

CONSTRUCTING GALOIS REPRESENTATIONS WITH  
PRESCRIBED SATO–TATE DISTRIBUTION

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Abstract here.

## BIOGRAPHICAL SKETCH

Daniel Miller was born in St. Paul, Minnesota. He completed his Bachelor of Science at the University of Nebraska–Omaha, during which he attended Cornell’s Summer Mathematics Institute in 2011. He started his Ph.D. at Cornell planning on a career in academia, but halfway through had a change of heart, and will be joining Microsoft’s Analysis and Experimentation team as a Data Scientist after graduation.

This thesis is dedicated to my undergraduate thesis advisor, Griff Elder. He is the reason I considered a career in math, his infectious enthusiasm for number theory has inspired me more than I can say.

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# CHAPTER 1

## INTRODUCTION

Todo: statistical heuristics. Statistical behavior of the Kolomogorov–Smirnov statistic.

### 1.1 Motivation from classical analytic number theory

We start with a problem central to the history of number theory: counting prime numbers. As usual, let  $\pi(x)$  be the number of rational primes  $\leq x$  and  $\text{Li}(x) = \int_2^x \frac{dt}{\log t}$  be the logarithmic integral. For any  $x$ , there is a (normalized) empirical measure capturing the distribution of those primes  $\leq x$ :

$$P_x = \frac{1}{\pi(x)} \sum_{p \leq x} \delta_{p/x},$$

which is supported on the unit interval  $[0, 1]$ . The prime number theorem tells us that as  $x \rightarrow \infty$ , these empirical measures approach the “true” measure  $L_x = \frac{\text{Li}(tx)}{\text{Li}(x)} dt$ . Traditionally, the prime number theorem is proven by showing that the Riemann  $\zeta$ -function has a meromorphic continuation past  $\Re = 1$ .

**Theorem 1.1.1.** *The function  $\zeta(s)$  admits a meromorphic continuation past  $\Re = 1$  with at most a simple pole at  $s = 1$  if and only if  $P_x \rightarrow L_x$  in the weak sense as  $x \rightarrow \infty$ .*

Since  $\zeta(s)$  has the desired property, the prime number theorem is true. It is natural to try to quantify the rate of converge of  $P_x$  to  $L_x$ . One natural way to do this is via the discrepancy

$$D(P_x, L_x) = \sup_{t \in [0, 1]} |P_x[0, t] - L_x[0, t]| = \sup_{t \in [0, 1]} \left| \frac{\pi(tx)}{\pi(x)} - \frac{\int_2^{tx} \frac{ds}{\log s}}{\int_2^x \frac{ds}{\log s}} \right|.$$

Numerical experiments suggest that  $D(P_x, L_x) \ll x^{-\frac{1}{2} + \epsilon}$ , and in fact we have the following result.

**Theorem 1.1.2.** *The Riemann Hypothesis holds if and only if  $D(P_x, L_x) \ll x^{-\frac{1}{2}+\epsilon}$ .*

Of course, neither side of this equivalence is known for certain to be true!

The above discussion finds a natural generalization in Artin  $L$ -functions. Let  $K/\mathbf{Q}$  be a finite Galois extension with group  $G = \text{Gal}(K/\mathbf{Q})$ . For any irreducible representation  $\rho: G \rightarrow \text{GL}_d(\mathbf{C})$ , there is a corresponding  $L$ -function defined as

$$L(\rho, s) = \prod_p \frac{1}{\det(1 - \rho(\text{fr}_p)p^{-s})},$$

where here (and for the remainder of this thesis) we tacitly omit those primes  $p$  at which  $\rho$  is ramified. Given a cutoff  $x$ , there is a natural empirical measure

$$P_x = \frac{1}{\pi(x)} \sum_{p \leq x} \delta_{\text{fr}_p},$$

where  $\text{fr}_p$  is a conjugacy class in  $G$ . Let

$$D(P_x) = \sup_{c \in G^\natural} \left| P_x(c) - \frac{1}{\#G^\natural} \right|,$$

where  $G^\natural$  is the set of conjugacy classes in  $G$ .

**Theorem 1.1.3.** *The measure  $P_x$  converge weakly to the uniform measure on  $G^\natural$  if and only if the function  $L(\rho, s)$  admits analytic continuation past  $\Re = 1$  for all nontrivial  $\rho$ .*

Both sides of this equivalence are true, and known as the Chebotarev density theorem. Moreover, there is a version of the strong Prime Number Theorem in this context.

**Theorem 1.1.4.** *The bound  $D(P_x) \ll x^{-\frac{1}{2}+\epsilon}$  holds if and only if each  $L(\rho, s)$ ,  $\rho$  nontrivial, satisfies the Riemann Hypothesis.*

This whole discussion generalizes to a more complicated set of Galois representations—those arising from elliptic curves.

## 1.2 Discrepancy and Riemann Hypothesis for elliptic curves

Let  $E/\mathbf{Q}$  be a non-CM elliptic curve. For any prime  $l$ , there is an  $l$ -adic Galois representation  $T_l E$  associated to  $E$ , known as the Tate module. This is a rank-2  $\mathbf{Z}_l$ -module with continuous  $G_{\mathbf{Q}}$ -action, so it induces a continuous representation  $\rho_{E,l}: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{Z}_l)$ . It is known [Sil09, Th. V.1.1] that the quantities  $a_p(E) = \mathrm{tr} \rho_l(\mathrm{fr}_p)$  lie in  $\mathbf{Z}$  and satisfy the Hasse bound  $|a_p(E)| \leq 2\sqrt{p}$ . Thus we can define, for each unramified prime  $p$ , the corresponding Satake parameter for  $E$ :

$$\theta_p(E) = \cos^{-1} \left( \frac{a_p(E)}{2\sqrt{p}} \right) \in [0, \pi).$$

The Satake parameters are packaged into an  $L$ -function as follows:

$$L^{\mathrm{an}}(E, s) = \prod_p \frac{1}{(1 - e^{i\theta_p(E)} p^{-s})(1 - e^{-i\theta_p(E)} p^{-s})} = \prod_p \frac{1}{1 - \det \begin{pmatrix} e^{i\theta_p} & \\ & e^{-i\theta_p} \end{pmatrix} p^{-s}}.$$

More generally we have, for each irreducible representation of  $\mathrm{SU}(2)$ , which will be  $\mathrm{sym}^k$  for some  $k \geq 1$ , the  $k$ -th symmetric power  $L$ -function

$$L^{\mathrm{an}}(\mathrm{sym}^k E, s) = \prod_p \prod_{j=0}^k \frac{1}{1 - e^{i(k-2j)\theta_p(E)} p^{-s}} = \prod_p \frac{1}{1 - \det \mathrm{sym}^k \begin{pmatrix} e^{i\theta_p} & \\ & e^{-i\theta_p} \end{pmatrix} p^{-s}}.$$

Numerical experiments suggest that the Satake parameters are distributed with respect to the Sato–Tate distribution  $\mathrm{ST} = \frac{2}{\pi} \sin^2 \theta \, d\theta$ . Indeed, for any cutoff  $x$ , let  $P_x$  be the empirical measure

$$P_x = \frac{1}{\pi(x)} \sum_{p \leq x} \delta_{\theta_p}.$$

The convergence of the  $P_x$  to the Sato–Tate measure is closely related to the analytic properties of the  $L(\mathrm{sym}^k E, s)$ . First, here is the famous Sato–Tate conjecture (now a theorem) in our language.

**Theorem 1.2.1** (Sato–Tate conjecture). *The measures  $P_x$  converge weakly to ST.*

**Theorem 1.2.2.** *Let Sato–Tate conjecture holds for  $E$  if and only if each of the functions  $L(\mathrm{sym}^k E, s)$  have analytic continuation past  $\Re = 1$ .*

The stunning recent proof of the Sato–Tate conjecture [CHT08; Tay08; HSBT10] in fact showed that the functions  $L(\mathrm{sym}^k E, s)$  were potentially automorphic, which gives analytic continuation.

The “usual” Riemann Hypothesis, and its generalization to Artin  $L$ -functions, have a natural generalization to elliptic curves. In this context, the discrepancy of the set  $\{\theta_p\}_{p \leq x}$  is

$$D(\{\theta_p\}_{p \leq x}, \mathrm{ST}) = \sup_{t \in [0, \pi]} |P_x[0, t] - \mathrm{ST}[0, t]|.$$

The following conjecture is first made in [AT99]: for  $E/\mathbf{Q}$  a non-CM elliptic curve, the bound  $D(\{\theta_p\}_{p \leq x}, \mathrm{ST}) \ll x^{-\frac{1}{2} + \epsilon}$  holds. The authors go on to prove what is essentially the following theorem (fully fleshed out in [Maz08]).

**Theorem 1.2.3.** *If  $D(\{\theta_p\}_{p \leq x}, \mathrm{ST}) \ll x^{-\frac{1}{2} + \epsilon}$ , then all the functions  $L(\mathrm{sym}^k E, s)$  satisfy the Riemann Hypothesis.*

It is natural to assume that the converse to this theorem holds. David Zywinia first suggested to the author that it might now. In this thesis, we construct a range of counterexamples to the implication “strong Sato–Tate implies Riemann” and explore why the two are in fact equivalent for Artin  $L$ -functions. We also construct a broader conjectural framework generalizing Akiyama–Tanigawa’s conjecture to more general motives. Moreover, we generalize the results of [Pan11] to show that there can be no purely Galois-theoretic proof of the Sato–Tate conjecture, for there are Galois representations with arbitrary Sato–Tate distributions! We also show that some of the results of [Sar07] about sums of the form  $\sum_{p \leq x} \frac{a_p}{\sqrt{p}}$  cannot be generalized to general Galois representations.

### 1.3 Notational conventions

The symbol  $f = \Omega(g)$  (in the convention of Hardy–Littlewood) means the negation of  $f = O(g)$ , and  $f = \Theta(g)$  means  $C_1g \leq f \leq C_2f$ .

If  $\mu$  is a measure, then  $\mu[a, b] = \mu([a, b])$  and etc.

If  $\mu$  is a measure,  $\text{cdf}_\mu(x) = \mu[-\infty, x]$ .

## CHAPTER 2

### DISCREPANCY

#### 2.1 Equidistribution

The discrepancy (also known as the Kolmogorov–Smirnov statistic) is a way of measuring how closely sample data fits a predicted distribution. It has many applications in computer science and statistics, but here we will focus on only the basic known properties, as well as how discrepancy changes when sequences are tweaked and/or combined.

First, recall that the discrepancy is a way of sharpening the “soft” convergence results of, say [Ser89, A.1]. Let  $X$  be a compact topological space,  $\{x_p\}$  a sequence of points in  $X$  indexed by the prime numbers.

