

Negative Nominal Interest Rates and Monetary Policy*

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Abstract

Can central banks reduce the effective lower bound on interest rates? If so, what would be the welfare implications? I construct a model with multiple means of payment where the costs of holding paper currency effectively reduce its rate of return, creating a negative effective lower bound on interest rates. I find that central banks can reduce this lower bound with a non-par exchange rate between currency and reserves, but it raises currency-holding costs for individuals, leading to welfare losses. Moreover, implementing a negative rate by reducing the lower bound has no welfare benefits because this policy combination lowers both the rate of return on currency and the interest rate on financial assets, leaving relative interest rates unchanged.

Keywords: Negative interest rate; Effective lower bound; Money; Banking; Monetary policy

JEL Codes: E4; E5

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1 Introduction

Since the global financial crisis, negative interest rates have become a part of the monetary policy toolkit for some prominent central banks in the world. There is a perspective that advocates for expanding the boundary of negative interest rates by reducing the effective lower bound on interest rates. Understanding how and why a central bank could control this effective lower bound requires uncovering its determinants. What economic fundamentals or frictions determine the effective lower bound? If a central bank attempts to reduce the effective lower bound by manipulating the frictions, what would be the welfare implications? This paper shows that while central banks can indeed reduce the effective lower bound, such efforts result in welfare losses without corresponding gains.

In mainstream macroeconomic theory, monetary policy is constrained by the zero lower bound on nominal interest rates.¹ This constraint arises because negative nominal interest rates are infeasible due to arbitrage: borrowing at a negative rate and investing in zero-interest paper currency (*currency*, hereafter) would always be profitable if nominal interest rates were negative. However, negative nominal short-term interest rates are implementable in practice.² This observation suggests the presence of substantial frictions that inhibit arbitrage, implying that the lower bound on nominal interest rates is effectively negative.

What are the frictions that create a negative effective lower bound? It is generally recognized that these frictions emerge mainly from the costs associated with storing, transporting, and exchanging large quantities of currency. For example, if a private bank faces a negative nominal interest rate on reserves, it might contemplate holding currency instead of having reserve balances with the central bank. However, such a decision would entail the costs of installing a sizable vault and hiring security guards to watch it. Moreover, the currency would be of little or no use in making interbank and online transactions.

Despite the feasibility of implementing negative nominal interest rates, some researchers have made the case that the effective lower bound can still be an important binding constraint on monetary policy, and have suggested potential policy tools aimed at reducing the effective lower bound.³ According to their arguments, the frictions that determine the effective lower bound are not welfare-reducing impediments, but frictions that should be enhanced. That is, greater friction leads to a lower effective lower bound, which permits welfare-enhancing monetary policy. In this paper, I for-

¹See, for example, [Woodford \(2003\)](#) and [Curdia and Woodford \(2010\)](#) for standard New Keynesian models. Also, the zero lower bound is a standard property in mainstream monetary theory. See, for example, [Lagos and Wright \(2005\)](#) and [Lagos et al. \(2017\)](#).

²The European Central Bank, the Swiss National Bank, the Swedish Riksbank, the Bank of Japan, and the National Bank of Denmark have all demonstrated the feasibility of negative nominal short-term interest rates.

³See, for example, [Goodfriend \(2016\)](#), [Rogoff \(2017a,b\)](#), and [Agarwal and Kimball \(2015, 2019\)](#). The effective lower bound was a binding constraint for some central banks during the 2010s, as they have been cautious about implementing negative or substantially negative interest rates. Although the Swiss National Bank and the National Bank of Denmark successfully set the record-low policy rate of -0.75 percent, it remains uncertain whether they could further reduce their policy rates and whether other central banks could set their policy rates at that level. For example, [Witmer and Yang \(2016\)](#) find that the effective lower bound for the Bank of Canada policy rate ranges between -0.25 percent and -0.75 percent.

mally examine this potential tradeoff by proposing a model that helps us understand how to reduce the effective lower bound and evaluate the associated welfare implications.

First, to uncover the determinants of the effective lower bound, it is crucial to explicitly consider the costs of handling currency and the imperfect role of currency as a means of payment. So, I introduce currency-holding costs into a general equilibrium model where currency and bank deposits serve as means of payment for transactions. Second, to demonstrate how a central bank can reduce the effective lower bound, I consider a non-traditional, market-based reserve policy that has been most favorably discussed in the literature. Specifically, this reserve policy involves altering the one-to-one exchange rate between currency and bank reserves (Eisler, 1932; Buiter, 2010; Agarwal and Kimball, 2015).⁴ As the central bank can set a different exchange rate for private banks' current currency withdrawals from the one for their future currency deposits, they can create a negative nominal rate of return on currency faced by private banks. Therefore, this approach has the potential to reduce the effective lower bound on nominal interest rates.

With this theoretical framework, I show how the central bank can lower the effective lower bound through this unconventional reserve policy—a *non-par* exchange rate for currency withdrawals. The effectiveness of this policy crucially depends on its ability to reduce the nominal rate of return on currency faced by private individuals. My finding is that a non-par exchange rate can indeed reduce the nominal rate of return on currency. This occurs due to the presence of currency-holding costs, which incentivize individuals to continue using their bank deposits despite facing a negative nominal return on currency. As a result, central banks can successfully lower the effective lower bound on nominal interest rates.

The ability of a non-par exchange rate to reduce the effective lower bound, however, can be limited in some cases. If individuals choose not to deposit their currency at banks, the central bank loses its ability to reduce the effective lower bound through a non-par exchange rate. This situation can happen because a non-par exchange rate creates incentives for side trading of currency in the private sector. Specifically, private banks are willing to acquire currency at a price (in units of reserves) lower than the non-par exchange rate, while private individuals are willing to unload their currency at a price higher than one unit of reserves. In cases where the marginal cost of holding currency does not increase with currency holdings, individuals deposit currency at banks only when the marginal benefit from side trading is sufficiently low, i.e., when the non-par exchange rate for currency withdrawals is sufficiently low.

This result can be reversed if the marginal cost of holding currency increases endogenously with currency holdings. For instance, a larger amount of currency held by individuals can promote currency theft. Then, the cost of holding currency can increase at the margin either because theft occurs more frequently or because individuals must incur higher security costs to prevent theft. In the model, individuals can steal currency, at a cost, from those engaged in side trades, so that

⁴In particular, Agarwal and Kimball (2019) consider this non-traditional reserve policy as “the first-best approach with the fewest undesirable side-effects” because the nominal rate of return on currency in units of reserves, created by the central bank, can be naturally transmitted to the rate of return on currency in units of other financial assets.

theft arises endogenously.⁵ Endogenous theft acts to increase the cost of holding currency, which in turn encourages individuals to deposit their currency at banks. Consequently, the central bank can always reduce the effective lower bound on nominal interest rates.

Turning to the consequences of reducing the effective lower bound, the reserve policy enables the central bank to implement a lower nominal interest rate. Lowering the nominal interest rate serves to mitigate inefficiency in ordinary transactions by increasing the rate of return on currency relative to other financial assets. However, the central bank faces a tradeoff, as the reserve policy also leads to welfare losses. This tradeoff arises because a non-par exchange rate induces more individuals to engage in socially useless and costly theft, making others bear a higher cost of holding currency.

Despite the apparent tradeoff, I find that it is never optimal to reduce the effective lower bound, as doing so only increases the cost of holding currency without yielding any welfare gains. Intuitively, lowering the nominal interest rate on reserves encourages private banks to hold more currency relative to interest-bearing assets. However, reducing the effective lower bound, which involves lowering the nominal rate of return on currency, does the opposite: it incentivizes private banks to hold less currency relative to interest-bearing assets. So, if the central bank reduces both the interest rate and the lower bound by the same magnitude, the effect of reducing the lower bound completely offsets that of reducing the interest rate. In other words, it is the interest rate on reserves relative to currency that determines the monetary policy stance. Since the central bank cannot further reduce this relative interest rate, there are no welfare gains from reducing the effective lower bound.

I also explore how potential disintermediation can affect the effective lower bound on nominal interest rates. Disintermediation happens when more individuals opt to use currency rather than bank deposits to make ordinary transactions given a sufficiently low interest rate. This is a practical concern because bank deposits serve as a primary and stable funding source for financing bank loans, and thus, disintermediation could lead to long-run inefficiency in the financial system. In an extended model where individuals can opt out of the banking system, I find that a non-par exchange rate policy enables central banks to set a negative nominal interest rate without causing disintermediation. However, this reserve policy also encourages costly theft and results in welfare losses, as in the baseline model.

In an extreme case where individuals can make all transactions with currency, the nominal interest rate must be sufficiently high to support banking activities.⁶ I show that the lower bound on nominal interest rates preventing complete disintermediation can be higher than the one preventing arbitrage. Since interest rates leading to complete disintermediation cannot be sustained in the economy, the effective lower bound is determined by the former rather than the latter. Although my analysis abstracts from the conflict between these two distinct lower bounds, which is a driving force for the imperfect pass-through of negative policy rates in [Eggertsson et al. \(2022\)](#), my findings

⁵An endogenous cost of holding currency in a form of endogenous theft is also modeled in [He et al. \(2005, 2008\)](#) and [Sanches and Williamson \(2010\)](#).

⁶This case is considered extreme because, in practice, there are transactions (e.g., online transactions) that cannot be made with paper currency.

suggest that a non-par exchange rate policy can reduce both lower bounds by increasing currency-holding costs faced by individuals.

This paper contributes to the literature on how to expand the boundary of negative nominal interest rates. Among others, [Agarwal and Kimball \(2019\)](#) provide a comprehensive survey and discuss the pros and cons of policy tools suggested in the literature. The idea of a non-par exchange rate between currency and reserves was first proposed by [Eisler \(1932\)](#) in the form of a dual currency system where one currency (physical currency) is used as a means of payment, and the other (electronic money) plays a unit-of-account role. [Buiter \(2010\)](#) revived Eisler’s proposal with a simple model illustrating how the central bank could frictionlessly reduce the lower bound by adjusting the exchange rate between the two currencies. More recently, [Agarwal and Kimball \(2015\)](#), [Goodfriend \(2016\)](#), and [Rogoff \(2017a,b\)](#) have favorably discussed a non-par exchange rate between currency and reserves as a potential policy tool. However, these discussions lacked a formal model to substantiate the proposed mechanisms. My paper complements existing discussions by formalizing the insights presented in the literature and examining the frictions that determine the effective lower bound. Furthermore, it proposes a novel mechanism demonstrating how a non-par exchange rate can effectively reduce the rates of return on currency faced by private banks and individuals.

This paper is also related to theoretical studies examining the implications of negative nominal interest rates, including [He et al. \(2008\)](#), [Rognlie \(2016\)](#), [Jung \(2019\)](#), [Ulate \(2021\)](#), [Eggertsson et al. \(2022\)](#), and [Abadi et al. \(2023\)](#). Among others, [He et al. \(2008\)](#) develop a model of currency and bank deposits, similar to mine, where using currency is relatively less safe due to the risk of theft. They show that a negative nominal interest rate can be optimal under certain conditions. My paper departs from theirs by distinguishing between private banks’ deposits with the central bank (reserves) and private individuals’ bank deposits. This framework enables an examination of how a non-par exchange rate between currency and reserves impacts the actual rate of return on currency and the terms of bank deposit contracts. Another related study is conducted by [Eggertsson et al. \(2022\)](#), which show that the lower bound on disintermediation-free interest rates (deposit rates) can exceed the lower bound on arbitrage-free policy rates (short-term interest rates). They find that when the former lower bound is binding while the latter is not, implementing a negative nominal interest rate policy can lead to contractionary effects due to the breakdown in the pass-through of the policy rate. Differently, my paper focuses specifically on examining how and why a central bank can reduce the lower bound on nominal interest rates.

The rest of the paper is organized as follows. I construct the baseline model in Section 2 and define and characterize an equilibrium in Section 3. Optimal monetary policy is analyzed in Section 4, while the policy implication for disintermediation is discussed in Section 5. Lastly, Section 6 is a conclusion.

2 Baseline Model

The basic structure of the model is similar to [Lagos and Wright \(2005\)](#) and [Rocheteau and Wright \(2005\)](#). Time is indexed by $t = 0, 1, 2, \dots$, and there are three subperiods in each period. The theft market (TM) opens in the first subperiod, the centralized market (CM) opens in the following subperiod, and the decentralized market (DM) opens in the last subperiod. I incorporate the theft market into the standard framework to reflect frictions limiting the size of the side trades of currency, as explained later. There is a continuum of buyers with unit mass, each of whom maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[-\tilde{H}_t - H_t + u(x_t) \right], \quad (1)$$

where $0 < \beta < 1$, \tilde{H}_t and H_t denote the buyer's labor supply in the TM and the CM respectively and x_t denotes his or her consumption in the DM. Assume that $u(\cdot)$ is strictly increasing, strictly concave, and twice continuously differentiable with $u'(0) = \infty$, $u'(\infty) = 0$, and $-\frac{xu''(x)}{u'(x)} < 1$. There is also a continuum of sellers with unit mass, each of whom maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[-\tilde{H}_t^s + X_t^s - h_t^s \right], \quad (2)$$

where X_t^s denotes the seller's consumption in the CM, and \tilde{H}_t^s and h_t^s denote his or her labor supply in the TM and the DM. Finally, there is a continuum of private banks each of which maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[X_t^b - H_t^b \right], \quad (3)$$

where X_t^b is the bank's consumption in the CM and H_t^b is its labor supply in the CM. Private banks are agents who are active only in the CM. In the CM or the DM, one unit of perishable consumption good can be produced with one unit of labor supply while no production takes place in the TM. Buyers cannot produce goods in the DM, while sellers cannot produce in the CM.

At the beginning of the TM, sellers holding currency can deposit the currency with the central bank in exchange for reserve balances.⁷ At this stage, currency trades one-to-one for reserve balances, and then sellers can exchange reserve balances for goods in the following CM. After sellers make currency deposits with the central bank, buyers can incur κ units of labor to acquire a theft technology (e.g., producing a weapon). Then, buyers and sellers are randomly matched. If a seller with currency meets a buyer with the theft technology, the buyer steals all of the seller's currency.

At the beginning of the CM, debts are paid off, then production, consumption, and exchange

⁷In practice, private individuals cannot have a reserve account with the central bank. This assumption could be interpreted that an individual deposits currency with a private bank and then the private bank deposits the currency in exchange for reserve balances. Assume that the central bank can enforce private banks' currency deposits at the beginning of the CM. An alternative interpretation is that when an individual deposits the currency with the central bank, the central bank credits the payment to the corresponding private bank which in turn credits it to the individual's bank account.

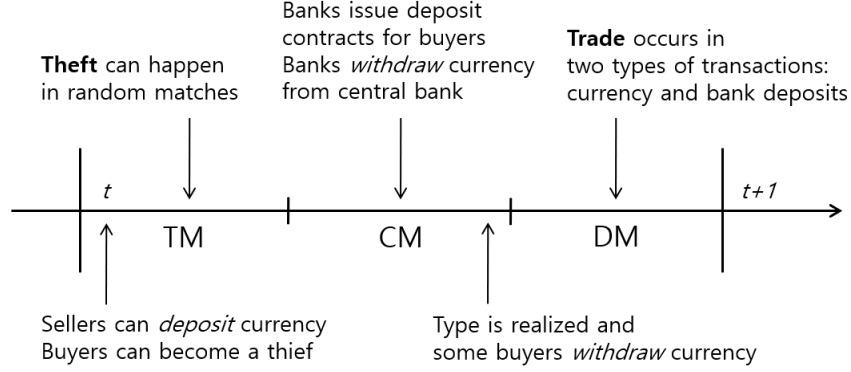


Figure 1: Timing of events

take place in a perfectly competitive market. Private banks can obtain currency in three different ways—from a seller, from a buyer (stolen currency), or by acquiring reserves with the central bank and exchanging the reserves for currency. Also, private banks write deposit contracts with buyers before buyers learn their types. A type for a buyer is the type of seller he or she will meet in the following DM, as specified in what follows. Bank deposit contracts provide insurance, by allowing buyers to withdraw currency at the end of the CM when they learn their types. Buyers' types are publicly observable.

