Sellers can exploit arbitrage by purchasing currency at price η and selling it at the same price with no risk of theft in the next period. Given a fixed real interest rate, an increase in η only increases the market price of currency, making arbitrage more profitable. Therefore, the ELB can even increase in response to an increase in η.

An interpretation is that, for the realized nominal rate of return on currency to be negative, there must be currency deposits and withdrawals at different exchange rates in equilibrium. If either one of those two activities does not occur, a nonpar exchange rate can fail to reduce the rate of return on currency and the ELB. In an equilibrium with no theft studied here, the absence of sellers' currency deposits breaks the link between the nonpar exchange rate and the nominal rate of return on currency. In contrast, if there is deflation in equilibrium, it is possible that private banks do not withdraw currency from the central bank cash window as can be seen in Appendix A.1.

Given the above equilibrium conditions, I can solve the model as follows. First, equations (31) and (33) solve for (x^c, x^d), given monetary policy (R^m, η) and fiscal policy v. Then, equation (30) solves for π, equation (32) solves for r^m and r^b, and inequalities (34) and (35) give necessary conditions for this equilibrium to exist.

(For a sufficiently low v)

	∂x^c	∂x^d	$\partial \pi$	∂r^m	∂ELB
∂R^m	$-$	$+$	$+$	$+$	$\cdot$
$\partial \eta$	$+$	$-$	$+$	$-$	$-$

(For a sufficiently high v)

	∂x^c	∂x^d	$\partial \pi$	∂r^m	∂ELB
∂R^m	$-$	$+$	$+$	$+$	$\cdot$
$\partial \eta$	$+$	$+$	$-$	$+$	$-$

Table 1: Effects of monetary policy (R^m, η) in an equilibrium with no theft

Effects of Monetary Policy Note that the size of the central bank's balance sheet ω is irrelevant to asset prices or consumption quantities. This occurs because an expansion in the size of the balance sheet involves central bank swaps of reserves for government bonds. As those assets are perfect substitutes for private banks at the margin, this only changes the composition of government bonds and reserves in bank asset portfolios, with no effects on other variables.

In what follows, I analyze the effects of monetary policy interventions given that the net nominal interest rate on reserves is close to zero and the exchange rate between currency and reserves is close to one. This implies that the ELB on the nominal interest rate is determined by the first argument in (35).

Proposition 1 *Suppose that inequalities (34) and (35) hold in equilibrium, (R^m, η) is sufficiently close to $(1,1)$, and the fixed cost of holding currency μ is sufficiently close to zero. Then, an increase in R^m results in a decrease in x^c, an increase in x^d, an increase in real interest rates (r^m, r^b), and an increase in π, with no effect on the ELB. In contrast, an increase in η results in an increase in x^c and a decrease in the ELB. Furthermore, there exists $\hat{v}$ such that an increase in η decreases x^d and (r^m, r^b) and increases π for $v \in (0, \hat{v}]$, while it increases x^d and (r^m, r^b) and decreases π for $v \in (\hat{v}, \bar{v})$ where $\bar{v}$ is the upper bound of the values of consolidated government debt that support an equilibrium with a binding collateral constraint.*

Proof See Appendix $\square$

With the exchange rate between currency and reserves η held constant, an increase in the nominal interest rate on reserves R^m affects bank asset portfolios since it becomes more profitable to hold more reserves or government bonds rather than currency. With a larger quantity of reserves or government bonds, banks can provide a larger quantity of claims to buyers, so x^d rises. However, a smaller quantity of currency outstanding, in real terms, leads to a smaller quantity of consumption in DM trades using currency x^c. Also, the decrease in the real quantity of currency outstanding must be accompanied by a decrease in the real rate of return on currency in equilibrium, implying a rise in the inflation rate π with η held constant. As a larger quantity of reserves and government bonds makes collateral less scarce, a rise in R^m acts to increase real interest rates, r^m and r^b. But real rates increase by less than do nominal interest rates.

A novel finding is that an increase in the exchange rate between currency and reserves η itself has real effects. In particular, an increase in η leads to an increase in the consumption quantity in DM trades using currency x^c, with R^m held constant. This occurs because an increase in η increases the price of currency in the CM, which in turn increases the value of currency in the DM. As currency is exchanged for a larger quantity of goods in DM trades, the quantity of consumption in those trades increases. The effect of an increase in η on the consumption quantity in DM trades using bank claims x^d depends on the value of consolidated government debt v. If v is sufficiently low or collateralizable assets are sufficiently scarce, then an increase in η decreases x^d and real interest rates (r^m, r^b), implying that a larger quantity of currency outstanding effectively decreases the stock of government bonds and reserves held by private banks. In contrast, if v is sufficiently high but not too high, then an increase in η increases x^d and real interest rates (r^m, r^b). In this case, a higher price of currency acts to decrease the real quantity of currency c' (the income effect dominates the substitution effect) which effectively relaxes the collateral constraint from (22). Therefore, x^c and x^d both increase. These results are summarized in Table 1.

Corollary 1 *If η is sufficiently close to one, an increase in η leads to a decrease in the ELB on the nominal interest rate. But if η is sufficiently high, an increase in η can increase the ELB.*

As mentioned earlier, the ELB on the nominal interest rate is determined by the first argument in (35) if the exchange rate between currency and reserves η is sufficiently close to one. This implies that there is no arbitrage opportunity for sellers from holding currency across periods as long as there is no such arbitrage opportunity for private banks. However, if η is sufficiently high, then the second argument in (35) can determine the ELB. In this case, no arbitrage for private banks from investing in currency does not prevent the sellers' opportunistic behavior. This happens because sellers trade currency at the market price η without threat of theft in the TM. So, an increase in η does not effectively reduce the rate of return on currency. Instead, it is possible that an increase in η leads to an increase in the rate of return on currency because the cost of storing currency becomes relatively smaller as the price of currency η rises. This relation between the exchange rate η and the ELB is illustrated by Figure 2.

Corollary 2 *Given that (R^m, η) is sufficiently close to $(1, 1)$ and the fixed cost of holding currency μ is sufficiently close to zero, suppose the central bank increases η and decreases R^m to hold ηR^m constant. Then, this policy increases x^c and decreases the ELB. Furthermore, if $v \in (0, \hat{v}]$, then x^d decreases and (r^m, r^b) decrease, while the effect on π is ambiguous. If $v \in (\hat{v}, \bar{v})$, then π falls while the effects on x^d and (r^m, r^b) are ambiguous.*

Suppose the ELB on the nominal interest rate is binding. If the central bank wishes to reduce the nominal interest rate R^m below the ELB, it could reduce the nominal rate of return on currency in the same magnitude (in percentage) as the nominal interest rate. Corollary 2 shows what happens if the central bank increases the exchange rate η and reduce the nominal interest rate on reserves R^m in the same magnitude, so that the relative rate of return on currency $\frac{1}{\eta R^m}$ remains constant. The policy leads to an increase in x^c and a decrease in the ELB, while its effects on other variables such

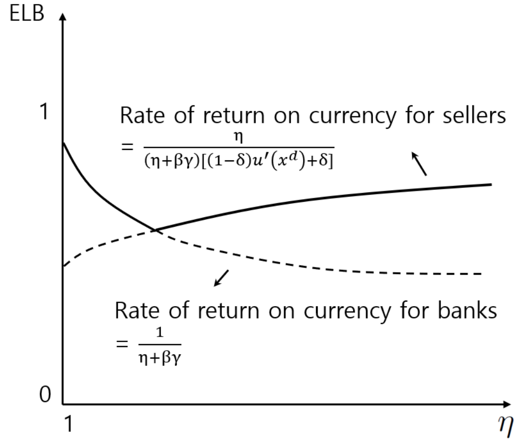

Figure 2: Exchange rate η and the effective lower bound (ELB)

as x^d, r^m, r^b, and π, depend on the value of consolidated government debt v. Most importantly, the policy is not neutral as it affects the DM consumption and real interest rates. These real effects arise in this equilibrium because the expected payoff for sellers from trading currency with private banks is higher than the payoff from depositing the currency with the central bank. In other words, the expected rate of return on currency perceived by sellers does not coincide with the target rate of return set by the central bank. Therefore, the policy serves to distort the allocation of currency and bank claims in DM transactions.

3.6 Equilibrium with Theft

In this section, I analyze an equilibrium where some sellers carry currency into the TM and some buyers steal currency. Suppose that the cost of acquiring the theft technology κ is sufficiently low, so that condition (29) holds in equilibrium. Then, from (17)-(20) and (29), I obtain

$$\pi = \frac{\beta}{\eta}\left[u'(x^c) - \delta u'(x^d) + \delta\right], \tag{36}$$

$$R^m = R^b = \frac{u'(x^c) - \delta u'(x^d) + \delta}{\eta\left[u'(x^d) - \delta u'(x^d) + \delta\right]}, \tag{37}$$

and the real interest rate is given by

$$r^m = r^b = \frac{1}{\beta\left[u'(x^d) - \delta u'(x^d) + \delta\right]}. \tag{38}$$

From (6), (24)-(26), and (36)-(37), the binding collateral constraint (15) can be rewritten as

$$\left[u'(x^c) + \frac{\delta}{1-\delta}\right]\rho(x^c + \beta\mu) + \left[u'\left(x^d\right) + \frac{\delta}{1-\delta}\right](1-\rho)x^d = v, \tag{39}$$

From Lemma 1 and (29), I can write the fraction of buyers who choose to acquire the theft techonology α^b and the fraction of sellers who choose to carry currency in the TM α^s as

$$\alpha^b = \frac{\eta - 1}{\eta}, \tag{40}$$

$$\alpha^s = \frac{\beta\kappa}{\rho\eta(x^c + \beta\mu)}, \tag{41}$$

and I can derive a necessary condition for this equilibrium to exist, which is given by

$$\kappa < \frac{\rho\eta(x^c + \beta\mu)}{\beta}. \tag{42}$$

From (27) and (40), no arbitrage from holding currency across periods is given by

$$R^m \geq \frac{1}{\eta + \beta\gamma}. \tag{43}$$

Solving the model is straighforward, as with an equilibrium with no theft. First, equations (37) and (39) solve for (x^c, x^d), given monetary policy (R^m, η) and fiscal policy v. Then, equation (36) solves for π, equation (38) solves for (r^m, r^b), equations (40)-(41) solve for (α^b, α^s), and inequalities (42) and (43) give necessary conditions for this equilibrium to exist.

Effects of Monetary Policy The size of the central bank's balance sheet ω is irrelevant to asset prices or consumption quantities as in an equilibrium with no theft. So, in what follows I analyze the effects of monetary policy interventions with R^m (the nominal interest rate on reserves) and η (the nonpar exchange rate between currency and reserves).

