ALGEBRAIC GROUPS AND REPRESENTATIONS

by

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Abstract. — This is a collection of several talks from the summer school "Algebraic groups and representations" at the Institut Camille Jordan during the summer of 2014. Talks covered several aspects of the general theory of algebraic groups, especially in prime characteristic, as well as some specific examples to their represention theory over finite and local fields.

Contents

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1. Affine groups in positive characteristic

1.1. Introduction. — Everything to be discussed is well-known, in the sense that it can probably be found in one of the texts [**DG80, Mil12**]. An official set of notes for these lectures can be found at `http://math.univ-lyon1.fr/homes-www/gille/prenotes/lyon_pg2014.pdf`.

The classical theory of affine algebraic groups over $\mathbf{C}$ treats such groups as Zariski-closed subgroups of $\mathrm{GL}_n(\mathbf{C})$ for varying n. This theory is closely connected to that of compact Lie groups. For example, the torus $\mathbf{G}_{\mathrm{m}} = \mathrm{GL}_1 = \mathrm{Spec}(\mathbf{C}[t^{\pm 1}])$ corresponds to the one-dimensional real torus S^1. Everything can be treated as varieties over $\mathbf{C}$.

Let k be an arbitrary field, e.g. $\mathbf{F}_q$, $\mathbf{F}_q(t)$, $\mathbf{R}$, $\ldots$. The natural approach to algebraic groups over k is a "schematic" one, i.e. using schemes. For fields of positive characteristics, there are some recent improvements to the theory of algebraic groups over k. On the one hand, there is theory of [**CGP10**] on so-called pseudo-reductive groups, and for commutative groups, there is recent work of Brion and others.

New phenomenon occur even over $k = \overline{\mathbf{F}_p}$. For example, the group $\mathrm{GL}(2)_k$ is not "linearly reductive." That is, its category of representations is *not* semisimple. Also, there are extensions of $\mathbf{G}_{\mathrm{a},k} = \mathrm{Spec}(k[t])$ by itself which are not split (coming from Witt vectors). For example, one could put $(x_0, x_1) + (y_0, y_1) = (x_0 + y_0, x_1 + y_1 - S_1(x_0, y_0))$, where

$$S_1(x, y) = \frac{(x+y)^p - x^p - y^p}{p} \in \mathbf{Z}[x, y].$$

This is the group W_2 of "additive Witt vectors of length two." It fits into an exact sequence $0 \to \mathbf{G}_{\mathrm{a}} \to W_2 \to \mathbf{G}_{\mathrm{a}} \to 0$ which is not split. Indeed, evaluated at $\mathbf{F}_p$, the sequence is $0 \to \mathbf{Z}/p \to \mathbf{Z}/p^2 \to \mathbf{Z}/p \to 0$.

Also, in positive characteristic, the intersection of (smooth) groups is not necessarily smooth! For example, let $G_1 = \{x^p + x = y\}$ and $G_2 = \{x = y\}$ as subgroups of $\mathbf{G}_{\mathrm{a}}^2$. The intersection $G_1 \times_{\mathbf{G}_{\mathrm{a}}^2} G_2 = \{x^p = 0, x = y\}$, which is not even reduced.

Over non-perfect fields of characteristic p (for example function fields or local fields) things can be even worse. The following example over $k = \mathbf{F}_p(t)$ is due to Tits. Let $G = \{x + tx^p + y^p = 0\} \subset (\mathbf{G}_{\mathrm{a},k})^2$. One can check that G is smooth and connected. However, if $p \geqslant 3$, then $G(k) = \{(0,0)\}$. In contrast, over an infinite perfect field F, the set $G(F)$ is Zariski dense in G for all affine connected G. To see that G is smooth, base-change to $k(\sqrt[p]{t}) = \mathbf{F}_p(t')$. In that field,

$$x + tx^p + y^p = x + (t'x)^p + y^p = x + (t'x + y)^p,$$

so $\mathbf{G}_{k(\sqrt[p]{t})} \simeq \mathbf{G}_{\mathrm{a},k(\sqrt[p]{t})}$. Now we show that $G(k) = 1$. If there is $(x, y) \in G(k)$ with $x = 0$, then one easily sees that $y = 0$. Suppose $x(t) = \frac{P(t)}{Q(t)}$. We can assume P and Q are relatively prime. Some simple algebra gives $P'Q - PQ' + P^p Q^{2-P} = 0$, which cannot be.

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1.2. Generalities. — The theory over positive characteristic is already complicated enough to make it no additional effort to work over an arbitrary commutative ring. For example, if k'/k is a purely inseparable extension of the form $k(\sqrt[p]{a})$, then $k' \otimes_k k' = k'[t]/(t - \sqrt[p]{a})^p$ is not even a product of fields. But it is an Artinian local ring. One could restrict to this (as is done in [**SGA 3**, VIa]).

For the rest of this section, fix a (commutative, unital) ring R. An R-*functor* is a covariant functor $F : R\text{-}\mathsf{Alg} \to \mathsf{Set}$. If X is an affine scheme over R, we write $R[X] = \Gamma(X)$; note that $X = \operatorname{Spec}(R[X])$. Let $h_X(S) = X(S) = \hom_{R\text{-}\mathsf{Alg}}(R[X], S)$. We say that an R-functor F is *representable* by an affine scheme X over R if there is an equivalence $h_X \xrightarrow{\sim} F$. The Yoneda Lemma tells us that such an X is unique up to unique isomorphism. Even better,

***1.2.1 Lemma* (Yoneda).** — *The map* $\hom_{R\text{-}\mathsf{Fun}}(h_X, F) \to F(X)$ *that sends* η *to* $\eta_X(1_X)$ *is a bijection.*

In light of this, we will usually define group schemes via their functor of points. It makes sense to speak of "group-valued R-functors," and we can define an *affine group scheme over* R to be an affine scheme X, together with the structure of a group-valued functor on h_X. In other words, for each S, we have a "multiplication" $X(S) \times X(S) \to X(S)$, "inverse" $X(S) \to X(S)$, and "identity" $1 \in X(S)$. These are required to be compatible with maps $S \to S'$ in the obvious way. To give X the structure of a group scheme is equivalent to giving $R[X]$ the structure of a Hopf algebra.

***1.2.2 Example* (constant groups).** — Let Γ be a finite (abstract) group. Let $G = \coprod_{\sigma\in\Gamma} \operatorname{Spec} R$, with multiplication induced by Γ. People sometimes write $G = \Gamma_R$, or $G = \underline{\Gamma}_R$. The same definition works if Γ is not finite, but $\underline{\Gamma}$ will no longer be affine.

***1.2.3 Example* (vector groups).** — For N an R-module, define $\mathbf{V}(N)(S) = \hom_S(N\otimes_R S, S)$. This is clearly a R-group functor, represented by $\operatorname{Spec}(\operatorname{Sym}_R(N^\vee))$.

***1.2.4 Example*.** — If M is an R-module, we put $\mathbf{W}(M)(S) = M \otimes_R S$. This is not always representable. If R is noetherian, $\mathbf{W}(M)$ is representable if and only if M is locally free.

***1.2.5 Example* (linear groups).** — Let A be an (associative) R-algebra. We define $\mathrm{GL}_1(A)(S) = (A \otimes_R S)^\times$. This functor is not always representable. However, if A is finitely-generated, locally free (as an R-module), then $\mathrm{GL}_1(A)$ is representable. Sometimes we will just write $A^\times$ for $\mathrm{GL}_1(A)$ (especially if A is a division ring over a field).

***1.2.6 Example* (diagonalizable groups).** — Let M be an (abstract) commutative group. Let $R[M]$ be the corresponding group ring. This is naturally a Hopf algebra via $\Delta(m) = m \otimes m$. Put $\mathrm{D}_R(M) = \operatorname{Spec}(R[M])$. For example, if $M = \mathbf{Z}$, then

$R[M] = R[t^{\pm 1}]$, so $\mathrm{D}_R(M) = \mathbf{G}_{\mathrm{m},R}$. If $M = \mathbf{Z}/n$, then $R[M] = R[t]/(t^n - 1)$, so $\mathrm{D}_R(M) = \mu_{n,R}$. A homomorphism $f : M_1 \to M_2$ induces $f_* : R[M_1] \to R[M_2]$, hence $f^* : \mathrm{D}_R(M_2) \to \mathrm{D}_R(M_1)$.

1.2.7 Exercise. — The functor $\mathrm{D}_R : \mathsf{Ab}^\circ \to \{\text{diagonalisable groups over } R\}$ is an equivalence of categories. [See SGA 3, exp.8]

If D is a group scheme, put $\widehat{D} = \hom(D, \mathbf{G}_\mathrm{m})$; this is the group of *characters* of D.

1.2.8 Exercise. — Show that $\hom(\mathrm{D}(M), \mathbf{G}_{\mathrm{a},R}) = 0$ for all M.

If $R = k$ is a field, then any closed subgroup $G \subset \mathrm{D}(M)$ is of the form $\mathrm{D}_k(M/M')$ for some $M' \subset M$. In other words, over a field, subgroups (and qotients, in fact) of diagonalisable groups are diagonalisable.

1.3. Basic facts on affine groups over k. — Throughout, let k be a field.

1.3.1 Definition. — An *affine algebraic group over k* is a group scheme G over k, such that G is affine of finite type.

1.3.2 Definition. — An affine algebraic group over k is *linear* if it is also smooth.

It is a non-trivial fact that if G is an affine group over k, then there is a closed embedding $G \hookrightarrow \mathrm{GL}_n$. The following result is from [**GP03**]. Supposing k is infinite and G is smooth, there exists $G \hookrightarrow \mathrm{GL}(V)$ and $f \in V^\vee \otimes_k V^\vee \otimes_k V$, such that $G = \{g \in \mathrm{GL}(V) : g^\vee \circ f = f\}$. Moreover, there exists a (possibly non-associative, non-unital) k-algebra A such that $G = \mathrm{Aut}(A) \subset \mathrm{GL}_k(A)$. (The two statements are the clearly equivalent.)

If G an affine algebraic group over k, then the associated reduced scheme G_{red} is not necessarily a k-group. Essentially, the problem is that $(-)_{\mathrm{red}}$ does not commute with products. If k is perfect, then $G_{\mathrm{red}} \times_k G_{\mathrm{red}}$, and $G_{\mathrm{red}} \subset G$ is a closed k-subgroup of G.

1.3.3 Example. — Let $G = \mu_3 \rtimes \mathbf{Z}/2$ over $\mathbf{F}_3$, the action being via inversion. Then $G_{\mathrm{red}} = \mathbf{Z}/2$ is not a normal subgroup.

1.3.4 Lemma. — *Let G be an affine k-group. Then the following are equivalent:*

1. *G is smooth*
2. *G is geometrically reduced*
3. *$\mathscr{O}_{G,e} \otimes \bar{k}$ is reduced*
4. *G admits a nonempty smooth open $U \subset G$.*

Proof. — The only nontrivial part is 4⇒1g. But one can base-change to $\bar{k}$, and note that $G_{\bar{k}} = \bigcup_{g \in G(\bar{k})} gU_{\bar{k}}$. □

1.3.5 Proposition. — *Let G be an affine k-group. Let $\{f_i : V \to G\}$ be morphisms from geometrically reduced affine k-schemes. Then there is a unique smallest closed k-subgroup, written $\Gamma_G(\{f_i\})$, such that the f_i factor through $\Gamma_G(\{f_i\})$.*

Proof. — First we suppose we have a single $f : H \to G$, where H is a smooth group. Then the schematic image $\mathrm{im}(f) \subset G$ works.

Next, we suppose that G admits a maximal smooth k-subgroup $G^\dagger$. (If k is perfect, then $G^\dagger = G_{\mathrm{red}}$.) □

If $f : G \times G \to G$ is $(g, h) \mapsto [g, h]$, then $\Gamma_G(f) = \mathscr{D}G$, the derived subgroup of G.

1.3.1. Maximal smooth subgroups. — Let k be a field, A a k-algebra. Recall we say that A is *geometrically reduced* (Bourbaki says *separable*) if $A \otimes_k K$ is reduced for all extensions K/k. If A is of finite type, one only needs $A \otimes_k \bar{k}$ to be reduced.

If A is of finite type and geometrically reduced, then for $X = \mathrm{Spec}\, A$,

1. X is generically smooth over k (i.e. there exists an affine dense open smooth subscheme of X over k)
2. $X(k^s)$ is dense in X

Now, as before, if G is an affine algebraic k-group, then it admits a maximal smooth k-subgroup $G^\dagger$. This subgroup enjoys the following properties:

1. If G is connected, so is $G^\dagger$.
2. $G^\dagger$ is the largest geometrically reduced closed k-subscheme of G.
3. $G^\dagger(k) = G(k)$.
4. If k is separably closed, $G^\dagger$ is the schematic closure of $G(k^s)$ in G.
5. If k is perfect, then $G^\dagger = G_{\mathrm{red}}$.
6. If K/k is a separable field extension, then $(G^\dagger) \times_k K \xrightarrow{\sim} (G_K)^\dagger$.

We will discuss 2 and 3 in greater depth. For 2, suppose $i : X \hookrightarrow G$ and X is affine and geometrically reduced. We can attach to this data a smooth group $\Gamma_G(i) \subset G$. But $\Gamma_G(i) \subset G^\dagger$, so $X \subset G^\dagger$.

Regarding 3: the inclusion $G^\dagger(k) \subset G(k)$ is obvious. Let $g \in G(k)$. Then $X = gG^\dagger \subset G$ is also smooth, whence $gG^\dagger \subset G^\dagger$, which yields $g \in G(k)$.

1.3.6 Example. — Here we show that $(-)^\dagger$ does not necessarily commute with inseparable base-change. Let $k = \mathbf{F}_p(t)$, $G = \{x^p + ty^p = 0\} \subset (\mathbf{G}_{\mathrm{a},k})^2$. We claim that $G^\dagger = 0$, but $(G_{k'})^\dagger = \mathbf{G}_{\mathrm{a},k'}$ (hence $(G_{k'})^\dagger \neq 0$) if $k' = k(\sqrt[p]{t})$. First we show that $G^\dagger = 0$. It suffices to show that $G(k^s) = 0$. Let $(x, y) \in G(k^s)$ be nonzero. But $x^p + ty^p = 0$ tells us that t is a p-power in k^s, a contradiction. Over k', the equation defining G is $(x + t'y)^p = 0$, which clearly has maximal smooth subscheme $\{x + t'y = 0\} \simeq \mathbf{G}_{\mathrm{a},k'}$.

1.3.7 Proposition. — *Let $f : H \to G$ be a homomorphism of affine algebraic k-groups. Assume that G is reduced. Then the following are equivalent:*

1. *f is faithfully flat*

2. *f is surjective on $\bar{k}$-points*
3. *f is dominant*

Recall that a ring homomorphism $f : A \to B$ is *flat* (or that B is flat over A) if the tensor-product functor $- \otimes_A B$ is exact. We say that B is *faithfully flat* if it is flat over A, and moreover the map $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

1.3.8 Proposition. — *Let $f : H \to G$ be a homomorphism of algebraic k-groups. Then the following are equivalent:*

1. *f is a closed immersion*
2. *f is an immersion*
3. *f is a monomorphism (i.e. for all $A \in k\text{-}\mathsf{Alg}$, the map $f_* : H(A) \to G(A)$ is injective)*

This fails over rings that are not fields (e.g. the ring $\mathbf{Z}_2$ of 2-adic integers). One might ask whether injectivity on $\bar{k}$-points suffices. The answer is no. Let $F : \mathbf{G}_{\mathrm{a}} \to \mathbf{G}_{\mathrm{a}}$ be Frobenius over a field of characteristic p. It is not an immersion, but is injective on $\bar{k}$-valued points. If we take $\mathbf{F}_p[t]/(t^p)$-valued points, then Frobenius is no longer injective. One puts $\boldsymbol{\alpha}_p = \operatorname{Spec}(\mathbf{F}_p[t]/t^p)$.

1.4. Using group functors. — Throughout, let R be our base ring. We are given three R-group functors G_1, G_2, G_3, and maps $\alpha : G_1 \to G_2$, $\beta : G_2 \to G_3$. We say that the sequence of R-group functors

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

is *exact* if for all $S \in R\text{-}\mathsf{Alg}$, the sequence

$$1 \to G_1(S) \to G_2(S) \to G_3(S) \to 1$$

is exact as a sequence of abstract groups.

Let $f : G \to H$ be a morphism of R-group functors. We define an R-group functor $\ker(f)$ by $\ker(f)(S) = \ker(G(S) \to H(S))$. There is an exact sequence

$$1 \to \ker(f) \to G \to H.$$

If G and H are (representable by) affine algebraic groups, so is $\ker(f)$.

***1.4.1 Example* (Witt vectors)**. — Over $\mathbf{F}_p$, we have seen that there is an exact sequence $0 \to \mathbf{G}_{\mathrm{a}} \to W_2 \to \mathbf{G}_{\mathrm{a}} \to 0$. In fact, this sequence splits on the right.

Note that if $1 \to G_1 \to G_2 \xrightarrow{f} G_3 \to 1$ is an exact sequence of affine algebraic groups, then $f : G_2 \to G_3$ admits a section. To see this, just look at the surjection $G_2(R[G_3]) \twoheadrightarrow G_3(R[G_2])$.

1.4.1. Semidirect products of group schemes. — Suppose G and H are affine R-group schemes, and $\theta : G \to \mathrm{Aut}(H)$ is a morphism of R-group functors. Here, $\mathrm{Aut}(H)$ is defined by

$$\mathrm{Aut}(H)(S) = \mathrm{Aut}_{S\text{-}\mathsf{gp}}(H_S).$$

So for each S, we have a homomorphism $\theta_S : G(S) \to \mathrm{Aut}(H_S)$. We can define the semidirect product (on the scheme $H \times G$) by $(h_1, g_1) \cdot (h_2, g_2) = (h_1\theta(g)h_2, g_1g_2)$ as usual. This gives us a group scheme $H \rtimes^\theta G$ that fits into an exact sequence of R-functors:

$$1 \to H \to H \rtimes^\theta G \to G \to 1.$$

For example, we earlier saw the scheme $\boldsymbol{\mu}_n \rtimes \mathbf{Z}/2$. Also, one can form $\mathbf{G}_\mathrm{a}^n \rtimes \mathrm{GL}_n$.

Let A be a commutative affine R-group scheme, and G an R-group scheme with an action on A via $\theta : G \to \mathrm{Aut}(A)$. We wish to classify R-group extensions of G by A with respect to this action. It turns out that there is a group $\mathrm{H}^2(G, A)$ (second Hochschild cohomology) which does this.

Let's look at the special case where $A = \mathbf{V}(N)$ for N an R-module.

***1.4.2 Theorem* (Grothendieck).** — *If $G = \mathrm{D}_R(M)$ acts on $\mathbf{V}(N)$, then $\mathrm{H}^i_0(G, \mathbf{V}(N)) = 0$ for all $i > 0$. In particular, extensions ofG by $\mathbf{V}(N)$ are split.*

1.4.2. Actions, centralizers, etc.— Let G be an affine R-group scheme, X an affine R-scheme. An *action* of G on X is a homomorphism $\theta : G \to \mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ is the R-group functor defined by

$$\mathrm{Aut}(X)(S) = \mathrm{Aut}_{S\text{-}\mathsf{Sch}}(X_S).$$

In other words, for each S, we have a homomorphism $\theta_S : G(S) \to \mathrm{Aut}_{S\text{-}\mathsf{Sch}}(X_S)$.

Warning: the functor $\mathrm{Aut}(X)$ is very rarely representable. In characteristic zero, $\mathrm{Aut}(\mathbf{G}_\mathrm{a})$ is represented by $\mathbf{G}_\mathrm{m}$, i.e. for all $\mathbf{Q}$-algebras S, the natural map $S^\times \to \mathrm{Aut}_{S\text{-}\mathsf{gp}}(\mathbf{G}_{\mathrm{a},S})$ is an isomorphism. But in characteristic $p > 0$, $\mathrm{Aut}_{S\text{-}\mathsf{gp}}(\mathbf{G}_{a,S})$ consists of non-commutative polynomials $a_0 + a_1t^p + \cdots + a_rt^{p^r}$, with $a_0 \in S^\times$ and the a_i nilpotent.

Let G act on X, and let $X_1, X_2 \subset X$ be closed R-subschemes. We define various functors:

(transporter)

$$\mathrm{Transp}_G(X_1, X_2)(S) = \{g \in G(S) : \theta(g)X_1(S') \subset X_2(S') \text{ for all } S'/S\}$$

(strict transporter)

$$\mathrm{Transpst}_G(X_1, X_2)(S) = \{g \in G(S) : \theta(g) \text{ induces } X_{1,S} \xrightarrow{\sim} X_{2,S}\}$$

(normalizer) $\quad \mathrm{Norm}_G(X_1) = \mathrm{Transp}_G(X_1, X_1)$

(centralizer) $\quad \mathrm{Cent}_G(X_1) = \{g \in G(S) : \theta(g) = 1 \text{ on } X_{1,S}\}.$

***1.4.3 Theorem* (Grothendieck).** — *Let $R = k$ be a field, X an affine scheme, G an affine algebraic k-group acting on X, and $X_1, X_2 \subset X$ closed k-subschemes. Then all the above functors are representable by closed k-subschemes of G.*

***1.4.4 Example*.** — Consider the action of G on itself by inner automorphisms. We see that if $H \subset G$ is a k-subgroup, then $\mathrm{Norm}_G(H)$ and $\mathrm{Cent}_G(H)$ exist as subgroup schemes of G.

The functors we have defined are the *schematic* transporter, normalizer, ... These can be different from the classical objects of the same name.

Let G be a reductive group, $P \subset G$ a parabolic subgroup, and $U \subset P$ its unipotent radical. Classically, one says that $P = N_G(U)$. This is true in the language of reduced varieties.

***1.4.5 Example*.** — Let k be a field of characteristic 2 ($\overline{\mathbf{F}_2}$ for example). Let $G = \mathrm{PGL}(2)_k$, $B \subset G$ the standard Borel, and $U \subset B$ the unipotent radical. We claim that $B \subsetneq N_G(U)$. Indeed, writing $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for projective coordinates of G, we have $B = \{c = 0\}$. But $J = \{c^2 = 0\}$ is an R-subgroup of G strictly containing B, but $J \subset N_G(U)$. Indeed, if $x^2 = 0$, then

$$\left(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1+bx & b \\ & 1+bx \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & b(1+x) \\ & 1 \end{smallmatrix}\right)$$

within $\mathrm{PGL}(2)$.

1.4.3. Weil restriction. — Recall the extension $\mathbf{C}/\mathbf{R}$ given by adding a root of $z^2 = 2 + i$. Writing $(x + iy)^2 = 2 + i$, we get two equations with coefficients in $\mathbf{R}$. This corresponds to Weil restriction.

Let $R \to S$ be a ring extension. Let F be an S-functor. Following Grothendieck (Weil writes $\mathrm{R}_{S/R}$) we define an R-functor $\Pi_{S/R}F$ by

$$(\Pi_{S/R}F)(R') = F(R' \otimes_R S).$$

If F is representable, it is natural ask under what conditions the restriction of scalars $\Pi_{S/R}F$ is representable.

***1.4.6 Theorem*.** — *Assume $R \to S$ is finite and locally free. Let Y be an affine S-scheme. Then $\Pi_{S/R}Y$ is representable by an affine R-scheme. Moreoveer, if Y is of finite presentation, so is $\prod_{S/R} Y$.*

***1.4.7 Example* (vector groups).** — If N is an S-module, then recall $\mathbf{V}(N)(T) = \hom(N \otimes_S T, T)$. Let M be the "scalar restriction" of N to R (i.e. N considered as an R-module). Then $\Pi_{S/R}\mathbf{V}(N) = \mathbf{V}(M)$.

If $S = \overbrace{R \times \cdots \times R}^{d}$, then an S-scheme Y is just a d-tuple of R-schemes $Y_1, \ldots, Y_d$. One has $\Pi_{S/R}Y = Y_1 \times_R \times \cdots \times_R Y_d$.

Weil restriction does not transform open covers into open covers! Consider $G = \mathbf{G}_{\mathrm{a},\mathbf{C}\times\mathbf{C}}$. Then $\Pi_{\mathbf{C}\times\mathbf{C}/\mathbf{C}}G = \mathbf{G}_\mathrm{a} \times \mathbf{G}_\mathrm{a}$. If $U_0 = \{t \neq 0\}$ and $U_1 = \{t \neq 1\}$, then $\{\Pi U_0, \Pi U_1\}$ is *not* an open cover of $\mathbf{G}_\mathrm{a}^2$.

The Weil restriction functor takes affine group schemes to affine group schemes. (That $\Pi_{S/R}$ preserves smoothness follows trivially once we know that formally smooth $\Rightarrow$ smooth.) However, as we have seen, it does *not* preserve surjectivity. For example, let k be a field of characteristic $p > 0$, and let $F : \mathbf{G}_{\mathrm{a},k} \to \mathbf{G}_{\mathrm{a},k}$ be the Frobenius $x \mapsto x^p$. Let k'/k be a purely inseparable extension. We have an induced morphism $\Pi_{k'/k}\mathbf{G}_\mathrm{a} \to \Pi_{k'/k}\mathbf{G}_\mathrm{a}$. Extend scalars once again via $k' \otimes_k k' \supset k'$. The ring $k' \otimes_k k'$ is a local Artinian ring over k. In the commutative diagram

$$\begin{tikzcd}
\text{infinitesimal} \arrow[r] \arrow[d] & \Pi_{k'\otimes k'/k'}\mathbf{G}_\mathrm{a} \arrow[r] \arrow[d] & \mathbf{G}_\mathrm{a} \arrow[d] \\
\text{infinitesimal} \arrow[r] & \Pi_{k'\otimes k'/k'}\mathbf{G}_\mathrm{a} \arrow[r] & \mathbf{G}_\mathrm{a}
\end{tikzcd}$$

The first vertical arrow is not surjective.

1.4.8 Example. — Let $p = 2$, and let $k' = k(\sqrt{a})$ be a purely inseparable extension. Then $k' \otimes_k k' \simeq k'[t]/(t^2)$. Write $A = k[t]/t^2$. Then for X/k affine, $Y = \Pi_{A/k}X_A$ is the tangent bundle of X. [...more that I didn't understand...]

Again, let k'/k be a purely inseparable extension. The functor $\Pi_{k'/k}$ transforms affine covers into affine covers. Moreover, $\Pi_{k'/k}$ extends to a functor on quasi-compact separated k'-schemes. Let $G = \Pi_{k'/k}\mathbf{G}_{\mathrm{a},k'}$; this is an interesting group. If $[k' : k] = p^r$, then G embeds into $\mathrm{GL}_{p^r} = \mathrm{GL}(k')$. This is one of the simplest possible examples of a pseudo-reductive group. It is not a torus – the base-change $G_{k'}$ contains a unipotent part $\mathbf{G}_{a,k'}$.

1.5. Descent and quotients. — Throughout, we will use faithfully flat descent as a kind of "black box" to prove certain results. Vistoli's notes [**Vis05**] are a fantastic reference for this topic.

1.5.1. Embedded descent. — Let k be a field, K/k a field extension. Let X_0 be an affine k-scheme of finite type. Write $X = X_0 \times_k K$. We are given a closed subscheme $Z \hookrightarrow X$. We say that Z *descends* to k if there is a closed subscheme $Z_0 \subset X_0$ such that $Z = Z_0 \times_k K$. If such a Z_0 exists, it is unique. Indeed, to define $Z_0 \subset X_0$, it is equivalent to define the associated ideal $I_{Z_0} \subset k[X_0]$. If Z_0 and Z_0' are both descents of Z to k, then we have I_{Z_0} and $I_{Z_0'}$, ideals in $k[X_0]$ such that $I_{Z_0} \otimes_k K = I_{Z_0'} \otimes_k K$. This implies $I_{Z_0} = I_{Z_0'}$ (special case of faithfully flat descent).

1.5.2. The Galois case. — Suppose K/k is a (possibly infinite) Galois extension. Then Z descends to k if and only if for all $\gamma \in \mathrm{Gal}(K/k)$, we have ${}^\gamma(I_Z) = I_Z$ in $K[X_0] = K \otimes_k k[X_0]$. One direction is easy, and for the other one uses Galois descent (via Speiser's lemma). Define $I_0 = \mathrm{H}^0(k, I_Z)$. Speiser's lemma tells us that $K \otimes_k I_0 = I_Z$.

***1.5.1 Corollary* (Galois case).** — *There exists a minimal "field of definition" of Z.*

Proof. — The field of definition F is by definition the field $F \subset K$ for which $\mathrm{Gal}(K/F) = \{\gamma \in \mathrm{Gal}(K/k) : \gamma(I_Z) = I_Z\}$. □

In [**EGA 4**, 4.8] there is a section called "field of definition." It depends on [**Bou03**, II.8]. Let V_0 be a k-vector space, $V = K \otimes V_0$. Let $W \subset V$ be a K-vector space. Then there is a minimal field of definition for W. The proof is an exercise in linear algebra.

Back to a general field extension $K \supset k$. For $Z \subset X$, we have a field of definition F of the subspace $I_Z \subset K[X_0]$. If $F + k$, then $I_Z = (I_0) \otimes_k K$ for a k-space I_0 which defines Z_0.

If $K^p \subset k \subset K$ (i.e. K is a height $\leqslant 1$ extension of k) there are some tools, due to Cartan and Jacobson, to replace Galois theory.

1.5.3. Flat sheaves. — The notion of "exact sequence of algebraic groups" that we defined earlier is too restrictive to be very useful. The correct context is that of flat sheaves.

An fppf (or flat for short, from "fidèlement plat de présantation finie," or "faithfully flat of finite presentation") cover is a ring map $R \to S$ which is faithfully flat and of finite presentation. In other words, S is the form $R[t_1, \ldots, t_n]/(g_1, \ldots, g_m)$. We say that an R-functor $F : R\text{-}\mathsf{Alg} \to \mathsf{Set}$ is an *fppf sheaf* if

1. $F(S_1 \times S_2) \xrightarrow{\sim} F(S_1) \times F(S_2)$
2. For each flat cover $S \to S'$, the following sequence is exact:

$$F(S) \to F(S') \rightrightarrows F(S' \otimes_S S').$$

   The two maps $F(S') \to F(S' \otimes_S S')$ come from the two maps $S' \to S' \otimes_S S'$. By "exact" we mean that the sequence is an equalizer.

Zariski covers (and étale covers as well) are fppf. Write $\mathrm{Sh}(R_{\mathrm{fppf}})$ for the category of fppf sheaves over R.

***1.5.2 Example*.** — Our groups $\mathbf{V}(N)$ are fppf sheaves. More generally, for any affine R-scheme X, the representable functor h_X is fppf due to the following result.

***1.5.3 Proposition* (Grothendieck).** — *If $R \to S$ is faithfully flat, then the sequence*

$$0 \to R \to S \to S \otimes_R S \to S \otimes_R S \otimes_R S \to \cdots$$

is exact.

Just as with sheaves on topological spaces, there is an operation of "fppf sheafification" for R-functors. It is a functor $R\text{-}\mathsf{Fun} \to \mathrm{Sh}(R_{\mathrm{fppf}})$.