In the DM, each buyer is randomly matched with a seller and makes a take-it-or-leave-it offer to the seller. In any DM matches, a matched buyer and seller do not know each others' histories (no memory or record keeping) and are subject to limited commitment. This implies that no buyers' IOUs can be traded in the DM. There are two types of sellers. Fraction ρ of sellers accepts only currency, and fraction $1 - \rho$ accepts only claims on banks. The timing of events is summarized in Figure 1.

Some sellers who acquire currency in the DM may want to exchange the currency for goods in the following CM, instead of safely depositing it with the central bank. Let α_t^s denote the fraction of sellers who carry currency into the CM conditional on having acquired the currency in the previous DM and let α_t^b denote the fraction of buyers who acquire the technology to steal currency. Then, the probability that each seller meets a buyer with the theft technology is α_t^b and the probability that each buyer meets a seller with currency is $\rho\alpha_t^s$.

There are three underlying assets in this economy — currency, reserves, and nominal government bonds. Currency and reserves are issued by the central bank. Currency is perfectly divisible, portable, and storable, and bears a nominal interest rate of zero. Reserves are private banks' account balances with the central bank, and one unit of reserves acquired in the CM of period t pays off R_{t+1}^m units of reserves at the beginning of the CM in period $t + 1$. Private banks visit the central bank if they want to withdraw currency from their reserve accounts. Following the ideas presented in [Eisler \(1932\)](#), [Buiter \(2010\)](#) and [Agarwal and Kimball \(2015\)](#) among others, the central bank can set an exchange rate between currency and reserves off par to create a negative

nominal rate of return on currency. The exchange rate, denoted by $\eta_t \geq 1$, measures the unit of reserves exchanged to withdraw one unit of currency in period t while deposited currency is always exchanged one-to-one for reserves. Nominal government bonds, issued by the fiscal authority, are one-period bonds with a gross nominal interest rate of R_t^b .

In addition to the underlying assets issued by the consolidated government, there are bank deposit claims that private banks create endogenously. I assume that private banks have a collateral technology that allows creditors to seize at least part of the asset if they default. This implies that private banks can issue asset-backed deposit claims that can be accepted as a means of payment in the DM. A bank could potentially hold currency in its asset portfolio from the CM of period t until the next. This incurs a storage cost γc_t in the CM of period $t + 1$, where $0 < \gamma < 1$ and c_t is the real quantity of currency acquired in the CM of period t . This proportional cost of storing currency across periods has implications for the lower bound on nominal interest rates, as will be shown later. Sellers have the same storage technology as banks, but buyers cannot hold currency across periods.⁸ Also, any individuals holding currency at the end of the DM bear a fixed cost μ . Eventually, sellers who receive currency in the DM will pay this cost, which will generate inefficiency in transactions using currency.⁹

2.1 Government

Confine attention to stationary equilibria where all real variables and government policies are constant across periods. Assume that the consolidated government starts issuing its liabilities with no unsettled debt outstanding in period 0. Then, the consolidated government budget constraint at $t = 0$ can be written as

$$\eta \bar{c} + \bar{m} + \bar{b} = \tau_0, \quad (4)$$

where \bar{c} , \bar{m} , and \bar{b} denote the real quantities of currency, reserves, and nominal government bonds outstanding at the end of period 0 (and in every following period). Also, τ_0 is the real quantity of lump-sum transfers to each buyer. Assume that the fiscal authority can levy lump-sum taxes on buyers in equal amounts. So, the consolidated government budget constraint at $t = 1, 2, \dots$ can be written as

$$\eta \bar{c} + \bar{m} + \bar{b} = \frac{\bar{c} + R^m \bar{m} + R^b \bar{b}}{\pi} + \tau, \quad (5)$$

where π is the gross inflation rate and τ is the real quantity of lump-sum transfers (or lump-sum taxes if $\tau < 0$) to each buyer. The left-hand side of (5) represents the total revenue of the consolidated government from issuing new liabilities, and the right-hand side represents its total expenditure,

⁸To make the model analytically tractable, I assume that buyers cannot store currency across periods. Relaxing this assumption would only complicate the model without adding any useful insights.

⁹Introducing a fixed cost is sufficient to generate inefficiency in currency-involved transactions, which has implications for optimal monetary policy. Adding a proportional cost to those holding currency at the end of the DM would make the model analytically intractable.

i.e., the sum of the repayments of government debt issued in the previous period and the transfers to buyers.

As in [Andolfatto and Williamson \(2015\)](#) and [Williamson \(2016\)](#), I assume that the fiscal authority determines the real value of the consolidated government debt outstanding, v , where

$$v = \eta \bar{c} + \bar{m} + \bar{b}, \quad (6)$$

for all periods.¹⁰ That is, the fiscal authority adjusts the level of lump-sum transfers, in response to a change in monetary policy, to achieve the fiscal policy goal. Given the fiscal policy that targets the total value of the consolidated government debt, the central bank's monetary policy serves to change its composition. This fiscal policy rule is a useful specification that creates a low real interest rate in the economy. As will be shown in what follows, a low value of consolidated government debt (or a low v) is translated into a low supply of collateralizable assets, resulting in binding collateral constraints, a liquidity premium on those assets, and low real interest rates.

3 Equilibrium

In this section, I will describe how buyers and sellers make their decisions in the TM and how private banks choose their deposit contracts and asset portfolios in the CM. Then, I will define and characterize an equilibrium and show why the effective lower bound on nominal interest rates can be negative.

3.1 Side Trading and Theft

In the CM, private banks can withdraw one unit of currency from the central bank by paying η units of their reserve balances. If there are would-be sellers of currency in the CM, private banks would be willing to participate in side trades of currency to reduce the cost of acquiring currency. In equilibrium, private banks must be indifferent between withdrawing currency from the central bank and obtaining currency from any would-be sellers of currency. This implies that the price of currency in terms of reserves in the CM must be η in equilibrium.¹¹

¹⁰The central bank purchases government bonds by issuing currency and reserves in period 0 and transfers its profits to the fiscal authority in every following period. This implies that the real value of the central bank's assets must be equal to that of its liabilities in every period, that is,

$$\eta \bar{c} + \bar{m} = \hat{b},$$

where \hat{b} denotes the real quantity of government bonds held by the central bank. Therefore, the fiscal authority determines the real quantity of total government bonds issued by the fiscal authority because

$$v = \hat{b} + \bar{b}.$$

¹¹This scenario arises when the real quantity of currency traded in the CM does not exceed the real quantity demanded by private banks. Specifically, I focus on stationary equilibria with sufficiently low values of v and γ , ensuring that there is inflation $\pi > 1$. Due to inflation, the real quantity of currency traded continually falls short of the real quantity demanded. Therefore, private banks must withdraw some currency from the central bank every

Side trading in currency between private banks and currency holders can take place only when there are sellers who choose not to deposit their currency with the central bank. However, in the TM, some buyers may incur κ units of labor supply to acquire the theft technology. Suppose that the representative currency-holding seller carries c^s units of currency in real terms into the TM. As acquiring the theft technology is costly, each buyer's decision on stealing currency must be incentive compatible in equilibrium, that is,

$$\text{if } \kappa > \rho\alpha^s\eta c^s, \quad \text{then } \alpha^b = 0, \quad (7)$$

$$\text{if } \kappa = \rho\alpha^s\eta c^s, \quad \text{then } 0 \leq \alpha^b \leq 1, \quad (8)$$

$$\text{if } \kappa < \rho\alpha^s\eta c^s, \quad \text{then } \alpha^b = 1, \quad (9)$$

where $\rho\alpha^s$ is the probability of meeting a currency-holding seller and ηc^s is the real value of currency held by the seller. Conditions (7)-(9) state that no theft takes place if the buyer strictly prefers not to steal currency, the buyer sometimes steals if he or she is indifferent between two options, and the buyer always steals if theft is strictly preferred. Also, each seller's decision on carrying currency into the TM must be incentive compatible, that is,

$$\text{if } (1 - \alpha^b)\eta < 1, \quad \text{then } \alpha^s = 0, \quad (10)$$

$$\text{if } (1 - \alpha^b)\eta = 1, \quad \text{then } 0 \leq \alpha^s \leq 1, \quad (11)$$

$$\text{if } (1 - \alpha^b)\eta > 1, \quad \text{then } \alpha^s = 1. \quad (12)$$

The seller does not carry the currency into the TM if the expected payoff from carrying one unit of currency $(1 - \alpha^b)\eta$ is less than the payoff from depositing it with the central bank and obtaining one unit of reserves. If the seller is indifferent between two choices, he or she sometimes carries the currency into the TM. Otherwise, the seller always carries the currency into the TM.

3.2 Deposit Contracts

Private banks write deposit contracts for buyers in the CM before buyers learn their types. Deposit contracts provide insurance to buyers as in [Williamson \(2016, 2022\)](#) by giving them an option to withdraw currency when they learn their types. Those buyers who do not exercise the option will use bank claims as a means of payment in DM meetings. Suppose a bank proposes a deposit contract (k, c', d) , where k is the quantity of CM goods deposited by each buyer at the beginning of the CM, c' is the real quantity of currency that the buyer can withdraw at the end of the CM, and d is the quantity of claims to CM goods in the following period that the buyer can exchange in the DM if currency has not been withdrawn. Also, the bank acquires an asset portfolio (b, m, c) , where b is the quantity of government bonds, m is the quantity of reserves, and c is the quantity of currency

period. See Appendix A.1 for the details on stationary equilibria with deflation.

in real terms. In equilibrium, the bank's problem can be written as

$$\max_{k, c', d, b, m, c} \left\{ -k + \rho u \left([1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta] \frac{\beta c'}{\pi} - \beta \mu \right) + (1 - \rho)u(\beta d) \right\} \quad (13)$$

subject to

$$k - b - m - \eta c + \beta \left[-(1 - \rho)d + \frac{R^m m + R^b b + c - \rho c'}{\pi} - \gamma(c - \rho c') \right] \geq 0, \quad (14)$$

$$-(1 - \rho)d + \frac{R^m m + R^b b + c - \rho c'}{\pi} \geq \frac{\delta (R^m m + R^b b + c)}{\pi}, \quad (15)$$

$$k, c', d, b, m, c, c - \rho c' \geq 0. \quad (16)$$

The objective function (13) is the representative buyer's expected utility, implying that the bank chooses a contract that maximizes the buyer's expected utility in equilibrium. With probability ρ , the buyer realizes that, in the following DM, he or she will be matched with a seller who accepts only currency. In this case, the buyer visits the bank to withdraw c' units of currency at the end of the CM. In the following DM, the buyer makes a take-it-or-leave-it offer to the matched seller and acquires $[1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta] \frac{\beta c'}{\pi} - \beta \mu$ units of goods.¹² With probability $1 - \rho$, the buyer learns that he or she will meet a seller who accepts a claim on the bank. As the buyer does not withdraw currency in this case, he or she receives a claim to d units of goods in the next CM. So, the buyer's take-it-or-leave-it offer implies that the buyer trades d deposit claims for βd units of goods.

Constraint (14) states that the bank earns a nonnegative discounted net payoff in equilibrium. In the CM, the bank receives k deposits from the buyer and acquires a portfolio of government bonds b , reserves m , and currency c . At the end of the CM, the bank pays off currency to the fraction ρ of buyers, each of whom withdraws c' currency. The remaining fraction $1 - \rho$ of buyers exchange their deposit claims in the DM. So, in the following CM, the bank pays off d units of goods to each holder of the deposit claims. Notice that the bank stores the remaining $c - \rho c'$ units of currency until the next CM which incurs $\gamma(c - \rho c')$ units of labor supply. At the beginning of the next CM, the bank must deposit the remaining currency with the central bank at a one-to-one exchange rate.

As for any agents in the economy, the bank is subject to limited commitment. So, the bank's deposit liabilities must be backed by collateral and (15) is a collateral constraint. Assume that the bank can abscond with a fraction δ of its assets, pledged as collateral, when it defaults. Then, the

¹²As the buyer makes a take-it-or-leave-it offer to the seller, the buyer can extract all the surplus from trade. By accepting the buyer's offer, the seller receives c' units of currency (in real terms) from the buyer in the DM. Then, in the next period, the seller will be holding $\frac{c'}{\pi}$ units of currency and bear μ units of fixed cost (a disutility from supplying labor) at the beginning of the TM. The ex-ante expected payoff per unit of currency is $1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta$ because with probability $1 - \alpha^s$ the seller deposits the currency to receive one unit of reserves, and with probability $\alpha^s(1 - \alpha^b)$ the seller successfully sells the currency in the CM at price η . Since the seller's surplus from trade is zero, the quantity of goods produced by the seller and transferred to the buyer (or equivalently, the disutility from producing goods) is equal to the seller's discounted expected net payoff from acquiring c' units of currency in the DM.

collateral constraint tells us that the bank must weakly prefer to repay its deposit liabilities in the CM and in the next CM rather than absconding with collateral. If the bank were to default, it would not let buyers withdraw currency. Finally, constraint (16) demonstrates that all real quantities must be nonnegative.

3.3 Definition of Equilibrium

Any contract that provides a positive discounted net payoff for the bank cannot be supported in equilibrium. If private banks were to earn a positive discounted net payoff, a bank would design an alternative contract that provides a slightly lower payoff per contract, but a higher total payoff by attracting all buyers. So, constraint (14) must hold with equality in equilibrium.

Let λ denote the Lagrange multiplier associated with the collateral constraint (15). Then, I can derive the first-order conditions for the bank's maximization problem, (13) subject to (14)-(16), as follows.

$$\frac{\beta[1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta]}{\pi} u' \left([1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta] \frac{\beta c'}{\pi} - \beta \mu \right) - \eta - \frac{\lambda \delta}{\pi} = 0, \quad (17)$$

$$\beta u'(\beta d) - \beta - \lambda = 0, \quad (18)$$

$$-1 + \frac{\beta R^m}{\pi} + \frac{\lambda R^m (1 - \delta)}{\pi} = 0, \quad (19)$$

$$-1 + \frac{\beta R^b}{\pi} + \frac{\lambda R^b (1 - \delta)}{\pi} = 0, \quad (20)$$

$$-\eta + \frac{\beta}{\pi} - \beta \gamma + \frac{\lambda (1 - \delta)}{\pi} \leq 0, \quad (21)$$

$$\lambda \left[-(1 - \rho)d + \frac{(1 - \delta) (R^m m + R^b b + c)}{\pi} - \frac{\rho c'}{\pi} \right] = 0. \quad (22)$$

A necessary condition for an equilibrium to exist is that sellers do not hold currency from the CM to the next CM. So, the expected payoff from holding currency across periods must be nonpositive at the margin. That is,

$$-\eta + \beta \left[\frac{1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta}{\pi} - \gamma \right] \leq 0. \quad (23)$$

Also, in equilibrium, asset markets clear in that the demand for each asset is equal to the supply. That is,

$$c = \bar{c}; \quad m = \bar{m}; \quad b = \bar{b}. \quad (24)$$

For convenience, let x^c and x^d denote the consumption quantities in DM meetings, respectively,

with currency and deposit claims being traded, i.e.,

$$x^c = [1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta] \frac{\beta c'}{\pi} - \beta \mu, \quad (25)$$

$$x^d = \beta d. \quad (26)$$

I assume that the central bank conducts monetary policy under a floor system where a sufficiently large quantity of reserve balances are held by private banks and the central bank sets the nominal interest rate on reserves R^m . Under this system, private banks treat reserves and government bonds as identical assets at the margin, so the nominal interest rate on reserves pegs the nominal interest rate on government bonds in equilibrium, i.e., $R^m = R^b$ from (19) and (20).¹³ Also, the central bank can expand its balance sheet through swaps of reserves for government bonds. Let $\omega = \eta \bar{c} + \bar{m}$ denote the size of the balance sheet, which is equivalent to the real value of the central bank's liabilities.¹⁴

Then, I can define an equilibrium as follows.