Proposition 2 *Suppose that inequalities (42) and (43) hold in equilibrium. Then, an increase in R^m or η results in a decrease in x^c, an increase in x^d, and an increase in real interest rates (r^m, r^b). In addition, an increase in R^m increases π and α^s with no effects on the ELB and α^b. An increase in η decreases π and the ELB, and increases α^b, but its effect on α^s is ambiguous.*

Proof See Appendix □

With the exchange rate between currency and reserves η held constant, the effects of an increase in the nominal interest rate on reserves R^m on consumption quantities, real interest rates, and the inflation rate are qualitatively identical to those in an equilibrium with no theft, although the fraction of sellers who carry currency into the TM α^s increases in this equilibrium. In response to an increase in R^m, sellers receive a smaller quantity of real currency from buyers in the DM. As buyers have a lower incentive to invest in the theft technology when sellers hold a smaller quantity of real currency, sellers can increase the probability of carrying currency into the TM until the fraction of buyers with the theft technology α^b remains the same.

	∂x^c	∂x^d	$\partial \pi$	∂r^m	∂ELB	$\partial \alpha^b$	$\partial \alpha^s$
∂R^m	$-$	$+$	$+$	$+$	$\cdot$	$\cdot$	$+$
$\partial \eta$	$-$	$+$	$-$	$+$	$-$	$+$	?

Table 2: Effects of monetary policy (R^m, η) in an equilibrium with theft

A key result is that an increase in the exchange rate between currency and reserves η (a decrease in the nominal/real rate of return on currency) leads to a decrease in the inflation rate π (an increase in the real rate of return on currency). So, the fall in the real rate of return on currency due to an increase in η is mitigated by a decrease in π in equilibrium. Also, an increase in η decreases the ELB. These results are consistent with Eisler (1932), Buiter (2010), and Agarwal and Kimball (2015), in that an increase in the exchange rate between currency and reserves reduces the real rate of return on currency and the ELB on nominal interest rates.

Note that a higher η, or a higher price of currency in the CM, induces buyers to invest in the theft technology more often. Then, due to a higher risk of theft, sellers become indifferent between carrying currency into the TM and safely depositing it with the central bank, although they can sell it in the CM at a higher price.[22] Therefore, the central bank can successfully reduce the rate of return on currency and the ELB on nominal interest rates in this equilibrium owing to endogenous theft. However, reducing the ELB is costly because a larger fraction of buyers investing in the theft technology implies a larger welfare loss.

As in an equilibrium with no theft, an increase in the exchange rate η itself has real effects as it decreases x^c and increases x^d, r^m and r^b. These effects occur because, with R^m held constant, an increase in η leads to a decrease in the rate of return on currency relative to government bonds and reserves $\frac{1}{\eta R^m}$. Note that the effects of an increase in the exchange rate η on consumption quantities and real interest rates are qualitatively the same as those of an increase in R^m. These results are summarized in Table 2.

Corollary 3 *Suppose the central bank increases η and decreases R^m to hold ηR^m constant. Then, this policy decreases π one-for-one, decreases α^s and the ELB, and increases α^b. However, consumption quantities (x^c, x^d) and real interest rates (r^m, r^b) remain unchanged.*

Suppose that the central bank increases the exchange rate η and reduces the interest rate on reserves R^m with $\frac{1}{\eta R^m}$, the relative rate of return on currency, held constant. In Corollary 3, the policy acts to decrease the inflation rate π one-for-one with an increase in η, implying no effect on the real rate of return on currency. Also, real interest rates do not change because a decrease in π offsets the decrease in R^m one-for-one (a pure Fisher effect). Notice that, although there are no effects on consumption quantities (x^c, x^d), the fraction of buyers who acquire the theft technology α^b increases in equilibrium and this has welfare implications.

[22]The effect of an increase in η on the sellers' behavior is ambiguous. Although a higher η implies a higher payoff from selling currency in the CM, a higher risk of theft tends to reduce the sellers' expected payoff. Therefore, the effect of an increase in η on the fraction of sellers who carry currency into the CM α^s is ambiguous.

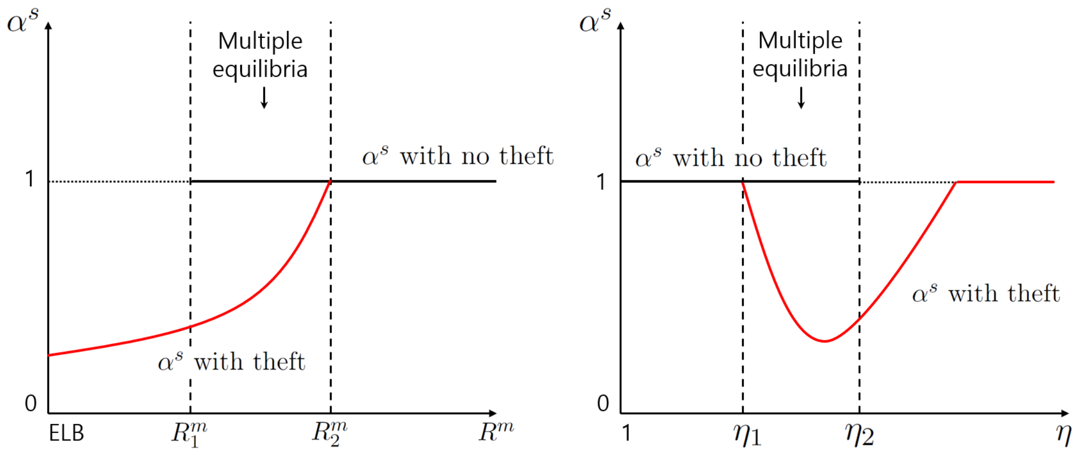

Figure 3: Equilibrium α^s with monetary policy (R^m, η)

With necessary conditions (34) and (42), Propositions 1 and 2 help us understand what type of equilibrium (or equilibria) may arise given monetary policies (R^m, η). With η held constant, an increase in R^m decreases x^c, so an equilibrium with no theft is more likely to arise for high R^m. Suppose η is sufficiently high or the cost of theft κ is sufficiently low, so that there exists theft for sufficiently low R^m. Then, as illustrated in the left panel of Figure 3, there may exist a range of nominal interest rates R^m that support equilibria with theft and with no theft. That is, for some parameters, multiple equilibria arise. In the figure, the upper bound on nominal interest rates that support an equilibrium with theft R_2^m is higher than the lower bound on nominal interest rates that support an equilibrium with no theft R_1^m. So, multiple equilibria exist for $R^m \in [R_1^m, R_2^m]$. However, for some parameters, the upper bound R_2^m can be lower than the lower bound R_1^m, so there does not exist an equilibrium for $R^m \in (R_2^m, R_1^m)$ in that case.

Propositions 1 and 2 also state that, with R^m held constant, an increase in η decreases x^c in an equilibrium with theft and increases x^c in an equilibrium with no theft. This implies that an equilibrium with no theft is more likely to arise for low η and high κ (the cost of theft). Suppose either R^m or κ is sufficiently high so that there is no theft for sufficiently low η. Then, for η sufficienly high but not too high, there may exist multiple equilibria as illustrated in the right panel of Figure 3. Notice that the effect of an increase in η on α^s (the fraction of sellers carrying currency in the TM) is ambiguous in an equilibrium with theft. The right panel of Figure 3 displays a special case where an increase in η decreases α^s for low η and increases α^s for high η in an equilibrium with theft. If η is sufficiently high, it is possible that all currency-holding sellers carry currency in the TM ($\alpha^s = 1$) although some buyers still currency in equilibrium ($0 < \alpha^b < 1$). In this section, I have focused on analyzing an equilibrium with theft where sellers are indifferent between depositing currency and not depositing, i.e., $0 < \alpha^b < 1$ and $0 < \alpha^s < 1$. Finally, Figure 4 shows how the nominal interest rate on reserves R^m and the nonpar exchange rate between currency and reserves η determine the existence of particular equilibria.

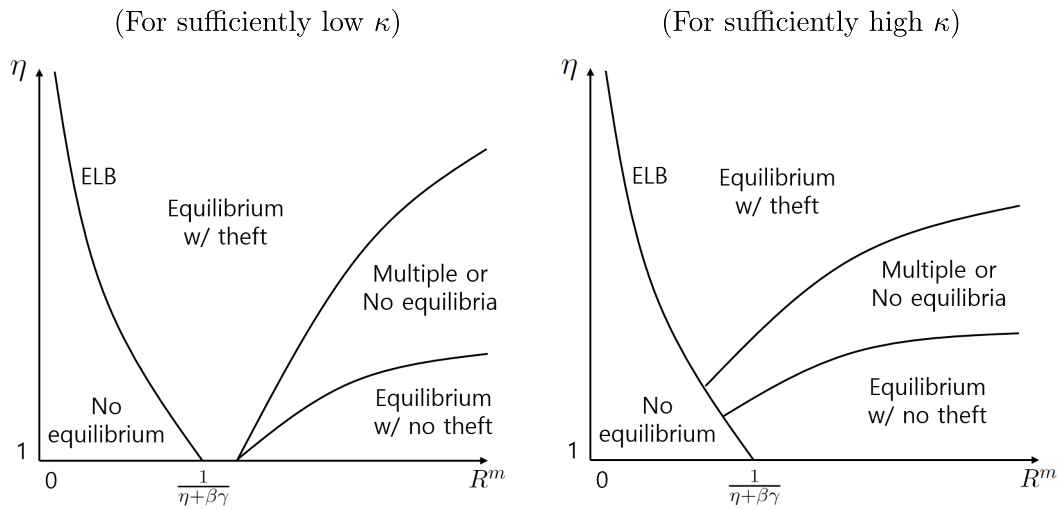

Figure 4: Equilibria with monetary policy (R^m, η)

4 Optimal Monetary Policy

I define welfare as

$$\mathcal{W} = \underbrace{0}_{\text{CM surpluses}} + \underbrace{\rho\left[u(x^c) - x^c + \beta\mu\right] + (1-\rho)\left[u(x^d) - x^d\right]}_{\text{DM surpluses}} - \underbrace{(\rho\beta\mu + \alpha^b\kappa)}_{\text{total cost in TM}}, \tag{44}$$

which is the sum of surpluses from trade in the CM and the DM, net of the total cost incurred in the TM. Welfare defined here is also equivalent to the sum of period utilities in equilibrium. I will discuss the optimal monetary policy using this measure in what follows.

