Binomial Expansions and Pascal’s Triangle

The binomial theorem, which uses Pascal’s triangles to determine coefficients, describes the algebraic expansion of powers of a binomial.

LEARNING OBJECTIVES

Use the Binomial Formula and Pascal’s Triangle to expand a binomial raised to a power and find the coefficients of a binomial expansion

KEY TAKEAWAYS

Key Points

- According to the theorem, it is possible to expand the power $(x+y)^n$ into a sum involving terms of the form ax^by^c , where the exponents b and c are nonnegative integers with $b+c=n$, and the coefficient a of each term is a specific positive integer depending on n and b .
- Using summation notation, the binomial theorem can be expressed as: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
- The rows of Pascal’s triangle contain the coefficients of binomial expansions and provide an alternate way to expand binomials. The rows are conventionally enumerated starting with row $n=0$ at the top, and the entries in each row are numbered from the left beginning with $k=0$.

Key Terms

- **integer**: An element of the infinite and numerable set $\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$
- **binomial**: A polynomial consisting of two terms, or monomials, separated by an addition or subtraction symbol.
- **binomial coefficient**: A coefficient of any of the terms in the expansion of the binomial power $(x+y)^n$

Binomial Theorem

The binomial theorem is an algebraic method of expanding a binomial expression. Essentially, it demonstrates what happens when you multiply a binomial by itself (as many times as you want). For example, consider the expression $(4x+y)^7$. It would take quite a long time to multiply the binomial $(4x+y)$ out seven times. The binomial theorem provides a short cut, or a formula that yields the expanded form of this expression.

According to the theorem, it is possible to expand the power $(x+y)^n$ into a sum involving terms of the form ax^by^c , where the exponents b and c are nonnegative integers with $b+c=n$, and the coefficient a of each term is a specific positive integer depending on n and b . When an exponent is zero, the corresponding power is usually omitted from the term (so that $3x^2y^0$ would be written as $3x^2$).

For example, consider the following expansion:

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Any coefficient a in a term ax^by^c of the expanded version is known as a binomial coefficient. The binomial coefficient also arises in combinatorics, where it gives the number of different combinations of b elements that can be chosen from a set of n elements. Recall that this could be written with the notation $\binom{n}{b}$, or "n choose b ."

According to the binomial theorem, it is possible to expand any power of $x + y$ into a sum of the form:

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1$$
$$+ \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n$$

where each value $\binom{n}{k}$ is a specific positive integer known as binomial coefficient. This formula is referred to as the Binomial Formula. Using summation notation, it can be written as:

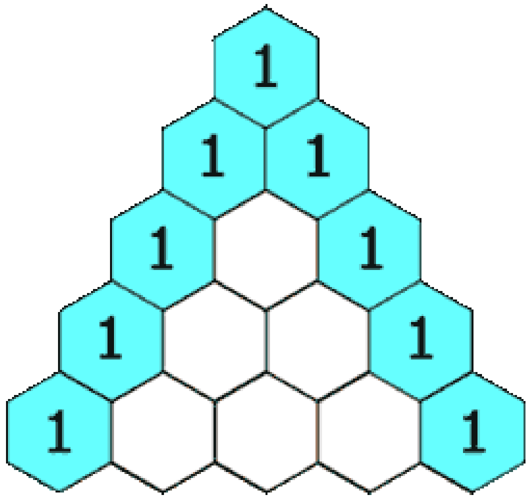
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

A significant amount of time may be required to apply the binomial theorem and perform all of the calculations in the above formula, particularly for high values of n . Therefore, what follows is a shortcut for finding binomial expansions using a visual tool.

Pascal's Triangle

Pascal's triangle is an alternative way of determining the coefficients that arise in binomial expansions, using a diagram rather than algebraic methods. For a binomial expansion with a relatively small exponent, this can be a straightforward way to determine the coefficients.

In the diagram below, notice that each number in the triangle is the sum of the two directly above it. This pattern continues indefinitely.



Pascal's Triangle: Each number in the triangle is the sum of the two directly above it.