Let $f : F \to F'$ be a morphism of flat fppf sheaves (in groups). We can define $\ker(f)$ "pointwise" as before:

$$\ker(f)(S) = \ker(f_S : F(S) \to F(S')).$$

This is a flat sheaf. However, we also want to define the image $\operatorname{im}(f)$ and coimage $\operatorname{coim}(f)$. These are the sheafifications of $S \mapsto \operatorname{im}(F(S) \to F'(S))$ and $S \mapsto F(S)/\ker(f)(S)$, respectively. Fortunately, it is still the case that the natural map $\operatorname{coim}(f) \to \operatorname{im}(f)$ is an isomorphism. We say that a sequence $1 \to F_1 \to F_2 \to F_3 \to 1$ of fppf sheaves (of groups) is *exact* if

1. for all $R \to S$, the sequence $1 \to F_1(S) \to F_2(S) \to F_3(S)$ is exact
2. for all $R \to S$ and all $\alpha \in F_3(S)$, there is a flat cover S'/S such that $\alpha \in \operatorname{im}(F_2(S') \to F_3(S'))$.

Instead of property 2 we could have required that the natural map $\operatorname{im}(F_2 \to F_3) \to F_3$ be an isomorphism.

1.5.4 Example. — The sequence $1 \to \boldsymbol{\mu}_n \to \mathbf{G}_m \xrightarrow{n} \mathbf{G}_m \to 1$ is not an exact sequence of R-functors for any R. However, it is fppf-exact. This is easily seen via lifting over the fppf cover $S \to S[\sqrt[n]{s}]$ for $s \in S^\times$.

In other words, we have an "equality of fppf sheaves" $\mathbf{G}_m = \mathbf{G}_m/\boldsymbol{\mu}_n$.

Let $H \subset G$ be a subgroup (as closed affine R-schemes). Using fppf sheafification, we can talk about $Q = G/H$. This is, *a priori*, simply an fppf sheaf. Is it representable by an R-scheme? Unfortunately, this is rarely the case. See [**Ray70**] for a counterexample.

1.5.5 Theorem. — *Let k be a field. If $H \subset G$ are affine algebraic groups over k, then the fppf sheaf G/H is representable by a quasi-projective k-scheme.*

Note that $(G/H) \times_k R$ will be an fppf quotient for each R/k.

One annoying feature of fppf sheafification is that it is non-trivial to find the points of a quotient G/H. Fortunately, if $H \subset G$ are over a field k, then for $X = G/H$, we have $X(\bar{k})) = G(\bar{k})/H(\bar{k})$. To see this, use the fact that if $\bar{k} \to S$ is a flat cover, there is a section $S \to \bar{k}$. Moreover, one has

$$X(k) = \{[g] \in G(\bar{k})/H(\bar{k}) : d_{1,*}(g) = d_{2,*}(g) \mod H(\bar{k} \otimes_k \bar{k})\}.$$

If k is perfect, then $X(K) = (G(k^s)/H(k^s))^{G_k}$.

1.5.6 Proposition. — *Let G be an affine algebraic group over k. Suppose G acts on a separated k-scheme X of finite type. For $x \in X(k)$, let $G_x = \operatorname{Cent}_G(x) \subset G$ be the scheme-theoretic centralizer. Then the fppf quotient G/G_x is representable by a quasi-projective k-scheme, and the orbit map $G \to X$, $g \mapsto g \cdot x$, induces an immersion $G/G_x \hookrightarrow X$.*

That this proposition implies the theorem comes from the Chevalley construction. Given $H \subset G$, Chevalley shows that there is a representation $G \to \mathrm{GL}(V)$ such that there is a point $x \in \mathbf{P}(V)$ such that $G_x = H$.

One indication that this approach is "correct" is that it generalizes. There is a generalization to the ring case by Grothendieck-Murre. This is despite the fact that there is no Chevalley theorem over general base rings.

1.5.4. Some properties. — Let $X = G/H$. Then $X \times_X H_X \to G \times_X G$ is an isomorphism. In other words, $G \to G/H$ is an H-torsor. If G is smooth, so is G/H. Finally, $G \to G/H$ is affine. If $H \subset G$ is normal, then G/H has the natural structure of a (affine) group scheme, and the sequence $1 \to H \to G \to G/H \to 1$ is exact (as fppf sheaves).

The converse is not true (i.e. if G/H is affine, it is not necessarily the case that H is normal). For example, let $T \subset \mathrm{GL}(n)$ be a maximal torus. The quotient $\mathrm{GL}(n)/T$ is affine, but T is not normal.

We call a *k-orbit in* $G = X/H$ a locally closed k-subscheme such that $Z(\bar{k})$ is an $H(\bar{k})$-orbit in $G(\bar{k})$.

Errata: 1. We had $f : F_1 \to F_2$ of flat sheaves. The map $f : \mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism a) if f is monic or b) if f is epic.

2. If G is an affine algebraic k-group, $G^\dagger$ was the largest smooth k-subgroup. It is *false* that G connected implies $G^\dagger$ connected.

1.5.5. Frobenius kernels and quotients. — Let k be a field of characteristic $p > 0$. Let $F : k \to k$ be the Frobenius $x \mapsto x^p$. If X is a k-scheme, we can define $X^{(p)} = X \times_k^F k$ – the base change of X to k via $F : k \to k$. One has $X^{(p)}(A) = X(A^{(p)})$, where $A^{(p)}$ has the same ring structure as A, but now k acts via $a \cdot x = a^p x$. We have a morphism $X \to X^{(p)}$ coming from the ring map $A \to A^{(p)}$. If G is a k-group, then $G \to G^{(p)}$ is a group homomorphism. Write ${}_F G = \ker(G \to G^{(p)})$, and similarly for powers of F. There is an exact sequence

$$1 \to {}_{F^n} G \to G \to G/{}_{F^n} G \to 1.$$

It turns out that for $n \gg 0$, th quotient $G/{}_{F_n} G$ is smooth.

1.6. Unipotent radicals and Levi subgroups.

— Let k be a field, k^s (resp. $\bar{k}$) a choice of separable (resp. algebraic) closure.

An affine algebraic k-group U is *unipotent* if $U_{\bar{k}}$ admits a composition series over $\bar{k}$ for which each successive quotient is a subgroup of $\mathbf{G}_{\mathrm{a},\bar{k}}$. It is not hard to show that if K/k is a field extension, then G is unipotent if and only if G_K is unipotent. In characteristic zero, then $\mathbf{G}_{\mathrm{a},\bar{k}}$ has only itself and 0 as $\bar{k}$-subgroups. However, in characteristic p, each of the groups $\mathbf{Z}/p^n = \ker(x \mapsto x^{p^n} - x)$ and $\boldsymbol{\alpha}_{p^n} = {}_{F^n}\mathbf{G}_{\mathrm{a}}$ are subgroups of $\mathbf{G}_{\mathrm{a}}$.

In the definition of a unipotent group, we could have equivalently required that over $\bar{k}$, each successive quotient in the composition series to be one of $\mathbf{G}_{\mathrm{a},\bar{k}}$, $\mathbf{Z}/p$, or $\boldsymbol{\alpha}_p$.

1.6.1 Lemma. — *1. An extension of a unipotent group is unipotent.*

2. Subgroups and quotients of unipotent groups are unipotent.

3. If U is unipotent, then $\hom_{k\text{-gp}}(U, \mathbf{G}_{\mathrm{m}}) = 0$.

If k has characteristic p, then commutative unipotent k-groups are actually quite interestin (e.g. Witt vectors of finite length). Classifying these groups is essentially impossible.

1.6.2 Proposition. — *Let G be an affine algebraic k-group. Then the following are equivalent:*

1. *G is unipotent*
2. *G can be embedded into the group of strictly upper-triangular matrices in some* $\mathrm{GL}(n)$
3. *for each representation $G \to \mathrm{GL}(V)$,* $\mathrm{H}^0(G, V) \neq 0$

Moreover, if G is smoooth over k, then the following are equivalent:

1. *G is unipotent*
2. *G admits a central composition series with successive quotients k-forms of* $\mathbf{G}_{\mathrm{a}}$.

In general, we say that G_0 is a *k-form* of G if $G_0 \times_k \bar{k} \simeq G \times_k \bar{k}$. If k is perfect, we know that the only k-frms of $\mathbf{G}_{\mathrm{a}}^n$ are trivial.

Let $T \subset B \subset G$ be the "standard setup" with a reductive k-group. Then not every unipotent subgroup of G can be conjugated into the unipotent radical $U \subset B$.

1.6.3 Example. — Let k be a field of characteristic 2. Let $k' = k(\sqrt{a})$ be a nontrivial extension. Let $u = \left(\begin{smallmatrix} & a \\ 1 & \end{smallmatrix}\right) \in \mathrm{PGL}_2(k)$. Then u generates a subgroup $\mathbf{Z}/2 \hookrightarrow \mathrm{PGL}(2)$. But this does not fit into any Borel subgroup.

1.6.4 Corollary. — *If k is perfect and G is smooth connected unipotent, then G admits a central decomposition series such that $G_{1+i}/G_i \simeq \mathbf{G}_{\mathrm{a}}^{m_i}$.*

We say that G is a *k-split unipotent group* if it has a filtration such that $G_{i+1}/G_i \simeq \mathbf{G}_{\mathrm{a}}$. An extension of k-split unipotent groups is k-split. Quotients (but not kernels) of k-split unipotent groups are k-split.

1.6.1. Commutative affine k-groups. — We say that an affine algebraic k-group G is of *multipliative type* if $G \times_k \bar{k} \simeq \mathrm{D}_{\bar{k}}(M)$ for M some abelian group. We say that G is a *torus* if it is multiplicative type, and also connected.

1.6.5 Theorem. — *Let G be a commutative affine algebraic k-group.*

1. *G is of multiplicative type if and only if* $\hom_{k\text{-gp}}(G, \mathbf{G}_{\mathrm{a}}) = 0$
2. *G admits a largest k-subgroup of multiplicative type G^{m}, and the quotient G/G^{m} is unipotent,*

So every commutative k-group G fits into a canonical exact sequence

$$0 \to G^{\mathrm{m}} \to G \to G/G^{\mathrm{m}} \to 0,$$

i.e. it is the extension of a unipotent group by a multiplicative group. Unfortunately, the subgroup $G^{\mathrm{m}} \subset G$ is not characteristic. If k is split, the above sequence splits isn a canonical way.

We can deduce 2 from 1 in the above theorem by setting $G^{\mathrm{m}} = \bigcap \ker(G \to$ unipotent).

1.6.6 Exercise. — Let $K = k(\sqrt[p]{a})$ be a purely inseparable exxtension. Let $G = \Pi_{K/k}\mathbf{G}_{\mathrm{m}}$. Show that $G^{\mathrm{m}} = \mathbf{G}_{\mathrm{m}} \hookrightarrow \Pi_{K/k}\mathbf{G}_{\mathrm{m}}$. However, show that $G \to G/G^{\mathrm{m}}$ does not split.

To see this, assume $1 \to G^{\mathrm{m}} \to G \to U = G/G^{\mathrm{m}} \to 1$ splits. Then

$$\hom_{k\text{-}\mathsf{gp}}(U, \Pi_{K/k}\mathbf{G}_{\mathrm{m}}) \simeq \hom_{K\text{-}\mathsf{gp}} U_K, \mathbf{G}_{\mathrm{m},K}) = 0.$$

***1.6.7 Lemma* (unipotent radical)**. — *Let G be a smooth affine connected k-group. Then*

1. *G admits a largest smooth connected unipotent normal k-subgroup*
2. *G admits a largest smooth connected unipotent normal k-split subgroup*

Denote by $\mathscr{R}_{\mathrm{u}}(G)$ and $\mathscr{R}_{\mathrm{u},s}(G)$ the groups given by the Lemma. Clearly $\mathscr{R}_{\mathrm{u},s}G \subset \mathscr{R}_{\mathrm{u}}G$. One calls $\mathscr{R}_{\mathrm{u}}G$ the *unipotent radical* of G and $\mathscr{R}_{\mathrm{u},s}G$ the *k-split unipotent radical* of G.

Proof. — We prove only part 1. Let U_1, U_2 be two maximal connected unipotent normal k-subgroups of G. Consider the sequence $1 \to U_1 \to G \to G/U_1 \to 1$. We claim that G/U_1 has no normal unipotent subgroup. For if $U_0 \subset G/U_1$ was such a subgroup, then its preimage in G would be a unipotent normal k-subgroup strictly containing U_1. This cannot be, so $\mathscr{R}_{\mathrm{u}}(G/U_1) = 1$. The group $U_1/(U_1 \cap U_2) \subset G/U_1$ is smooth (being the quotient of a smooth group) and must be trivial. This tells us that $U_1 = U_2 = U$. □

A corollary of our proof is that $\mathscr{R}_{\mathrm{u}}(G/\mathscr{R}_{\mathrm{u}}G) = 1$.

1.6.8 Lemma. — *The formation of $\mathscr{R}_{\mathrm{u}}G$ and $\mathscr{R}_{\mathrm{u},s}G$ commute with separable field extensions. In particular, the "geometric unipotent radical" $\mathscr{R}_{\mathrm{u}}(G_{\bar{k}})$ is defined over $k^{p^{-\infty}}$.*

1.6.9 Example. — Let $K = k(\sqrt[p]{a})$ be a purely inseparable field extension. Let $G = \Pi_{K/k}\mathbf{G}_{\mathrm{m}}$. Then the field of definition of $\mathscr{R}_{\mathrm{u}}G_{\bar{k}}$ is K, i.e. $\mathscr{R}_{\mathrm{u}}(G_{\bar{k}})$ is not defined over k.

Let G be a smooth connected affine k-group. We say that G is *reductive* if $\mathscr{R}_{\mathrm{u}}(G_{\bar{k}}) = 1$. We say that G is *pseudo-reductive* if $\mathscr{R}_{\mathrm{u}}G = 1$.

Clearly reductive groups are pseudo-reductive. The converse holds over perfect fields, but fails in general, as is witnessed by the above example. If K/k is an arbitrary finite field extension and G is a reductive K-group, then the Weil restriction $\Pi_{K/k}G$ is pseudo-reductive (but not necessarily reductive).

1.6.2. Levi subgroups. — Let G be a smooth connected affine k-group. We say that a closed k-subgroup $L \subset G$ is a *Levi subgroup* if the composite $L_{\bar{k}} \to G_{\bar{k}} \to G_{\bar{k}}/\mathscr{R}_{\mathrm{u}}(G_{\bar{k}})$ is an isomorphism. Equivalently, $\mathscr{R}_{\mathrm{u}}(G_{\bar{k}}) \rtimes L \xrightarrow{\sim} G_{\bar{k}}$.

***1.6.10 Theorem* (Mostow).** — *Suppose k has characteristic zero. Then Levi subgroups exist and are $(\mathscr{R}_{\mathrm{u}}G)(k)$-conjugate.*

The following theorem has a recent proof in [**Dem**] which is very nice.

***1.6.11 Theorem* (SGA 3).** — *An extension $1 \to U \to E \to T \to 1$ with U smooth connected unipotent and T a k-torus is split.*

In other words, $\mathrm{H}^1(T, U) = 0$ for such U and T. We are interested in the special case where $G = G_0 \times_k K$ for a field extension K/k. One has $G_0 \hookrightarrow \Pi_{K/k}G_{0,K}$.

***1.6.12 Exercise*.** — If G is reductive over k, then $G_0 \hookrightarrow \Pi_{K/k}G_{0,K}$ is a Levi subgroup.

Existence of Levi subgroups (in the general case) is a hard questions. Recent papers of McNinch. It is possible for Levi subgroups to exist after a separable extension, but still not be defined over the base field. Counterexamples over algebraically closed fields are constructed using an analogue of Weil restriction called the *Greenberg functor*. There is G/k such that $G(k) = \mathrm{SL}_2(W_2(k))$. The exact sequence $0 \to k \to W_2(k) \to k$ gives us an exact sequence

$$1 \to \mathscr{R}_{\mathrm{u}}G \to G \to \mathrm{SL}_2 \to 1,$$

but this sequence has no section on the right. For example, with $k = \mathbf{F}_p$, the map $\mathrm{SL}_2(\mathbf{Z}/p^2) \to \mathrm{SL}_2(\mathbf{F}_p)$ has no section.

1.6.3. Structure theorems. — There are structure theorems (due to Tits) for smooth connected unipotent k-groups. We have seen the "split" ("déployé" in French) case with each subquotient $\mathbf{G}_{\mathrm{a}}$. The "ployé" case is when $\hom(\mathbf{G}_{\mathrm{a}}, G) = 1$. This notion is insensitive to separable field extensions. [k-wand here (sp?)]

***1.6.13 Theorem* (Tits).** — *Let G be a smooth unipotent connected k-group. Then*

1. *G admits a largest normal k-wand unipotent subgroup G_w and G/G_w is unipotent k-split*
2. *if G is k-wand, then G admits a composition series $G_0 = 1 \subset G_1 \subset \cdots \subset G_n = G$ such that G_{i+1}/G_i is k-wand, and "k-fou" of $\mathbf{G}_{\mathrm{a}}^{n_i}$.*

2. Possibly non-affine algebraic groups

Our topic is the structure of algebraic groups (possibly non-affine) over a field (possibly non-perfect).

2.1. Introduction. —

***2.1.1 Theorem A* (Chevalley, Barsotti, Rosenlicht)**. — *Let G be a connected algebraic group over a perfect field. Then there exists a largest connected linear subgroup $L \subset G$. Moreover, L is normal in G and G/L is an abelian variety.*

In other words, every algebraic group is the extension of an abelian variety by an affine algebraic group.

***2.1.2 Theorem B* (Rosenlicht, Demazure-Gabriel)**. — *Let G be a group scheme of finite type over a field k. Then G has a largest linear quotient L', i.e. a smallest normal subgroup scheme H such that G/H is linear. Moreover, H is smooth, connected, commutative, and $\mathscr{O}(H) = \Gamma(H, \mathscr{O}_H) = k$.*

These theorems are "dual" in some informal sense. Note that the assumptions of Theorem A are much weaker. The group H in Theorem B is an abelian variety when k is perfect. A good reference is [**Mil13**].

2.1.3 Definition. — A *k-group scheme* is a scheme G over k, equipped with morphisms $m : G \times G \to G$ (multiplication), $i : G \to G$ (inverse) and $e \in G(k)$ (identity) such that for all k-schemes S, these give $G(S)$ the structure of a group. An *algebraic group* is a smooth group scheme of finite type.

Warning: this definition is less restrictive than Gille's.

If $k = \bar{k}$, then this is the same notion as in [**Bor91**] or [**Spr09**]. They identify G with the set $G(k)$ of k-points. Every group scheme is separated, i.e. the diagonal $\Delta_G \subset G \times G$ is closed.

Let K/k be a field extension, G a k-group scheme. Then $G_K = G \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$ is the base-extension of G to K. The functor $(-)_K$ commutes with products, so G_K is a K-group scheme.

Our main examples of algebraic groups are the *additive group* $\mathbf{G}_{\mathrm{a}}$, the *multiplicative group* $\mathbf{G}_{\mathrm{m}}$, and the *general linear group* GL_n. These are easily defined via their functors of points. A *linear group scheme* is a (closed) subgroup scheme of some GL_n. It turns out that for k-groups, linear $\Leftrightarrow$ affine of finite type. We can regard GL_n as a closed subgroup of SL_{n+1} in the usual way.

Recall that closed subgroup schemes of $\mathbf{G}_{\mathrm{a}}$ correspond to (certain) ideals $I \subset \mathscr{O}(\mathbf{G}_{\mathrm{a}}) = k[t]$. Such ideals will be generated by a polynomial $f \in k[t]$. Such a polynomial needs to satisfy $f(x+y) \in \langle f(x), f(y)\rangle \subset k[x,y]$, $f(0) = 0$, and $f(-t) \in \langle f \rangle$.

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Some elementary manipulations show that if k has characteristic zero, such an f must be of the form at for $a \in k^\times$. In characteristic $p > 0$, such an f can be any of the polynomials $\sum_{i \geqslant 1} a_i t^{p^i}$. These are called *p-polynomials.* For a p-polynomial f, the subgroup scheme $V(I) \subset \mathbf{G}_{\mathrm{a}}$ will be nonreduced. For each $n \geqslant 1$, we get a scheme $\boldsymbol{\mu}_n \subset \mathbf{G}_{\mathrm{m}}$ defined by

$$\boldsymbol{\mu}_n(A) = \{a \in A^\times : a^n = 1\}.$$

Our final example of group schemes are *elliptic curves.* For us, an elliptic curve over k is a smooth projective curve E of genus 1 with a distinguished point $0 \in E(k)$. Using Riemann-Roch, such a curve embeds into $\mathbf{P}^2_k$. If $6 \in k^\times$, then E has affine form $y^2 = x^3 + ax + b$, the standard equation for an elliptic curve. One defines a group law on E in the usual way.

For a group scheme G, locally of finite type over k, the following conditions are equivalent:

1. G is smooth
2. G is geometrically reduced
3. $\mathscr{O}_{G,e} \otimes_k \bar{k}$ is reduced

If k has characteristic zero, then *all* k-groups locally of finite type are smooth. In general, we have a largest reduced subscheme $G_{\mathrm{red}} \subset G$; this is a group scheme if k is perfect.

2.2. Homogeneous spaces and quotients. — The reference here is [**SGA 3**, $\mathrm{VI_A}$ §3]. Let G be a k-group scheme, locally of finite type, and $H \subset G$ a subgroup scheme. Then there is a morphism of schemes $q : G \to G/H$ such that q is H-invariant (i.e. if $m, p : G \times H \to G$ are the multiplication and projection morphisms, we have $qm = qp$) and any H-invariant morphism $G \to X$ factors uniquely through q. Moreover, q is faithfully flat, and the diagram

$$\begin{tikzcd} G \times H \arrow[r, "m"] \arrow[d, "p"] & G \arrow[d, "q"] \\ G \arrow[r, "q"] & G/H \end{tikzcd}$$

is cartesian. It follows that $q : G \to G/H$ is an H-torsor (for the fppf topology). The map q is locally of finite type.

If $H \subset G$ is normal, then the quotient G/H has the structure of a group scheme, uniquely characterized by the property that $q : G \to G/H$ is a morphism of group schemes.

2.2.1 Example. — *Let $G = \mathbf{G}_{\mathrm{m}}$ and $H = \boldsymbol{\mu}_n$. Then $G/H = \mathbf{G}_{\mathrm{m}}$ via the n-th power map $x \mapsto x^n$. But the map $G(k) \to (G/H)(k)$ is usually not surjective. But the map $G(\bar{k}) \to (G/H)(\bar{k})$ is surjective.*

If we worked over an arbitrary base ring k, we would need to pass to fppf extensions in order for $G \to G/H$ to be pointwise surjective. In other words, for $s \in (G/H)(A)$, there is an fppf cover $A \to B$ and $\widetilde{s} \in G(B)$ such that $\widetilde{s}|_A = s$.

If G is smooth, so is G/H.

if G and H are affine, $\mathscr{O}(G/H) = \mathscr{O}(G)^H$. (Here $\mathscr{O}(G)^H$ consists of H-invariant morphisms $G \to \mathbf{A}^1$.)

2.3. Lie algebras. — Let G be a k-group scheme. Let $k[\tau] = k[t]/(t^2)$. Consider the base change $G_{k[\tau]}$ as a k-group scheme. The maps $k \rightleftarrows k[\tau]$ induce morphisms $G_{k[\tau]} \leftrightarrows G$. It turns out that $G_{k[\tau]} = G \ltimes \ker(\pi)$, where $\pi : G_{k[\tau]} \to G$ is the canonical projection. Now $\pi : G_{k[\tau]} \to G$ is the "tangent bundle" of G, so its fiber $\pi^{-1}(g)$ for $g \in G(k)$ is the tangent space $T_g G$. In particular, $\pi^{-1}(0) = \operatorname{Lie} G$. Since $\operatorname{Lie} G$ lives in the semidirect product $G_{k[\tau]} = G \ltimes \ker(\pi)$, it has an action of G by conjugation. If we put $\mathfrak{g} = \operatorname{Lie} G$, this gives a homomorphism $\operatorname{ad} : G \to \operatorname{GL}(\mathfrak{g})$. Differentiating this gives a representation (also called the adjoint representation) $\operatorname{ad} : \mathfrak{g} \to \operatorname{Lie}(\operatorname{GL} \mathfrak{g}) = \operatorname{End}(\mathfrak{g})$. Define $[x, y] = \operatorname{ad}(x) y$; this is the usual bracket on $\mathfrak{g}$.

2.4. Neutral component. — Let G be a k-group scheme, locally of finite type. Let G° be the connected component of e in G. Then G° is a normal subgroup scheme of G, and the quotient G/G° is étale (smooth of dimension 0). The group G° is of finite type, and is geometrically irreducible.

***2.4.1 Example* (automorphism group schemes).** — Let X be a k-scheme. Define a functor Aut_X on k-schemes by $\operatorname{Aut}_X(S) = \operatorname{Aut}_{S\text{-}\mathsf{sch}}(X_S)$. Grothendieck proved that if X is proper, then Aut_X is represented by a k-group scheme, locally of finite type. One has $\operatorname{Lie}(\operatorname{Aut}_X) = \operatorname{Der}_k(\mathscr{O}_X) = \mathrm{H}^0(X, T_X)$. Suppose X is projective. Then to $f \in \operatorname{Aut}_X(S)$, we can associate its graph $\Gamma_f \subset (X \times X)_S$. So the automorphisms of X are parameterized by an open subscheme of the Hilbert scheme of X. The Hilbert scheme is projective, so Aut_X is quasi-projective.

When X is proper, then there is an alternative construction of Matsumura-Oort using a general representability criterion. For the simplest types of projective varieties (projective space) everything can be written down. One has $\operatorname{Aut}_{\mathbf{P}(V)} = \operatorname{GL}(V)$. If X is a smooth projective curve of genus 0, then we can write $X \simeq V(q) \subset \mathbf{P}^2$ for a quadratic form q. One has $\operatorname{Aut}_X = \operatorname{PO}(q)$; this is a k-form of PGL_2.

For X a nice curve of genus 1, then $\operatorname{Aut}_X^\circ$ is an elliptic curve, and the component group $\operatorname{Aut}_X / \operatorname{Aut}_X^\circ$ is finite. If X is a nice curve of genus $g \geqslant 2$, then Aut_X is finite.

For smooth projective surfaces X, the quotient group $\operatorname{Aut}_X / \operatorname{Aut}_X^\circ$ may be infinite. For example, if $X = E \times E$ for E an elliptic curve, then $\operatorname{Aut}_X^\circ = E \times E$. But $\pi_0(\operatorname{Aut}_X) \supset \operatorname{GL}_2(\mathbf{Z})$.

[Here I write $\pi_0 G$ for the quotient G/G°.]

It is currently an open problem whether $\pi_0(\mathrm{Aut}_X)$ is finitely generated for X a smooth projective variety.

Even if X is a smooth projective surface, the group Aut_X may not be reduced. A good example is an Igusa surface in characteristic 2. Let E, F be elliptic curves. Let $X = (E \times F)/\sigma$, where σ is the involution defined by $\sigma(x, y) = (-x, y + y_0)$, where $y_0 \in F(k)$ is chosen to have order 2. In characteristic 2, T_X is trivial! Now $\mathrm{Lie}(\mathrm{Aut}_X) = k^2$. Let $\pi : E \times F \to X$ be the quotient map. This is étale, so $\pi^* T_X = (T_{E\times F})^\sigma = (\mathscr{O}^2_{E\times F})^\sigma = \mathscr{O}^2_{E\times F}$, in characteristic 2.

An open question: which group schemes occur as Aut°_X for X proper, normal? Over perfect fields, all connected algebraic groups occur.

2.4.2 Example. — Let E be an elliptic curve with origin $0 \in E(k)$. Let $L \to E$ be a line bundle. Then $L \smallsetminus \{\text{zero section}\}$ is a $\mathbf{G}_{\mathrm{m}}$-bundle on E. The map $\pi : L \smallsetminus 0 \to E$ is in fact a (Zariski-locally trivial) $\mathbf{G}_{\mathrm{m}}$-torsor. This induces a class $[L] \in \mathrm{H}^1(E, \mathbf{G}_{\mathrm{m}})$. We claim that if $\deg(L) = 0$, then $G = L \smallsetminus 0$ has the unique structure of a commutative group such that $1 \to \mathbf{G}_{\mathrm{m}} \to G \xrightarrow{\pi} E \to 0$ is exact. This result is due to Serre. This is an example of the sequence produced by Theorem A. We would like to see how this fits into Theorem B. To do this, we need to compute $\mathscr{O}(G)$. We compute

$$\begin{aligned}\mathscr{O}(G) &= \mathrm{H}^0(G, \mathscr{O}_G)\\ &= \mathrm{H}^0(E, \pi_*\mathscr{O}_G)\\ &= \mathrm{H}^0\left(E, \bigoplus_{n\in\mathbf{Z}} L^n\right)\\ &= \bigoplus_{n\in\mathbf{Z}} \mathrm{H}^0(E, L^n).\end{aligned}$$

As an exercise, show that if $M \to E$ is a line bundle of degree 0, then $\mathrm{H}^0(E, M) \neq 0$ if and only if $M \simeq \mathscr{O}_E$. If L has infinite order in $\mathrm{Pic}\, E$, then all $L^n \not\simeq M$, so $\mathscr{O}(G) = k$. If L has order m, then $\mathscr{O}(G) =\simeq k[t^{\pm 1}]$, where $\deg t = m$. We have obtained a morphism $G \to \mathbf{G}_{\mathrm{m}}$. We claim that this fits into an exact sequence

$$1 \to F \to G \to \mathbf{G}_{\mathrm{m}} \to 1,$$

where F is an elliptic curve. The map $\pi : G \to E$ induces an isogeny $F \twoheadrightarrow E$ of degree m.

2.5. Proof of “Theorem B”. —

2.5.1 Lemma. — *Let G be a group scheme of finite type over k. Then the following are equivalent:*

1. *G is linear*
2. *G is affine*
3. *$G_{\bar{k}}$ is affine*

4. $(G_{\bar{k}})_{\mathrm{red}}$ *is affine*

Proof. — Clearly $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$. Moreover, $2 \Rightarrow 1$ exactly as for algebraic groups (as in Borel's book). The non-trivial part is $4 \Rightarrow 3$, which holds for any scheme of finite type (also noetherian). This is an exercise in Hartshorne. The implication $3 \Rightarrow 2$ is a type of descent. Recall that a scheme X is affine if and only if the functor $\mathrm{H}^0 : \mathsf{qc}(X) \to \mathsf{Vect}_k$ is exact. But $\mathrm{H}^0(G_{\bar{k}}, \mathscr{F}) = \mathrm{H}^0(G, \mathscr{F})_{\bar{k}}$ (just use the Čech complex). This is exact, so we're done. □

2.5.2 Lemma. — *Let G be a linear group scheme, $\pi : X \to Y$ a G-torsor. Then the morphism π is affine.*

Proof. — We know that π is faithfully flat, G acts on X, π is G-invariant, and the diagram

$$\begin{tikzcd} G \times X \arrow[r, "m"] \arrow[d, "p"] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

is Cartesian (this is what it means for X to be a G-torsor over Y). Let $V \subset Y$ be affine open, $U = \pi^{-1}(V)$. We need to prove that U is also affine. We do this by proving that $\mathrm{H}^0(U, -)$ is exact. But this is $\mathrm{H}^0(V, \pi_* -)$. But V is affine, so $\mathrm{H}^0(V, -)$ is exact. All that remains is to prove that $(\pi|_U)_*$ is exact. We have a cartesian diagram

$$\begin{tikzcd} G \times U \arrow[r, "m"] \arrow[d, "p"'] & U \arrow[d, "\pi|_U"] \\ U \arrow[r, "\pi|_U"] & V. \end{tikzcd}$$

One has $\pi_U^* \pi_{U*} \simeq p_* m^*$. Since π_U^* is exact, it suffices to show that $\pi_U^* \pi_{U*} = p_* m^*$ is exact. But m^* is exact by faithful flatness, and p_* is exact because p is affine. □

2.5.3 Proposition. — *Let G be a group scheme of finite type, $H \subset G$ a normal subgroup scheme. Then G is affine if and only if H and G/H are affine.*

Proof. — We may assume $k = \bar{k}$.