Definition *Given exogenous fiscal policy v and monetary policy (R^m, ω, η) , a stationary equilibrium consists of DM consumption quantities (x^c, x^d) , asset quantities (k, c', d, b, m, c) , the fraction of buyers who choose to steal currency in the TM α^b , the fraction of sellers who choose to carry currency into the TM conditional on having acquired the currency in the previous DM α^s , transfers (τ_0, τ) , gross inflation rate π , and gross nominal interest rate on government bonds R^b , satisfying the consolidated government budget constraints (4) and (5), the fiscal policy rule (6), the first-order conditions for the bank's problem (17)-(22), no arbitrage condition for sellers (23), the incentive compatibility conditions for buyers and sellers (7)-(12), and market clearing conditions (24).*

Notice that, according to the definition, the fiscal and monetary policies are given exogenously. While the fiscal authority determines the total value of consolidated government debt, in real terms, the central bank has three policy targets: (i) the nominal interest rate on reserves, (ii) the size of the central bank's balance sheet, and (iii) the exchange rate between currency and reserves. Also, note that the fiscal authority adjusts transfers, in response to monetary policy, so as to satisfy its fiscal policy target and the government budget constraints.

¹³In practice, reserves are considered as a useful means of payment in intraday trading in the banking system. However, a key property of the U.S. financial system in the post-financial crisis period is that a large volume of reserves has been held by private banks without being used in intraday financial transactions. This observation allows us to abstract from the role of reserves in intraday transactions. Although only reserves can be turned into currency through the central bank cash window, this does not create an interest rate differential between reserves and government bonds.

¹⁴The real value of liabilities is equal to that of assets because the central bank is assumed to transfer any profits/losses to the fiscal authority and the central bank's net worth is zero.

3.4 Characterization of Equilibrium

I first characterize the effective lower bound (ELB) on the gross nominal interest rate R^m . Note that inequality (21) represents no arbitrage for private banks from acquiring currency in the CM, holding it across periods, and depositing it with the central bank in the next CM. Substituting (19) into (21) gives

$$R^m \geq \frac{1}{\eta + \beta\gamma}.$$

Also, using (19), the no arbitrage condition for sellers from holding currency across periods (23) can be rewritten as

$$R^m \geq \frac{1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta}{(\eta + \beta\gamma)[(1 - \delta)u'(x^d) + \delta]}.$$

The gross nominal interest rate R^m must be sufficiently high to prevent arbitrage for both private banks and sellers. So, the ELB on the gross nominal interest rate in equilibrium can be defined as follows:

$$R^m \geq \max \left\{ \frac{1}{\eta + \beta\gamma}, \frac{1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta}{(\eta + \beta\gamma)[(1 - \delta)u'(x^d) + \delta]} \right\} \equiv ELB. \quad (27)$$

Suppose the ELB is determined by the first argument in the above maximization problem. Then, the gross nominal interest rate on reserves R^m can be less than one, or the net nominal interest rate $R^m - 1$ can be negative, for two reasons. If holding currency across periods is costly and the cost is proportional to the quantity of currency, i.e., $\gamma > 0$, a negative net nominal interest rate on reserves can be supported in equilibrium. This result can explain why some central banks could implement negative nominal interest rates in practice without causing a flight to currency. Another reason why the net nominal interest rate can be negative comes from a non-par exchange rate for currency withdrawals $\eta > 1$. Because the exchange rate of reserves for currency withdrawals is higher than that for currency deposits, negative net nominal interest rates can be sustained without triggering arbitrage, consistent with the idea presented by [Eisler \(1932\)](#), [Buiter \(2010\)](#), and [Agarwal and Kimball \(2015\)](#).

Now, suppose the second argument in the above maximization problem determines the ELB on $R^m - 1$. Then, there is a nonstandard reason for a negative ELB. As I will show later, the term $(1 - \delta)u'(x^d) + \delta$ is higher than one due to low real interest rates on collateralizable assets. Real interest rates are low because those assets are useful as collateral and therefore bear a liquidity premium. However, sellers do not use currency or any interest-bearing assets as collateral, and thus, do not earn a non-monetary payoff from holding currency. This implies that the rate of return on currency perceived by sellers is lower than the one perceived by banks, making a negative nominal interest rate feasible. For analytical convenience, I will focus on cases where (27) holds with strict inequality. That is, the nominal interest rate on reserves is not constrained by the ELB.

Suppose the fraction ρ of sellers, who accept currency in the DM, holds c^s currency each in real terms at the beginning of the TM. Then, the following lemma shows the values of α^b and α^s that are consistent with optimal decisions of buyers and sellers in equilibrium.

Lemma 1 *Suppose that $\eta = 1$. Then, no theft occurs in equilibrium, i.e., $\alpha^b = 0$. Furthermore, $\alpha^s \in [0, 1]$ for all $\kappa \geq \rho\eta c^s$ and $\alpha^s \in [0, \bar{\alpha}^s]$ for all $\kappa < \rho\eta c^s$ where $\bar{\alpha}^s = \frac{\kappa}{\rho\eta c^s}$. Alternatively, suppose that $\eta > 1$. Then, no theft occurs in equilibrium with $\alpha^b = 0$ and $\alpha^s = 1$ for all $\kappa \geq \rho\eta c^s$ while theft exists in equilibrium with $\alpha^b = \frac{\eta-1}{\eta}$ and $\alpha^s = \frac{\kappa}{\rho\eta c^s}$ for all $\kappa < \rho\eta c^s$.*

Proof See Appendix \square

According to Lemma 1, theft does not arise in equilibrium if private banks can withdraw currency from their reserve accounts at par. In this case, the market price of currency is identical to the price of reserves in units of goods, making sellers indifferent between depositing the currency with the central bank and trading the currency with a private bank. Also, theft does not exist in equilibrium if the exchange rate for currency withdrawals is non-par ($\eta > 1$) but the cost of theft for buyers κ is too high. In this case, all currency-holding sellers carry their currency in the TM rather than depositing it with the central bank, i.e., $\alpha^s = 1$. Finally, if the cost of theft for buyers is sufficiently low, a non-par exchange rate for currency withdrawals induces sellers to trade currency with private banks. However, sellers become indifferent between trading it with a bank and depositing it with the central bank in equilibrium. This occurs because a low cost of theft induces some buyers to steal the currency, balancing out between the marginal cost and the marginal benefit of trading currency.

From (25) and Lemma 1, I can show that

$$x^c = \frac{\beta c' \eta}{\pi} - \beta \mu, \quad \forall \kappa \geq \frac{\rho \eta c'}{\pi} \quad (28)$$

$$x^c = \frac{\beta c'}{\pi} - \beta \mu. \quad \forall \kappa < \frac{\rho \eta c'}{\pi} \quad (29)$$

In what follows, I will consider the case where the real value of the consolidated government debt outstanding v is sufficiently low, so as to confine attention to an equilibrium with binding collateral constraints.

3.5 Equilibrium with No Theft

In this section, I characterize an equilibrium where all currency-holding sellers trade currency with private banks in the CM with no threat of theft in the TM. Suppose that the cost of theft for buyers κ is sufficiently high, so that condition (28) holds, $\alpha^b = 0$, and $\alpha^s = 1$ in equilibrium.¹⁵ Then, from (17)-(20) and (28), the inflation rate π and the nominal interest rates on reserves and government

¹⁵Analyzing this case is equivalent to analyzing an equilibrium in the model without theft or endogenous costs of holding currency.

bonds R^m and R^b are given by

$$\pi = \frac{\beta}{\eta} \left[\eta u'(x^c) - \delta u'(x^d) + \delta \right], \quad (30)$$

$$R^m = R^b = \frac{\eta u'(x^c) - \delta u'(x^d) + \delta}{\eta [u'(x^d) - \delta u'(x^d) + \delta]}, \quad (31)$$

and the corresponding real interest rates are given by

$$r^m = r^b = \frac{1}{\beta [u'(x^d) - \delta u'(x^d) + \delta]}. \quad (32)$$

In equilibrium, the consumption quantity in DM trades involving deposit claims x^d is inefficiently low due to a binding collateral constraint, leading to low real interest rates.

From (6), (24)-(26), and (30)-(31), the binding collateral constraint (15) can be rewritten as

$$\left[u'(x^c) + \frac{\delta}{(1-\delta)\eta} \right] \rho(x^c + \beta\mu) + \left[u'(x^d) + \frac{\delta}{1-\delta} \right] (1-\rho)x^d = v. \quad (33)$$

Equation (33) implies that the aggregate demand for collateral (the left-hand side) must be equal to the aggregate supply (the right-hand side) in equilibrium. From (28), a necessary condition for buyers to not invest in the theft technology is given by

$$\kappa \geq \frac{\rho(x^c + \beta\mu)}{\beta}. \quad (34)$$

Finally, no arbitrage condition from holding currency across periods is given by

$$R^m \geq \max \left\{ \frac{1}{\eta + \beta\gamma}, \frac{\eta}{(\eta + \beta\gamma)[(1-\delta)u'(x^d) + \delta]} \right\} \equiv ELB. \quad (35)$$

If the non-par exchange rate for currency withdrawals η is sufficiently close to one, and if the real interest rate $r^m = r^b$ is sufficiently low so that the term $(1-\delta)u'(x^d) + \delta$ is sufficiently high, then the ELB is determined by the first argument. In this case, a higher η implies a lower ELB, consistent with the claims made by [Eisler \(1932\)](#), [Buiter \(2010\)](#), and [Agarwal and Kimball \(2015\)](#). However, if η and $r^m = r^b$ are sufficiently high, then the second argument governs the ELB on the nominal interest rate. In this case, an increase in η does not necessarily lower the ELB, and it can even increase the ELB. This happens because there are insufficient frictions to prevent sellers from holding currency across periods. In other words, the marginal cost of holding currency across periods faced by sellers does not catch up with its marginal benefit. Sellers can exploit arbitrage by purchasing currency at price η and selling it at the same price with no risk of theft in the next period. Given a fixed real interest rate, an increase in η only increases the market price of currency, making arbitrage more profitable. Therefore, the ELB can even increase in response to an increase in η .

An interpretation is that, for the nominal rate of return on currency to be negative, there must

be currency deposits and withdrawals at different exchange rates in equilibrium. If either one of those two activities does not occur, a non-par exchange rate can fail to reduce the rate of return on currency and the ELB. In an equilibrium with no theft studied here, the absence of sellers' currency deposits breaks the link between the non-par exchange rate and the nominal rate of return on currency. In contrast, if there is deflation in equilibrium, it is possible that private banks do not withdraw currency from the central bank cash window as can be seen in Appendix A.1.

Given the above equilibrium conditions, I can solve the model as follows. First, equations (31) and (33) solve for (x^c, x^d) , given monetary policy (R^m, η) and fiscal policy v . Then, equation (30) solves for π , equation (32) solves for r^m and r^b , and inequalities (34) and (35) give necessary conditions for this equilibrium to exist.

Effects of Monetary Policy Note that the size of the central bank's balance sheet ω is irrelevant to asset prices or consumption quantities. This occurs because an expansion in the size of the balance sheet only involves the central bank's swaps of reserves for government bonds. As those assets are perfect substitutes for private banks at the margin, this only changes the composition of government bonds and reserves in bank asset portfolios, with no effects on other variables.

In what follows, I analyze the effects of monetary policy interventions given that the net nominal interest rate on reserves is close to zero and the exchange rate between currency and reserves is close to one. This implies that the ELB on the nominal interest rate is determined by the first argument in (35).

Proposition 1 *Suppose that inequalities (34) and (35) hold in equilibrium, (R^m, η) is sufficiently close to $(1, 1)$, and the fixed cost of holding currency μ is sufficiently close to zero. Then, an increase in R^m results in a decrease in x^c , an increase in x^d , an increase in real interest rates (r^m, r^b) , and an increase in π , with no effect on the ELB. In contrast, an increase in η results in an increase in x^c and a decrease in the ELB. Furthermore, there exists \hat{v} such that an increase in η decreases x^d and (r^m, r^b) and increases π for $v \in (0, \hat{v}]$, while it increases x^d and (r^m, r^b) and decreases π for $v \in (\hat{v}, \bar{v})$ where \bar{v} is the upper bound of the values of consolidated government debt that support an equilibrium with a binding collateral constraint.*

Proof See Appendix \square

With the exchange rate between currency and reserves η held constant, an increase in the nominal interest rate on reserves R^m affects bank asset portfolios since it becomes more profitable to hold more reserves or government bonds rather than currency. With a larger quantity of reserves or government bonds, banks can provide a larger quantity of claims to buyers, so x^d rises. However, a smaller quantity of currency outstanding, in real terms, leads to a smaller quantity of consumption in DM trades using currency x^c . Also, the decrease in the real quantity of currency outstanding must be accompanied by a decrease in the real rate of return on currency in equilibrium, implying

(For a sufficiently low v)						(For a sufficiently high v)					
	∂x^c	∂x^d	$\partial \pi$	∂r^m	∂ELB		∂x^c	∂x^d	$\partial \pi$	∂r^m	∂ELB
∂R^m	—	+	+	+	·	∂R^m	—	+	+	+	·
$\partial \eta$	+	—	+	—	—	$\partial \eta$	+	+	—	+	—

Table 1: Effects of monetary policy (R^m, η) in an equilibrium with no theft

a rise in the inflation rate π with η held constant. As a larger quantity of reserves and government bonds makes collateral less scarce, a rise in R^m acts to increase real interest rates, r^m and r^b . But real rates increase by less than do nominal interest rates.

A novel finding is that an increase in the exchange rate between currency and reserves η itself has real effects. In particular, an increase in η leads to an increase in the consumption quantity in DM trades using currency x^c , with R^m held constant. This occurs because an increase in η increases the price of currency in the CM, which in turn increases the value of currency in the DM. As currency is exchanged for a larger quantity of goods in DM trades, the quantity of consumption in those trades increases. The effect of an increase in η on the consumption quantity in DM trades using bank claims x^d depends on the value of consolidated government debt v . If v is sufficiently low or collateralizable assets are sufficiently scarce, then an increase in η decreases x^d and real interest rates (r^m, r^b) , implying that a larger quantity of currency outstanding effectively decreases the stock of government bonds and reserves held by private banks. In contrast, if v is sufficiently high but not too high, then an increase in η increases x^d and real interest rates (r^m, r^b) . In this case, a higher price of currency acts to decrease the real quantity of currency c' (the income effect dominates the substitution effect) which effectively relaxes the collateral constraint from (22). Therefore, x^c and x^d both increase. These results are summarized in Table 1.

Corollary 1 *If η is sufficiently close to one, an increase in η leads to a decrease in the ELB on the nominal interest rate. But if η is sufficiently high, an increase in η can increase the ELB.*

As mentioned earlier, the ELB on the nominal interest rate is determined by the first argument in (35) if the exchange rate between currency and reserves η is sufficiently close to one. This implies that there is no arbitrage opportunity for sellers from holding currency across periods as long as there is no such arbitrage opportunity for private banks. In this case, the central bank can reduce the ELB to the level below the base lower bound, the level of the ELB prevalent with a typical one-to-one exchange rate between currency and reserves. However, if η is sufficiently high, then the second argument in (35) can determine the ELB. In this case, no arbitrage for private banks from investing in currency does not prevent the sellers' opportunistic behavior. This happens because sellers trade currency at the market price η without threat of theft in the TM. So, an increase in η does not effectively reduce the rate of return on currency. Instead, it is possible that an increase in η leads to an increase in the rate of return on currency because the cost of storing currency becomes relatively smaller as the price of currency η rises. This relation between the exchange rate η and

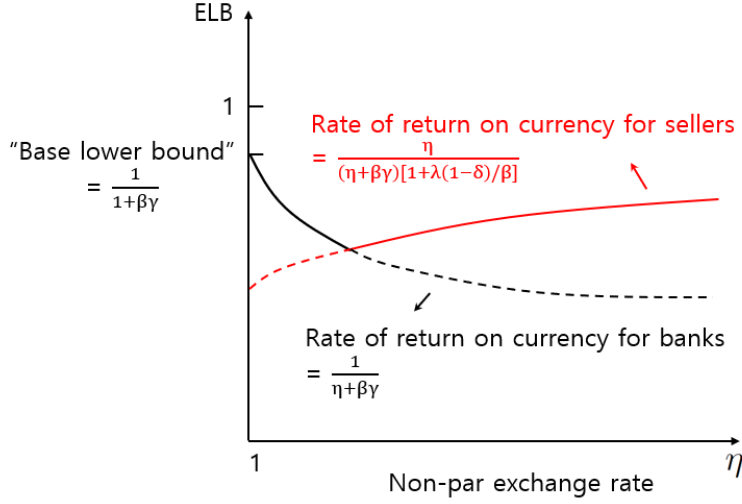


Figure 2: Exchange rate η and the effective lower bound (ELB)

the ELB is illustrated by [Figure 2](#).

