2.3 Last Example

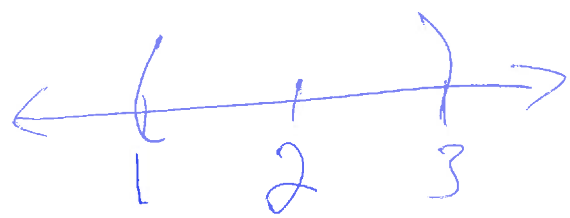
$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = \lim_{x \to 2} (x+2) = 4$$

Given $\varepsilon = 1$. Find δ.

$$\left| \frac{x^2-4}{x-2} - 4 \right| < 1$$

$$|(x+2) - 4| < 1$$

$$|x-2| < 1$$



$$-1 < x-2 < 1$$

$1 < x < 3$, so $x \in (1,3)$ $\quad \delta = 1$

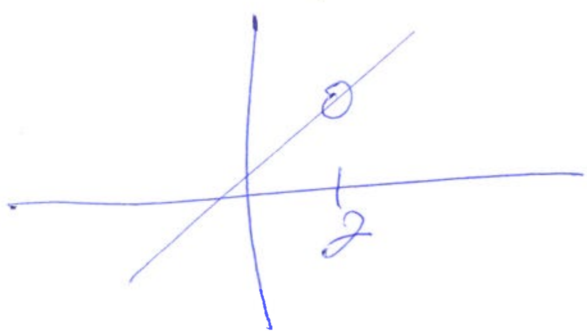
2.5 Continuity

Def We say that $f(x)$ is continuous at the point $x=c$ if $\lim_{x\to c} f(x) = f(c)$.

Remark $f(c)$ has to be defined

↳ $\frac{x^2-4}{x-2}$, $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$. However,

$\frac{x^2-4}{x-2}$ is not defined at $x=2$.



Rmk In terms of δ-ε: We say that $f(x)$ is continuous at $x=c$ if for every $\varepsilon>0$, there exists $\delta>0$ such that if:

$$|x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon$$

↳ our limit

Def We say that $f(x)$ is <u>continuous</u> if $f(x)$ is continuous at every point on its domain.

Ex $f(x) = x^2$ is continuous
$f(x) = 0$ is continuous

Def We say that $f(x)$ is:

↳ Left-continuous at $x=c$ if $\lim_{x\to c^-} f(x) = f(c)$.

↳ Right-continuous at $x=c$ if $\lim_{x\to c^+} f(x) = f(c)$.

Ex $f(t) = \sqrt{4-t^2}$

↳ $f(t)$ is continuous on $(-2,2)$.

↳ $f(t)$ is left-continuous at $\cancel{t}=2$

↳ $f(t)$ is right-continuous at $t=-2$

↳ $f(t)$ is left-continuous at $t=2$

↳ $f(2) = 0$

↳ $\lim\limits_{t \to 2^-} f(t) = 0$

↳ $f(t)$ is right-continuous at $t=-2$

↳ $f(-2) = \sqrt{4-(-2)^2} = 0$

↳ $\lim\limits_{t \to -2^+} f(t) = 0$

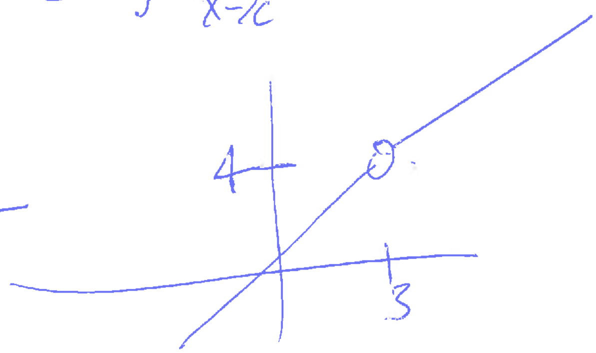
Classifying Discontinuities

Def We say that $f(x)$ has a removable discontinuity at $x=c$ if:

(i) $f(x)$ is not continuous at $x=c$

(ii) $\lim\limits_{x \to c} f(x)$ exists and is finite

Ex



↳ $f(x)$ is not defined at $x=3$

↳ $\lim\limits_{x \to 3} f(x) = 4$

So $f(x)$ has a removable discont at $x=3$.

Q How can we extend $f(x)$ to be continuous?

$$g(x) = \begin{cases} f(x) & ; \ x \neq 3 \\ 4 & ; \ x = 3. \end{cases}$$

Ex $\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} (x+2) = 4$

↳ $\frac{x^2-4}{x-2}$ is not defined at $x=2$

So $\frac{x^2-4}{x-2}$ has a removable dis cont. at $x=2$.