**Definition 2.1.1.** *Let  $\mu$  be a continuous probability measure on  $X$ . The sequence  $\{x_p\}$  is equidistributed with respect to  $\mu$  if for all  $f \in C(X)$ , we have*

$$\lim_{x \rightarrow \infty} \frac{1}{\pi(x)} \sum_{p \leq x} f(x_p) \rightarrow \int f \, d\mu.$$

In other words,  $\{x_p\}$  is  $\mu$ -equidistributed if the empirical measures  $P_x = \frac{1}{\pi(x)} \sum_{p \leq x} \delta_{x_p}$  converge to  $\mu$  in the weak topology. It is easy to see that  $\{x_p\}$  is  $\mu$ -equidistributed if and only if  $\left| \sum_{p \leq x} f(x_p) \right| = o(x)$  for all continuous  $f$  having  $\int f \, d\mu = 0$ . In fact, one can restrict to a set of  $f$  which generate a dense subspace of  $C(X)^{\mu=0}$ .

In the discussion in [Ser89, A.1],  $X$  is the space of conjugacy classes in a compact Lie group, and  $f$  is allowed to range over the characters of irreducible, nontrivial representations of the group. In this section, we will show that the entire discussion can be generalized to a much broader class of *strange Dirichlet series*,

which are of the form

$$L_f(\{x_p\}, s) = \prod_p \frac{1}{1 - f(x_p)p^{-s}}.$$

A useful, but not too well known, result, is that we in fact can consider functions  $f$  which are only continuous almost everywhere.

**Theorem 2.1.2.** *Let  $X$  be a compact separable metric space with no isolated points. Let  $\mu$  be a Borel measure on  $X$  and let  $f: X \rightarrow \mathbf{C}$  be bounded and measurable. Then  $f$  is continuous almost everywhere if and only if*

$$\lim_{x \rightarrow \infty} \frac{1}{\pi(x)} \sum_{p \leq x} f(x_p) = \int f \, d\mu$$

for all  $\mu$ -equidistributed sequences  $\{x_p\}$ .

*Proof.* This follows immediately from the proof of [Maz95, Th. 1] □

## 2.2 Definitions and first results

We will define discrepancy for measures on the  $d$ -dimensional half-open box  $[0, \infty)^d$ .

For vectors  $x, y \in [0, \infty)^d$ , we say  $x < y$  if  $x_1 < y_1, \dots, x_d < y_d$ , and in that case write  $[x, y)$  for the half-open box  $[x_1, y_1) \times \dots \times [x_d, y_d)$ .

**Definition 2.2.1.** *Let  $\mu, \nu$  be probability measures on  $[0, \infty)^d$ . The discrepancy of  $\mu$  with respect to  $\nu$  is*

$$D(\mu, \nu) = \sup_{x < y} |\mu[x, y) - \nu[x, y)|,$$

where  $x < y$  range over  $[0, \infty)^d$ .

The star discrepancy of  $\mu$  with respect to  $\nu$  is

$$D^*(\mu, \nu) = \sup_{0 \leq y} |\mu[0, y) - \nu[0, y)|,$$

where  $y$  ranges over  $[0, \infty)^d$ .

**Lemma 2.2.2.** *Let  $\mu, \nu$  be Borel measures on  $\mathbf{R}^d$ . Then*

$$D^*(\mu, \nu) \leq D(\mu, \nu) \leq 2^d D^*(\mu, \nu).$$

*Proof.* The first inequality holds because the supremum defining the discrepancy is taken over a larger set than that defining star discrepancy. To prove the second inequality, let  $x < y$  be in  $[0, \infty)^d$ . For  $S \subset \{1, \dots, d\}$ , let

$$I_S = \{t \in [0, y) : t_i < x_i \text{ for all } i \in S\}.$$

The inclusion-exclusion principle for measures tells us that:

$$\mu[x, y) = \sum_{S \subset \{1, \dots, d\}} (-1)^{\#S} \mu(I_S),$$

and similarly for  $\nu$ . Since each of the  $I_S$  are “half-open boxes” we know that  $|\mu(I_S) - \nu(I_S)| \leq D^*(\mu, \nu)$ . It follows that

$$|\mu[x, y) - \nu[x, y)| \leq \sum_{S \subset \{1, \dots, d\}} |\mu(I_S) - \nu(I_S)| \leq 2^d D^*(\mu, \nu).$$

For a discussion and related context, see [KN74, Ch. 2 Ex. 1.2].  $\square$

We are usually interested in comparing empirical measures and their conjectured distribution. Namely, let  $\mathbf{x} = \{x_p\}$  be a sequence in  $[0, \infty)^d$  indexed by the prime numbers, and  $\mu$  a Borel measure on  $[0, \infty)^d$ . For any real number  $N \geq 2$ , we write  $\mathbf{x}^N$  for the empirical measure given by

$$\mathbf{x}^N(S) = \frac{1}{\pi(N)} \sum_{p \leq N} \delta_{x_p}(S) = \frac{\#\{p \leq N : x_p \in S\}}{\pi(N)}.$$

Also, we write  $\mathbf{x}_{\geq N}$  for the truncated sequence  $(x_p)_{p \geq N}$ , and similarly for  $\mathbf{x}_{\leq N}$ , etc.

In this context,

$$D^*(\mathbf{x}^N, \nu) = \sup_{y \in [0, \infty)^d} \left| \frac{\#\{p \leq N : x_p \in [0, y)\}}{\pi(N)} - \int_{[0, y)} d\nu \right|.$$



If the measure  $\nu$  is only defined on a subset of  $[0, \infty)^d$ , we will tacitly extend it by zero. Moreover, if the sequence  $\mathbf{x}$  actually lies in a torus  $(\mathbf{R}/a\mathbf{Z})^d$ , we identify that torus with the  $[0, a)^d \subset [0, \infty)^d$ . If  $\nu$  is the Lebesgue measure (on  $[0, \infty)^d$ ) or the normalized Haar measure on the torus, we write  $D^*(\mathbf{x}^N)$  in place of  $D^*(\mathbf{x}^N, \nu)$ .

Sometimes the sequence  $\mathbf{x}$  will not be indexed by the prime numbers, but rather by some other discrete subset of  $\mathbf{R}^+$ . In that case we will still use the notations  $\mathbf{x}^N$ ,  $\mathbf{x}_{\geq N}$ , etc., keeping in mind that  $\pi(N)$  is replaced by  $\#\{\text{indices} \leq N\}$ .

## 2.3 Examples

One of the first examples of equidistributed sequences is translates of an irrational quantity modulo one.

**Theorem 2.3.1.** *Let  $a \in \mathbf{R}$  be irrational. Then the sequence  $\mathbf{x} = (a, 2a \bmod 1, 3a \bmod 1, \dots)$  is equidistributed in  $[0, 1]$ .*

*Proof.* This follows from the more precise results of Chapter 4. □

Sequences of this form will have discrepancy that decays like  $N^{-\alpha \pm \epsilon}$ , for  $\alpha \in (0, 1/2)$ . It can be useful to have a sequence whose discrepancy decays faster. The best known decay is achieved by the following sequence.

**Definition 2.3.2.** *The van der Corput sequence is given by  $\{\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \dots\}$ . More precisely, write  $n$  in base 2 as  $n = \sum a_i 2^i$ . Then  $v_n = \sum a_i 2^{-(i+1)}$ .*

The van der Corput sequence has generalizations to other bases and higher dimensions. It is well-known for being “very equidistributed”—i.e., its discrepancy has extremely fast convergence to zero.

**Lemma 2.3.3.** *Let  $\mathbf{v} = \{v_n\}$  be the van der Corput sequence. Then  $D(\mathbf{v}^N) \leq \frac{\log(N+1)}{N \log 2}$ .*

*Proof.* This is [KN74, Ch. 2 Th. 3.5]. In particular, we will use often that  $D(\mathbf{v}^N) \ll \frac{\log N}{N}$ .  $\square$

Now, the van der Corput sequence is uniformly distributed, but there is a convenient trick to construct sequences equidistributed with respect to more general measures.

**Definition 2.3.4.** *Let  $\mu$  be a probability measure on  $[a, b]$ . We call  $\mu$  good if  $\text{cdf}_\mu$  is continuous, strictly increasing, and sends  $a \mapsto 0$ .*

Note that if  $\mu$  is a good measure, then  $\text{cdf}_\mu$  is an order isomorphism from  $[a, b]$  to  $[0, 1]$ .

**Theorem 2.3.5.** *Let  $\mu$  be a good measure on a closed interval. Then there exists a sequence  $\mathbf{x} = (x_1, x_2, \dots)$  such that  $D(\mathbf{x}^N, \mu) \ll \frac{\log(N)}{N}$ .*

*Proof.* Since  $\text{cdf}_\mu$  is a continuous bijection and its domain is a compact set,  $\text{cdf}_\mu$  is an order isomorphism. Then Lemma 2.5.3 tells us that for  $\mathbf{v}$  the van der Corput sequence on  $[0, 1]$ , we have  $D(\text{cdf}_\mu^{-1}(\mathbf{v})^N, \mu) = D(\mathbf{v}^N, \mu)$ , which gives us the desired result with  $\mathbf{x} = \text{cdf}_\mu^{-1}(\mathbf{v})$ .  $\square$

**Theorem 2.3.6.** *Let  $\mu$  be a good measure on a closed interval. Fix  $\alpha \in (0, 1)$ . Then there exists a sequence  $\mathbf{x} = (x_1, x_2, \dots)$  such that  $D^*(\mathbf{x}^N, \mu) = \Theta(N^{-\alpha})$ .*

*Proof.* If  $\mathbf{x}_{\leq N}$  is a sequence of length  $N$ , let  $\mathbf{x}_{\leq N} : a_{\leq M}$  be the sequence  $x_1, \dots, x_N, a, \dots, a$  ( $M$  copies of  $a$ ). Then

$$D^*(\mathbf{x}^N : a^M, \mu) \geq \left| \frac{\#\{n \leq N + M : x_n = a\}}{N + M} - \mu\{a\} \right| \geq \frac{M}{N + M}.$$

On the other hand,

$$\begin{aligned} |\text{cdf}_{N,M}(t) - \text{cdf}_N(t)| &\leq \frac{|\#\{n \leq N : x_n \leq t\} + M - \frac{M+N}{N} \#\{n \leq N : x_n \leq t\}|}{M+N} \\ &\leq \frac{2M}{M+N}, \end{aligned}$$

which implies that  $D^*(\mathbf{x}^N : a^M, \mu) \leq D^*(\mathbf{x}^N, \mu) + \frac{2M}{M+N}$ . Let  $\mathbf{v}$  be the  $\mu$ -equidistributed van der Corput sequence of Theorem 2.3.5, possibly transformed linearly to lie in  $[a, b]$ . We know that  $D(\mathbf{v}^N, \mu) \ll N^{-\alpha}$ , with the constant in question depending only on  $\alpha$ .

