Computer Aided Geometric Design 7 (1990) 33–41
North-Holland

Good approximation of circles by curvature-continuous Bézier curves

Tor DOKKEN and Morten DÆHLEN

Center for Industrial Research, P.O. Box 124, Blindern, 0314 Oslo 3, Norway

Tom LYCHE and Knut MØRKEN

University of Oslo, Institute for Informatics, P.O. Box 1080, Blindern, 0316 Oslo 3, Norway

Presented at Oberwolfach April 1989 by M. Dæhlen
Received July 1989; revised October 1989

Abstract. We provide a surprisingly simple cubic Bézier curve which gives a very accurate approximation to a segment of a circle. Joining the Bézier segments we obtain an approximation to the circle with continuous tangent and curvature. For 45° segments the error is approximately $2 \cdot 10^{-6}$, and in general the approximation is sixth order accurate.

Keywords. Bézier curves, circle, geometric continuity, parametric approximation, accuracy.

1. Introduction

In this paper we present some methods for approximating circular arcs with parametrically defined piecewise cubic polynomial curves. The methods are based on the Bézier representation of piecewise polynomials and yield order six convergence.

The main application of our method is for representing circular segments in a computer based geometry system with a circle primitive. Especially in a spline based system such an approximation is convenient since it is desirable to be able to represent all geometric objects in the same format.

Let $p(t) = (x(t),\ y(t))$ be a planar parametric curve. The methods are all based on using the quantity $\psi(t) = |x(t)^2 + y(t)^2 - 1|$ as a measure of the error in the circle approximation. Note that the degenerate constant curve $p(t) = (\cos a,\ \sin a)$ for some $a \in \mathbb{R}$ gives $\psi(t) \equiv 0$. To avoid this degeneracy, we constrain the curve by requiring it to interpolate a circular arc at the start and end points and have tangent vectors parallel to the arc at the same points. The approximations are then obtained by making $x(t)^2 + y(t)^2 - 1$ oscillate in the characteristic manner of the error in best uniform approximation of functions, and this turns out to yield very high accuracy. The constraints ensure that if we use the same method to approximate two neighbouring arcs, the two approximations will join with continuous tangent direction and curvature. In fact, if the two arcs have the same length, the parametrization is such that even the magnitude of the tangent vector is continuous.

The key point in the methods is to use ψ as a measure of the error since this is independent of any particular parametrization of the circle. If we compare our approximations componentwise with the standard trigonometric parametrizations of the circle, the sixth order convergence disappears, as is to be expected from well-known results in Approximation Theory.

In Section 2 we will review the basics about Bézier curves and geometric continuity. In Section 3 we describe our approach to circle approximation, and in Sections 4–6 we give the three methods. In Section 7 we give the results of some numerical tests.

2. Some preliminaries

We will work with piecewise polynomial planar curves in Bézier form. This means that each polynomial piece is on the form

$$p(t)=\sum_{i=0}^{3} P_i\binom{3}{i}(1-t)^{3-i}t^i, \quad \text{for } t\in[0,1],$$

where the control points $(P_i)_0^3$ are points in $\mathbb{R}^2$.

Two curves $p:[a, b]\to\mathbb{R}^2$ and $q:[c, d]\to\mathbb{R}^2$ are G^2-continuous at $p(b)=q(c)$ if

$$p'(b)=\beta_1 q'(c), \qquad p''(b)=\beta_1^2 q''(c)+\beta_2 q'(c), \tag{1}$$

where β_1 and β_2 are real numbers with β_1 nonzero. Note that if $\beta_1=1$, then p and q join with C^1-continuity for these particular parametrizations. By using the definition above it can be shown that the two cubic Bézier segments

$$p(t)=\sum_{i=0}^{3} P_i\binom{3}{i}(1-t)^{3-i}t^i, \quad t\in[0,1]$$

and

$$q(t)=\sum_{i=0}^{3} Q_i\binom{3}{i}(1-t)^{3-i}t^i, \quad t\in[0,1]$$

join with G^2-continuity at $p(1)=q(0)$ if

$$P_3=Q_0, \qquad P_3-P_2=Q_1-Q_0, \qquad P_2-Q_1=\hat{\beta}_2(P_1-Q_2), \tag{2}$$

where $\hat{\beta}_2\neq 0$. The two first conditions follow from continuity of the position and the first derivative, while the third condition is a consequence of the preceding two and (1) adapted to cubic Bézier curves with $\beta_1=1$, cf. Fig. 1. Note that the second of these conditions even ensure that the size of the tangent is continuous. For more details about Bézier curves and geometric continuity, see [Farin '88].