Proposition 3 *If the cost of theft κ is sufficiently low and the fixed cost of holding currency μ is sufficiently close to zero, then the optimal monetary policy consists of $\eta = 1$ and $R^m \leq 1$ for given $\mu \geq 0$. However, if the optimal nominal interest rate on reserves is constrained by the ELB, then the optimal monetary policy is $\eta = 1$ and $R^m = ELB$.*

Proof See Appendix □

To understand the intuition behind the results, consider the case where there is no fixed cost of holding currency at the beginning of the TM ($\mu = 0$) as a benchmark. In Appendix A.2, I provide the proof of Proposition 3 in two steps. Taking α^b (the fraction of buyers who choose to steal) as given, an optimal monetary policy can be characterized by a *modified* Friedman rule. That is, the nominal interest rate on reserves relative to currency is zero (or equivalently, $\eta R^m = 1$) at the optimum. A modified Friedman rule achieves a social optimum, given α^b, by allowing buyers to perfectly smooth their consumption across different states of the world. Then, I argue that, among those policy alternatives, the optimal monetary policy consists of $\eta = 1$ and $R^m = 1$ because this eliminates costly investment in the theft technology.

Optimality is achieved when the exchange rate between currency and reserves is one-to-one, as in a traditional central banking system, and the nominal interest rate on reserves is zero (a Friedman rule). Since a zero nominal interest rate is optimal even though negative interest rates are available, the Friedman rule policy rate can be thought of as the "reversal interest rate", the interest rate at which lowering interest rate becomes contractionary (Brunnermeier and Koby, 2019).

Now, consider the case where there is a fixed cost of storing currency ($\mu > 0$). In this case, a negative nominal interest rate is optimal ($R^m < 1$) given a one-to-one exchange rate $\eta = 1$. As currency-holding buyers need to compensate for the storage cost incurred by sellers, they carry a larger quantity of currency than in an economy with no storage costs ($\mu = 0$). Then, it is welfare-improving to reduce the cost of holding currency by lowering the nominal interest rate R^m and the inflation rate π further from the one characterized by the Friedman rule.

Note that, even if the optimal nominal interest rate is constrained by the ELB, introducing a nonpar exchange rate to further reduce the nominal interest rate does not improve welfare. From Corollary 3, such a policy only increases costly theft without increasing surpluses from trade in the DM and the CM. Therefore, it is optimal to set the nominal interest rate at the ELB in this case.

Proposition 4 *If the cost of theft κ is sufficiently high, then the optimal monetary policy is given by $\eta = \bar{\eta}$ where $\bar{\eta}$ consists of the solution to (31) and (33) for $x^c = \beta(\frac{\kappa}{\rho} - \mu)$. In addition, if the fixed cost of holding currency μ is sufficiently close to zero, then the optimal nominal interest rate on reserves R^m satisfy that $R^m > 1/\bar{\eta}$.*

Proof See Appendix □

Interestingly, if the cost of theft κ is sufficiently high and theft does not take place in equilibrium, the optimal nominal interest rate can be higher than the one satisfying a modified Friedman rule. In Appendix A.2, I show that with a modified Friedman rule, or $\eta R^m = 1$, an increase in the interest rate on reserves R^m or the exchange rate η improves social welfare. An increase in R^m increases the level of welfare for the same reason as in the case with a sufficiently low cost of theft, by smoothing consumption quantities across states. However, increasing η from a modified Friedman rule improves welfare for a nonstandard reason. Note that in an equilibrium with no theft an increase in η does not effectively reduce the nominal rate of return on currency perceived by sellers but does increase the price of currency in the CM. A higher price of currency increases the quantity of consumption in DM meetings using currency, and the resulting increase in the buyer's utility from those DM meetings exceeds the potential decrease in the utility from DM meetings using bank claims. Therefore, a modified Friedman rule does not achieve a social optimum and the optimal monetary policy can be characterized by $\eta R^m > 1$. Also, as the level of welfare increases with η, it is optimal to set the exchange rate at the highest possible level that does not cause theft in equilibrium.

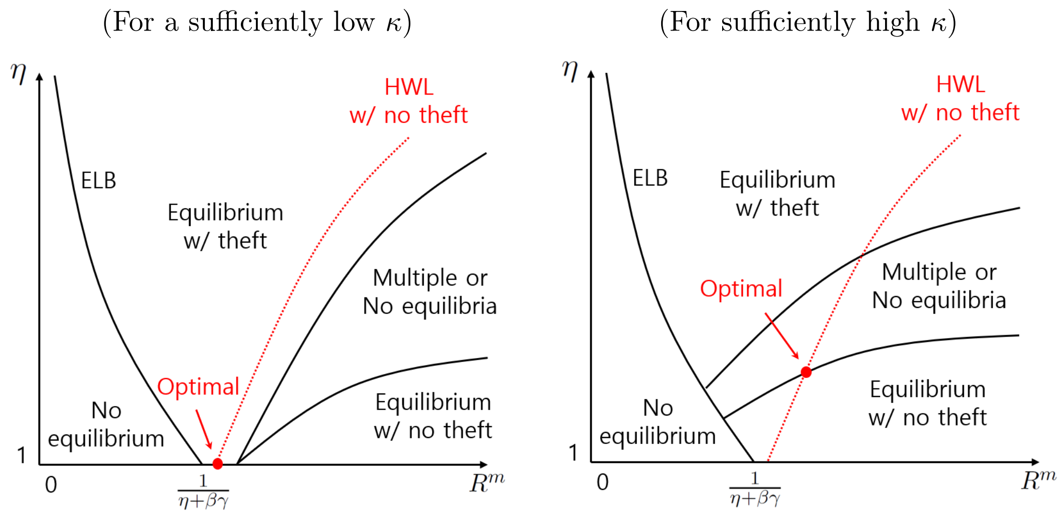

Figure 5: Optimal monetary policy

These results can be illustrated by Figure 5. In the figure, the HWL curve in each panel depicts the locus of nominal interest rates R^m that deliver the highest welfare given an exchange rate η in an equilibrium with no theft. If the cost of theft κ is sufficiently low, then welfare can be maximized at $\eta = 1$ and $R^m < 1$ as in the left panel. In contrast, if κ is sufficiently high, then welfare can be maximized at the highest possible level of η, with the corresponding R^m on the HWL curve, that supports an equilibrium with no theft as a unique equilibrium.

5 Quantitative Analysis

Theoretically, introducing a nonpar exchange rate between currency and reserves can decrease welfare by encouraging costly theft and distorting the equilibrium allocation. To understand the magnitude of the welfare cost, I calibrate the baseline model to the U.S. economy, and conduct a counterfactual analysis to evaluate the welfare cost of introducing a nonpar exchange rate between currency and reserves.

5.1 Calibration

I consider an annual model and assume that the utility function in the DM takes the form $u(x) = \frac{x^{1-\sigma}}{1-\sigma}$. When calibrating the baseline model to data, I exclude the cost of theft κ because a one-to-one exchange rate between currency and reserves implies no theft in the model.[23] Then, there are eight parameters to calibrate: σ (the curvature of DM consumption), β (discount factor), ρ (the fraction of currency transactions in the DM), μ (the fixed cost of storing currency), γ (the proportional cost

[23]To quantify the welfare cost arising from an increase in theft, the cost of theft κ needs to be calibrated. Since theft does not occur in the model given a one-to-one exchange rate, parameter κ must be directly calibrated outside the model. However, to the best of my knowledge, there is no data that allows measuring the cost of theft.

Parameters	Values	Calibration targets	Sources
β	0.96	Standard in literature	
R^m	1.0025	Avg. interest rate on reserves: 0.25%	FRED
γ	0.00	Lowest target range for fed funds rate: 0-0.25%	FRED
σ	0.16	Money demand elasticity (1959-2007): -4.19	FRED
ρ	0.16	Currency to M1 ratio: 17.22%	FRED; Lucas and Nicolini (2015)
v	1.11	Avg. locally-held public debt to GDP: 66.73%	FRED
δ	0.44	Avg. inflation rate: 1.06%	FRED
μ	0.03	Fixed storage cost: 5% of currency payments	Author's assumption

Table 3: Calibration results

of storing currency), δ (the fraction of assets private banks can abscond with), R^m (the nominal interest rate on reserves), and v (the value of government liabilities held by the public).

Table 3 summarizes the calibration results along with the target moments. Most of the target moments are constructed from the U.S. data for 2013-2015. I consider the period 2013-2015 because it is proper to consider a time period when the policy rate was close to zero as the purpose of this exercise is to evaluate the welfare cost of reducing the ELB. Also, key variables such as the nominal interest rate on reserves and domestically-held public debt to GDP were stable during this period.

There are three parameters calibrated externally. The discount factor β is given by $\beta = 0.96$. From Federal Reserve Economic Data (FRED), the nominal interest rate on reserves was 0.25 percent over period 2013-2015 ($R^m = 1.0025$). Finally, the lowest target range for the federal funds rate has been between 0 and 0.25 percent since 1954. Although this does not imply that the proportional cost of storing currency is zero, the proportional cost γ is assumed to be zero for convenience.[24]

Calibrating σ (the curvature of DM consumption) involves matching the elasticity of money demand in the model with the empirical money demand elasticity obtained from the data. Estimating the money demand elasticity requires a longer time-series data, so I choose the time period from 1959 to 2007.[25] Using data on currency in circulation and nominal GDP from FRED, I calculate the currency-to-GDP ratios. Then, the money demand elasticity can be estimated using Moody's AAA corporate bond yields from FRED and the currency-to-GDP ratios, and the estimated elasticity is -4.19.[26]

Then, I jointly calibrate four parameters: the curvature of DM consumption σ, the fraction of currency transactions in the DM ρ, the value of government liabilities held by the public v, the fraction of assets that can be absconded δ, and the fixed cost of storing currency μ. The curvature

[24]The proportional cost of storing currency implies that the ELB on the nominal interest rate can be negative. However, the Federal Reserve might have faced some legal and political issues of implementing negative nominal interest rates. As a negative rate has not been explored in the U.S., it seems difficult to calibrate the proportional cost of storing currency with this model.

[25]In the aftermath of Global Financial Crisis in 2007-2008, the demand for currency has increased possibly due to non-transactional purposes. To calculate the elasticity of money demand only for transactions, I exclude post-crisis data as in Chiu et al. (2022) and Altermatt and Wang (2022), for example.

[26]The interest rate on liquid bonds (e.g., 3-month Treasury Bill rate) can fluctuate due to liquidity premium. I consider AAA corporate bond yield as the nominal interest rate on illiquid bonds and $\frac{\pi}{\beta} - 1$ as its theoretical counterpart.