$\Rightarrow$. Clearly closed subgroups $H \subset G$ are affine. The fact that G/H is affine is harder – see [**Bor91**] for a proof that $G_{\mathrm{red}}/H_{\mathrm{red}}$ is affine. But we need to show that $(G/H)_{\mathrm{red}}$ is affine. There is a map $G_{\mathrm{red}}/H_{\mathrm{red}} \to (G/H)_{\mathrm{red}}$, which is a group homomorphism. It is a bijection on $\bar{k}$-points. Let K be its (scheme-theoretic) kernel. The scheme K is of finite type, and has a unique k-point. This implies that K is a finite group scheme. In [**Mum08**] for a proof of the fact that if X is an affine group scheme, and K is a finite group scheme acting on X then X/K is affine. □

Equivalently, affine groups are stable under extensions.

2.5.1. Affinization. — Let X be a k-scheme. Then there is a universal morphism from X to an affine scheme – namely $\varphi_X : X \to \operatorname{Spec} \mathscr{O}(X)$. One has

$$\hom_{k\text{-}\mathsf{sch}}(X, \operatorname{Spec} A) = \hom_{k\text{-}\mathsf{alg}}(A, \mathscr{O}(X)).$$

Warning: if X is of finite type, then $\mathscr{O}(X)$ may not be!

2.5.4 Example. — *Let $L \to E$ be a line bundle of infinite order as before. Let $M \to E$ be a line bundle of negative degree. Consider the vector bundle $X = L \oplus M$ over E. One has*

$$\mathscr{O}(X) = \bigoplus_{m,n \geqslant 0} L^{-m} \otimes M^{-n}.$$

[. . . stuff I didn't understand. . .]

Note that φ_X commutes with products (since $\mathscr{O}(X \times Y) = \mathscr{O}(X) \otimes_k \mathscr{O}(Y)$). Thus, if G is a group scheme, then $\operatorname{Spec} \mathscr{O}(G)$ is an affine group scheme, and φ_G is a homomorphism.

2.5.5 Theorem. — *If G is of finite type, then $H = \ker(\varphi)$ is the smallest normal subgroup scheme of G such that G/H is affine. Moreover, $\mathscr{O}(H) = k$ and $\mathscr{O}(G) = \mathscr{O}(G/H)$. In particular, $G/H = \operatorname{Spec} \mathscr{O}(G$ is finitely generated.*

Proof. — Let $H_1 \subset G$ be normal such that G/H_1 is affine. Then $G \to G/H_1$ factors through φ_X as in

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[dr, "q"'] & \operatorname{Spec} \mathscr{O}(G) \arrow[d, dotted] \\ & G/H_1. \end{tikzcd}$$

Then $H = \varphi^{-1}(1) \subset H_1 = q^{-1}(1)$. It remains to show that G/H is affine. Write $\mathscr{O}(G) = \bigcup V_i$, where each V_i is a G-stable finite-dimensional subspace of $\mathscr{O}(G)$. Then H is the kernel of the right G-action on $\mathscr{O}(G)$, i.e.

$$H = \bigcap_i \ker(G \to \mathrm{GL}(V_i)).$$

By noetherianness, $H = \ker(G \to \mathrm{GL}(V_{i_0}))$ for some i_0, so snce $G/H \to \mathrm{GL}(V_{i_0})$ has trivial kernel, it is a closed immersion by a theorem proved by Gille.

[. . . didn't follow whole proof. . .] $\square$

***2.5.6 Theorem* (Demazure-Gabriel).** — *Let G be a k-group scheme of finite type such that $\mathscr{O}(G) = k$. Then G is smooth, connected, and every homomorphism $G \to H$ (the latter a connected group scheme) factors through the center of H.*

Proof. — Since $\mathscr{O}(G_{\bar{k}}) = \mathscr{O}(G)_{\bar{k}}$, we may assume $k = \bar{k}$. Note that $\pi_0 G = G/G^\circ$ is a finite group, and $\mathscr{O}(G/G^\circ) \hookrightarrow \mathscr{O}(G) = k$, so $G = G^\circ$. Also, G/G_{red} is finite, and $\mathscr{O}(G/G_{\mathrm{red}}) = k$, so G is reduced (hence smooth). We use a rigidity lemma:

let $\mathscr{O}(X) = k$, Y be irreducible, and if $f : X \times Y \to Z$ is a morphism of schemes of finite type such that $f(X \times y_0) = z_0$ for some $y_0 \in Y(k)$ and $z_0 \in Z(k)$, then $f(x, y) = f(x_0, y)$ for all $(x, y) \in X \times Y$. In the literature, one often replaces "$\mathscr{O}(X) = k$" with "X proper."

In our case, define $G \times H \to H$ by $(g, h) \mapsto f(g)hf(g)^{-1}h^{-1}$. Note that this map is identically $= 1$ if and only if our Theorem holds. But $G \times 1_H \mapsto 1_H$, so our rigidity lemma gives the result. □

In particular, G itself is commutative.

Proof of rigidity lemma. — We use infinitesimal neighborhoods. The point $y_0 \in Y(k)$ can be written as $y_0 = \operatorname{Spec}(\mathscr{O}_{Y,y_0}/\mathfrak{m})$. For each n, define $y_n = \operatorname{Spec}(\mathscr{O}_{Y,y_0}/\mathfrak{m}^{n+1})$. Then $\{y_n : n \geqslant 0\}$ is an increasing sequence of closed subschemes of Y, and the union $\bigcup y_n$ is dense in Y. (This is a reformulation of Krull's intersection theorem, which says that $\bigcap \mathfrak{m}^n = 0$.) We claim that $f : X \times y_n \to z_n$ for all n, where n is defined in the obvious way. So $f_n(x, y) = f_n(x_0, y)$ identically. Let $W \subset X \times Y$ be defined by $f(x, y) = f(x_0, y)$. If $g : X \times Y \to Z \times Z$ is $(f, f(x_0, -))$, then $W = g^{-1}(\Delta_Z)$, and contains $X \times y_n$ for all n. By our density argument, $W = X \times Y$. □

By our rigidity lemma, abelian varieties are commutative. Moreover, if $f : A \to G$ is a morphism of schemes, G is a connected algebraic group, and $f(0) = 1$, then f is a homomorphism of group schemes. To see this, consider the morphism $(x, y) \mapsto f(x + y)f(y)^{-1}f(x)^{-1}$ from $A \times A$ to G. It sends $A \times 0$ and $0 \times A$ to 1, so it is identically 1, whence the result.

We are trying to prove:

2.5.7 Theorem. — *Let G be a conneced algebraic group over a perfect field k. Then there is a exact sequence $1 \to L \to G \to A \to 1$, where L is connected linear and A is an abelian variety.*

First we need some preliminaries on abelian varieties. If A is an abelian variety, we will denote the group law on A by $+$ and the neutral element by 0. We have shown that every morphism (of k-schemes) $A \to G$ which preserves neutral elements is a homomorphism with central image. It is also true that (scheme-theoretic) morphisms from affine groups to abelian varieties are morphisms

2.5.8 Proposition. — *An abelian variety contains no rational curve.*

Proof. — We need to prove that morphisms $f : \mathbf{P}^1_k \to A$ are constant. Let C be the image of f in A. We have $k(C) \subset k(\mathbf{P}^1)$, so $\widetilde{C} \simeq \mathbf{P}^1$. We may assume f is generically smooth on its image. We have $df : T_{\mathbf{P}^1} = \mathscr{O}_{\mathbf{P}^1}(2) \to f^*T_A = \mathscr{O}_{\mathbf{P}^1} \otimes \operatorname{Lie} A$, non-zero by our assumption. This cannot happen: twist both sides by -1 and we get $\mathscr{O}_{\mathbf{P}^1}(1)$ (which has sections) embedding into $\mathscr{O}_{\mathbf{P}^1}(-1)$ (which has none). □

***2.5.9 Theorem* (Weil extension).** — *Let $f : X \to A$ be a rational map, where X is a smooth variety and A is an abelian variety. Then f is a morphism.*

[BLR] has a proof.

Proof. — Our rational map may be factored via $p_2 = f \circ p_1$, where p_1 is birational and proper. We do this via the graph of f:

$$\Gamma_f = \text{Zar. cl.}\{(x, f(x)) : f \text{ is defined at } x\} \subset X \times A.$$

Then $\Gamma_f \to X$ is birational and proper. There is a rational curve $C|subsetY = \Gamma_f$ contracted by p_1 [O. Debarre: Higher-dimensional algebraic geometry, 1.4.3]. Thus $C \subset x \times A$, a contradiction. □

2.5.10 Proposition. — *Let $f : X \times Y \to A$ be a morphism. Let $x_0 \in X(k)$, $y_0 \in Y(k)$ be smooth. Then $f(x, y) = f(x_0, y) + f(x, y_0) - f(x_0, y_0)$.*

Proof. — We want to reduce things to a rigidity lemma from earlier. Without loss of generality, X and Y are smooth. Let $X \hookrightarrow \overline{X}$ be a Nagata compactification, and let $\overline{X}_{\rm sm}$ be its smooth locus (in positive characteristic, we only know $\overline{X}$ is normal). We have $\mathscr{O}(\overline{X}_{\rm sm}) = \mathscr{O}(\overline{X}) = k$. We can extend f to a morphism $\overline{X}_{\rm sm} \times Y \to A$. Let $g(x, y) = f(x, y) - f(x_0, y) - f(x, y_0) + f(x_0, y_0)$. One has $g(X, y_0) = 0$, so the rigidity lemma tells us that g only depends on y. But setting $x = x_0$, we see that $g = 0$. □

2.5.11 Corollary. — *Let G be a connected algebraic group, $f : G \to A$ a morphim of varieties preserving neutral elements. Then G is a homomorphism.*

Proof. — Consider $G \times G \to A$, $(x, y) \mapsto f(xy)$ and apply the previous proposition. □

***2.5.12 Theorem* (Rosenlicht).** — *Let G be a connected algebraic group, $A \subset G$ an abelian subvariety. Then there is normal connected subgroup scheme $H \subset G$ such that $G = A \cdot H$ and $A \cap H$ is finite. If k is perfect, we may take H reduced.*

***2.5.13 Proposition* (Weil, Rosenlicht).** — *Let G be a connected commutative algebraic group, and let X be a (fppf-) G-torsor over k. Then there exists a morphism $\varphi : X \to G$ such that $\varphi(g \cdot x) = \varphi(x) + ng$ for some $n \in \mathbf{Z} \smallsetminus 0$.*

Proof. — If $X(k) \ni x_1$, then $G \to X, g \mapsto g \cdot x_1$ is an isomorphism. In general, we can find $x_1 \in X(K)$, where K/k is finite Galois. Let $\Gamma = \text{Gal}(K/k)$. Let $x_1, \dots, x_n$ be the Galois conjugates of x_1. For $x \in X(\bar{k})$, we can write $x = g_i x_i$ for some $g_i \in G(\bar{k})$. Define $\varphi : X_{\bar{k}} \to G_{\bar{k}}$ by $x \mapsto \sum g_i$. This is defined over K, is Γ-equivariant, and $\varphi(g \cdot x) = ng + \varphi(x)$. This descends to the desired φ. □

If we use Galois cohomology, things become easy. We know that isomorphism classes of G-torsors over K are classified by the commutative group $\mathrm{H}^1(k, G)$. The group operation corresponds to $X_1 + X_2 = (X_1 \times X_2)/(g, -g)$. The set of isomorphism classes of G-torsors trivial over K is the kernel of $\mathrm{H}^1(k, G) \to \mathrm{H}^1(K, G)$, traditionally denoted $\mathrm{H}^1(\Gamma, G(K))$. This is killed by $\#\Gamma$.

Proof of "main theorem". — If k is perfect, we may take H reduced. Let $q : G \to G/A$ be the quotient map. Then q is an A-torsor over the generic point of G/A. Thus there exists $\varphi : G_{k(G/A)} \to A_{k(G/A)}$, hence a rational map $G \to A$, such that $\varphi(ag) = \varphi(g) + na$ when defined. By Weil's extension theorem, φ is a morphism. Also, we may assume $1_G \mapsto 0$. It turns out that φ is a homomorphism. Let $H = (\ker\varphi)^\circ$. Then $A \cap H \subset \ker(\varphi|_A) = \ker(n_A)$. The last group is finite. Also, $\varphi|_A$ is surjective. Thse facts easily imply that $G = A \cdot \ker(\varphi) = A \cdot H$. □

We used the fact that $n_A : A \to A$ is an isogeny for $n \in \mathbf{Z} \smallsetminus 0$. If n is invertible in k, things are easy. One has $\mathrm{Lie}({}_nA) = \ker(n : \mathrm{Lie}\, A \to \mathrm{Lie}\, A)$, and this is zero if $n \in k^\times$. So n_A is étale at 0, hence everywhere by translation. By dimension considerations, n_A is finite and surjective. For the general case, see [**Mum08**].

2.6. Applications. —

2.6.1 Corollary **(Poincaré reducibility)**. — *Let G be an abelian variety, $A \subset G$ an abelian subvariety. Then $G = A + B$, where B is an abelian variety with $A \cap B$ finite.*

Proof. — We have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & A' \arrow[r] \arrow[d] & G/A \arrow[r] \arrow[d] \arrow[dl, dotted] & 0 \\
0 \arrow[r] & A \arrow[r] & G \arrow[r, "q"] & G/A \arrow[r] & 0
\end{tikzcd}$$

The square on the right is Cartesian, and φ yields the dotted map, whence $G' \simeq A \times G/A$. □

2.6.2 Example. — This example is due to Raynaud, in [**SGA 3**, XVIII.A3]. Let k be an imperfect field, $k \subsetneq K$ a purely inseparable field extension. Let H be a connected algebraic group over K. Consider $G = R_{K/k}(H)$. If H is quasi-projective, this exists [**CGP10**, A]. There is an exat sequence $1 \to U \to G_K \to H \to 1$, where U is connected unipotent. If $U \subset V_K$ for some subgroup scheme $V \subset G$, then $V = G$. If H is a nontrivial abelian variety, then G_K satisfies Theorem A, but not G.

It follows from this example that Theorem A fails over any imperfect field. If $H = A_K$ for an abelian k-variety A, we have a map $A \to R_{K/k}(A_K) = G$. This is the inclusion of a closed subgroup. But the image of $A \to G$ has no quasi-complement L, because otherwise $G/L = A/(A \cap L)$, an abelian variety. This cannot be.

Proof of Theorem A. — We follow [**Ros56**], except in modern language. Let G be a connected algebraic group. Then G has a largest connected linear *normal* subgroup $L(G)$. Indeed, let $L \subset G$ be connected linear normal maximal, and let $L_1 \subset G$ satisfy the same properties. Then $L \cdot L_1/L \simeq L - 1/(L_1 \cap L)$, a linear group. So $L \cdot L_1$ is linear. By maximality, $L_1 \subset L$.

It is easy to see that $L(G/L(G)) = 1$. So we may assume that G has no nontrivial connected linear normal subgroups. Such groups are called *pseudo-abelian varieties*, after Totaro. We need to show that G is proper, i.e. pseudo-abelian varieties are abelian. We assume $k = \bar{k}$. See [Totaro, 2013, Ann. ENS] for details. Consider $1 \to H \to G \to L' \to 1$, where L' is linear and $\mathscr{O}(H) = k$. If H is proper, we are done by the existence of a quasi-complement: $G = H \cdot L''$, where $H \cap L''$ is finite. The map $L'' \to L$ is surjective with finite kernel. Since L'' is an extension of a linear group by a finite one, it is linear. So $G/L'' = H/(H \cap L')$ is an abelian variety.

If H is not proper, we use the following

***2.6.3 Lemma* (Rosenlicht).** — *Any non-proper connected algebraic group contains a nontrivial linear subgroup.*

Idea of proof. — If a group scheme acts faithfully on a variety X, then $\mathrm{Stab}_G(x)$ is linear for any $x \in X(k)$. Indeed, $\mathrm{Stab}_G(x)$ acts on $X_n = \mathrm{Spec}(\mathscr{O}_{X,x}/\mathfrak{m}^{n+1})$ for all n. There must be a faithful action of $\mathrm{Stab}_G(x)$ on some X_n. But X_n is affine, so $\mathrm{Stab}_G(x)$ acts faithfully on the finite-dimensional k-vector space $\mathscr{O}(X_n)$. Let $G \hookrightarrow X$ be a normal completion so that $D = X \smallsetminus G$ is a Cartier divisor. Then G acts rationally on X by left multiplication. if $x \in D$ and $g \cdot x$ is defined, then $g \cdot x \in D$. Since $\dim D = \dim G - 1$, all points in D have isotropy groups of positive dimension. After positively modifying D, we assume G acts rationlly on some irreducible component E.

[...didn't understand the end...] □

□

Rosenlicht's proof has been rewritten in [**BSU13**]. Also, see [**Mil13**].

2.6.4 Corollary. — *Every algebraic group is a quasi-projective variety.*

Proof. — Work over $\bar{k}$: use $1 \to L \to G \xrightarrow{\pi} A \to 1$. The map π is an affine morphism, and we know that A is projective. Let $\mathscr{L}$ be an ample line bndle on A; then $\pi^*\mathscr{L}$ is ample on G. The bundle $\pi^*\mathscr{L}$ is defined over some finite extension K/k. Then $N_{K/k}(\pi^*\mathscr{L})$ is also ample (and defined over k). □

Let G be a connected algebraic group over a perfect field k. Then we have an exact sequence

$$1 \to L = G_{\mathrm{aff}} \to G \xrightarrow{\pi} A \to 1,$$

where L is affine and A is an abelian variety.

2.6.5 Proposition. — *1. G_{aff} is the largest connected affine subgroup of G.*

2. π is the universal morphism (of varieties) to an abelian variety.

Proof. — 1. Let $H \subset G$ be connected affine. Then $H \to A$ is constant.

2. Let $f : G \to B$ be a morphism of varieties sending 1 to 0. Then $f : G_{\mathrm{aff}} \to B$ is constant, and $(G_{\mathrm{aff}})_{\bar{k}} \to B_{\bar{k}}$ has no rational curve. But $(G_{\mathrm{aff}})_{\bar{k}}$ is rational by general theory. □

We also have an exact sequence

$$1 \to H \to G \xrightarrow{\varphi} L' = \operatorname{Spec} \mathscr{O}(G) \to 1,$$

where $\mathscr{O}(H) = k$. We call a group scheme H with $\mathscr{O}(H) = k$ *anti-affine*. We write $H = G_{\mathrm{ant}}$. It is easy to see that G_{ant} is the largest anti-affine subgroup of G.

***2.6.6 Proposition* (Rosenlicht)**. — *$G = G_{\mathrm{aff}} \cdot G_{\mathrm{ant}}$ and $G_{\mathrm{aff}} \cap G_{\mathrm{ant}} \supset (G_{\mathrm{ant}})_{\mathrm{aff}}$ with finite quotient.*

We can reformulate this theorem. There is an exact sequence

$$1 \longrightarrow \tfrac{G_{\mathrm{ant}} \cap G_{\mathrm{aff}}}{(G_{\mathrm{ant}})_{\mathrm{aff}}} \longrightarrow \tfrac{G_{\mathrm{aff}} \times G_{\mathrm{ant}}}{(G_{\mathrm{ant}})_{\mathrm{aff}}} \longrightarrow G \longrightarrow 1.$$

The group on the left is finite. The group G_{aff} is connected linear, G_{ant} can be completely
form $T \times \mathbf{G}_{\mathrm{a}}^n$ (T a torus) in characteristic zero.

2.6.7 Proposition. — *Let G be an anti-affine group over any field k of characteristic $p > 0$. Then G is a* semi-abelian variety, *i.e. an extension of an abelian variety by a torus.*

Proof (Rosenlicht). — If $k = \bar{k}$, we can apply Chevalley's theorem to get

$$0 \to T \times U \to G \to A \to 0,$$

where T is a torus and U is connected commutative unipotent. We want to prove that $U = 0$. Mod out by T to get

$$0 \to U \to G/T \to A \to 0,$$

where G/T is still anti-affine because $\mathscr{O}(G/T) = \mathscr{O}(G)^T$. We have $p^n U = 0$ for $n \gg 0$, and p_A^n is an isogeny. Thus $\ker(p_{G/T}^n)$ is an extension of a finite group scheme by U. Thus $p^n(G/T) \subset G/T$ is an abelian variety, so we can write $G/T = (p^n G/T) \cdot H$ for a quasi-complement H. But then $G/T \twoheadrightarrow H/(H \cap p^n G/T)$, and H is linear (isogenous to U). This can only happen if $H/(H \cap p^n G/T) = 1$, whence the result.

For arbitrary k, the maximal torus $T_{\bar{k}} \subset G_{\bar{k}}$ descends to a maximal torus of G. This follows from a very general theorem of Grothendieck on tori. In $T(\bar{k})$, we may find g such that $\overline{\langle g \rangle} = T_{\bar{k}}$ (unless k is finite). Then g^{p^n} is regular for all n. We may assume $g \in T(k^s)$. The rest is easy. □

2.6.8 Definition. — A *pseudo-abelian variety* is a connected algebraic group without nontrivial connected linear subgroups.

If k is perfect, then Chevalley's theorem tells us that pseudo-abelian varieties are abelian. If k is not perfect, these definitions do not coincide (see an earlier example of Raynaud). One starts with an abelian variety of A over k, and puts $G = R_{K/k}(A_K)$ for K/k purely inseparable. Basic properties of Weil restriction give us an exact sequence

$$0 \to A \to G \to U \to 0,$$

where U is unipotent. If A is simple, then G is pseudo-abelian.

***2.6.9 Theorem* (Totaro)**. — *Any pseudo-abelian variety G is part of an extension*

$$0 \to A \to G \to U \to 0$$

with A abelian and U connected unipotent. Moreover, G is commutative.

... didn't write down. ... — □

Any connected unipotent U with $pU = 0$ occurs in such an extension.

2.7. Structure theory for semi-abelian varieties. — Fix a torus T and an abelian variety A. We study extensions $0 \to T \to G \to A \to 0$. Start with the case $T = \mathbf{G}_{\mathrm{m}}$. Such an extension will be a $\mathbf{G}_{\mathrm{m}}$-torsor (= line bundle) over A. So $\mathrm{Ext}^1(A, \mathbf{G}_{\mathrm{m}}) \hookrightarrow \mathrm{H}^1(A, \mathbf{G}_{\mathrm{m}}) = \mathrm{Pic}(A)$. It turns out that the image lies in $\mathrm{Pic}^\circ(A)$, the subgroup of $\mathrm{Pic}(A)$ consisting of algebraically trivial line bundles. Recall that $L \in \mathrm{Pic}(A)$ is *algebraically trivial* if there exists a line bundle $\mathscr{L}$ on $X \times S$ for S a connected variety, such that $\mathscr{L}_{s_1} \simeq L$ for some $s_1 \in S(k)$ and $\mathscr{L}_{s_0} \simeq \mathscr{O}_A$ for some other $s_0 \in S(k)$.

2.7.1 Exercise. — *If A is an elliptic curve, then a line bundle is algebraically trivial if and only if it has degree zero.*

***2.7.2 Theorem* (Weil, Barsotti)**. — *The map* $\mathrm{Ext}^1(A, \mathbf{G}_{\mathrm{m}}) \to \mathrm{Pic}^\circ(A)$ *is an isomorphism. Moreover,* $\mathrm{Pic}^\circ(A)$ *is naturally the group of k-rational points of an abelian variety $A^\vee$.*

If we define $\mathrm{Ext}^1(A, \mathbf{G}_{\mathrm{m}})$ as a functor by $\mathrm{Ext}^1(A, \mathbf{G}_{\mathrm{m}})(S) = \mathrm{Ext}^1_S(A_S, \mathbf{G}_{\mathrm{m},S})$, then $\mathrm{Ext}^1(A, \mathbf{G}_{\mathrm{m}}) = A^\vee$. The functor $(-)^\vee$ is an anti-equivalence from the category of abelian varieties to itself.

Let's see how things work. From our exact sequence $0 \to \mathbf{G}_{\mathrm{m}} \to G \xrightarrow{\pi}\!\!\to A \to 0$, the sheaf $\pi_* \mathscr{O}_G$ is a sheaf of $\mathscr{O}_A$-algebras with $\mathbf{G}_{\mathrm{m}}$-action, i.e. $\pi_*\mathscr{O}_G = \bigoplus_{n \in \mathbf{Z}} L^n$, where $L \in \mathrm{Pic}^\circ(A)$ is the image of $\mathrm{Ext}^1(A, \mathbf{G}_{\mathrm{m}}) \to \mathrm{Pic}^\circ(A)$.

Next we'll look at extensions of split tori $T \simeq \mathbf{G}_{\mathrm{m}}^n$. More invariantly, $T = \hom(\widehat{T}, \mathbf{G}_{\mathrm{m}})$. We get $\mathrm{Ext}^1(A, T) = \hom(\widehat{T}, A^\vee(k))$. If we think of things in terms of functors, $\mathrm{Ext}^1(A, T) = \hom(\widehat{T}, A^\vee)$. We get $\pi_*\mathscr{O}_G = \bigoplus_{\chi \in \widehat{T}} L_{c(\chi)}$, where $c : \widehat{T} \to$

$A^\vee(k)$. From this, we get that G is anti-affine if and only if c is injective. Indeed, $\mathscr{O}(G) = \bigoplus_{\chi\in\widehat{T}} \mathrm{H}^0(A, L_{c(\chi)})$ and for $L \in \mathrm{Pic}^\circ(A)$, $\mathrm{H}^0(A, L) \neq 0$ if and only if L is trivial.

For arbitrary torus T, since T splits over k^s, we have

$$\mathrm{Ext}^1(A, T) \xrightarrow{\sim} \hom(\widehat{T}(k^s), A^\vee(k^s))^\Gamma,$$

where $\Gamma = \mathrm{Gal}(k^s/k)$.

2.7.3 Corollary. — *In characteristic $p > 0$, anti-affine groups with abelian part A are classified by Γ-stable lattices in $A^\vee(k^s)$.*

If $k = \mathbf{F}_q$ or $\overline{\mathbf{F}_q}$, then $k = k^s$, and $A^\vee(k^s) = \bigcup A^\vee(\mathbf{F}_{q^{nm}})$, a torsion group. So there are no anti-affine groups over finite fields other than abelian varieties:

***2.7.4 Corollary* (Arima '61)**. — *Every connected algebraic group G over a finite field is isogenous to $G_{\mathrm{aff}} \times A$, where G_{aff} is linear and A is abelian.*

The kernel of $G_{\mathrm{aff}} \times A \times A \to G$ is a finite group scheme, but there are few other restrictions.

Let's study anti-affine groups in characteristic zero. By considering the exact sequence $0 \to T \times U \to G \to A \to 0$, we get that G is anti-affine if and only if both G/T and G/U are. The group G/U is a semi-abelian variety. Since we are in characteristic zero, U is a vector group, so G/T is a vector extension of A.

[. . . didn't record proof of this. . .]

Any abelian variety has a universal vector extension:

$$0 \to \mathrm{H}^1(A, \mathscr{O}_A)^\vee \to E(A) \to A \to 0.$$

It is a nontrivial result that $\mathrm{Ext}^1(A, \mathbf{G}_\mathrm{a}) \xrightarrow{\sim} \mathrm{H}^1(A, \mathscr{O})$. This is a theorem of Rosenlicht and Serre.

2.7.5 Proposition. — *Suppose the base field has characteristic zero. Then $E(A)$ is anti-affine, and the anti-affine vector extensions of A are exactly the quotients of $E(A)$.*

In other words, any extension fits into a commutative diagram with pushout on the left:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(A, \mathscr{O})^\vee \arrow[r] \arrow[d] & E(A) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U \arrow[r] & G \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

Future problems are to find analogues of this theory over discrete valuation rings. There is a version of the affinization theorem by Raynaud, [**SGA 3**, VI]. There is no clear analogue of Chevalley's "theorem A." A powerful result is Grothendieck's semistable reduction theorem.

3. Geometric representation theory

3.1. Introduction. — The goal is to compute character tables for "finite groups of Lie type." Let G be a connected reductive (linear algebraic) group over $\mathbf{F}_q$. Then $G(\mathbf{F}_q)$ is a finite group. Our goal is to compute the (complex) character table of $G(\mathbf{F}_q)$.

There are two difficulties here. First, we need to classify isomorphism classes of irreducible representations $\rho : G(\mathbf{F}_q) \to \mathrm{GL}(V)$, where V is a finite-dimensional $\mathbf{C}$-vector space. Then, we need to compute the values (on conjugacy classes) of $\chi_\rho : G(\mathbf{F}_q) \to \mathbf{C}$ given by $g \mapsto \mathrm{tr}(\rho(g))$.

papers	result	tools
[**Jor07**]	$\mathrm{GL}_2(\mathbf{F}_q)$	combinatorics
[**Sch07**]	$\mathrm{SL}_2(\mathbf{F}_q)$	combinatorics
[**Gre55**]	$\mathrm{GL}_n(\mathbf{F}_q)$	combinatorics
[**DL76**]	general G	ℓ-adic cohomology
Lusztig 1981–1990	character-sheaf theory	perverse sheaves
[**Sho95b**], [**Sho95a**]	Lusztig conjecture if $Z = Z^\circ$	
[**Wal04b**], [**Wal04a**]	for SO_n and Sp_{2n}	
Bonnalé 2003	for SL_n	

Green's computation was highly combinatorial. On the other hand, Deligne and Lusztig constructed the representations "directly" using ℓ-adic étale cohomology. Lusztig conjectured that there was a relation between "character sheaves" and irreducible characters of $G(\mathbf{F}_q)$.

We will illustrate the general machinery using the specific example of GL_n.

