Time series analysis – ch. 10.1 – 10.2

10.1

time series – 2D scalar data $\{(t_1, y_1) \ldots (t_N, y_N)\}$

different treatment than regression b/c

1) sense of directionality
2) y_i's are not independent – y_{i+1} depends on y_i

signal detection in time series data

equivalent to testing null hypothesis of flat signal + noise

consider: $y(t) = A \sin(\omega t)$

sampled by N points w/ Gaussian errors with std dev. σ

variance $V = \sigma^2 + \frac{1}{2}A^2$

for $A = 0$,

$$\chi^2_{dof} = \frac{1}{N}\sum_j \frac{y_j^2}{\sigma^2} \sim \frac{V}{\sigma^2} = 1 \pm \sqrt{\frac{2}{N}} \quad \text{if } A=0 \text{ true}$$

$$\text{Prob.}\left(\chi^2_{dof} > 1 + 3\sqrt{\tfrac{2}{N}}\right) \sim 10^{-3}$$

min. detectable A: for 10^{-3} false pos. rate

$$\chi^2_{d.o.f.} > 1 + 3\sqrt{\frac{2}{N}} = \frac{V}{\sigma^2} = \frac{1}{\sigma^2}\left(\sigma^2 + \tfrac{1}{2}A^2\right)$$

$$\Rightarrow A > 2.9\,\sigma\, N^{-1/4} = \begin{cases} 0.92\sigma, & N = 10^2 \\ 0.52\sigma, & N = 10^3 \\ 0.29\sigma, & N = 10^4 \end{cases}$$

even lower A_{min} for simple harmonic model

signal does not have to be periodic to be picked out by model. 2 types of non-periodic variability:

- stochastic – always variable but not predictable
  e.g. quasars
- temporally localized / one-time
  e.g. SNe, γ ray burst, grav. waves from BH merger

$\boxed{10.2}$

methods in time series analysis

time domain - can apply previous chapters' tools

frequency domain - Fourier analysis, discrete Fourier transform, wavelet analysis, digital filtering

parameter estimation, model selection, + classification already covered so will skip here

Fourier analysis

Fourier transform of function $h(t)$:

$$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-i2\pi f t}\, dt$$

N.B. writing in terms of f [Hz] instead of usual ω, (-) sign goes in time integral (engineering / scipy convention)

inverse transform

$$h(t) = \int_{-\infty}^{\infty} H(f)\, e^{i2\pi f t}\, df$$

if $h(t)$ is a real function, $H(f)$ is generally complex

* exception - $h(t)$ is even $\Rightarrow$ $H(f)$ also real + even

shift time axis of $h(t) \longrightarrow h(t+\Delta t)$:

$$H(f) \longrightarrow \int_{-\infty}^{\infty} h(t+\Delta t)\, e^{-i2\pi f t}\, dt = H(f)\, e^{i2\pi f \Delta t}$$

for unshifted $h(t)$

a zero-mean Gaussian is an even real function so

$$H_{Gaussian}(f) = e^{-2\pi^2\sigma^2 f^2}$$

if mean is shifted to μ, then

$$H_{Gaussian}(f) = e^{-2\pi^2\sigma^2 f^2}\, e^{i2\pi f\mu} = e^{-2\pi^2\sigma^2 f^2}\left[\cos 2\pi f\mu + i \sin 2\pi f\mu\right]$$

power spectral density (PSD) function / power spectrum

one-sided: defined for $0 \leq f < \infty$

$$PSD(f) \equiv |H(f)|^2 + |H(-f)|^2$$

measures power contained in freq. interval $f \to f + df$

ex. for $h(t) = \sin \frac{2\pi t}{T}$, $P(f) = \delta(f - \frac{1}{T})$

can fully describe $h(t)$ with a single freq. $f = \frac{1}{T}$

Parseval's theorem:

$$P_{tot} = \int_0^\infty PSD(f)\, df = \int_{-\infty}^{\infty} |h(t)|^2\, dt$$

can compute total power in either time or freq. domain

convolution theorem

convolution of functions $a(t)$ + $b(t)$:

$$(a * b)(t) = \int_{-\infty}^{\infty} a(t')\, b(t - t')\, dt'$$

observations are always some convolution of the true measurement + measurement process

e.g. space-based image + telescope diffraction

true spectrum + spectral resolution limit

we measure $(\underbrace{a}_{\text{convolving pattern}} * \underbrace{b}_{\text{signal}})$, hopefully have knowledge of a so we can extract b

convolution theorem: if $h = a * b$, then their Fourier transforms are related by

$$H(f) = A(f) \cdot B(f)$$

which makes obtaining $b(t)$ simple: Fourier transform to find $B(f) = H(f) / A(f)$ then inverse transform back to get $b(t)$

