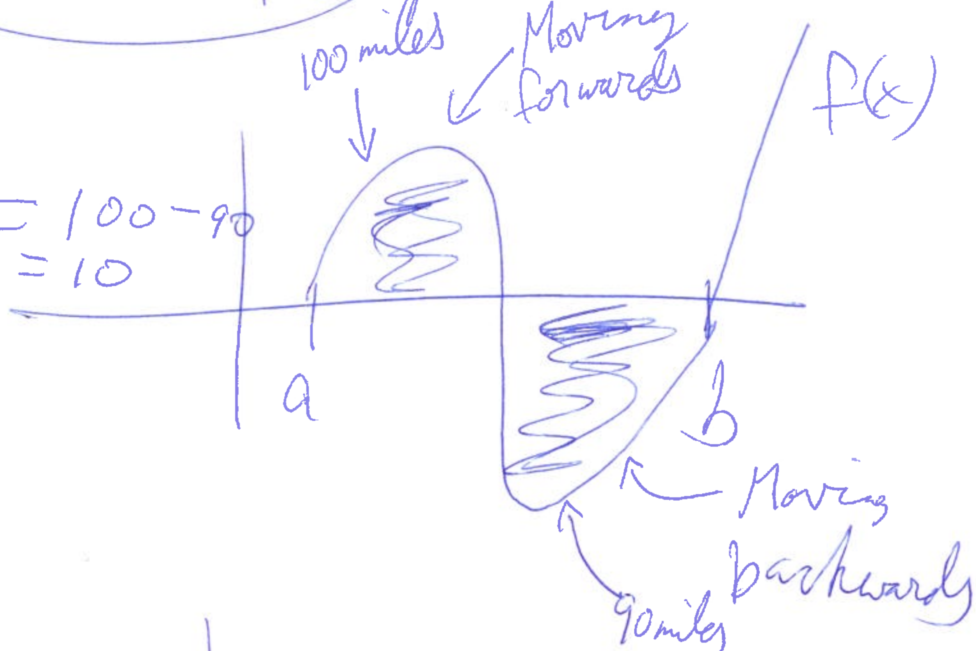
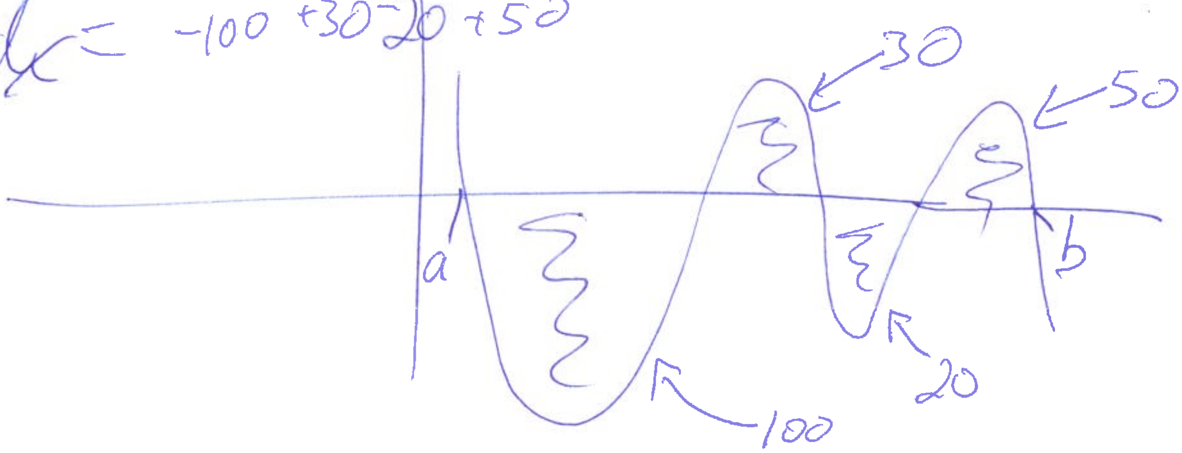
5.3 Definite Integrals.

$$\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = 100 - 90$$
$$= 10$$



$$\int_a^b f(x)\,dx = -100 + 30 - 20 + 50$$

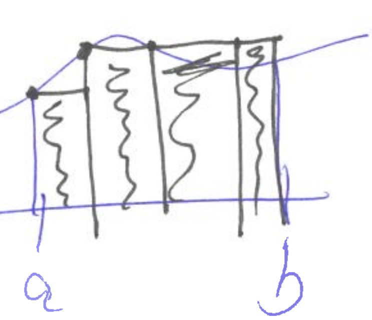


$$\int_a^b f(x)\,dx$$

height

infinitesimal width of rect

Add up area of narrow infinitesimally ~~small~~ rectangles



Area of Rect: width * ht

<u>Thm (Fundamental Thm of Calculus - I)</u> Let $f(x)$ be <u>continuous</u> function on $[a,b]$, and let $F(x)$ be an anti-derivative of $f(x)$ on $[a,b]$. Then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