3.2. Combinatorial background. — A good combinatorial reference for all of this is [**Mac95**].

3.2.1. The ring of symmetric functions. — For $r \geqslant 1$, put $\Lambda_r = \mathbf{Z}[x_1, \dots, x_r]^{S_r} = \bigoplus_{k \geqslant 0} \Lambda_r^k$, where Λ_r^k is the set of homogeneous symmetric polynomials of degree k. For $m \geqslant r$, we have a map $\rho_r^m : \mathbf{Z}[x_1, \dots, x_m] \to \mathbf{Z}[x_1, \dots, x_r]$, given by

$$x_i \mapsto \begin{cases} x_i & i \leqslant r \\ 0 & i > r \end{cases}$$

This induces a map $\rho_r^m : \Lambda_m^k \to \Lambda_r^k$. Put $\Lambda^k = \varprojlim_r \Lambda_r^k$ and $\Lambda = \bigoplus_{k \geqslant 0} \Lambda^k$. We call Λ the *ring of symmetric functions in infinitely many variables.* Elements of Λ look like e.g. $P_2(x) = x_1^2 + x_2^2 + \cdots$.

A *partition* is an infinite non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers, with finitely many nonzero terms. We put $|\lambda| = \sum \lambda_i$ and $\ell(\lambda) = \max\{i : \lambda_i \neq 0\}$. Let $\mathcal{P}$ be the set of all partitions, and $\mathcal{P}_n$ the set of partitions of size n.

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3.2.2. Monomial symmetric functions. — For $\lambda = (\lambda_1, \dots, \lambda_r, 0, \dots)$, let

$$m_\lambda(x_1, \dots, x_r) = \sum x_1^{\alpha_1} \cdots x_r^{\alpha_r},$$

where the sum runs over all distinct permutations $(\alpha_1, \dots, \alpha_r)$ of $(\lambda_1, \dots, \lambda_r)$. For example,

$$m_{(2,2,1)}(x_1, x_2, x_3) = x_1^2 x_2^2 x_3 + x_1^2 x_2 x_3^2 + x_1 x_2^2 x_3^2. \tag{1}$$

This defines an element $m_\lambda \in \Lambda$ such that $m_\lambda(x_1, \dots, x_r)$ is the right-hand side of (1) if $\ell(\lambda) \leqslant r$, and 0 otherwise. If $\lambda = 0$, put $m_\lambda = 1$. Then $\{m_\lambda : \lambda \in \mathcal{P}\}$ is a $\mathbf{Z}$-basis for Λ.

3.2.3. Elementary symmetric functions. — For each $n \geqslant 0$, define $e_n \in \Lambda$ by

$$e_n = \begin{cases} m_{(1^n)} & n \geqslant 1 \\ 1 & n = 0 \end{cases}$$

For $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P}$, we define $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$. It turns out that $\{e_\lambda : \lambda \in \mathcal{P}\}$ is a $\mathbf{Z}$-basis for Λ, and $\Lambda = \mathbf{Z}[e_1, e_2, \dots]$.

3.2.4. Power-sums. — For $n \geqslant 0$, define $p_n \in \Lambda$ by

$$p_n = \begin{cases} m_{(n^1)} & n \geqslant 1 \\ 1 & n = 0 \end{cases}$$

For example,

$$p_3 = x_1^3 + x_2^3 + x_3^3 + x_4^3 + \cdots.$$

For $\lambda = (\lambda_1, \lambda_2, \dots)$, put $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$. The p_λ form a $\mathbf{Q}$-basis of $\Lambda_{\mathbf{Q}} = \Lambda \otimes \mathbf{Q}$. Thus $\Lambda_{\mathbf{Q}} = \mathbf{Q}[p_1, p_2, \dots]$.

3.2.5. Shur functions. — Given r variables $x_1, \dots, x_r$, put $\delta_r = (r-1, r-2, \dots, 1, 0)$. For any $\lambda = (\lambda_1, \lambda_2, \dots)$, put

$$a_{\lambda+\delta_r} = \det \begin{pmatrix} x_1^{\lambda_1+\delta_1} & x_2^{\lambda_2+\delta_2} & \cdots & x_1^{\lambda_r} \\ \vdots & \vdots & & \vdots \\ x_r^{\lambda_1+\delta_1} & x_r^{\lambda_2+\delta_2} & \cdots & x_r^{\lambda_r} \end{pmatrix} = \sum_{w \in S_r} \varepsilon(w) w(x_1^{\lambda_1+\delta_1} \cdots x_r^{\lambda_r+\delta_r}).$$

This is skew-symmetric, and we have $a_{\delta_r} \mid a_{\lambda+\delta_r}$ in $\mathbf{Z}[x_1, \dots, x_r]$. The *Schur function* is

$$S_\lambda(x_1, \dots, x_r) = \frac{a_\lambda + \delta_r}{a_{\delta_r}} \in \mathbf{Z}[x_1, \dots, x_r]^{S_r}. \tag{2}$$

This defines an element $S_\lambda \in \Lambda$, determined by

$$S_\lambda(x_1, \dots, x_r) = \begin{cases} \text{right-hand side of (2)} & \ell(\lambda) \leqslant r \\ 0 & \text{otherwise} \end{cases}$$

Moreover, $\{S_\lambda : \lambda \in \mathcal{P}\}$ is a $\mathbf{Z}$-basis for Λ.

3.2.6. Relationship with representations of the symmetric group. — There is a natural bijection

$$S_n^\natural = \{\text{conjugacy classes in } S_n\} \xrightarrow{\sim} \mathcal{P}_n,$$

sending a conjugacy class c to the partition coming from lengths of cycles in the decomposition of an element of c into disjoint cycles. For example, an n-cycle maps to the partition $(n, 0, 0, \dots)$, and the identity maps to $(1^n, 0, \dots)$. Let $\mu \mapsto c_\mu$ be the inverse of this bijection. So we have a bijection $\mathcal{P}_n \xrightarrow{\sim} \mathrm{Irr}(S_n)$, written $\lambda \mapsto \chi^\lambda$. One has

$$S_\lambda = \sum_\mu \frac{1}{z_\mu} \chi_\mu^\lambda p_\mu,$$

where $z_\mu = \# C_{S_n}(\sigma_\mu)$, and χ_μ^λ is the value of χ^λ at c_μ. A trivial corollary is that

$$\chi^\lambda = \sum_\mu \frac{1}{z_\mu} \chi_\mu^\lambda 1_{c\mu}.$$

So $\{\chi^\lambda\}$ and $\{1_{c_\mu}\}$ are both $\mathbf{C}$-bases for $\hom(S_n^\natural, \mathbf{C})$.

We would like to do something similar for $\mathrm{GL}_n(\mathbf{F}_q)$.

3.2.7. Hall-Littlewood symmetric functions. — Let t be an extra variable. Consider the ring $\Lambda_{\mathbf{Z}[t]} = \Lambda[t] = \Lambda \otimes \mathbf{Z}[t]$. For $\lambda \in \mathcal{P}$ with $\ell(\lambda) \leqslant r$, put

$$\begin{aligned} R_\lambda(x_1, \dots, x_r; t) &= \sum_{w \in S_r} w \left(x_1^{\lambda_1} \cdots x_r^{\lambda_r} \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \right) \\ &= \frac{1}{a_{\delta_r}} \sum_{w \in S_r} \varepsilon(w) w \left(x_1^{\lambda_1} \cdots x_r^{\lambda_r} \prod_{i<j} x_i - t x_j \right). \end{aligned}$$

These are elements of $\Lambda_r[t]$. Unfortunately, $R_\lambda(x_1, \dots, x_r, 0; t) \neq R_\lambda(x_1, \dots, x_r; t)$ as elements of $\Lambda_{r+1}[t]$, so we need to "fudge things" a bit to get a well-defined element of $\Lambda[t]$. For $m \geqslant 1$, put

$$v_m(t) = \sum_{w \in S_m} w \left(\prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \right).$$

For $\lambda = (\lambda_1, \dots, \lambda_r, 0, \dots)$, denote by m_i the multiplicity of i in λ if $i \geqslant 1$, and by $m_0 = \#\{i \leqslant r : \lambda_i = 0\}$. Note that m_0 depends heavily on r. Define

$$v_{\lambda,r}(t) = \prod_{i \geqslant 0} v_{m_i}(t).$$

Then

$$R_\lambda(x_1, \dots, x_r; t) = v_\lambda(t) \sum_{w \in S_r / S_r^\lambda} w \left(x_1^{\lambda_1} \dots x_r^{\lambda_r} \prod_{\lambda_i > \lambda_j} \frac{x_i - t x_j}{x_i - x_j} \right),$$

where as before $S_r^\lambda = \{w \in S_r : \lambda_{w(i)} = \lambda_i \text{ for } 1 \leqslant i \leqslant r\} \simeq \prod_{i\geqslant 0} S_{m_i}$. Define

$$P_\lambda(x_1,\dots,x_r;t) = \frac{1}{v_\lambda(t)} R_\lambda(x_1,\dots,x_r;t);$$

this is the *Hall-Littlewood symmetric function*.

3.2.1 Proposition. — *1.* $P_\lambda(x_1,\dots,x_r;0) = S_\lambda(x_1,\dots,x_r)$.

2. $P_\lambda(x_1,\dots,x_r;1) = m_\lambda(x_1,\dots,x_r)$.

3. $P_\lambda(x_1,\dots,x_r,0;t) = P_\lambda(x_1,\dots,x_r;t)$.

By 3, we have a well-defined element $P_\lambda \in \Lambda[t]$. Moreover, $\{P_\lambda : \lambda \in \mathcal{P}\}$ is a $\mathbf{Z}[t]$-basis for $\Lambda[t]$.

The *Kostka-Foulkes polynomials* $\{K_{\lambda\mu}(t)\}$ are by the relation

$$S_\lambda = \sum_\mu K_{\lambda\mu}(t) P_\mu.$$

***3.2.2 Theorem* (Lascoux-Schützenberger, Lusztig)**. — $K_{\lambda\mu}(t) \in \mathbf{Z}_{\geqslant 0}[t]$.

3.3. The character table of $\mathrm{GL}_n(\mathbf{F}_q)$. —

3.3.1. The conjugacy classes. — Fix $n \geqslant 1$. Let $M = \overline{\mathbf{F}_q}^\times$, considered as an abstract group. Let

$$\widetilde{\mathbf{T}}_n = \left\{\varphi : M \to \mathcal{P} : \|\varphi\| := \sum_{m\in M} |\varphi(m)| = n\right\}.$$

We claim that $\mathrm{GL}_n(\overline{\mathbf{F}_q})^\natural \xrightarrow{\sim} \widetilde{\mathbf{T}}_n$. We illustrate this with an example. If $n = 3$, then the conjugacy class of

$$\begin{pmatrix} \alpha & 1 & & \\ & \alpha & & \\ & & \beta & \\ & & & \beta \end{pmatrix}$$

is sent to the map $M \to \mathcal{P}$ given by

$$\gamma \mapsto \begin{cases} (2,0,0,\dots) & \gamma = \alpha \\ (1,1,0,0,\dots) & \gamma = \beta \\ 0 & \text{otherwise} \end{cases}$$

For the general case, let $F : \overline{\mathbf{F}_q} \to \overline{\mathbf{F}_q}$ be the Frobenius $x \mapsto x^q$, and let $\Phi = \{F\text{-orbits of } M\}$. Every F-orbit is of the form $\{x, x^q, \dots, x^{q^{d-1}}\}$, for the minimal d with $x^{q^d} = x$. If $f \in \Phi$, put $d(f) = \#f$ (cardinality of f as an orbit set).

3.3.1 Proposition. — *Put* $\mathbf{T}_n = \{\varphi : \Phi \to \mathcal{P} : \|\varphi\| = \sum_{f\in\Phi} d(f)|\varphi(f)| = n\}$. *Then* $\mathrm{GL}_n(\mathbf{F}_q)^\natural \xrightarrow{\sim} \mathbf{T}_n$.

Proof. — Write $\mathrm{GL}_n = \mathrm{GL}_n(\overline{\mathbf{F}_q})$, and let $F : \mathrm{GL}_n \to \mathrm{GL}_n$ be the Frobenius $(a_{ij}) \mapsto (a_{ij}^q)$. We will show that

$$\{F\text{-stable conjugacy classes of } \mathrm{GL}_n\} \xrightarrow{\sim} \mathbf{T}_n.$$

Indeed, in any conjugacy class C, there is an element of the form

$$\begin{pmatrix} \Delta_1 & & \\ & \Delta_2 & \\ & & \ddots \end{pmatrix},$$

where

$$\Delta_i = \begin{pmatrix} \alpha_i & 1 & \\ & \alpha_i & 1 \end{pmatrix}$$

If $F(C) = C$, then Δ and $F(\Delta)$ are GL_n-conjugate. Thus F permutes the blocks Δ_i, so $F(\alpha_i) = \alpha_j$ for some j. Thus Δ_i and Δ_j have the same Jordan form. $\square$

We claim that

$$\mathrm{GL}_n(\mathbf{F}_q)^\natural \xrightarrow{\sim} \{F\text{-stable conjugacy classes of } \mathrm{GL}_n\}.$$

Note that this fails for general algebraic groups. In general, suppose G is a connected affine algebraic group acting on an algebraic variety X (all over $\overline{\mathbf{F}_q}$). Assume G, X, and the action of G on X are all defined over $\mathbf{F}_q$. Let $F : G \to G$ and $F : X \to X$ be the relative Frobenii. Since the action of G on X is defined over $\mathbf{F}_q$, we have $F(g \cdot x) = F(g) \cdot F(x)$.

1. If O is an F-stable G-orbit of X, then $O^F = \{x \in O : F(x) = x\} = O(\mathbf{F}_q)$ is not empty (this uses the fact that $g \mapsto g^{-1}F(g)$, $G \to G$ is surjective). Thus the above map is surjective for any G.
2. If $x \in X^F = X(\mathbf{F}_q)$ and $\mathrm{Stab}_G(x) = \mathrm{Stab}_G(x)^\circ$, then $(G \cdot x)^F = G^F \cdot x$.

In GL_n, the stabilizers $\mathrm{Stab}_{\mathrm{GL}_n}(x)$ are *always* connected ($x \in \mathrm{GL}_n$).

3.3.2 Exercise. — Prove that $\mathrm{Stab}_{\mathrm{SL}_2}\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$ has two connected components if $2 \nmid q$.

Each $g \in \mathrm{GL}_n(\mathbf{F}_q)$ defines an $\mathbf{F}_q[t]$-structure on $V = (\mathbf{F}_q)^n$ as follows: $t \cdot v = g(v)$ for all $v \in V$. This structure depends only on the conjugacy class of g. If $C \subset \mathrm{GL}_n(\mathbf{F}_q)$ is a conjugacy class corresponding to $\varphi : \Phi \to \mathcal{P}$, denote by V_φ the corresponding $\mathbf{F}_q[t]$-structure on V. From commutative algebra, we know that $V_\varphi \simeq \bigoplus_{f \in \Phi} \bigoplus_{i \geqslant 1} \mathbf{F}_q[t]/(P_f)^{\varphi(f)_i}$, and $P_f = \prod_{i=1}^{d-1}(t - x^{q^i})$.

3.3.3 Example **($\mathrm{GL}_2(\mathbf{F}_q)$).** — Consider the following table, in which $\varphi(f) = 0$ unless it is defined to be otherwise:

conjugacy class	$\varphi : \Phi \to \mathcal{P}$
$\begin{pmatrix} a & \\ & a \end{pmatrix}$, $a \in \mathbf{F}_q^\times$	$\{a\} \mapsto (1,1,0,\dots)$
$\begin{pmatrix} a & \\ & b \end{pmatrix}$, $a \neq b \in \mathbf{F}_q^\times$	$\{a\}, \{b\} \mapsto (1,0,\dots)$
$\begin{pmatrix} a & 1 \\ & a \end{pmatrix}$, $a \in \mathbf{F}_q^\times$	$\{a\} \mapsto (2,0,\dots)$
$\begin{pmatrix} x & \\ & x^q \end{pmatrix}$, $x \in \mathbf{F}_{q^2} \smallsetminus \mathbf{F}_q^\times$	$\{x, x^q\} \mapsto (1,0,\dots)$

3.3.2. Parabolic induction. — For any finite group H, write $\mathcal{C}H = \mathbf{C}^{H^\natural}$ for the space of functions $H \to \mathbf{C}$ that are constant on conjugacy classes.

Let $\lambda = (n_1, n_2, \dots, n_r, 0, \dots)$ be a partition of n. Let $\mathrm{GL}_\lambda \subset \mathrm{GL}_n$ be the group consisting of matrices of the form $A_1 \oplus \cdots \oplus A_r$, with each A_i an invertible $n_i \times n_i$ matrix. Note that $\mathrm{GL}_\lambda \simeq \prod_i \mathrm{GL}_{n_i}$. If we write $L = \mathrm{GL}_\lambda$, we will define functions $R_L^{\mathrm{GL}_n} : \mathcal{C}(L(\mathbf{F}_q)) \to \mathcal{C}(\mathrm{GL}_n(\mathbf{F}_q))$.

Let P be the standard parabolic of "block upper-triangular matrices" associated to λ, and let U_P be the unipotent radical of P. One has $P = \mathrm{GL}_\lambda \cdot U_P$. If $f \in \mathcal{C}(L(\mathbf{F}_q))$, define $R_L^{\mathrm{GL}_n}(f) = \mathrm{ind}_P^{\mathrm{GL}_n}(\widetilde{f})$, where $\widetilde{f} : P(\mathbf{F}_q) \to \mathbf{C}$ is given by $\ell u \mapsto f(\ell)$ for $\ell \in L$, $u \in U_P$. If $f \in \mathbf{N}[\mathrm{Irr}(L(\mathbf{F}_q))]$, then $R_L^{\mathrm{GL}_n}(f) \in \mathbf{N}[\mathrm{Irr}(\mathrm{GL}_n(\mathbf{F}_q))]$. It turns out that $R_L^{\mathrm{GL}_n}(f)$ does not depend on the choice of a rational P having L as a Levi factor.

Put $A_n = \mathcal{C}(\mathrm{GL}_n(\mathbf{F}_q)$ and $A = \bigoplus_{n \geqslant 0} A_n$. We define an inner product on A: if $f \in A_{n_1}$ and $g \in A_{n_2}$, put

$$f \circ g = R_{\mathrm{GL}(n_1) \times \mathrm{GL}(n_2)}^{\mathrm{GL}(n_1+n_2)}(f, g).$$

This is commutative and associative.

Let $\varphi : \Phi \to \mathcal{P}$ have finite support. Denote by $\pi_\varphi \in \mathcal{C}(\mathrm{GL}_{\|\varphi\|}(\mathbf{F}_q))$ the characteristic function of the associated conjugacy class c_φ. Then $\{\pi_\varphi\}$ is a $\mathbf{C}$-basis for A, and π_0 is the multiplicative unit of A.

3.3.4 Lemma. — *If* $\|\varphi\| = \sum \|\varphi_i\|$, *then* $\pi_{\varphi_1} \circ \pi_{\varphi_2} \circ \cdots \circ \pi_{\varphi_r}(c_\varphi)$ *is the number of sequences* $0 = W^{(0)} \subset W^{(1)} \subset \cdots \subset W^{(r)} = V_\varphi$ *of submodules of* V_φ *such that* $W^{(i)}/W^{(i-1)} \simeq V_{\varphi_i}$.

Here V_φ is the $\mathbf{F}_q[t]$-structure on $(\mathbf{F}_q)^n$ defined by φ.

3.3.3. The characteristic map. — For each $f \in \Phi$, let $\{x_{1,f}, x_{2,f}, \dots\}$ be infinitely many variables. Let $x = \{x_1, x_2, \dots\}$ be a set of infinitely many variables. Put $\Lambda_{\mathbf{Q}(q)} = \Lambda(x) \otimes \mathbf{Q}(q)$. If $u \in \Lambda_{\mathbf{Q}(q)}$, we denote by $u(f)$ the corresponding function in $\Lambda_{\mathbf{Q}(q)}(x_f)$.

Put $B = \mathbf{C}[e_n(f) : n \geqslant 0, f \in \Phi]$. Then $B \subset \mathbf{Q}(q) \otimes \bigotimes_{f \in \mathbf{Q}} \Lambda(x_f)$. The Hall pairing on $\Lambda(x)$ makes the Schur functions an orthonormal basis. We grade B by $\deg(x_i, f) = d(f)$. For $\lambda \in \mathcal{P}$, $f \in \Phi$, put $\widetilde{P}_\lambda(f) = q^{-n(\lambda)d(f)} P_\lambda(f; q^{-d(f)})$. Recall that $n(\lambda) = \sum_{i \geqslant 1}(i-1)\lambda_i$. For $\varphi : \Phi \to \mathcal{P}$ with finite support, we put

$$\widetilde{P}_\varphi = \prod_{f \in \Phi} \widetilde{P}_{\varphi(f)}(f) \in B.$$

Since the Hall-Littlewood symmetric functions form a $\mathbf{Z}$-basis of $\Lambda(x)$, the $\widetilde{P}_\varphi$ form a $\mathbf{C}$-basis of B.

The *characteristic map* $\mathrm{Ch} : A \to B$, given by $\pi_\varphi \mapsto \widetilde{P}_\varphi$, is an (isometric) isomorphism of graded $\mathbf{C}$-algebras.

3.3.4. Construction of the irreducible characters of $\mathrm{GL}_n(\mathbf{F}_q)$. — Let $M_n = \mathbf{F}_{q^n}^\times$. Let $\widehat{M_n} = \hom(M_n, \mathbf{C}^\times)$, and let $L = \varinjlim_n \widehat{M_n}$ and $L_n = \mathrm{im}(\widehat{M_n} \to L)$. The Frobenius F acts on L; let Θ be the set of F-orbits in L. We want $\mathrm{Irr}(\mathrm{GL}_n(\mathbf{F}_q))$ to be in bijection with

$$\left\{ \psi : \Theta \to \mathcal{P} \text{ such that } \|\psi\| := \sum_{\theta\in\Theta} d(\theta)\cdot|\psi(\theta)| = n \right\}.$$

The set $\mathrm{Irr}(\mathrm{GL}_n(\mathbf{F}_q))$ is an orthonormal basis for $\langle\cdot,\cdot\rangle$. We look for an orthonormal basis of B. For $\chi \in \mathrm{Irr}(\mathrm{GL}_n(\mathbf{F}_q))$, we expect $\mathrm{Ch}(\chi)$ to be "related to" Schur functions.

For $f \in \Phi$ and $x \in f$, put

$$\widetilde{P}_n(x) = \begin{cases} P_{n/d}(f) & d = d(f) \mid n \\ 0 & \text{otherwise} \end{cases}$$

Then for each $\xi \in L$, define

$$\widetilde{P}_n(\xi) = \begin{cases} (-1)^{n-1}\sum_{x\in M_n}\langle x,\xi\rangle_n \widetilde{P}_n(x) & \xi \in L_n \\ 0 & \text{otherwise} \end{cases}$$

Note that $\widetilde{P}_n(\xi)$ depends only on the F-orbit of ξ. For $\theta\in\Theta$, put $\widetilde{P}_r(\theta) = \widetilde{P}_{rd}(\xi)$, for any $\xi\in\theta$, and $d = d(\theta)$.

3.3.5 Example. — If $\xi = 1 \in L$ and $n = 2$, then

$$\widetilde{P}_n(\xi) = -\sum_{\substack{f\in\Phi\\ d(f)=1}}\sum_i x_{i,f}^2 - 2\sum_{\substack{f\in\Phi\\ d(f)=2}}\sum_i x_{i,f}.$$

Make the change of variables $\{x_{i,f}\} \to \{y_{i,\theta}\}$, such that $\widetilde{P}_r(\theta)$ are power-sums in $\{y_{i,\theta}\}$.

For $\lambda \in \mathcal{P}$, put $\widetilde{P}_\lambda(\theta) = P_{\lambda_1}(\theta)P_{\lambda_2}(\theta)\cdots$ and $S_\lambda(\theta) = \sum_\mu \frac{1}{z_\mu}\chi_\mu^{-1}\widetilde{P}_\mu(\theta)$. For $\psi : \Theta \to \mathcal{P}$ with finite support, put $S_\psi = \prod_{\theta\in\Theta} S_{\psi(\theta)}(\theta)$.

3.3.6 Proposition. — *1.* $\{S_\psi\}$ *is an orthornormal* $\mathbf{C}$*-basis of* B.

2. Assume that $|\lambda| = |\mu|$*. Then*

$$S_\lambda(1) = \sum_\mu \widetilde{K}_{\lambda\mu}(q)\widetilde{P}_\mu(1).$$

Here we are using the definition $\widetilde{K}_{\lambda\mu}(q) = q^{n(\mu)}K_{\lambda\mu}(q^{-1})$.

3.3.7 Theorem. — *The map* $A \to B$ *induces* $\mathrm{Irr}(\mathrm{GL}_n(\mathbf{F}_q)) \to \{S_\psi : \|\psi\| = n\}$.

Proof. — If $\theta \in \Theta$ and $n \geqslant 0$, then $e_n(\theta)$ is the characteristic function of a character of $\mathrm{GL}_{nd(\theta)}(\mathbf{F}_q)$.For $\psi : \Theta \to \mathcal{P}$, the S_ψ are polynomials in the $e_n(\theta)$, hence have coefficients in $\mathbf{Z}$. This implies that $S_\psi = \mathrm{Ch}(\text{virtual character})$, say $\mathscr{X}^\psi$. One has $\langle S_\psi, S_\psi \rangle = \langle \mathscr{X}^\psi, \mathscr{X}^\psi \rangle$, so $\mathscr{X}^\psi$ or $-\mathscr{X}^\psi$ is irreducible. Thus $\mathscr{X}^\psi(1) > 0$. $\square$

To conclude, call "unipotent character" the irreducible characters $\mathscr{X}^\psi$ with

$$\psi \in \{\Theta \xrightarrow{\psi} \mathcal{P} : \text{support of } \psi = 1\}.$$

Later on, we will give a geometric construction of the $\mathscr{X}^\psi$.

3.4. ℓ-adic sheaves and perverse sheaves. — The $\widetilde{K}_{\lambda\mu}(q)$ can be realized as Khazdan-Lusztig polynomials using perverse sheaves.

3.4.1. Sheaves on topological spaces. — Let X be a topological space. A *presheaf* on X is a (contravariant) functor $F : \mathrm{Op}(X) \to \mathcal{C}$, where $\mathcal{C}$ is the category of groups, sets, rings, modules, ... Morphisms of presheaves are natural transformations of functors. A presheaf is a *sheaf* if for any family $\{U_i\}$ of open subsets of X, and any family $\{s_i \in FU_i\}$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there exists a unique $s \in FU$ such that $s|_{U_i} = s_i$. We often use script letters for sheaves, e.g. $\mathscr{F}, \mathscr{G}, \ldots$. If $\mathscr{F}$ is a sheaf on X and $x \in X$, write $\mathscr{F}_x = \varinjlim_{U \ni x} \mathscr{F}(U)$ for the *stalk* of $\mathscr{F}$ at x.

[...I know what sheaves are...]

3.4.2. ℓ-adic analogue. — If X is a variety over $\mathbf{C}$, we can give $X(\mathbf{C})$ the structure of a topological space using the topology on $\mathbf{C}$. In characteristic p, a variety has no natural topology that gives a good sheaf theory. Fortunately, a very deep theory of Grothendieck gives us something called the "étale topology," so we can talk about sheaves on $X_{\text{ét}}$.

We would like to use the étale topology to define spaces $\mathrm{H}^i(X_{\text{ét}}, k)$ (k a field of characteristic zero) for X over $\mathbf{F}_q$. Unfortunately, if we try to define $\mathrm{H}^i(X_{\text{ét}}, k)$ in the naive manner, we get trivial cohomology.

[...Grothendieck's insight $\to \mathrm{H}^i(X_{\text{ét}}, \mathbf{Q}_\ell)$...]

3.4.3. $\overline{\mathbf{Q}_\ell}$-sheaves. — Let k be an algebraically closed field, X a k-variety. Let ℓ be a prime invertible in k. Start with a sheaf $\mathscr{F}$ of $\mathbf{Z}/\ell^n$-modules on $X_{\text{ét}}$. We say that $\mathscr{F}$ is *constructible* if there is a finite partition $X = \bigsqcup X_\alpha$ of locally closed subsets such that $\mathscr{F}|_{X_\alpha}$ is locally constant, and moreover the stalks $\mathscr{F}_{\bar{x}}$ be finite for all geometric points $\bar{x}$ of X. Denote by $\mathrm{Sh}_c(X_{\text{ét}}, \mathbf{Z}/\ell^n)$ the category of constructible $\mathbf{Z}/\ell^n$-sheaves.

If $k = \mathbf{C}$, then there is a morphism of topoi $\varepsilon : X(\mathbf{C}) \to X_{\text{ét}}$, inducing an equivalence of categories $\varepsilon^* : \mathrm{Sh}_c(X_{\text{ét}}, \mathbf{Z}/\ell^n) \xrightarrow{\sim} \mathrm{Sh}_c(X(\mathbf{C}), \mathbf{Z}/\ell^n)$.

A *constructible sheaf of $\mathbf{Z}_\ell$-modules* is a family $(\mathscr{F}_n, f_{n+1} : \mathscr{F}_{n+1} \to \mathscr{F}_n)$ such that

1. $\mathscr{F}_n \in \mathrm{Sh}_c(X_{\text{ét}}, \mathbf{Z}/\ell^n)$
2. f_{n+1} induces an isomorphism $\mathscr{F}_{n+1} \otimes \mathbf{Z}/\ell^n \xrightarrow{\sim} \mathscr{F}_n$

Note that we could have just talked about sheaves of $\mathbf{Z}_\ell$-modules, but this does not give us an interesting category. [pro-étale topology lets you avoid this] We define

$$\hom_{\mathbf{Z}_\ell}(\mathscr{F},\mathscr{G}) = \varprojlim_n \hom(\mathscr{F}_n,\mathscr{G}_n).$$

This gives us a perfectly good category $\mathrm{Sh}_c(X_{\text{ét}},\mathbf{Z}_\ell)$.

Next we define a category of constructible $\mathbf{Q}_\ell$-sheaves. Its objects are constructible $\mathbf{Z}_\ell$-sheaves, but morphisms are

$$\hom_{\mathbf{Q}_\ell}(\mathscr{F},\mathscr{G}) = \hom_{\mathbf{Z}_\ell}(\mathscr{F},\mathscr{G}) \otimes \mathbf{Q}.$$

In other words, $\mathrm{Sh}_c(X_{\text{ét}},\mathbf{Q}_\ell)$ is the localization of $\mathrm{Sh}_c(X_{\text{ét}},\mathbf{Z}_\ell)$ at all morphisms of the form "multiply ℓ."

