# Dodgson Winner Problem

Dhamma Kimpara

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## 1 Introduction

Suppose you are a theoretical computer scientist, wandering in the wilds of theory-land and you come upon a complexity class. It has not been seen before. It seems intuitive and natural. So you decide to submit it to the complexity zoo [Aaronson et al., 2005]. But the worry is whether this class will turn out to capture the complexity of important real-world problems. In other words, is anyone or any problems going to come around to your new section of the zoo and join your party? After all, there are 545 classes and counting in the zoo!

This was the case for the class $\Theta_2^p$ in the mid-1990s. $\Theta_2^p$ or $\mathsf{P}_{||}^{\mathsf{NP}}$, the class of languages decidable by a polynomial time Turing machine with parallel access to $\mathsf{NP}$, was introduced in 1983 by [Papadimitriou and Zachos, 1982]. The location of this class in the polynomial hierarchy is:

By the mid 1990s, the theoretical importance of $\Theta_2^p$ was recognized in complexity theory. Klaus W. Wagner established half a dozen characterizations of $\Theta_2^p$ [Wagner, 1990], several complete problems and a toolkit for establishing $\Theta_2^p$ -hardness. Hemachandra [1987] and Köbler, Johannes et al. [1987] showed that $\Theta_2^p$ was equivalent to the class of problems that can be solved by $\mathcal{O}(\log(n))$ sequential Turing queries to $\mathsf{NP}$. Furthermore, if $\mathsf{NP}$contains some $\Theta_2^p$ -hard problem, then the polynomial hierarchy collapses to $\mathsf{NP}$. However, these connections and complete problems lived in the pure theory section of the zoo and did not yet have the appeal of problems from "real world" settings.

Along came the Dodgson winner problem, invented in 1876 by Charles Lutwidge Dodgson, better known under the pen name of Lewis Carroll. Hemaspaandra et al. [1999] proved that this problem was $\Theta_2^p$ -complete. This was the first "real-world" problem proven complete for the class $\Theta_2^p$ .

Dodgson's election system takes the following form. An election is a finite number of voters, who each cast a linear order over a common finite set of candidates. Note that linear orders are "tie-free". The winner is determined by whichever candidate is closest to being a Cordorcet winner, a criteria used by other election systems. A Condorcet winner is a candidate $a$ who for every other candidate $b$, is preferred to $b$ by strictly more than half of the voters. We naturally want election systems to be Condorcet-consistent, i.e. the system has the property where if $a$ is a Condorcet winner, $a$ is the one and only winner in the election. Dodgson's election system is Condorcet-consistent [Brandt et al., 2016]

The winner(s) in a Dodgson election is defined as the candidate(s) who are the "closest" to being Condorcet winners. The winners are the candidates that have the lowest Dodgson score. The Dodgson score of a candidate $a$ is the smallest number of sequential exchanges of adjacent candidates in preference orders such that after those exchanges $a$ is a Cordorcet winner.

Note that it is remarkable that we find $\Theta_2^p$ -complete problems that were defined 100 years before complexity theory itself existed. Dodgson winner is also extremely natural when compared with other complete problems in this class such as determining if the maximum size clique in a graph is of odd size (odd-max-clique). In this project we will use Comp-SAT, which was recently shown to be complete for this class [Lukasiewicz and Malizia, 2017]

The rest of this project presents the theory of and a practical algorithm for the Dodgson winner problem. We present slightly modified proof of completeness of the Dodgson winner problem based on new results for the class $\Theta_2^p$ . Then we present, implement, and examine a heuristic algorithm that is self-knowingly correct for most practical instances of the problem.

# 2 Preliminaries

## 2.1 Problem Setting

**Definition 1.** Dodgson Election

**Definition 2.** Condorcet Winner

**Definition 3.** Dodgson Score

**Decision Problem.** DodgsonWinner
Instance: A Dodgson Election and Candidate $\langle C, c, V \rangle$
Decide: Is $c$ a winner of the election? In other words, does $c$ have the minimum Dodgson Score in the election?

**Decision Problem.** DodgsonScore
**Instance:** $k \in \mathbb{N}$, A Dodgson Election and Candidate $\langle C, c, V \rangle$
**Decide:** Is the Dodgson Score of candidate $c$ less than or equal to $k$?

**Decision Problem.** Comp-SAT [Lukasiewicz and Malizia, 2017]
Instance: A pair $\langle A, B \rangle$ of sets of 3CNF formulas.
Decide: Is the number of satisfiable formulas in $A$ greater than the number of satisfiable formulas in $B$.

## 2.2 Complexity Classes and Definitions

**Definition 4.** $\Theta_2^p$

**Corollary 1.** If DodgsonWinner is $\mathsf{NP}$-complete then $\mathsf{PH} = \mathsf{NP}$.

**Theorem 1.** [Lukasiewicz and Malizia, 2017] Comp-SAT is $\Theta_2^p$ - complete.

# 3 The complexity of the Dodgson Winner Problem

**Theorem 2.** DodgsonWinner $\in \Theta_2^p$ .

Proof:

**Theorem 3.** DodgsonWinner is $\Theta_2^p$ - hard

First we briefly outline the reduction from Comp-SAT to DodgsonWinner. Let us start with a Comp-SAT instance $< A, B >$, for each 3CNF $F$ formula in $A$ and $B$, reduce $F$ into the corresponding instance of ThreeDimensionalMatching (3DM), which is possible because 3DM is $\mathsf{NP}$-complete. Now, for each 3DM problem, we reduce it into an instance of DodgsonScore. Using a merger lemma, we will merge these into 2 elections, one for $A$ and one for $B$, thus an instance of 2ElectionRanking (2ER). Then, 2ER will be reduced to DodgsonWinner. Our reduction avoids needing to prove that 2ER is $\Theta_2^p$ -hard.

A reference diagram is provided:

$$Comp - SAT \leq_{tt} 3DM \leq_m^P DodgsonScore \rightarrow_{merge} 2ER \leq_m^P DodgsonScore$$

### 3.1 Outline of relevant lemmas

**Lemma 1.** 3.4

**Lemma 2.** 3.5

**Decision Problem.** TwoElectionRanking (2ER)

**Lemma 3.** 2ER is $\Theta_2^p$ - hard.

**Lemma 4.** 3.7

**Decision Problem.** ThreeDimensionalMatching (3DM)

# 4 Practical Greedy Algorithm

# 5 Experiments

# 6 Conclusion

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