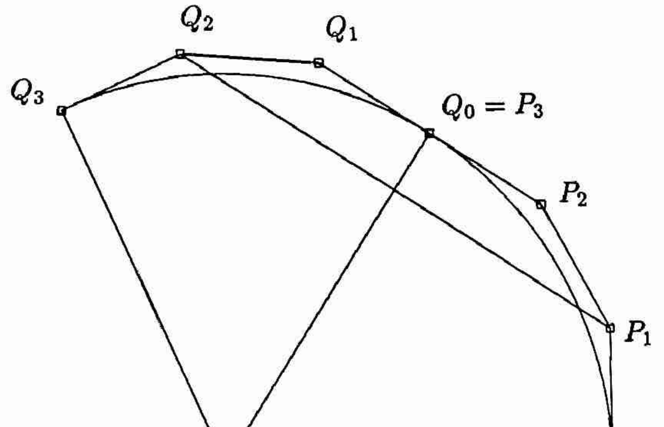

Fig. 1. Two Bézier segments which are C^1- and curvature-continuous at $p(1)=q(0)$.

3. Circle approximation

In this section we describe our approach to Bézier approximation of circular segments. Given a circular arc of angular width α, we want to find a cubic Bézier approximation $p(t)$ to this circle segment. Without loss of generality we assume the circular segment to be of the form

$$f(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \text{for } \theta \in [0, \alpha], \tag{3}$$

with $\alpha \in [0, 2\pi]$, since an approximation to a general circle segment of width α can be obtained from an approximation to f by a radial scaling and a rotation.

Initially, we consider approximations to f of the form

$$p(t) = (x(t), y(t)) = \sum_{i=0}^{3} P_i \binom{3}{i} (1-t)^{3-i} t^i, \tag{4}$$

where

$$\begin{aligned} &P_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + L\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ &P_2 = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} - L\begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}, \quad P_3 = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \end{aligned} \tag{5}$$

and where L is a real number, the Euclidean length of the two vectors $P_1 - P_0$ and $P_3 - P_2$. This means that p interpolates the value and the direction of the tangent of the given segment at the beginning and end,

$$\begin{aligned} p(0) &= f(0), \qquad p'(0) = 3Lf'(0), \\ p(1) &= f(\alpha), \qquad p'(1) = 3Lf'(\alpha). \end{aligned} \tag{6}$$

Thus, we have a one-parameter family of curves controlled by the parameter L. Our aim is to find a value of L that makes the 'error'

$$\psi(t) = |\phi(t)| = |x(t)^2 + y(t)^2 - 1|, \quad \text{for } t \in [0, 1]$$

small. The Euclidean distance from the point $(x(t), y(t))$ to the unit circle is given by

$$|e(t)| = |\sqrt{x(t)^2 + y(t)^2} - 1|.$$

If $\psi(t)$ is small, then $|e(t)|$ is also small since

$$\psi(t) = |\sqrt{x(t)^2 + y(t)^2} - 1|\left(\sqrt{x(t)^2 + y(t)^2} + 1\right) \approx 2|e(t)|.$$

In this paper we will refer to all of ψ, ϕ and e as the error.

By letting $q(t)$ be a rotation of $p(t)$ through an angle α we obtain a similar approximation on a neighbouring segment. Such a rotation is easily seen to satisfy the continuity constraints (2) at the join, see Fig. 1. Indeed, in this way we can build up an approximation to a circular arc of arbitrary width with good precision by splitting the arc into a suitable number of segments of equal angular width and then gluing together the individual Bézier segments.

Given α, we seek to find L as a function of α such that the distance between the curve $p(t)$ given by (4) and (5) and the unit circle segment given by (3) is small. From the experience of best uniform approximation of functions, we will attempt to do this by making the function

$$\phi(t) = x(t)^2 + y(t)^2 - 1$$

oscillate in certain ways.

