3.3 Derivative Rules

Power Rule Let $f(x) = x^n$. ← variable is in **base**

The $f'(x) = nx^{n-1}$.

Ex $f(x) = x^2$, $f'(x) = 2x$

$f(x) = \frac{1}{x} = x^{-1}$, $f'(x) = -1 \cdot x^{-2} = -x^{-2} \left(= \frac{-1}{x^2}\right)$

Constant Multiple Rule Let $c \in \mathbb{R}$. Let $f(x)$ be a differentiable function. If $g(x) = c \cdot f(x)$, then $g'(x) = c \cdot f'(x)$.

Ex $f(x) = 5x^4$, $f'(x) = 5(4x^3) = 20x^3$

$f(x) = \underset{c}{7}\,\underset{g(x)}{x^5}$, $f'(x) = \underset{c}{7}\,\underset{g'(x)}{(5x^4)} = 35x^4$

Q Which grows faster: x^2 or $5x^2$?

A $5x^2$ grows faster

↳ $f(x) = x^2$, then $f'(x) = 2x$

↳ $g(x) = 5x^2$, then $g'(x) = 5(2x) = 10x$

Sum Rule Let $f(x), g(x)$ be differentiable at $c \in \mathbb{R}$.
If $h(x) = f(x) + g(x)$, then $h(x)$ is also differentiable at c, with derivative

$$h'(c) = f'(c) + g'(c)$$

Ex $f(x) = 5x^4 + 7x^3 + 11x^2 + 13x + 17$

$$f'(x) = 20x^3 + 21x^2 + 22x + 13 + 0$$

Proof of Sum Rule Given $K(x) = f(x) + g(x)$, where $f(x), g(x)$ are differentiable at $x = c$.

Then

$$\lim_{h \to 0} \frac{K(c+h) - K(c)}{h} = \lim_{h \to 0} \frac{[f(c+h) + g(c+h)] - [f(c) + g(c)]}{h}$$

$$= \underbrace{\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}}_{} + \lim_{h \to 0} \frac{g(c+h) - g(c)}{h}$$

$$= f'(c) + g'(c).$$

Exponentials Let $a > 0$, $a \neq 1$. Let $f(x) = a^x$.
Then $f'(x) = \ln(a)\, a^x$.

Ex $f(x) = 2^x$ ← variable in exponent

$$f'(x) = \ln(2)\, 2^x$$

Ex $f(t) = e^t$, $f'(t) = \ln(e)\, e^t = e^t$

Ex $f(x) = x^2 + 2^x$

$$f'(x) = 2x + \ln(2)\, 2^x$$

Log Rules Let $a > 0$, $a \neq 1$. Let $f(x) = \log_a(x)$.

Then: $f'(x) = \dfrac{1}{x \ln(a)}$

Ex $f(x) = \log_3(x)$, $f'(x) = \dfrac{1}{x \ln(3)}$

$f(x) = \underbrace{\ln(x)}_{\log_e(x)}$, $f'(x) = \dfrac{1}{x \ln(e)} = \dfrac{1}{x}$

Derivatives of Sin and Cos

Thm $\displaystyle\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$

<u>Thm</u> $\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0.$

<u>Pf</u> $\lim_{x\to 0} \frac{1-\cos(x)}{x} = \lim_{x\to 0} \left(\frac{1-\cos(x)}{x}\right) \cdot \frac{(1+\cos(x))}{(1+\cos(x))}$

$$= \lim_{x\to 0} \frac{1-\cos^2(x)}{x(1+\cos(x))} = \lim_{x\to 0} \frac{\sin^2(x)}{x(1+\cos(x))}$$

$$= \lim_{x\to 0} \left(\frac{\sin(x)}{x} \cdot \frac{\sin(x)}{1+\cos(x)}\right)$$

$$= \left(\lim_{x\to 0} \frac{\sin(x)}{x}\right)\left(\lim_{x\to 0} \frac{\sin(x)}{1+\cos(x)}\right)$$

$$= 1 \cdot \frac{0}{2} = 0.$$

Thm Let $f(x) = \sin(x)$. Then $f'(x) = \cos(x)$.

Pf $f'(x) = \lim_{h \to 0} \dfrac{\sin(x+h) - \sin(x)}{h}$

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \sin(x)\frac{(\cos(h) - 1)}{h} + \lim_{h \to 0} \frac{\cos(x)\sin(h)}{h}$$

$$= \sin(x) \lim_{h \to 0} \frac{(\cos(h) - 1)}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h}$$

$$= \sin(x) \cdot 0 + \cos(x) \cdot 1$$

$$= \cos(x). \quad \square$$

Thm $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$.

Pf Exercise for students.

Thm (Product Rule). Suppose $f(x), g(x)$ are differentiable at the point c. Let $h(x) = f(x) \cdot g(x)$. Then:

$$h'(c) = f'(c)g(c) + f(c)g'(c)$$

Ex $h(x) = (x^2+1)(x^3+3)$.

$$h'(x) = 2x(x^3+3) + (x^2+1)(3x^2).$$

Ex $f(x) = x^2 \cdot 2^x$

$$f'(x) = 2x \cdot 2^x + x^2\left(\ln(2)\, 2^x\right)$$

Ex $f(x) = \dfrac{e^x}{x} = e^x x^{-1}$

$$f'(x) = e^x x^{-1} + e^x(-x^{-2})$$

<u>Thm (Chain Rule).</u> Suppose $f(x)$ is differentiable at $x=c$, and suppose $g(x)$ is differentiable at $f(c)$. If $h(x) = g(f(x))$, then $h(x)$ is differentiable at $x=c$ with derivative:

$$h'(c) = g'(\underbrace{f(c)}_{\text{inside}}) \cdot f'(c).$$

<u>Ex</u> $f(x) = (x^2+5)^{100}$ $\qquad f'(x) = 100(x^2+5)^{99} \cdot 2x$

↳ x^2+5 is inside

↳ u^{100} as outside

<u>Ex</u> $f(t) = \ln(t^3+3t+5)$ $\qquad$ ~~$f'(t) = \ln$~~

↳ t^3+3t+5 is inside.

↳ $\ln(u)$ is outside; $g(u) = \ln(u)$, $g'(u) = \frac{1}{u}$

$$f'(t) = \underbrace{\frac{1}{t^3+3t+5}}_{g'(t^3+3t+5)}(3t^2+3)$$

Ex $f(t) = 2^{3^t}$ $f(u) = 2^u$ $f(g(t)) = 2^{3^t}$

$g(t) = 3^t$

$$\boxed{f'(t) = \ln(2)\, 2^{3^t} \ln(3)\, 3^t}$$

Ex $f(x) = \sin(\cos(x))$

$$\boxed{f'(x) = \cos(\cos(x)) \cdot (-\sin(x))}$$