3.9

Thm $e := \lim_{x \to 0} (1+x)^{1/x}$

Pf Let $f(x) = \ln(x)$. So $f'(x) = \frac{1}{x}$, and $f'(1) = 1$.

By the def of the derivative:

$$f'(1) = 1 = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x} = \lim_{x \to 0} \frac{\ln(1+x) - \ln(1)}{x}$$

$$= \lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \frac{1}{x}\left(\ln(1+x)\right)$$

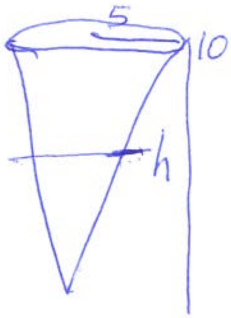
$$= \lim_{x \to 0} \left(\ln\left((1+x)^{1/x}\right)\right)$$

So $e^1 = \lim_{x \to 0} e^{\ln\left((1+x)^{1/x}\right)} = \lim_{x \to 0} (1+x)^{1/x}$. $\square$

Goal

3.10 Related Rates

Ex Suppose we have a conical tank pointed downwards.



Suppose water runs into tank at rate of 9 ft^3/min.

↳ The tank has ht of 10 ft
↳ Base radius of 5 f

Want How fast is water rising when water is 6 ft deep?

$$V = \frac{1}{3}\pi r^2 h \qquad \frac{dV}{dt} = 9 \qquad \frac{r}{h} = \frac{5}{10} = \frac{1}{2}$$

↳ So $r = \frac{h}{2}$

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

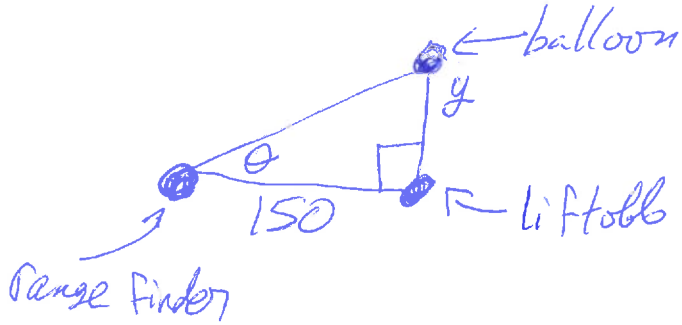
$$\frac{dV}{dt} = \frac{\pi}{12}(3h^2)\frac{dh}{dt} = \frac{\pi}{4}h^2 \cdot \frac{dh}{dt}$$

← Want to solve for

$$9 = \frac{\pi}{4}(6^2)\frac{dh}{dt} = \frac{36\pi}{4}\frac{dh}{dt} = 9\pi\frac{dh}{dt}$$

$$\boxed{\text{So } \frac{dh}{dt} = \frac{1}{\pi} \text{ ft/min}}$$

Ex A balloon rising straight up from a level field is tracked by a range finder 150 ft away from liftoff point.



↳ When $\theta = \frac{\pi}{4}$, the angle is changing @ rate of 0.14 rad/min.

So $\frac{d\theta}{dt} = 0.14$.

Want How fast is balloon rising when $\theta = \frac{\pi}{4}$.

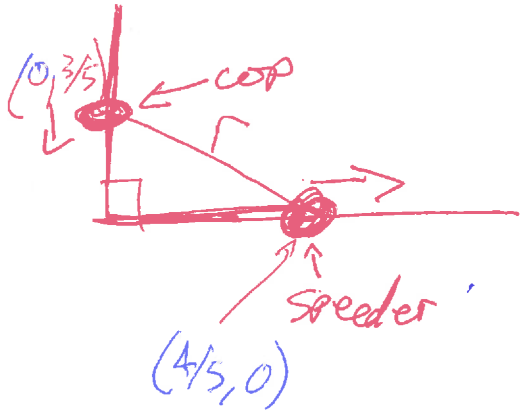
$$\tan(\theta) = \frac{y}{150} \Rightarrow y = 150\tan(\theta)$$

Want $\frac{dy}{dt} = 150\sec^2(\theta)\frac{d\theta}{dt}$

$$= 150\sec^2(\pi/4)(0.14) = 150(\sqrt{2})^2(0.14)$$

So $\frac{dy}{dt} = 150(2)(0.14) = \boxed{300(0.14) \text{ ft/min}}$

Ex Police cruiser approaching right-angled intersection from North. Cop is chasing speeding car that turned corner and is moving straight east



↳ Cop is $\frac{3}{5}$ miles North

↳ Speeder $\frac{4}{5}$ miles East

↳ Radar indicates dist btwn them is incr @ rate of 60 mph

So $\frac{dr}{dt} = 20$

↳ Cruiser moving at rate of 60 mph

$\frac{dy}{dt} = -60$

Want Velocity of speeder at this instant.

$$x^2 + y^2 = r^2$$

$$\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = r^2$$

$$2x \cdot \frac{dx}{dt} + 2y\left(\frac{dy}{dt}\right) = 2r\,\frac{dr}{dt}$$

$$\frac{9}{25} + \frac{16}{25} = r^2$$

$$r^2 = 1 \Rightarrow r = 1.$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}$$

$$\left(\frac{4}{5}\right)\frac{dx}{dt} + \underbrace{\left(\frac{3}{5}\right)(-60)}_{-36} = 1 \cdot 20$$

$$\frac{4}{5}\left(\frac{dx}{dt}\right) = 20 + 36 = 56$$

$$\boxed{\frac{dx}{dt} = \frac{5(56)}{4} = 70 \text{ mph}}$$

Ex Vol of sphere $V = \frac{4}{3}\pi r^3$

↳ Air pumped @ rate of 5 cm^3/min

$$\frac{dV}{dt} = 5$$

↳ Want Rate @ which radius incr when diameter = 20 cm.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$5 = 4\pi(10^2)\frac{dr}{dt}$$

$$5 = 400\pi \frac{dr}{dt} \Rightarrow \boxed{\frac{dr}{dt} = \frac{1}{80\pi} \text{ cm/min}}$$