Exercises for the lecture course Algebraic Topology II – Sheet 10

University of Bonn, summer term 2025

Exercise 37. Compute the first Chern class of the principal S^1-bundle over S^2 given by the Hopf fibration.

Exercise 38. The infinite dihedral group D_∞ is defined by the presentation $\langle s, t \mid sts = t^{-1}, s^2 = 1\rangle$.

(a) Show that there is a fibration $S^1 \to BD_\infty \to \mathbb{RP}^\infty$;

(b) Compute $H_n(BD_\infty; \mathbb{Z})$ for $n \in \mathbb{Z}^{\geq 0}$.

Exercise 39. Let M be a closed connected 3-manifold whose fundamental group is perfect, i.e. $\pi = [\pi, \pi]$. Consider a prime p. Let $f\colon M \to S^3$ be a map of a degree which is prime to p. Consider a pullback

$$\begin{tikzcd} \overline{E} \arrow[r, "\overline{f}"] \arrow[d] & E \arrow[d, "q"] \\ M \arrow[r, "f"] & S^3 \end{tikzcd}$$

where q is a fibration. Let $\mathcal{H}_*$ be a homology theory with values in $\mathbb{F}_p$-modules satisfying the disjoint union axiom.

Prove or disprove that $\mathcal{H}_n(\overline{f})\colon \mathcal{H}_n(\overline{E}) \to \mathcal{H}_n(E)$ is an isomorphism for all $n \in \mathbb{Z}$.

Exercise 40. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration of path connected spaces with a CW-complex B as basis. Suppose that the action of $\pi_1(B)$ on $H_q(F)$ given by the fiber transport is trivial. Prove or disprove:

(a) The map $H^n(i)\colon H^n(E) \to H^n(F)$ factorizes as the composite $H^n(E) \xrightarrow{\alpha} E_\infty^{0,n} \xrightarrow{\beta} H^n(F)$ for an epimorphism α and a monomorphism β;

(b) The map $H^n(p)\colon H^n(B) \to H^n(E)$ factorizes as the composite $H^n(B) \xrightarrow{\alpha} E_\infty^{n,0} \xrightarrow{\beta} H^n(E)$ for an epimorphism α and a monomorphism β;

[0]Hand-in Monday 23.06.