Corollary 2 *Given that (R^m, η) is sufficiently close to $(1, 1)$ and the fixed cost of holding currency μ is sufficiently close to zero, suppose the central bank increases η and decreases R^m to hold ηR^m constant. Then, this policy increases x^c and decreases the ELB. Furthermore, if $v \in (0, \hat{v}]$, then x^d decreases and (r^m, r^b) decrease, while the effect on π is ambiguous. If $v \in (\hat{v}, \bar{v})$, then π falls while the effects on x^d and (r^m, r^b) are ambiguous.*

If the central bank wishes to lower the nominal interest rate R^m to the level below the base lower bound, it could do so by reducing the nominal rate of return on currency and the ELB. Corollary 2 shows what happens if the central bank increases the exchange rate η and reduces the nominal interest rate on reserves R^m in the same magnitude, so that the relative rate of return on currency $\frac{1}{\eta R^m}$ remains constant. The policy leads to an increase in x^c and a decrease in the ELB, while its effects on other variables such as x^d , r^m , r^b , and π , depend on the value of consolidated government debt v . Most importantly, the policy is not neutral as it affects the DM consumption and real interest rates. These real effects arise in this equilibrium because the expected payoff for sellers from trading currency with private banks is higher than the payoff from depositing the currency with the central bank. In other words, the expected rate of return on currency perceived by sellers does not coincide with the target rate of return set by the central bank. Therefore, the policy serves to distort the allocation of currency and bank claims in DM trades.

3.6 Equilibrium with Theft

In this section, I analyze an equilibrium where some sellers carry currency into the TM and some buyers steal currency. Suppose that the cost of acquiring the theft technology κ is sufficiently low,

so that condition (29) holds in equilibrium. Then, from (17)-(20) and (29), I obtain

$$\pi = \frac{\beta}{\eta} \left[u'(x^c) - \delta u'(x^d) + \delta \right], \quad (36)$$

$$R^m = R^b = \frac{u'(x^c) - \delta u'(x^d) + \delta}{\eta [u'(x^d) - \delta u'(x^d) + \delta]}, \quad (37)$$

and real interest rates are given by

$$r^m = r^b = \frac{1}{\beta [u'(x^d) - \delta u'(x^d) + \delta]}. \quad (38)$$

From (6), (24)-(26), and (36)-(37), the binding collateral constraint (15) can be rewritten as

$$\left[u'(x^c) + \frac{\delta}{1-\delta} \right] \rho(x^c + \beta\mu) + \left[u'(x^d) + \frac{\delta}{1-\delta} \right] (1-\rho)x^d = v, \quad (39)$$

From Lemma 1 and (29), I can write the fraction of buyers who choose to acquire the theft technology α^b and the fraction of sellers who choose to carry currency in the TM α^s as

$$\alpha^b = \frac{\eta - 1}{\eta}, \quad (40)$$

$$\alpha^s = \frac{\beta\kappa}{\rho\eta(x^c + \beta\mu)}, \quad (41)$$

and I can derive a necessary condition for this equilibrium to exist, which is given by

$$\kappa < \frac{\rho\eta(x^c + \beta\mu)}{\beta}. \quad (42)$$

From (27) and (40), no arbitrage from holding currency across periods is given by

$$R^m \geq \frac{1}{\eta + \beta\gamma} \equiv ELB. \quad (43)$$

Solving the model is straightforward, as with an equilibrium with no theft. First, equations (37) and (39) solve for (x^c, x^d) , given monetary policy (R^m, η) and fiscal policy v . Then, equation (36) solves for π , equation (38) solves for (r^m, r^b) , equations (40) and (41) solve for (α^b, α^s) , and inequalities (42) and (43) give necessary conditions for this equilibrium to exist.

Effects of Monetary Policy The size of the central bank's balance sheet ω is irrelevant to asset prices or consumption quantities as in an equilibrium with no theft. So, in what follows I analyze the effects of monetary policy interventions with R^m (the nominal interest rate on reserves) and η (the non-par exchange rate between currency and reserves).

	∂x^c	∂x^d	$\partial \pi$	∂r^m	∂ELB	$\partial \alpha^b$	$\partial \alpha^s$
∂R^m	−	+	+	+	·	·	+
$\partial \eta$	−	+	−	+	−	+	?

Table 2: Effects of monetary policy (R^m, η) in an equilibrium with theft

Proposition 2 *Suppose that inequalities (42) and (43) hold in equilibrium. Then, an increase in R^m or η results in a decrease in x^c , an increase in x^d , and an increase in real interest rates (r^m, r^b) . In addition, an increase in R^m increases π and α^s with no effects on the ELB and α^b . An increase in η decreases π and the ELB, and increases α^b , but its effect on α^s is ambiguous.*

Proof See Appendix \square

With the exchange rate between currency and reserves η held constant, the effects of an increase in the nominal interest rate on reserves R^m on consumption quantities, real interest rates, and the inflation rate are qualitatively identical to those in an equilibrium with no theft, although the fraction of sellers who carry currency into the TM α^s increases in this equilibrium. In response to an increase in R^m , sellers receive a smaller quantity of real currency from buyers in the DM. As buyers have a lower incentive to invest in the theft technology when sellers hold a smaller quantity of real currency, sellers can increase the probability of carrying currency into the TM until the fraction of buyers with the theft technology α^b remains the same.

A key result is that an increase in the exchange rate between currency and reserves η (a decrease in the nominal/real rate of return on currency) leads to a decrease in the inflation rate π (an increase in the real rate of return on currency). So, the fall in the real rate of return on currency due to an increase in η is mitigated by a decrease in π in equilibrium. Also, an increase in η decreases the ELB. These results are consistent with [Eisler \(1932\)](#), [Buiter \(2010\)](#), and [Agarwal and Kimball \(2015\)](#), in that an increase in the exchange rate between currency and reserves reduces the real rate of return on currency and the ELB on nominal interest rates.

Note that a higher η , or a higher price of currency in the CM, induces buyers to invest in the theft technology more often. Then, due to a higher risk of theft, sellers become indifferent between carrying currency into the TM and safely depositing it with the central bank, although they can sell it in the CM at a higher price.¹⁶ Therefore, the central bank can successfully reduce the rate of return on currency and the ELB on nominal interest rates in this equilibrium owing to endogenous theft. However, reducing the ELB is costly because a larger fraction of buyers investing in the theft technology implies a larger welfare loss.

As in an equilibrium with no theft, an increase in the exchange rate η itself has real effects as it decreases x^c and increases x^d , r^m and r^b . These effects occur because, with R^m held constant, an increase in η leads to a decrease in the rate of return on currency relative to government bonds and

¹⁶The effect of an increase in η on the sellers' behavior is ambiguous. Although a higher η implies a higher payoff from selling currency in the CM, a higher risk of theft tends to reduce the sellers' expected payoff. Therefore, the effect of an increase in η on the fraction of sellers who carry currency into the CM α^s is ambiguous.

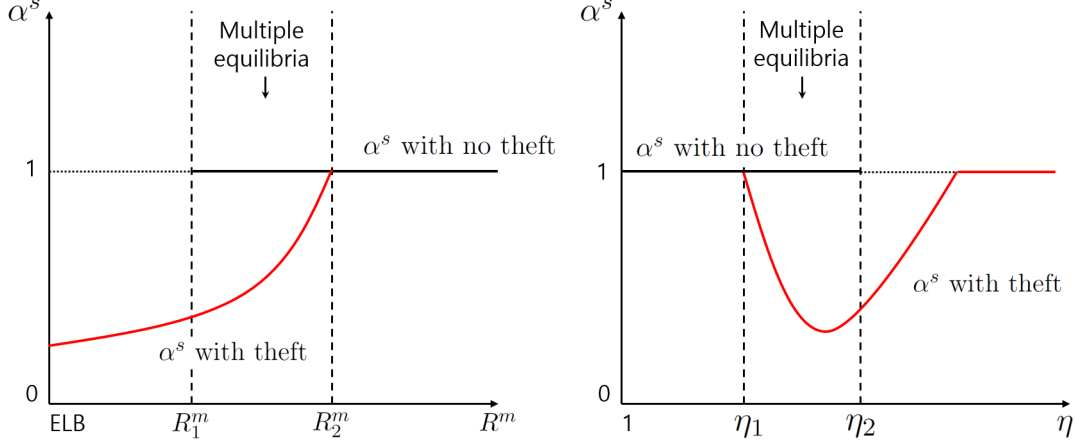


Figure 3: Equilibrium α^s with monetary policy (R^m, η)

reserves $\frac{1}{\eta R^m}$. Note that the effects of an increase in the exchange rate η on consumption quantities and real interest rates are qualitatively the same as those of an increase in R^m . These results are summarized in [Table 2](#).

Corollary 3 *Suppose the central bank increases η and decreases R^m to hold ηR^m constant. Then, this policy decreases π one-for-one, decreases α^s and the ELB, and increases α^b . However, consumption quantities (x^c, x^d) and real interest rates (r^m, r^b) remain unchanged.*

Suppose that the central bank increases the exchange rate η and reduces the interest rate on reserves R^m with $\frac{1}{\eta R^m}$, the relative rate of return on currency, held constant. In Corollary 3, the policy acts to decrease the inflation rate π one-for-one with an increase in η , implying no effect on the real rate of return on currency. Also, real interest rates do not change because a decrease in π offsets the decrease in R^m one-for-one (a pure Fisher effect). Notice that, although there are no effects on consumption quantities (x^c, x^d) , the fraction of buyers who acquire the theft technology α^b increases in equilibrium and this has welfare implications.

With necessary conditions (34) and (42), Propositions 1 and 2 help us understand what type of equilibrium (or equilibria) may arise given monetary policies (R^m, η) . With η held constant, an increase in R^m decreases x^c , so an equilibrium with no theft is more likely to arise for high R^m . Suppose η is sufficiently high or the cost of theft κ is sufficiently low, so that there exists theft for sufficiently low R^m . Then, as illustrated in the left panel of [Figure 3](#), there may exist a range of nominal interest rates R^m that support equilibria with theft and with no theft. That is, for some parameters, multiple equilibria arise. In the figure, the upper bound on nominal interest rates that support an equilibrium with theft R_2^m is higher than the lower bound on nominal interest rates that support an equilibrium with no theft R_1^m . So, multiple equilibria exist for $R^m \in [R_1^m, R_2^m]$. However, for some parameters, the upper bound R_2^m can be lower than the lower bound R_1^m , so there does not exist an equilibrium for $R^m \in (R_2^m, R_1^m)$ in that case.

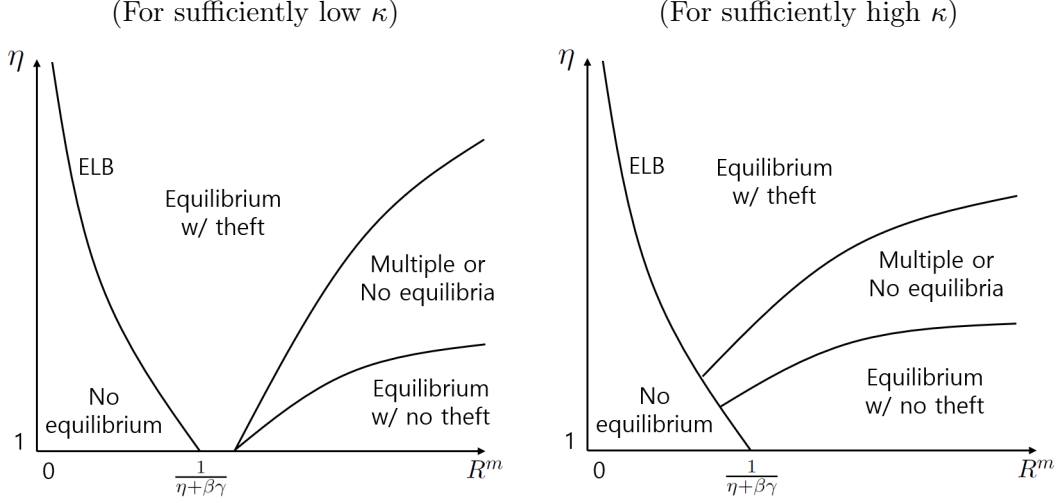


Figure 4: Equilibria with monetary policy (R^m, η)

Propositions 1 and 2 also state that, with R^m held constant, an increase in η decreases x^c in an equilibrium with theft and increases x^c in an equilibrium with no theft. This implies that an equilibrium with no theft is more likely to arise for low η and high κ (the cost of theft). Suppose either R^m or κ is sufficiently high so that there is no theft for sufficiently low η . Then, for η sufficiently high but not too high, there may exist multiple equilibria as illustrated in the right panel of Figure 3. Notice that the effect of an increase in η on α^s (the fraction of sellers carrying currency in the TM) is ambiguous in an equilibrium with theft. The right panel of Figure 3 displays a special case where an increase in η decreases α^s for low η and increases α^s for high η in an equilibrium with theft. If η is sufficiently high, it is possible that all currency-holding sellers carry currency in the TM ($\alpha^s = 1$) although some buyers steal currency in equilibrium ($0 < \alpha^b < 1$). In this section, I have focused on analyzing an equilibrium with theft where sellers are indifferent between depositing currency and not depositing, i.e., $0 < \alpha^b < 1$ and $0 < \alpha^s < 1$. Finally, Figure 4 shows how the nominal interest rate on reserves R^m and the non-par exchange rate between currency and reserves η determine the existence of particular equilibria.

4 Optimal Monetary Policy

I define welfare as

$$\mathcal{W} = \underbrace{0}_{\text{CM surpluses}} + \underbrace{\rho [u(x^c) - x^c] + (1 - \rho) [u(x^d) - x^d]}_{\text{DM surpluses}} - \underbrace{\alpha^b \kappa}_{\text{total cost in TM}}, \quad (44)$$

which is the sum of surpluses from trade in the CM and the DM, net of the total cost incurred in the TM. Welfare defined here is also equivalent to the sum of period utilities in equilibrium. I will discuss the optimal monetary policy using this measure in what follows.

Proposition 3 *If the cost of theft κ is sufficiently low and the fixed cost of holding currency μ is sufficiently close to zero, then the optimal monetary policy consists of $\eta = 1$ and $R^m \leq 1$ for given $\mu \geq 0$. However, if the optimal nominal interest rate on reserves is constrained by the base lower bound, then the optimal monetary policy is $\eta = 1$ and $R^m = \text{base lower bound}$.*

Proof See Appendix \square

To understand the intuition behind the results, consider the case where there is no fixed cost of holding currency at the end of the DM ($\mu = 0$) as a benchmark. In Appendix A.3, I provide the proof of Proposition 3 in two steps. Taking α^b (the fraction of buyers who choose to steal) as given, an optimal monetary policy can be characterized by a *modified* Friedman rule. That is, the nominal interest rate on reserves relative to currency is zero (or equivalently, $\eta R^m = 1$) at the optimum. A modified Friedman rule achieves a social optimum, given α^b , by allowing buyers to perfectly smooth their consumption across different states of the world. Then, I argue that, among those policy alternatives, the optimal monetary policy consists of $\eta = 1$ and $R^m = 1$ because this policy combination eliminates costly investment in the theft technology.

Optimality is achieved when the exchange rate between currency and reserves is one-to-one, as in a traditional central banking system, and the nominal interest rate on reserves is zero (a Friedman rule). Since a zero nominal interest rate is optimal even though negative interest rates are available, the Friedman rule policy rate can be thought of as the “reversal interest rate”, the interest rate at which lowering the interest rate becomes contractionary (Abadi et al., 2023).