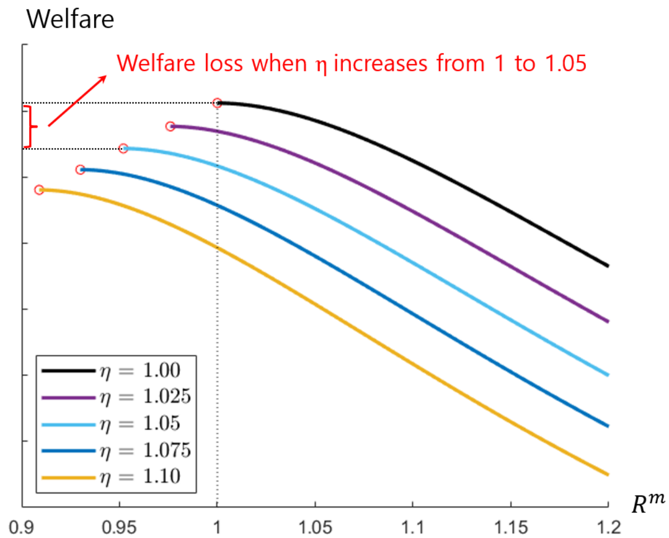

Figure 6: Monetary policy (R^m, η) and welfare

parameter σ is calibrated to match the estimated money demand elasticity. Using the currency-in-circulation data from FRED and the new M1 series from Lucas and Nicolini (2015), I calibrate the fraction of currency transactions in the DM ρ until the model generates the currency-to-M1 ratio. I use domestically-held public debt to GDP from FRED to calibrate the value of publicly-held government liabilities v.[27] Another variable I use to calibrate parameters is the inflation rate. Together with other parameters, the fraction of assets that can be absconded δ is calibrated so that the model generates an inflation rate consistent with the observed rate of 1.06 percent. Finally, I calibrate the fixed cost of storing currency μ to be 5 percent of cash payments.[28]

5.2 Counterfactual Analysis

I consider three different environments where the fixed cost of theft κ is (i) 2.5 percent, (ii) 5 percent, and (iii) 10 percent of the current consumption level. Given the calibrated parameters and each κ, I vary the nonpar exchange rate η and find the corresponding nominal interest rate on reserves that maximizes welfare, denoted by R^*_η. As illustrated by Figure 6, an increase in η decreases the ELB on the nominal interest rate. But, the welfare level under the optimal nominal interest rate R^*_η decreases as η increases.

My approach to quantifying the welfare cost of increasing η is to measure how much consumption private individuals would need to be compensated to endure the nonpar exchange rate η. For any

[27] I define domestically-held public debt by total public debt net of public debt held by foreign and international investors.

[28] This assumption implies that each buyer pays approximately 5 percent more to purchase goods in currency transactions, to compensate the seller's storage cost. Alternatively, it could be interpreted that buyers get around 5 percent of discounts when purchasing goods in bank deposit transactions.

η	ELB	R^*_η	$(\Delta_\eta - 1) \times 100$		
			$\kappa = 2.5\%$	$\kappa = 5\%$	$\kappa = 10\%$
1.00	1.000	1.000	-	-	-
1.025	0.976	0.976	0.0559	0.1118	0.2240
1.05	0.952	0.952	0.1091	0.2183	0.4374
1.075	0.930	0.930	0.1599	0.3198	0.6409
1.10	0.909	0.909	0.2084	0.4168	0.8353

Table 4: ELB, optimal interest rate, and the welfare cost of reducing the ELB

(R^m, η), the welfare measure is given by

$$\mathcal{W}(R^m, \eta) = \rho[u(x^c) - x^c] + (1-\rho)[u(x^d) - x^d] - \alpha^b \kappa.$$

If I choose a nonpar exchange rate η but also adjust the quantities of consumption in the DM by a factor Δ, welfare is expressed as

$$\mathcal{W}_\Delta(R^m = R^*_\eta, \eta) = \rho[u(\Delta x^c) - x^c] + (1-\rho)[u(\Delta x^d) - x^d] - \alpha^b \kappa.$$

Then, I can obtain the value Δ_η that solves $\mathcal{W}_{\Delta_\eta}(R^m = R^*_\eta, \eta > 1) = \mathcal{W}(R^m = R^*_\eta, \eta = 1)$. The welfare cost of introducing η can be measured as $\Delta_\eta - 1$ percent of consumption. If private individuals are compensated with this amount of consumption, they would be indifferent between the two policy choices: a one-to-one exchange rate and a nonpar exchange rate.

Table 4 presents the ELB, the optimal nominal interest rate R^*_η, and the welfare cost of introducing a nonpar exchange rate η given a fixed cost of theft κ. Specifically, an increase in η reduces both the ELB and the optimal interest rate regardless of the cost of theft. Recall that, given a fixed cost of storing currency close to zero ($\mu \approx 0$), the optimal monetary policy can be characterized by a modified Friedman rule ($\eta R^m \approx 1$). So, an increase in η decreases the optimal nominal interest rate R^*_η. As monetary policy is conducted optimally given a nonpar exchange rate η, there would be no distortion in the equilibrium prices and allocations.

Introducing a nonpar exchange rate η, however, increases the aggregate cost of theft in equilibrium. If the fixed cost of theft κ is 2.5 percent of the current consumption level, increasing η by 5 percent and 10 percent costs, respectively, 0.11 percent and 0.21 percent of consumption. If κ is 5 percent of the current consumption level, increasing η by the same magnitudes costs, respectively, 0.22 percent and 0.42 percent of consumption. Finally, if the value of κ is the same as 10 percent of the current consumption level, increasing η by, respectively, 5 percent and 10 percent, costs 0.44 percent and 0.84 percent of consumption.

How large is the welfare cost of introducing a nonpar exchange rate? To better understand its magnitude, the welfare cost computed here can be compared with estimates for the welfare cost of another policy that have been frequently discussed in the literature: the welfare cost of 10 percent inflation. As the estimates for the welfare cost of 10 percent inflation are typically around 1 percent of consumption, the welfare cost of introducing a nonpar exchange rate seems significant.[29] Note that the welfare cost of using a nonpar exchange rate critically depends on the fixed cost of investing in the theft technology κ. As κ increases, the welfare cost also increases proportionally.

6 Disintermediation

In the baseline model, only currency is accepted in some transactions while only bank claims are accepted in other transactions. In this case, no arbitrage conditions from holding currency across periods determine the effective lower bound (ELB) on nominal interest rates, as shown earlier. Another concern about implemening a negative interest rate is a possibility of disintermediation: consumers can choose to withdraw all currency from their deposit accounts.[30] However, as currency and bank deposits are not substitutable, disintemediation does not occur in the baseline model. To understand the implications of introducing a nonpar exchange rate for potential disintermediation, I modify the baseline model by assuming that currency can be acceptable in all DM transactions. Although this is an extreme assumption, it will help us understand how the ELB can be determined to prevent disintermediation.

As it is possible to use only currency in DM transactions, buyers might benefit from opting out of banking arrangements. In the previous sections, I assumed that a fraction $1-\rho$ of sellers accept only bank claims as a means of payment in DM transactions. Here, I will assume that those sellers accept both currency and bank claims, while a fraction ρ of sellers accept only currency as in the baseline model. Other than that, the model is the same as the one described in Section 2.

Let θ denote the fraction of buyers who choose to deposit with private banks in the CM. Each private bank's contracting problem and the first-order conditions for the bank's problem are identical to those in the baseline model. A fraction $1-\theta$ of buyers choose to opt out of banking arrangements and use only currency in DM transactions. Each of these buyers solves

$$\max_{c^o \geq 0} \left\{ -\eta c^o + u\left(\frac{\beta c^o \left[1-\alpha^s+\alpha^s(1-\alpha^b)\eta\right]}{\pi} - \beta\mu \right) \right\}, \tag{45}$$

where c^o is the quantity of currency in real terms. Although private banks do not write a deposit contract with these buyers, I assume that private banks withdraw currency from the central bank to sell it to these buyers whenever necessary. Then, a no arbitrage condition implies that the price of currency is η in equilibrium. The first-order condition for each of these buyer's problem is given

[29] The welfare cost of increasing inflation from 0 percent to 10 percent is 0.62 percent of consumption in Chiu and Molico (2010), 0.87 percent in Lucas Jr (2000), and 1.32 percent (take-it-or-leave-it offer) in Lagos and Wright (2005), for example.

[30] Disintermediation is a practical concern. As a negative deposit rate might lead to massive cash withdrawals, private banks may not want to reduce their deposit rates below zero. See Eggertsson et al. (2019) for empirical evidence on the malfunction of the transmission mechanism of monetary policy at negative territory.

by

$$-\eta + \frac{\beta\left[1-\alpha^s+\alpha^s(1-\alpha^b)\eta\right]}{\pi}u'\left(\frac{\beta c^o\left[1-\alpha^s+\alpha^s(1-\alpha^b)\eta\right]}{\pi}-\beta\mu\right)=0, \tag{46}$$

and let x^o denote the consumption quantity in DM meetings for the buyer. Asset market clearing conditions are given by

$$\theta c+(1-\theta)c^o=\bar{c}; \quad \theta m=\bar{m}; \quad \theta b=\bar{b}. \tag{47}$$

Let U^b denote the expected utility for buyers who write banking contracts and let U^o denote the expected utility for buyers who opt out of banking contracts. Then, using (14), (17)-(19), and (46), U^b and U^o can be written as

$$U^b=\rho\left[u(x^c)-(x^c+\beta\mu)u'(x^c)\right]+(1-\rho)\left[u(x^d)-x^d u'(x^d)\right], \tag{48}$$

$$U^o=u(x^o)-(x^o+\beta\mu)u'(x^o). \tag{49}$$

In equilibrium, the fraction θ must be the solution to the following problem:

$$\max_{0\leq\theta\leq 1}\left[\theta U^b+(1-\theta)U^o\right], \tag{50}$$

which implies that θ must be consistent with each buyer's utility maximization problem.

Depending on who the seller meets in the DM, the quantity of currency can differ across currency-holding sellers in the TM. Some buyers write deposit contracts and withdraw c' units of real currency in equilibrium. This implies that sellers who meet these buyers in the DM hold $\frac{c'}{\pi}$ units of real currency in the following TM. Some buyers use only currency in the DM, so sellers who meet these buyers in the DM hold $\frac{c^o}{\pi}$ units of real currency in the following TM. For convenience, I assume that each seller decides whether to carry currency into the TM before realizing the type of buyer he or she meets in the DM.

