1.5/1.6 Inverse, Exponential, Logarithmic Functions

Def A function $f(x)$ is one-to-one if for every two distinct x_1, x_2 in the domain, $f(x_1) \neq f(x_2)$.

Ex $f(x) = mx + b$ $(m \neq 0)$, then $f(x)$ is one-to-one

$f(x) = 2$ is not one-to-one

$f(x) = x^3$ is one-to-one

$f(x) = x^2$ is not one-to-one

Def Let $f(x)$ be one-to-one with domain D and range R. The inverse function f^{-1} is defined such that if $f(a) = b$, then $f^{-1}(b) = a$.

Note $f^{-1}(x)$ is (usually) not $\frac{1}{f(x)}$

Ex

x	0	1	2	3
$f(x)$	3	4.5	7	10.5

x	3	4.5	7	10.5
$f^{-1}(x)$	0	1	2	3

Q What is $f(f^{-1}(3))$?

$f^{-1}(3) = 0$. So $f(f^{-1}(3)) = 3$

$f(0) = 3$

Q What is $f^{-1}(f(2))$?

$f(2) = 7$
$f^{-1}(7) = 2$

So $f^{-1}(f(2)) = 2$

Key Takeaway

$$f(f^{-1}(y)) = y.$$
$$f^{-1}(f(x)) = x$$

Ex $f(x) = x^3$

$$f^{-1}(x) = x^{1/3}.$$
$$f^{-1}(f(x)) = f^{-1}(x^3) = (x^3)^{1/3} = x.$$

Ex $f(x) = 2x+1$

Want $f^{-1}(x)$

To find $f^{-1}(x)$, swap y and x: Then solve for y.

$$x = 2y+1$$
$$x - 1 = 2y$$
$$y = \frac{x-1}{2}$$
$$\boxed{f^{-1}(x) = \frac{x-1}{2}}$$

Ex $f(x) = \sqrt{x+2} - 3$

Want $f^{-1}(x)$

$$x = \sqrt{y+2} - 3$$

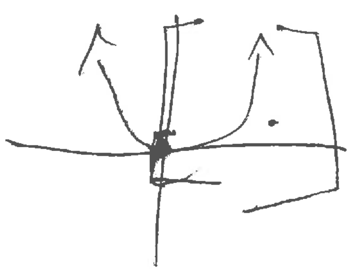
Q $(x+3)^2 = (x+3)(x+3) = x^2 + 6x + 9$

$$x+3 = \sqrt{y+2}$$

$$(x+3)^2 = y+2$$

$$\boxed{y = (x+3)^2 - 2} = f^{-1}(x)$$

Ex Recall that $f(x) = x^2$ is not one-to-one,



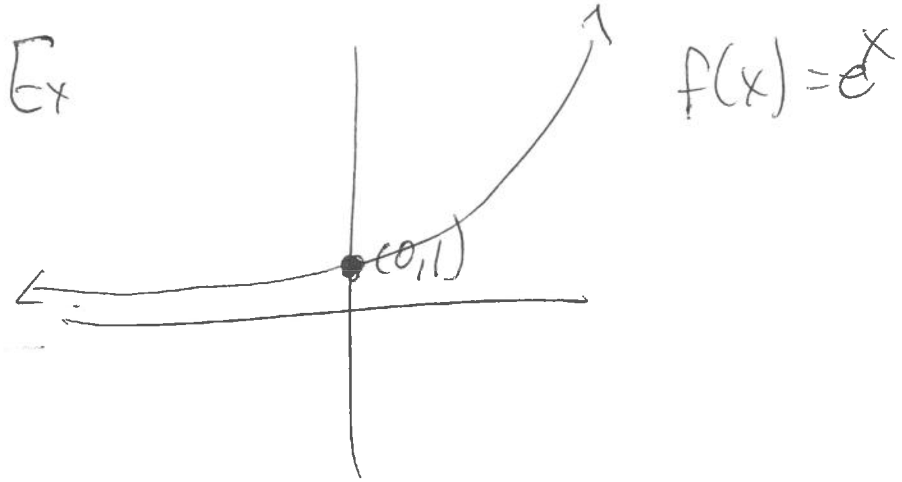
on $[0,\infty)$, $f^{-1}(x) = \sqrt{x}$

Def Let $a > 0$, $a \neq 1$. An exponential function is of the form $f(x) = a^x$.

