8.3 Trig Integrals

Ex $\int \sin^3(x)\cos^2(x)\,dx = \int \sin(x)\,\sin^2(x)\,\cos^2(x)\,dx$

$$= \int \sin(x)\left(1-\cos^2(x)\right)\cos^2(x)\,dx$$

$$= \int \sin(x)\left(\cos^2(x) - \cos^4(x)\right)dx$$

$$= \int \sin(x)\cos^2(x)\,dx - \int \sin(x)\cos^4(x)\,dx$$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$= -\int u^2\,du - \left(-\int u^4\,du\right)$$

$$= \frac{-u^3}{3} + \frac{u^5}{5} + C$$

$$\boxed{= \frac{-\cos^3(x)}{3} + \frac{\cos^5(x)}{5} + C}$$

Ex

$$\int \sin^{11}(x)\cos^2(x)\,dx$$

$$= \int \sin(x)\,\underbrace{\sin^{10}(x)}\,\cos^2(x)\,dx$$

$$= \int \sin(x)\left(1-\cos^2(x)\right)^5 \cos^2(x)\,dx$$

$$= \int \sin(x)\Big[\binom{5}{0}1^0\left(\cos^2(x)\right)^5 + \binom{5}{1}1^1\cdot\left(\cos^2(x)\right)^4 + \binom{5}{2}1^2\left(\cos^2(x)\right)^3 + \binom{5}{3}1^3\left(\cos^2(x)\right)^2 + \binom{5}{4}1^4\cos^2(x) + \binom{5}{5}\cdot 1^5\Big]\cos^2(x)\,dx$$

$$= \int \sin(x)\cos^{12}(x)\,dx + 5\int \sin(x)\cos^{10}(x)\,dx + 10\int \sin(x)\cos^8(x)\,dx + 10\int \sin(x)\cos^6(x)\,dx + 5\int \sin(x)\cos^4(x)\,dx + \int \sin(x)\,dx$$

Ex $\int \cos^5(x)\,dx = \int \cos^4(x)\cos(x)\,dx$

$= \int (1-\sin^2(x))^2 \cos(x)\,dx$

$u = \sin(x)$
$du = \cos(x)\,dx$
$\int (1-u^2)^2\,du$

$= \int (\sin^4(x) - 2\sin^2(x) + 1)\cos(x)\,dx$

$= \int \sin^4(x)\cos(x)\,dx - 2\int \sin^2(x)\cos(x)\,dx + \int \cos(x)\,dx$

$u = \sin(x)$
$du = \cos(x)\,dx$

$= \int u^4\,du - 2\int u^2\,du + \int \cos(x)\,dx$

$= \frac{u^5}{5} - \frac{2 \cdot u^3}{3} + \sin(x) + C$

$$\boxed{= \frac{\sin^5(x)}{5} - \frac{2\sin^3(x)}{3} + \sin(x) + C}$$

Ex $\int \sin^7(x)\,dx = \int \sin^6(x)\sin(x)\,dx$

$$= \int (1-\cos^2(x))^3 \sin(x)\,\cancel{dx}$$

$u = \cos(x)$

$du = -\sin(x)\,\cancel{dx}$

$$= -\int (1-u^2)^3\,du$$

$$= -\int \left(-\binom{3}{0} u^6 + \binom{3}{1} u^4 - \binom{3}{2} u^2 + \binom{3}{3} 1 \right) du$$

$$= -\left[\frac{-u^7}{7} + \frac{3u^5}{5} - \frac{3u^3}{3} + u \right] + C$$

$$= \frac{u^7}{7} - \frac{3u^5}{5} + u^3 - u + C$$

$$\boxed{= \frac{\cos^7(x)}{7} + \frac{-3}{5}\cos^5(x) + \cos^3(x) - \cos(x) + C}$$

Recall $\cos(2x) = \cos^2(x) - \sin^2(x)$

In terms of cos: $= \cos^2(x) - (1 - \cos^2(x))$

$= 2\cos^2(x) - 1$

In terms of sin: $= (1 - \sin^2(x)) - \sin^2(x)$

$= 1 - 2\sin^2(x)$

Since $\cos(2x) = 2\cos^2(x) - 1$

$$\boxed{\Rightarrow \cos^2(x) = \frac{1 + \cos(2x)}{2}}$$

$\cos(2x) = 1 - 2\sin^2(x)$

$$\Rightarrow \boxed{\sin^2(x) = \frac{1 - \cos(2x)}{2}}$$

Ex $\int \sin^2(x)\cos^4(x)\,dx$

$$= \int \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right)^2 dx$$

$$= \frac{1}{8}\int (1-\cos(2x))(1+\cos(2x))^2\,dx$$

$$= \frac{1}{8}\int \left(1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\right)dx$$

$$= \frac{1}{8}\left[x+\frac{\sin(2x)}{2} - \int \cos^2(2x)\,dx - \int \cos^3(2x)\,dx\right]$$

$$= \frac{1}{8}\left[x+\frac{\sin(2x)}{2} - \frac{1}{2}\int (1+\cos(4x))\,dx - \int \cos(2x)\cos^2(2x)\,dx\right]$$

$$= \frac{1}{8}\left[x+\frac{\sin(2x)}{2} - \frac{x}{2} - \frac{1}{2}\left(\frac{\sin(4x)}{4}\right) - \int \cos(2x)(1-\sin^2(2x))\,dx\right]$$

$u = \sin(2x)$

$du = 2\cos(2x)$

$$\int \cos(2x)(1-\sin^2(2x))\,dx = \frac{1}{2}\int (1-u^2)\,du = \frac{1}{2}u + \frac{1}{2}\left(\frac{u^2}{3}\right) + C$$

$$= \frac{1}{2}\sin(2x) - \frac{1}{6}\sin^2(2x) + C$$

$$\boxed{= \frac{1}{8}\left[x+\frac{\sin(2x)}{2} - \frac{x}{2} - \frac{1}{8}\sin(4x) - \frac{1}{2}\sin(2x) - \frac{1}{6}\sin^2(2x)\right] + C}$$