The fraction of buyers who acquire the theft technology α^b and the fraction of sellers who choose to participate in side trades of currency α^s must be incentive compatible in equilibrium. Then,

$$\text{if} \quad \kappa>\frac{\left[\theta\rho c'+(1-\theta)c^o\right]\alpha^s\eta}{\pi}, \quad \text{then} \quad \alpha^b=0, \tag{51}$$

$$\text{if} \quad \kappa=\frac{\left[\theta\rho c'+(1-\theta)c^o\right]\alpha^s\eta}{\pi}, \quad \text{then} \quad 0\leq\alpha^b\leq 1, \tag{52}$$

$$\text{if} \quad \kappa<\frac{\left[\theta\rho c'+(1-\theta)c^o\right]\alpha^s\eta}{\pi}, \quad \text{then} \quad \alpha^b=1. \tag{53}$$

With probability $\theta\rho\alpha^s$, each buyer is matched with a seller holding $\frac{c'}{\pi}$ units of real currency and with probability $(1-\theta)\alpha^s$ each buyer is matched with a seller holding $\frac{c^o}{\pi}$ units of real currency. Conditions (51)-(53) state that theft does not occur if theft is too costly, theft sometimes takes place if the buyer is indifferent between stealing and not stealing, and theft always takes place if theft is profitable. Also, each seller's decision on whether to carry currency into the TM must be incentive compatible. That is,

$$\text{if} \quad (1-\alpha^b)\eta < 1, \quad \text{then} \quad \alpha^s = 0, \tag{54}$$

$$\text{if} \quad (1-\alpha^b)\eta = 1, \quad \text{then} \quad 0 \leq \alpha^s \leq 1, \tag{55}$$

$$\text{if} \quad (1-\alpha^b)\eta > 1, \quad \text{then} \quad \alpha^s = 1. \tag{56}$$

In this section, I confine attention to an equilibrium with a sufficiently low cost of theft κ, implying that theft takes place in equilibrium.[31] Also, notice that I have focused on cases where collateralized assets are sufficiently scarce. Specifically in this section, I will assume that

$$v < x^* + \beta\mu, \tag{57}$$

where x^* is the efficient quantity of consumption in DM transactions that solves $u'(x) = 1$. This assumption implies that the consumption quantities in DM transactions for buyers choosing bank contracts are inefficiently low due to the shortage of collateral. But, this also implies that the central bank cannot support the efficient quantity of consumption for those opting out of banking contracts. This is because the quantity of currency outstanding is constrained by the size of the central bank's balance sheet, which can only be increased by purchasing scarce government debt. Then, the following proposition characterizes the effective lower bound (ELB) on nominal interest rates R^m.

Proposition 5 *Suppose that both the cost of theft κ and the fixed cost of holding currency μ are sufficiently low, and that the value of consolidated government debt outstanding v satisfies (57). Then, an equilibrium exists if and only if*

$$R^m \geq \frac{u'(x^o)}{\eta[(1-\delta)u'(x^d)+\delta]}, \tag{58}$$

where (x^o, x^d), together with x^c, are the solutions to $(x^o+\beta\mu)u'(x^o) = v$, $u'(x^o) = u'(x^c) - \delta u'(x^d) + \delta$, and $U^b = U^o$ from (48)-(49). Furthermore,

$$\frac{u'(x^o)}{(1-\delta)u'(x^d)+\delta} > 1.$$

Proof See Appendix $\square$

Proposition 5 shows that nominal interest rates R^m below some threshold cannot be supported in equilibrium. This is because a sufficiently low R^m induces buyers to use only currency in DM

[31] I can provide equilibrium conditions given a sufficiently high cost of theft, but analyzing the effects of monetary policy in that case appears not to be straightforward.

transactions but there is a shortage of government debt to back the required quantity of currency. That is, the central bank cannot issue the required quantity of currency to meet the public demand, if R^m is sufficiently low. The right-hand side of (58) can be interpreted as the ELB on nominal interest rates. Therefore, introducing a nonpar exchange rate $\eta > 1$ can reduce the ELB as in Eisler (1932), Buiter (2010), and Agarwal and Kimball (2015).

The proportional cost of storing currency γ becomes irrelevant here as nominal interest rates that effectively encourage buyers to participate in banking arrangements are sufficiently high to prevent arbitrage opportunities from storing currency across periods. Interestingly, the ELB on nominal interest rates can be positive given a one-to-one exchange rate ($\eta = 1$). When the nominal interest rate on reserves is zero ($R^m = 1$), there is an incentive for buyers to opt out of deposit contracts because $x^o > x^c = x^d$. That is, there is inefficiency in the banking system due to a shortage of government debt and a binding collateral constaint, leading to lower consumption in the DM for contracting buyers $x^c = x^d$ than noncontracting buyers x^o. As a complete flight to currency cannot be supported in equilibrium, the nominal interest rate must be higher than zero to prevent buyers from opting out of deposit contracts.

From now on, I will consider cases where $R^m > \frac{u'(x^o)}{\eta[u'(x^d)-\delta u'(x^d)+\delta]}$ and assume that $\mu = 0$ for analytical convenience.[32] That is, there is no fixed cost of holding currency at the beginning of the TM. Then, from (6), (17)-(22), and (46)-(47), I obtain

$$\eta R^m = \frac{u'(x^c) - \delta u'(x^d) + \delta}{u'(x^d) - \delta u'(x^d) + \delta}, \tag{59}$$

$$(1-\rho)\theta x^d \left[u'(x^d) + \frac{\delta}{1-\delta}\right] + \rho\theta(x^c)\left[u'(x^c) + \frac{\delta}{1-\delta}\right] + (1-\theta)(x^o)u'(x^o) = v, \tag{60}$$

$$\pi = \frac{\beta\left[u'(x^c) - \delta u'(x^d) + \delta\right]}{\eta}. \tag{61}$$

Equations (59)-(61) come from the first-order conditions for a private bank's problem in equilibrium and equation (60) is the collateral constraint. The collateral constraint here is somewhat different from (39), the collateral constraint in the baseline model, as some government debt must be additionally held by the central bank to back its liabilities—currency held by buyers who opt out of banking arrangements. From (17), (18), (46), and (48)-(50),

$$u'(x^o) = u'(x^c) - \delta u'(x^d) + \delta, \tag{62}$$

$$\rho\left[u(x^c) - x^c u'(x^c)\right] + (1-\rho)\left[u(x^d) - x^d u'(x^d)\right] \geq u(x^o) - x^o u'(x^o). \tag{63}$$

Equation (62) is a necessary condition that guarantees positive quantities of consumption in DM transactions for both types of buyers (buyers who choose to participate in banking arrangements and buyers who choose not to do so). This equation implies that $x^c < x^o < x^d$ in equilibrium. Notice that there is inefficiency in deposit contracts arising from the shortage of collateral, which

[32] The results obtained here can be applied to cases with a sufficiently low μ.

not only constrains x^d but also constrains x^c. So, bank deposit contracts are useful only when $x^c < x^d$ because otherwise buyers would strictly prefer opting out of deposit contracts.[33] Equation (63) shows that, in equilibrium, buyers must weakly prefer participating in a banking arrangement to not participating. Finally, from (52) and (55),

$$\alpha^b = \frac{\eta - 1}{\eta}, \tag{64}$$

$$\alpha^s = \frac{\beta\kappa}{\eta\left[\theta\rho x^c + (1-\theta)x^o\right]}, \tag{65}$$

$$\kappa < \frac{\eta\left[\theta\rho x^c + (1-\theta)x^o\right]}{\beta}. \tag{66}$$

Equations (64) and (65) determine the fraction of buyers who choose to acquire the theft technology α^b and the fraction of sellers who choose to carry currency in the TM α^s. Inequality (66) is a necessary condition for this equilibrium to exist.

I can solve the model differently depending on whether (63) holds with equality. If (63) holds with equality, equations (59), (62), and (63) solve for x^c, x^d, and x^o given monetary policy (R^m, η) and fiscal policy v. Then, equation (60) solves for θ, equation (61) solves for π, and equations (64) and (65) solve for α^b and α^s, respectively. If (63) holds with strict inequality, equations (59) and (60) with $\theta = 0$ solve for x^c and x^d. Then, equation (61) solves for π and equations (64) and (65) solve for α^b and α^s, respectively, with $\theta = 0$. Equation (62) solves for x^o, the would-be quantity of consumption in DM transactions if, off equilibrium, a buyer were to opt out of banking arrangements.

The following proposition shows how the fraction θ is determined in equilibrium and the effects of monetary policy (R^m, η) depending on the value of θ.

Proposition 6 *Suppose that the fixed cost of holding currency μ is zero. If ηR^m is sufficiently high but not too high, then $0 \le \theta < 1$ in equilibrium and (63) holds with equality. In this case, an increase in ηR^m (an increase in R^m or η or both) leads to decreases in x^c and x^o and increases in x^d and θ along with an increase in real interest rates (r^m, r^b). Furthermore, an increase in R^m increases π and α^s but does not affect α^b. An increase in η decreases π and increases α^b but its effect on α^s is ambiguous. If ηR^m is very high, then $\theta = 1$ in equilibrium and (63) holds with strict inequality.*

Proof See Appendix $\square$

In Proposition 6, a higher nominal interest rate R^m or exchange rate η implies a lower rate of return on currency relative to reserves and government bonds, leading to a substitution of bank claims for currency in DM transactions. If $0 \le \theta < 1$ in equilibrium, the consumption quantity

[33] However, using currency in transactions is also inefficient due to the fixed cost of holding currency μ. With a sufficiently high μ, buyers could choose to use deposit contracts even though $x^d \le x^c$.

in DM transactions using bank claims x^d increases along with a rise in the fraction of buyers who participate in banking arrangements θ, while the consumption quantities in DM transactions using currency x^c and x^o decrease. So, given a traditional central banking system with $\eta = 1$, lowering the nominal interest rate R^m can contribute to disintermediation in that the fraction of buyers participating in banking arrangements θ falls.

The central bank, however, can introduce an appropriate nonpar exchange rate between currency and reserves η that helps hold the relative rate of return on currency $\frac{1}{\eta R^m}$ constant. This implies that the central bank can implement a negative nominal interest rate without causing a disruptive effect on the banking system, consistent with the conventional view, since the fraction θ as well as x^c, x^d, and x^o would remain unchanged with ηR^m held constant. But, the fraction of buyers who invest in the costly theft technology α^b would increase, which has welfare implications.

I can define a welfare measure for this economy as

$$\mathcal{W} = \rho\theta \left[u\left(x^c\right) - x^c\right] + \left(1-\rho\right)\theta[u(x^d) - x^d] + \left(1-\theta\right)\left[u\left(x^o\right) - x^o\right] - \alpha^b\kappa, \tag{67}$$

which is the sum of surpluses from trade in the CM and the DM, net of the total cost of theft in the TM. Then, the following proposition characterizes the optimal monetary policy.