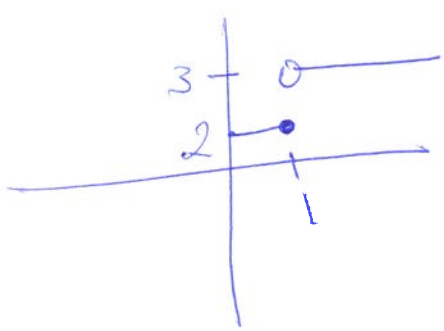
Def We say that $f(x)$ has a jump discont at $x=c$ if:

↳ $\lim_{x \to c^+} f(x)$ exists and is finite

2↳ $\lim_{x \to c^-} f(x)$ exists and is finite

↳ $\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)$

Ex



$\lim_{x \to 1^-} f(x) = 2$

$\lim_{x \to 1^+} f(x) = 3$

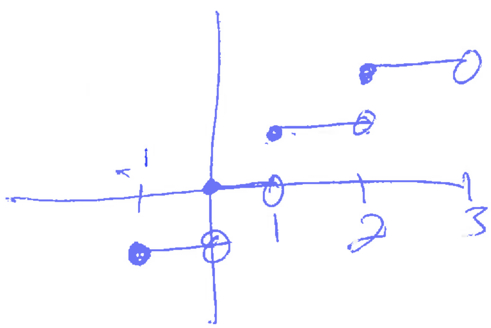
So $f(x)$ has a jump discontinuity at $x=1$.

Ex $f(x) = \lfloor x \rfloor$ (Floor function)

$\lfloor x \rfloor = x$ (if x is an integer)

Otherwise $\lfloor x \rfloor$ rounds down.

$\hookrightarrow \lfloor 1.01 \rfloor = 1$ $\hookrightarrow \lfloor 1 \rfloor = 1$



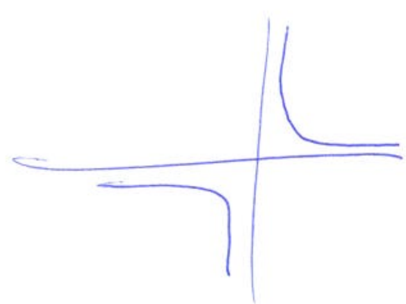
$\lim_{x \to 1^-} \lfloor x \rfloor = 0$

$\lim_{x \to 1^+} \lfloor x \rfloor = 1$

So $\lfloor x \rfloor$ has a jump discont at $x=1$

Def We say that $f(x)$ has infinite discontinuity at $x=c$ if $\lim_{x \to c^+} f(x) = \pm\infty$ and $\lim_{x \to c^-} f(x) = \pm\infty$, but $\lim_{x \to c^+} f(x)$ need not equal $\lim_{x \to c^-} f(x)$.

Ex $f(x) = \frac{1}{x}$



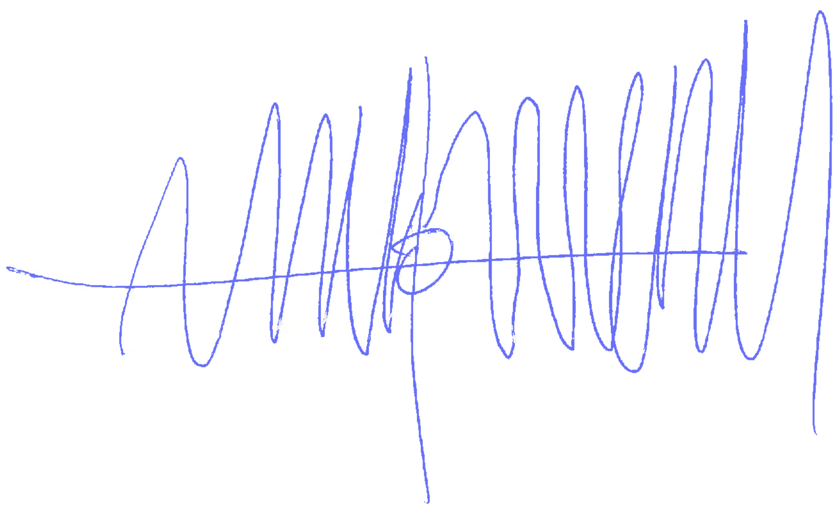
$\lim_{x \to 0^+} f(x) = \infty$

$\lim_{x \to 0^-} f(x) = -\infty$

So $\frac{1}{x}$ has infinite discont at $x=0$.

~~Def~~ Oscillating Discontinuities

Ex $f(x) = \sin\left(\frac{1}{x}\right)$



Thm (Intermediate Value Theorem) Suppose $f(x)$ is continuous on $[a,b]$. If y is btwn $f(a)$ and $f(b)$, then there exists $c \in [a,b]$ such that ~~f(x)~~ $f(c) = y$.

Ex $f(x) = x^3 - x - 1$. Does $f(x)$ have a root in $[1,2]$?

A $f(1) = 1^3 - 1 - 1 = -1$

$f(2) = 2^3 - 2 - 1 = 5$

Take $y_0 = 0$, which is btwn -1 and 5

$f(x)$ is continuous

So by IVT, there exists $c \in [1,2]$ s.t. $f(c) = 0$.

Ex Is there a solution to:

$$\sqrt{2x+5} = 4 - x^2 ?$$

Rewrite $x^2 + \sqrt{2x+5} = 4$

Label $f(x) = x^2 + \sqrt{2x+5}$

↳ **Note** Domain of $f(x)$ is $[\frac{-5}{2}, \infty)$

$$f(\tfrac{-5}{2}) = (\tfrac{-5}{2})^2 + 0 = \frac{25}{4} > 4$$

$$f(0) = 0 + \sqrt{5} = \sqrt{5} < 3$$

↳ So 4 is btwn $\sqrt{5}$ and $\frac{25}{4}$

↳ $f(x)$ is continuous.

↳ So by IVT, there exists $c \in [-5/2, \infty)$ s.t. $f(c) = 4$.