Exercises for the lecture course Algebraic Topology II – Sheet 11

University of Bonn, summer term 2025

Exercise 41. Let R be a torsionfree commutative ring. Prove or disprove that the divided power R-algebra $R\left[y, \frac{y^2}{2!}, \frac{y^3}{3!}, \frac{y^4}{4!} \cdots\right]$ and the R-algebra $R[x]$ for $|x|$ and $|y|$ even are isomorphic as graded R-algebras if and only if $|x| = |y|$ and $\mathbb{Q} \subseteq R$ hold.

Exercise 42. Let R be a commutative ring. Prove or disprove:

(a) The full subcategory of R-Mod given by finitely generated R-modules is a Serre class, if and only if R is Noetherian;

(b) The full subcategory of R-Mod given by R-modules whose underlying set is finite is a Serre class;

(c) The full subcategory of R-Mod given projective R-modules is a Serre class if and only if R is a semisimple, i.e., every R-module is projective.

(d) The full subcategory of R-Mod given by free R-modules is a Serre class if and only if R is a field.

Exercise 43. (a) Prove that there is an isomorphism $H^*(BU(n), \mathbb{Z}) \simeq \mathbb{Z}[c_1, \ldots, c_n]$ of graded rings with generators in degrees $|c_i| = 2i$. *Hint: Induction on n.*

(b) Consider a complex rank n vector bundle $\zeta\colon E \to B$ over a CW-complex B and denote its classifying map by $f\colon B \to BU(n)$. We can define its kth *Chern class* by $c_k(\zeta) = f^*c_k \in H^{2k}(B, \mathbb{Z})$ for $k \leq n$ and $c_k(\zeta) = 0$ for $k > n$. Prove the following:

  (a) $c_0(\zeta) = 1$.

  (b) $c_k(\zeta) = 0$ for $k \geq 1$ if ζ is trivial.

  (c) Compute $c_k(\gamma_n)$ where γ_n is the universal rank n bundle over $BU(n)$.

  (d) Show that this definition of $c_1(\zeta)$ agrees with the one from the lecture for $n = 1$.

  (e) Explain how $c_n(\zeta)$ can be identified with the Thom class of the associated sphere bundle $S(\zeta)$, sometimes also called its Euler class.

Exercise 44. (a) Show that the map

$$\mathrm{edge}_{n,0}(S^n) \times \Omega_n(\mathrm{pr})\colon \Omega_n(S^n) \to \Omega_n(\{\bullet\}) \times H_n(S^n; \mathbb{Z})$$

is bijective, where $\mathrm{pr}\colon S^n \to \{\bullet\}$ is the projection;

(b) Show that $\mathrm{edge}_{n,0}(X)$ sends the bordism class of $f\colon M \to X$ to the image of the fundamental class $[M]$ under the map $H_n(f; \mathbb{Z})\colon H_n(M; \mathbb{Z}) \to H_n(X; \mathbb{Z})$, provided this claim holds for $X = S^n$, without using exercise 36.

[0] Hand-in Monday 30.06.