If E is a finite extension of $\mathbf{Q}_\ell$, we can repeat the whole process for E (start with $O_E/\mathfrak{p}^n$, take a projective limit, then tensor with $\mathbf{Q}$). This gives us a category $\mathrm{Sh}_c(X_{\text{ét}},E)$. If $E \subset E'$ are finite extensions of $\mathbf{Q}_\ell$, we get a functor $\mathrm{Sh}_c(X_{\text{ét}},E) \to \mathrm{Sh}_c(X_{\text{ét}},E')$, written $\mathscr{F} \mapsto \mathscr{F} \otimes_E E'$. One has

$$\hom_{E'}(\mathscr{F} \otimes_E E', \mathscr{G} \otimes_E E') = \hom(\mathscr{F},\mathscr{G}) \otimes_E E'.$$

Finally we construct the category of $\overline{\mathbf{Q}_\ell}$-sheaves. Its objects consist of constructible E-sheaves for varying finite extensions E of $\mathbf{Q}_\ell$. If $\mathscr{F}$ is an E-sheaf and $\mathscr{G}$ is an E'-sheaf, choose a finite extension F containing E and E', and define

$$\hom_{\overline{\mathbf{Q}_\ell}}(\mathscr{F},\mathscr{G}) = \hom(\mathscr{F} \otimes_E F, \mathscr{G} \otimes_{E'} F) \otimes_F \overline{\mathbf{Q}_\ell}.$$

Our category $\mathrm{Sh}_c(X_{\text{ét}},\overline{\mathbf{Q}_\ell})$ should be thought of the analogue of the category of constructible $\mathbf{C}$-sheaves on the space $X(\mathbf{C})$ for X a complex variety. When there is no danger of confusion, write $\mathrm{Sh}_c(X)$ instead of $\mathrm{Sh}_c(X_{\text{ét}},\overline{\mathbf{Q}_\ell})$.

If $\mathscr{F} \in \mathrm{Sh}_c(X)$, then $\mathscr{F}$ is a *local system* (or a smooth sheaf) if each $\mathscr{F}_n$ is locally constant. *Warning*: $\mathscr{F}$ is not necessarily globally locally constant, i.e. there may not exist a partition that simultaneously trivializes each $\mathscr{F}_n$. If $\mathscr{F} \in \mathrm{Sh}_c(X)$ and $\bar{x}$ is a geometric point of X, then we define the stalk of $\mathscr{F}$ at $\bar{x}$ by

$$\mathscr{F}_{\bar{x}} = (\varprojlim_n \mathscr{F}_{n,\bar{x}}) \otimes \overline{\mathbf{Q}_\ell}.$$

These are finite-dimensional $\overline{\mathbf{Q}_\ell}$-vector spaces.

As in the classical case, when can define a "bounded derived category of constructible sheaves" $\mathsf{D}^b_c(X)$. Its objects are complexes $\mathscr{K}^\bullet$ with $\mathscr{H}^\bullet\mathscr{K}^\bullet$ being constructible $\overline{\mathbf{Q}_\ell}$-sheaves. For any morphism $f : X \to Y$, we have functors

$$\begin{aligned} f_* &: \mathsf{D}^b_c(X) \to \mathsf{D}^b_c(Y) \\ f_! &: \mathsf{D}^b_c(X) \to \mathsf{D}^b_c(Y) \\ f^* &: \mathsf{D}^b_c(Y) \to \mathsf{D}^b_c(X) \\ f^! &: \mathsf{D}^b_c(Y) \to \mathsf{D}^b_c(X). \end{aligned}$$

If f is proper, then $f_* = f_!$, and if f is an open immersion, then $f^* = f^!$.

3.4.4. Analogy with functions. — Let $f : E \to F$ be a map of finite sets. Let k be a field, and write k^E, k^F for the spaces of functions $E \to k$, $F \to k$. We have functions

$$\begin{aligned} f^* &: k^F \to k^E & h &\mapsto h \circ f \\ f_* &: k^E \to k^F & h &\mapsto x \mapsto \sum_{y \in f^{-1}(x)} h(y) \end{aligned}$$

which should be thought of as analogies of our functors between derived categories.

3.4.5. Connection with cohomology. — Let X be a variety over an algebraically closed field k. Let $p : X \to \{\mathrm{pt}\}$ be the unique morphism. For $\mathscr{K} \in \mathsf{D}^b_c(X)$, we define

$$\mathbf{H}^i(X, \mathscr{K}) = \mathscr{H}^i(p_* \mathscr{K}^\bullet)$$
$$\mathbf{H}^i_c(X, \mathscr{K}^\bullet) = \mathscr{H}^i(p_! \mathscr{K}^\bullet)$$

In particular, for $\mathscr{K}^\bullet = \cdots \to 0 \to \overline{\mathbf{Q}_\ell} \to 0 \to \cdots$, we put

$$\mathrm{H}^i(X, \overline{\mathbf{Q}_\ell}) = \mathbf{H}^i(X, \mathscr{K}^\bullet)$$
$$\mathrm{H}^i_c(X, \overline{\mathbf{Q}_\ell}) = \mathbf{H}^i_c(X, \mathscr{K}^\bullet)$$

More directly, one has

$$\mathrm{H}^i_c(X, \overline{\mathbf{Q}_\ell}) = \left(\varprojlim \mathrm{H}^i_c(X_{\text{ét}}, \mathbf{Z}/\ell^n) \right) \otimes \overline{\mathbf{Q}_\ell}.$$

3.4.6. Verdier duality. — We have a functor $D_X : \mathsf{D}^b_c(X) \to \mathsf{D}^b_c(X)$ such that

1. $D_X^2 = 1$
2. if $f : X \to Y$ is a morphism, then $D_Y f_! = f_* D_X$ and $f^! D_Y = D_X f^*$
3. $\mathbf{H}^{-i}_c(X, \mathscr{K}^\bullet) = \mathbf{H}^i(X, D_X \mathscr{K}^\bullet)$
4. if $\mathscr{E}$ is a local system and X is smooth, then $D_X(\mathscr{E}[\dim X]) = \mathscr{E}^\vee[\dim X]$

In particular, if $\mathscr{E} \simeq \mathscr{E}^\vee$, then $\mathrm{H}^i_c(X, \mathscr{E}) \simeq \mathrm{H}^{2\dim X - i}(X, \mathscr{E})$.

3.4.7. Intersection cohomology. — Let Y be an irreducible nonsingular locally closed subset of X. Goresky, MacPherson and Deligne defined, for any local system $\mathscr{E}$ on Y, a complex $\mathrm{IC}(\overline{Y}, \mathscr{E}) \in \mathsf{D}^b_c(\overline{Y})$ such that

$$D_{\overline{Y}}(\mathrm{IC}(\overline{Y}, \mathscr{E})[\dim Y]) \simeq \mathrm{IC}(\overline{Y}, \mathscr{E}^\vee)[\dim Y].$$

The complex $\mathscr{K} = \mathrm{IC}(\overline{Y}, \mathscr{E})[\dim Y]$ is characterized by the following properties:

1. $\mathscr{H}^{-\dim Y} \mathscr{K}^\bullet|_Y \simeq \mathscr{E}$
2. $\mathscr{H}^i \mathscr{K}^\bullet = 0$ if $i < -\dim Y$
3. $\dim(\operatorname{supp} \mathscr{H}^i \mathscr{K}^\bullet) < -i$ for $i > -\dim Y$
4. $\dim(\operatorname{supp} \mathscr{H}^i D \mathscr{K}^\bullet) < -i$ if $i > -\dim Y$

If Y is smooth, then $\mathrm{IC}(Y, \mathscr{E}) \simeq \mathscr{E}$.

3.4.1 Exercise. — Assume Z is a nonsingular open subset of $\overline{Y}$. If $\mathscr{L}$ is a local system on Z such that $\mathscr{L}|_{Z \cap Y} \simeq \mathscr{E}|_{Z \cap Y}$, then $\mathrm{IC}(\overline{Y}, \mathscr{E}) \simeq \mathrm{IC}(\overline{Y}, \mathscr{L})$.

Exercise

3.4.8. Perverse sheaves. — We define $\mathsf{Pv}(X)$ to be the full subcategory of $\mathscr{K}^\bullet \in \mathsf{D}_c^b(X)$ such that

1. $\dim(\operatorname{supp} \mathscr{H}^i \mathscr{K}^\bullet) \leqslant -i$
2. $\dim(\operatorname{supp} \mathscr{H}^i D\mathscr{K}^\bullet) \leqslant -i$

Clearly $\mathsf{Pv}(X)$ contains all $\mathrm{IC}(\overline{Y}, \mathscr{E})[\dim Y]$ extended by zero on $X \smallsetminus \overline{Y}$.

***3.4.2 Theorem* (Beilinson, Bernstein, Deligne, Gabber)**
Any simple perverse sheaf is of the form $\mathrm{IC}(\overline{Y}, \mathscr{E})[\dim Y]$ *extended by zero on* $X \smallsetminus \overline{Y}$.

As in the classical case, $\mathsf{Pv}(X)$ is a semisimple abelian category in which all objects have finite length. The inclusion $\mathsf{Pv}(X) \hookrightarrow \mathsf{D}_b^c(X)$ induces an equivalence between the derived category of $\mathsf{Pv}(X)$ and $\mathsf{D}_b^c(X)$.

Warning: The functors $f_!, f^!, f_*, f^*$ do not always carry one category of perverse sheaves into another (i.e. they do not preserve "perversity").

3.4.9. Stratifications. — We say that $X = \bigsqcup X_\alpha$ is a *stratification* of X if it is a finite partition of X into equidimensional nonsingular locally closed subsets of X, such that if $X_\beta \cap \overline{X_\alpha}$, then $X_\beta \subset \overline{X_\alpha}$.

3.4.3 Proposition. — *Let* $f : X \to Y$ *be a proper surjective map with* X *irreducible (or just equidimensional), and let* $X = \bigsqcup X_\alpha$ *be a stratification of* X*. For* $y \in Y$*, put* $f^{-1}(y)_\alpha = f^{-1}(y) \cap X_\alpha$*. Assume that*

$$\dim \left\{ y \in Y : \dim f^{-1}(y)_\alpha \geqslant \frac{1}{2}(i - \operatorname{codim} X_\alpha) \right\} \leqslant \dim Y - i$$

for all α *and* i*. Then* $f_* : \mathsf{Pv}(X) \to \mathsf{Pv}(Y)$ *is well-defined.*

3.4.4 Corollary. — *If* $f : X \to Y$ *is surjective, proper, and* X *is nonsingular irreducible, then if for all* i*,*

$$\dim \left\{ y \in Y : \dim f^{-1}(y) \geqslant i \right\} \leqslant \dim Y - 2i \tag{3}$$

the functor $f_* : \mathsf{Pv}(X) \to \mathsf{Pv}(Y)$ *is well-defined.*

A surjective proper map satisfying (3) is called *semi-small.*

3.4.5 Exercise. — Let k be an algebraically closed field. Consider the varieties

$$\begin{aligned} \mathfrak{gl}_2(k) &= \text{affine space over } k \\ W &= \{\text{nilpotent matrices}\} \\ B &= \begin{pmatrix} * & * \\ & * \end{pmatrix} \end{aligned}$$

Let

$$\widetilde{W} = \left\{ (x, gB) \in \mathfrak{gl}_2 \times \mathrm{GL}_2 / B : g^{-1} x g \in \begin{pmatrix} & * \\ & \end{pmatrix} \right\}.$$

Show that the map $\widetilde{W} \to W$, $(x, gB) \mapsto x$ is semi-small.

3.4.6 Theorem **(special case of the decomposition theorem)**
If $f : X \to Y$ is a proper smooth map, then $f_*(\overline{\mathbf{Q}_\ell})$ is semisimple.

3.4.7 Corollary. — *If $f : X \to Y$ is semi-small, X nonsingular irreducible, then*

$$f_*(\overline{\mathbf{Q}_\ell}[\dim X]) \simeq \mathrm{IC}(Y, \overline{\mathbf{Q}_\ell})[\dim Y] \oplus \bigoplus_{\substack{(Z,\xi) \\ \overline{Z} \subsetneq Y}} V_{Z,\xi} \otimes \mathrm{IC}(\overline{Z}, \xi)[\dim Z]$$

In terms of cohomology, this tells us that

$$\mathrm{H}^i_c(X, \overline{\mathbf{Q}_\ell}) \simeq \mathrm{IH}^i_c(\overline{Y}, \overline{\mathbf{Q}_\ell}) \oplus \bigoplus_{(Z,\xi)} V_{Z,\xi} \otimes \mathrm{IH}^{i+\bullet}_c(\overline{Z}, \xi).$$

3.4.10. F-equivariance. — Suppose X is defined over $\mathbf{F}_q$ with (relative) Frobenius $F : X \to X$. We have a functor $F^* : \mathsf{D}^b_c(X) \to \mathsf{D}^b_c(X)$. We say that $\mathscr{K} \in \mathsf{D}^b_c(X)$ is *F-stable* if $F^*\mathscr{K} \simeq \mathscr{K}$. An *$F$-equivariant complex* on X is a pair $(\mathscr{K}, \phi)$ with $\phi : F^*\mathscr{K} \xrightarrow{\sim} \mathscr{K}$. Morphisms $\varphi : (\mathscr{K}, \phi) \to (\mathscr{K}', \phi')$ are defined via the commutative diagram

$$\begin{tikzcd} F^*\mathscr{K} \arrow[r, "F^*\varphi"] \arrow[d, "\phi"] & F^*\mathscr{K}' \arrow[d, "\phi'"] \\ \mathscr{K} \arrow[r, "\varphi"] & \mathscr{K}' \end{tikzcd}$$

Write $\mathsf{D}^b_c(X)_F$ for the category of F-equivariant complexes.

3.4.11. Characteristic function of $(\mathscr{K}, \phi)$. — We define $\chi_{\mathscr{K},\phi} : X^F = X(\mathbf{F}_q) \to \overline{\mathbf{Q}_\ell}$ by

$$x \mapsto \sum_i (-1)^i \operatorname{tr}\left(\phi^i_x, \mathscr{H}^i_x\mathscr{K}\right).$$

3.4.8 Lemma. — *If $\mathscr{K} \simeq \mathscr{K}'$ are simple perverse sheaves, and if $\phi : F^*\mathscr{K} \xrightarrow{\sim} \mathscr{K}$ and $\phi' : F^*\mathscr{K}' \xrightarrow{\sim} \mathscr{K}'$, then there is a unique $c_{\phi\phi'} \in \overline{\mathbf{Q}_\ell}$ such that $\chi_{\mathscr{K},\phi} = c_{\phi\phi'}\chi_{\mathscr{K}',\phi'}$. If $c_{\phi\phi'} = 1$, then $(\mathscr{K}, \phi) \simeq (\mathscr{K}', \phi')$.*

Any $\phi : F^*\mathscr{E} \to \mathscr{E}$, with $\mathscr{E}$ a local system on Y (an F-stable, nonsingular, locally closed subset of X) induces a canonical $\phi : F^*\,\mathrm{IC}(\overline{Y}, \mathscr{E})) \xrightarrow{\sim} \mathrm{IC}(\overline{Y}, \mathscr{E})$.

3.4.9 Proposition. — *If $f : X \to Y$ commutes with Frobenius F, then $f_! : \mathsf{D}^b_c(X)_F \to \mathsf{D}^b_c(Y)_F$ and $f^* : \mathsf{D}^b_c(Y)_F \to \mathsf{D}^b_c(X)_F$.*

Proof. — Let $(\mathscr{K}, \phi) \in \mathsf{D}^b_c(X)_F$. From $\phi : F^*\mathscr{K} \xrightarrow{\sim} \mathscr{K}$ we get $f_!F^*\mathscr{K} \xrightarrow{f_!\phi} f_!\mathscr{K}$. We apply the proper base change theorem to $f : X \to Y$ to get a canonical isomorphism $F^*f_! \simeq f_!F^*$. From all this we get $\widetilde{\phi} : F^*f_!\mathscr{K} \xrightarrow{\sim} f_!\mathscr{K}$. Our functor is $(\mathscr{K}, \phi) \mapsto (f_!\mathscr{K}, \widetilde{\phi})$. □

3.4.10 Theorem **(trace formula).** — *Assume $f : X \to Y$ commutes with F. This gives $f : X^F \to Y^F$ between finite sets. Then*

1. $f_*\chi_{\mathscr{K},\phi} = \chi_{f_!(\mathscr{K},\phi)}$, i.e. the following diagram commutes:

$$\begin{tikzcd} \mathrm{Fun}(X^F, \overline{\mathbf{Q}_\ell}) \arrow[r, "f_!"] & \mathrm{Fun}(Y^F, \overline{\mathbf{Q}_\ell}) \\ \mathsf{D}^b_c(X)_F \arrow[u, "\chi"] \arrow[r, "f_!"] & \mathsf{D}^b_c(Y)_F \arrow[u, "\chi"'] \end{tikzcd}$$

2. The following diagram commutes:

$$\begin{tikzcd} \mathrm{Fun}(Y^F, \overline{\mathbf{Q}_\ell}) \arrow[r, "f^*"] & \mathrm{Fun}(X^F, \overline{\mathbf{Q}_\ell}) \\ \mathsf{D}^b_c(Y)_F \arrow[u, "\chi"] \arrow[r, "f^*"] & \mathsf{D}^b_c(X)_F \arrow[u, "\chi"'] \end{tikzcd}$$

The proof uses Grothendieck's trace formula. For any $(\mathscr{K}, \phi) \in \mathsf{D}^b_c(X)_F$, ϕ gives us an action of Frobenius on $\mathbf{H}^i_c(X, \mathscr{K})$. The theorem is that

$$\sum_{x \in X^F} \chi_{\mathscr{K},\phi}(x) = \sum_i (-1)^i \operatorname{tr}(F, \mathbf{H}^i_c(X, \mathscr{K})).$$

In particular, if $\mathscr{K} = \overline{\mathbf{Q}_\ell}$ and $\phi : F^*\overline{\mathbf{Q}_\ell} \to \overline{\mathbf{Q}_\ell}$ induces the identity on stalks of X^F, then

$$\# X^F = \sum (-1)^i \operatorname{tr}(F^*, \mathrm{H}^i_c(X, \overline{\mathbf{Q}_\ell})).$$

The website of Alberto Arabia has some great notes on all of this. The following fixes an error.

3.4.11 Proposition. — *Let $f : X \to Y$ be a surjective proper map with X irreducible, and let $X = \bigsqcup X_\alpha$ be a stratification of X. For $y \in Y$, put $f^{-1}(y)_\alpha = X_\alpha \cap f^{-1}(y)$. Assume*

$$\dim \left\{ y \in Y : \dim f^{-1}(y)_\alpha \geqslant \frac{1}{2}(i - \operatorname{codim} X_\alpha) \right\} \leqslant \dim Y - i$$

for all α and i. Then $f_*(\mathrm{IC}(X, \mathscr{E})[\dim X]) \in \mathsf{Pv}(Y)$ for $\mathscr{E}$ a local system on X_0.

Let $\mathscr{K} \in \mathsf{Pv}(X)$ obtained as $\mathscr{K} = i_*\mathscr{K}'$ for $\mathscr{K}' \in \mathsf{Pv}(X')$, where $X' \subset X$ is closed. [... stuff I didn't understand...]

3.4.12 Example. — Let $G = \mathrm{GL}_2(\mathbf{C})$, $B \subset G$ the standard Borel, and $U \subset B$ the unipotent radical. Let N be the (2-dimensional) variety of nilpotent matrices in $\mathfrak{gl}_2$. Let $\widetilde{W} = \{(x, gB) \in \mathfrak{gl}_2 \times G/B : g^{-1}xg \in \mathrm{Lie}(U)\}$. There is a map $\widetilde{W} \to W$, called the Springer resolution. It fits into a cartesian diagram

$$\begin{tikzcd} \mathbf{P}^1 \simeq G/N \arrow[r, "\pi^0"] \arrow[d] & \{0\} \arrow[d] \\ \widetilde{W} \arrow[r, "\pi"] & W \end{tikzcd}$$

The complex $\overline{\mathbf{Q}_\ell}[1]$ is a pervrse sheaf on G/B. The Springer resolution is semismall. However, $\pi^0(\overline{\mathbf{Q}_\ell}[1]) = \overline{\mathbf{Q}_\ell} \oplus \overline{\mathbf{Q}_\ell}[2]$, which is not perverse. On the other hand, $\overline{\mathbf{Q}_\ell}[2] \in \mathsf{Pv}(\widetilde{M})$, and $\pi_*\overline{\mathbf{Q}_\ell}[2] = \mathrm{IC}(W, \overline{\mathbf{Q}_\ell}) \oplus \mathrm{IC}(\{0\}, \overline{\mathbf{Q}_\ell})$. In other words, $\pi_*(\overline{\mathbf{Q}_\ell}[2])|_{\{0\}} \simeq \pi_*^0(\overline{\mathbf{Q}_\ell}[1])[1]$.

3.4.12. G-equivariance. — Let G be a connected linear algebraic group defined over $\mathbf{F}_q$. Let X be a variety defined over $\mathbf{F}_q$. Assume we are given an action of G on X, also defined over $\mathbf{F}_q$. Let $\pi : G \times X \to X$ be the projection map, and let $\rho : G \times X \to X$ be the map encoding the action of G on X. We say that a perverse sheaf $\mathscr{K} \in \mathsf{Pv}(X)$ is *G-equivariant* if $\pi^*\mathscr{K} \simeq \rho^*\mathscr{K}$. We would like to create a category $\mathsf{Pv}_G(X)$ of "G-equivariant perverse sheaves" on X. Let $\alpha : G \times G \to G$ be the multiplication morphism, let $p_2 : G \times G \to G$ be projection onto the second coordinate, and let $i : X \to G \times X$ be the inclusion $x \mapsto (1, x)$.

3.4.13 Lemma. — *Assume $\mathscr{K} \in \mathsf{Pv}(X)$ is G-equivariant. Then there is a unique isomorphism $\phi : \pi^*\mathscr{K} \xrightarrow{\sim} \rho^*\mathscr{K}$ such that*

1. *$i^*\phi = 1_{\mathscr{K}}$*
2. *$(\alpha \times 1_X)^*\phi = (1_G \times \rho)^*\phi \circ (p_2 \times 1_X)^*\phi$*

With this under our belt, we can define the category $\mathsf{Pv}_G(X)$ to be the subcategory of $\mathsf{Pv}(X)$ whose objects are G-equivariant perverse sheaves on X. Morphisms in $\mathsf{Pv}_G(X)$ are isomorphisms $\psi : \mathscr{K} \to \mathscr{K}'$ such that the following diagram commutes:

$$\begin{tikzcd}
\pi^*\mathscr{K} \arrow[r, "\pi^*\psi"] \arrow[d, "\phi_{\mathscr{K}}"] & \pi^*\mathscr{K}' \arrow[d, "\phi_{\mathscr{K}'}"] \\
\rho^*\mathscr{K} \arrow[r, "\rho^*\psi"] & \rho^*\mathscr{K}'
\end{tikzcd}$$

3.4.14 Lemma. — *The category $\mathsf{Pv}_G(X)$ is a full subcategory of $\mathsf{Pv}(X)$.*

3.4.15 Proposition. — *1. The duality functor D_X induces an auto-equivalence $D_X : \mathsf{Pv}_G(X) \xrightarrow{\sim} \mathsf{Pv}_G(X)$.*

2. If $\mathscr{K} \in \mathsf{Pv}_G(X)$, then so are all subquotients of $\mathscr{K}$.

3. The simple objects of $\mathsf{Pv}_G(X)$ are the $\mathrm{IC}(\overline{Y}, \mathscr{E})[\dim Y]$, for $\mathscr{E}$ a G-equivariant local system on a G-stable closed subset $Z \subset Y$.

If $\mathscr{K} \in \mathsf{Pv}_G(X)$ is F-stable with $\phi : F^*\mathscr{K} \xrightarrow{\sim} \mathscr{K}$, then the characteristic function $\chi_{\mathscr{K},\phi} : X(\mathbf{F}_q) \to \overline{\mathbf{Q}_\ell}$ is constant on $G(\mathbf{F}_q)$-orbits in $X(\mathbf{F}_q)$. We will use this fact to construct two natural bases for $\mathcal{C}_{G^F}(X^F)$, the space of G^F-equivariant functions $X^F \to \overline{\mathbf{Q}_\ell}$. Consider pairs (O, ξ), where O is a G-orbit in X, and ξ is an irreducible G-equivariant local system on O. We say that $(O, \xi) \simeq (O', \xi')$ if $O = O'$ and $\xi \simeq \xi'$. Let I be the set of isomorphism classes of F-stable pairs (O, ξ). For each $i \in I$, choose a representative (O_i, ξ_i), together with an isomorphism $\phi_i : F^*\xi_i \to \xi_i$.

For each $i \in I$, define a functions $X_i, Y_i : X^F \to \overline{\mathbf{Q}_\ell}$ by $x \mapsto \chi_{\mathrm{IC}(\overline{O_i},\xi_i),\phi_i}$ and $x \mapsto \mathrm{tr}(\phi_i, (\xi_i)_x)$. Then $\{X_i\}$ and $\{Y_i\}$ are $\overline{\mathbf{Q}_\ell}$-bases of $\mathcal{C}_{G^F}(X^F)$. If $\mathrm{Stab}_G(O) = \mathrm{Stab}_G(O)^\circ$, then there is only one irreducible G-equivariant local system (up to isomorphism) on O – namely $\overline{\mathbf{Q}_\ell}$. In that case, we can choose ϕ_i so that $\phi_i : F^*\overline{\mathbf{Q}_\ell} \to \overline{\mathbf{Q}_\ell}$ induces the identity on stalks. In that case, $Y_i = 1_{O_i}$. In general, O_i^F is a disjoint union of G^F-orbits.

3.5. Geometric realization of the unipotent characters of $\mathrm{GL}_n(\mathbf{F}_q)$. —

3.5.1. Parabolic induction. — Throughout, everything lives over $\overline{\mathbf{F}_q}$.

First we do this with functions. Let $L \subset \mathrm{GL}_n$ be a standard Levi associated to a partition $\lambda \vdash n$. Let P be the associated parabolic; note that $P = L \ltimes U_P$, where U is the unipotent radical of P. Let $F : \mathrm{GL}_n \to \mathrm{GL}_n$ be the Frobenius map (over $\mathbf{F}_q$). Then $\mathrm{GL}_n^F = \mathrm{GL}_n(\mathbf{F}_q)$, and $\mathcal{C}(\mathrm{GL}_n^F)$ consists of functions $\mathrm{GL}_n(\mathbf{F}_q) \to \overline{\mathbf{Q}_\ell}$ that are constant on conjugacy classes. We define $R_L^{\mathrm{GL}_n} : \mathcal{C}(L^F) \to \mathcal{C}(\mathrm{GL}_n^F)$ via the following. Let $\pi_P : P \to L$ be the projection $\ell u \mapsto u$. We define $R_L^{\mathrm{GL}_n}$ to be the composite

$$\mathcal{C}(L^F) \xrightarrow{\pi_P^*} \mathcal{C}(P^F) \xrightarrow{\mathrm{ind}_{P^F}^{\mathrm{GL}_n^F}} \mathcal{C}(\mathrm{GL}_n^F).$$

We also define

$$\begin{aligned} V_1 &= \{(x,g) \in \mathrm{GL}_n^2 : g^{-1}xg \in P\} \\ V_2 &= \{(x,gP) \in \mathrm{GL}_n \times \mathrm{GL}_n / P : g^{-1}xg \in P\} \end{aligned}$$

We define maps

$$\begin{aligned} \pi : V_1 &\to L & (x,g) &\mapsto \pi_P(g^{-1}xg) \\ \rho_1 : V_1 &\to V_2 & (x,g) &\mapsto (x,gP) \\ \rho_2 : V_2 &\to \mathrm{GL}_n & (x,gP) &\mapsto x. \end{aligned}$$

Note that $R_L^{\mathrm{GL}_n} = (\rho_2) + *\widetilde{\rho}_1\pi^*$, where $\widetilde{\rho}_1 : \mathcal{C}_{\mathrm{GL}_n \times P}(V_1^F) \to \mathcal{C}_{\mathrm{GL}_n}(V_2^F)$ sends f to the unique map h on V_2^F such that $\rho_1^* h = f$.

Now we repeat the above, but with perverse sheaves instead of functions. We will define a functor $R_L^{\mathrm{GL}_n} : \mathsf{Pv}_L(L) \to \mathsf{D}_c^b(\mathrm{GL}_n)$. First, note that our morphism π is smooth with connected fibers of dimension $m = \dim \mathrm{GL}_n + \dim P$. Thus $\pi^*[m] : \mathsf{Pv}_L(L) \to \mathsf{Pv}_{\mathrm{GL}(n)\times P}(V_1)$ is well-defined, where $\mathrm{GL}(n)$ acts on V_1 by $g \cdot (x,h) = (gx, gh)$.

The map ρ_1 is a ocally trivial fibration for the Zariski topology. Thus $\rho_1^*[\dim P] : \mathsf{Pv}_{\mathrm{GL}(n)}(V_2) \xrightarrow{\sim} \mathsf{Pv}_{\mathrm{GL}(n)\times P}(V_1)$. It follows that for each $\mathscr{K} \in \mathsf{Pv}_L(L)$, there is a unique $\widetilde{\mathscr{K}} \in \mathsf{Pv}_{\mathrm{GL}(n)}(V_2)$ such that $\rho_1^*[\dim P]\widetilde{\mathscr{K}} \simeq \pi^*[m]\mathscr{K}$. Define $R_L^{\mathrm{GL}(n)}(\mathscr{K}) = (\rho_2)_*\widetilde{\mathscr{K}}$; note that this is "exactly the same" as our definition of $R_L^{\mathrm{GL}(n)}$ for functions. Again, note that $R_L^{\mathrm{GL}(n)}\mathscr{K}$ is *not* generally perverse, even when $\mathscr{K}$ is.

For simplicity, write $G = \mathrm{GL}(n)_{\overline{\mathbf{F}_q}}$.

If $\phi : F^*\mathscr{K} \xrightarrow{\sim} \mathscr{K}$, then R_L^G induces a canonical $\widetilde{\phi} : F^* R_L^G \mathscr{K} \xrightarrow{\sim} R_L^G \mathscr{K}$. So the "Harish-Chandra induction" $R_L^G : \mathcal{C}(L^F) \to \mathcal{C}(G^F)$ has an analogue $R_L^G : \mathsf{Pv}_L(L)_F \to \mathsf{D}_c^b(G)_F$. If we let χ denote "take characteristic function" then the following diagram commutes:

$$\begin{tikzcd} \mathsf{Pv}_L(L)_F \arrow[r, "R_L^G"] \arrow[d, "\chi"] & \mathsf{D}_c^b(G)_F \arrow[d, "\chi"] \\ \mathcal{C}(L^F) \arrow[r, "R_L^G"] & \mathcal{C}(G^F) \end{tikzcd}$$

3.5.2. Characters of $\mathrm{GL}_n(\mathbf{F}_q)$. — Let $R_{(1^n)} = \sum_{\lambda \in \mathcal{P}_n} \chi_{(1^n)}^{\lambda'} \mathscr{X}^\lambda = R_L^G(\mathrm{id})$. Here $\mathscr{X}^\lambda$ is the unipotent character of $\mathrm{GL}_n(\mathbf{F}_q)$ associated to λ, and λ' is the dual partition of λ. For $\mu \in \mathcal{P}_n$, let $R_\mu = \sum_{\lambda \in \mathcal{P}_n} \chi_\mu^{\lambda'} \mathscr{X}^\lambda = \mathrm{Ch}^{-1}(P_\mu(\mathrm{id}))$. We can invert things to give $\mathscr{X}^\lambda = \sum_\mu \frac{1}{z_\mu} \chi_\mu^{\lambda'} R_\mu$. Recall that $\{R_\mu\}$ and $\{\mathscr{X}^\lambda\}$ are both bases of the same subspace of $\mathcal{C}(G^F)$.