Rmk The domain of an exponential function is $\mathbb{R}$. The range is $(0,\infty)$.

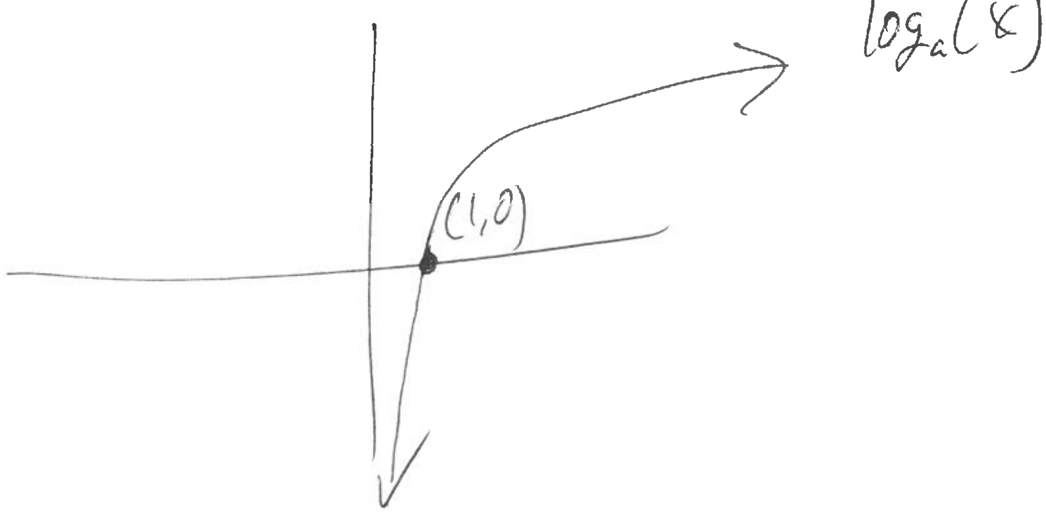
Ex 2^x, e^x, 3^x, $\left(\frac{1}{2}\right)^x$ are all exponential functions

Ex



<u>Def</u> Let $a > 0$, $a \neq 1$. The logarithm base-a is the inverse function of $f(x) = a^x$. Denote $\log_a(x)$.

<u>Rmk</u> The domain of $\log_a(x)$ is $(0, \infty)$
The range is $\mathbb{R}$



<u>Rmk</u> $\ln(x) := \log_e(x)$

<u>Rmk</u> Rules of Exponents and Logs should be review.

- Ex Find exponential function to fit the points $(1,2)$ and $(3,4)$

Recall General form exponential function is

$P(t) = P_0 e^{rt}$

$r = \frac{\ln(2)}{2}$

$P(1) = 2 = P_0 e^{r}$

$P(3) = 4 = P_0 e^{3r}$

$\frac{4}{2} = \frac{\cancel{P_0} e^{3r}}{\cancel{P_0} e^{r}} = e^{2r}$

$e^{2r} = 2$

$\ln(e^{2r}) = \ln(2)$

$2r = \ln(2)$

$r = \frac{\ln(2)}{2}$

$P(1) = 2 = P_0 e^{\ln(2)/2}$

$2 = P_0 \left(e^{\ln(2)}\right)^{1/2}$

$2 = P_0 2^{1/2}$

$\frac{2^1}{2^{1/2}} = P_0$

$P_0 = 2^{1/2} = \sqrt{2}$

$P(t) = \sqrt{2}\, e^{\ln(2)t/2}$

$= \sqrt{2}\left(2^{t/2}\right)$

Ex $(4,5)$ and $(7,9)$. Find exponential function.