We construct the sequence  $\mathbf{x}$  via the following recipe. Start with  $(x_1 = v_1, x_2 = v_2, \dots)$  until, for some  $N_1$ ,  $D^*(\mathbf{x}^{N_1}, \mu) < N_1^{-\alpha}$ . Then set  $x_{N_1+1} = a$ ,  $x_{N_1+2} = a$ ,  $\dots$ , until  $D^*(\mathbf{x}^{N_1+M_1}, \mu) > (N_1 + M_1)^{-\alpha}$ . Then set  $x_{N_1+M_1+1} = v_{N_1+1}$ ,  $x_{N_1+M_1+2} = v_{N_1+2}$ ,  $\dots$ , until once again  $D^*(\mathbf{x}^{N_1+M_1+N_2}, \mu) < (N_1 + M_1 + N_2)^{-\alpha}$ . Repeat indefinitely. We will show first, that the two steps are possible, and that nowhere does  $D^*(\mathbf{x}^N, \mu)$  differ by too much from  $N^{-\alpha}$ .

Note that  $\frac{M+1}{N+M+1} - \frac{M}{N+M} \leq N^{-1}$ . This tells us that when we are adding  $a$ 's at the end of  $\mathbf{x}^N$ , the discrepancy of  $\mathbf{x}_{\leq N} : a_{\leq M}$  increases by at most  $N^{-1}$  at each step. So if  $D^*(\mathbf{x}^N, \mu) < N^{-\alpha}$ , we can ensure that  $D^*(\mathbf{x}^N : a^M, \mu)$  is at most  $N^{-1}$  greater than  $N^{-\alpha}$ .

Moreover, we know that  $D^*(\mathbf{x}^N : a, \mu)$  is at most  $\frac{2}{N+1}$  away from  $D^*(\mathbf{x}^N, \mu)$ . So when adding van der Corput elements to the end of the sequence, its' discrepancy cannot decay any faster than by  $\frac{2}{N+1}$  per  $a$  added. This yields

$$|D^*(\mathbf{x}^N, \mu) - N^{-\alpha}| \ll N^{-1},$$

which is even stronger than we need. □

## 2.4 The Koksma–Hlawka inequality

Here we summarize the results of the paper [Ö99], generalizing them as needed for our context. Recall that a function  $f$  on  $[0, \infty)^d$  is said to be of *bounded variation* if there is a finite Radon measure  $\nu$  such that  $f(x) - f(0) = \nu[0, x]$ . In such a case we write  $\text{Var}(f) = |\nu|$ . If the appropriate differentiability conditions are satisfied, then

$$\text{Var}(f) = \int_{[0, \infty)^d} \left| \frac{d^d f}{dx_1 \dots dx_d} \right|.$$

**Theorem 2.4.1** (Koksma–Hlawka). *Let  $\mu$  be a probability measure on  $[0, \infty)^d$ ,  $f$  a function of bounded variation. Then for any sequence  $\mathbf{x}$  in  $[0, \infty)^d$ , we have*

$$\left| \frac{1}{\pi(x)} \sum_{p \leq x} f(x_p) - \int f \, d\mu \right| \leq \text{Var}(f) D(\mathbf{x}^N, \mu).$$

*Proof.* By our assumptions there is a Radon measure  $\nu$  such that  $f(y) - f(0) = \nu[0, y]$ . What follows is essentially trivial, noting that  $1_{[0, x]}(y) = 1_{[y, \infty)^d}(x)$ .

$$\begin{aligned} \frac{1}{\pi(x)} \sum_{p \leq x} f(x_p) - \int f \, d\mu &= \frac{1}{\pi(x)} \sum_{p \leq x} (f(x_p) - f(0)) - \int (f - f(0)) \, d\mu \\ &= \frac{1}{\pi(x)} \sum_{p \leq x} \int 1_{[y, \infty)^d}(x_p) \, d\nu(y) - \int \int 1_{[0, y]} \, d\nu \, d\mu(y) \\ &= \int \frac{1}{\pi(x)} \sum_{p \leq x} 1_{[y, \infty)^d}(x_p) - \int 1_{[y, \infty)^d} \, d\mu \, d\nu(y) \end{aligned}$$

It follows that

$$\left| \frac{1}{\pi(x)} \sum_{p \leq x} f(x_p) - \int f \, d\mu \right| \leq \sup_{y \in [0, \infty)} \left| \frac{1}{\pi(x)} \sum_{p \leq x} 1_{[y, \infty)^d}(x_p) - \int 1_{[y, \infty)^d} \, d\mu \right| \cdot |\nu|.$$

The supremum in question is clearly bounded above by  $D(\mathbf{x}^N, \mu)$ , so the proof is complete.  $\square$

## 2.5 Comparing sequences

**Lemma 2.5.1.** *Let  $\mathbf{x}$  and  $\mathbf{y}$  be sequences in  $[0, \infty)$ . Suppose  $\mu$  is an absolutely continuous probability measure on  $[0, \infty)$  with continuous bounded Radon–Nikodym derivative  $\frac{d\mu}{d\lambda}$ , where  $\lambda$  is the Lebesgue measure. Then*

$$|D^*(\mathbf{x}^N, \nu) - D^*(\mathbf{y}^N, \nu)| \leq \left\| \frac{d\mu}{d\lambda} \right\|_{\infty} \epsilon + \frac{\#\{n \leq N : |x_n - y_n| \geq \epsilon\}}{N}.$$

*Proof.* Let  $\epsilon > 0$  and  $t \in [0, \infty)$  be arbitrary. For all  $n \leq N$  such that  $y_n < t$ , either  $x_n < t + \epsilon$  or  $|x_n - y_n| \geq \epsilon$ . It follows that

$$\mathbf{y}^N[0, t) \leq \mathbf{x}^N[0, t + \epsilon) + \frac{\#\{n \leq N : |x_n - y_n| \geq \epsilon\}}{N}.$$

Moreover, we trivially have  $|\mathbf{x}^N[0, t + \epsilon) - \nu[0, t + \epsilon)| \leq D^*(\mathbf{x}^N, \nu)$ . Putting these together, we get:

$$\begin{aligned} \mathbf{y}^N[0, t) - \nu[0, t) &\leq \mathbf{x}^N[0, t + \epsilon) - \nu[0, t) + \frac{\#\{n \leq N : |x_n - y_n| \geq \epsilon\}}{N} \\ &\leq \nu[t, t + \epsilon) + D^*(\mathbf{x}^N, \nu) + \frac{\#\{n \leq N : |x_n - y_n| \geq \epsilon\}}{N} \\ &\leq \left\| \frac{d\mu}{d\lambda} \right\|_{\infty} \epsilon + D^*(\mathbf{x}^N, \nu) + \frac{\#\{n \leq N : |x_n - y_n| \geq \epsilon\}}{N} \end{aligned}$$

This tells us that

$$D^*(\mathbf{y}^N, \nu) \leq \left\| \frac{d\mu}{d\lambda} \right\|_{\infty} \epsilon + D^*(\mathbf{x}^N, \nu) + \frac{\#\{n \leq N : |x_n - y_n| \geq \epsilon\}}{N}.$$

Reversing the roles of  $\mathbf{x}$  and  $\mathbf{y}$ , we obtain the desired result.  $\square$

**Lemma 2.5.2.** *Let  $\sigma$  be an isometry of  $\mathbf{R}$ , and  $\mathbf{x}$  a sequence in  $[0, \infty)$  such that  $\sigma(\mathbf{x})$  is also in  $[0, \infty)$ . Let  $\nu$  be an absolutely continuous measure on  $[0, \infty)$  such that  $\sigma_*\nu$  is also supported on  $[0, \infty)$ . Then*

$$|D(\mathbf{x}^N, \nu) - D(\sigma_*\mathbf{x}^N, \sigma_*\nu)| \leq \frac{2}{\pi(N)}.$$

*Proof.* Every isometry of  $\mathbf{R}$  is a combination of translations and reflections. The statement is clear with translations (the two discrepancies are equal). So, suppose  $\sigma(t) = a - t$  for some  $a > 0$ . Since  $\nu$  is absolutely continuous,  $\nu\{t\} = 0$  for all  $t \geq 0$ . In particular,  $\nu[s, t] = \nu(s, t]$ . In contrast,  $\mathbf{x}^N\{t\} \leq \pi(N)^{-1}$ . For any interval  $[s, t]$  in  $[0, \infty)$ , we know that

$$|\mathbf{x}^N[s, t] - \mathbf{x}^N(s, t]| \leq \frac{2}{\pi(N)},$$

hence

$$|\mathbf{x}^N[s, t] - \nu[s, t] - (\sigma_*\mathbf{x}^N)[a - t, a - s] - (\sigma_*\nu)[a - t, a - s]| \leq \frac{2}{\pi(N)}.$$

This proves the result.  $\square$

A trick we will use throughout this thesis involves comparing the discrepancy of a sequence with the discrepancy of a pushforward sequence, with respect to the pushforward measure.

