2.5 Continuity

~~Thm Let f(x), g(x) be continuous. Then g∘f is continuous. Recall g∘f = g(f(x))~~

Continuity We say that $f(x)$ is continuous at c if $\lim_{x\to c} f(x) = f(c) = f\left(\lim_{x\to c} x\right)$

Ex $\lim_{x\to \pi/2} \cos\left[2x + \sin\left(\frac{3\pi}{2} + x\right)\right]$

$$= \cos\left[\lim_{x\to\pi/2}\left(2x + \cancel{\sin\left(\tfrac{3\pi}{2}\right)}\ \sin\left(\tfrac{3\pi}{2}+x\right)\right)\right]$$

$$= \cos\left[\pi + \lim_{x\to\pi/2} \sin\left(\tfrac{3\pi}{2}+x\right)\right]$$

$$= \cos\left[\pi + \sin\left(\lim_{x\to\frac{\pi}{2}}\left(\tfrac{3\pi}{2}+x\right)\right)\right]$$

$$= \cos\left[\pi + \sin(2\pi)\right]$$

$$= \cos(\pi) = -1.$$

Ex $\lim_{x \to 1} \sin^{-1}\left(\frac{1-x}{1-x^2}\right)$

$$= \sin^{-1}\left(\lim_{x\to 1} \frac{1-x}{1-x^2}\right)$$

$$= \sin^{-1}\left(\lim_{x\to 1} \frac{1-x}{(1-x)(1+x)}\right)$$

$$= \sin^{-1}\left(\lim_{x\to 1} \frac{1}{1+x}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Ex $\lim_{x\to 1} \ln(x-1) = \ln\left(\lim_{x\to 1} (x-1)\right)$

$\lim_{x\to 1} (x-1) = 0$.

Note $\ln(0)$ is undefined

So $\lim_{x\to 1} \ln(x-1)$ DNE

Thm Let $f(x)$, $g(x)$ be functions. Suppose $f(x)$ is continuous at $x=a$, and $g(x)$ is continuous at $x=f(a)$. Then $g \circ f$ is continuous at $x=a$. ($g(f(x))$ is continuous at a).

Remark As we saw above:

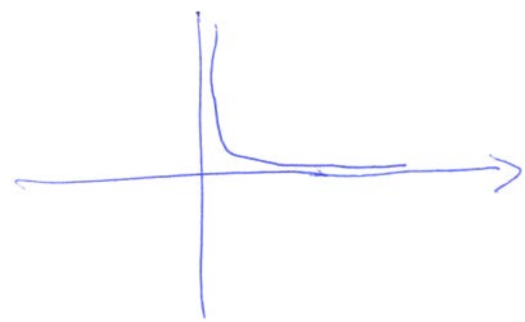
↳ $f(x) = x-1$ is continuous at $x=1$

↳ $g(x) = \ln(x)$ is not continuous (or even defined) at $x=0=f(1)$.

This is why $\lim_{x \to 1} \ln(x-1)$ DNE.

2.6 Limits at Infinity

Ex $\lim_{x \to \infty} \frac{1}{x} = 0$



Ex $\lim_{x \to \infty} \frac{1}{e^x} = 0$

Ex $\lim_{x \to \infty} \frac{x^2}{x^4} = \lim_{x \to \infty} \frac{1}{x^2} = 0$

Ex $\lim_{x \to \infty} \frac{5x^2+8x-3}{3x^2+2} = \lim_{x \to \infty} \frac{5x^2/x^2 + 8x/x^2 - 3/x^2}{3x^2/x^2 + 2/x^2}$

$= \lim_{x \to \infty} \frac{5 + \frac{8}{x} - \frac{3}{x^2}}{3 + \frac{2}{x^2}} = \frac{5+0+0}{3+0} = \frac{5}{3}$

<u>Ex</u> $\lim\limits_{x\to-\infty} \dfrac{x^3+1}{x^2} = \lim\limits_{x\to-\infty} \left(\dfrac{x^3}{x^2} + \dfrac{1}{x^2}\right)$