Discrete Fourier transform – for use on uniformly discretely sampled data

pro: very fast algorithms

can extend concepts to nonuniform sampling

con: uniform temporal spacing rare (at least in astro)

discretized data, so Fourier transform integrals $\rightarrow \sum_j$

$h(t)$ is continuous + real, we make N measurements over total time T at constant intervals Δt and have measurement vector $\vec{h}$

discrete Fourier transform of $\{h_j\}$; $j = 0, \ldots, N-1$

$$H_k = \sum_{j=0}^{N-1} h_j \exp[-i 2\pi j k / N]$$

inverse " "

$$h_j = \frac{1}{N} \sum_{k=0}^{N-1} H_k \exp[i 2\pi j k / N]$$

given vector $\vec{H}$, $h_j = h(t_j)$ can be described as sum of sinusoids

compare to integral transforms:

- units of H_k + h_j same, unlike continuous transf.
- sum over sampled data vs. integral over $(-\infty, \infty)$

Nyquist sampling theorem: if $h(t)$ is 'band-limited' and $\Delta t \leq t_c$, then $h(t)$ can be exactly reconstructed from evenly sampled data

defn. of band-limited: $H(f) = 0$ for $|f| > f_c$

f_c: band limit / Nyquist critical freq.

$t_c = \frac{1}{2 f_c}$: resolution limit in t-space

$$h(t) = \frac{\Delta t}{t_c} \sum_{k=-\infty}^{\infty} h_k \frac{\sin[2\pi f_c (t - k\Delta t)]}{2\pi f_c (t - k \Delta t)}$$

aliasing effect occurs if $h(t)$ not band-limited or $\Delta t > t_c$

all of the power in freq.s $|f| > f_c$ falsely transfers into $|f| < f_c$

shows up as $H(f = \frac{1}{2\Delta t}) \neq 0$

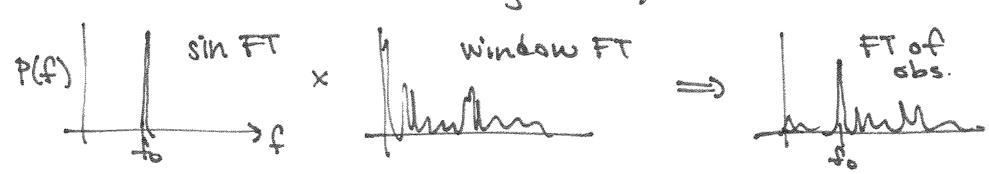
window functions + uneven sampling

for time domain measurements, window function is sum of delta functions

uniform sampling: δ's evenly spaced in t, so Fourier transf. is δ's spaced at $1/\Delta t$

non uniform sampling: FT of window func. is more complex

e.g. for sinusoidal $h(t) = \sin 2\pi f t$, $H(f) = \delta(f - f_0)$
but convolving $H(f)$ with messier FT of window function results in PSD with power in freq.s other than f_0 (book Fig. 10.4)



wavelets

for events localized in time, better to use localized basis function in Fourier transf. instead of sin/cos

wavelets - localized in t + f

can construct complete orthonormal set of basis functions by scaling + translating wavelet

allows PSD that depends on both f + t

i.e. localized freq. information

digital filtering

low pass filtering

e.g. data band-limited to $f < f_c$ with white Gaussian noise extending to Nyquist limit $f_N = \frac{1}{2\Delta t} > f_c$

can remove a lot of noise at $|f| > f_c$ w/o affecting data too much

take inverse FT of

$$\hat{y}(f) = y(f)\,\Phi(f)$$

discrete FT of data — filter in f space

using $\Phi(f) = \begin{cases} 0 & , |f| > f_c \\ 1 & \text{otherwise} \end{cases}$ would produce ringing in signal

optimal filter: Wiener filter - minimizes mean integrated square error between $\hat{y}$ + y

$$\Phi_W(f) = \frac{P_S(f)}{P_S(f) + P_N(f)}$$

* requires even sampling

P_S, P_N : 'signal' + 'noise' components of 2-component fit to PSD of input data