**Lemma 2.5.3.** *Let  $f$  be an order isomorphism  $f: [a, b] \rightarrow [c, d]$ . If  $\mathbf{x}$  is a sequence on  $[a, b]$  and  $\mu$  is a probability measure on  $[a, b]$ , then  $D(\mathbf{x}^N, \mu) = D(f(\mathbf{x})^N, f_*\mu)$ , and likewise for star discrepancy.*

*Proof.* This is a simple computation, which we only check for star discrepancy:

$$\begin{aligned} D^*(f(\mathbf{x})^N, f_*\mu) &= \sup_{t \in [c, d]} \left| \frac{\#\{n \leq N : f(x_n) \leq t\}}{N} - (f_*\mu)[a, t] \right| \\ &= \sup_{t \in [a, b]} \left| \frac{\#\{n \leq N : x_n \leq f^{-1}(t)\}}{N} - \mu[a, f^{-1}(t)] \right| \\ &= D(\mathbf{x}^N, \mu). \end{aligned}$$

$\square$

**Lemma 2.5.4.** *Let  $f$  be an order anti-automorphism  $[a, b] \rightarrow [c, d]$ . If  $\mathbf{x}$  is a sequence on  $[a, b]$  and  $\mu$  is a probability measure on  $[a, b]$ , then  $D(\mathbf{x}^N, \mu)$ ?*

*Proof.* Repeat the proof of Lemma 2.5.3, except we have  $s \leq f(x_n) \leq t$  if and only if  $f^{-1}(t) \leq f(x_n) \leq f^{-1}(s)$ , and likewise  $(f_*\mu)[s, t] = \mu[f^{-1}(t), f^{-1}(s)]$ .  $\square$

## 2.6 Combining sequences

**Definition 2.6.1.** *Let  $\mathbf{x}$  and  $\mathbf{y}$  be sequences in  $[0, \infty)^d$ . We write  $\mathbf{x} \wr \mathbf{y}$  for the interleaved sequence*

$$(x_2, y_2, x_3, y_3, x_5, y_5, \dots, x_p, y_p, \dots).$$

For the interleaved sequence  $\mathbf{x} \wr \mathbf{y}$ , we write  $(\mathbf{x} \wr \mathbf{y})^N$  for the empirical measure

$$(\mathbf{x} \wr \mathbf{y})^N = \frac{1}{2\pi(N)} \sum_{p \leq N} \delta_{x_p} + \delta_{y_p}.$$

**Theorem 2.6.2.** *Let  $I$  and  $J$  be disjoint open boxes in  $[0, \infty)^d$ , and let  $\mu, \nu$  be absolutely continuous probability measures on  $I$  and  $J$ , respectively. Let  $\mathbf{x}$  be a sequence in  $I$  and  $\mathbf{y}$  be a sequence in  $J$ . Then*

$$\max\{D(\mathbf{x}^N, \mu), D(\mathbf{y}^N, \nu)\} \leq D((\mathbf{x} \wr \mathbf{y})^N, \mu + \nu) \leq D(\mathbf{x}^N, \mu) + D(\mathbf{y}^N, \nu)$$

*Proof.* Any half-open box in  $[0, \infty)^d$  can be split by a coordinate hyperplane into two disjoint half-open boxes  $[a, b) \sqcup [s, t)$ , each of which intersects at most one of  $I$  and  $J$ . We may assume that  $[a, b) \cap J = \emptyset$  and  $[s, t) \cap I = \emptyset$ . Then

$$\begin{aligned} |(\mathbf{x} \wr \mathbf{y})^N([a, b) \sqcup [s, t)) - (\mu + \nu)([a, b) \sqcup [s, t))| &\leq |\mathbf{x}^N[a, b) - \mu[a, b)| + |\mathbf{y}^N[s, t) - \nu[s, t)| \\ &\leq D(\mathbf{x}^N, \mu) + D(\mathbf{y}^N, \nu). \end{aligned}$$

This yields the second inequality in the statement of the theorem. To see the first, assume that the maximum discrepancy is  $D(\mathbf{x}^N, \mu)$ , and let  $[s, t)$  be a half-open

box such that  $|\boldsymbol{x}^N[s, t] - \mu[s, t]|$  is within an arbitrary  $\epsilon$  of  $D(\boldsymbol{x}^N, \mu)$ . We can assume that  $[s, t)$  does not intersect  $J$ , and thus

$$|(\boldsymbol{x} \wr \boldsymbol{y})^N[s, t] - (\mu + \nu)[s, t]| = |\boldsymbol{x}^N[s, t] - \mu[s, t]|,$$

which yields the result. □



## CHAPTER 3

### STRANGE DIRICHLET SERIES

To-do: show that [Ser89, A.1] works for  $L_f(\mathbf{z}, s)$ ,  $f$  almost-everywhere continuous.

### 3.1 Definitions

We start by considering a very general class of Dirichlet series. In fact, they are all Dirichlet series that admit a product formula with degree-1 factors, but in this thesis they will be called strange Dirichlet series. The motivating example was suggested by Ramakrishna. Let  $E/\mathbf{Q}$  be an elliptic curve and let

$$L_{\text{sgn}}(E, s) = \prod_p \frac{1}{1 - \text{sgn}(a_p)p^{-s}}.$$

How much can we say about the behavior of  $L_{\text{sgn}}(E, s)$ ? For example, does it “know” the rank of  $E$ ?

**Definition 3.1.1.** *Let  $\mathbf{z} = (z_2, z_3, z_5, \dots)$  be a sequence of complex numbers indexed by the primes. The associated strange Dirichlet series is*

$$L(\mathbf{z}, s) = \prod_p \frac{1}{1 - z_p p^{-s}}.$$

If  $z_p$  is only defined for all but finitely many primes, then we tacitly set  $z_p = 0$  for all primes for which  $z_p$  is not defined.

**Lemma 3.1.2.** *Let  $\mathbf{z}$  be a sequence with  $\|\mathbf{z}\|_\infty \leq 1$ . Then  $L(\mathbf{z}, s)$  defines a holomorphic function on the region  $\{\Re s > 1\}$ . Moreover, on that region,*

$$\log L(\mathbf{z}, s) = \sum_{p^r} \frac{z_p^n}{np^{ns}}.$$

*Proof.* Expanding the product for  $L(\mathbf{z}, s)$  formally, we have

$$L(\mathbf{z}, s) = \sum_{n \geq 1} \frac{\prod_p z_p^{v_p(n)}}{n^s}.$$

An easy comparison with the Riemann zeta function tells us that this sum is holomorphic on  $\{\Re s > 1\}$ . By [Apo76, Th. 11.7], the product formula holds in the same region. The formula for  $\log L(\mathbf{z}, s)$  comes from [Apo76, 11.9 Ex.2].  $\square$

**Lemma 3.1.3** (Abel summation). *Let  $\mathbf{z} = (z_2, z_3, z_5, \dots)$  be a sequence of complex numbers,  $f$  a smooth complex-valued function on  $\mathbf{R}$ . Then*

$$\sum_{p \leq N} f(p) z_p = f(N) \sum_{p \leq N} z_p - \int_2^N f'(x) \sum_{p \leq x} z_p dx.$$

*Proof.* Simply note that if  $p_1, \dots, p_n$  is an enumeration of the primes  $\leq N$ , we have

$$\begin{aligned} \int_2^N f'(x) \sum_{p \leq x} z_p dx &= \sum_{p \leq N} z_p \int_{p_n}^N f' + \sum_{i=1}^{n-1} \sum_{p \leq p_{i+1}} z_p \int_{p_i}^{p_{i+1}} f' \\ &= (f(N) - f(p_n)) \sum_{p \leq N} z_p + \sum_{i=1}^{n-1} (f(p_{i+1}) - f(p_i)) \sum_{p \leq p_{i+1}} z_p \\ &= f(N) \sum_{p \leq N} z_p - \sum_{p \leq N} f(p) z_p, \end{aligned}$$

as desired.  $\square$

**Theorem 3.1.4.** *Assume  $|\sum_{p \leq x} z_p| \ll x^{\alpha+\epsilon}$  for some  $\alpha \in [\frac{1}{2}, 1]$ . Then the series for  $\log L(\mathbf{z}, s)$  converges to a holomorphic function on the region  $\{\Re s > \alpha\}$ .*

*Proof.* Formally split the sum for  $\log L(\mathbf{z}, s)$  into two pieces:

$$\log L(\mathbf{z}, s) = \sum_p \frac{z_p}{p^s} + \sum_p \sum_{r \geq 2} \frac{z_p^r}{r p^{rs}}.$$

For each  $p$ , we have

$$\left| \sum_{r \geq 2} \frac{z_p^r}{r p^{rs}} \right| \leq \sum_{r \geq 2} p^{-r \Re s} = p^{-2 \Re s} \frac{1}{1 - p^{-\Re s}}.$$

Elementary analysis gives

$$1 \leq \frac{1}{1 - p^{-\Re s}} \leq 2 + 2\sqrt{2},$$

so the second piece of  $\log L(\mathbf{z}, s)$  converges absolutely when  $\Re s > \frac{1}{2}$ . We could simply cite [Ten95, II.1 Th. 10]; instead we prove directly that  $\sum_p \frac{z_p}{p^s}$  converges absolutely to a holomorphic function on the region  $\{\Re s > \alpha\}$ .

By Lemma 3.1.3 with  $f(x) = x^{-s}$ , we have

$$\begin{aligned} \sum_{p \leq N} \frac{z_p}{p^s} &= N^{-s} \sum_{p \leq N} z_p + s \int_2^N \sum_{p \leq x} z_p \frac{dx}{x^{s+1}} \\ &\ll N^{-\Re s + \alpha + \epsilon} + s \int_2^N x^{\alpha + \epsilon} \frac{dx}{x^{s+1}}. \end{aligned}$$

Since  $\alpha - \Re s < 0$ , the first term is bounded. Since  $s + 1 - \alpha > 1$  and  $\epsilon$  is arbitrary, the integral converges absolutely, and the proof is complete.  $\square$

**Theorem 3.1.5.** *Let  $\mathbf{z} = (z_2, z_3, \dots)$  be a sequence with  $\|\mathbf{z}\|_\infty \leq 1$ , and assume  $\log L(\mathbf{z}, s)$  has analytic continuation to  $\{\Re s > \alpha\}$  for some  $\alpha \in [\frac{1}{2}, 1]$ , and that for  $\sigma > \alpha$ , we have  $|\log L(\mathbf{z}, \sigma + it)| \ll |t|^{1-\epsilon}$  (implied constant independent of  $\sigma$ .) Then  $|\sum_{p \leq N} z_p| \ll N^{\alpha + \epsilon}$ .*

*Proof.* Recall that we can write

$$\log L(\mathbf{z}, s) = \sum_p \frac{z_p}{p^s} + \sum_p \sum_{r \geq 2} \frac{z_p^r}{r p^{rs}} = \sum_p \frac{z_p}{p^s} + O(\zeta(2\Re s)).$$

Thus, for any  $\epsilon > 0$ , analytic continuation and the bound on  $|\log L(\mathbf{z}, \sigma + it)|$  implies the same analytic continuation and bound for  $\sum \frac{z_p}{p^s}$  on  $\{\Re s > \alpha + \epsilon\}$ .