Proposition 7 *Suppose that the fixed cost of holding currency μ is zero. The optimal monetary policy consists of $\eta = 1$ and $R^m = \frac{u'(x^o)}{u'(x^d) - \delta u'(x^d) + \delta} > 1$, where (x^o, x^d), together with x^c, are the solutions to $x^o u'(x^o) = v$, (62), and (63) with equality.*

Proof See Appendix □

Setting a one-to-one exchange rate between currency and reserves ($\eta = 1$), as in a traditional central banking system, is optimal because the central bank can eliminate costly theft without reducing welfare. The central bank can avoid a reduction in welfare by choosing an appropriate policy (η, R^m). Also, optimality is obtained when the central bank sets the nominal interest rate on reserves R^m at the ELB. As the ELB is higher than one, the quantity of consumption in DM transactions using bank claims is higher than the quantities in other DM transactions using currency for any given R^m. So, lowering the nominal interest rate always improves welfare as it allows buyers to better smooth their consumption across DM transactions.

Which model specification would represent reality better, the baseline model or the modified one? In the baseline model, I assume that only currency is used in some transactions while only bank claims are used in other transactions. In contrast, in the modified model, currency is accepted everywhere while bank deposits can be accepted only in some transactions. In practice, some transactions including online transactions cannot be made with currency. In contrast, some transactions can be made only with currency either because the consumer or the retailer does not have access to banking system or because they value privacy. Also, there are transactions where both means of payment can be accepted. Therefore, reality may be somewhere between the two versions of the model.

7 Conclusion

In the literature, a nonpar exchange rate between currency and reserves has been proposed as a potential policy instrument that could reduce the effective lower bound (ELB) on nominal interest rates. I have constructed a model with two means of payment, currency and bank deposits, and frictions associated with the storage and transportation of currency, to study the implications of introducing a nonpar exchange rate. A key finding is that a nonpar exchange rate can indeed reduce the ELB on nominal interest rates if there exist sufficient frictions that induce agents to exchange currency and reserves rather than avoiding the central bank.

Introducing a nonpar exchange rate, however, can be costly because lowering the ELB must be accompanied by an enhancement in the frictions that determine the ELB. In particular, a nonpar exchange rate can increase the market value of currency, and thus encourage socially undesirable behavior and decrease welfare. Even if the optimal interest rate is constrained by the ELB, a nonpar exchange rate does not help increase welfare because introducing a nonpar exchange rate leads to a fall in the optimal interest rate. As this only increases the resource cost to support such undesirable activities, the optimal monetary policy is to set the nominal interest rate at the ELB and maintain the one-to-one exchange rate between currency and reserves. With a modified version of the model, I have also shown that a nonpar exchange rate can help the central bank implement a negative interest rate without causing disintermediation, although this can decrease welfare.

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A Appendix

A.1 Equilibrium with Deflation and No Theft

In this section, I confine attention to stationary equilibria where there is deflation and the cost of theft is sufficiently high. Due to deflation, the real value of currency increases over time and private banks can acquire a sufficient amount of currency in the CM. Private banks are no longer indifferent between acquiring currency from other private individuals and withdrawing currency from the central bank because they do not need to bear the cost of withdrawing currency. Instead, sellers must be indifferent between depositing their currency with the central bank and side trading with private banks. So, one unit of real currency must be exchanged for one unit of good in the CM in equilibrium. From (7) and (28), a necessary condition for theft to not take place is given by

$$\kappa \geq \frac{\rho(x^c + \beta\mu)}{\beta}.$$

As the price of real currency is one instead of η in equilibrium, η's in equations (6)-(12), (17)-(22) must be replaced by one. Then, from (19) and (21), no arbitrage condition for private banks from holding currency across periods can be expressed by

$$R^m \geq \frac{1}{1+\beta\gamma}.$$

That is, the nonpar exchange rate η is irrelevant to the effective lower bound (ELB). This occurs because there are no currency withdrawals from the central bank cash window. Also, it is obvious that the nonpar exchange rate does not affect equilibrium prices and allocations.

The inflation rate in this equilibrium can be written as

$$\pi = \beta \left[u'(x^c) - \delta u'(x^d) + \delta \right].$$

To observe deflation in equilibrium, β and $u'(x^c)$ must be sufficiently low while δ and $u'(x^d)$ must be sufficiently high. It turns out that v must be sufficiently high and R^m is sufficiently low to support deflation in equilibrium. Note that in the body of the paper I focus on cases with sufficiently low v and γ so that there is no deflation in equilibrium for any R^m that is higher than the ELB.

A.2 Omitted Proofs

Proof of Lemma 1: First, consider the case where $\eta = 1$. In this case, sellers are indifferent between depositing the currency with the central bank and trading it with a private bank only if there is no theft, i.e., $\alpha^b = 0$. Also, $\alpha^s = 0$ is optimal for all $\alpha^b \in (0, 1]$. Suppose that $\alpha^b \in (0, 1]$ in equilibrium. Then, sellers would always choose to deposit their currency with the central bank, implying $\alpha^s = 0$. Then, there would be no incentives for buyers to acquire the theft technology by incurring κ units of labor, which contradicts with $\alpha^b \in (0, 1]$. Therefore, there must be no theft in equilibrium, if an equilibrium exists. Now, suppose that $\alpha^b = 0$ in equilibrium. A necessary condition for this equilibrium to exist is $\kappa \geq \rho\alpha^s\eta c^s$. If $\kappa > \rho\eta c^s$, then $\alpha^b = 0$ is optimal for buyers for any given $\alpha^s \in [0,1]$. If $\kappa = \rho\eta c^s$, buyers are indifferent between acquiring the theft technology and not doing anything in the TM. In this case, an equilibrium exists only if $\alpha^b = 0$. If $\kappa < \rho\eta c^s$, then equilibrium exists only if the fraction of sellers who carry currency in the TM is sufficiently low. Since a necessary condition for the absence of theft is $\kappa \geq \rho\alpha^s\eta c^s$, an equilibrium exists with $\alpha^s \in [0, \bar{\alpha}^s]$ where $\bar{\alpha}^s = \frac{\kappa}{\rho\eta c^s}$. Therefore, given that $\eta = 1$, there exist a continuum of equilibria with $\alpha^b = 0$ and $\alpha^s \in [0,1]$ for $\kappa \geq \rho\eta c^s$ and a continuum of equilibria with $\alpha^b = 0$ and $\alpha^s \in [0, \bar{\alpha}^s]$ for $\kappa < \rho\eta c^s$ where $\bar{\alpha}^s = \frac{\kappa}{\rho\eta c^s}$.

Next, I consider the case where $\eta > 1$. Suppose that $\alpha^b = 0$ in equilibrium. Then, this leads to $\alpha^s = 1$ as sellers would strictly prefer to trade currency with a private bank rather than depositing it with the central bank. A necessary condition for this equilibrium to exist is $\kappa \geq \rho\eta c^s$. Therefore, a unique equilibrium exists with $\alpha^b = 0$ and $\alpha^s = 1$ for $\kappa \geq \rho\eta c^s$. If $\kappa < \rho\eta c^s$, then $\alpha^b = 0$ cannot be supported in equilibrium as there are incentives for buyers to acquire the theft technology, given that $\alpha^s = 1$. However, $\alpha^b = 1$ cannot be an equilibrium as well because $\alpha^b = 1$ would lead to $\alpha^s = 0$, and then, there would be no incentives for buyers to acquire the theft technology. An equilibrium exists if and only if buyers and sellers are both indifferent between their own options. This implies that, from (8) and (11),

$$\alpha^b = \frac{\eta - 1}{\eta}, \tag{68}$$

$$\alpha^s = \frac{\kappa}{\rho\eta c^s}. \tag{69}$$

Therefore, there exists a unique equilibrium with (68) and (69) for $\kappa < \rho\eta c^s$.

Proof of Proposition 1: Note that, in equilibrium, equations (31) and (33) solve for x^c and x^d. Confine attention to the comparative statics analysis with respect to R^m. I totally differentiate (31) and (33) and evaluate the derivatives of x^c and x^d for $(R^m, \eta) = (1,1)$ and $\mu = 0$ to obtain

$$\frac{dx^c}{dR^m} = \frac{(1-\rho)[(1-\delta)u'(x)+\delta][(1-\delta)(1-\sigma)u'(x)+\delta]}{u''(x)[(1-\delta)(1-\sigma)u'(x)+\delta+\rho\beta\mu(1-\delta)u''(x)]} < 0,$$
$$\frac{dx^d}{dR^m} = \frac{-\rho[(1-\delta)u'(x)+\delta][(1-\delta)(1-\sigma)u'(x)+\delta+\beta\mu(1-\delta)u''(x)]}{u''(x)[(1-\delta)(1-\sigma)u'(x)+\delta+\rho\beta\mu(1-\delta)u''(x)]} > 0,$$

where $x^c = x^d = x$. Then, it is immediate that from (30) and (32) r^m, r^b, and π increase, and from the first argument in (35) the ELB remains unchanged.

Now, I turn my attention to the comparative statics analysis with respect to η. For convenience, let $\sigma = -\frac{xu''(x)}{u'(x)}$ so that $\sigma \in (0,1)$. Then, evaluating the derivatives of x^c and x^d with respect to η for $(R^m, \eta) = (1,1)$ and $\mu = 0$ yields

$$\frac{dx^c}{d\eta} = -\frac{\delta\{\rho\sigma u'(x) + (1-\rho)[(1-\delta)(1-\sigma)u'(x) + \delta][u'(x) - 1] - \beta\mu\rho u''(x)\}}{u''(x)[(1-\delta)(1-\sigma)u'(x) + \delta + \rho\beta\mu(1-\delta)u''(x)]} > 0,$$
$$\frac{dx^d}{d\eta} = \frac{\rho\delta[(1-\sigma)u'(x) - 1 + \beta\mu u''(x)][(1-\delta)u'(x) + \delta]}{u''(x)[(1-\delta)(1-\sigma)u'(x) + \delta + \rho\beta\mu(1-\delta)u''(x)]}.$$

Also, I evaluate equation (33) for $(R^m, \eta) = (1, 1)$ and $\mu = 0$ to obtain

$$\left[u'(x) + \frac{\delta}{1-\delta}\right](x + \rho\beta\mu) = v, \tag{70}$$

where x is increasing in v. Collateral constraint (70) does not bind in equilibrium if

$$v \geq \frac{x^* + \rho\beta\mu}{1-\delta}. \tag{71}$$

Let $\bar{v}$ denote the right-hand side of inequality (71), $\hat{x}$ denote the solution to $u'(x) = \frac{1}{1-\sigma}$, and $\hat{v}$ denote the solution to (70) when $x = \hat{x}$. Then, I can write the derivatives of x^d with respect to η as

$$\frac{dx^d}{d\eta} \leq 0, \qquad \text{if} \quad v \in (0, \hat{v}]$$
$$\frac{dx^d}{d\eta} > 0. \qquad \text{if} \quad v \in (\hat{v}, \bar{v})$$

So, from (30), an increase in η decrease r^m and r^b for $v \in (0, \hat{v}]$ and increase r^m and r^b for $v \in (\hat{v}, \bar{v})$. From (32) or

$$\pi = \beta R^m \left[u'(x^d) - \delta u'(x^d) + \delta\right],$$

an increase in η increases π for $v \in (0, \hat{v}]$ and decreases π for $v \in (\hat{v}, \bar{v})$. Finally, from the first argument in (35), the ELB falls.

