Exercises for the lecture course Algebraic Topology II – Sheet 3

University of Bonn, summer term 2025

Aufgabe 9. Let (P) be a property of $\mathbb{Z}$-modules. We say that a connected CW-complex X has property (P) if $H_n(X)$ has (P) for $n \in \mathbb{Z}^{\geq 1}$. We call property (P) compatible with products if for two connected finite CW-complexes X and Y all three spaces X, Y, and $X \times Y$ have property (P) if two of them have property (P).

Decide which of the following properties (P) is compatible with products:

(a) The $\mathbb{Z}$-module is trivial;

(b) The $\mathbb{Z}$-module is finite;

(c) The $\mathbb{Z}$-module is finitely generated;

(d) The $\mathbb{Z}$-module is finitely generated free.

Aufgabe 10. Let M be a closed smooth manifold of dimension d. Let $\{U_1, U_2, \ldots, U_n\}$ be a finite set of open subsetes U_i of M such that every U_i is diffeomorphic to $\mathbb{R}^d$.

Construct an injective smooth map

$$f\colon M \to \mathbb{R}^{dn+n}$$

whose differential T_xf is injective for every $x \in M$.

Aufgabe 11. Let μ be an n-dimensional system of vector bundles over the CW-complex X. Let μ' be the $n+1$-dimensional system of vector bundles over X obtained from μ whose vector bundle in degree k is $\xi_k \oplus \underline{\mathbb{R}}$ if ξ_k is the vector bundle in degree k of μ, and whose structure maps are the obvious ones.

Give and prove a formula how to compute $\Omega_*(\mu')$ from $\Omega_*(\mu)$.

Aufgabe 12. Is there a fiber bundle $F \to S^4 \to B$ for which F and B are closed connected orientable manifolds of dimension ≥ 1?

[0]Hand-in Monday 28.04.