For any  $T > 0$ , let  $\gamma_T = \gamma_{1,T} + \gamma_{2,T} + \gamma_{3,T} + \gamma_{4,T}$  be the following contour:

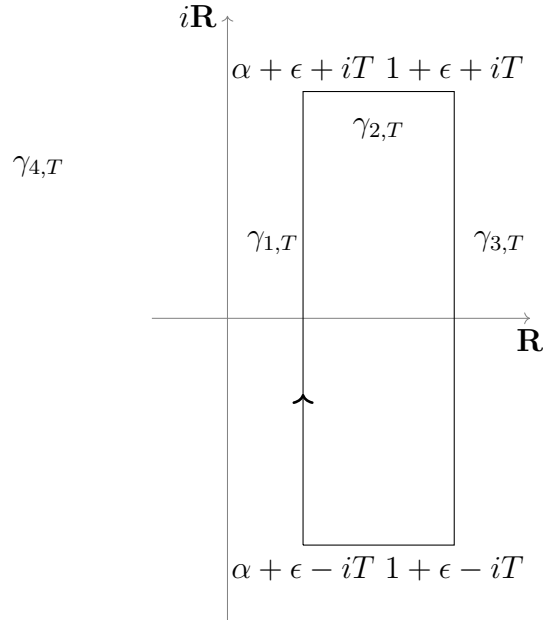
$$\gamma_{1,T}(t) = (\alpha + \epsilon) + it \quad t \in [-T, T]$$

$$\gamma_{2,T}(t) = t + iT \quad t \in [\alpha + \epsilon, 1 + \epsilon]$$

$$\gamma_{3,T}(t) = (1 + \epsilon) + it \quad t \in [T, -T]$$

$$\gamma_{4,T}(t) = t - iT \quad t \in [1 + \epsilon, \alpha + \epsilon].$$

Graphically, the contour looks like this:



By Perron's formula [Apo76, Th. 11.18],

$$\lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{-\gamma_{3,T}} \sum_p \frac{z_p}{p^s} N^z \frac{dz}{z} = \frac{1}{2} \sum_{p \leq N} z_p.$$

for  $N \in \mathbf{Z}$ , and the same without the  $\frac{1}{2}$  on the right-hand side when  $N \notin \mathbf{Z}$ .

Let  $h(s)$  be the analytic continuation of  $\sum z_p p^{-s}$  to  $\{\Re s > \alpha\}$ . Since  $\int_{\gamma_T} h(s) \frac{ds}{s} = 0$ , we obtain

$$\left| \sum_{p \leq N} z_p \right| \ll \lim_{T \rightarrow \infty} \left( \left| \int_{\gamma_{1,T}} h(s) N^s \frac{ds}{s} \right| + \left| \int_{\gamma_{2,T}} h(s) N^s \frac{ds}{s} \right| + \left| \int_{\gamma_{4,T}} h(s) N^s \frac{ds}{s} \right| \right).$$

We know that  $|h(\sigma + it)| \ll |t|^{1-\epsilon}$ , so we can bound

$$\left| \int_{\gamma_{2,T}} h(s) N^s \frac{ds}{s} \right| = \left| \int_{\alpha+\epsilon}^{1+\epsilon} \frac{h(t + iT) N^{t+iT}}{t + iT} dt \right| \ll \frac{N^{1+\alpha}}{T^\epsilon},$$

and similarly for  $\gamma_{4,T}$ . Finally, note that

$$\left| \int_{\gamma_{1,T}} h(s) N^s \frac{ds}{s} \right| \ll \int_{-T}^T |t|^{1-\epsilon} \frac{N^{\alpha+\epsilon}}{(\alpha + \epsilon)^2 + t^2} dt \ll N^{\alpha+\epsilon}.$$

Letting  $T \rightarrow \infty$  we obtain the desired result.  $\square$

In this thesis, we are interested in the following sort of strange Dirichlet series. Let  $X$  be a space,  $f: X \rightarrow \mathbf{C}$  a function with  $\|f\|_\infty \leq 1$ , and  $\mathbf{x} = (x_2, x_3, \dots)$  a sequence in  $X$ . Write

$$L_f(\mathbf{x}, s) = \prod_p \frac{1}{1 - f(x_p) p^{-s}},$$

for the associated strange Dirichlet series.

## 3.2 Relation to automorphic and motivic $L$ -functions

## 3.3 The Riemann Hypothesis

## 3.4 Discrepancy of sequences and the Riemann Hypothesis

## 3.5 Strange Dirichlet series over function fields

## CHAPTER 4

### IRRATIONALITY EXPONENTS

#### 4.1 Definitions and first results

We follow the notation of [Lau09]. Let  $x = (x_1, \dots, x_d) \in \mathbf{R}^d$  be such that the  $x_i$  are  $\mathbf{Q}$ -linearly independent.

**Definition 4.1.1.** Let  $\omega_0(x)$  (resp.  $\omega_{d-1}(x)$ ) be the supremum of the set of real numbers  $\omega$  for which there exist infinitely many  $m = (m_0, \dots, m_d) \in \mathbf{Z}^{r+1}$  such that

$$\begin{aligned} \max\{|m_0 x_i - m_i|\} &\leq \|m\|_\infty^{-\omega} \quad (\text{resp.} \\ |m_0 + m_1 x_1 + \dots + m_r x_r| &\leq \|m\|_\infty^{-\omega}). \end{aligned}$$

These two quantities are related by Khintchine's Transference Principle, namely

$$\frac{\omega_{d-1}(x)}{(d-1)\omega_{d-1}(x) + d} \leq \omega(x) \leq \frac{\omega_{d-1}(x) - d + 1}{d}.$$

Moreover, these inequalities are sharp in a very strong sense.

**Theorem 4.1.2** (Jarník). Let  $w \geq 1/d$ . Then there exists  $x \in \mathbf{R}^d$  such that  $\omega_0(x) = w$  and  $\omega_{d-1}(x) = dw + d - 1$ .

*Proof.* Do this. □

**Theorem 4.1.3.** When  $d = 1$ , relate  $\omega_0(x)$  to the irrationality measure.

*Proof.* Recall that the irrationality measure  $\mu(x)$  is the infimum of the set of positive reals  $\mu$  such that

$$0 < \left| x - \frac{p}{q} \right| < q^{-\mu}$$

has only finitely many solutions  $p/q$  with  $p, q$  integers. □

Mention Roth's theorem... generalize to higher dimension?

Now given  $x \in \mathbf{R}^d$ , we write  $d(x, \mathbf{Z}^d) = \min_{m \in \mathbf{Z}^d} |x - m|$ , where  $|\cdot|$  is any fixed norm on  $\mathbf{R}^d$ . Note that  $d(x, \mathbf{Z}^d) = 0$  if and only if  $x \in \mathbf{Z}^d$ .

**Lemma 4.1.4.** *Let  $x \in \mathbf{R}^d$  with  $\|x\|_\infty \leq 1$  and  $\omega_0(x)$  (resp.  $\omega_{d-1}(x)$ ) finite. Then*

$$\begin{aligned} \frac{1}{d(nx, \mathbf{Z}^d)} &\ll |n|^{\omega_0(x)+\epsilon} && (\text{resp.} \\ \frac{1}{d(\langle m, x \rangle, \mathbf{Z})} &\ll |m|^{\omega_{d-1}(x)+\epsilon} && \text{for } m \in \mathbf{Z}^d). \end{aligned}$$

*Proof.* Let  $\epsilon > 0$ . Then there are only finitely many  $n \in \mathbf{Z}$  (resp.  $m \in \mathbf{Z}^d$ ) such that the inequalities in Definition 4.1.1 hold with  $\omega_0(x) + \epsilon$  (resp.  $\omega_{d-1}(x) + \epsilon$ ). In other words, there exist constants  $C_0, C_{d-1} > 0$  such that

$$\begin{aligned} \max\{|m_0 x_i - m_i|\} &\geq C_0 \|m\|_\infty^{-\omega_0(x)-\epsilon}, \\ |m_0 + m_1 x_1 + \cdots + m_d x_d| &\geq C_{d-1} \|m\|_\infty^{-\omega_{d-1}(x)-\epsilon} \end{aligned}$$

for all  $m \neq 0$ .

Start with the first inequality in the statement of the result, where up to constant, we may assume that  $|\cdot| = \|\cdot\|_\infty$  in the definition of  $d(nx, \mathbf{Z}^d)$ . Let  $m = (m_1, \dots, m_d)$  be the lattice point achieving the minimum  $|nx - m|$ . Then we know that

$$d(nx, \mathbf{Z}^d) \geq C_0 \|(m_1, \dots, m_d)\|_\infty^{-\omega_0(x)-\epsilon}.$$

Moreover, since  $|nx - m| < 1$ , there exists a constant  $C'_0$  such that

$$d(nx, \mathbf{Z}^d) \geq C'_0 |n|^{-\omega_0(x)-\epsilon}.$$

It follows that

$$\frac{1}{d(nx, \mathbf{Z}^d)} \ll |n|^{\omega_0(x)+\epsilon},$$

the implied constant depending on  $x$ ,  $\epsilon$ , and the choice of norm  $|\cdot|$ .

Now let's consider the second inequality in the statement of the result. Note that  $d(m_1x_1 + \dots + m_dx_d, \mathbf{Z}) = |m_0 + m_1x_1 + \dots + m_dx_d|$  for some  $m_0$  with  $|m_0| \leq \|(m_1, \dots, m_d)\|_2 \|x\|_2 + 1$ . Thus  $\|(m_1, \dots, m_d)\|_\infty \ll \|x\|_2 \|(m_1, \dots, m_d)\|_2$ , which gives us

$$d(m_1x_1 + \dots + m_dx_d, \mathbf{Z}) \geq C_{d-1} \|(m_1, \dots, m_d)\|_2^{-\omega_{d-1}(x) - \epsilon}.$$

This implies

$$\frac{1}{d(\langle m, x \rangle, \mathbf{Z})} \ll |m|^{\omega_{r-1}(x) + \epsilon},$$

the implied constant depending on  $x$ ,  $\epsilon$ , and the choice of  $|\cdot|$ .  $\square$

## 4.2 Irrationality exponents and discrepancy

Let  $x \in \mathbf{R}^d$  with  $x_1, \dots, x_d$  linearly independent over  $\mathbf{Q}$ . We wish to control the discrepancy of the sequence  $\{x, 2x, 3x, \dots\}$  in  $(\mathbf{R}/\mathbf{Z})^d$ .

**Theorem 4.2.1** (Erdős–Turán–Koksma). *Let  $\mathbf{x}$  be a sequence in  $\mathbf{R}^d$  and  $h$  an arbitrary integer. Then*

$$D(\mathbf{x}^N) \ll \frac{1}{h} + \sum_{0 \leq \|m\|_\infty \leq h} \frac{1}{r(m)} \left| \frac{1}{N} \sum_{n \leq N} e^{2\pi i \langle m, x_n \rangle} \right|,$$

where the first sum ranges over  $m \in \mathbf{Z}^d$ ,  $r(m) = \prod \max\{1, |m_i|\}$ , and the implied constant depends only on  $d$ .