<u>Ex</u> $$\int_1^2 x\,dx = \frac{x^2}{2}\Bigg|_1^2 = \frac{2^2}{2} - \frac{1^2}{2}$$

<u>Ex</u> $$\int_0^\pi \sin(x)\,dx = -\cos(x)\Bigg|_0^\pi = \left[-(-1)\right] - \left[(-1)\right]$$

$$= 1 - {}^-1 = 2$$

Ex $\displaystyle\int_e^{e^2} \frac{1}{x}\,dx = \ln(|x|)\Big|_e^{e^2} = \ln(e^2) - \ln(e)$
$= 2 - 1 = 1$

$\displaystyle\int_{e^2}^{e} \frac{1}{x}\,dx = \ln(|x|)\Big|_{e^2}^{e} = \ln(e) - \ln(e^2)$
$1 - 2 = -1$

Observe $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Avg Value Suppose $f(x)$ is continuous on $[a,b]$.
The average value $f(x)$ takes on over $[a,b]$ is:

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Ex Avg value x^3 takes on over $[0,2]$.

$$\frac{1}{2-0}\int_0^2 x^3\,dx = \frac{1}{2}\left(\frac{x^4}{4}\right)\Big|_0^2$$
$$= \frac{1}{2}\left(\frac{2^4}{4} - \frac{0}{4}\right) = 2$$

Thm (Mean Value Thm for Integrals) Let $f(x)$ be continuous on $[a,b]$. Then there exists $c \in [a,b]$ s.t.:

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Ex $f(x) = x^2$ on $[0,2]$. As $f(x) = x^2$ is cont on $[0,2]$, there exists $c \in [0,2]$ s.t.

$$f(c) = \frac{1}{2-0} \int_0^2 x^2\,dx = \frac{1}{2}\left(\frac{x^3}{3}\right)\Big|_0^2 = \frac{1}{2}\left(\frac{8}{3} - \frac{0}{3}\right)$$

$$= \frac{4}{3}$$

I.e. There exists $c \in [0,2]$ s.t.

$$f(c) = c^2 = \frac{4}{3}. \quad \left(\text{Here, } c = \sqrt{\tfrac{4}{3}}\right)$$

Thm (Fund. Thm of Calculus - II) Suppose $f(t)$ is continuous on $[a,b]$. Define $F(x) = \int_a^x f(t)\,dt$, on $[a,b]$. We have that $F(x)$ is continuous on $[a,b]$ and differentiable on open interval (a,b). In particular:

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

Ex $G(x) = \int_a^{x} (t^3+1)\,dt$

Let $F(t)$ be an anti-deriv of t^3+1

$$\int_a^{x} (t^3+1)\,dt = \underbrace{F(x) - F(a)}$$

$$\frac{d}{dx}G(x) = \frac{d}{dx}\left[\int_a^x (t^3+1)\,dt\right] = \frac{d}{dx}\left[F(x) - \overset{\text{constant}}{F(a)}\right]$$

$$= F'(x) - 0 = (x^3+1) - 0 = x^3+1$$

<u>Ex</u> $G(x) = \int_3^x 3t\sin(t)\,dt$. <u>Want</u> $G'(x)$

<u>Declare</u> Let $F(t)$ be an anti-deriv of $3t\sin(t)$.

<u>Now</u> $\frac{d}{dx}G(x) = \frac{d}{dx}\left[\int_3^x 3t\sin(t)\,dt\right] = \frac{d}{dx}\left[F(x) - F(3)\right]$

$= F'(x) - 0 = \boxed{3x\sin(x)}$

<u>Ex</u> $G(x) = \int_x^0 t\,dt = -\int_0^x t\,dt$

<u>Want</u> $G'(x)$. <u>Let</u> $F(t)$ be an anti-deriv of t.

$\frac{d}{dx}G(x) = \frac{d}{dx}\left[-\int_0^x t\,dt\right] = \frac{d}{dx}\left[-\left[F(x) - F(0)\right]\right]$

$= -\left[F'(x) - 0\right] = -F'(x) = \boxed{-x}.$