Quotient Rule Suppose $f(x)$ and $g(x)$ are differentiable at $x=c$, and suppose $g(c) \neq 0$. Let $h(x) = \frac{f(x)}{g(x)}$, then $h(x)$ is differentiable at $x=c$, with derivative:

$$h'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{(g(c))^2}$$

Ex $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$

$$f'(x) = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

Ex $f(x) = \sec(x) = \frac{1}{\cos(x)}$

$$f'(x) = \frac{\cos(x)\cdot 0 - 1\cdot(-\sin(x))}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)}$$

$$= \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\cos(x)} = \tan(x)\sec(x)$$

3.7 Implicit Differentiation

<u>Previously</u> $y = f(x)$

<u>Ex</u> $y^2 = x$ <u>Want</u> $y' \left(= \frac{dy}{dx}\right)$

$$\underbrace{\frac{d}{dx} y^2}_{} = \frac{d}{dx} x = \frac{dx}{dx} = 1 \qquad 1 = 2y \frac{dy}{dx}$$

$$\frac{d}{dx} y^2 = 2y \cdot \frac{dy}{dx} \leftarrow \text{Want} \qquad \text{So } \boxed{\frac{dy}{dx} = \frac{1}{2y}}$$

$\frac{d}{dx}$: differentiate with respect to x.

<u>Ex</u> $x^2 + y^2 = r^2$ (where $r > 0$ is a constant)

<u>Want</u> $\frac{dy}{dx}$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 2x\overbrace{\left(\frac{dx}{dx}\right)}^{1} + 2y\left(\frac{dy}{dx}\right) = 0 \quad \frac{d}{dx}(r^2) = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

Ex $y^2 = x^2 + \sin(xy)$

Find $\frac{dy}{dx}$

$$2y\left(\frac{dy}{dx}\right) = 2x + \cos(xy)\cdot\left(\frac{d}{dx}(xy)\right)$$

$$2y\left(\frac{dy}{dx}\right) = 2x + \cos(xy)\left[1\cdot y + x\frac{dy}{dx}\right]$$

$$2y\left(\frac{dy}{dx}\right) = 2x + \cos(xy)\cdot y + \underline{\cos(xy)\,x\,\frac{dy}{dx}}$$

$$2y\frac{dy}{dx} - \cos(xy)\cdot\left(x\frac{dy}{dx}\right) = 2x + \cos(xy)\cdot y$$

$$\frac{dy}{dx}\boxed{\left(2y - x\cos(xy)\right)} = 2x + y\cos(xy)$$

$$\boxed{\frac{dy}{dx} = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}}$$

Ex $2x^3 - 3y^2 = 8$

Find $\frac{dy}{dx}$

$$6x^2 - 6y\left(\frac{dy}{dx}\right) = 0$$

$$6x^2 = 6y\,\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{x^2}{y}}$$

Want Second derivative (y'' or $\frac{d^2y}{dx^2}$)

$$\frac{d^2y}{dx^2} = \frac{y(2x) - x^2\left(1\cdot\frac{dy}{dx}\right)}{y^2} = \frac{y\cdot 2x - x^2\left(\frac{dy}{dx}\right)}{y^2}$$

$$\boxed{= \frac{y(2x) - x^2\left(\frac{x^2}{y}\right)}{y^2}}$$

$\leftarrow \frac{dy}{dx}$

Tangent Lines Need

↳ Point (x_0, y_0)
↳ Slope m (evaluate $\left.\frac{dy}{dx}\right|_{(x_0,y_0)}$)

$y = mx + b$, then $y' = m$

Ex $f(x) = e^x$, tangent line at $x_0 = 0$

$y_0 = f(x_0) = f(0) = e^0 = 1$

$f'(x) = e^x$, so $m = f'(0) = e^0 = 1$

$y - 1 = 1(x - 0)$

$y - 1 = x$

Ex $x^3 + y^3 - 9xy = 0$, want tangent line @ $(2,4)$

↳ Check Is $(2,4)$ on $x^3 + y^3 - 9xy = 0$?

$2^3 + 4^3 - 9(2)(4)$
$= 8 + 64 - 72 = 0$ ✓

↳ Find $\frac{dy}{dx}$

↳ Find $\frac{dy}{dx}$:

$$3x^2 + 3y^2\left(\frac{dy}{dx}\right) - 9\left(1\cdot y + x\frac{dy}{dx}\right) = 0$$

$$3x^2 - 9y = 9x\left(\frac{dy}{dx}\right) - 3y^2\left(\frac{dy}{dx}\right)$$

$$3x^2 - 9y = 3\left(\frac{dy}{dx}\right)(3x - y^2)$$

$$\frac{dy}{dx} = \frac{3x^2 - 9y}{3(3x - y^2)} = \frac{x^2 - 3y}{3x - y^2}$$

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \frac{2^2 - 3(4)}{3(2) - 4^2} = \frac{4}{5}$$

Tangent Line $\boxed{y - 4 = \frac{4}{5}(x-2)}$