$= \lim\limits_{x\to-\infty} \left(x + \dfrac{1}{x^2}\right) = -\infty + 0 = -\infty$

<u>Ex</u> $\lim\limits_{x\to\infty} \left(x - \sqrt{x^2+16}\right)$

$= \lim\limits_{x\to\infty} \left(x - \sqrt{x^2+16}\right) \dfrac{\left(x+\sqrt{x^2+16}\right)}{\left(x+\sqrt{x^2+16}\right)}$

$= \lim\limits_{x\to\infty} \dfrac{x^2-(x^2+16)}{x+\sqrt{x^2+16}}$

$= \lim\limits_{x\to\infty} \dfrac{-16}{x+\sqrt{x^2+16}} = 0$

Ch. 3 3.1/3.2 Derivative

Def Let $f(x)$ be a function, and let $c \in \mathbb{R}$. We say that $f(x)$ is differentiable (has a derivative) at $x=c$ if the following limit exists and is finite:

$$\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$$

dist$(c+h, c)$

Ex What is the derivative of $f(x)=x^2$ at $x=7$?

$$\lim_{h \to 0} \frac{(7+h)^2 - 7^2}{h} = \lim_{h \to 0} \frac{49+14h+h^2-49}{h}$$

$$= \lim_{h \to 0} \frac{14h+h^2}{h} = \lim_{h \to 0} (14+h) = 14.$$

The derivative of x^2 @ $x=7$ is 14.

~~Ex Find deri~~

Def (Derivative as a function). Let $f(x)$ be a function. The derivative of $f(x)$ is the function $f'(x)$ whose value at x is:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Ex $f(x) = x^2$

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2hx + h^2}{h} = \lim_{h\to 0} (2x+h) = 2x$$

So $f'(x) = 2x$.

Recall $f'(7) = 14$.

Ex ~~[illegible]~~ Find the derivative of $\frac{1}{x}$ using the limit def.

$$\frac{\left(\frac{a}{b}\right)}{c} = \frac{1}{c}\left(\frac{a}{b}\right)$$

$$\lim_{h \to 0} \frac{\left(\frac{1}{x+h} - \frac{1}{x}\right)}{h} = \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{x}{x(h+x)} - \frac{(x+h)}{x(h+x)}\right)$$

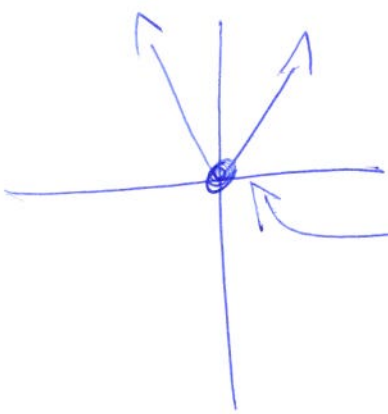
$$= \lim_{h \to 0} \frac{1}{\cancel{h}}\left(\frac{-\cancel{h}}{x(h+x)}\right) = \lim_{h \to 0} \frac{-1}{x(h+x)} = \frac{-1}{x^2}$$

So the derivative of $\frac{1}{x}$ is $f'(x) = \frac{-1}{x^2}$

<u>Ex</u> $f(x) = |x|$ is <u>not</u> differentiable at $x = 0$.

$\hookrightarrow \lim\limits_{h \to 0^-} \dfrac{|0+h| - |0|}{h} = \lim\limits_{h \to 0^-} \dfrac{|h|}{h} = \lim\limits_{h \to 0^-} \dfrac{-h}{h} = -1$

$\hookrightarrow \lim\limits_{h \to 0^+} \dfrac{|0+h| - |0|}{h} = \lim\limits_{h \to 0^+} \dfrac{|h|}{h} = \lim\limits_{h \to 0^+} \dfrac{h}{h} = 1$



cusp at $x=0$