Now, consider the case where there is a fixed cost of storing currency at the end of the DM ($\mu > 0$). In this case, a negative nominal interest rate is optimal ($R^m < 1$) given a one-to-one exchange rate $\eta = 1$. As currency-holding buyers need to compensate for the storage cost incurred by sellers, they carry a larger quantity of currency than in an economy with no storage costs ($\mu = 0$). Then, it is welfare-improving to reduce the cost of holding currency by lowering the nominal interest rate R^m and the inflation rate π further from the one characterized by the Friedman rule.

Note that, even if the optimal nominal interest rate is constrained by the base lower bound, introducing a non-par exchange rate to further reduce the nominal interest rate does not improve welfare. From Corollary 3, such a policy only increases costly theft without increasing surpluses from trade in the DM and the CM. Therefore, it is optimal to set the nominal interest rate at the base lower bound in this case.

Proposition 4 *If the cost of theft κ is sufficiently high, then the optimal monetary policy is given by $\eta = \bar{\eta}$ where $\bar{\eta}$ consists of the solution to (31) and (33) for $x^c = \beta(\frac{\kappa}{\rho} - \mu)$. In addition, if the fixed cost of holding currency μ is sufficiently close to zero, then the optimal nominal interest rate on reserves R^m satisfy that $R^m > 1/\bar{\eta}$.*

Proof See Appendix \square

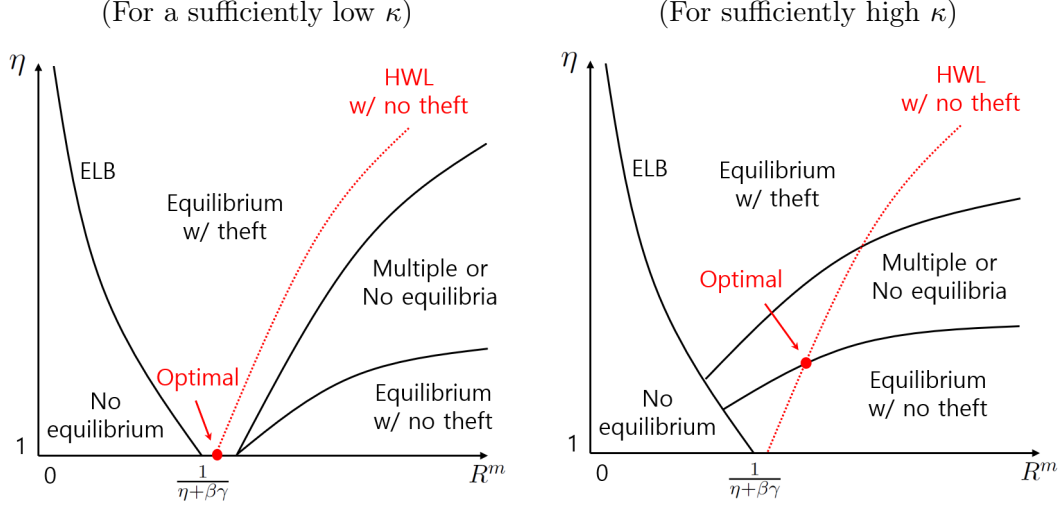


Figure 5: Optimal monetary policy

Interestingly, if the cost of theft κ is sufficiently high and theft does not take place in equilibrium, the optimal nominal interest rate can be higher than the one satisfying a modified Friedman rule. In Appendix A.3, I show that at a modified Friedman rule, or $\eta R^m = 1$, an increase in the interest rate on reserves R^m or the exchange rate η improves social welfare. An increase in R^m increases the level of welfare for the same reason as in the case with a sufficiently low cost of theft, by smoothing consumption quantities across states. However, increasing η from a modified Friedman rule improves welfare for a nonstandard reason. Note that in an equilibrium with no theft an increase in η does not effectively reduce the nominal rate of return on currency perceived by sellers but does increase the price of currency in the CM. A higher price of currency increases the quantity of consumption in DM meetings using currency, and the resulting increase in the buyer's utility from those DM meetings exceeds the potential decrease in the utility from DM meetings using bank claims. Therefore, a modified Friedman rule does not achieve a social optimum and the optimal monetary policy can be characterized by $\eta R^m > 1$. Also, as the level of welfare increases with η , it is optimal to set the exchange rate at the highest possible level that does not cause theft in equilibrium.

These results can be illustrated by Figure 5. In the figure, the HWL curve in each panel depicts the locus of nominal interest rates R^m that deliver the highest welfare given an exchange rate η in an equilibrium with no theft. If the cost of theft κ is sufficiently low, then welfare can be maximized at $\eta = 1$ and $R^m < 1$ as in the left panel. In contrast, if κ is sufficiently high, then welfare can be maximized at the highest possible level of η , with the corresponding R^m on the HWL curve, that supports an equilibrium with no theft as a unique equilibrium.

5 Disintermediation

In the baseline model, only currency is accepted in some transactions while only bank deposit claims are accepted in other transactions. In this case, no arbitrage conditions from holding currency across

periods determine the effective lower bound (ELB) on nominal interest rates, as shown earlier. Another concern about implementing a negative interest rate is a possibility of disintermediation: consumers can choose to withdraw all currency from their deposit accounts.¹⁷ However, as currency and bank deposits are not substitutable, disintermediation does not occur in the baseline model. To understand the implications of introducing a non-par exchange rate for potential disintermediation, I modify the baseline model by assuming that currency can be acceptable in all DM transactions. Although this is an extreme assumption, it will allow us to examine how a non-par exchange rate for currency withdrawals helps prevent disintermediation.

As it is possible to use only currency in DM transactions, buyers might benefit from opting out of banking arrangements. In the previous sections, I assumed that a fraction $1 - \rho$ of sellers accept only bank claims as a means of payment in DM transactions. Here, I assume that those sellers accept both currency and bank claims, while a fraction ρ of sellers accept only currency as in the baseline model. Other than that, the model is the same as the one described in Section 2.

Let θ denote the fraction of buyers who choose to deposit with private banks in the CM. Each private bank's contracting problem and the first-order conditions for the bank's problem are identical to those in the baseline model. A fraction $1 - \theta$ of buyers choose to opt out of banking arrangements and use only currency in DM transactions. Each of these buyers solves

$$\max_{c^o \geq 0} \left\{ -\eta c^o + u \left(\frac{\beta c^o [1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta]}{\pi} - \beta \mu \right) \right\}, \quad (45)$$

where c^o is the real quantity of currency acquired by each buyer in the CM. Although private banks do not write a deposit contract with these buyers, I assume that private banks withdraw currency from the central bank to sell it to these buyers whenever necessary. Then, a no arbitrage condition implies that the price of currency is η in equilibrium. The first-order condition for each of these buyer's problem is given by

$$-\eta + \frac{\beta [1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta]}{\pi} u' \left(\frac{\beta c^o [1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta]}{\pi} - \beta \mu \right) = 0, \quad (46)$$

and let x^o denote the consumption quantity in DM meetings for the buyer. Asset market clearing conditions are given by

$$\theta c + (1 - \theta)c^o = \bar{c}; \quad \theta m = \bar{m}; \quad \theta b = \bar{b}. \quad (47)$$

Let U^b denote the expected utility for buyers who write banking contracts and let U^o denote the expected utility for buyers who opt out of banking contracts. Then, using (14), (17)-(19), and

¹⁷Disintermediation is a practical concern. As a negative deposit rate might lead to massive cash withdrawals, private banks may not want to reduce their deposit rates below zero. See Eggertsson et al. (2022) for empirical evidence on the breakdown of the monetary policy transmission when the policy rate is negative.

(46), U^b and U^o can be written as

$$U^b = \rho [u(x^c) - (x^c + \beta\mu)u'(x^c)] + (1 - \rho) [u(x^d) - x^d u'(x^d)], \quad (48)$$

$$U^o = u(x^o) - (x^o + \beta\mu)u'(x^o). \quad (49)$$

In equilibrium, the fraction θ must be the solution to the following problem:

$$\max_{0 \leq \theta \leq 1} [\theta U^b + (1 - \theta)U^o], \quad (50)$$

which implies that θ must be consistent with each buyer's utility maximization problem.

Depending on who the seller meets in the DM, the quantity of currency can differ across currency-holding sellers in the TM. Some buyers write deposit contracts and withdraw c' units of real currency in equilibrium. This implies that sellers who meet these buyers in the DM hold $\frac{c'}{\pi}$ units of real currency in the following TM. Some buyers use only currency in the DM, so sellers who meet these buyers in the DM hold $\frac{c^o}{\pi}$ units of real currency in the following TM. For convenience, I assume that each seller decides whether to carry currency into the TM before realizing the type of buyer he or she meets in the DM.

The fraction of buyers who acquire the theft technology α^b and the fraction of sellers who choose to participate in side trades of currency α^s must be incentive compatible in equilibrium. Then,

$$\text{if } \kappa > \frac{[\theta\rho c' + (1 - \theta)c^o] \alpha^s \eta}{\pi}, \text{ then } \alpha^b = 0, \quad (51)$$

$$\text{if } \kappa = \frac{[\theta\rho c' + (1 - \theta)c^o] \alpha^s \eta}{\pi}, \text{ then } 0 \leq \alpha^b \leq 1, \quad (52)$$

$$\text{if } \kappa < \frac{[\theta\rho c' + (1 - \theta)c^o] \alpha^s \eta}{\pi}, \text{ then } \alpha^b = 1. \quad (53)$$

With probability $\theta\rho\alpha^s$, each buyer is matched with a seller holding $\frac{c'}{\pi}$ units of real currency and with probability $(1 - \theta)\alpha^s$ each buyer is matched with a seller holding $\frac{c^o}{\pi}$ units of real currency. Conditions (51)-(53) state that theft does not occur if theft is too costly, theft sometimes takes place if the buyer is indifferent between stealing and not stealing, and theft always takes place if theft is profitable. Also, each seller's decision on whether to carry currency into the TM must be incentive compatible. That is,

$$\text{if } (1 - \alpha^b)\eta < 1, \text{ then } \alpha^s = 0, \quad (54)$$

$$\text{if } (1 - \alpha^b)\eta = 1, \text{ then } 0 \leq \alpha^s \leq 1, \quad (55)$$

$$\text{if } (1 - \alpha^b)\eta > 1, \text{ then } \alpha^s = 1. \quad (56)$$

In this section, I confine attention to an equilibrium with a sufficiently low cost of theft κ , implying that theft takes place in equilibrium.¹⁸ Also, notice that I have focused on cases where

¹⁸I can provide equilibrium conditions given a sufficiently high cost of theft, but analyzing the effects of monetary

collateralizable assets are sufficiently scarce. Specifically in this section, I will assume that

$$v < x^* + \beta\mu, \quad (57)$$

where x^* is the efficient quantity of consumption in DM transactions that solves $u'(x) = 1$. This assumption implies that the consumption quantities in DM transactions for buyers choosing bank contracts are inefficiently low due to the shortage of collateral. But, this also implies that the central bank cannot support the efficient quantity of consumption for those opting out of banking contracts. This is because the quantity of currency outstanding is constrained by the size of the central bank's balance sheet, which can only be increased by the central bank's purchase of scarce government debt. Then, the following proposition characterizes the effective lower bound (ELB) on nominal interest rates R^m .

Proposition 5 *Suppose that both the cost of theft κ and the fixed cost of holding currency μ are sufficiently low, and that the value of consolidated government debt outstanding v satisfies (57). Then, an equilibrium exists if and only if*

$$R^m \geq \frac{u'(x^o)}{\eta[(1-\delta)u'(x^d) + \delta]}, \quad (58)$$

where (x^o, x^d) , together with x^c , are the solutions to $(x^o + \beta\mu)u'(x^o) = v$, $u'(x^o) = u'(x^c) - \delta u'(x^d) + \delta$, and $U^b = U^o$ from (48)-(49). Furthermore,

$$\frac{u'(x^o)}{(1-\delta)u'(x^d) + \delta} > 1.$$

Proof See Appendix \square

Proposition 5 shows that any nominal interest rate R^m below the threshold presented in (58) cannot be supported in equilibrium. This is because a sufficiently low R^m induces all buyers to use only currency in DM transactions ($\theta = 0$) but there is a shortage of government debt to back the required quantity of currency. That is, the central bank cannot issue the required quantity of currency to meet the public demand, if R^m is sufficiently low. The right-hand side of (58) can be interpreted as the ELB on nominal interest rates that prevents complete disintermediation.¹⁹ Therefore, introducing a non-par exchange rate $\eta > 1$ can reduce the ELB as in [Eisler \(1932\)](#), [Buiter \(2010\)](#), and [Agarwal and Kimball \(2015\)](#).

policy in that case appears not to be straightforward.

¹⁹This finding also implies that the lower bound on nominal interest rates preventing complete disintermediation is always higher than the one preventing arbitrage. Although I abstract from the conflict between the two different lower bounds, a driving force for the imperfect pass-through of negative policy rates in [Eggertsson et al. \(2022\)](#), my findings suggest that a non-par exchange rate policy can reduce both lower bounds by increasing currency-holding costs faced by individuals.

The proportional cost of storing currency γ becomes irrelevant here as nominal interest rates that effectively encourage buyers to participate in banking arrangements are sufficiently high to prevent arbitrage opportunities from storing currency across periods. Interestingly, the ELB on nominal interest rates can be positive given a one-to-one exchange rate ($\eta = 1$). When the nominal interest rate on reserves is zero ($R^m = 1$) and the fixed cost of holding currency μ is sufficiently low, there is an incentive for buyers to opt out of deposit contracts because $x^o > x^c = x^d$. That is, there is inefficiency in the banking system due to a shortage of government debt and a binding collateral constraint, leading to lower consumption in the DM for contracting buyers $x^c = x^d$ than non-contracting buyers x^o . As a complete flight to currency cannot be supported in equilibrium, the nominal interest rate must be higher than zero to prevent buyers from opting out of deposit contracts.²⁰

From now on, I will consider cases where $R^m > \frac{u'(x^o)}{\eta[u'(x^d) - \delta u'(x^d) + \delta]}$ and assume that $\mu = 0$ for analytical convenience.²¹ That is, there is no fixed cost of holding currency at the beginning of the TM. Then, from (6), (17)-(22), and (46)-(47), I obtain

$$\eta R^m = \frac{u'(x^c) - \delta u'(x^d) + \delta}{u'(x^d) - \delta u'(x^d) + \delta}, \quad (59)$$

$$(1 - \rho)\theta x^d \left[u'(x^d) + \frac{\delta}{1 - \delta} \right] + \rho\theta(x^c) \left[u'(x^c) + \frac{\delta}{1 - \delta} \right] + (1 - \theta)(x^o)u'(x^o) = v, \quad (60)$$

$$\pi = \frac{\beta [u'(x^c) - \delta u'(x^d) + \delta]}{\eta}. \quad (61)$$

Equations (59)-(61) come from the first-order conditions for a private bank's problem in equilibrium and equation (60) is the collateral constraint. The collateral constraint here is somewhat different from (39), the collateral constraint in the baseline model, as some government debt must be purchased by the central bank to issue currency for buyers who opt out of banking arrangements. From (17), (18), (46), and (48)-(50),

$$u'(x^o) = u'(x^c) - \delta u'(x^d) + \delta, \quad (62)$$

$$\rho [u(x^c) - x^c u'(x^c)] + (1 - \rho) [u(x^d) - x^d u'(x^d)] \geq u(x^o) - x^o u'(x^o). \quad (63)$$

Equation (62) is a necessary condition that guarantees positive quantities of consumption in DM transactions for both types of buyers (buyers who choose to participate in banking arrangements and buyers who choose not to do so). This equation implies that $x^c < x^o < x^d$ in equilibrium. Notice that there is inefficiency in deposit contracts arising from the shortage of collateral, which not only constrains x^d but also constrains x^c . So, bank deposit contracts are useful only when

²⁰One can imagine that a higher fixed storage cost μ would increase inefficiency in DM transactions using currency. This would reduce the incentive for buyers to opt out of deposit contracts, which would serve to lower the ELB. While analyzing the qualitative effect of an increase in μ appears to be complicated, the intuition suggests that a sufficiently high μ would generate a negative ELB on nominal interest rates.