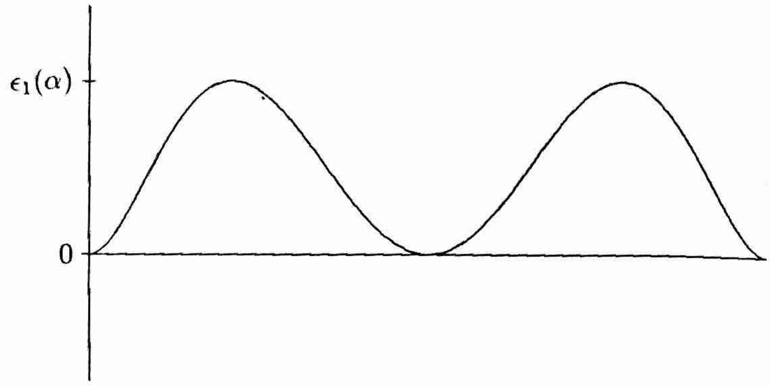

Fig. 2. The function $\phi_1(t)$.

The components $x(t)$ and $y(t)$ of the Bézier segment given in (4) and (5) are

$$x(t) = (1-t)^3 + 3(1-t)^2 t + 3(\cos\alpha + L\sin\alpha)(1-t)t^2 + t^3\cos\alpha,$$

$$y(t) = 3L(1-t)^2 t + 3(\sin\alpha - L\cos\alpha)(1-t)t^2 + t^3\sin\alpha.$$

Since $x(t)$ and $y(t)$ are cubic polynomials, we have that $\phi(t)$ is a polynomial of degree six, and direct computation gives

$$\phi(t) = 15a(1-t)^4 t^2 + 20b(1-t)^3 t^3 + 15a(1-t)^2 t^4, \tag{7}$$

where

$$a = \tfrac{1}{5}\left(3L^2 + 2L\sin\alpha - 2(1-\cos\alpha)\right), \tag{8}$$

$$b = \tfrac{1}{10}\left(-9L^2\cos\alpha + 18L\sin\alpha - 10(1-\cos\alpha)\right). \tag{9}$$

Thus, the error function ϕ is the polynomial of degree six with Bézier control points $(0, 0, a, b, a, 0, 0)$. Note that $\phi(\frac{1}{2} - t) = \phi(\frac{1}{2} + t)$, which reflects the natural symmetry in the construction. This means that ϕ always has a local extremum at $t = \frac{1}{2}$.

4. A one-sided Hermite interpolant

In our first approximation we choose L in (7) so that $\phi(\frac{1}{2}) = 0$, and end up with an approximation p_1 with $\phi(t) \geqslant 0$ for all $t \in [0, 1]$, see Fig. 2. In addition, p_1 satisfies (6), so that this is a one-sided Hermite interpolant.

Theorem 1. *The Bézier curve $p_1(t) = (x_1(t), y_1(t))$ obtained by setting L in (5) equal to*

$$L_1 = \tfrac{4}{3}\tan\tfrac{1}{4}\alpha$$

interpolates the circle segment at $(1, 0)$, $(\cos\frac{1}{2}\alpha, \sin\frac{1}{2}\alpha)$, and $(\cos\alpha, \sin\alpha)$ and never enters inside the circle. The error is given by

$$\epsilon_1(\alpha) = \max_{t\in[0,1]}\left\{x_1(t)^2 + y_1(t)^2 - 1\right\} = \frac{4}{27}\frac{\sin^6\frac{1}{4}\alpha}{\cos^2\frac{1}{4}\alpha}$$

Proof. By some algebra and trigonometry one finds that a and b reduce to

$$a_1 = \frac{16}{15}\frac{\sin^6\frac{1}{4}\alpha}{\cos^2\frac{1}{4}\alpha}, \qquad b_1 = -\frac{8}{5}\frac{\sin^6\frac{1}{4}\alpha}{\cos^2\frac{1}{4}\alpha}$$

for this value of L. This simplifies ϕ to

$$\phi_1(t) = 16t^2(1-t)^2(1-2t)^2 \frac{\sin^6 \frac{1}{4}\alpha}{\cos^2 \frac{1}{4}\alpha}. \tag{10}$$

This expression is clearly always non-negative which means that the approximation always lies on or outside the circle. By definition the Bézier curve interpolates the circle segment at the beginning and end. Since $\phi_1(\frac{1}{2}) = 0$ it must also interpolate at a third point, and by symmetry this must be $(\cos \frac{1}{2}\alpha, \sin \frac{1}{2}\alpha)$.

