2.2

Thm (Squeeze Thm): Let $f(x), g(x), h(x)$ are functions, and let $c \in \mathbb{R}$. Suppose: $f(x) \le g(x) \le h(x)$ on some interval containing c. If:

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L,$$

then $\lim_{x \to c} g(x) = L$.

Ex $\lim_{x \to 0} x^2 e^{\sin(1/x)}$. Hint What is range of $\sin(t)$?

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

Recall e^t is increasing function

$$e^{-1} \le e^{\sin(\frac{1}{x})} \le e^1 \quad (*)$$

$$x^2 e^{-1} \le x^2 e^{\sin(\frac{1}{x})} \le x^2 e$$

$$\lim_{x \to 0} x^2 e^{-1} = 0 \qquad \lim_{x \to 0} x^2 e = 0$$

$$\boxed{\lim_{x \to 0} x^2 e^{\sin(1/x)} = 0}$$ By Squeeze Thm.

Ex Suppose $g(x)$ is a function such that

$$1 - \frac{x^2}{4} \le g(x) \le 1 + \frac{x^2}{4}, \quad \text{for all } x \in \mathbb{R}.$$

What is $\lim_{x \to 0} g(x)$?

↳ $\lim_{x \to 0} \left(1 - \frac{x^2}{4}\right) = 1$ ↳ $\lim_{x \to 0} \left(1 + \frac{x^2}{4}\right) = 1$

$\boxed{\lim_{x \to 0} g(x) = 1}$ By Squeeze Thm.

Thm Let $f(x)$, $g(x)$ be functions. Let $c \in \mathbb{R}$.

Let $L := \lim_{x \to c} f(x)$, $M := \lim_{x \to c} g(x)$.

The following hold:

↳ $\lim_{x \to c} (f(x) + g(x)) = L + M$

↳ $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$

↳ $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ $(M \neq 0)$

↳ $\lim_{x \to c} [f(x)]^n = L^n$ (n is positive integer)

↳ $\lim_{x\to c} [f(x)]^{1/n} = L^{1/n}$ (n is ~~an~~ a positive integer and $L^{1/n}$ is well-defined).

Ex $\lim_{x\to -1} \sqrt{x}$ DNE. So above rule does not apply.

2.3 δ-ε definition of a limit.

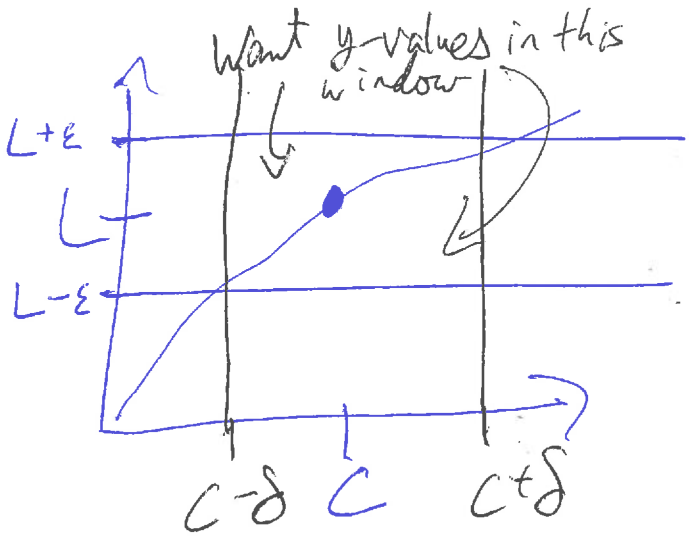
Def We say that $\lim_{x\to c} f(x) = L$ if:

For every $\varepsilon > 0$ (ε as error-term/margin of error)

There exists $\delta_\varepsilon > 0$ depending only on ε such that:

if $0 < |x-c| < \delta$, then $|f(x) - L| < \varepsilon$.

important b/c $f(c)$ may not be defined or $f(c) \neq L$.



y-value needs to be within ε of L

x-values need to be within δ of c

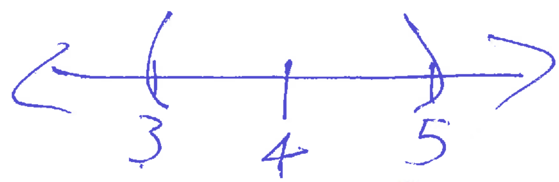
Ex $\lim_{x \to 4} (2x-1) = 7$

Given $\varepsilon = 2$ (Want $2x-1$ to be distance 2 from 7: $\text{dist}(2x-1, 7) < 2$).

Find $\delta > 0$ so that if $|x-4| < \delta$ ($\text{dist}(x,4) < \delta$), then $|(2x-1)-7| < \varepsilon$ ($\text{dist}(2x-1, 7) < \varepsilon$).

Start with desired bound and work backwards:

$$|(2x-1)-7| < 2$$
$$|2x-8| < 2$$
$$|2(x-4)| < 2$$
$$2|x-4| < 2$$
$$|x-4| < 1$$



Take $\boxed{\delta = 1}$. So if $|x-4| < 1$, then $|(2x-1)-7| < 2$

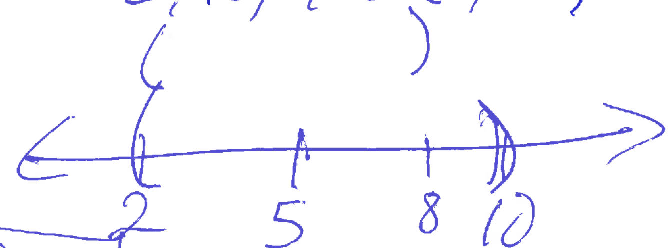
Ex $\lim_{x \to 5} \sqrt{x-1} = 2$. Given $\varepsilon = 1$.

Find $\delta > 0$.

Start $|\sqrt{x-1} - 2| < 1$

$$-1 < \sqrt{x-1} - 2 < 1$$
$$1 < \sqrt{x-1} < 3$$
$$1 < x-1 < 9$$

$2 < x < 10$
ie, $x \in (2, 10)$



$\boxed{\delta = 3}$

Ex $\lim\limits_{x \to 10} \sqrt{19-x} = 3$. Given $\varepsilon = 1$.

Find δ.

$|\sqrt{19-x} - 3| < 1$

$-1 < \sqrt{19-x} - 3 < 1$

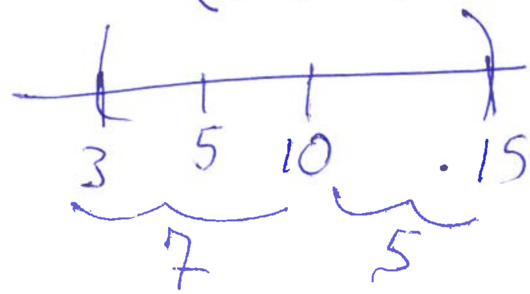
$2 < \sqrt{19-x} < 4$

$4 < 19-x < 16$

$-15 < -x < -3$

$3 < x < 15$

$(5,15) \subset (3,15)$



$\delta = 5$