*Proof.* This is [DT97, Th. 1.21].  $\square$

**Lemma 4.2.2.** *Let  $x \in \mathbf{R}$ . Then*

$$\left| \sum_{n \leq N} e^{2\pi i n x} \right| \ll \frac{1}{d(x, \mathbf{Z})}.$$



*Proof.* We begin with an easy bound:

$$\left| \sum_{n \leq N} e^{2\pi i n x} \right| = \frac{|e^{2\pi i (N+1)x} - 1|}{|e^{2\pi i x} - 1|} \leq \frac{2}{|e^{2\pi i x} - 1|}.$$

Since  $|e^{2\pi i m x} - 1| = \sqrt{2 - 2\cos(2\pi x)}$  and  $\cos(2\theta) = 1 - 2\sin^2 \theta$ , we obtain

$$\left| \sum_{n \leq N} e^{2\pi i n x} \right| \leq \frac{1}{|\sin(\pi x)|}.$$

It is easy to check that  $|\sin(\pi x)| \geq d(x, \mathbf{Z})$ , whence the result.  $\square$

**Corollary 4.2.3.** *Let  $x \in (\mathbf{R}/\mathbf{Z})^d$  with  $(x_1, \dots, x_d)$  linearly independent over  $\mathbf{Q}$ .*

*Then for  $\mathbf{x} = (x, 2x, 3x, \dots)$ , we have*

$$D(\mathbf{x}^N) \ll \frac{1}{h} + \frac{1}{N} \sum_{0 < \|m\|_\infty \leq h} \frac{1}{r(m)d(\langle m, x \rangle, \mathbf{Z})}$$

*for any integer  $h$ , with the implied constant depending only on  $d$ .*

*Proof.* Apply the Erdős–Turán–Koksma inequality and bound the exponential sums using Lemma 4.2.2.  $\square$

**Theorem 4.2.4.** *Let  $\mathbf{x} = (x, 2x, 3x, \dots)$  in  $(\mathbf{R}/\mathbf{Z})^d$ . Then*

$$D(\mathbf{x}^N) \ll N^{-\frac{1}{\omega_{d-1}(x)+1} + \epsilon}.$$

*Proof.* Choose  $\delta > 0$  such that  $\frac{1}{\omega_{d-1}(x)+1+\delta} = \frac{1}{\omega_{d-1}(x)+1} - \epsilon$ .

By Corollary 4.2.3, we know that

$$D(\mathbf{x}^N) \ll \frac{1}{h} + \frac{1}{N} \sum_{0 < \|m\|_\infty \leq h} \frac{1}{r(m)d(\langle m, x \rangle, \mathbf{Z})},$$

and by Lemma 4.1.4, we know that  $d(\langle m, x \rangle, \mathbf{Z})^{-1} \ll |m|^{\omega_{d-1}(x)+\delta}$ . It follows that

$$D(\mathbf{x}^N) \ll \frac{1}{h} + \frac{1}{N} \sum_{0 < \|m\|_\infty \leq h} \frac{|m|^{\omega_{d-1}(x)+\delta}}{r(m)}.$$

The only tricky part is bounding the sum.

$$\begin{aligned}
\sum_{0 < \|m\|_\infty \leq h} \frac{|m|_\infty^{\omega_{d-1}(x)+\delta}}{r(m)} &\ll \int_1^h \int_1^{t_d} \cdots \int_1^{t_2} \frac{t_d^{\omega_{d-1}(x)+\delta}}{t_1 \cdots t_d} dt_1 \cdots dt_d \\
&\ll \int_1^h t^{\omega_{d-1}(x)+\delta-1} dt \prod_{j=1}^{d-1} \int_1^h \frac{dt}{t} \\
&\ll (\log h)^{d-1} h^{\omega_{d-1}(x)+\delta}.
\end{aligned}$$

It follows that

$$D(\mathbf{x}^N) \ll \frac{1}{h} + \frac{1}{N} (\log h)^{d-1} h^{\omega_{d-1}(x)+\delta}.$$

Setting  $h \approx N^{\frac{1}{1+\omega_{d-1}(x)+\delta}}$ , we see that

$$D(\mathbf{x}^N) \ll N^{-\frac{1}{\omega_{d-1}(x)+1+\delta}} = N^{-\frac{1}{\omega_{d-1}(x)+1}+\epsilon}.$$

For a slightly different proof of a similar result (given as a sequence of exercises), see [KN74, Ch. 2, Ex. 3.15, 16, 17].  $\square$

**Theorem 4.2.5.** *Let  $x \in \mathbf{R}$  be such that  $x_1, \dots, x_d$  are linearly independent over  $\mathbf{Q}$ , and let  $\mathbf{x} = (x, 2x, 3x, \dots)$  in  $(\mathbf{R}/\mathbf{Z})^d$ . Then*

$$D(\mathbf{x}^N) = \Omega \left( N^{-\frac{d}{\omega_0(x)}-\epsilon} \right).$$

*Proof.* Here  $f = \Omega(g)$  in the sense of Hardy, namely that  $\limsup \frac{f}{g} > 0$ . We follow the proof of [KN74, Ch. 2, Th. 3.3]. Given  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $\frac{d}{\omega_0(x)-\delta} = \frac{d}{\omega_0(x)} + \epsilon$ .

By the definition of  $\omega_0(x)$ , there exist infinitely many  $(q, m_1, \dots, m_d)$  with  $q > 0$  such that

$$\|qx - m\|_\infty \leq \|(q, m_1, \dots, m_d)\|_\infty^{-\omega_0(x)+\delta/2}.$$

Since  $\|(q, m_1, \dots, m_d)\|_\infty \geq q$ , we derive the stronger statement that for infinitely many  $q \rightarrow \infty$ , there exists  $m = (m_1, \dots, m_d) \in \mathbf{Z}^d$  such that  $\|qx - m\|_\infty \leq$

$q^{-\omega_0(x)+\delta/2}$  or, equivalently,  $|x - \frac{m}{q}| \leq q^{-1-\omega_0(x)+\delta/2}$ . Pick such a  $q$ , and let  $N = \lfloor q^{\omega_0(x)-\delta} \rfloor$ . Then for each  $n \leq N$ , we have  $\|nx - \frac{n}{q}m\|_\infty \leq q^{-1-\delta/2}$ . Thus, for each  $n \leq N$ , each  $nx$  is within  $q^{-1-\delta/2}$  of the grid  $\frac{1}{q}\mathbf{Z}^d \subset (\mathbf{R}/\mathbf{Z})^d$ . Thus, they miss a box with side lengths  $q^{-1} - 2q^{-1-\delta/2}$ . For  $q$  sufficiently large,  $q^{-1} - 2q^{-1-\delta/2} \geq 1/2q$ , so the discrepancy of  $\mathbf{x}^N$  is bounded below by  $2^{-d}q^{-d}$ . Since  $q^{\omega_0(x)-\delta} \leq 2N$ , the discrepancy at  $N$  is bounded below by

$$2^{-d} \left( (2N)^{-\frac{1}{\omega_0(x)+\delta}} \right)^{-d} = 2^{-d-\frac{d}{\omega_0(x)+\delta}} N^{-\frac{d}{\omega_0(x)+\delta}} = 2^{-d(1+\frac{1}{\omega_0(x)})-\epsilon} N^{-\frac{d}{\omega_0(x)}-\epsilon}.$$

□

## CHAPTER 5

### DEFORMATION THEORY

#### 5.1 Category of test objects

This section summarizes the theory in [SGA 3<sub>1</sub>, VII<sub>B</sub>, §0–1], adapting it to the deformation theory of Galois representations. All rings are commutative with unit.

**Definition 5.1.1.** *Let  $\Lambda$  be a ring. A topological  $\Lambda$ -module  $M$  is pseudocompact if it is a filtered inverse limit of discrete finite-length  $\Lambda$ -modules. The ring  $\Lambda$  is pseudocompact if it is pseudocompact as a module over itself.*

Let  $\Lambda$  be a topological ring. Given a pseudocompact  $\Lambda$ -algebra  $A$ , write  $\mathbf{C}_\Lambda$  for the opposite of the category of  $\Lambda$ -algebras which have finite length as  $\Lambda$ -modules. Given such a  $\Lambda$ -algebra  $A$ , write  $X = \mathrm{Spf}(A)$  for the corresponding object of  $\mathbf{C}_\Lambda$ , and we put  $A = \mathcal{O}(X)$ .

**Lemma 5.1.2.** *Let  $\Lambda$  be a pseudocompact ring,  $\mathbf{C}_\Lambda$  as above. Then  $\mathbf{C}_\Lambda$  is closed under finite limits and colimits.*

*Proof.* That  $\mathbf{C}_\Lambda$  is closed under finite colimits follows from the fact that finite-length  $\Lambda$ -algebras are closed under finite limits (the underlying modules are closed under finite limits). Moreover, since the tensor product of finite length modules also has finite length, and quotients of length modules have finite length,  $\mathbf{C}_\Lambda$  is closed under finite limits. □

**Lemma 5.1.3.** *Let  $\Lambda$  be a pseudocompact local ring. Then  $\Lambda$  is henselian, in any of the following senses:*

1. *Every finite  $\Lambda$ -algebra is a product of local  $\Lambda$ -algebras.*

2. The first condition is satisfied for  $\Lambda$ -algebras of the form  $\Lambda[t]/f$ , where  $f$  is monic.

3. Let  $\mathfrak{m}$  be the maximal ideal of  $\Lambda$ . Then  $A \mapsto A/\mathfrak{m}$  is an equivalence of categories from finite étale  $\Lambda$ -algebras to finite étale  $\Lambda/\mathfrak{m}$ -algebras.

*Proof.* The conditions are equivalent by [EGA 4<sub>4</sub>, 18.5.11]. Recall that  $\Lambda = \varprojlim \Lambda/\mathfrak{a}$ , where  $\mathfrak{a}$  ranges over closed ideals of finite index. Let  $A$  be a pseudo-compact  $\Lambda$ -algebra. For any ideal  $\mathfrak{a} \subset \Lambda$ , the ring  $\Lambda/\mathfrak{a}$  is henselian by [EGA 4<sub>4</sub>, 18.5.14], so  $A/\mathfrak{a}$  is a product of local  $\Lambda/\mathfrak{a}$ -algebras. Moreover, by [EGA 4<sub>4</sub>, 18.5.4], the map  $A/\mathfrak{a} \rightarrow A/\mathfrak{m}$  is a bijection on idempotents. The inverse limit of these compatible systems of idempotents decompose  $A$  into a product of local  $\Lambda$ -algebras.  $\square$

Following Grothendieck, if  $\mathcal{C}$  is an arbitrary category, we write  $\widehat{\mathcal{C}} = \text{hom}(\mathcal{C}^\circ, \mathbf{Set})$  for the category of contravariant functors  $\mathcal{C} \rightarrow \mathbf{Set}$ . We regard  $\mathcal{C}$  as a full subcategory of  $\widehat{\mathcal{C}}$  via the Yoneda embedding, so for  $X, Y \in \mathcal{C}$ , we write  $X(Y) = \text{hom}_{\mathcal{C}}(Y, X)$ . With this notation, the Yoneda Lemma states that  $\text{hom}_{\widehat{\mathcal{C}}}(X, P) = P(X)$  for all  $X \in \mathcal{C}$ .

