3.8/3.9

Derivatives of Inverse Functions

Thm Suppose that $f(x)$ has the interval I as its domain, and suppose $f'(x) \neq 0$ on I. Suppose $f(x)$ is one-to-one on I (so $f^{-1}(x)$ exists on range of $f(x)$). Then $f^{-1}(x)$ is differentiable at every point in the range of $f(x)$, where

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Pf We have that $f(f^{-1}(x)) = x$. So:

$$\frac{d}{dx} f(f^{-1}(x)) = \frac{d}{dx} x = 1$$

Apply Chain rule

Note that $\frac{d}{dx} f(f^{-1}(x)) = f'(f^{-1}(x)) \cdot (f^{-1})'(x)$.

$$\text{So } (f^{-1})'(x) \cdot f'(f^{-1}(x)) = 1$$

Thus, $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$. $\square$

Ex $f(x)=x^2$ on $(0,\infty)$. So $f^{-1}(x)=\sqrt{x}$.

Using power rule $(f^{-1})'(x)=\dfrac{1}{2\sqrt{x}}$

Inverse Function Rule $(f^{-1})'(x)=\dfrac{1}{f'(f^{-1}(x))}$

$(f^{-1})'(x)=\dfrac{1}{2(\sqrt{x})}$

Ex $f(x)=x^3-2$ on $(0,\infty)$. Note $f(2)=6$

Want $(f^{-1})'(6)$.

Recall $(f^{-1})'(x)=\dfrac{1}{f'(f^{-1}(x))}$

$\hookrightarrow f'(x)=3x^2$

$\hookrightarrow f^{-1}(6)=2$

$(f^{-1})'(6)=\dfrac{1}{3(f^{-1}(6))^2}$

$=\dfrac{1}{3(2^2)}=\dfrac{1}{12}$

Ex $f(x) = \sin^{-1}(x)$

Want $f'(x)$.

$$f'(x) = \frac{1}{\cos(\sin^{-1}(x))}$$

$$= \frac{1}{\sqrt{\cos^2(\sin^{-1}(x))}}$$

$$= \frac{1}{\sqrt{1-\sin^2(\sin^{-1}(x))}} = \frac{1}{\sqrt{1-x^2}}$$

$$\sin^2(\sin^{-1}(x)) = \left(\sin(\sin^{-1}(x))\right)\cdot\left(\sin(\sin^{-1}(x))\right)$$

Ex $f(x) = \cos^{-1}(x)$

Want $f'(x)$. $= \frac{1}{-\sin(\cos^{-1}(x))} = -\frac{1}{\sin(\cos^{-1}(x))}$

$$= \frac{-1}{\sqrt{1-\cos^2(\cos^{-1}(x))}} = \frac{-1}{\sqrt{1-x^2}}.$$

$$\cos^2(\cos^{-1}(x)) = \left(\cos(\cos^{-1}(x))\right)^2$$

Ex $g(x) = \tan^{-1}(x)$

Find $g'(x) = \dfrac{1}{\sec^2(\tan^{-1}(x))}$.

Recall $\sin^2(x) + \cos^2(x) = 1$. (Div by $\cos^2(x)$)

$\tan^2(x) + 1 = \sec^2(x)$.

$$\frac{1}{\sec^2(\tan^{-1}(x))} = \frac{1}{1 + \tan^2(\tan^{-1}(x))} = \boxed{\frac{1}{1+x^2}}$$

Idea $f(x) = \ln(u(x))$.

Want $u'(x)$.

$$f'(x) = \frac{u'(x)}{u(x)} \Rightarrow u'(x) = u(x) f'(x)$$

Ex $f(x) = \dfrac{(x^2+1)(x+3)^{1/2}}{(x-1)}$ Want $f'(x)$

$$\ln(f(x)) = \ln\left((x^2+1)(x+3)^{1/2}\right) - \ln(x-1)$$

$$= \ln(x^2+1) + \tfrac{1}{2}\ln(x+3) - \ln(x-1)$$

$$\frac{f'(x)}{f(x)} = \frac{2x}{x^2+1} + \frac{1}{2}\left(\frac{1}{x+3}\right) - \frac{1}{x-1}.$$

$$f'(x) = f(x)\left[\frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1}\right]$$

$$= \frac{(x^2+1)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1}\right)$$

Ex $f(x) = x^x$ Want $f'(x)$

$$\ln(f(x)) = x\ln(x)$$

$$\frac{f'(x)}{f(x)} = \ln(x) + x\left(\frac{1}{x}\right) = \ln(x) + 1$$

$$f'(x) = f(x)(\ln(x)+1) = x^x(\ln(x)+1)$$