4.5

IF: 0^0, 1^∞, ∞^0

↳ Take log first

Ex $L := \lim_{x\to\infty} x^{1/x}$ (IF: ∞^0)

Goal Eval $\lim_{x\to\infty} \ln(x^{1/x}) = \ln(L)$

$$\ln(L) = \lim_{x\to\infty} \frac{1}{x}\ln(x) \quad (\text{IF: } 0\cdot\infty)$$

$$= \lim_{x\to\infty} \frac{\ln(x)}{x} \quad \left(\text{IF: } \frac{\infty}{\infty}\right)$$

By assump

$$= \lim_{x\to\infty} \left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{1}{x} = 0 \; (= \ln(L))$$

$$L = \boxed{e^0 = 1}$$

Ex $L := \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n$ (I.F. 1^∞)

Goal Compute $\ln(L)$.

$$\lim_{n\to\infty} n \ln\left(1+\frac{x}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{x}{n}\right)}{1/n} \quad \left(\text{I.F. } \frac{0}{0}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{-xn^{-2}}{\left(1+\frac{x}{n}\right)}}{-1/n^2} = \lim_{n\to\infty} \cancel{-n^2}\left(\frac{\cancel{-xn^{-2}}}{1+\frac{x}{n}}\right)$$

$$= \lim_{n\to\infty} \frac{+x}{1+\frac{x}{n}} = \frac{x}{1+0} = x \quad (= \ln(L))$$

$$L = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$$

$$e^{5x} = \lim_{n\to\infty} \left(1+\frac{5x}{n}\right)^n$$

$$e^{x^2} = \lim_{n\to\infty} \left(1+\frac{x^2}{n}\right)^n$$

(Exam ?)

4.8 Anti-derivatives / Indefinite Integrals and Diff Eqs.

Def Let $f(x)$ be a continuous function on $[a,b]$. We say that $F(x)$ is an *anti-derivative* of $f(x)$ provided $F'(x) = f(x)$.

Ex $f(x) = 2x$

$F(x) = x^2$ is an anti-derivative of $f(x) = x^2$

Observe $F'(x) = 2x = f(x)$

$G(x) = x^2 + 3$ is also an *anti-deriv.* of $f(x) = 2x$

In particular, for any constant C, $x^2 + C$ is an anti-deriv. of $2x$

The indefinite integral $\int f(x)\,dx$ returns family of all anti-derivatives of $f(x)$.

$$\int f(x)\,dx$$

integral operator

measure/ width of tiny rectangles/ tell us variable to integrate

Basic Integral Rules

Power Rule For $n \neq -1$, $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

Check $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = \frac{1}{n+1}\left((n+1)x^n\right) + 0$

$= x^n$

Ex $\int x^3\,dx = \frac{x^4}{4} + \underbrace{C}_{}$

NEED $+C$

Rule $\int \frac{1}{x}\,dx \quad \left(= \int x^{-1}\,dx\right)$

$= \ln(|x|) + C$

Exponentials For $a > 0$, $a \neq 1$,

$$\int a^x \, dx = \frac{1}{\ln(a)} a^x + C$$

Check $$\frac{d}{dx}\left(\frac{1}{\ln(a)} a^x + C\right) = \frac{1}{\ln(a)}\left(\ln(a) a^x\right) + 0$$
$$= a^x$$

Ex $$\int e^x \, dx = e^x + C$$

Ex $$\int \sin(x) \, dx = -\cos(x) + C$$

$$\int \cos(x) \, dx = \sin(x) + C$$

$$\int \sec^2(x) \, dx = \tan(x) + C$$

$$\int \csc^2(x) \, dx = -\cot(x) + C$$

$$\int \sec(x)\tan(x) \, dx = \sec(x) + C$$

$$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$$

Sum Rule $\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

Scalar Mult For constant c, $\int c\,f(x)\,dx = c\int f(x)\,dx$

Ex $\int 2x^2\,dx = 2\int x^2\,dx = 2\left(\frac{x^3}{3}\right) + C$

(a) (b)

Diff Eqs

Ex $\frac{dy}{dx} = 2x-7$, $y(2)=0$. Want $y(x)$

initial/boundary condition

$$\int dy = \int (2x-7)\,dx$$

$$y = x^2 - 7x + C$$

$$y(2) = 2^2 - 7(2) + c = 0$$

$$4 - 14 + c = 0 \Rightarrow c = 10$$

$$\boxed{y(x) = x^2 - 7x + 10}$$

Ex $\frac{ds}{dt} = 1 + \cos(t)$, $s(0) = 4$

$$\int ds = \int (1 + \cos(t))\,dt$$

$$s = t + \sin(t) + C$$

$$s(0) = 4 = 0 + 0 + c \Rightarrow c = 4$$

$$\boxed{s(t) = t + \sin(t) + 4}$$

First-Order, Separable diff eq.