3.5.3. Induction picture for $L = T$. — Let $T \subset G$ be the standard maximal torus. We have maps

$$T \leftarrow \{(g, hB) \in G \times G/B : h^{-1}gh \in B\} \to G.$$

Note that $R_L^G(\overline{\mathbf{Q}_\ell}[\dim T]) = (\rho_2)_* \overline{\mathbf{Q}_\ell}[\dim G]$.

3.5.1 Theorem. — *The map ρ_2 is semi-small (small, in fact). Moreover $(\rho_2)_*(\overline{\mathbf{Q}_\ell}[\dim G])$ is a semisimple G-equivariant perverse sheaf on G.*

In fact, we can prove that $(\rho_2)_*(\overline{\mathbf{Q}_\ell}[\dim G]) \simeq \mathrm{IC}(G, \xi)[\dim G]$. We need to describe ξ. Note that $g \in G$ is semisimple regular if and only if $C_G(g)$ is conjugate to T. Consider G_{reg}, the subset of G consisting of semisimple regular elements. This is a nonsingular irreducible Zariski-open subset of G. Let

$$X_2 = \{(g, hT) \in G \times G/T : h^{-1}gh \in T_{\mathrm{reg}}\}.$$

We have the following diagram:

$$\begin{tikzcd} V_2 \arrow[r] & G \\ X_2 \arrow[u] \arrow[r, "\alpha"] & G_{\mathrm{reg}} \arrow[u] \end{tikzcd} \tag{4}$$

Recall that $S_n = N_G(T)/T$, and this acts on X_2 by $w \cdot (g, hT) = (g, hw^{-1}T)$. The map α is a Galois covering with Galois group S_n. One has

$$\alpha_* \overline{\mathbf{Q}_\ell} = \bigoplus_{\chi \in \mathrm{Irr}(S_n)} \mathscr{L}_\chi^{\chi(1)}$$

It turns out that the diagram (4) is Cartesian. The base-change theorem tells us that $\rho_2(\overline{\mathbf{Q}_\ell}[\dim G])|_{G_{\mathrm{reg}}} \simeq \alpha_*(\overline{\mathbf{Q}_\ell}[\dim G])$. We know that $\rho_2(\overline{\mathbf{Q}_\ell}[\dim G]) \simeq \mathrm{IC}(G, \xi)[\dim G]$.

We can take ξ to be $\alpha_*\overline{\mathbf{Q}_\ell}$ on G_{reg}. As a result, we get

$$(\rho_2)_*(\overline{\mathbf{Q}_\ell}[\dim G]) \simeq \bigoplus_{\chi\in\mathrm{Irr}(S_n)} V_\chi \otimes \mathrm{IC}(G, \mathscr{L}_\chi)[\dim G].$$

Beause S_n acts on $X-2$, we can define an action of S_n on $\alpha_*\overline{\mathbf{Q}_\ell}$. For $w \in S_n$, we get $\theta_w : R_T^G(\overline{\mathbf{Q}_\ell}) \xrightarrow{\sim} R_T^G(\overline{\mathbf{Q}_\ell})$ and $F^* R_T^G\overline{\mathbf{Q}_\ell} \xrightarrow{\widetilde{\phi}} R_T^G\overline{\mathbf{Q}_\ell} \to \theta_w R_T^G\overline{\mathbf{Q}_\ell}$.

***3.5.2 Theorem* (Lusztig, 1998).** — *If $w \in S_n$ is of cycle type and $\mu \in \mathcal{P}_n$, then*

$$(-1)^* X_{R_T^G(\overline{\mathbf{Q}_\ell}),\theta_w\widetilde{\phi}} = R_\mu.$$

***3.5.3 Theorem* (Lusztig, 1981).** — *1.* $\mathrm{IC}(G, \mathscr{L}_{X^\lambda})|_{G_{\mathrm{reg}}} \simeq \mathrm{IC}(\overline{C_\lambda}, \overline{\mathbf{Q}_\ell})$.

2. $q^{n(\lambda)} X_{\mathrm{IC}(\overline{C_\lambda},\overline{\mathbf{Q}_\ell})}(c_\mu) = \widetilde{K}_{\lambda\mu}(q)$.

3. $X_{\mathrm{IC}(\overline{C_\chi},\overline{\mathbf{Q}_\ell})}(c_\mu) = \sum_i \dim \mathscr{H}^{2i}_{c_\mu} \mathrm{IC}(\overline{C_\lambda}) q^i$.

From 2 and 3, we get that $\widetilde{K}_{\lambda\mu} \in \mathbf{N}[q]$.

[here I write $X_?$ instead of $\chi_?$ for the character associated to a sheaf ?]

4. Bruhat-Tits theory

4.1. Introduction. — Bruhat-Tits theory is the non-archimedean analogue of the theory of symmetric spaces for semisimple real Lie groups

4.1.1 Example. — Put $G = \mathrm{SL}_2(\mathbf{R})$. This is a semisimple Lie group. To understand the structure of G, it is useful to use the G-action on its symmetric space $X = G/K$, where K is a maximal compact subgroup. The symmetric space X is naturally a C^∞ Riemannian manifold with nice metric properties. In fact, it is non other than the well-known upper-half plane $\{z \in \mathbf{C} : \Im z > 0\}$.

The starting point for Bruhat-Tits theory is a non-archimedean valued field k (instead of the *archimedean* valued field $\mathbf{R}$) and G, a semisimple algebraic k-group. We would like there to be a useful / interesting $G(k)$-space X such that the $G(k)$-action allows us to derive structure properties for the group $G(k)$ of rational points. This action should have certain properties:

1. transitivity, or some reasonable weakening of it
2. non-positive curvature properties for X

The group $G(k)$ and its homogeneous spaces are all totally disconnected. Usually, the maximal compact subgroups of $G(k)$ are *not* all conjugate. So we really need new ideas (rather than taking a homogeneous space) to attach a nice metric space with $G(k)$-action to G.

4.2. Euclidean buildings. —

4.2.1. Simplicial definition. — This definition is due to Jacques Tits, from the late 1950s. See the exercises in [**Bou02**, IV]. A good general reference is [**AB08**].

Roughly speaking, a Euclidean building is a simplicial complex covered by subcomplexes isomorphic to a given Coxeter tiling, with some incidence properties for the copies of the tiling. The “slices” in the building are called Coxeter complexes.

Let (W, S) be a Coxeter system. Then there (always) exists a polysimplicial complex (product of simplicial complexes) Σ, on the maximal cells of which W acts freely and transitively. As an ordered set, $\Sigma = \{wW_I : I \subset S, W_I = \langle I\rangle, w \in W\}$.

4.2.1 Example. — Let $W = D_\infty$, $S = \{s_0, s_1\}$ (reflections about 0 and 1). Then Σ is the real line tessellated by the integers.

4.2.2 Example. — This is the $\widetilde{A}_2$ case. Here Σ is $\mathbf{R}^2$, tiled by regular triangles.

The complex Σ can also be a spherical tiling (e.g. a tiling of the circle by similar segments) or a hyperbolic tiling. Here, we will only be interested in Euclidean tilings, and maybe spherical ones.

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4.2.3 Definition. — Let (W, S) be a Euclidean reflection group (i.e. an affine Coxeter group) with Coxeter tiling $\Sigma \subset \mathbf{R}^r$). Then X is said to be a Euclidean building of type (W, S) if it is a polysimplicial complex covered by copies of Σ (called the *apartments*) so that

SEB1) Any two cells (called *facets*) are contained in a suitable apartment.
SEB2) Given any two apartments $\mathbf{A}$, $\mathbf{A}'$, there is a simplicial automorphism $\mathbf{A} \simeq \mathbf{A}'$ fixing $\mathbf{A} \cap \mathbf{A}'$.

4.2.4 Example. — Buildings of type D_∞ correspond to trees in which all vertices have valence $\geqslant 2$. In other words, these are graphs without loops or leaves. Apartments are bi-infinite geodesics in the tree.

4.2.5 Example. — For $\widetilde{A}_2$, we now have the tiling of $\mathbf{R}^2$ by regular hexagons. Imagine gluings of half-tilings along codimension-1 walls.

We will see that the group $\mathrm{SL}_3(\mathbf{Q}_p)$ acts on an $\widetilde{A}_2$-tiling strongly transitively (i.e. the group action is transitive on the inclusions of a chamber (maximal facet) into apartments. In other words, for any $C \subset \mathbf{A}$ and $C' \subset \mathbf{A}'$, chambers living in apartments, there is $g \in \mathrm{SL}_2(\mathbf{Q}_3)$ such that $g\mathbf{A} = \mathbf{A}'$ and $gC = C'$. This is the substitute for homogeneity of symmetric spaces. The main outcome of Bruhat-Tits theory is that "to any reductive group G over a local field k, is attached a Euclidean building X on which $G(k)$ acts strongly transitively." (This is the so-called *geometric half* of Bruhat-Tits theory. There is another part, which investigates models for G over the valuation ring of k.)

4.2.2. Non-simplicial version of Euclidean buildings. — As a general convention, a *local field* is a locally compact topological field, endowed with a non-archimedean absolute value. Such fields are classified: they are

- finite extensions of $\mathbf{Q}_p$
- $\mathbf{F}_q((t))$ for q a prime power

What happens if k is not discretely valued? This case is also covered by Bruhat-Tits theory. However, when the valuation is not discrete, the building is no longer a simplicial complex. But it still admits a metric, and is still a complete metric space with non-positive curvature whenever k is complete. Why should we care about the case where k is not discretely valued?

1. Bruhat and Tits did.
2. When studying the space of (linear) representations of a given finitely-generated discrete group Γ: $X_n(\Gamma) = \{\varphi : \Gamma \to \mathrm{GL}(\mathbf{R})\}/\mathrm{conj}$, then $\varphi : \Gamma \to \mathrm{GL}_n(\mathbf{R})$ corresponds to an action of $\mathrm{GL}_n(\mathbf{R})/\mathrm{SO}(n) = X$. There exists a compactification procedure for $X_n(\Gamma)$, such that the added points at ∞ correspond to Γ-actions on non-simplicial buildings.

3. There is a connection between Bruhat-Tits theory over arbitrary valued fields and analytic geometry in the sense of Berkovich.

4.2.3. Metric properties of buildings. — We would like to motivate the axioms SEB1 and SEB2. Essentially, they exist in order for us to be able to define a metric on the whole building. We already have a natural Euclidean metric on each apartment (because they are, by definition, subsets of Euclidean spaces). We want to have a metric on X which gives the Euclidean one by restriction to any apartment. Essentially, the axioms SEB1-2 are "desinged" for the local metrics to patch together. For any $x, x' \in X$, choose an apartment $\mathbf{A} \ni x, x'$. We can define the distance between x and x' from the metric on $\mathbf{A}$. This is well-defined by the second axiom (after some work).

***4.2.6 Theorem* (Bruhat-Tits).** — *Any Euclidean building X admits a distance d such that (X, d) is a complete,* CAT(0) *metric space.*

The property "CAT(0)" essentially captures "non-positively curved and simply connected." More precisely, a metric space (X, d) is said to be CAT(0) if

1. it is *geodesic* (for all $x, x' \in X$, there is a continuous path $\gamma : [0, d(x, x')] \to X$ such that $\gamma(0) = x$, $\gamma(d(x, x')) = x'$, and $d(\gamma(s), \gamma(t)) = |s - t|$)
2. geodesic triangles as thin as in the Euclidean plane (if $x, y, z \in X$, draw the geodesic triangle in $\mathbf{R}^2$. The length of the paths between a point and half-edge are $\leqslant$ what we get in the Euclidean case)

The notion of a CAT(0) space is due to Gromov.

***4.2.7 Lemma* (Bruhat-Tits fixed point).** — *Let G act by isometries on a complete* CAT(0) *metric space. If G has a bounded orbit, then it has a fixed point.*

Proof (Serre). — Any nonempty bounded subset in X admits a *unique* metrically characterized barycenter. □

For example, whenever a compact group acts by isometries on a complete CAT(0) space, then it admits a fixed point.

Why is this useful? If G is a reductive k-group, we have a Euclidean building X with a strongly transitive action of $G(k)$. In the archimedean case, all maximal compacts are conjugate. This was proved by Emil Cartan by convexity arguments that amounted to the negative(?) curvature of the associated symmetric space. We can use the action of $G(k)$ on X to classify maximal compact subgroups of Gk, which is useful for studying unitary representations of $G(k)$. Also, the Bruhat-Tits fixed point lemma is used to prove the geometric part of Bruhat-Tits theory.

In general, if G is an algebraic group over an arbitrary field F, in order to understand $G(F)$, one looks at $G(F^s)$ with its $\mathrm{Gal}(F^s/F)$-action. In other words, $G(F) = \mathrm{H}^0(F, G)$. The idea to attach a Euclidean building to $G(k)$ is

1. use a field extension L/k which splits G

2. attach a building X_L to $G(L)$
3. try to prove that $(X_L)^{\mathrm{Gal}(L/k)}$ is a smaller building

To show that the space $(X_L)^{\mathrm{Gal}(L/k)}$ is "large enough" we use the Bruhat-Tits fixed point lemma.

For classical groups seen as fixed-point sets for involutions on linear groups, suitable Bruhat-Tits buildings are often fixed-point sets on the building of the ambient GL_n for the natural action of the involution.

4.3. Bruhat-Tits buildings for $\mathrm{GL}(n)$**.** — The spaces for $\mathrm{GL}(n)$ were introduced by Goldman and Iwahori in [**GI63**] before the general notion of a building existed. Start with $\mathrm{SL}_n(\mathbf{R})$. The Archimedean symmetric space is $X = \mathrm{SL}_n(\mathbf{R})/\mathrm{SO}(n)$. To generalize this, we see X as the space of normalized scalar products on $\mathbf{R}^n$. Now let k be an ultrametric field, and let V be a d-dimensional k-vector space. Let $\mathscr{N} = \mathscr{N}_V$ be the space of non-archimedean norms on V. Our goal is to show that $\mathscr{N}$ is a Bruhat-Tits building.

4.3.1. Examples of norms. — If we are willing to make choices, this is easy. Choose an ordered basis $\boldsymbol{e} = (e_1, \ldots, e_d)$ for V. Let $\boldsymbol{c} = (c_1, \ldots, c_d)$ be an ordered d-tuple of real numbers. Then we have an ultrametric norm $\|\cdot\|_{\boldsymbol{e},\boldsymbol{c}}$ on V defined by

$$\Big\|\sum \lambda_i e_i\Big\|_{\boldsymbol{e},\boldsymbol{c}} = \max\{e^{c_i}|\lambda_i| : 1 \leqslant i \leqslant d\}.$$

Given a norm $\|\cdot\|$ on V, we say that a basis $\boldsymbol{e}$ is *adapted to* (or *diagonalizes* $\|\cdot\|$ if there is a tuple $\boldsymbol{c}$ such that $\|\cdot\| = \|\cdot\|_{\boldsymbol{e},\boldsymbol{c}}$.

Weil and Goldman-Iwahori proved that if k is a local field, then given any two norms $\|\cdot\|$, $\|\cdot\|'$, there is a basis $\boldsymbol{e}$ and parameters $\boldsymbol{c}$, $\boldsymbol{c}'$ such that

$$\begin{aligned} \|\cdot\| &= \|\cdot\|_{\boldsymbol{e},\boldsymbol{c}} \\ \|\cdot\|' &= \|\cdot\|_{\boldsymbol{e},\boldsymbol{c}'}. \end{aligned}$$

The proof is similar to that of the classical Grahm-Schmidt theorem. Essentially, one uses the compactness of $\mathbf{P}^d(k)$.

We have

$$\mathscr{N} = \bigcup_{\boldsymbol{e} \text{ basis of } V} \mathbf{A}_{\boldsymbol{e}},$$

where

$$\mathbf{A}_{\boldsymbol{e}} = \{\|\cdot\|_{\boldsymbol{e},\boldsymbol{c}} : \boldsymbol{c} \in \mathbf{R}^d\} \simeq \mathbf{R}^d.$$

Fix a basis $\boldsymbol{e}$. We want to see a Euclidean reflection group act on $\mathbf{A}_{\boldsymbol{e}}$. This group will be $S_d \ltimes \mathbf{Z}^d$. If k is a local field, write k° for the valuation ring of k, and $v : k^\times \to \mathbf{Z}$ for the valuation. Then

$$k^\circ = \{x \in k : |x| \leqslant 1\} = \{x \in k : v(x) \geqslant 0\}.$$

The ring k° is a valuation ring, whose maximal ideal is denoted k^+. (Rémy writes $k^{\circ\circ}$.) Let $\kappa = k^\circ/k^+$ be the residue field of k; since k is locally compact, this is a finite field of order $q = p^e$. Lastly, choose a uniformiser π satisfying $k^+ = \pi k^\circ$. If $k = \mathbf{Q}_p$, then $k^\circ = \mathbf{Z}_p$, $\pi = p$, and $k^+ = p\mathbf{Z}_p$. If $k = \mathbf{F}_q(\!(t)\!)$, then $k^\circ = \mathbf{F}_q[\![t]\!]$ and $\pi = t$.

Back to $\mathbf{A}_{\boldsymbol{e}}$. The action by S_d is given by permutation of the indices. For $\sigma \in S_d$, $\sigma \cdot \|\cdot\|_{\boldsymbol{e},\boldsymbol{c}} = \|\cdot\|_{\boldsymbol{e},\sigma(\boldsymbol{c})}$, where $\sigma(\boldsymbol{c}) = (c_{\sigma(1)}, \ldots, c_{\sigma(d)})$. If $\boldsymbol{m} \in \mathbf{Z}^d$, then $\boldsymbol{m}$ acts by $\boldsymbol{m} \cdot \|\cdot\|_{\boldsymbol{e},\boldsymbol{c}} = \|\cdot\|_{\boldsymbol{e},\boldsymbol{c}+\boldsymbol{m}}$.

This action of $S_d \ltimes \mathbf{Z}^d$ lifts to an action by $N_{\boldsymbol{e}}$, the group of monomial matrices (i.e. each row and column has a unique nonzero entry) where the matrix inducing an isomorphism $\mathrm{GL}(V) \xrightarrow{\sim} \mathrm{GL}_n(k)$ is given by $\boldsymbol{e}$. In general, the group $\mathrm{GL}(V)$ acts on $\mathscr{N}_V$ by precomposition, i.e.

$$g\|\cdot\| = \|\cdot\| \circ g^{-1}.$$

For this action, $N_{\boldsymbol{e}}$ stabilizes $\mathbf{A}_{\boldsymbol{e}}$ and gives the previous action by $S_d \ltimes \mathbf{Z}^d$. At last, the group $\mathrm{GL}(V)$ acts transitively on the set of subspaces $\mathbf{A}_{\boldsymbol{e}}$.

For $\boldsymbol{m} \in \mathbf{Z}^d$, write $\pi^{\boldsymbol{m}}$ for the diagonal matrix (a_{ij}) with $a_{ii} = \pi^{m_i}$. The group $\pi^{\mathbf{Z}^d}$ is the translation part of $S_d \times \mathbf{Z}^d$.

4.3.2. Connection with Bruhat-Tits theory. — The following theorem is originally due to [**GI63**], but was reformulated by Bruhat and Tits.

4.3.1 Theorem. — *Assume k is a local field. Then the quotient space of $\mathscr{N}_V$ by homothety is a Euclidean building with apartments the $\mathbf{A}_{\boldsymbol{e}}$ and Weyl group $S_d \times \mathbf{Z}^{d-1}$.*

For us, *homothety* is the equivalence relation where two norms are identified whenever they are proportional.

The identification $\mathbf{R}^d \xrightarrow{\sim} \mathbf{A}_{\boldsymbol{e}}$, $\boldsymbol{c} \mapsto \|\cdot\|_{\boldsymbol{e},\boldsymbol{c}}$ identifies $\mathbf{A}_{\boldsymbol{e}}/\sim$ with $\mathbf{R}^d/\Delta$, where $\Delta = \mathbf{R} \cdot (1, \ldots, 1)$ is the diagonal.

Recall $g\|\cdot\| = \|\cdot\| \circ g^{-1}$. The action of diagonalizable matrices at least ono the apartment is fixed by the basis diagonalizing the matrix. For the case $d = 2$, see Serre's book on trees. But be careful! In Serre's book only the vertices of the building (here a tree) are taken into account. The vertices correspond to homothety classes of k°-lattices in V.

4.3.3. Maximal compact subgroups of $\mathrm{GL}(V)$. — Using the Bruhat-Tits fixed point lemma, one knows that a compact subgroup $K \subset \mathrm{GL}(V)$ has to fix some point in $\mathscr{N}_V/\sim$. Thus it is interesting to have a description of stabilizers of norms.

4.3.2 Proposition. — *We have*

$$\mathrm{Stab}_{\mathrm{SL}(V)}(\|\cdot\|_{\boldsymbol{e},\boldsymbol{c}}) = \{g \in \mathrm{SL}(V) : \log|g_{ij}| \leqslant c_j - c_i\}.$$

where (g_{ij}) is the matrix of g with respect to $\boldsymbol{e}$. In particular, $\mathrm{Stab}_{\mathrm{SL}(V)}(\|\cdot\|_{\boldsymbol{e},0}) \simeq \mathrm{SL}_d(k^\circ)$.

[Warning: Rémy wrote $|x| = e^{-v(x)}$ rather than $|x| = (\#\kappa)^{-v(x)}$.]

Using concrete computations, one can show that a fundamental domain for the $\mathrm{SL}(V)$ action on $\mathscr{N}_V/\sim$ is given

$$\{\|\cdot\|_{\boldsymbol{e},\boldsymbol{c}} : 0 \leqslant c_1 \leqslant \cdots \leqslant c_d \leqslant 1\}$$

for $\boldsymbol{e}$ fixed.

For $d = 3$, a fundamental domain looks like an equilateral triangle. For $d = 2$, a fundamental domain is an edge in the tree $\mathscr{N}_V/\sim$.

For $d = 2$, one can color the vertices with 2 colors so that any two neighbors don't have the same color (or type). The $\mathrm{SL}(V)$-action is type preserving. As a conseqence, using the Bruhat-Tits lemma, we see that there are exactly two conjugacy classes of maximal compact subgroups in $\mathrm{SL}(V)$. (For arbitrary V, there are d conjugacy classes.)

The group $\mathrm{GL}_d(k)$ is generated by diagonal matrices and elementary unipotent matrices. We have described the action of diagonal matrices on $\mathscr{N}_V$. What remains is to describe the action of elementary unipotent matrices on $\mathscr{N}_V$. First we introduce some notation. Fix a basis $\boldsymbol{e}$ of V. Then E_{ij} is the matrix (with respect to $\boldsymbol{e}$) with zeros everywhere but the (i, j)-th component, where there is a 1. Put $\mu_{i,j}(\lambda) = 1 + \lambda E_{i,j}$ for $\lambda \in k$.

***4.3.3 Proposition* (“folding” by unipotent matrices)**

The intersection $\mu_{ij}(\lambda)\mathbf{A}_{\boldsymbol{e}} \cap \mathbf{A}_{\boldsymbol{e}}$ is the half-space of $\mathbf{A}_{\boldsymbol{e}}$ defined by $c_j - c_i \geqslant \log|\lambda|$. The isometry given by $\mu_{ij}(\lambda)$ fixes (pointwise) this intersection, and sends the remaining (open) halfspace of $\mathbf{A}_{\boldsymbol{e}}$ onto a disjoint complement.

4.3.4 Example. — Let $d = 2$. Then our matrix is $\left(\begin{smallmatrix} 1 & \lambda \\ & 1 \end{smallmatrix}\right) = \mu_+(\lambda)$. [...stuff in terms of a picture I couldn't repeat...]

4.4. Borel-Tits' theory. — For our purposes, this is the theory which describes the abstract group structure of $G(k)$, for G a semisimpile group and k an arbitrary field; this will be done in purely combinatorial terms. The final statement more or less amounts to the existence of a spherical building on which $G(k)$ acts strongly transitively.

4.4.1. Algebraic terminology and conjugacy theorems. — The main references for this section are the books of Borel, Springer, and Waterhouse. Let G be a linear algebraic group over some field k. We say that $g \in G(k)$ is *unipotent* if in some (a posteriori, any) linear representation ρ, its image is unipotent (i.e. $\rho(g) - 1$ is nilpotent). A group G is called *unipotent* if all its' elements are unipotent. Alternatively, we could require that any finite-dimensional representation have a non-zero fixed vector.

We say that a connected subgroup $T \subset G$ is a *torus* if $\bar{k}[T] \simeq \bar{k}[X^*(T)]$. If T is defined over k, we say that T is *k-split* if $k[T] \simeq k[X_k^*(T)]$.

Let G be a connected linear algebraic group over k. The *unipotent radical* of G is the (unique) maximal subgroup for the following properties:

1. closed
2. connected
3. unipotent
4. normal

The *rational unipotent radical* of G is the (unique) maximal unipotent of subgroup satisfying all the above properties, and additionally being defined over k. We denote it by $\mathscr{R}_{u,k}(G)$.

We say that G is *reductive* if $\mathscr{R}_u(G) = 1$. We say that G is *pseudo-reductive* if $\mathscr{R}_{u,k}(G) = 1$. The *radical* of G is defined by replacing "unipotent" by "solvable" in the above definition. Let $\mathscr{R}(G)$ be the radical of G. We say that G is *semisimple* if $\mathscr{R}(G) = 1$.

It is a fact that "a group is reductive if and only if it is geometrically pseudo-reductive." The goal in Lie theory is to extract elementary combinatorics out of algebraic differential geometric situations. Here, we start with a reductive group G defined over k. At least in the split case, the outcome of our method is a root system. The well-known situation is where $k = \bar{k}$. The main tool is the adjoint representation, given by conjugation on the Lie algebra. Put $\mathfrak{g} = \mathrm{Lie}(G) = \ker(G(k[\varepsilon]) \to G(k))$, where $k[\varepsilon] = k[X]/X^2$. The action of G on $\mathrm{Lie}\, G$ induces a representation $\mathrm{ad} : G \to \mathrm{GL}(\mathfrak{g})$. Let $T \subset G$ be a maximal k-torus (split, given our assumptions). Then the representation $\mathrm{ad}\,|_T$ decomposes as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\psi \in X^*(T) \smallsetminus \{0\}} \mathfrak{g}_\psi,$$

where $\mathfrak{g}_\psi = \{v \in \mathfrak{g} : \mathrm{ad}(t)v = \psi(t)v\}$. The non-zero characters occuring non-trivially in this sum are called the *roots* and they form a root system in vector space $X^*(T) \otimes \mathbf{R}$ endowed with a W-invariant scalar product. Here $W = N_G(T)/T$. (For all of this we assumed $k = \bar{k}$.) When $k = \bar{k}$, the root system is *reduced* (i.e. the proportionality relation between two roots except opposition).

Now back to arbitrary k. The idea is to replace T by a "useful" torus defined over k (useful in the sense that it has "as many" characters defined over k as possible so that the weight decomposition of $\mathfrak{g}$ will be as subtle as possible). More precisely, take T to be a maximal k-split torus. We have a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\psi \in X_k^*(T) \smallsetminus \{0\}} \mathfrak{g}_\psi.$$

This is a (possibly non-reduced) root system. We claim that this decomposition of the Lie algebra induces a decomposition of G.

[*Warning*: this assumes there is a non-central k-split torus $T \subset G$.]

The characters $\alpha \in X^*(T) \smallsetminus 0$ with $\mathfrak{g}_\alpha \neq 0$ are called *roots* with respect to T. Let $\Phi = \Phi(G, T)$ be the set of roots.

The group $N_G(T)$ acts on $X_k^*(T)$ via conjugation on T, and it stabilizes Φ. It acts via the finite (by rigidity of tori) quotient $W = N_G(T)/Z_G(T)$. The $\mathbf{R}$-linear span $V = X_k^*(T)_{\mathbf{R}}$ admits a W-invariant scalar product (because W is finite) for which Φ is a (possibly non-reduced) root system. The Weyl group of the root system Φ is W.

For each root α, we denote by U_α the closed connected subgroup of G whose Lie algebra is $\mathfrak{g}_\alpha$.

4.4.1 Definition. — Let G be an abstract group. Let T be a subgroup and Φ a root system. We assume we are given for each $\alpha \in \Phi$ a subgroup U_α and a class moduo T, say M_α. We say that $(\{U_\alpha\}_{\alpha\in\Phi}, \{M_\alpha\}_{\alpha\in\Phi})$ is a *root group datum of type* Φ if the following axioms hold:

1. for all $\alpha \in \Phi$, $U_\alpha \neq 1$
2. for all $\alpha, \beta \in \Phi$, $[U_\alpha, U_\beta] \subset \langle U_\gamma : \gamma\mathbf{N}\alpha + \mathbf{N}\beta\rangle$
3. if $\alpha, 2\alpha \in \Phi$, then $U_{2\alpha} \subsetneq U_\alpha$
4. for all $\alpha \in \Phi$, $U_{-\alpha} \setminus 1 \subset U_\alpha M_\alpha U_\alpha$
5. for all $\alpha, \beta \in \Phi$, $m \in M_\alpha$, $mU_\beta m^{-1} \subset U_{r_\alpha(\beta)}$
6. for all choices of positive subsystems for Φ^+, we have $U^+T \cap U^- = 1$, where $U^+ = \langle U_\gamma : \gamma \in \pm\Phi_+\rangle$

It is a good exercise to check these axioms for $\mathrm{GL}_n(F)$, where Φ is of type A_{n-1}.

We now give an "unfair summary" of Borel-Tits theory.

***4.4.2 Theorem* (Borel-Tits '65)**. — *Let G be a reductive group over k. We assume that G contains a non-central k-split torus, say T. Denote by $\{U_\alpha\}$ the corresponding root groups. Then $G(k)$ admits a root group datum in which the root groups are given by the $U_\alpha(k)$, and $T = Z_G(T)(k)$.*

What is this good for?

1. The existence of a root group datum implies the existence of a weaker combinatorial structure called a *Tits system*. This provided a uniform way to prove projective simplicity for rational points of isotropic simple groups.
2. The axioms imply the existence of a spherical building on which $G(k)$ acts strongly transitively.

If the group is not isotropic, things become much more difficult.

4.4.2. Conjugacy results and anisotropic kernel. — So far, the combinatorial objects we constructed have depended on the choice of a maximal k-split torus T. In fact, we have the following theorem.

***4.4.3 Theorem* (Borel-Tits '65)**. — *Let G be as above. Then*

1. *Maximal k-split tori are conjugate under the $G(k)$-action.*
2. *Minimal parabolic k-subgroups are $G(k)$-conjugate. In fact, $G(k)$ acts transitively on inclusions $T \subset B$ of maximal k-split tori into minimal parabolics B.*

We introduce some terminology. Grops of the form $Z_G(T)$ are reductive anisotropic over k. In other words, $Z_G(T)$ contains no non-central k-split torus. (For k a local field, a semisimple reductive k-group is anisotropic if and only if $G(k)$ is compact.) The groups $Z_G(T)$ are also $G(k)$-conjugate. This conjugacy class is called the *anisotropic kernel* of G. In general, semisimple groups over k are classified by data consisting of the anisotropic kernel, together with a Galois action on the Dynkin diagram. For details, see Tit's Boulder lecture notes, or Satake's book.

