Exercises for the lecture course Algebraic Topology I – Sheet 10

University of Bonn, winter term 24/25

Aufgabe 37. Let ξ be an n-dimensional vector bundle over the space B. For $l \in \mathbb{Z}^{\geq 0}$ an l-*framing* of ξ is a bundle isomorphism $(\mathrm{id}_B, \overline{u})\colon \underline{\mathbb{R}^{n+l}} \xrightarrow{\cong} \xi \oplus \underline{\mathbb{R}^l}$ over B. We call an l_0-framing $(\mathrm{id}_B, \overline{u}_0)\colon \underline{\mathbb{R}^{n+l_0}} \xrightarrow{\cong} \xi \oplus \underline{\mathbb{R}^{l_0}}$ and an l_1-framing $(\mathrm{id}_B, \overline{u}_1)\colon \underline{\mathbb{R}^{n+l_1}} \xrightarrow{\cong} \xi \oplus \underline{\mathbb{R}^{l_1}}$ equivalent if there exists $l \in \mathbb{Z}^{\geq 0}$ with $l \geq l_0, l_1$ such that for $i = 0, 1$ the two bundle isomorphisms over B

$$\underline{\mathbb{R}^{n+l}} = \underline{\mathbb{R}^{n+l_i}} \oplus \underline{\mathbb{R}^{l-l_i}} \xrightarrow{(\mathrm{id}_B, \overline{u}_i) \oplus \mathrm{id}_{\underline{\mathbb{R}^{l-l_i}}}} \xi \oplus \underline{\mathbb{R}^{l_i}} \oplus \underline{\mathbb{R}^{l-l_i}} = \xi \oplus \underline{\mathbb{R}^l}$$

are homotopic through bundle isomorphisms over B. A stable framing on ξ is an equivalences class of l-framings.

(a) Prove that the group $[B, \mathrm{SO}]$ acts transitively and freely on the set of stable framing of ξ if there exists a stable framing on ξ;

(b) Show that the tangent bundle TS^2 has precisely one stable framing;

(c) Show that the tangent bundle TS^1 has precisely two stable framings;

(d) Construct explicite representatives for these stable framings on TS^2 and TS^1.

Aufgabe 38. Prove that $\pi_0^s \cong \mathbb{Z}$ and that there is a surjection $\mathbb{Z} \to \pi_1^s$.

Aufgabe 39. Construct a natural isomorphism

$$\pi_n^s(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} H_n(X; \mathbb{Q})$$

for any space X using the fact that π_n^s is finite for every $n \in \mathbb{Z}^{\geq 1}$.

Aufgabe 40. Consider $n \geq 2$ and $X = S^1 \vee S^n$. Show that the $\mathbb{Z}[\pi_1(X)]$-module $\pi_n(X)$ is free of rank 1.

[0]Hand-in Monday 16.12.