²¹The results obtained here can be applied to cases with a sufficiently low μ .

$x^c < x^d$ because otherwise buyers would strictly prefer opting out of deposit contracts.²² Equation (63) shows that, in equilibrium, buyers must weakly prefer participating in a banking arrangement to not participating. Finally, from (52) and (55),

$$\alpha^b = \frac{\eta - 1}{\eta}, \quad (64)$$

$$\alpha^s = \frac{\beta \kappa}{\eta [\theta \rho x^c + (1 - \theta) x^o]}, \quad (65)$$

$$\kappa < \frac{\eta [\theta \rho x^c + (1 - \theta) x^o]}{\beta}. \quad (66)$$

Equations (64) and (65) determine the fraction of buyers who choose to acquire the theft technology α^b and the fraction of sellers who choose to carry currency in the TM α^s . Inequality (66) is a necessary condition for this equilibrium to exist.

I can solve the model differently depending on whether (63) holds with equality. If (63) holds with equality, equations (59), (62), and (63) solve for x^c, x^d , and x^o given monetary policy (R^m, η) and fiscal policy v . Then, equation (60) solves for θ , equation (61) solves for π , and equations (64) and (65) solve for α^b and α^s , respectively. If (63) holds with strict inequality, equations (59) and (60) with $\theta = 0$ solve for x^c and x^d . Then, equation (61) solves for π and equations (64) and (65) solve for α^b and α^s , respectively, with $\theta = 0$. Equation (62) solves for x^o , the would-be quantity of consumption in DM transactions if, off equilibrium, a buyer were to opt out of banking arrangements.

The following proposition shows how the fraction θ is determined in equilibrium and the effects of monetary policy (R^m, η) depending on the value of θ .

Proposition 6 *Suppose that the fixed cost of holding currency μ is zero. If ηR^m is sufficiently high but not too high, then $0 \leq \theta < 1$ in equilibrium and (63) holds with equality. In this case, an increase in ηR^m (an increase in R^m or η or both) leads to decreases in x^c and x^o and increases in x^d and θ along with an increase in real interest rates (r^m, r^b) . Furthermore, an increase in R^m increases π and α^s but does not affect α^b . An increase in η decreases π and increases α^b but its effect on α^s is ambiguous. If ηR^m is very high, then $\theta = 1$ in equilibrium and (63) holds with strict inequality.*

Proof See Appendix \square

In Proposition 6, a higher nominal interest rate R^m or exchange rate η implies a lower rate of return on currency relative to reserves and government bonds, leading to a substitution of bank claims for currency in DM transactions. If $0 \leq \theta < 1$ in equilibrium, the consumption quantity in DM transactions using bank claims x^d increases along with a rise in the fraction of buyers who

²²However, using currency in transactions is also inefficient due to the fixed cost of holding currency μ . With a sufficiently high μ , buyers could choose to use deposit contracts even though $x^d \leq x^c$.

participate in banking arrangements θ , while the consumption quantities in DM transactions using currency x^c and x^o decrease. So, given a traditional central banking system with $\eta = 1$, lowering the nominal interest rate R^m can contribute to disintermediation in that the fraction of buyers participating in banking arrangements θ falls.

The central bank, however, can introduce an appropriate non-par exchange rate between currency and reserves η that helps hold the relative rate of return on currency $\frac{1}{\eta R^m}$ constant. This implies that the central bank can implement a negative nominal interest rate without causing a disruptive effect on the banking system, consistent with the conventional view, since the fraction θ as well as x^c , x^d , and x^o would remain unchanged with ηR^m held constant. But, the fraction of buyers who invest in the costly theft technology α^b would increase, which has welfare implications.

I can define a welfare measure for this economy as

$$\mathcal{W} = \rho \theta [u(x^c) - x^c] + (1 - \rho) \theta [u(x^d) - x^d] + (1 - \theta) [u(x^o) - x^o] - \alpha^b \kappa, \quad (67)$$

which is the sum of surpluses from trade in the CM and the DM, net of the total cost of theft in the TM. Then, the following proposition characterizes the optimal monetary policy.

Proposition 7 *Suppose that the fixed cost of holding currency μ is zero. The optimal monetary policy consists of $\eta = 1$ and $R^m = \frac{u'(x^o)}{u'(x^d) - \delta u'(x^d) + \delta} > 1$, where (x^o, x^d) , together with x^c , are the solutions to $x^o u'(x^o) = v$, (62), and (63) with equality.*

Proof See Appendix \square

The central bank can maximize the sum of surpluses from trade in the CM and the DM by choosing an appropriate policy (η, R^m) . However, setting a one-to-one exchange rate between currency and reserves ($\eta = 1$), as in a traditional central banking system, is optimal because the central bank can eliminate costly theft without reducing welfare. Also, optimality is obtained when the central bank sets the nominal interest rate on reserves R^m at the base lower bound. As the base lower bound is higher than one provided that there is no fixed cost of holding currency, the quantity of consumption in DM transactions using bank claims is higher than the quantities in other DM transactions using currency for any given R^m . Therefore, lowering the nominal interest rate always improves welfare as it allows buyers to better smooth their consumption across DM transactions.

Which model specification would better capture real-world payment systems, the baseline model or the modified one? In the baseline model, only currency is used in some transactions while only bank claims are used in other transactions. In contrast, in the modified model, currency is widely accepted but bank deposits are not. In practice, payment systems exhibit a mix of characteristics from both model specifications. Some transactions such as online transactions cannot be made with currency, while some transactions can be made only with currency either because the consumer or the retailer does not have access to the banking system or because they prioritize privacy. Also, there are transactions where both means of payment can be accepted.

6 Conclusion

In the literature, a non-par exchange rate between currency and reserves has been proposed as a potential policy instrument that could reduce the effective lower bound (ELB) on nominal interest rates. I have constructed a model with two means of payment, currency and bank deposits, and frictions associated with the storage and security costs of currency to study the implications of introducing a non-par exchange rate. A key finding is that a non-par exchange rate can indeed reduce the ELB on nominal interest rates if there exist sufficient frictions that induce agents to exchange currency and reserves rather than avoiding the central bank.

Introducing a non-par exchange rate, however, can be costly because lowering the ELB must be accompanied by an enhancement in the frictions that determine the ELB.²³ Specifically, a non-par exchange rate can increase the market value of currency, which encourages socially undesirable behavior such as currency side trading and theft. Even if the optimal interest rate is constrained by the ELB, a non-par exchange rate does not help increase welfare because the effect of reducing the ELB offsets the effect of lowering the nominal interest rate. As this policy only increases the resource cost associated with the undesirable activities, the optimal monetary policy is to set the nominal interest rate at the current level of the ELB and maintain the one-to-one exchange rate between currency and reserves. With a modified version of the model, I have also shown that a non-par exchange rate can help the central bank implement a negative interest rate without causing disintermediation, although this can decrease welfare.

Replacing physical currency with central bank digital currency (CBDC) can also help reduce the ELB. This is because the central bank can directly set a negative nominal interest rate on CBDC, which is impossible with physical currency. Although the central bank can easily reduce the ELB, the effect of implementing a negative interest rate on reserves with a negative interest rate on CBDC remains ambiguous. This paper suggests that such a policy would not improve welfare because the nominal interest rate on reserves relative to CBDC, which determines the monetary policy stance, would remain unchanged.²⁴

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²³In Appendix A.2, I calibrate the model to the U.S. economy to quantify the magnitude of the welfare cost incurred by a nonpar exchange rate. Compared to the welfare cost of inflation, I find that the welfare cost of reducing the ELB is not negligible.

²⁴This conjecture is consistent with the work by Williamson (2022) in that, in an economy with CBDC, government bonds, and private capital, the relevant interest rate is the interest rate on government bonds relative to CBDC rather than the interest rate on government bonds relative to zero interest currency.

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A Appendix

A.1 Equilibrium with Deflation and No Theft

In this section, I confine attention to stationary equilibria where there is deflation and the cost of theft is sufficiently high. Due to deflation, the real value of currency increases over time and private banks can acquire a sufficient amount of currency in the CM. Private banks are no longer indifferent between acquiring currency from other private individuals and withdrawing currency from the central bank because they do not need to bear the cost of withdrawing currency. Instead, sellers must be indifferent between depositing their currency with the central bank and side trading with private banks. So, one unit of real currency must be exchanged for one unit of good in the CM in equilibrium. From (7) and (28), a necessary condition for theft to not take place is given by

$$\kappa \geq \frac{\rho(x^c + \beta\mu)}{\beta}.$$

As the price of real currency is one instead of η in equilibrium, η ’s in equations (6)-(12), (17)-(22) must be replaced by one. Then, from (19) and (21), no arbitrage condition for private banks from holding currency across periods can be expressed by

$$R^m \geq \frac{1}{1 + \beta\gamma}.$$

That is, the non-par exchange rate η is irrelevant to the effective lower bound (ELB). This occurs because there are no currency withdrawals from the central bank cash window. Also, it is obvious that the non-par exchange rate does not affect equilibrium prices and allocations.

The inflation rate in this equilibrium can be written as

$$\pi = \beta \left[u'(x^c) - \delta u'(x^d) + \delta \right].$$

To observe deflation in equilibrium, β and $u'(x^c)$ must be sufficiently low while δ and $u'(x^d)$ must be sufficiently high. It turns out that v must be sufficiently high and R^m is sufficiently low to support deflation in equilibrium. Note that in the body of the paper I focus on cases with sufficiently low v and γ so that there is no deflation in equilibrium for any R^m that is higher than the ELB.

A.2 Quantitative Analysis

Theoretically, introducing a non-par exchange rate between currency and reserves can decrease welfare by encouraging costly theft and distorting the equilibrium allocation. To understand the magnitude of the welfare cost, I calibrate the baseline model to the U.S. economy, and conduct a counterfactual analysis to evaluate the welfare cost of introducing a non-par exchange rate between currency and reserves.

A.2.1 Calibration

I consider an annual model and assume that the utility function in the DM takes the form $u(x) = \frac{x^{1-\sigma}}{1-\sigma}$. When calibrating the baseline model to data, I exclude the cost of theft κ because a one-to-one exchange rate between currency and reserves implies no theft in the model.²⁵ Then, there are eight parameters to calibrate: σ (the curvature of DM consumption), β (discount factor), ρ (the fraction of currency transactions in the DM), μ (the fixed cost of storing currency), γ (the proportional cost of storing currency), δ (the fraction of assets private banks can abscond with), R^m (the nominal interest rate on reserves), and v (the value of government liabilities held by the public).

Table 3 summarizes the calibration results along with the target moments. Most of the target moments are constructed from the U.S. data for 2013-2015. I consider the period 2013-2015 because it is proper to consider a time period when the policy rate was close to zero as the purpose of this exercise is to evaluate the welfare cost of reducing the ELB. Also, key variables such as the nominal interest rate on reserves and domestically-held public debt to GDP were stable during this period.

There are three parameters calibrated externally. The discount factor β is given by $\beta = 0.96$. From Federal Reserve Economic Data (FRED), the nominal interest rate on reserves was 0.25 percent over period 2013-2015 ($R^m = 1.0025$). Finally, the lowest target range for the federal funds rate has

²⁵To quantify the welfare cost arising from an increase in theft, the cost of theft κ needs to be calibrated. Since theft does not occur in the model given a one-to-one exchange rate, parameter κ must be directly calibrated outside the model. However, to the best of my knowledge, there is no data that allows measuring the cost of theft.

Parameters	Values	Calibration targets	Sources
β	0.96	Standard in literature	
R^m	1.0025	Avg. interest rate on reserves: 0.25%	FRED
γ	0.00	Lowest target range for fed funds rate: 0-0.25%	FRED
σ	0.17	Money demand elasticity (1959-2007): -4.19	FRED
ρ	0.17	Currency to M1 ratio: 17.22%	FRED; Lucas and Nicolini (2015)
v	1.13	Avg. locally-held public debt to GDP: 66.73%	FRED
δ	0.45	Avg. inflation rate: 1.06%	FRED
μ	0.01	Fixed storage cost: 2% of currency payments	Author's assumption

Table 3: Calibration results

been between 0 and 0.25 percent since 1954. Although this does not imply that the proportional cost of storing currency is zero, the proportional cost γ is assumed to be zero for convenience.²⁶

Calibrating σ (the curvature of DM consumption) involves matching the elasticity of money demand in the model with the empirical money demand elasticity obtained from the data. Estimating the money demand elasticity requires a longer time-series data, so I choose the time period from 1959 to 2007.²⁷ Using data on currency in circulation and nominal GDP from FRED, I calculate the currency-to-GDP ratios. Then, the money demand elasticity can be estimated using Moody's AAA corporate bond yields from FRED and the currency-to-GDP ratios, and the estimated elasticity is -4.19.²⁸

Then, I jointly calibrate four parameters: the curvature of DM consumption σ , the fraction of currency transactions in the DM ρ , the value of government liabilities held by the public v , the fraction of assets that can be absconded δ , and the fixed cost of storing currency μ . The curvature parameter σ is calibrated to match the estimated money demand elasticity. Using the currency-in-circulation data from FRED and the new M1 series from Lucas and Nicolini (2015), I calibrate the fraction of currency transactions in the DM ρ until the model generates the currency-to-M1 ratio. I use domestically-held public debt to GDP from FRED to calibrate the value of publicly-held government liabilities v .²⁹ Another variable I use to calibrate parameters is the inflation rate. Together with other parameters, the fraction of assets that can be absconded δ is calibrated so that the model generates an inflation rate consistent with the observed rate of 1.06 percent. Finally, I calibrate the fixed cost of storing currency μ to be 2 percent of cash payments.³⁰

²⁶The proportional cost of storing currency implies that the ELB on the nominal interest rate can be negative. However, the Federal Reserve might have faced some legal and political issues of implementing negative nominal interest rates. As a negative rate has not been explored in the U.S., it seems difficult to calibrate the proportional cost of storing currency with this model.

²⁷In the aftermath of global financial crisis in 2007-2008, the demand for currency has increased possibly due to non-transactional purposes. To calculate the elasticity of money demand only for transactions, I exclude post-crisis data as in Chiu et al. (2022) and Altermatt and Wang (2022), for example.

²⁸The interest rate on liquid bonds (e.g., 3-month Treasury Bill rate) can fluctuate due to liquidity premium. I consider AAA corporate bond yield as the nominal interest rate on illiquid bonds and $\frac{\pi}{\beta} - 1$ as its theoretical counterpart.

²⁹I define domestically-held public debt by total public debt net of public debt held by foreign and international investors.

³⁰This assumption implies that each buyer pays approximately 2 percent more to purchase goods in currency

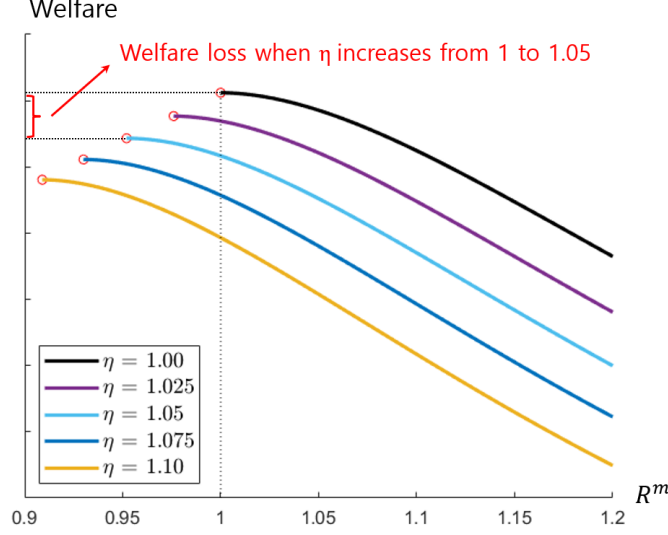


Figure 6: Monetary policy (R^m, η) and welfare

A.2.2 Counterfactual Analysis

I consider three different environments where the fixed cost of theft κ is (i) 2.5 percent, (ii) 5 percent, and (iii) 10 percent of the current consumption level. Given the calibrated parameters and each κ , I vary the non-par exchange rate η and find the corresponding nominal interest rate on reserves that maximizes welfare, denoted by R_η^* . As illustrated by Figure 6, an increase in η decreases the ELB on the nominal interest rate. But, the welfare level under the optimal nominal interest rate R_η^* decreases as η increases.