4.4.3. Example of a non-reduced root system. — This is in the context of real Lie groups. Let F be one of $\mathbf{R}, \mathbf{C}$, and the quaternions. Let $\bar{\cdot}$ be the conjugation (if F is the complex numbers or the quaternions). Let $\mathrm{SU}(n,1) = \{g \in \mathrm{SL}_n(F) : g^* J g = J\}$, where J is an anti-diagonal matrix with -1 everywhere along the reverse diagonal. The Lie algebra $\mathfrak{su}(n,1)$ consists of matrices $\left(\begin{smallmatrix} w & Z^* \\ Z & Y \end{smallmatrix}\right)$, where Y is an $n \times n$ antihermetian matrix (i.e. $Y + Y^* = 0$), $w \in F$, and $Z \in F^n$.

The points of a maximal $\mathbf{R}$-split torus can be described as follows: let

$$A = \left\{ a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} : t \in \mathbf{R} \right\}.$$

The weight space decomposition for $\mathfrak{g} = \mathfrak{su}(n,1))$ under the action of A is

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}.$$

The 2α occur if and only if $F \supsetneq \mathbf{R}$. Here,

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} w & 0 & t \\ 0 & m' & 0 \\ t & 0 & w \end{pmatrix} : m \in \mathfrak{u}(n-1, F), t \in \mathbf{R} \text{ and } w + \overline{w} = 0 \right\}.$$

One has

$$\mathfrak{g}_\alpha = \left\{ \begin{pmatrix} 0 & z* & 0 \\ z & 0 & -z \\ 0 & z^* & 0 \end{pmatrix} : z \in F^{n-1} \right\}$$

of dimension $d(n-1$, where $d = [F : \mathbf{R}]$. Finally,

$$\mathfrak{g}_{2\alpha} = \left\{ \begin{pmatrix} w & 0 & -w \\ 0 & 0 & 0 \\ -w & 0 & w \end{pmatrix} : w + \overline{w} = 0 \right\};$$

this has dimension $d-1$. The anisotropic kernel $\mathfrak{g}_\alpha$ can be arbitrarily large, even though the root system is the (non-reduced) root system of rank one called BC_1 with roots $\{\pm 2\alpha, \pm\alpha\} \subset \mathbf{Z} \cdot \alpha$. For $F = \mathbf{C}$, the groups we get are $\mathrm{SU}(n,1)$, and for $F = \mathbf{H}$, the groups are $\mathrm{Sp}(n,1)$.

4.5. Construction of Bruhat-Tits buildings. — The Euclidean buildings we will construct will be obtained by gluing together copies of a Euclidean tessellation attached to the affinization of the Weyl group given by Borel-Tits. The gluing will be done by an equivalence relation which (eventually) prescribes point stabilizers. These stabilizers are defined by using a filtration on each root group, which comes from the valuation on the ground field.

4.5.1. Gluings and foldings. — We need:

1. a model for the apartments
2. a gluing / equivalence relation

Let G be a semisimple grop over a local field k. We assume that G is k-isotropic (i.e. it contains a non-central k-split torus). So $G(k)$ is noncompact. (This is due to Bruhat-Tits-Rousseau, Prasad.)

First we construct the apartment. Let T be a maximal k-split torus and $N = N_G(T)$. We denote by $\Sigma_{\text{vect}} = X_{*,k}(T) \otimes \mathbf{R}$, where $X_{*,k}(T) = \hom_{k\text{-}\mathsf{gp}}(\mathbf{G}_{\text{m}}, T)$. This should be seen as an analogue of $\mathbf{A}_{\boldsymbol{e}}$. There is a (well-defined) affine space Σ under Σ_{vect} admitting an action by $N(k)$, say $\nu : N(k) \to \text{Aff}(\Sigma)$, such that

1. There exists a scalar product on Σ_{vect} such that $\nu(N) \subset \text{Isom}(\Sigma)$.
2. The vectorial part of $\nu(N)$ is the Weyl group of Φ.
3. The translation (normal) subgroup in $\nu(N)$ has compact fundamental domain. In fact, $\nu(N)$ is an affine Coxeter group.

This is a generalization of the action on $\mathbf{A}_{\boldsymbol{e}}$ in Section 4.3 by the permutation matrices and the diagonal matrices $\pi^{\boldsymbol{m}}$. There, we had $\nu(N) \simeq S_d \ltimes \mathbf{Z}^{d-1}$.

Now we want to define an equivalence relation in order to glue infinitely many copies of Σ. The idea is to associate a compact subgroup P_x (a "parahoric" subgroup) to each point $x \in \Sigma$. This is done using the filtration on the root groups U_α given by Borel-Tits theory.

One should keep the example $\text{SL}_2(\mathbf{Q}_p)$ in mind. Its maximal torus T consists of diagonal matrices $\left(\begin{smallmatrix} t & \\ & t^{-1} \end{smallmatrix}\right)$, and this give a geodesic in the tree. The subgroup $\left(\begin{smallmatrix} 1 & \mathfrak{p}^m \\ & 1 \end{smallmatrix}\right)$ fixes a ray in one direction whose "'depth" depends on m. A group $\left(\begin{smallmatrix} 1 & \\ \mathfrak{p}^\ell & 1 \end{smallmatrix}\right)$ fixes a ray in the opposite direction. Once the building is obtained, the filtration corresponds to the size of the fixed ray.

If G is split, everything is easy. For general G, things are more difficult. The split case is easy because there exist compatible parameterizations of the root groups U_α (all one-dimensional). One calls these parameterizations *pinnings* ("épinglages" in French). We will base-change to make our group split, then use a (2-step) descent argument.

The abstract abstract context used by Bruhat-Tits consists in formalizing the existence of compatible filtrations $U_{\alpha,\bullet}$ on groups admitting an abstract root group datum. This is done in [**BT72**], which is basically just combinatorics and buildings. The existence by descent and use of integral forms (over k°) for G is in [**BT84**].

From now on, we assume we have suitable filtrations $U_{\alpha,\bullet}$ on root groups. In particular, $\bigcup_{r\in\mathbf{Z}} U_{\alpha,r} = U_\alpha(k)$ and $\bigcap_{r\in\mathbf{Z}} U_{\alpha,r} = 1$. Choose $x \in \Sigma$. We define $N_x = \mathrm{Stab}_{N(k)}(x)$ for the ν-action. For all $\alpha \in \Phi$, we define $U_{\alpha,x}$ to be the largest subgroup $U_{\alpha,r}$ such that the positive subspace associated to the affine linear form $\alpha + r$ contains x, i.e. $(\alpha + r)(x) \geqslant 0$. At last, we define $P_x = \langle N_x, U_{\alpha,x} : \alpha \in \Phi\rangle$. Then, we define the relation $\sim$ on $G(k) \times \Sigma$ by $(g,x) \sim (h,y)$ if and only if there exists $n \in N(k)$ such that $y = \nu(n)\cdot x$ and $g^{-1}hn \in P_n$.

***4.5.1 Theorem* (Bruhat-Tits).** — *The relation $\sim$ is an equivalence, and the quotient space $G(k)\times\Sigma/\sim$ is a building of type Σ on which $G(k)$ acts strongly transitively.*

4.6. Gluings and foldings. — To sum up. “Let G be a group with a root group datum $((U_\alpha)_{\alpha\in\Phi}, (M_\alpha)_{\alpha\in\Phi})$. Assume that this datum admits a compatible system of valuations. Then there exists a Euclidean building on which G acts strongly transitively. Moreover, the model for the apartment is explicit.” As mentioned above, this was by Bruhat and Tits in their paper [**BT72**]. This is almost a necessary and sufficient condition. In other words, suitably transitive actions on buildings imply the existence of a root group datum with valuation.

We briefly treat descent. Let k be a local field (or possibly a more general valued field). Let G be a semisimple group over k. The point is to find a Galois extension F/k which splits G. Since G_F is the base-change of a Chevalley-Demazure group scheme (which are defined over $\mathbf{Z}$) we get a pinning, which can be used to construct the valuation on our root group datum.

At this stage we have a building $\mathscr{B}(X,F)$ by the previous gluing procedure. We have an action of $\Gamma = \mathrm{Gal}(F/k)$ on $G(F)$. Thanks to the connection between the combinatorics of $G(F)$ and the geometry of $\mathscr{B}(G,F)$, we can define an action of Γ on $\mathscr{B}(G,F)$. *Hint*: let p be the residue characteristic of k, and assume G is simply connected. Then the chamber stabilizers are the normalizers of the pro-p Sylow subgroups of $G(F)$.

Suitably decompose F/k into two extensions so that $G(k) = G(F)^\Gamma$ admits a root group datum with valuation so that the associated building is $\mathscr{B}(X,F)^\Gamma$. This doesn't always work. One needs some assumptions on the ramification of the extension F/k. See [**BT84**] for details. One may need to choose an intermediate extension $k \subset F' \subset F$ so that G is quasi-split over F' (i.e. contains a Borel subgroup over F'). Equivalently, the centralizer of an F'-split torus is a torus. Quasi-split groups have no anisotropic kernel. In good cases, the extension F/F' is unramified, and we use integral structure (group schemes over valuation rings).

***4.6.1 Theorem* (Bruhat-Tits).** — *Let k be a complete, discretely-valued field with perfect residue field. Then the group G admits a Euclidean building over k on which $G(k)$ acts strongly transitively.*

In [**Rou77**], it is proved that the building $\mathscr{B}(X, -)$ behaves functorially with respect to the ground field. In good cases, one has $\mathscr{B}(G, F)^\Gamma = \mathscr{B}(G, k)$.

4.7. Compact subgroups. — This is closely related to integral structures for G, as well as decomposition theorems for $G(k)$, which are used in the study of representations of $G(k)$.

As before, assume G is simply connected and k is local. Then the building $\mathscr{B}(G, k)$ is *locally finite*, i.e. given any facet σ, there are only finitely many facets whose closure contains σ. For any $x \in \mathscr{B}(G, k)$, the stabilizer P_x is compact and open in $G(k)$. If p is the residue characteristic of k, then P_x is virtually pro-p. In other words, each P_x contains a pro-p subgroup of finite index.

4.7.1 Theorem. — *1. For all $x \in \mathscr{B}(G, k)$, there is a smooth group scheme over k°, $\mathscr{G}_x$, such that $\mathscr{G}_x \otimes_{k^\circ} k = G$.*

2. The set of facets whose closure contains x is a spherical finite building $\mathscr{B}_x$.

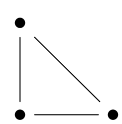
3. $\mathscr{B}_x$ is the spherical building of the (semisimple quotient of) $\mathscr{G}_x \otimes_{k^\circ} \kappa$.

If $x \in \mathbf{A}$ is a vertex, we say that x is *special* if $\mathrm{Stab}_W(x)$ is the full vectorial part of the affine Weyl group W.

4.7.2 Complement. — *Up to passing to a huge non-archimedean extension K/k, any point $x \in \mathscr{B}(G, k)$ can be seen as a special vertex in $\mathscr{B}(G, K)$.*

4.7.3 Example. — Let $x \in \mathscr{B}(\mathrm{SL}_3, \mathbf{Q}_p)$ which is the barycenter of some chamber. Up to a totally ramified extension K of degree 3, x can be seen as a special vertex in $\mathscr{B}(\mathrm{SL}_2, K)$.

4.7.4 Example. — Here we give a vertex which is not special. The group is Sp_4. A typical chamber looks like



The vertices on the ends of the long edge are special, but the common vertex of the short edges is not special.

4.7.5 Theorem. — *Let G be a simply connected group over a local field k. Then the $G(k)$-conjugacy classes of maximal compact subgroups of $G(k)$ are in bijection with vertices in the closure of any chamber in $\mathscr{B}(G, k)$.*

Note the contrast with the case of real groups, where all maximal compacts are conjugate.

4.8. Two decompositions. — Recall the Cartan and Iwasawa decompositions. If G is a real Lie group, we have

$$G = K\overline{A}K \qquad \text{“Cartan decomposition”}$$
$$G = KAN \qquad \text{“Iwasawa decomposition”}$$

Let $\mathbf{A}$ be an apartment, and let $c \subset \mathbf{A}$ be a chamber. We pick a vertex $o \in \overline{c}$. From our gluing relation, we get that walls in the Euclidean tesselation $\mathbf{A}$ correspond to the zero-sets of the affine roots $\alpha + r$ for $\alpha \in \Phi$, $r \in \mathbf{Z}$. Half-spaces bounded by walls correspond to roots (at least, when Φ is reduced). The group $U_{\alpha,r}$ fixes $\alpha + r$ (the latter seen as a half-space in $\mathbf{A}$) and folds the other half of $\mathbf{A}$.

The $G(k)$-action is transitive enough on $\mathscr{B}(G,k)$ to show that for any $x \in \mathscr{B}(G,k)$, there exists $g \in G(k)$ fixing C, such that $gx \in \mathbf{A}$. *Hint*: draw a geodesic segmentfrom c to x, and fold (by finite induction) along the walls which are crossed by the segments or its intermediate transforms. This is almost a proof of the Cartan decomposition.

Take K to be $P_0 = \mathrm{Stab}_{G(k)}(o)$. Pick $g \in G(k)$. Fold $x = g \cdot o$ onto $\mathbf{A}$. Then $g \in P_c \subset K$. Then $k^{-1}g \cdot o \in \mathbf{A}$ for some k. Then use a translation in the maximal split torus corresponding to $\mathbf{A}$.

4.8.1 Theorem. — *Let G, k be as before. If o is a special vertex, then $G(k) = K\overline{T}^+K$, where $\overline{T}^+$ is the subsemigroup of T (the maximal k-split torus corresponding to $\mathbf{A}$) defined by the condition that for $t \in T(k)$, we have $t \in \overline{T}^+$ if and only if $t \cdot o$ is an element of the some fixed Weyl chamber in $\mathbf{A}$.*

We say that K is special if the vertex defining it is. So K is special if and only if K contains elements lifting all the elements in the spherical Weyl group, i.e. “K contains all the non-commutative part of the affine Weyl group.”

In harmonic analysis, one looks at spaces $\mathscr{H}(G,K) = C_c^\infty(K\backslash G(k)/K)$ of locally constant, compactly supported, K-bi-invariant functions. We say that $(G(k),K)$ is a *Gelfand pair* if convolution gives $\mathscr{H}(G,K)$ the structure of a commutative ring. The Hecke algebra $\mathscr{H}(G,K)$ is commutative if and only if K is special. See Macdonald's book [**Mac71**].

There is also a p-adic analogue of the Iwasawa decomposition: it looks like $G(k) = KT(k)U^+(k)$. It is based on the same “folding” idea, except that here the foldings are stepwise required to fix the boundary at infinity of the Weyl chamber $Q \subset \mathbf{A}$ defined by o and c.

4.9. Unitary groups in three variables. — Let F/k be a quadratic extension of local fields. Let V be a three-dimensional k vector space. Choose a basis $e_0, e_{\pm 1}$, and define the Hermitian form

$$h(z_{-1}, z_0, z_1) = z_{-1}\sigma(z_0) + z_0\sigma(z_1) + z_1\sigma(z_{-1}),$$

where $\sigma \in \mathrm{Gal}(F/k) \smallsetminus 1$. We want to understand the group $\mathrm{SU}(V,h) = \mathrm{SL}(V)^{\sigma}$, where σ is (another involution) defined by

$$\sigma(M) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} {}^{\mathrm{t}}M^{-1} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The apartment of $\mathrm{SL}(V)$ defined by $\theta = (e_{-1}, e_0, e_1)$ consists of the homothety classes of lattices $\sum \mathfrak{p}^{m_i} e_i$. We have three positive root groups in SL_3:

$$U_{12} = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U_{1,3} = ?$$

$$U_{2,3} = ?$$

The group U_{13} is strongly σ-stable, and σ interchanges U_{12} and U_{23}.

What are maximal split tori in $\mathrm{SU}(V,h)$? Let

$$\tau = \begin{pmatrix} \pi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi^{-1} \end{pmatrix}.$$

The Zariski closure of $\langle \tau \rangle$ is a maximal split torus. According to whether F/k is ramified or not we don't get the same tree. For example, if $F = \mathbf{F}_q(\!(X)\!)$ and $k = \mathbf{F}_q(\!(X^2)\!)$, then for $\pi = X$, we have $\sigma(X) = -\sigma$. Then $\tau \notin \mathrm{SU}(V,H)(k)$, but τ^2 is in this group. If on the other hand we look at the unramified extension $\mathbf{F}_{q^2}(\!(X)\!)/\mathbf{F}_q(\!(X)\!)$, then $\sigma(\pi) = \pi$, so $\tau \in \mathrm{SU}(V,h)(k)$. Be careful with keeping or not all the intersections of walls with fixed point sets as vertices in the non-split building. When F/k is ramified, the building is a homogeneous tree (all valences are the same). But when F/k is unramified, the building is only semi-homogeneous (valences are q or $q+1$).

5. Pseudo-reductive groups

5.1. Motivation and examples. — Suppose we have some problem concerning a "general" affine algebraic group over a field k. If k is not perfect, it becomes quite difficult to reduce problems to the reductive case.

5.1.1 Example. — Let $G = \mathrm{GL}_n$. Let X be the variety of non-degenerate quadratic forms on k^n. We assume the action of $G(\bar{k})$ on $X(\bar{k})$ is transitive. Then X is a "homogeneous space" for G. So any choice of $x \in X(\bar{k})$ induces $G/\operatorname{Stab}_{G_{\bar{k}}}(x) \xrightarrow{\sim} X_{\bar{k}}$. Now choose $x_0 \in X(\bar{k})$. Assume k is a global field, S is a finite set of places of k, and $x \in X(k)$. We want

$$\#G(k)\backslash\{x \in X(k) : x \in G(k_v) \cdot x_0 \text{ for all } x \notin S\} < \infty.$$

To study this set, introduce the transporter scheme $T_x = \mathrm{Transp}(x_0, x)$ for $x \in X(\bar{k})$. Our set above injects into $Ш^1_S(k, G_{x_0})$. So the question is: is $Ш^1_S(k, G)$ finite for all affine algebraic k-groups G?

If G is finite étale, finiteness of $Ш^1_S(k, G)$ follows from the Čebotarev Density Theorem. This can be applied to G/G° in general. If G is connected reductive, Borel-Serre (in characteristic zero) and Harder / Borel-Prasad (in positive characteristic) have shown that these Tate-Shavarevich sets are finite.

In characteristic p, we have the problem that G_{red} is often not a subgroup scheme of G, and even when it is, it may not be smooth. By a trick, using the fact that the extensions k_v/k are separable, we can still replace G with its maximal smooth subgroup. So we can reduce to the case where G is smooth and connected. How far is such a G from being reductive? For perfect k, not too far. Over perfect fields, $\mathscr{R}_{\mathrm{u}}(G_{\bar{k}}) \subset G_{\bar{k}}$ descends to $U \subset G$. The unipotent group U has a canonical filtration with successive quotients isomorphic to $\mathbf{G}_{\mathrm{a}}$. So we have an exact sequence

$$1 \to U \to G \to G/U \to 1$$

where the group on the left is "manageable" and the group on the right is reductive.

If k is imperfect, such k-descent of $\mathscr{R}_{\mathrm{u}}(G_{\bar{k}})$ may not exist in G.

5.1.2 Definition. — $\mathscr{R}_{\mathrm{u},k}(G)$ is the maximal smooth connected unipotent normal k-subgroup of G. Similarly $\mathscr{R}_k(G)$ is the maximal smooth connected solvable normal k-subgroup of G.

If K/k is an extension, then $\mathscr{R}_{\mathrm{u},k}(G) \subset \mathscr{R}_{\mathrm{u},K}(G_K)$ and $\mathscr{R}_k(G)_k \subset \mathscr{R}_K(G_K)$. If K/k is separable (e.g k^s/k, $k(V)/k$ for smooth V, or k_v/k for global k) then we have equalities, i.e. formation of radical commutes with separable field extension. This uses a "smooth specialization" argument.

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5.1.3 Definition. — Say G is *pseudo-reductive* if $\mathscr{R}_{\mathrm{u},k}(G) = 1$. Say G is *pseudo-semisimple* if $\mathscr{R}_{\mathrm{u},k}(G) = 1$ and $G = \mathscr{D}G$.

We will see later that pseudo-semisimple groups have $\mathscr{R}_k G = 1$, but the converse fails. We can make examples of commutative pseudo-reductive groups C with $C(k^s)[p] \neq 1$. These are very "far" from being tori, and there is no nice classification (akin to Galois lattices) for such groups.

Note that for general smooth connected affine G, in the following exact sequence:

$$1 \to \mathscr{R}_{\mathrm{u},k}(G) \to G \to G/\mathscr{R}_{\mathrm{u},k}(G) \to 1$$

the group on the right is pseudo-reductive. The unipotent group $\mathscr{R}_{\mathrm{u},k}G$ can be pretty horrible. Tits developed useful filtrations for them. See Appendix B in [**CGP10**] for details.

We will develop a structure theory for pseudo-reductive groups. This can be used to show that $\text{Ш}^1_S(k, G)$ is finite for function fields.

5.2. Weil restriction. — Let k be a field, and k' a finite k-algebra, i.e. $\dim_k k' < \infty$. Let X' be an affine algebraic k'-scheme. Note that the algebra k' is a finite product of local artinian k-algebras. We define the *Weil restriction* of X', $X = R_{k'/k}(X')$ as "X viewed as a k-scheme using the k-basis of k' to make coordinates." So if X' is smooth of pure dimension d over a field k', then X should be k-smooth of dimension $d[k' : k]$. The functor of points of X is

$$X(A) = X'(k' \otimes_k A).$$

This should be viewed as analogous to viewing a d-dimensional complex manifold as a $2d$-dimensional real manifold.

Some people write $\Pi_{k'/k}$ instead of $R_{k'/k}$ for Weil restriction.

5.2.1 Example. — $R_{k'/k}\mathbf{A}^n_{k'} = \mathbf{A}^{n[k':k]}_k$ via a k-basis of k'.

5.2.2 Example. — Note that $R_{k'/k}(\mathrm{SL}_n)(A) = \mathrm{SL}_n(k' \otimes_k A)$. This is given by "$\det = 1$" on $M_n(k' \otimes_k A) \subset M_{n[k':k]}(A)$. This group can be pretty messy.

5.2.3 Example. — Suppose $k = \prod k'_i$. Then $\mathrm{Spec}(k') = \coprod \mathrm{Spec}(k'_i)$, so $X' = \coprod X'_i$ for k'_i-schemes X'_i. One has $X = R_{k'/k} = \prod_i R_{k'_i/k}(X'_i)$. Checking this is a good exercise.

5.2.4 Example. — Let G' be a smooth algebraic k'-group, $\mathfrak{g}' = \mathrm{Lie}\, G'$. Then $\mathrm{Lie}\, G$ is $\mathfrak{g}'$ viewed as a (p-) Lie algebra over k. See [**CGP10**, A.7.6, A.7.13] for details.

Why should we care about Weil restriction? Suppose k'/k is a (finite) field extension, and K/k is a "big" field extension (e.g. $K = k^s$). Then $K' = k' \otimes_k K$ will be a finite K-algebra. If k'/k is a separable extension, K' will be a product of fields. If k'/k is inseparable, K' may have nilpotents. One always has $R_{k'/k}(X')_K = R_{K'/K}(X'_{K'})$. This is easily checked by looking at the functors of points on K'-algebras.

5.2.5 Example. — Let k'/k be a separable field extension. Then let $A = \prod_{k' \hookrightarrow k^s} k^s$. We have

$$R_{k'/k}(X)_{k^s} = \prod_{A/k^s} (X' \otimes_{k'} A) = \prod_\sigma \sigma^* X'.$$

This explains the $\prod$ notation for Weil restriction.

5.2.6 Example. — Suppose $k' = k(\sqrt[p]{a})$ for $a \in k \smallsetminus k^p$. Consider $R_{k'/k}(X')_{\bar{k}} = R_{(k' \otimes \bar{k})/\bar{k}}(X' \otimes_{k'} \bar{k})$. This lives over the (non-reduced) ring $\bar{k}[t]/t^p$.

5.2.7 Example. — Let k'/k be a non-separable extension. Then the group $R_{k'/k}(\mathrm{GL}_n)$ is *never* reductive, but is always pseudo-reductive. For $\bar{k}$-points, $\mathrm{GL}_n(\bar{k} \otimes k') = \mathrm{GL}_n(\bar{k}[t]/t^p)$, which has a "huge p^∞-torsion" subgroup. In general, for connected reductive $G' \neq 1$ over k', the Weil restriction $R_{k'/k}(G')$ is never reductive, but always pseudo-reductive. (Again, k'/k is non-separable here.)

***5.2.8 Theorem* (Borel-Tits IHES 27, 6.21iii)**. — *Let G be a connected, semisimple, simply-connected k-group. Then there exists (unique up to unique isomorphism) a pair $(k'/k, G')$ where k' is a finite étale k-algebra, $G' = \coprod G'_i$, where the G'_i are absolutely simple, simply connected semisimple groups over the k'_i such that $G \simeq R_{k'/k}G$.*

The uniqueness in this theorem means the following. If (k'', G'') is another such pair, then any k-isomorphism $R_{k'/k}(G') \simeq R_{k''/k}(G'')$ arises from a unique compatible pair $(\varphi : k' \xrightarrow{\sim} k'', \alpha : G' \xrightarrow{\sim} G'')$

Proof. — By Galois descent, we can assume $k = k^s$. Then $k' = \prod k^s$, so the theorem just says that G is a product of simple factors, and that isomorphisms between such products arise from bijections of their index sets, and compatible group isomorphisms. $\square$

5.2.9 Lemma. — *1. Suppose k' is a field, and X' is smooth over k'. Then X is smooth over k.*

2. If X' is geometrically connected and smooth over k', then the same are true for X over k.

Proof. — 1. Use the infinitesimal criterion. It's enough to use finite algebras.

2. In the group setting: the key case is $k = k^s$, k' a field. We'll look at $k = \mathbf{F}_p(t, u)$ and $k' = k(\sqrt[p]{t})$. Then $G(\bar{k}) = G'(k' \otimes \bar{k}) \xrightarrow{q} G'(\bar{k})$. The kernel of q has a filtration by $U_i = \{g \equiv 1 \mod \mathfrak{m}^i\}$; these are normal in $G_{\bar{k}}$, and $U_i/U_{i+1} \simeq \mathfrak{g}' \otimes_{\bar{k}} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. So "we can conclude" that $G_{\bar{k}}$ has a composition series with connected successive quotients. $\square$

Part 2 is false without the smoothness assumption! You can make examples of a geometrically integral curve $X' \subset \mathbf{A}^2_{k'}$, smooth away from one point, regular at the missing point, such that X has two connected components.

5.2.10 Proposition. — *Let k'/k be a non-zero finite reduced k-algebra, G' a pseudo-reductive k'-group with connected fibers. Then $G = R_{k'/k}(G')$ is a pseudo-reductive k-group.*

Proof. — Without loss of generality, $k = k^s$ and k' is a field. We know that G is smooth and connected. We need to show that it has no nontrivial smooth connected unipotent normal k-subgroups. Let $U \subset G$ be such a subgroup; we want to show $U = 1$. The definition of Weil restriction means that $U \hookrightarrow G$ corresponds to a map $f : U_{k'} \to G'$ over k'. This map is equivariant for the $G(k) = G'(k')$-action, and $G(k) \subset G$ is Zariski-dense (since G is smooth and $k = k^s$). Thus $f(U_{k'})$ is normal in $G_{k'}$, so $f = 1$. Thus the map $U \hookrightarrow G$ is trivial, so $U = 1$. □

5.3. Some disorienting examples. —

5.3.1 Example. — Let $k' = k(\sqrt[p]{a})$ for $a \in k \smallsetminus k^p$ and k of characteristic $p > 0$. Consider the fppf-exact sequence

$$1 \to \boldsymbol{\mu}_p \to \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}} \to 1$$

over k'. Restrict to k (and note that Weil restriction is left exact):

$$\begin{tikzcd} 1 \arrow[r] & R_{k'/k}(\boldsymbol{\mu}_p) \arrow[r] \arrow[dr, dotted] & R_{k'/k}(\mathbf{G}_{\mathrm{m}}) \\ & & \mathbf{G}_{\mathrm{m}} \arrow[u, hook] \end{tikzcd}$$

We get a fppf-exact sequence

$$1 \to R_{k'/k}(\boldsymbol{\mu}_p) \to R_{k'/k}\mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}} \to 1$$

where $R_{k'/k}(\boldsymbol{\mu}_p)$ has dimension $p-1$. However $R_{k'/k}(\boldsymbol{\mu}_p)$ has no k^s-points. Moreover, $R_{k'/k}(\mathbf{G}_{\mathrm{m}})/\mathbf{G}_{\mathrm{m}}$ is killed by p. (We can check this on k^s-points, because $(H_1/H_2)(k^s) = H_1(k^s)/H_2(k^s)$ if H_2 is smooth)

5.3.2 Example. — Consider $1 \to \boldsymbol{\mu}_p \to \mathrm{SL}_p \xrightarrow{f} \mathrm{PGL}_p \to 1$ over k' and Weil restrict:

$$1 \to R_{k'/k}(\boldsymbol{\mu}_p) \to R_{k'/k}(\mathrm{SL}_p) \to R_{k'k}(\mathrm{PGL}_p) \to 1.$$

It turns out that

$$R_{k'/k}(\mathrm{SL}_p)/R_{k'/k}(\boldsymbol{\mu}_p) \hookrightarrow R_{k'/k}(\mathrm{PGL}_p)$$

is normal with abelian (unipotent) cokernel of dimension $p-1$. Look at an open cell: $U^- \times \widetilde{T} \times U^+ \xrightarrow{f} U^- \times T \times U^+$. On tori (in suitable coordinates) f looks like $(t_1, \ldots, t_{p-1}) \mapsto (t_1^p, t_2, \ldots, t_{p-1})$. On Weil restriction, we can write down the image directly to get that the cokernel is $R_{k'/k}(\mathbf{G}_{\mathrm{m}})/\mathbf{G}_{\mathrm{m}}$.

Using $\widetilde{T} = \mathbf{G}_{\mathrm{m}}^{\Delta^\vee}$, we can prove the following theorem:

5.3.3 Theorem. — *The Weil restriction of a simply connected group is perfect.*

One reduces to SL_2 via open cells.

[Note: the map Michel constructed can be seen as the composite

$$X \hookrightarrow R_{K/k}(X_K) \xrightarrow{\sim} R_{K/k}(G_K) \xrightarrow{trace} G$$

Also, think of Weil restriction as pushforward of fppf sheaves.]

5.4. Elementary properties and examples. —

5.4.1 Lemma. — *If G is pseudo-reductive, every smooth connected normal k-subgroup $N \subset G$ is pseudo-reductive.*

Proof. — We can assume k is separably closed. We claim that $\mathscr{R}_{\mathrm{u},k}(N)$ is also normal in G, hence $\mathscr{R}_{\mathrm{u},k}(N) = 1$. It suffices to check that $\mathscr{R}_{\mathrm{u},k}(N)$ is stable under $G(k)$-conjugation (because $G(k) \subset G$ is Zariski-dense, which works because $k = k^s$ and G is smooth). But $G(k)$-conjugation preserves N, so it certainly preserves $\mathscr{R}_{\mathrm{u},k}(N)$, because it's stable under all k-automorphisms of N. $\square$

So, for example the derived subgroup $\mathscr{D}G$ is pseudo-reductive. We've already seen that this fails for quotients.