**Lemma 5.1.4.** *Let  $\mathcal{X} \in \widehat{\mathcal{C}_\Lambda}$ . Then  $\mathcal{X}$  is left exact if and only if there exists a filtered system  $\{X_i\}_{i \in I}$  in  $\mathcal{C}_\Lambda$  together with a natural isomorphism  $\mathcal{X}(\cdot) \simeq \varinjlim X_i(\cdot)$ . Write  $\text{Ind}(\mathcal{C}_\Lambda)$  for the category of such functors. Then  $\text{Ind}(\mathcal{C}_\Lambda)$  is closed under colimits, and the Yoneda embedding  $\mathcal{C}_\Lambda \hookrightarrow \text{Ind}(\mathcal{C}_\Lambda)$  preserves filtered colimits.*

*Proof.* This follows from the results of [KS06, 6.1].  $\square$

**Lemma 5.1.5.** *The functors  $\mathcal{C}_\Lambda \rightarrow \text{Ind}(\mathcal{C}_\Lambda) \rightarrow \widehat{\mathcal{C}_\Lambda}$  are left exact.*

*Proof.* This is [KS06, 6.1.17].  $\square$

If  $R$  is a pseudocompact  $\Lambda$ -algebra, write  $\mathrm{Spf}(R)$  for the object of  $\widehat{\mathbf{C}}_\Lambda$  defined by  $\mathrm{Spf}(R)(A) = \mathrm{hom}_{\mathrm{cts}/\Lambda}(R, A)$ , the set of continuous  $\Lambda$ -algebra homomorphisms.

**Lemma 5.1.6.** *The functor  $\mathrm{Spf}$  induces an (anti-)equivalence between the category of pseudocompact  $\Lambda$ -algebras and  $\mathrm{Ind}(\mathbf{C}_\Lambda)$ .*

*Proof.* This is [SGA 3<sub>1</sub>, VII<sub>B</sub> 0.4.2 Prop.].  $\square$

So  $\mathrm{Ind}(\mathbf{C}_\Lambda)$  is the category of pro-representable functors on finite length  $\Lambda$ -algebras. *Warning:* in many papers, for example the foundational [Maz97], one reserves the term *pro-representable* for functors of the form  $\mathrm{Spf}(R)$ , where  $R$  is *noetherian*. We do not make this restriction.

**Lemma 5.1.7.** *The category  $\mathrm{Ind}(\mathbf{C}_\Lambda)$  is an exponential ideal in  $\widehat{\mathbf{C}}_\Lambda$ .*

*Proof.* By this we mean the following. Let  $\mathcal{X} \in \mathrm{Ind}(\mathbf{C}_\Lambda)$ ,  $P \in \widehat{\mathbf{C}}_\Lambda$ . Then the functor  $\mathcal{X}^P$  defined by

$$\mathcal{X}^P(S) = \mathrm{hom}_{\widehat{\mathbf{C}}_{\Lambda/S}}(P/S, \mathcal{X}/S)$$

is also in  $\mathrm{Ind}(\mathbf{C}_\Lambda)$ . Given the characterization of  $\mathrm{Ind}(\mathbf{C}_\Lambda)$  as left exact functors, this is easy to prove, see e.g. [Joh02, 4.2.3].  $\square$

If  $\mathcal{C}$  is a category, we write  $\mathbf{Gp}(\mathcal{C})$  for the category of group objects in  $\mathcal{C}$ .

**Corollary 5.1.8.** *Let  $\Gamma \in \mathbf{Gp}(\widehat{\mathbf{C}}_\Lambda)$  and  $\mathcal{G} \in \mathbf{Gp}(\mathrm{Ind}(\mathbf{C}_\Lambda))$ , then the functor  $[\Gamma, \mathcal{G}]$  defined by*

$$[\Gamma, \mathcal{G}](S) = \mathrm{hom}_{\mathbf{Gp}/S}(\Gamma/S, \mathcal{G}/S)$$

*is in  $\mathrm{Ind}(\mathbf{C}_\Lambda)$ . In particular, if  $\Gamma$  is a profinite group, then the functor*

$$[\Gamma, \mathcal{G}](S) = \mathrm{hom}_{\mathrm{cts}/\mathbf{Gp}}(\Gamma, \mathcal{G}(S))$$

*is in  $\mathrm{Ind}(\mathbf{C}_\Lambda)$ .*

*Proof.* The first claim follows easily from Lemma 5.1.7 and Lemma 5.1.5. Just note that  $[\Gamma, \mathcal{G}]$  is the equalizer:

$$[\Gamma, \mathcal{G}] \longrightarrow \mathcal{G}^\Gamma \underset{m_{\mathcal{G}*}}{\overset{m_\Gamma^*}{\rightrightarrows}} \mathcal{G}^{\Gamma \times \Gamma},$$

that is, those  $f: \Gamma \rightarrow \mathcal{G}$  such that  $f \circ m_\Gamma = m_{\mathcal{G}} \circ (f \times f)$ . The latter claim is just a special case.  $\square$

## 5.2 Quotients in the flat topology

If  $\Lambda$  is a pseudocompact ring, the category  $\mathbf{Ind}(\mathbf{C}_\Lambda)$  has nice “geometric” properties. However, for operations like taking quotients, we will embed it into the larger category  $\mathbf{Sh}_\flat(\mathbf{C}_\Lambda)$  of flat sheaves. We call a collection  $\{U_i \rightarrow X\}$  of morphisms in  $\mathbf{C}_\Lambda$  a *flat cover* if each ring map  $\mathcal{O}(X) \rightarrow \mathcal{O}(U_i)$  is flat, and moreover  $\mathcal{O}(X) \rightarrow \prod \mathcal{O}(U_i)$  is faithfully flat. By [SGA 3<sub>I</sub>, IV 6.3.1], this is a subcanonical Grothendieck topology on  $\mathbf{C}_\Lambda$ . We call it the *flat topology*, even though finite presentation comes for free because all the rings are finite length.

**Lemma 5.2.1.** *Let  $\mathbf{Sh}_\flat(\mathbf{C}_\Lambda)$  be the category of sheaves (of sets) on  $\mathbf{C}_\Lambda$  with respect to the flat topology. Then a presheaf  $P \in \widehat{\mathbf{C}_\Lambda}$  lies in  $\mathbf{Sh}_\flat(\mathbf{C}_\Lambda)$  if and only if  $P(\coprod U_i) = \prod P(U_i)$  and moreover, whenever  $U \rightarrow X$  is a flat cover where  $\mathcal{O}(U)$  and  $\mathcal{O}(X)$  are local rings, the sequence*

$$P(X) \longrightarrow P(U) \rightrightarrows P(U \times_X U).$$

*is exact. Moreover,  $\mathbf{Ind}(\mathbf{C}_\Lambda) \subset \mathbf{Sh}_\flat(\mathbf{C}_\Lambda)$ .*

*Proof.* The first claim is the content of [SGA 3<sub>I</sub>, IV 6.3.1(ii)]. For the second, note that any  $\mathcal{X} \in \mathbf{Ind}(\mathbf{C}_\Lambda)$  will, by 5.1.4, convert (arbitrary) colimits into limits. Thus  $\mathcal{X}(\coprod U_i) = \prod \mathcal{X}(U_i)$ . If  $U \rightarrow X$  is a flat cover, then by (loc. cit.),  $U \times_X U \rightrightarrows$

$U \rightarrow X$  is a coequalizer diagram in  $\mathbf{C}_\Lambda$ , hence  $\mathcal{X}(X) \rightarrow \mathcal{X}(U) \rightrightarrows \mathcal{X}(U \times_X U)$  is an equalizer.  $\square$

Our main reason for introducing the category  $\mathbf{Sh}_\flat(\mathbf{C}_\Lambda)$  is that, as a (Grothendieck) topos, it is closed under arbitrary colimits. Recall that in an *equivalence relation* in  $\widehat{\mathbf{C}_\Lambda}$  is a morphism  $R \rightarrow X \times X$  such that, for all  $S$ , the map  $R(S) \rightarrow X(S) \times X(S)$  is an injection whose image is an equivalence relation on  $X(S)$ . We define the quotient  $X/R$  to be the coequalizer

$$R \rightrightarrows X \longrightarrow X/R.$$

By Giraud's Theorem [MLM94, App.], for any  $S \in \mathbf{C}_\Lambda$ , the natural map  $X(S)/R(S) \rightarrow (X/R)(S)$  is injective. It will not be surjective in general.

We let  $\mathbf{Sh}_\flat(\mathbf{C}_\Lambda)$  inherit definitions from  $\mathbf{C}_\Lambda$  as follows. If  $P$  is a property of maps in  $\mathbf{C}_\Lambda$  (for example, “flat,” or “smooth,”) and  $f: X \rightarrow Y$  is a morphism in  $\mathbf{Sh}_\flat(\mathbf{C}_\Lambda)$ , we say that  $f$  has  $P$  if for all  $S \in \mathbf{C}_\Lambda$  and  $y \in Y(S)$ , the pullback  $X_S = X \times_Y S$  lies in  $\mathbf{C}_\Lambda$ , and the pullback map  $X_S \rightarrow S$  has property  $P$ . For example, if  $X = \mathrm{Spf}(R')$  and  $Y = \mathrm{Spf}(R)$ , then  $X \rightarrow Y$  has property  $P$  if and only if for all finite length  $A$  and continuous  $\Lambda$ -algebra maps  $R \rightarrow A$ , the induced map  $A \rightarrow R' \otimes_R A$  has  $P$ .

**Theorem 5.2.2.** *Let  $\mathcal{R} \rightarrow \mathcal{X} \times \mathcal{X}$  be an equivalence relation in  $\mathbf{Ind}(\mathbf{C}_\Lambda)$  such that one of the maps  $\mathcal{R} \rightarrow \mathcal{X}$  is flat. Then the quotient  $\mathcal{X}/\mathcal{R}$  lies in  $\mathbf{Ind}(\mathbf{C}_\Lambda)$ , and  $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$  is a flat cover.*

*Proof.* This is [SGA 3<sub>I</sub>, VII<sub>B</sub> 1.4].  $\square$

By [Mat89, 29.7], if  $k$  is a field and  $R$  is a complete regular local  $k$ -algebra, then  $R \simeq k[[t_1, \dots, t_n]]$ . In particular,  $R$  admits an augmentation  $\epsilon: R \rightarrow k$ . There is a general analogue of this result, but first we need a definition.