Proof of Proposition 2: Notice that the consumption quantities in the DM, x^c and x^d, are determined by equations (37) and (39). I can rewrite equation (39) in the form

$$F(x^c, x^d) = v, \tag{72}$$

and show that the function $F(\cdot, \cdot)$ is strictly increasing in both $0 \leq x^c < x^*$ and $0 \leq x^d < x^*$ because $-x\frac{u''(x)}{u'(x)} < 1$. This property implies that equation (72) can be depicted by a downward-sloping locus in (x^c, x^d) space, given v. Also, equation (37) can be depicted by an upward-sloping locus in (x^c, x^d) space, given (R^m, η).

For the comparative statics, suppose there is an increase in R^m with η held constant. It is straightforward that an increase in R^m decreases x^c and increases x^d from (37) and (39). Then, from (36) and (38), π rises and real interest rates (r^m, r^b) rise. From (27), (40), and (41), α^s increases but α^b and the ELB do not change. Next, suppose that there is an increase in η with

R^m remaining constant. Then, from (37) and (39), x^c decreases and x^d increases. From (38), real interest rates (r^m, r^b) rise. Using (37), (36) can be written as

$$\pi = \beta R^m \left[u'(x^d) - \delta u'(x^d) + \delta \right],$$

so π falls. From (27) and (40), α^b increases and the ELB falls. However, from (41), the effect on α^s is ambiguous since ηx^c can increase or decrease depending on parameters.

Proof of Proposition 3: The proof involves two steps. First, I will search for monetary policies that maximize the welfare measure $\mathcal{W}$, taking α^b as exogenously given. Then, I will determine the optimal monetary policy considering that α^b is endogenously determined in response to a change in monetary policy.

In the first step, I solve the following maximization problem given $\alpha^b \in [0, 1]$:

$$\max_{(R^m, \eta)} \rho \left[u\left(x^c\right) - x^c \right] + (1-\rho) \left[u\left(x^d\right) - x^d \right] - \alpha^b \kappa \tag{73}$$

subject to

$$\eta R^m = \frac{u'\left(x^c\right) - \delta u'\left(x^d\right) + \delta}{u'\left(x^d\right) - \delta u'\left(x^d\right) + \delta}, \tag{74}$$

$$\left[u'(x^c) + \frac{\delta}{1-\delta} \right] \rho(x^c + \beta\mu) + \left[u'\left(x^d\right) + \frac{\delta}{1-\delta} \right] (1-\rho) x^d = v, \tag{75}$$

$$R^m \geq \frac{1}{\eta + \beta\gamma}, \quad \eta \geq 1 \tag{76}$$

Note that a monetary policy measure that is relevant to welfare in equilibrium is ηR^m. Let $\Omega \equiv \eta R^m$ denote the policy measure. Then, from (76), Ω must satisfy that $\Omega \geq \frac{\eta}{\eta+\beta\gamma}$. Differentiating the objective (73) with respect to Ω gives

$$\frac{d\mathcal{W}}{d\Omega} = \rho \left[u'(x^c) - 1 \right] \frac{dx^c}{d\Omega} + (1-\rho) \left[u'(x^d) - 1 \right] \frac{dx^d}{d\Omega}. \tag{77}$$

Let $\sigma = -\frac{x u''(x)}{u'(x)}$. Then, from totally differentiating (74) and (75), I obtain

$$\frac{dx^c}{d\Omega} = \frac{(1-\rho) \left[(1-\sigma) u'(x^d) + \frac{\delta}{1-\delta} \right] \left[(1-\delta) u'(x^d) + \delta \right]}{\Phi} < 0, \tag{78}$$

$$\frac{dx^d}{d\Omega} = \frac{-\rho \left[(1-\sigma) u'(x^c) + \frac{\delta}{1-\delta} + \beta\mu u''(x^c) \right] \left[(1-\delta) u'(x^d) + \delta \right]}{\Phi} > 0, \tag{79}$$

where

$$\Phi = (1-\rho)u''(x^c)\left[(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}\right] \\ + \rho u''(x^d)\left[(1-\delta)\Omega + \delta\right]\left[(1-\sigma)u'(x^c) + \frac{\delta}{1-\delta} + \beta\mu u''(x^c)\right] < 0,$$

for a sufficiently low μ. Note that a monetary policy Ω attains a local optimum if the resulting consumption allocation x^c and x^d satisfy $\frac{d\mathcal{W}}{d\Omega} = 0$. From (77)-(79), I can characterize the optimal allocation x^c and x^d as follows:

$$\begin{aligned}
&\frac{d\mathcal{W}}{d\Omega} = 0, \\
&\Leftrightarrow \left[u'(x^c) - 1\right]\left[(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}\right] - \left[u'(x^d) - 1\right]\left[(1-\sigma)u'(x^c) + \frac{\delta}{1-\delta} + \beta\mu u''(x^c)\right] = 0, \\
&\Rightarrow \left[u'(x^c) - 1\right]\left[(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}\right] \leq \left[u'(x^d) - 1\right]\left[(1-\sigma)u'(x^c) + \frac{\delta}{1-\delta}\right], \\
&\Leftrightarrow \frac{u'(x^c) - 1}{(1-\sigma)u'(x^c) + \frac{\delta}{1-\delta}} \leq \frac{u'(x^d) - 1}{(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}}.
\end{aligned}$$

Since the function $F(x) = \frac{u'(x)-1}{(1-\sigma)u'(x)+\frac{\delta}{1-\delta}}$ is strictly decreasing in x, the above inequality is equivalent to $x^c \geq x^d$. Note that $x^c = x^d$ if $\Omega = 1$ and that x^c decreases and x^d increases as Ω rises. Therefore, the optimal monetary policy Ω must satisfy $\Omega \leq 1$ where the inequality holds with equality if and only if $\mu = 0$.

From the first step, I have shown that an optimal monetary policy must be a combination of (R^m, η) such that $\eta R^m \leq 1$. All the optimal combinations of monetary policy lead to the same gains from trade in the DM, that is, $\rho\left[u\left(x^c\right) - x^c\right] + (1-\rho)\left[u\left(x^d\right) - x^d\right]$. However, from (40), the fraction of buyers who choose to steal currency in the TM α^b increases as the exchange rate η rises. This implies that the welfare measure $\mathcal{W}$ is maximized if and only if $\eta = 1$. Therefore, the optimal monetary policy is given by $\eta = 1$ and $R^m \leq 1$.

Proof of Proposition 4: Suppose that the cost of theft κ is sufficiently high. Then, there is no theft or $\alpha^b = 0$ in equilibrium. To show that social welfare is increasing in η, suppose that the central bank sets the nominal interest rate on reserves R^m to obtain $x^c = x^d = x$ given an exchange rate between currency and reserves η. Such policy needs not be optimal but it helps understand the optimal level of the exchange rate η. Equations (31) and (33) can be rewritten as

$$R^m = R^b = \frac{\eta u'(x) - \delta u'(x) + \delta}{\eta\left[u'(x) - \delta u'(x) + \delta\right]}, \tag{80}$$

$$u'(x)\left[x + \rho\beta\mu\right] + \frac{\left[\rho + (1-\rho)\eta\right]\delta x + \rho\delta\beta\mu}{(1-\delta)\eta} = v. \tag{81}$$

In this case, equation (81) solves for x and then equation (80) solves for R^m. If the value of the consolidated government debt is sufficiently low, or

$$v \leq \frac{[(1-\delta\rho)\eta + \delta\rho]\, x^* + \rho\beta\mu\, [(1-\delta)\eta + \delta]}{(1-\delta)\eta},$$

then x increases with η for a sufficiently low μ. Since the level of welfare is given by $\mathcal{W} = u(x) - x$ in this equilibrium, an increase in η effectively increases the level of welfare as long as the nominal interest rate R^m can be chosen to achieve $x^c = x^d = x$. But, from Proposition 1, an increase in η must be accompanied by an increase in R^m to attain the same consumption quantities across two types of DM transactions, implying that choosing R^m is not constrained by the ELB. Although the optimal R^m may not satisfy $x^c = x^d$, that the social welfare is increasing in η remains unchanged. Finally, η must be sufficiently low so that buyers do not have incentives to steal currency. Therefore, at the optimum, η is chosen so that buyers are indifferent between stealing currency and not stealing.

Now, suppose that there is no fixed cost of holding currency at the beginning of the TM, i.e., $\mu = 0$. Consider the following maximization problem given $\eta \geq 1$:

$$\max_{(R^m,\eta)} \rho\left[u\left(x^c\right) - x^c\right] + (1-\rho)\left[u\left(x^d\right) - x^d\right] \tag{82}$$

subject to

$$\eta R^m = \frac{\eta u'\left(x^c\right) - \delta u'\left(x^d\right) + \delta}{u'\left(x^d\right) - \delta u'\left(x^d\right) + \delta}, \tag{83}$$

$$\left[u'(x^c) + \frac{\delta}{(1-\delta)\eta}\right]\rho x^c + \left[u'\left(x^d\right) + \frac{\delta}{1-\delta}\right](1-\rho)x^d = v, \tag{84}$$

$$R^m \geq \max\left\{\frac{1}{\eta+\beta\gamma}, \frac{\eta}{(\eta+\beta\gamma)[(1-\delta)u'(x^d)+\delta]}\right\}. \tag{85}$$

I differentiate the objective (82) with respect to R^m to obtain

$$\frac{d\mathcal{W}}{dR^m} = \rho\left[u'(x^c) - 1\right]\frac{dx^c}{dR^m} + (1-\rho)\left[u'(x^d) - 1\right]\frac{dx^d}{dR^m}. \tag{86}$$

Letting $\sigma = -\frac{xu''(x)}{u'(x)}$ and totally differentiating (83) and (84) gives

$$\frac{dx^c}{dR^m} = \frac{\eta(1-\rho)\left[(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}\right]\left[(1-\delta)u'(x^d) + \delta\right]}{\Lambda} < 0,$$

$$\frac{dx^d}{dR^m} = \frac{-\eta\rho\left[(1-\sigma)u'(x^c) + \frac{\delta}{(1-\delta)\eta}\right]\left[(1-\delta)u'(x^d) + \delta\right]}{\Lambda} > 0,$$

where

$$\Lambda = (1-\rho)\eta u''(x^c)\left[(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}\right] \\ + \rho u''(x^d)\left[(1-\sigma)u'(x^c) + \frac{\delta}{(1-\delta)\eta}\right][\eta R^m(1-\delta)+\delta] < 0.$$

Then, I can evaluate the derivative of $\mathcal{W}$ or equation (86) for $\eta R^m = 1$. Noting that $\eta u'(x^c) = u'(x^d)$ from (83), I obtain

$$\left.\frac{d\mathcal{W}}{dR^m}\right|_{\eta R^m=1} = \frac{\eta(1-\eta)\rho(1-\rho)\left[(1-\delta)u'(x^d)+\delta\right]}{\eta^2(1-\rho)u''(x^c)+\rho u''(x^d)} \geq 0, \tag{87}$$

implying that $\eta R^m \geq 1$ at an optimum.