Differentiating (10) and solving $\phi_1'(t) = 0$, we find that ϕ_1 attains its maximum at the two points $t_1 = (3 + \sqrt{3})/6$ and $t_2 = (3 - \sqrt{3})/6$, and the value at these points is

$$\epsilon_1(\alpha) = \frac{4}{27} \frac{\sin^6 \frac{1}{4}\alpha}{\cos^2 \frac{1}{4}\alpha} \qquad \square$$

Recall that for small angles α, we have $\cos \alpha \approx 1$ and $\sin \alpha \approx \alpha$. The preceding theorem therefore shows that the error in the proposed approximation decreases with the sixth power of the length of the circular segment. The general approximation method described in [de Boor, Höllig & Sabin '87] has the same convergence rate, and this is two orders more than what can usually be achieved in cubic approximation of functions. As was mentioned above, such a convergence rate is possible since a point on the approximation is compared with the nearest point on the circle segment so that no particular parametrization of the circle is used.

Note that as we approach a full circle, i.e., as $\alpha \to 2\pi$, the error $\epsilon_1(\alpha)$ grows without bounds.

5. An approximation with equioscillating error

The fact that the above approximation never enters inside the circle immediately suggests an alternative approximation with smaller error. If the Bézier curve in Theorem 1 is pulled slightly in towards the center of the circle, the outside deviation from the circle is reduced and the maximum error will be reduced until the approximation has been pulled in by roughly half the original error. This is summarized in the next result.

Theorem 2. *Let $p_2(t) = (x_2(t), y_2(t)) = (\rho x_1(t), \rho y_1(t))$ be the Bézier curve obtained by multiplying the Bézier curve $p_1(t)$ in Theorem 1 by*

$$\rho = \sqrt{\frac{2}{2 + \epsilon_1(\alpha)}}.$$

The error $\epsilon_2(\alpha)$ in this approximation is given by

$$\epsilon_2(\alpha) = \max_{t \in [0,1]} \left\{ x_2(t)^2 + y_2(t)^2 - 1 \right\} = \frac{2 \sin^6 \frac{1}{4}\alpha}{27 \cos^2 \frac{1}{4}\alpha + 2 \sin^6 \frac{1}{4}\alpha}. \tag{11}$$

Proof. Since $\phi_2(t) = x_2(t)^2 + y_2(t)^2 - 1 = \rho^2 \phi_1(t) + \rho^2 - 1$, we see that ϕ_2 attains its extreme values at the same points as ϕ_1. We therefore have

$$\phi_2(0) \leqslant \phi_2(t) \leqslant \phi_2(t_1), \quad \text{for all } t \in [0, 1],$$

where t_1 is the first maximum of ϕ_1, see the proof of Theorem 1. But now

$$\phi_2(0) = \rho^2 - 1 \quad \text{and} \quad \phi_2(t_1) = \rho^2 \epsilon_1(\alpha) + \rho^2 - 1.$$

For the given value of ρ, we find

$$-\phi_2(0) = \phi_2(t_1) = \epsilon_2(\alpha) = \frac{\epsilon_1(\alpha)}{2 + \epsilon_1(\alpha)},$$