$P(4) = P_0 e^{4r} = 5$

$P(7) = P_0 e^{7r} = 9$

$\frac{9}{5} = \frac{P_0 e^{7r}}{P_0 e^{4r}} = e^{3r}$

$\ln(9/5) = 3r$

$r = \frac{\ln(9/5)}{3}$

$P(4) = P_0 e^{4r} = 5$ $\qquad r = \frac{\ln(9/5)}{3}$

$P(7) = P_0 e^{7r} = 9$

$P(4) = 5 = P_0 \underbrace{e^{4\ln(9/5)/3}}$

$5 = P_0 \left(e^{\ln(9/5)}\right)^{4/3}$

$5 = P_0 (9/5)^{4/3}$

$P_0 = 5 \cdot \left(\frac{5}{9}\right)^{4/3}$

$P(t) = 5\left(\frac{5}{9}\right)^{4/3}\left(e^{\ln(9/5)t/3}\right)$

Ex Radioactive particle decays at continuous rate of 5%. What is half-life?

Recall $P(t) = P_0 e^{rt}$ $\qquad P(t) = 1 \cdot e^{-.05t}$

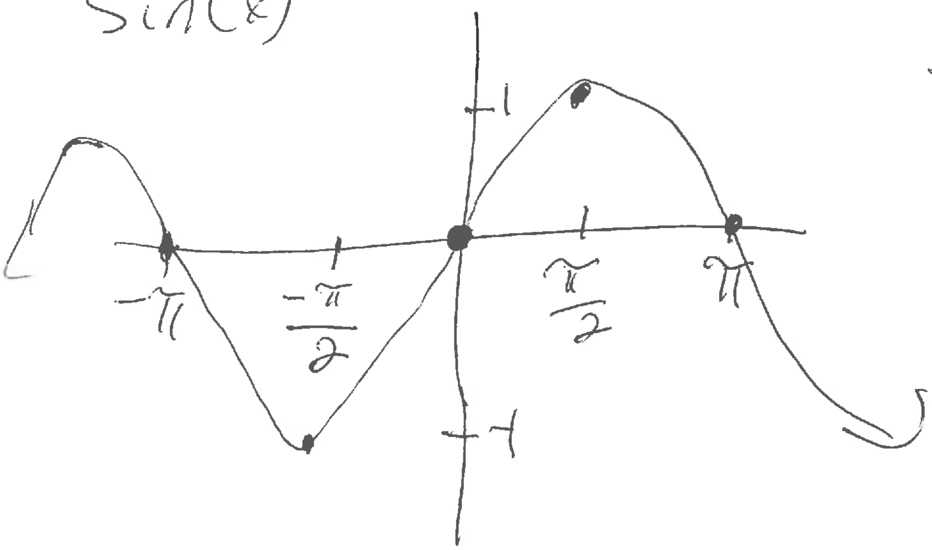
Want to solve $\frac{1}{2} = e^{-.05t}$ $\qquad -.05 = \frac{-1}{20}$

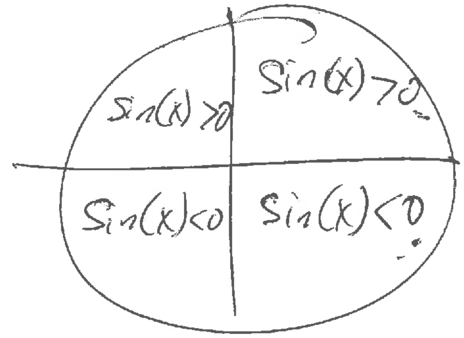
$\ln\left(\frac{1}{2}\right) = -.05t$

$\boxed{t = -20 \ln\left(\frac{1}{2}\right)}$

Inverse Trig Functions

$\sin(x)$





First + 4th quadrants

$\sin^{-1}(x)$ has domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \pi/2]$

Ex $\sin^{-1}\left(\frac{-1}{2}\right) = \frac{-\pi}{6}$

$\left(\text{ie, } \sin(\theta) = \frac{1}{2}\right)$

$\cos^{-1}(x)$ has domain $[-1, 1]$ and range $[0, \pi]$

↳ Ex $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$\tan^{-1}(x)$ has domain $[\frac{-\pi}{2}, \frac{\pi}{2}]$, range is $\mathbb{R}$

Ex $\tan^{-1}(\sqrt{3}) = \pi/3$