Next, differentiate the objective (82) with respect to η to obtain

$$\frac{d\mathcal{W}}{d\eta} = \rho\left[u'(x^c)-1\right]\frac{dx^c}{d\eta} + (1-\rho)\left[u'(x^d)-1\right]\frac{dx^d}{d\eta}. \tag{88}$$

Totally differentiate (83) and (84) to get $\frac{dx^c}{d\eta}$ and $\frac{dx^d}{d\eta}$ and then evaluate the derivatives for $\eta R^m = 1$. This gives

$$\frac{dx^c}{d\eta} = \frac{\delta\rho x^c u''(x^d) - \delta\eta(1-\rho)\left[u'(x^d)-1\right]\left[(1-\delta)(1-\sigma)u'(x^d)+\delta\right]}{\eta\left[(1-\delta)(1-\sigma)u'(x^d)+\delta\right]\left[\rho u''(x^d)+\eta^2(1-\rho)u''(x^c)\right]} > 0,$$
$$\frac{dx^d}{d\eta} = \frac{\delta\rho\left\{\eta x^c u''(x^c) + \left[u'(x^d)-1\right]\left[(1-\delta)(1-\sigma)u'(x^d)+\delta\right]\right\}}{\eta\left[(1-\delta)(1-\sigma)u'(x^d)+\delta\right]\left[\rho u''(x^d)+\eta^2(1-\rho)u''(x^c)\right]}.$$

Using (88), I obtain

$$\left.\frac{d\mathcal{W}}{d\eta}\right|_{\eta R^m=1} = \frac{\mathit{\Gamma} + \rho\delta x^c\left\{\rho u''(x^d)\left[u'(x^c)-1\right] + \eta(1-\rho)u''(x^c)\left[u'(x^d)-1\right]\right\}}{\eta\left[(1-\delta)(1-\sigma)u'(x^d)+\delta\right]\left[\rho u''(x^d)+\eta^2(1-\rho)u''(x^c)\right]}, \tag{89}$$

where

$$\mathit{\Gamma} = \rho\delta(1-\rho)(\eta-1)\left[u'(x^d)-1\right]\left[(1-\delta)(1-\sigma)u'(x^d)+\delta\right] \geq 0.$$

From (89), the derivative of $\mathcal{W}$ is strictly positive if $\eta = R^m = 1$, i.e., $\left.\frac{d\mathcal{W}}{d\eta}\right|_{\eta=R^m=1} > 0$. This implies that the monetary policy at $\eta = R^m = 1$ is not optimal. Therefore, from (87) and (89), I conclude that the optimal monetary policy is away from a modified Friedman rule or $\eta R^m > 1$.

Proof of Proposition 5: Suppose that $\theta = 0$ in equilibrium. From (17)-(19) and (46),

$$\eta R^m \left[u'(x^d) - \delta u'(x^d) + \delta\right] = u'(x^o), \tag{90}$$

$$u'(x^o) = u'(x^c) - \delta u'(x^d) + \delta, \tag{91}$$

where (x^c, x^d) are the off-equilibrium consumption quantities in DM transactions, if a buyer were to participate in banking contracts. It can be shown that $\left|\frac{d[u'(x^d)-\delta u'(x^d)+\delta]}{d[\eta R^m]}\right| < 1$, so from (90) x^o increases with a decrease in ηR^m. However, the limited quantity of collateral $v < x^* + \beta\mu$ implies that, from (60), the highest possible quantity for x^o is $\bar{x}$ that solves $(\bar{x} + \beta\mu)u'(\bar{x}) = v$ and $\bar{x} < x^*$. So, any ηR^m that leads to x^o higher than $\bar{x}$ cannot be supported in equilibrium, implying that, from (90),

$$R^m \geq \frac{u'(\bar{x})}{\eta\left[u'(x^d) - \delta u'(x^d) + \delta\right]}, \tag{92}$$

where x^d is the off-equilibrium consumption quantity in DM transactions using bank claims that is consistent with (91). Also, any R^m higher than the right-hand side of (92) implies that $0 < \theta \leq 1$ and $U^b \geq U^o$. So, by continuity, the off-equilibrium consumption quantities (x^c, x^d) when $R^m = \frac{u'(\bar{x})}{\eta\left[u'(x^d)-\delta u'(x^d)+\delta\right]}$ must satisfy (91) and $U^b = U^o$ from (48)-(49) given $x^o = \bar{x}$.

Recall that the nominal interest rate R^m must satisfy (43). That is, there must be no arbitrage opportunities from carrying currency across periods in equilibrium. To prove that the lower bound on the nominal interest rate in inequality (92) is always higher than the lower bound in (43), I claim that the following condition holds in equilibrium if the fixed cost of holding currency μ is close to zero:

$$\frac{u'(\bar{x})}{u'(x^d) - \delta u'(x^d) + \delta} > 1. \tag{93}$$

Suppose $\eta R^m = 1$ in equilibrium, so that deposit contracts effectively allow buyers to consume the same quantity of goods across two types of DM transactions, i.e., $x^c = x^d = x$. Then, the quantity of DM consumption for buyers opting out of deposit contracts is higher than the quantity for buyers holding deposit contracts ($x^o > x$) since, from (90)-(91),

$$u'(x^o) = (1-\delta)u'(x) + \delta.$$

This implies that the expected utility for buyers opting out of contracts is higher than the expected utility for buyers opting in because from (48)-(49),

$$U^o - U^b = \left[u(x^o) - x^o u'(x^o)\right] - \left[u(x) - xu'(x)\right] - \beta\mu\left[u'(x^o) - \rho u'(x)\right] > 0,$$

for a sufficiently low μ. So, $\eta R^m = 1$ cannot be supported in equilibrium (a contradiction) as this policy would lead to a complete disintermediation, i.e., $\theta = 0$. To encourage banking activities, $\eta R^m > 1$ must be satisfied so that (93) must hold in equilibrium. Also, $\frac{u'(\bar{x})}{\eta\left[u'(x^d)-\delta u'(x^d)+\delta\right]} > \frac{1}{\eta+\beta\gamma}$ for any $\eta \geq 1$ because $\eta + \beta\gamma$ increases more than $\eta\left[u'(x^d) - \delta u'(x^d) + \delta\right]$ as η rises. Therefore, the effective lower bound on the nominal interest rate is determined by (92).

Proof of Proposition 6: First, note that an equilibrium with $\theta = 0$ exists only if R^m is set at the effective lower bound (ELB) defined in (58). As mentioned in Proof of Proposition 5, any R^m higher than the ELB implies that $x^o < \bar{x}$ where $\bar{x}u'(\bar{x}) = v$. Since $x^o u'(x^o) < v$ and the collateral constraint must bind in an equilibrium where v is sufficiently low, there must be some buyers participating in banking contracts, i.e., $\theta > 0$. So, for any R^m that is higher than the ELB, $\theta > 0$ in equilibrium.

Next, consider an equilibrium with $0 < \theta < 1$. Then, (x^c, x^d, x^o, θ) must satisfy:

$$\eta R^m = \frac{u'(x^c) - \delta u'(x^d) + \delta}{u'(x^d) - \delta u'(x^d) + \delta}, \tag{94}$$

$$(1-\rho)\theta x^d \left[u'(x^d) + \frac{\delta}{1-\delta}\right] + \rho\theta x^c \left[u'(x^c) + \frac{\delta}{1-\delta}\right] + (1-\theta)x^o u'(x^o) = v, \tag{95}$$

$$u'(x^o) = u'(x^c) - \delta u'(x^d) + \delta, \tag{96}$$

$$\rho\left[u(x^c) - x^c u'(x^c)\right] + (1-\rho)\left[u(x^d) - x^d u'(x^d)\right] = u(x^o) - x^o u'(x^o). \tag{97}$$

Suppose that there is an increase in ηR^m. Then, from (94), x^c decreases and x^d increases as ηR^m rises. From (96), x^o decreases as x^c decreases and x^d increases. Also, from (96), a necessary condition for this equilibrium to exist is $x^c < x^d$, which implies that U^b decreases as x^c falls and x^d rises. Since the left-hand side of (97) decreases as x^c falls and x^d rises, x^o must fall in equilibrium. Then, from (95), θ must rise in equilibrium. The effects of an increase in R^m or an increase in η on $(\pi, \alpha^b, \alpha^s)$ are straightforward from (61), (64), and (65).

As an increase in ηR^m decreases x^c and x^o and increases x^d and θ, there exists $\Omega = \eta R^m$ that satisfies equation (94) where x^c and x^d are the solutions to equations (95)-(97) when $\theta = 1$. Therefore, I can conclude that, in equilibrium, $0 \leq \theta < 1$ if $\frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta} \leq \eta R^m < \Omega$ and $\theta = 1$ if $\eta R^m \geq \Omega$ where $\bar{x}$ and $\underline{x}^d$ are the quantities defined in Proof of Proposition 5.

Proof of Proposition 7: In Proof of Proposition 6, I have shown that, for any $\eta R^m > \frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta}$, the fraction θ is positive and $x^c < x^d$ in equilibrium. This implies that the left-hand side and the right-hand side of (97) both increase as x^c and x^o rise and x^d falls. So, given η, lowering R^m increases the welfare measure $\mathcal{W}$ because it increases x^c and x^o and decreases x^d and θ. Then, by continuity, the maximum $\mathcal{W}$ can be obtained when $\eta R^m = \frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta}$ given η. However, if the central bank conducts monetary policy (R^m, η) such that $\eta R^m = \frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta}$, a higher η only implies a higher α^b without increasing the sum of surpluses from trade in the CM and the DM. As a higher α^b leads to a larger total cost of theft, the welfare measure $\mathcal{W}$ is maximized if and only if $\eta = 1$ and $R^m = \frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta}$.