My approach to quantifying the welfare cost of increasing η is to measure how much consumption private individuals would need to be compensated to endure the non-par exchange rate η . For any (R^m, η) , the welfare measure is given by

$$\mathcal{W}(R^m, \eta) = \rho[u(x^c) - x^c] + (1 - \rho)[u(x^d) - x^d] - \alpha^b \kappa.$$

If I choose a non-par exchange rate η but also adjust the quantities of consumption in the DM by a factor Δ , welfare is expressed as

$$\mathcal{W}_\Delta(R^m = R_\eta^*, \eta) = \rho[u(\Delta x^c) - x^c] + (1 - \rho)[u(\Delta x^d) - x^d] - \alpha^b \kappa.$$

Then, I can obtain the value Δ_η that solves $\mathcal{W}_{\Delta_\eta}(R^m = R_\eta^*, \eta > 1) = \mathcal{W}(R^m = R_\eta^*, \eta = 1)$. The welfare cost of introducing η can be measured as $\Delta_\eta - 1$ percent of consumption. If private individuals are compensated with this amount of consumption, they would be indifferent between

transactions, to compensate the seller's storage cost. The fixed storage cost μ could be lower or higher than 2 percent of cash payments, but varying μ from 0 percent to 10 percent does not make much difference for counterfactual analysis.

η	ELB	R_η^*	$(\Delta_\eta - 1) \times 100$		
			$\kappa = 2.5\%$	$\kappa = 5\%$	$\kappa = 10\%$
1.00	1.000	1.000	-	-	-
1.025	0.976	0.976	0.0561	0.1118	0.2236
1.05	0.952	0.952	0.1095	0.2183	0.4367
1.075	0.930	0.930	0.1604	0.3199	0.6399
1.10	0.909	0.909	0.2091	0.4168	0.8339

Table 4: ELB, optimal interest rate, and the welfare cost of reducing the ELB

the two policy choices: a one-to-one exchange rate and a non-par exchange rate.

Table 4 presents the ELB, the optimal nominal interest rate R_η^* , and the welfare cost of introducing a non-par exchange rate η given a fixed cost of theft κ . Specifically, an increase in η reduces both the ELB and the optimal interest rate regardless of the cost of theft. Recall that, given a fixed cost of storing currency close to zero ($\mu \approx 0$), the optimal monetary policy can be characterized by a modified Friedman rule ($\eta R^m \approx 1$). So, an increase in η decreases the optimal nominal interest rate R_η^* . As monetary policy is conducted optimally given a non-par exchange rate η , there would be no distortion in the equilibrium prices and allocations.

Introducing a non-par exchange rate η , however, increases the aggregate cost of theft in equilibrium. If the fixed cost of theft κ is 2.5 percent of the current consumption level, increasing η by 5 percent and 10 percent costs, respectively, 0.11 percent and 0.21 percent of consumption. If κ is 5 percent of the current consumption level, increasing η by the same magnitudes costs, respectively, 0.22 percent and 0.42 percent of consumption. Finally, if the value of κ is the same as 10 percent of the current consumption level, increasing η by, respectively, 5 percent and 10 percent, costs 0.44 percent and 0.84 percent of consumption.³¹

How large is the welfare cost of introducing a non-par exchange rate? To better understand its magnitude, the welfare cost computed here can be compared with estimates for the welfare cost of another policy that has been frequently discussed in the literature: the welfare cost of 10 percent inflation. As the estimates for the welfare cost of 10 percent inflation are typically around 1 percent of consumption, the welfare cost of introducing a non-par exchange rate seems significant.³² Note that the welfare cost of using a non-par exchange rate critically depends on the fixed cost of investing in the theft technology κ . As κ increases, the welfare cost also increases proportionally.

A.3 Omitted Proofs

Proof of Lemma 1: First, consider the case where $\eta = 1$. In this case, sellers are indifferent between depositing the currency with the central bank and trading it with a private bank only if

³¹Assuming a different fixed cost of storing currency μ does not significantly change the result. For example, if μ is 10 percent of cash payments and κ is 5 percent of the current consumption level, increasing η by 5 percent and 10 percent costs, respectively, 0.2185 percent and 0.4172 percent of consumption.

³²The welfare cost of increasing inflation from 0 percent to 10 percent is 0.62 percent of consumption in Chiu and Molico (2010), 0.87 percent in Lucas (2000), and 1.32 percent (take-it-or-leave-it offer) in Lagos and Wright (2005), for example.

there is no theft, i.e., $\alpha^b = 0$. Also, $\alpha^s = 0$ is optimal for all $\alpha^b \in (0, 1]$. Suppose that $\alpha^b \in (0, 1]$ in equilibrium. Then, sellers would always choose to deposit their currency with the central bank, implying $\alpha^s = 0$. Then, there would be no incentives for buyers to acquire the theft technology by incurring κ units of labor, which contradicts with $\alpha^b \in (0, 1]$. Therefore, there must be no theft in equilibrium, if an equilibrium exists. Now, suppose that $\alpha^b = 0$ in equilibrium. A necessary condition for this equilibrium to exist is $\kappa \geq \rho\alpha^s\eta c^s$. If $\kappa > \rho\eta c^s$, then $\alpha^b = 0$ is optimal for buyers for any given $\alpha^s \in [0, 1]$. If $\kappa = \rho\eta c^s$, buyers are indifferent between acquiring the theft technology and not doing anything in the TM. In this case, an equilibrium exists only if $\alpha^b = 0$. If $\kappa < \rho\eta c^s$, then equilibrium exists only if the fraction of sellers who carry currency in the TM is sufficiently low. Since a necessary condition for the absence of theft is $\kappa \geq \rho\alpha^s\eta c^s$, an equilibrium exists with $\alpha^s \in [0, \bar{\alpha}^s]$ where $\bar{\alpha}^s = \frac{\kappa}{\rho\eta c^s}$. Therefore, given that $\eta = 1$, there exist a continuum of equilibria with $\alpha^b = 0$ and $\alpha^s \in [0, 1]$ for $\kappa \geq \rho\eta c^s$ and a continuum of equilibria with $\alpha^b = 0$ and $\alpha^s \in [0, \bar{\alpha}^s]$ for $\kappa < \rho\eta c^s$ where $\bar{\alpha}^s = \frac{\kappa}{\rho\eta c^s}$.

Next, I consider the case where $\eta > 1$. Suppose that $\alpha^b = 0$ in equilibrium. Then, this leads to $\alpha^s = 1$ as sellers would strictly prefer to trade currency with a private bank rather than depositing it with the central bank. A necessary condition for this equilibrium to exist is $\kappa \geq \rho\eta c^s$. Therefore, a unique equilibrium exists with $\alpha^b = 0$ and $\alpha^s = 1$ for $\kappa \geq \rho\eta c^s$. If $\kappa < \rho\eta c^s$, then $\alpha^b = 0$ cannot be supported in equilibrium as there are incentives for buyers to acquire the theft technology, given that $\alpha^s = 1$. However, $\alpha^b = 1$ cannot be an equilibrium as well because $\alpha^b = 1$ would lead to $\alpha^s = 0$, and then, there would be no incentives for buyers to acquire the theft technology. An equilibrium exists if and only if buyers and sellers are both indifferent between their own options. This implies that, from (8) and (11),

$$\alpha^b = \frac{\eta - 1}{\eta}, \quad (68)$$

$$\alpha^s = \frac{\kappa}{\rho\eta c^s}. \quad (69)$$

Therefore, there exists a unique equilibrium with (68) and (69) for $\kappa < \rho\eta c^s$.

Proof of Proposition 1: Note that, in equilibrium, equations (31) and (33) solve for x^c and x^d . Confine attention to the comparative statics analysis with respect to R^m . I totally differentiate (31) and (33) and evaluate the derivatives of x^c and x^d for $(R^m, \eta) = (1, 1)$ and $\mu = 0$ to obtain

$$\begin{aligned} \frac{dx^c}{dR^m} &= \frac{(1 - \rho)[(1 - \delta)u'(x) + \delta][(1 - \delta)(1 - \sigma)u'(x) + \delta]}{u''(x)[(1 - \delta)(1 - \sigma)u'(x) + \delta + \rho\beta\mu(1 - \delta)u''(x)]} < 0, \\ \frac{dx^d}{dR^m} &= \frac{-\rho[(1 - \delta)u'(x) + \delta][(1 - \delta)(1 - \sigma)u'(x) + \delta + \beta\mu(1 - \delta)u''(x)]}{u''(x)[(1 - \delta)(1 - \sigma)u'(x) + \delta + \rho\beta\mu(1 - \delta)u''(x)]} > 0, \end{aligned}$$

where $x^c = x^d = x$. Then, it is immediate that from (30) and (32) r^m , r^b , and π increase, and from the first argument in (35) the ELB remains unchanged.

Now, I turn my attention to the comparative statics analysis with respect to η . For convenience,

let $\sigma = -\frac{xu''(x)}{u'(x)}$ so that $\sigma \in (0, 1)$. Then, evaluating the derivatives of x^c and x^d with respect to η for $(R^m, \eta) = (1, 1)$ and $\mu = 0$ yields

$$\begin{aligned}\frac{dx^c}{d\eta} &= -\frac{\delta\{\rho\sigma u'(x) + (1-\rho)[(1-\delta)(1-\sigma)u'(x) + \delta][u'(x) - 1] - \beta\mu\rho u''(x)\}}{u''(x)[(1-\delta)(1-\sigma)u'(x) + \delta + \rho\beta\mu(1-\delta)u''(x)]} > 0, \\ \frac{dx^d}{d\eta} &= \frac{\rho\delta[(1-\sigma)u'(x) - 1 + \beta\mu u''(x)][(1-\delta)u'(x) + \delta]}{u''(x)[(1-\delta)(1-\sigma)u'(x) + \delta + \rho\beta\mu(1-\delta)u''(x)]}.\end{aligned}$$

Also, I evaluate equation (33) for $(R^m, \eta) = (1, 1)$ and $\mu = 0$ to obtain

$$\left[u'(x) + \frac{\delta}{1-\delta}\right](x + \rho\beta\mu) = v, \quad (70)$$

where x is increasing in v . Collateral constraint (70) does not bind in equilibrium if

$$v \geq \frac{x^* + \rho\beta\mu}{1-\delta}. \quad (71)$$

Let \bar{v} denote the right-hand side of inequality (71), \hat{x} denote the solution to $u'(x) = \frac{1}{1-\sigma}$, and \hat{v} denote the solution to (70) when $x = \hat{x}$. Then, I can write the derivatives of x^d with respect to η as

$$\begin{aligned}\frac{dx^d}{d\eta} &\leq 0, & \text{if } v \in (0, \hat{v}] \\ \frac{dx^d}{d\eta} &> 0, & \text{if } v \in (\hat{v}, \bar{v})\end{aligned}$$

So, from (30), an increase in η decrease r^m and r^b for $v \in (0, \hat{v}]$ and increase r^m and r^b for $v \in (\hat{v}, \bar{v})$. From (32) or

$$\pi = \beta R^m [u'(x^d) - \delta u'(x^d) + \delta],$$

an increase in η increases π for $v \in (0, \hat{v}]$ and decreases π for $v \in (\hat{v}, \bar{v})$. Finally, from the first argument in (35), the ELB falls.

Proof of Proposition 2: Notice that the consumption quantities in the DM, x^c and x^d , are determined by equations (37) and (39). I can rewrite equation (39) in the form

$$F(x^c, x^d) = v, \quad (72)$$

and show that the function $F(\cdot, \cdot)$ is strictly increasing in both $0 \leq x^c < x^*$ and $0 \leq x^d < x^*$ because $-x\frac{u''(x)}{u'(x)} < 1$. This property implies that equation (72) can be depicted by a downward-sloping locus in (x^c, x^d) space, given v . Also, equation (37) can be depicted by an upward-sloping locus in (x^c, x^d) space, given (R^m, η) .

For the comparative statics, suppose there is an increase in R^m with η held constant. It is straightforward that an increase in R^m decreases x^c and increases x^d from (37) and (39). Then,

from (36) and (38), π rises and real interest rates (r^m, r^b) rise. From (27), (40), and (41), α^s increases but α^b and the ELB do not change. Next, suppose that there is an increase in η with R^m remaining constant. Then, from (37) and (39), x^c decreases and x^d increases. From (38), real interest rates (r^m, r^b) rise. Using (37), (36) can be written as

$$\pi = \beta R^m \left[u'(x^d) - \delta u'(x^d) + \delta \right],$$

so π falls. From (27) and (40), α^b increases and the ELB falls. However, from (41), the effect on α^s is ambiguous since ηx^c can increase or decrease depending on parameters.

Proof of Proposition 3: The proof involves two steps. First, I will search for monetary policies that maximize the welfare measure \mathcal{W} , taking α^b as exogenously given. Then, I will determine the optimal monetary policy considering that α^b is endogenously determined in response to a change in monetary policy.

In the first step, I solve the following maximization problem given $\alpha^b \in [0, 1]$:

$$\max_{(R^m, \eta)} \rho [u(x^c) - x^c] + (1 - \rho) \left[u(x^d) - x^d \right] - \alpha^b \kappa \quad (73)$$

subject to

$$\eta R^m = \frac{u'(x^c) - \delta u'(x^d) + \delta}{u'(x^d) - \delta u'(x^d) + \delta}, \quad (74)$$

$$\left[u'(x^c) + \frac{\delta}{1 - \delta} \right] \rho (x^c + \beta \mu) + \left[u'(x^d) + \frac{\delta}{1 - \delta} \right] (1 - \rho) x^d = v, \quad (75)$$

$$R^m \geq \frac{1}{\eta + \beta \gamma}, \quad \eta \geq 1 \quad (76)$$

Note that a monetary policy measure that is relevant to welfare in equilibrium is ηR^m . Let $\Omega \equiv \eta R^m$ denote the policy measure. Then, from (76), Ω must satisfy that $\Omega \geq \frac{\eta}{\eta + \beta \gamma}$. Differentiating the objective (73) with respect to Ω gives

$$\frac{d\mathcal{W}}{d\Omega} = \rho [u'(x^c) - 1] \frac{dx^c}{d\Omega} + (1 - \rho) [u'(x^d) - 1] \frac{dx^d}{d\Omega}. \quad (77)$$

Let $\sigma = -\frac{xu''(x)}{u'(x)}$. Then, from totally differentiating (74) and (75), I obtain

$$\frac{dx^c}{d\Omega} = \frac{(1 - \rho) \left[(1 - \sigma)u'(x^d) + \frac{\delta}{1 - \delta} \right] [(1 - \delta)u'(x^d) + \delta]}{\Phi} < 0, \quad (78)$$

$$\frac{dx^d}{d\Omega} = \frac{-\rho \left[(1 - \sigma)u'(x^c) + \frac{\delta}{1 - \delta} + \beta \mu u''(x^c) \right] [(1 - \delta)u'(x^d) + \delta]}{\Phi} > 0, \quad (79)$$

where

$$\begin{aligned}\Phi = (1 - \rho)u''(x^c) & \left[(1 - \sigma)u'(x^d) + \frac{\delta}{1 - \delta} \right] \\ & + \rho u''(x^d) [(1 - \delta)\Omega + \delta] \left[(1 - \sigma)u'(x^c) + \frac{\delta}{1 - \delta} + \beta\mu u''(x^c) \right] < 0,\end{aligned}$$

for a sufficiently low μ . Note that a monetary policy Ω attains a local optimum if the resulting consumption allocation x^c and x^d satisfy $\frac{d\mathcal{W}}{d\Omega} = 0$. From (77)-(79), I can characterize the optimal allocation x^c and x^d as follows:

$$\begin{aligned}\frac{d\mathcal{W}}{d\Omega} &= 0, \\ \Leftrightarrow [u'(x^c) - 1] & \left[(1 - \sigma)u'(x^d) + \frac{\delta}{1 - \delta} \right] - [u'(x^d) - 1] \left[(1 - \sigma)u'(x^c) + \frac{\delta}{1 - \delta} + \beta\mu u''(x^c) \right] = 0, \\ \Rightarrow [u'(x^c) - 1] & \left[(1 - \sigma)u'(x^d) + \frac{\delta}{1 - \delta} \right] \leq [u'(x^d) - 1] \left[(1 - \sigma)u'(x^c) + \frac{\delta}{1 - \delta} \right], \\ \Leftrightarrow \frac{u'(x^c) - 1}{(1 - \sigma)u'(x^c) + \frac{\delta}{1 - \delta}} & \leq \frac{u'(x^d) - 1}{(1 - \sigma)u'(x^d) + \frac{\delta}{1 - \delta}}.\end{aligned}$$

Since the function $F(x) = \frac{u'(x) - 1}{(1 - \sigma)u'(x) + \frac{\delta}{1 - \delta}}$ is strictly decreasing in x , the above inequality is equivalent to $x^c \geq x^d$. Note that $x^c = x^d$ if $\Omega = 1$ and that x^c decreases and x^d increases as Ω rises. Therefore, the optimal monetary policy Ω must satisfy $\Omega \leq 1$ where the inequality holds with equality if and only if $\mu = 0$.