5.4.2 Example. — Let k'/k be a finite field extension. Let G' be a connected, simply connected semisimple group over k'. Let $\widetilde{G}' \to G'$ be the simply-connected cover, and $\boldsymbol{\mu} = \ker(\widetilde{G}' \to G')$; this is of multiplicative type, because a maximal torus $\widetilde{T}' \subset \widetilde{G}'$ is its own centralizer, hence it contains all central subgroups. In fact, $\boldsymbol{\mu} \subset \widetilde{T}'[n]$ for some $n \geqslant 1$. Let $G = R_{k'/k}(\widetilde{G}')/R_{k'/k}(\boldsymbol{\mu}) \hookrightarrow R_{k'/k}(G')$. This injection has commutative cokernel (because this cokernel lives inside $\mathsf{R}^1_{\mathrm{fppf}} f_* \boldsymbol{\mu}$, where $f : \operatorname{Spec} k' \to \operatorname{Spec} k$). The group G is perfect! We see that $G = \mathscr{D}(R_{k'/k}G)$, so G is pseudo-reductive. By open cell consideration, $(R_{k'/k}(G')/G)_{k^s}$ can be described in terms of "Weil-restricted $\mathbf{G}_{\mathrm{m}}$ modulo $\mathbf{G}_{\mathrm{m}}$," which is unipotent. Precisely, it is $\operatorname{cok}(R_{k'/k}(\widetilde{T}') \to R_{k'/k}(T'))$. If k'/k is purely inseparable, the cokernel has no nontrivial tori, so it's unipotent.

Warning: over every imperfect k, there exists pseudo-semisimple G and normal pseudo-semisimple $N \subset G$ such that G/N is *not* pseudo-reductive.

Suppose a pseudo-reductive group G is *pseudo-split* if it has a split maximal k-torus. Does G have a pseudo-split k^s-form? (i.e. does there exist pseudo-split H/k such that $G_{k^s} \simeq H_{k^s}$) This fails even if k is perfect! When such a Chevalley group exists, it is unique.

The good news is that there is a good structure theory of (high-dimensional) root groups, root system (which could be non-reduced), open-cell, Bruhat decomposition on rational points, …for pseudo-reductive groups. Also, Cartan subgroups are commutative (i.e. if $T \subset G$ is a maximal torus, then $Z_G(T)$ is commutative and pseudo-reductive). Also, $\mathscr{D}(\mathscr{D}G) = \mathscr{D}G$.

5.4.3 Magic Lemma. — *Let G be a pseudo-reductive k-group, and $X \subset G$ a geometrically integral closed subscheme that meets 1. If $X_{\bar{k}} \subset R_u(G_{\bar{k}})$, then $X = 1$.*

Proof. — Without loss of generality, $k = k^s$. Let $H \subset G$ be the k-subgroup normally generated by X (so H is the Zariski-closure of the smallest normal subgroup of $G(k)$ containing by $X(k)$). It is a basic fact that H is smooth connected normal. Also, formation of H commutes with arbitrary extension of the ground field. But $X_{\bar{k}} \subset \mathscr{R}_{\mathrm{u}}(G_{\bar{k}})$, so $H_{\bar{k}} \subset \mathscr{R}_{\mathrm{u}}(G_{\bar{k}})$. It follows that $H \subset \mathscr{R}_{\mathrm{u}}(G)$, so $H = 1$, whence $X = 1$. □

5.4.4 Corollary. — *1. If G is pseudo-semisimple, then $\mathscr{R}_k(G) = 1$.*

2. If $f_1, f_2 : G \twoheadrightarrow H$ are morphisms between pseudo-reductive groups such that $f_{1,\bar{k}} = f_{2,\bar{k}}$. Then $f_1 = f_2$.

Proof. — 1. Let $X = \mathscr{R}_k(G)$. Then $X_{\bar{k}} \subset R_{\bar{k}}(G_{\bar{k}}) = \mathscr{R}_{\bar{k}}(G_{\bar{k}})$. Apply the magic lemma.

2. Let X be the Zariski-closure of the map $g \mapsto f_1(g)f_2(g)^{-1}$. Note that $X = 1$ if and only if $f_1 = f_2$. But $X_{\bar{k}} \subset \mathscr{R}_{\bar{k}}(H_{\bar{k}})$, so apply the magic lemma. □

Warning: the converse to 1 is false!

5.4.5 Theorem. — *Suppose G is pseudo-reductive over k.*

1. If G is solvable, then it is commutative.

2. $\mathscr{D}G$ is perfect (i.e. pseudo-semisimple).

3. If $S \subset G$ is a k-torus, then $Z_G(S)$ is pseudo-reductive.

4. If $T \subset G$ is a maximal k-torus, then $Z_G(T)$ is commutative.

Proof. — 1. Let $X = \mathscr{D}G$. Note that $G_{\bar{k}}^{\mathrm{red}}$ is solvable and reductive, hence commutative. So $\mathscr{D}(G_{\bar{k}}^{\mathrm{red}}) = 1$. But $\mathscr{D}(G)_{\bar{k}} \to G_{\bar{k}}^{\mathrm{red}}$ has image in $\mathscr{D}(G_{\bar{k}}^{\mathrm{red}}) = 1$. But $X_{\bar{k}} = \mathscr{D}(G)_{\bar{k}} = \mathscr{D}(G_{\bar{k}})$, so we can apply the magic lemma.

2. This is much trickier.

3-4. [...didn't write down proof...] □

Later on, we will work with root groups over k. We will show that if $N' \subset N \subset G$ are all smooth connected normal (in the next), then $N' \subset G$ is normal. (So normality is transitive for pseudo-reductive groups.) The first step in the proof is to pass to derived groups. A reference for this is [**CGP10**, 1.2.7,3.1.10].

5.5. Standard construction. — Let's start with motivation via reductive groups built from standard semisimple groups. If G is connected reductive, then $G = \mathscr{D}G \cdot Z$, where Z is a maximal k-torus and $Z \cap \mathscr{D}G$ is finite central. So $G = (Z \times \mathscr{D}G)/\boldsymbol{\mu}$.

Let $T \subset G$ be a maximal k-torus. Then $S = T \cap \mathscr{D}G$ is a maximal k-torus of $\mathscr{D}G$ (this is valid with $\mathscr{D}G$ replaced by any smooth connected normal k-subgroup).

Moreover, $T = ZS$ is an almost direct product. We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \boldsymbol{\mu} & \longrightarrow & \mathscr{D}G \times Z & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & S & \longrightarrow & \mathscr{D}G \rtimes T & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

The square on the left is a central pushout. We can write $G = (\mathscr{D}G \times T)/S$. This is better because Weil restriction commutes with quotients by *smooth* subgroups.

[...more stuff I didn't understand...stopped taking notes...]

Suppose G is a pseudo-reductive group over k. We have $\mathscr{R}_{\mathrm{u}}(G_{\bar{k}}) \subset G_{\bar{k}}$. By Galois descent, this comes from $U \subset G_{k^{\mathrm{perf}}}$. This descends to some $U_0 \subset G_K$, for a finite purely inseparable extension K/k.

5.5.1 Fact. — *If K/k is a field extension and X is a k-scheme, then any closed $Z \subset X_K$ desends to $Z_0 \subset X_0$, for a unique $k_0 \subset K$ and X_0 over k_0.*

This is [**EGA 4**, 4.8].

In the discussion above, take K/k minimal such that $\mathscr{R}_{\mathrm{u},K}(G_K)$ is a K-descent of $\mathscr{R}_{\mathrm{u}}(G_{\bar{k}})$. We can form $G' = G_K^{\mathrm{red}}$. This corresponds to $i_G : G \to R_{K/k}(G')$. Unfortunately, i_G is very far from being an isomorphism. First problem: if we pass to $G_{\bar{k}} \twoheadrightarrow G_{\bar{k}}^{\mathrm{red}} \twoheadrightarrow G_{\bar{k}}^{\mathrm{ad}} = \prod G_i$, where the G_i are adjoint semisimple. The kernel of $G \to G_i$ has a minimal field of definition k_i'/k. But K lumps together all the k_i'.

Now we can outline the construction. Let k' be a finite reduced k-algebra. Let $G' = \coprod G_i'$ be a reductive k'-group with each fiber G_i' connected. The main case we're interested in is when all G_i' are absolutely simple and simply connected semisimple. Let $T = \coprod T_i'$ be a maximal k'-torus in G'. The Weil restriction $R_{k'/k}(T') \subset R_{k'/k}G'$ is a Cartan subgroup – see [**CGP10**, A.5.15(i)]. Consider a factorization $R_{k'/k}(T') \xrightarrow{\phi} C \to R_{k''/k}(\overline{T}') \hookrightarrow R_{k'/k}(G^{\mathrm{ad}})$, where C is commutative pseudo-reductive. Let

$$G = \frac{R_{k'/k}(G') \rtimes C}{R_{k'/k}(T')}.$$

The natural map $C \to G$ turns out to be an inclusion, and it makes C a Cartan subgroup of G.

5.5.2 Theorem. — *1. G is pseudo-reductive.*

2. If G is not commutative, then for any maximal torus $\mathscr{T} \subset G$ and $\mathscr{C} = Z_G(\mathscr{T})$, we can find a four-tuple (G'', k'', T'', C'') such that

- *$C'' = \mathscr{C}$*
- *all fibers G_j'' over factor fields k_j'' are absolutely simple, simply-connected, semisimple.*

3. This "better 4-tuple" is unique up to unique isomorphism

We call the G as in 2 "standard." For a proof of pseudo-reductivity of the standard construction: $(\mathscr{G} \rtimes C)/\mathscr{C}$ ($\mathscr{C} \subset \mathscr{G}$ self-centralizing, $\mathscr{G}$ pseudo-reductive, and $\mathscr{C} \to C$ acts on $\mathscr{G}$ compatible with the $\mathscr{G}$-action) see [**CGP10**, 1.4.3].

5.6. Fields of definition and splitting of central extensions. — Last time we "massaged" our 4-tuple (G', k', T', C) in standard construction to have the extra property that all the fibers G'_i over factor fields are absolutely simple and simply connected. With these extra properties, we claimed a very strong uniqueness result for the 4-tuple, given (G, T) ($C = Z_G(T)$). Uniqueness reduces existence proofs for "standardness" to the case of a separably closed ground field.

The key issue for the uniqueness aspect is: suppose k'/k is a finite field extension, G' is an absolutely simple connected semisimple k'-group, and $G = R_{k'/k}(G')$. In what sense does G determine $(G', k'/k)$ uniquely up to unique isomorphism?

5.6.1 Example. — Suppose k'/k is separable, and G' is simply connected. We saw that $(G', k'/k)$ is unique via an indirect reason: if $(G'', k''/k)$ is another pair, any k-isomorphism $R_{k'/k}(G') \xrightarrow{\sim} R_{k''/k}(G'')$ arises from a unique pair (φ, α), where $\varphi : k' \xrightarrow{\sim} k''$ and $\alpha : G' \xrightarrow{\sim} G''$ are compatible.

We want a more "explicit" way to extract (G', k') from G. There is a natural map $q : G_{k'} = R_{k'/k}(G')_{k'} \to G'$ (much like $f^* f_* \mathscr{F} \to \mathscr{F}$ for sheaves). For a k'-algebra A', when evaluated at A' this is $G'(k' \otimes_k A') \to G'(A')$, induced from $k' \otimes_k A' = (k' \otimes_k k') \otimes_{k'} A' \twoheadrightarrow k' \otimes_{k'} A' = A'$. Over k', q gives an absolutely simple conected semisimple quotient of $G_{k'}$. If $k' \otimes_k k'$ has several copies of k' as quotient fields (e.g. $k' \otimes_k k' \twoheadrightarrow k'$ given by $a \otimes b \mapsto a\sigma(b)$, for $\sigma \in \mathrm{Aut}(k'/k)$), then we get quotients $q_\sigma : G_{k'} \twoheadrightarrow \sigma^*(G')$, with *different* kernels than q. The morphism q is a surjection with smooth connected kernel [**CGP10**, A.5.11(1,3)].

5.6.2 Example. — Suppose k'/k is purely inseparable. Then $\ker(q)$ is *unipotent*. So $\ker(q) = \mathscr{R}_{\mathrm{u},k'}(G_{k'})$ is a k'-descent of $\mathscr{R}_{\mathrm{u}}(G_{\bar{k}})$. It turns out that this k'/k is the *minimal* field of definition over k of $\mathscr{R}_{\mathrm{u}}(G_{\bar{k}}) \subset G_{\bar{k}}$. Note that $G' = G_{k'}/\mathscr{R}_{\mathrm{u},k'}(G_{k'})$ recovers G' from G in such cases.

5.6.3 Proposition. — *Let H be a non-solvable smooth connected affine k-group. Consider pairs $(k'/k, q : H_{k'} \to \mathscr{H}')$ where*

- *k' is a nonzero finite reduced k-algebra ($= \prod k'_i$)*
- *$\mathscr{H}' = \coprod \mathscr{H}'_i$ is a k'-group where each $\mathscr{H}'_i$ is an absolutely simple connected semisimple k'_i-group of adjoint type.*
- *$q : H_{k'_i} \to \mathscr{H}'_i$ is a maximal absolutely simple adjoint semisimple quotient.*

There is an initial such pair, which we denote by $(K/k, q_H : H_K \to \overline{\mathscr{H}})$. That is, for any $(k'/k, q : H_{k'} \twoheadrightarrow \mathscr{H}')$, there is a unique $\varphi : K \to k'$ as k-algebras such that there *is a (unique) isomorphism making the following diagram commute:*

$$\begin{tikzcd} H_{k'} \arrow[r, two heads, "q"] \arrow[dr, "\varphi^*(q_H)"'] & \mathscr{H}' \\ & \overline{\mathscr{H}}_K \arrow[u, "\sim"] \end{tikzcd}$$

Proof. — This is [**CGP10**, 4.2.1]. It's enough to prove the existence and uniqueness under the assumption that $k = k^s$. Look at $H_{\bar{k}} \twoheadrightarrow H_{\bar{k}}^{\mathrm{ss}} \twoheadrightarrow H_{\bar{k}}^{\mathrm{ad}} = \prod H_i'$ consider some composite $q_i : H_{\bar{k}} \twoheadrightarrow H_i'$. Let k_i'/k be the minimal field of definition over k of $\ker(q_i) \subset H_{\bar{k}}$. Put $K = \prod k_i'$ and $q_i : H_{k_i'} \twoheadrightarrow H_{k_i'}/(\text{descent of } \ker(q_i'))$. $\square$

5.6.4 Example. — If $H = R_{k'/k}(G')$, where k'/k is Galois, then $K = k'$ and $q : H_{k'} \twoheadrightarrow G'$ is the canonical quotient. For any $\gamma \in \mathrm{Gal}(k'/k)$, we also have $q_\gamma : H_{k'} \twoheadrightarrow \gamma^*(G')$. This map is $\gamma^*(q_H)$. So by keeping track of the quotient map, we eliminate all ambiguity in the field maps.

5.6.5 Definition. — The *simply connected datum* attached to H is $(K/k, \widetilde{\mathscr{H}}, f)$, where $\widetilde{\mathscr{H}} \to \overline{\mathscr{H}}$ is the simply-connected simple cover and $f : H \to R_{K/k}(\overline{\mathscr{H}})$ corresponds to $q : H_K \twoheadrightarrow \overline{\mathscr{H}}$.

5.6.6 Example. — Suppose G is in standard pseudo-reductive form $(G', kk'/k, T', C)$, with all G_i' absolutely simple over k_i' and simply connected. The simply connected datum is $(k'/k, G', f)$, where

$$f : G = \frac{R_{k'/k}(G') \rtimes C}{R_{k'/k}(T')} \to \frac{R_{k'/k}(R^{\mathrm{ad}}) \rtimes R_{k'/k}(\overline{T}')}{R_{k'/k}(\overline{T}')} = R_{k'/k}(G^{\mathrm{ad}}).$$

Proof. — Without loss of generality, $k = k^s$. Chase fields of definition, and do a bit more work with adjoint quotients. $\square$

So if we want to prove some non-commutative (hence non-solvable) pseudo-reductive group G is standard, its simply connected datum gives a good candidate for $(k'/k, G')$. But how are we to relate G to $(R_{k'/k}(G') \rtimes C)/R_{k'/k}(T')$? All we have is a map $R_{k'/k}(G') \to R_{k'/k}(G'^{\mathrm{ad}})$. In the end, we need to exploit the minimality of k'.

Any pseudo-reductive G is generated by its derived subgroup and a single Cartan: $G = C \cdot \mathscr{D}G$. This is because $G \twoheadrightarrow G/\mathscr{D}G$ is a *commutative* quotient. The Cartan $C \subset G$ will have to map surjectively onto $G/\mathscr{D}G$, whence the claim. By fiddling around with C, we can show that G is standard if and only if $\mathscr{D}G$ is standard. So its enough to show that $\mathscr{D}G$ is standard. Thus we can assume $k = k^s$ and G is pseudo–semisimple.

5.6.7 Theorem. — *If $6 \in k^\times$, then G is standard. If k has characteristic 3, or k has characteristic 2 with $[k : k^2] = 2$, then G is "generalized standard."*

How to prove the main theorem? A key tool is root groups. We want to reduce to the case where G is an "irreducible root system" (akin to "simple factors" of a connected semisimple group).

5.6.8 Proposition. — *Let G be pseudo-semisimple. Let $\{N_i\}$ be a set of minimal nontrivial normal pseudo-semisimple k-subgroups. Then the N_i's pairwise commute, $\#\{N_i\} < \infty$, and $\prod N_i \to G$ is surjective with central kernel.*

Proof. — Without loss of generality, $k = k^s$. We have a split maximal torus. Fiddle around with "root groups." □

Warning: the kernel of $\prod N_i \twoheadrightarrow G$ could have positive dimension.

The upshot of this is that when $k = k^s$, we can reduce to the case where G has *no* nontrivial smooth connected normal k-subgroups $N \neq G$ (i.e. G pseudo-simple, which is equivalent to the root system of G being irreducible).

We reformulate the problem. Let G be an absolutely pseudo-simple (i.e. G_{k^s} is pseudo-simple) group. Let K be the minimal field of definition of $\mathscr{R}_{\mathrm{u}}(G_{\bar{k}}) \subset \mathscr{R}(G_{\bar{k}})$. We have $G_K \twoheadrightarrow G_K^{\mathrm{ss}} = G'$, and

$$\mathscr{D}(R_{K/k}(G')) = \frac{R_{K/k}(G')}{R_{K/k}(\boldsymbol{\mu})},$$

where $G' = \widetilde{G}'/\boldsymbol{\mu}$. We would like there to exist j_G as in the diagram:

$$\begin{tikzcd} & R_{K/k}(\widetilde{G}') \arrow[dl, dotted, "j_G"'] \arrow[d] \\ G \arrow[r, "\xi_G"'] & \frac{R_{K/k}(\widetilde{G}')}{R_{K/k}(\boldsymbol{\mu})}. \end{tikzcd}$$

If j_G exists, then it is unique. Any two are related through multiplication against a homomorphism $R_{K/k}(\widetilde{G}') \to R_{K/k}(\boldsymbol{\mu})$. But the first group is perfect and the second is commutative, so such homomorphisms are trivial.

5.6.9 Lemma. — *G is standard if and only if j_G exists.*

5.6.10 Theorem. — *G is standard if and only if ξ_G is surjective and* $\ker(\xi_G) \subset G$.

Proof. — This is [**CGP10**, 5.3.8]. The important direction is $\Leftarrow$. □

We have an exact sequence

$$1 \longrightarrow G \longrightarrow Z \xrightarrow{\xi_G} \frac{R_{K/k}(\widetilde{G}')}{R_{K/k}(\boldsymbol{\mu})} \longrightarrow 1.$$

The group Z is central with no nontrivial connected smooth k-subgroups. Extend this to a diagram

$$\begin{tikzcd} 1 \arrow[r] & Z \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & R_{K/k}(\widetilde{G}') \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & G \arrow[r] & Z \arrow[r, "\xi_G"] & \frac{R_{K/k}(\widetilde{G}')}{R_{K/k}(\boldsymbol{\mu})} \arrow[r] & 1. \end{tikzcd}$$

where the square on the right is a pullback. The miracle is: for Z a commutative affine algebraic k-group scheme with no nontrivial smooth connected k-subgroup (i.e. $Z(k^s)$ is finite)there is a criterion for central extensions $1 \to Z \to E \to \mathscr{G} \to 1$ to be split, at least for pseudo-reductive k-groups $\mathscr{G}$. This criterion is: "$Z_{\mathscr{G}}(T)_{k^s}$ is rationally expressed in terms of root groups." In other words, there are rational maps $h_i : Z_{\mathscr{G}}(T)_{k^s} \to U_{\alpha_i}$ such that $\prod h_i : Z_{\mathscr{G}}(T)_{k^s} \to \mathscr{G}$ is the inclusion.

5.6.11 Example. — Take $\mathscr{G} = R_{k'/k}(\widetilde{G}')$. Then $Z_{\mathscr{G}}(T) = R_{k'/k}(\widetilde{T}')$, where $\widetilde{T}' = \prod_{a \in \Delta} a^\vee \mathbf{G}_{\mathrm{a}}$. Here we use the formula

$$\begin{pmatrix} t & \\ & 1/t \end{pmatrix} = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ -1/t & 1 \end{pmatrix} \begin{pmatrix} 1 & t-1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}.$$

5.7. Dynamic method, root groups, and exotic constructions. — We start by giving some examples of non-standard absolutely pseudo-simple groups in characteristics 2 and 3. This can happen for several reasons:

Multiple edges in Dynkin diagrams of G_2, B_n, C_n, F_2, and A_1. In characteristic 3, G_2 and in characteristic 2, the other guys don't work correctly because of some extra triviality in the commutation relations within $\mathfrak{g}$. So $\mathfrak{g}$ as a G-module has "extra" subrepresentations. More specifically, let G be connected of one of these types. Then there is a unique G-subrepresentation $\mathfrak{n} \subset \mathfrak{g}$, defined by $\mathfrak{n} = \mathfrak{t}_< + \mathfrak{g}_<$, where $\mathfrak{t}_<$ is the span of the short roots and $\mathfrak{g}_<$ is the span of the short root space. This is a p-Lie subalgebra. We can define $N = \exp(\mathfrak{n}) \subset \ker(F_{G/k} : G \to G^{(p)})$, a normal k-subgroup scheme. So the Frobenius $F_{G/k} : G \to G^{(p)}$ factors through $\pi_G : G \to \overline{G} = G/N$. It turns out that $\overline{G}$ is simply connected of dual type, and we call π_G the *very special isogeny*. The map π_G carries short root groups to long ones via $x \mapsto x^p$, and it carries long root groups onto short ones via $x \mapsto x$.

5.7.1 Example. — Let (V, q) be a non-degenerate, odd-dimensional quadratic space in characteristic 2. We have an exact sequence

$$\mathrm{Spin}(q) \to \mathrm{SO}(q) \to \mathrm{Sp}(V/V^\perp, \overline{B}_q).$$

The composite is an isogeny $B_n \to C_n$.

5.7.2 Basic exotic construction. — Let $k \subsetneq k' \subset k^{1/p}$ for $p \in \{2, 3\}$. Let G' be a simply connected absolutely simple of type G_2 in characteristic 3, or F_4, B_n, C_n if $p = 2$. [...didn't understand...]

5.7.3 Theorem. — *In characteristic* 3*, and in characteristic* 2 *with* $[k : k^2] = 2$*, the above [left out] give "all" non-standard examples.*

When $[k : k^2] > 2$, things are even more complicated.

If $[k : k^p] = p$, then $k' = k^{1/p}$. So in the above constructions, a miracle happens: $\xi_G : G(k) \to G'(k')$ is bijective on $k^s \otimes k'_s$-points.

Recall our basis puzzle: G is a pseudo-simple group over a separably closed field k. We want to show that $\xi_G : G \to R_{K/k}(\widetilde{G}')/R_{K/k}(\boldsymbol{\mu})$ is surjective with central kernel. We will concentrate on surjectivity. For this we need a satisfactory theory of root groups.

Let's start by recalling a dynamic definition of parabolic subgroups. Let G be a connected reductive group over a field k, $P \subset G$ a parabolic subgroup. Then P arises from $\lambda : \mathbf{G}_\mathrm{m} \to G$ over k in the following sense. Put (informally):

$$P_G(\lambda) = \left\{ g \in G : \text{the value } \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists} \right\}.$$

More precisely, for $g \in G(R)$, we have $\mathbf{G}_{\mathrm{m},R} \to G_R$ via the action above. If this extends to $\mathbf{A}^1_R$, we say the limit exists, and call the image of 0 the limit.

5.7.4 Example. — Let $G = \mathrm{SL}_3$. Then

$$\lambda(t) = \begin{pmatrix} t^{13} & & \\ & t^{13} & \\ & & t^{13} \end{pmatrix} \Rightarrow P_G(\lambda) = \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}.$$

It is a nontrivial that such $P_G(\lambda)$ are parabolic k-subgroups, and that this construction gives all k-parabolics. We will use this construction to *define* the notion of a parabolic subgroup of a pseudo-reductive group. If G'/P' is a complete variety over a purely inseparable k'/k, then $R_{k'/k}(G'/P')$ is *never* complete. So completeness of the quotient is the wrong notion of parabolicity in the pseudo-reductive case. The unipotent radical $U_G(\lambda) = \mathscr{R}_\mathrm{u}(P_G(\lambda))$ also has a dynamic description:

$$U_G(\lambda) = \left\{ g \in G : \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} = 1 \right\}.$$

If we put $Z_G(\lambda) = Z_G(\lambda(\mathbf{G}_\mathrm{m}))$, then $P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda)$. The group $Z_G(\lambda)$ is a Levi subgroup of $P_G(\lambda)$.

5.7.5 Theorem. — *Let G be an affine algebraic k-group scheme. Let $\lambda : \mathbf{G}_\mathrm{m} \to G$ be a k-homomorphism. Put $t \cdot g = \lambda(t) g \lambda(t)^{-1}$. Define $P_G(\lambda) = \{g \in G : \exists \lim t \cdot g\}$ and $U_G(\lambda) = \ker(P_G(\lambda) \to G)$, and $Z_G(\lambda) = Z_G(\lambda) = P_G(\lambda) \cap P_G(-\lambda)$. Then*

1. *These are represented by closed k-subgroup schemes of G.*
2. $\mathrm{Lie}(P_G(\lambda)) = \mathfrak{g}_{\lambda \geqslant 0}$, $\mathrm{Lie}(U_G(\lambda 0)) = \mathfrak{g}_{\lambda > 0}$.
3. $P_G(\lambda) \simeq Z_G(\lambda) \ltimes U_G(\lambda)$.
4. *$U_G(-\lambda) \times P_G(\lambda) \to G$ is an open immersion.*
5. *In the smooth case, $U_G(\lambda)$ is split unipotent.*

Proof. — This is [**CGP10**, 2.1.8]. If $\mathbf{G}_{\mathrm{m}}$ acts on G, then $\mathbf{G}_{\mathrm{m}}$ acts on $k[G]$, so we get a $\mathbf{Z}$-grading on $k[G]$. Check that $k[G]_{<0}$ is an ideal which cuts out a quotient of $k[G]$ representing $P_G(\lambda)$. For part 4, $G \hookrightarrow \mathrm{GL}_n$. Then $P_G(\lambda) = G \cap P_{\mathrm{GL}_n}(\lambda)$, $U_G(\lambda) = G \cap U_{\mathrm{GL}_n}(\lambda)$, ... We know that $U_{\mathrm{GL}_n}(-\lambda) \times P_{\mathrm{GL}_n}(\lambda) \to \mathrm{GL}_n$ is an open immersion. Roughly speaking, we intersect this immersion with G on both sides. □

It follows that if G is smooth connected, then so are $P_G(\lambda)$, $U_G(\lambda)$, and $Z_G(\lambda)$.

5.7.6 Definition. — Let G be a smooth connected affine k-group. A *pseudo-parabolic subgroup* $H \subset G$ is a k-subgroup of the form $P_G(\lambda) \cdot \mathscr{R}_{\mathrm{u},k}(G)$ for some $\lambda : \mathbf{G}_{\mathrm{m}} \to G$ over k.

5.7.7 Example. — Suppose $G \twoheadrightarrow G'$, $\lambda \in X_*(G)$, and $\lambda' = \mathbf{G}_{\mathrm{m}} \xrightarrow{\lambda} G \to G'$. Then we get morphisms

$$\begin{aligned} P_G(\lambda) &\to P_{G'}(\lambda') \\ U_G(\lambda) &\to U_{G'}(\lambda') \\ Z_G(\lambda) &\to Z_{G'}(\lambda'). \end{aligned}$$

We claim these are all surjections. Why? Consider the following diagram:

$$\begin{tikzcd} U_G(-\lambda) \times Z_G(\lambda) \times U(\lambda) \arrow[r, hook] \arrow[d] & G \arrow[d, two heads] \\ U(-\lambda') \times Z(\lambda') \times U(\lambda') \arrow[r, hook] & G' \end{tikzcd}$$

in which the horizontal arrows are open immersions. The group on the lower left has closed and dense image hence it's surjective on each factor!

If G is pseudo-reductive, then each $Z_G(\lambda)$ is also pseudo-reductive.

Let G be a pseudo-reductive k-group, $T \subset G$ a split maximal k-torus, and $\Phi = \Phi(G,T)$ the induced set of roots. For $a \in \Phi$, let $T_a = (\ker a)^\circ_{\mathrm{red}} \subset T$ be the codimension-1 torus killed by a. Consider $\lambda_a : \mathbf{G}_{\mathrm{m}} \to T \subset Z_G(T_a)$ such that $\langle a, \lambda_a \rangle > 0$. Put $U_{(a)} = U_{Z_G(\lambda)}(\lambda_a)$.

5.7.8 Proposition. — *$U_{(a)}$ is independent of the choice of λ_a.*

We call $U_{(a)}$ the *root group* for a. The group $\mathrm{Lie}(U_{(a)})$ is the direct sum of weight spaces for $\mathbf{Q}_{>0}a \cap \Phi$ inside $X(T)_{\mathbf{Q}}$.

5.7.9 Proposition. — *$U_{(a)}$ is commutative, p-torsion in characteristic $p > 0$, and even a vector group. Moreover, it has a T-equivariant linear structure.*

Proof. — Use the magic lemma. □

5.7.10 Theorem. — *If k does not have characteristic 2, then $\Phi \subset X(T) = X(T_{\bar{k}})$ is $\Phi(G^{\mathrm{red}}_{\bar{k}}, T_{\bar{k}})$. Moreover, $(U_{(a)})_{\bar{k}} \twoheadrightarrow U_a \subset G^{\mathrm{red}}_{\bar{k}}$.*

Proof. — This is [**CGP10**, 2.3.10]. □

[. . . didn't get the end. . .]

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