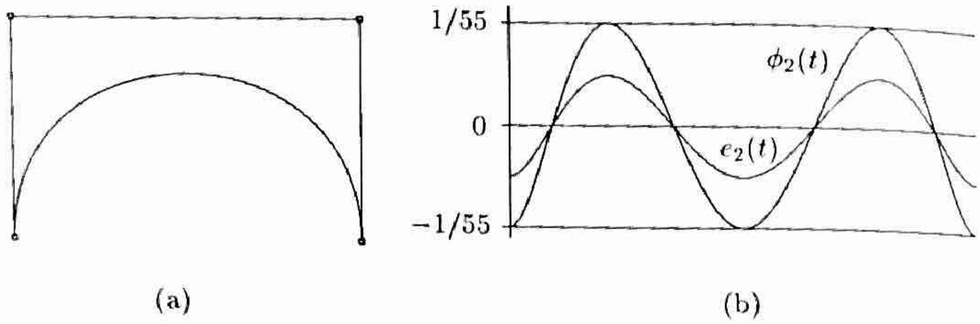

Fig. 3. (a) Cubic Bézier approximation to the semicircle. (b) The functions $\phi_2(t)$ and $e_2(t)$.

and from this, the required result (11) follows by inserting the value of $\epsilon_1(\alpha)$ from Theorem 1. □

Note that multiplying a Bézier curve by a scalar simply amounts to multiplying the control points by the scalar. If the control points of $p_1(t)$ are $(P_i^1)_{i=0}^3$, the control points of $p_2(t)$ will therefore be $(\rho P_i^1)_{i=0}^3$.

For small angles α, we know that $\epsilon_1(\alpha)$ is small and therefore

$$\epsilon_2(\alpha) \approx \tfrac{1}{2}\epsilon_1(\alpha).$$

We also note that $\epsilon_2(\alpha)$ remains bounded when α approaches 2π.

Fig. 3(a) shows an approximation to the semicircle with one cubic Bézier segment using the technique in Theorem 2. Fig. 3(b) shows the equioscillating function $\phi_2(t)$ and the radial error

$$e_2(t) = \frac{\phi_2(t)}{\sqrt{1+\phi_2(t)}+1}. \tag{12}$$

By (11) we have $\epsilon_2(\pi) = 1/55$, and therefore $\max_t |e_2(t)| \approx 1/110$ in this case. We observe that the radial error $e_2(t)$ also almost equioscillates five times.

Fig. 4(a) shows an approximation to the whole circle by four Bézier segments also constructed as in Theorem 2. Fig. 4(b) shows the error $e_2(t)$ given by (12). Fig. 4(c) shows the

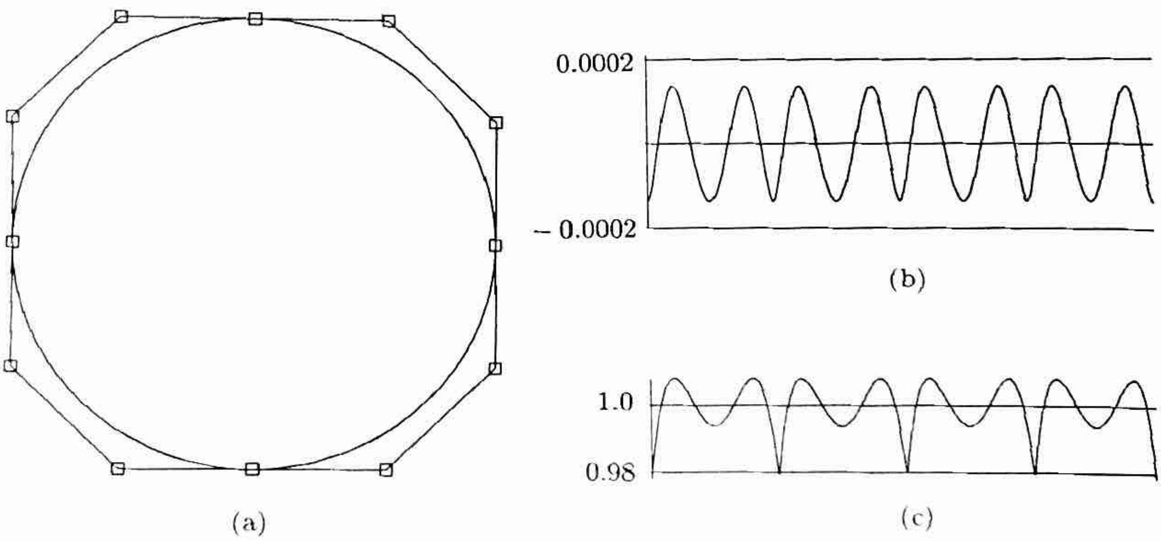

Fig. 4. A C^1- and curvature-continuous cubic Bézier approximation to the whole circle. (b) The error. (c) The signed curvature.

curvature in this case. We observe that on each segment, also the curvature oscillates five times around the constant 1, which is the curvature of the unit circle.