From the first step, I have shown that an optimal monetary policy must be a combination of (R^m, η) such that $\eta R^m \leq 1$. All the optimal combinations of monetary policy lead to the same gains from trade in the DM, that is, $\rho[u(x^c) - x^c] + (1 - \rho)[u(x^d) - x^d]$. However, from (40), the fraction of buyers who choose to steal currency in the TM α^b increases as the exchange rate η rises. This implies that the welfare measure \mathcal{W} is maximized if and only if $\eta = 1$. Therefore, the optimal monetary policy is given by $\eta = 1$ and $R^m \leq 1$.

Proof of Proposition 4: Suppose that the cost of theft κ is sufficiently high. Then, there is no theft or $\alpha^b = 0$ in equilibrium. To show that social welfare is increasing in η , suppose that the central bank sets the nominal interest rate on reserves R^m to obtain $x^c = x^d = x$ given an exchange rate between currency and reserves η . Such policy needs not be optimal but it helps understand the optimal level of the exchange rate η . Equations (31) and (33) can be rewritten as

$$R^m = R^b = \frac{\eta u'(x) - \delta u'(x) + \delta}{\eta [u'(x) - \delta u'(x) + \delta]}, \quad (80)$$

$$u'(x)[x + \rho\beta\mu] + \frac{[\rho + (1 - \rho)\eta]\delta x + \rho\delta\beta\mu}{(1 - \delta)\eta} = v. \quad (81)$$

In this case, equation (81) solves for x and then equation (80) solves for R^m . If the value of the consolidated government debt is sufficiently low, or

$$v \leq \frac{[(1 - \delta\rho)\eta + \delta\rho]x^* + \rho\beta\mu[(1 - \delta)\eta + \delta]}{(1 - \delta)\eta},$$

then x increases with η for a sufficiently low μ . Since the level of welfare is given by $\mathcal{W} = u(x) - x$ in this equilibrium, an increase in η effectively increases the level of welfare as long as the nominal interest rate R^m can be chosen to achieve $x^c = x^d = x$. But, from Proposition 1, an increase in η must be accompanied by an increase in R^m to attain the same consumption quantities across two types of DM transactions, implying that choosing R^m is not constrained by the ELB. Although the optimal R^m may not satisfy $x^c = x^d$, that the social welfare is increasing in η remains unchanged. Finally, η must be sufficiently low so that buyers do not have incentives to steal currency. Therefore, at the optimum, η is chosen so that buyers are indifferent between stealing currency and not stealing.

Now, suppose that there is no fixed cost of holding currency at the beginning of the TM, i.e., $\mu = 0$. Consider the following maximization problem given $\eta \geq 1$:

$$\max_{(R^m, \eta)} \rho [u(x^c) - x^c] + (1 - \rho) [u(x^d) - x^d] \quad (82)$$

subject to

$$\eta R^m = \frac{\eta u'(x^c) - \delta u'(x^d) + \delta}{u'(x^d) - \delta u'(x^d) + \delta}, \quad (83)$$

$$\left[u'(x^c) + \frac{\delta}{(1 - \delta)\eta} \right] \rho x^c + \left[u'(x^d) + \frac{\delta}{1 - \delta} \right] (1 - \rho)x^d = v, \quad (84)$$

$$R^m \geq \max \left\{ \frac{1}{\eta + \beta\gamma}, \frac{\eta}{(\eta + \beta\gamma)[(1 - \delta)u'(x^d) + \delta]} \right\}. \quad (85)$$

I differentiate the objective (82) with respect to R^m to obtain

$$\frac{d\mathcal{W}}{dR^m} = \rho [u'(x^c) - 1] \frac{dx^c}{dR^m} + (1 - \rho) [u'(x^d) - 1] \frac{dx^d}{dR^m}. \quad (86)$$

Letting $\sigma = -\frac{xu''(x)}{u'(x)}$ and totally differentiating (83) and (84) gives

$$\begin{aligned} \frac{dx^c}{dR^m} &= \frac{\eta(1 - \rho) \left[(1 - \sigma)u'(x^d) + \frac{\delta}{1 - \delta} \right] [(1 - \delta)u'(x^d) + \delta]}{\Lambda} < 0, \\ \frac{dx^d}{dR^m} &= \frac{-\eta\rho \left[(1 - \sigma)u'(x^c) + \frac{\delta}{(1 - \delta)\eta} \right] [(1 - \delta)u'(x^d) + \delta]}{\Lambda} > 0, \end{aligned}$$

where

$$\begin{aligned}\Lambda = (1 - \rho)\eta u''(x^c) \left[(1 - \sigma)u'(x^d) + \frac{\delta}{1 - \delta} \right] \\ + \rho u''(x^d) \left[(1 - \sigma)u'(x^c) + \frac{\delta}{(1 - \delta)\eta} \right] [\eta R^m(1 - \delta) + \delta] < 0.\end{aligned}$$

Then, I can evaluate the derivative of \mathcal{W} or equation (86) for $\eta R^m = 1$. Noting that $\eta u'(x^c) = u'(x^d)$ from (83), I obtain

$$\left. \frac{d\mathcal{W}}{dR^m} \right|_{\eta R^m=1} = \frac{\eta(1 - \eta)\rho(1 - \rho) \left[(1 - \delta)u'(x^d) + \delta \right]}{\eta^2(1 - \rho)u''(x^c) + \rho u''(x^d)} \geq 0, \quad (87)$$

implying that $\eta R^m \geq 1$ at an optimum.

Next, differentiate the objective (82) with respect to η to obtain

$$\frac{d\mathcal{W}}{d\eta} = \rho \left[u'(x^c) - 1 \right] \frac{dx^c}{d\eta} + (1 - \rho) \left[u'(x^d) - 1 \right] \frac{dx^d}{d\eta}. \quad (88)$$

Totally differentiate (83) and (84) to get $\frac{dx^c}{d\eta}$ and $\frac{dx^d}{d\eta}$ and then evaluate the derivatives for $\eta R^m = 1$. This gives

$$\begin{aligned}\frac{dx^c}{d\eta} &= \frac{\delta \rho x^c u''(x^d) - \delta \eta(1 - \rho) \left[u'(x^d) - 1 \right] \left[(1 - \delta)(1 - \sigma)u'(x^d) + \delta \right]}{\eta \left[(1 - \delta)(1 - \sigma)u'(x^d) + \delta \right] [\rho u''(x^d) + \eta^2(1 - \rho)u''(x^c)]} > 0, \\ \frac{dx^d}{d\eta} &= \frac{\delta \rho \left\{ \eta x^c u''(x^c) + \left[u'(x^d) - 1 \right] \left[(1 - \delta)(1 - \sigma)u'(x^d) + \delta \right] \right\}}{\eta \left[(1 - \delta)(1 - \sigma)u'(x^d) + \delta \right] [\rho u''(x^d) + \eta^2(1 - \rho)u''(x^c)]}.\end{aligned}$$

Using (88), I obtain

$$\left. \frac{d\mathcal{W}}{d\eta} \right|_{\eta R^m=1} = \frac{\Gamma + \rho \delta x^c \left\{ \rho u''(x^d) \left[u'(x^c) - 1 \right] + \eta(1 - \rho)u''(x^c) \left[u'(x^d) - 1 \right] \right\}}{\eta \left[(1 - \delta)(1 - \sigma)u'(x^d) + \delta \right] [\rho u''(x^d) + \eta^2(1 - \rho)u''(x^c)]}, \quad (89)$$

where

$$\Gamma = \rho \delta (1 - \rho)(\eta - 1) \left[u'(x^d) - 1 \right] \left[(1 - \delta)(1 - \sigma)u'(x^d) + \delta \right] \geq 0.$$

From (89), the derivative of \mathcal{W} is strictly positive if $\eta = R^m = 1$, i.e., $\left. \frac{d\mathcal{W}}{d\eta} \right|_{\eta=R^m=1} > 0$. This implies that the monetary policy at $\eta = R^m = 1$ is not optimal. Therefore, from (87) and (89), I conclude that the optimal monetary policy is away from a modified Friedman rule or $\eta R^m > 1$.

Proof of Proposition 5: Suppose that $\theta = 0$ in equilibrium. From (17)-(19) and (46),

$$\eta R^m \left[u'(x^d) - \delta u'(x^d) + \delta \right] = u'(x^o), \quad (90)$$

$$u'(x^o) = u'(x^c) - \delta u'(x^d) + \delta, \quad (91)$$

where (x^c, x^d) are the off-equilibrium consumption quantities in DM transactions, if a buyer were to participate in banking contracts. It can be shown that $\left| \frac{d[u'(x^d) - \delta u'(x^d) + \delta]}{d[\eta R^m]} \right| < 1$, so from (90) x^o increases with a decrease in ηR^m . However, the limited quantity of collateral $v < x^* + \beta\mu$ implies that, from (60), the highest possible quantity for x^o is \bar{x} that solves $(\bar{x} + \beta\mu)u'(\bar{x}) = v$ and $\bar{x} < x^*$. So, any ηR^m that leads to x^o higher than \bar{x} cannot be supported in equilibrium, implying that, from (90),

$$R^m \geq \frac{u'(\bar{x})}{\eta [u'(x^d) - \delta u'(x^d) + \delta]}, \quad (92)$$

where x^d is the off-equilibrium consumption quantity in DM transactions using bank claims that is consistent with (91). Also, any R^m higher than the right-hand side of (92) implies that $0 < \theta \leq 1$ and $U^b \geq U^o$. So, by continuity, the off-equilibrium consumption quantities (x^c, x^d) when $R^m = \frac{u'(\bar{x})}{\eta [u'(x^d) - \delta u'(x^d) + \delta]}$ must satisfy (91) and $U^b = U^o$ from (48)-(49) given $x^o = \bar{x}$.

Recall that the nominal interest rate R^m must satisfy (43). That is, there must be no arbitrage opportunities from carrying currency across periods in equilibrium. To prove that the lower bound on the nominal interest rate in inequality (92) is always higher than the lower bound in (43), I claim that the following condition holds in equilibrium if the fixed cost of holding currency μ is close to zero:

$$\frac{u'(\bar{x})}{u'(x^d) - \delta u'(x^d) + \delta} > 1. \quad (93)$$

Suppose $\eta R^m = 1$ in equilibrium, so that deposit contracts effectively allow buyers to consume the same quantity of goods across two types of DM transactions, i.e., $x^c = x^d = x$. Then, the quantity of DM consumption for buyers opting out of deposit contracts is higher than the quantity for buyers holding deposit contracts ($x^o > x$) since, from (90)-(91),

$$u'(x^o) = (1 - \delta)u'(x) + \delta.$$

This implies that the expected utility for buyers opting out of contracts is higher than the expected utility for buyers opting in because from (48)-(49),

$$U^o - U^b = [u(x^o) - x^o u'(x^o)] - [u(x) - x u'(x)] - \beta\mu [u'(x^o) - \rho u'(x)] > 0,$$

for a sufficiently low μ . So, $\eta R^m = 1$ cannot be supported in equilibrium (a contradiction) as this policy would lead to a complete disintermediation, i.e., $\theta = 0$. To encourage banking activities, $\eta R^m > 1$ must be satisfied so that (93) must hold in equilibrium. Also, $\frac{u'(\bar{x})}{\eta [u'(x^d) - \delta u'(x^d) + \delta]} > \frac{1}{\eta + \beta\gamma}$ for any $\eta \geq 1$ because $\eta + \beta\gamma$ increases more than $\eta [u'(x^d) - \delta u'(x^d) + \delta]$ as η rises. Therefore, the

effective lower bound on the nominal interest rate is determined by (92).

Proof of Proposition 6: First, note that an equilibrium with $\theta = 0$ exists only if R^m is set at the effective lower bound (ELB) defined in (58). As mentioned in Proof of Proposition 5, any R^m higher than the ELB implies that $x^o < \bar{x}$ where $\bar{x}u'(\bar{x}) = v$. Since $x^ou'(x^o) < v$ and the collateral constraint must bind in an equilibrium where v is sufficiently low, there must be some buyers participating in banking contracts, i.e., $\theta > 0$. So, for any R^m that is higher than the ELB, $\theta > 0$ in equilibrium.

Next, consider an equilibrium with $0 < \theta < 1$. Then, (x^c, x^d, x^o, θ) must satisfy:

$$\eta R^m = \frac{u'(x^c) - \delta u'(x^d) + \delta}{u'(x^d) - \delta u'(x^d) + \delta}, \quad (94)$$

$$(1 - \rho)\theta x^d \left[u'(x^d) + \frac{\delta}{1 - \delta} \right] + \rho\theta x^c \left[u'(x^c) + \frac{\delta}{1 - \delta} \right] + (1 - \theta)x^ou'(x^o) = v, \quad (95)$$

$$u'(x^o) = u'(x^c) - \delta u'(x^d) + \delta, \quad (96)$$

$$\rho [u(x^c) - x^cu'(x^c)] + (1 - \rho) [u(x^d) - x^du'(x^d)] = u(x^o) - x^ou'(x^o). \quad (97)$$

Suppose that there is an increase in ηR^m . Then, from (94), x^c decreases and x^d increases as ηR^m rises. From (96), x^o decreases as x^c decreases and x^d increases. Also, from (96), a necessary condition for this equilibrium to exist is $x^c < x^d$, which implies that U^b decreases as x^c falls and x^d rises. Since the left-hand side of (97) decreases as x^c falls and x^d rises, x^o must fall in equilibrium. Then, from (95), θ must rise in equilibrium. The effects of an increase in R^m or an increase in η on $(\pi, \alpha^b, \alpha^s)$ are straightforward from (61), (64), and (65).

As an increase in ηR^m decreases x^c and x^o and increases x^d and θ , there exists $\Omega = \eta R^m$ that satisfies equation (94) where x^c and x^d are the solutions to equations (95)-(97) when $\theta = 1$. Therefore, I can conclude that, in equilibrium, $0 \leq \theta < 1$ if $\frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta} \leq \eta R^m < \Omega$ and $\theta = 1$ if $\eta R^m \geq \Omega$ where \bar{x} and \underline{x}^d are the quantities defined in Proof of Proposition 5.

Proof of Proposition 7: In Proof of Proposition 6, I have shown that, for any $\eta R^m > \frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta}$, the fraction θ is positive and $x^c < x^d$ in equilibrium. This implies that the left-hand side and the right-hand side of (97) both increase as x^c and x^o rise and x^d falls. So, given η , lowering R^m increases the welfare measure \mathcal{W} because it increases x^c and x^o and decreases x^d and θ . Then, by continuity, the maximum \mathcal{W} can be obtained when $\eta R^m = \frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta}$ given η . However, if the central bank conducts monetary policy (R^m, η) such that $\eta R^m = \frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta}$, a higher η only implies a higher α^b without increasing the sum of surpluses from trade in the CM and the DM. As a higher α^b leads to a larger total cost of theft, the welfare measure \mathcal{W} is maximized if and only if $\eta = 1$ and $R^m = \frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta}$.