The rows of Pascal's triangle are numbered, starting with row $n = 0$ at the top. The entries in each row are numbered from the left beginning with $k = 0$ and are usually staggered relative to the numbers in the adjacent rows. A simple construction of the triangle proceeds in the following manner. On row 0 , write only the number 1 . Then, to construct the elements of following rows, add the two above numbers to find the new value. If either of the above numbers is not present, substitute a zero in its place. For example, each number in row one is $0 + 1 = 1$.

To understand how this pattern applies to the binomial formula, consider the expansion:

$$(x+y)^2 = x^2 + 2xy + y^2 = 1x^2y^0 + 2x^1y^1 + 1x^0y^2$$

Notice the coefficients are the numbers in row two of Pascal's triangle: $1, 2, 1$. In general, when a binomial like $x + y$ is raised to a positive integer power we have:

$$(x+y)^n = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots a_{n-1}xy^{n-1} + a_ny^n$$

Where the coefficients a_i in this expansion are precisely the numbers on row n of Pascal's triangle.

Notice that the entire right diagonal of Pascal's triangle corresponds to the coefficient of y^n in these binomial expansions, while the next diagonal corresponds to the coefficient of xy^{n-1} and so on.

Example: Find the expansion of $(x+y)^5$ using Pascal's triangle

Notice that $n=5$, and recall that this would correspond to row 5 of Pascal's triangle.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Pascal's Triangle: Pascal's triangle with 5 rows.

Recall that the binomial expansion of $(x+y)^5$ will be written in the following form, where the coefficients are the numbers in row 5 of Pascal's triangle:

$$(x+y)^5 = a_0x^5 + a_1x^4y + a_2x^3y^2 + a_3x^2y^3 + a_4xy^4 + a_5y^5$$

It can be observed in the triangle that row 5 is $1, 5, 10, 10, 5, 1$. Applying these numbers to the binomial expansion, we have:

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

Binomial Expansion and Factorial Notation

The binomial theorem describes the algebraic expansion of powers of a binomial.

LEARNING OBJECTIVES

Use factorial notation to find the coefficients of a binomial expansion

KEY TAKEAWAYS

Key Points

- According to the theorem, it is possible to expand the power $(x+y)^n$ into a sum involving terms of the form ax^by^c , where the exponents b and c are nonnegative integers with $b+c=n$, and the coefficient a of each term is a specific positive integer depending on n and b .
- The factorial of a non-negative integer n , denoted by $n!$, is the product of all positive integers less than or equal to n .
- Binomial coefficients can be written as $\binom{n}{k}$ or ${}_nC_k$ and are defined in terms of the factorial function $n!$.

Key Terms

- **factorial**: The result of multiplying a given number of consecutive integers from 1 to the given number. In equations, it is symbolized by an exclamation mark ($!$). For example, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.
- **binomial coefficient**: A coefficient of any of the terms in the expansion of the binomial power $(x+y)^n$.

Recall that the binomial theorem is an algebraic method of expanding a binomial that is raised to a certain power, such as $(4x+y)^7$. The theorem is given by the formula:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

The coefficients that appear in the binomial expansion are called binomial coefficients. These are usually written $\binom{n}{k}$ or ${}_nC_k$, and pronounced “n choose k “.

The coefficient of a term $x^{n-k}y^k$ in a binomial expansion can be calculated using the combination formula. Recall that the combination formula represents the number of ways to choose k objects from among n , where order does not matter. The formula consists of factorials:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Note that although this formula involves a fraction, the binomial coefficient $\binom{n}{k}$ is actually an integer.

In calculating coefficients, recall that the factorial of a non-negative integer n , denoted by $n!$, is the product of all positive integers less than or equal to n . For example,

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. The value of $0!$ is 1 , according to the convention for an empty product.

Finally, you may recall that the factorial $n!$ also represents the number of possible permutations of n , or the number of ways in which n objects can be arranged or selected.

