Exercises for the lecture course Algebraic Topology II
– Sheet 5

University of Bonn, summer term 2025

Exercise 17. Let $\mathcal{H}_*$ be a homology theory with values in R-modules satisfying the disjoint union axiom. Let X be a 2-dimensional CW-complex.

Prove or disprove that for every $n \in \mathbb{Z}$ there is a filtration

$$\{0\} \subseteq A \subseteq B \subseteq \mathcal{H}_n(X)$$

satisfying

$$
\begin{aligned}
A &\cong_R H_0(X; \mathcal{H}_{n-1}(\{\bullet\})); \\
B/A &\cong_R H_1(X; \mathcal{H}_{n-1}(\{\bullet\})); \\
\mathcal{H}_n(X)/B &\cong_R H_2(X; \mathcal{H}_{n-2}(\{\bullet\})).
\end{aligned}
$$

Exercise 18. Let X be a connected finite CW-complex with $\pi_1(X) \cong \mathbb{Z}/3$ whose universal covering is homeomorphic to S^3.

(a) Compute $H_n(X, \mathbb{Z})$ and $H_n(X; \mathbb{F}_2)$ for $n \in \mathbb{Z}^{\geq 0}$;

(b) Compute $K_n(X)$ for $n \in \mathbb{Z}$;

(c) Compute $\Omega_n(X)$ for $n \in \{0, 1, 2, 3, 4, 5\}$.

Exercise 19. Consider a homology theory $\mathcal{H}_*$ such that $\mathcal{H}_q(\{\bullet\})$ is finitely generated free for $q \in \mathbb{Z}$ and and a finite CW-complex X with $H_p(X)$ finitely generated free for every $p \geq 0$.

Prove or disprove that the Atiyah-Hirzebruch spectral sequence strongly collapses and yields isomorphisms

$$\mathcal{H}_n(X) \cong \bigoplus_{p+q=n} H_p(X) \otimes_{\mathbb{Z}} \mathcal{H}_q(\{\bullet\}).$$

Exercise 20. Let X be a CW-complex such that $H_n(X; \mathbb{Z})$ is a finite group of odd order for all $n \in \mathbb{Z}^{\geq 1}$. Prove or disprove that there is an isomorphism $\mathcal{N}_*(X) \simeq H_0(X) \otimes \mathcal{N}_*(\{\bullet\})$.

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[0]Hand-in Monday 12.05.