6. A Hermite interpolant with equioscillating error

The Bézier curve given in Theorem 1 provides a one-sided Hermite interpolant to the circle segment. The error oscillates but is always positive. If we insist on the approximation interpolating the circle segment at the endpoints, it seems plausible that the error would be minimized if we make the error function equioscillate as suggested by Fig. 5. This is the essential feature of the third method.

It will be convenient to introduce the family of functions $h(t)$ given by

$$h(t) = 15(1-t)^4t^2 + 20K(1-t)^3t^3 + 15(1-t)^2t^4. \tag{13}$$

If we set $K = b/a$, then $\phi(t) = ah(t)$. The approximations in Sections 4 and 5 correspond to $K = -\frac{3}{2}$. The next lemma gives the value of K such that $h(t)$ has the desired shape.

Lemma 3. *With the constant K equal to*

$$K_3 = \tfrac{1}{2} - \sqrt[3]{3-2\sqrt{2}} - \sqrt[3]{3+2\sqrt{2}} \tag{14}$$

the function $h(t)$ equioscillates three times.

Proof. Because of symmetry, the function $h(t)$ must have a local extremum at $t = \frac{1}{2}$. In addition there may be two more interior extrema which can be found by solving $h'(t) = 0$. The first of these is

$$\mu = \frac{1 - \sqrt{1 - 4/(3-2K)}}{2}.$$

The value of K_3 is then found by solving the third degree equation $h(\mu) + h(\frac{1}{2}) = 0$. □

Our final result gives the value of L in (5) such that the error function $\phi(t)$ equioscillates as indicated by Lemma 3.

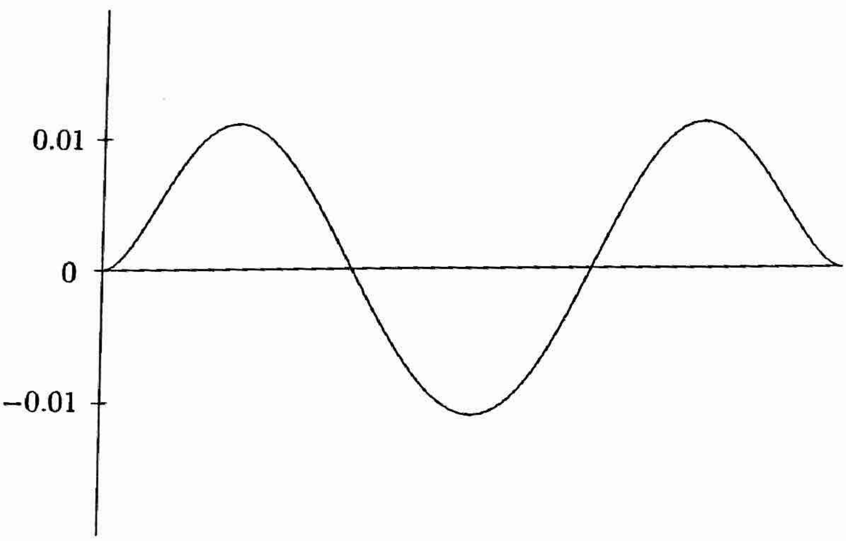

Fig. 5. The Bézier function $h(t)$ given by (13), with K given by (14).

