Study Guide 2.5-2.6

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Instructions: Complete the following problems. Justify all your answers in complete sentences, where appropriate.

1 Section 2.5

Problem 1) For each of the following functions $f(x)$, determine if $f(x)$ is continuous and justify your answer. If $f(x)$ is **not** continuous, identify and classify all discontinuities.

(a) $f(x) = \dfrac{1}{x-2} - 3x$.

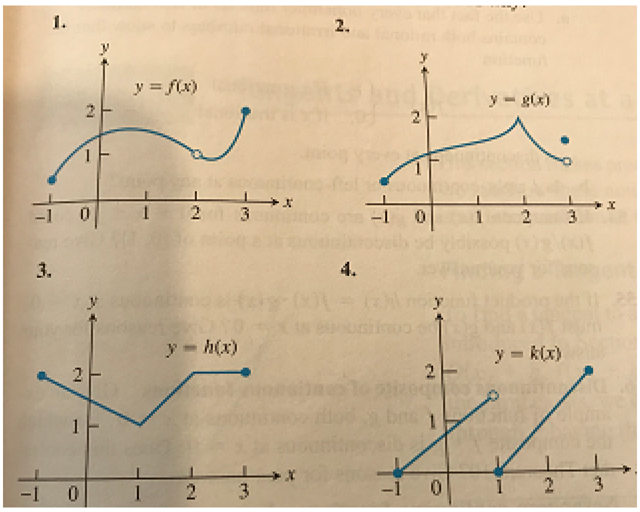
(b) $f(x) = \dfrac{x+1}{x^2 - 4x + 3}$.

(c) $f(x) = |x-1| + e^x$. [**Note:** e^x is continuous.]

(d) $f(x) = \dfrac{1}{|x|+1} + \dfrac{x^2}{2}$

(e) $f(x) = \lceil x \rceil$ [**Note:** This is known as the *ceil* function, which leaves integers unchanged and rounds up non-integer values. For example, $f(0.01) = 1$ and $f(-1) = -1$.]

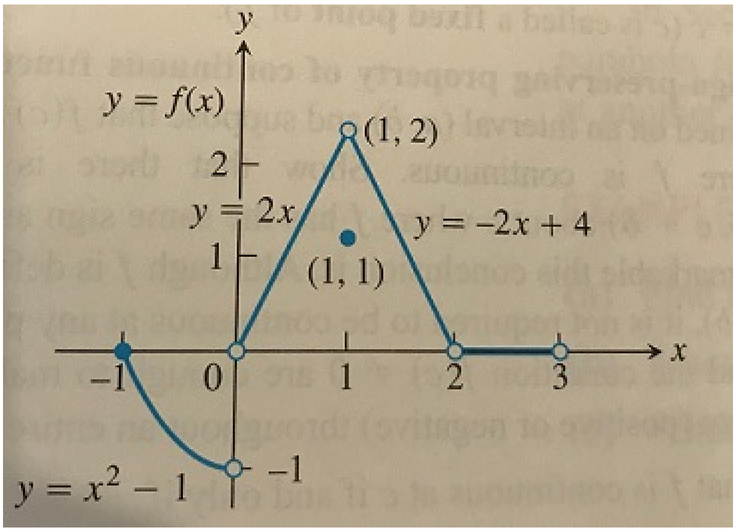
(f) The functions given by each of the following graphs.



Problem 2) Consider the following function:

$$f(x) = \begin{cases} x^2 - 1, & : -1 \leq x < 0, \\ 2x, & : 0 < x < 1 \\ 1 & : x = 1 \\ -2x + 4, & : 1 < x < 2 \\ 0, & : 2 < x < 3. \end{cases}$$

Whose graph is given by:



Answer the following questions:

(a) Does $f(-1)$ exist? What about $\lim_{x \to -1^+} f(x)$? Is it true that: $f(-1) = \lim_{x \to 1^+} f(x)$?

(b) Based on the previous part, justify whether $f(x)$ is continuous at $x = -1$.

(c) Repeat parts (a) and (b) for $x = 1$.

(d) Is $f(x)$ defined at $x = 2$? Is $f(x)$ continuous at $x = 2$? Justify your answer.

(e) On what intervals is $f(x)$ continuous?

(f) Extend $f(x)$ to a continuous function.

Problem 3) Let $h(t) = (t^2 + 3t - 10)/(t - 2)$. Define $h(2)$ in a way that extends $h(t)$ to be continuous at $t = 2$.

Problem 4) Is there a solution to $f(x) = x^5 - 2x^4 - x - 3 = 0$ on $[2, 3]$? Justify your answer.

Problem 5) Suppose $f(x) = \dfrac{-3x + 4}{-3x + 7}$. Find an interval such that $f(x)$ has a solution.

Problem 6) Suppose that $f(x)$ is a continuous function defined on the interval $[a, b]$, and the range of $f(x)$ also lies in the interval $[a, b]$. Show that $f(x)$ has a fixed point; that is, a point $x_0 \in [a, b]$ such that $f(x_0) = x_0$.[1] [**Hint:** Consider a function $g(x) = f(x) - x$ and apply the Intermediate Value Theorem.]

Problem 7) For what value of a is:

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3, \end{cases}$$

continuous for every x?

[1] This is a special case of the Brouwer Fixed Point Theorem. Fixed points are of particular interest in several areas of mathematics, including differential equations, scientific computing, game theory, and computational complexity. In his dissertation, John Nash used the Brouwer Fixed Point Theorem to establish the existence of Nash equilibria for finite, simultaneous play games. Computing Brouwer fixed points and computing Nash equilibria are both hard problems for the complexity class PPAD.

2 Section 2.6

Problem 8) For each function $f(x)$, find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

(a) $f(x) = 2x^{-1} - 3$.

(b) $f(x) = \pi - 2x^{-2}$

(c) $f(x) = \dfrac{1}{8 - (5/x)}$

(d) $f(x) = \sin(x)/x$

(e) $f(x) = \dfrac{2 - x + \sin(x)}{x + \cos(x)}$. [Hint: Still divide through by the largest power of x.]

(f) $f(x) = \dfrac{2x+3}{5x+7}$

(g) $f(x) = \dfrac{10x^5 + x^4 + 31}{x^6}$

(h) $f(x) = \dfrac{x^3 + 7x^2 - 2}{x^2 - x + 1}$

Problem 9) Evaluate the following limits.

(a) $\lim_{x\to\infty} \sqrt{\dfrac{8x^2 - 3}{2x^2 + x}}$

(b) $\lim_{x\to-\infty} \left(\dfrac{x^2 + x - 1}{8x^2 - 3}\right)^{1/3}$.