Example: Use the binomial formula to find the expansion of $(x+y)^4$

Start by substituting $n = 4$ into the binomial formula:

$$(x+y)^4 = \sum_{k=0}^{4} \binom{4}{k} x^{4-k} y^k$$

In order to solve this, we will need to expand the summation for all values of k .

$$\begin{aligned}(x+y)^4 &= \binom{4}{0} x^{4-0} y^0 + \binom{4}{1} x^{4-1} y^1 + \binom{4}{2} x^{4-2} y^2 + \binom{4}{3} x^{4-3} y^3 + \binom{4}{4} x^{4-4} y^4 \\ &= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x^1 y^3 + \binom{4}{4} y^4\end{aligned}$$

Recall that $\binom{4}{0}$ and $\binom{4}{4}$ are both equivalent to 1, as there is only one way to choose either 0 or 4 objects from among 4 . Therefore, we have:

$$= x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + y^4$$

Now we must evaluate each of the remaining combinations:

$$\binom{4}{1} = \frac{4!}{1!(4-1)!} = \frac{4!}{1!3!} = 4$$

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = 6$$

$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!1!} = 4$$

Substituting these integers into the expansion, we have:

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Finding a Specific Term

The rth term of the binomial expansion can be found with the equation: $\binom{n}{r-1} a^{n-(r-1)} b^{r-1}$.

Learning Objectives

Practice finding a specific term of a binomial expansion

Key Takeaways

Key Points

- Properties for the binomial expansion include: the number of terms is one more than n (the exponent), and the sum of the exponents in each term adds up to n .
- Applying $\binom{n}{r-1} a^{n-(r-1)} b^{r-1}$ and $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, one can find a particular term of a binomial expansion without going through every single term.

Key Terms

- **integer**: An element of the infinite and numerable set $\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$.

There may be instances when we want to identify a certain term in the expansion of $(x+y)^n$. It is straightforward to identify the terms where n is an integer with a low value. You might multiply each binomial out to identify the coefficients, or you might use Pascal's triangle. However, what if n has a greater value? How would you identify a particular term of $(3x-4)^{12}$? Luckily, there is a formula that can be used to calculate the terms in such situations.

Let's go through a few expansions of binomials, in order to consider any patterns that are present in the terms.

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

A few things should be noticed:

- The number of terms is one more than n (the exponent).
- The power of a starts with n and decreases by 1 each term.
- The power of b starts with 0 and increases by 1 each term.
- The sum of the exponents in each term adds up to n .
- The coefficients of the first and last terms are both 1 and they follow Pascal's triangle.

If the expansion is short, such as:

$$\begin{aligned}(x+2)^3 &= x^3 + 2x^2 2^1 + 2x^1 2^2 + 2^3 \\ &= x^3 + 4x^2 + 8x + 8\end{aligned}$$

Then it is easy to find a particular term. This becomes difficult and time consuming when the expansion is large. There is, luckily, a shortcut for identifying particular terms of longer expansions. The following formula yields the r th term in the expansion:

$$\binom{n}{r-1} a^{n-(r-1)} b^{r-1}$$

Recall that the combination formula provides a way to calculate $\binom{n}{k}$:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Example: Find the fifth term of $(3x-4)^{12}$

Note that the value of $n = 12$ in this case. Because we are looking for the fifth term, we use $r = 5$. Plugging these values into the formula, we have:

$$\binom{12}{5-1} (3x)^{12-(5-1)} (-4)^{5-1}$$

$$\binom{12}{4} (3x)^8 (-4)^4$$

Remember to evaluate $\binom{12}{4}$ using the combination formula:

$$\begin{aligned}\frac{n!}{(n-k)!k!} &= \frac{12!}{(12-4)!4!} \\ &= 495\end{aligned}$$

Subbing in $\binom{12}{4} = 495$ in the formula, we have:

$495(3x)^8(-4)^4$

When the power is applied to the terms, the result is:

$495 \cdot 6561x^8 \cdot 256 = 831409920x^8$

Thus, the fifth term of $(3x-4)^{12}$ is $831409920x^8$.

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