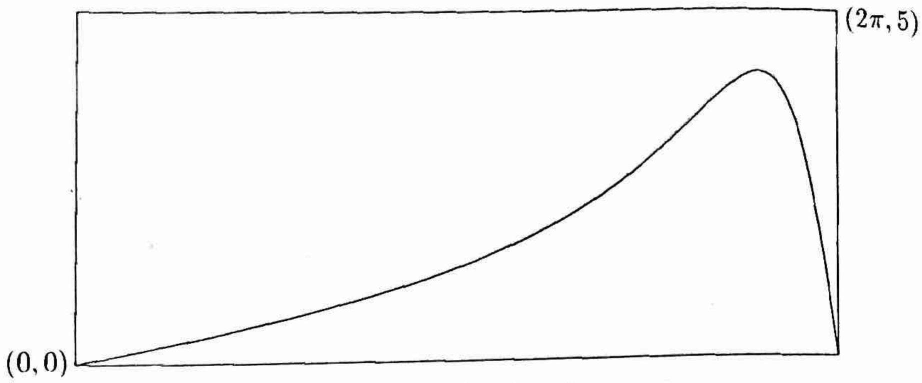

Fig. 6. L_3 as a function of α.

Theorem 4. *The Bézier curve $p_3(t) = (x_3(t),\ y_3(t))$ determined by setting the length L of the vectors $P_1 - P_0$ and $P_3 - P_2$ in (5) equal to*

$$L_3 = \left\{(9 - 2K_3)\sin\alpha - \sqrt{((9 - 2K_3)\sin\alpha)^2 - 6(2K_3 + 3\cos\alpha)(5 - 2K_3)(1 - \cos\alpha)}\right\} \times \{3(2K_3 + 3\cos\alpha)\}^{-1} \tag{25}$$

has an error function $x_3(t)^2 + y_3(t)^2 - 1$ that equioscillates three times.

Proof. The expression for L_3 is the positive root of the quadratic equation

$$b/a = K_3,$$

where a and b are given by (8) and (9). □

Fig. 6 shows L_3 given in Theorem 4 as a function of α. Note that the value of L_1 given in Theorem 1 and the value of L_3 given by (15) are only marginally different, at least for moderate α. Through numerical experiments, we have found that for small α, the range of values of L that yield the high order convergence is very narrow. For example, for $\alpha = \frac{1}{2}\pi$, a perturbation of L in the third digit may increase the error by several orders of magnitude.

7. Numerical results

Table 1 shows the errors when approximating the whole unit circle with $n = 4, 8, 16, 32$ cubic Bézier segments, or equivalently, the error in approximating circle segments of angular

Table 1
Errors

n	E_1	E_2	E_3
4	0.14E−3	0.19E−3	0.14E−2
8	0.21E−5	0.30E−5	0.21E−4
16	0.33E−7	0.47E−7	0.32E−6
32	0.52E−9	0.74E−9	0.49E−8

width $2\pi/n$. The error estimates shown in Table 1 have been computed by sampling the various approximations. The errors shown are the following.

- The column marked E_1 shows the radial error $\max_t |e_2(t)|$ given by (12), in the approximation given in Theorem 2.
- The column marked E_2 shows the radial error when using the Hermite interpolation approach given in Section 6.
- The column marked E_3 shows the radial error obtained by applying the general interpolation method described in [de Boor, Höllig & Sabin '87] to the case of a circle.

These numbers illustrate well the rapid convergence of the methods considered here. The method described in Section 5 stands out as being the most accurate, and together with the method in Section 4 undoubtedly the simplest.

Notes added in proof

(1) After this manuscript was submitted, we became aware of the paper

> Gossling, T.H. (1976), The 'Duct' system of design for practical objects, in: *Proc. World Congress on the Theory of Machines and Mechanisms*, Milan, 305–316.

Here the athor gives the approximation method described in Theorem 1, but only with a rough numerical error estimate.

(2) In this paper we have described some methods for approximating circle segments by parametric polynomials which give very small errors, but we have said nothing about how close we may be to the optimal solution. It has been pointed out to us by Dr. T. Goodman that the approximation given in Theorem 4 in fact minimizes $\max_{t\in[0,1]} |\phi(t)|$ with ϕ given by (7), at least for $\alpha \leqslant \pi$. This means that the error in this method is bounded by the error in the method in Theorem 1, and hence is also order six accurate.

References

de Boor, C., Höllig, K. and Sabin, M. (1987), High accuracy geometric Hermite interpolation, Computer Aided Geometric Design 4, 269–278.

Farin, G.E. (1988), *Curves and Surfaces in Computer Aided Geometric Design*, Academic Press, San Diego.