Regret Bounds for Online Gradient Descent with $d + 1$ point Bandit Feedback

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1 Introduction

Definition 1. Regret:

$$\mathbb{E}\frac{1}{k}\sum_{t=1}^{T}\sum_{i=1}^{k}\ell_t(y_{t,i}) - \mathbb{E}\sum_{t=1}^{T}\min_{x_t^*\in\mathcal{K}}\ell_t(x_t^*)$$

2 Preliminaries

2.1 Problem Setting

First we formally introduce the problem. Our objective is:

$$\min_{x_t\in\mathcal{K}}\ell_t(x_t)$$

over rounds $t = 1, ...T$. Where $\mathcal{K} \subset \mathbb{R}^d$ is our action set and ℓ_t are adversarially chosen time varying loss functions. The key in our setting is that we do not have first order gradient information $\nabla\ell_t$ but we are able to get zeroth-order (Bandit) feedback with $k = d + 1$ points. First we outline our assumptions.

We assume that $\mathcal{K}$ is compact and has a nonempty interior (otherwise project $\mathcal{K}$ to a lower dimensional space). For this work, when we implicitly or explicitly refer to norms, we will be using the euclidean norm. We also have the following assumptions for each loss function ℓ_t:

Assumption 1. Let $\mathcal{B}$ denote the unit ball centered at the origin. There exists $r, D > 0$ such that

$$r\mathcal{B} \subseteq \mathcal{K} \subseteq D\mathcal{B}$$

Assumption 2. The gradient of ℓ_t over $\mathcal{K}$ is bounded:

$$\|\nabla\ell_t(x)\| \leq G \quad \forall t, \forall x \in \mathcal{K}$$

Assumption 3. Strong Convexity. For $\mu \geq 0$, ℓ_t is μ-strongly convex over the set $\mathcal{K}$:

$$\ell_t(x) \geq \ell_t(y) + \nabla\ell_t(y)^\intercal(x - y) + \frac{\mu}{2}\|x - y\|^2, \quad \forall x, y \in \mathcal{K}$$

Assumption 4. ℓ_t is L-smooth on $\mathcal{K}$ if it is differentiable on an open set containing $\mathcal{K}$ and its gradient is Lipschitz continuous with constant L:

$$\|\nabla\ell_t(x) - \nabla\ell_t(y)\| \leq L\|x - y\|, \quad \forall x, y \in \mathcal{K}$$

.

We also use the notation $\mathbb{E}_t$ to denote the conditional expectation conditioned on all randomness in the first $t - 1$ rounds.

3 Projected Gradient Descent with k queries per round

We now present that main result, generic for k query feedback.

Lemma 1.

Lemma 2. For any point $x \in \mathcal{K}$,

$$\frac{1}{k}\sum_{i=1}^{k} \ell_t(y_{t,i}) - \ell_t(x) \leq \ell_t(x_t) - \ell_t((1-\xi)x) + G\delta + GD\xi.$$

Proof. proof □

Theorem 1. Assume that the assumptions hold. Suppose on round t the algorithm plays k random queries $y_{t,1}, ..., y_{t,k}$, constructs a gradient estimator $\tilde{g}_t$ and uses the the algorithmic step $x_{t+1} = proj_{(1-\xi)\mathcal{K}}(x_t - \eta\tilde{g}_t)$ with $1/\eta \geq L$, $\delta = \frac{\log(T)}{T}$, and $\xi = \frac{\delta}{r}$. If the gradient estimator satisfies the following conditions for all $t \geq 1$:

1. $\|x_t - y_{t,i}\| \leq \delta$ for $i = 1, ...k$.
2. $\|\tilde{g}_t\| \leq G_1$ for some constant G_1.
3. $\|\mathbb{E}_t\tilde{g}_t - \nabla\ell_t(x_t)\| \leq c\delta$ for some constant c.

Then for any sequence $\{x_t^*\}_{t=1}^{T} \in \mathcal{K}^T$ we have

$$\mathbb{E}\frac{1}{k}\sum_{t=1}^{T}\sum_{i=1}^{k} \ell_t(y_{t,i}) - \mathbb{E}\sum_{t=1}^{T}\min_{x_t^* \in \mathcal{K}} \ell_t(x_t^*) \leq G_1K_1\sum_{t=2}^{T}\left\|x_t^* - x_{t-1}^*\right\| + G_1K_2 + G_1\log(T)\big(1 + 2c + \frac{D}{r}\big).$$

where the constants K_1 and K_2 are explicitly given by

$$K_1 := \frac{\|x_1 - x_1^*\| - \rho\|x_T - x_T^*\|}{(1-\rho)}, \quad K_2 := \frac{1}{1-\rho}.$$

Where $0 \leq \rho := (1-\eta\mu)^{1/2} < 0$. Is our linear convergence constant.

Proof. Start by defining $h_t(x) = \ell_t(x) + (\tilde{g}_t - \nabla\ell_t(x))^\intercal x$. Then h_t has the same convexity properties as ℓ_t. Note also that $\nabla h_t(x_t) = \tilde{g}_t$. So the algorithm is actually performing gradient descent, as if with full information) on the functions h_t restricted to $(1-\xi)\mathcal{K}$. Using the regret bound from Lemma 1 we have that

$$\mathbb{E}\sum_{t=1}^{T}\frac{1}{k}\sum_{i=1}^{k} \ell_t(y_{t,i}) - \mathbb{E}\sum_{t=1}^{T} \ell_t(x_t^*) \leq G_1K_1\sum_{t=2}^{T}\left\|x_t^* - x_{t-1}^*\right\| + G_1K_2 := Regret_T^D(OGD).$$

Then taking expectations,

$$\begin{aligned}
\mathbb{E}\sum_{t=1}^{T}[\ell_t(x_t) - \ell_t(x_t^*)] &= \mathbb{E}\sum_{t=1}^{T}[h_t(x_t) - h_t(x_t^*)] + \mathbb{E}\sum_{t=1}^{T}[\ell_t(x_t) - h_t(x_t) - \ell_t(x_t^*) + h_t(x_t^*)] \\
&\leq Regret_T^D(OGD) + \mathbb{E}\sum_{t=1}^{T}(\mathbb{E}_t\tilde{g}_t - \nabla\ell_t(x_t))^\intercal(x_t - x_t^*) \\
&\leq Regret_T^D(OGD) + 2c\delta DT.
\end{aligned}$$

Where the first inequality is by the convexity of ℓ_t and h_t. Now we use Lemma 2 and obtain

$$\mathbb{E}\frac{1}{k}\sum_{t=1}^{T}\sum_{i=1}^{k}\ell_t(y_{t,i}) - \mathbb{E}\sum_{t=1}^{T}\min_{x_t^*\in\mathcal{K}}\ell_t(x_t^*) \leq Regret_T^D(OGD) + 2c\delta DT + TG\delta + GDT\xi.$$

To finish the proof, plug in the values for δ and ξ.

$\square$

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A Code

A.1 Algorithm

A.2 Data Collection