**Definition 5.2.3.** A map  $f: \mathcal{X} \rightarrow \mathcal{Y}$  in  $\text{Ind}(\mathbf{C}_\Lambda)$  is a residual isomorphism if for all  $S = \text{Spf}(k) \in \mathbf{C}_\Lambda$  where  $k$  is a field, the map  $f: \mathcal{X}(S) \rightarrow \mathcal{Y}(S)$  is a bijection.

**Lemma 5.2.4.** Let  $f: \mathcal{X} \rightarrow \mathcal{Y}$  be a smooth map in  $\text{Ind}(\mathbf{C}_\Lambda)$  that is a residual isomorphism. Then  $f$  admits a section.

*Proof.* By [SGA 3<sub>I</sub>, VII<sub>B</sub> 0.1.1], it suffices to prove the result when  $\mathcal{X} = \text{Spf}(R')$ ,  $\mathcal{Y} = \text{Spf}(R)$ , for local  $\Lambda$ -algebras  $R \rightarrow R'$  with the same residue field. Let  $k = R/\mathfrak{m}_R \xrightarrow{\sim} R'/\mathfrak{m}_{R'}$  be their common residue field. From the diagram

$$\begin{array}{ccc} R' & \cdots\cdots\cdots & R \\ \uparrow & \searrow & \downarrow \\ R & \longrightarrow & k, \end{array}$$

the definition of (formal) smoothness, and a limiting argument involving the finite length quotients  $R/\mathfrak{a}$ , we obtain the result.  $\square$

**Corollary 5.2.5.** Let  $\mathcal{R} \rightarrow \mathcal{X} \times \mathcal{X}$  be an equivalence relation satisfying the hypotheses of Theorem 5.2.2. Suppose further that

1. One of the maps  $\mathcal{R} \rightarrow \mathcal{X}$  is smooth, and
2. The projection  $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$  is a residual isomorphism.

Then  $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$  admits a section, so  $\mathcal{X}(S)/\mathcal{R}(S) \xrightarrow{\sim} (\mathcal{X}/\mathcal{R})(S)$  for all  $S \in \mathbf{C}_\Lambda$ .

*Proof.* By 5.2.4, it suffices to prove that  $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$  is smooth. By [EGA 4<sub>4</sub>, 17.7.3(ii)], smoothness can be detected after flat descent. So base-change with respect to the projection  $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$ . In the following commutative diagram

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{\quad} & \mathcal{X} \\ \parallel & \searrow & \downarrow \\ \mathcal{X} \times_{\mathcal{X}/\mathcal{R}} \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X}/\mathcal{R} \end{array}$$

we can ensure the smoothness of  $\mathcal{R} \rightarrow \mathcal{X}$  by our hypotheses. Since  $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$  is smooth after flat base-change, the original map is smooth.  $\square$

**Example 5.2.6.** The hypothesis on residue fields in 5.2.5 is necessary. To see this, let  $\Lambda = k$  be a field,  $k \hookrightarrow K$  a finite Galois extension with Galois group  $G$ . Then  $G \times \mathrm{Spf}(K) \rightrightarrows \mathrm{Spf}(K)$  has quotient  $\mathrm{Spf}(k)$ , but the map  $\mathrm{Spf}(K)(S) \rightarrow \mathrm{Spf}(k)(S)$  is *not* surjective for all  $S \in \mathbf{C}_k$ , e.g. it is not for  $S = \mathrm{Spf}(k)$ .

**Example 5.2.7.** The hypothesis of smoothness in Theorem 5.2.5 is necessary. To see this, let  $k$  be a field of characteristic  $p > 0$ . Then the formal additive group  $\widehat{\mathbf{G}}_{\mathrm{a}} = \mathrm{Spf}(k[[t]])$  has a subgroup  $\alpha_p$  defined by

$$\alpha_p(S) = \{s \in \mathcal{O}(S) : s^p = 0\}.$$

The quotient  $\widehat{\mathbf{G}}_{\mathrm{a}}/\alpha_p$  has as affine coordinate ring  $k[[t^p]]$ . In particular, the following sequence is exact in the flat topology:

$$0 \longrightarrow \alpha_p \longrightarrow \widehat{\mathbf{G}}_{\mathrm{a}} \xrightarrow{(\cdot)^p} \widehat{\mathbf{G}}_{\mathrm{a}} \longrightarrow 0.$$

It follows that  $\alpha_p \times \widehat{\mathbf{G}}_{\mathrm{a}} \rightrightarrows \widehat{\mathbf{G}}_{\mathrm{a}} \xrightarrow{(\cdot)^p} \widehat{\mathbf{G}}_{\mathrm{a}}$  is a coequalizer in  $\mathbf{Sh}_{\mathrm{fl}}(\mathbf{C}_k)$  satisfying all the hypotheses of 5.2.5 except smoothness. And indeed, as one sees by letting  $S = \mathrm{Spf}(A)$  for any non-perfect  $k$ -algebra  $A$ , the map  $(\cdot)^p : \widehat{\mathbf{G}}_{\mathrm{a}}(S) \rightarrow \widehat{\mathbf{G}}_{\mathrm{a}}(S)$  is *not* surjective for all  $S$ .

### 5.3 Deformations of group representations

Relate to [Bİ3].

Let  $\Gamma \in \mathbf{Gp}(\widehat{\mathbf{C}}_{\Lambda})$  and  $\mathcal{G} \in \mathbf{Ind}(\mathbf{C}_{\Lambda})$ . By 5.1.8, the functor

$$\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})(S) = \mathrm{hom}_{\mathbf{Gp}/S}(\Gamma_S, \mathcal{G}_S)$$

is in  $\text{Ind}(\mathbf{C}_\Lambda)$ . We would like to define an ind-scheme  $\text{Rep}(\Gamma, \mathcal{G})$  as “ $\text{Rep}^\square(\Gamma, \mathcal{G})$  modulo conjugation,” but this requires some care. The conjugation action of  $\mathcal{G}$  on  $\text{Rep}^\square(\Gamma, \mathcal{G})$  will have fixed points, so the quotient will be badly behaved. We loosely follow [Til96].

Assume  $\Lambda$  is local, with maximal ideal  $\mathfrak{m}$  and residue field  $\mathbf{k}$ . Fix  $\bar{\rho} \in \text{Rep}^\square(\Gamma, \mathcal{G})(\mathbf{k})$ , i.e. a residual representation  $\bar{\rho}: \Gamma \rightarrow \mathcal{G}(\mathbf{k})$ . Let  $\text{Rep}^\square(\Gamma, \mathcal{G})_{\bar{\rho}}$  be the connected component of  $\bar{\rho}$  in  $\text{Rep}^\square(\Gamma, \mathcal{G})$ . Assume that  $\mathcal{G}$  and  $Z(\mathcal{G})$  are smooth; then the quotient  $\mathcal{G}^{\text{ad}} = \mathcal{G}/Z(\mathcal{G})$  is also smooth. Let  $\mathcal{G}^{\text{ad}, \circ}$  be the connected component of 1 in  $\mathcal{G}^{\text{ad}}$ .

**Theorem 5.3.1.** *Suppose  $(\Lambda, \mathfrak{m}, \mathbf{k})$  is local. If  $\mathcal{X}, \mathcal{Y} \in \text{Ind}(\mathbf{C}_\Lambda)$  are connected and  $\mathcal{X}(\mathbf{k}) \neq \emptyset$ , then  $\mathcal{X} \times_\Lambda \mathcal{Y}$  is connected.*

*Proof.* We are reduced to proving the following result from commutative algebra: if  $R, S$  are local pro-artinian  $\Lambda$ -algebras and  $R$  has residue field  $\mathbf{k}$ , then  $R \widehat{\otimes}_\Lambda S$  is local. Since  $R \widehat{\otimes}_\Lambda S = \varprojlim (R/\mathfrak{r}) \otimes_\Lambda (S/\mathfrak{s})$ ,  $\mathfrak{r}$  (resp.  $\mathfrak{s}$ ) ranges over all open ideals in  $R$  (resp.  $S$ ), we may assume that both  $R$  and  $S$  are artinian. The rings  $R$  and  $S$  are henselian, so  $R \otimes S$  is local if and only if  $(R/\mathfrak{m}_R) \otimes (S/\mathfrak{m}_S) = S/\mathfrak{m}_S$  is local, which it is.  $\square$

We conclude that the action of  $\mathcal{G}^{\text{ad}, \circ}$  on  $\text{Rep}^\square(\Gamma, \mathcal{G})$  preserves  $\text{Rep}^\square(\Gamma, \mathcal{G})_{\bar{\rho}}$ . Thus we may put

$$\text{Rep}(\Gamma, \mathcal{G})_{\bar{\rho}} = \text{Rep}^\square(\Gamma, \mathcal{G})_{\bar{\rho}} / \mathcal{G}^{\text{ad}, \circ}.$$

If  $\mathcal{G}^{\text{ad}, \circ}$  acts faithfully on  $\text{Rep}^\square(\Gamma, \mathcal{G})_{\bar{\rho}}$ , then we recover the classical notion of the deformation functor.

**Theorem 5.3.2.** *Let  $\Gamma$  be a profinite group,  $\bar{\rho}: \Gamma \rightarrow \mathcal{G}(\mathbf{k})$  a representation with  $H^0(\Gamma, \text{Ad } \bar{\rho}) = 0$ . Then  $\text{Rep}(\Gamma, \mathcal{G})_{\bar{\rho}}$  exists and is what you expect.*

*Proof.* Need assumptions on  $Z(\mathcal{G})$ ,  $\mathcal{G}$  should be smooth.

Need  $Z(\mathcal{G}) = \ker(\mathcal{G} \rightarrow \mathrm{GL}(\mathfrak{g}))$  in connected case. This should use  $\mathfrak{g} = \mathrm{Lie}(\mathrm{Aut} \mathcal{G})$ , via deviations in [SGA 3<sub>1</sub>].

Recall first that  $\mathrm{Rep}^\square(\Gamma, \mathcal{G})_{\bar{\rho}} \dots$ . Main things: need a residual isomorphism (this one can check directly) and faithful action (do this!).  $\square$

[...local conditions]

## 5.4 Tangent spaces and obstruction theory

For  $S_0 \in \mathbf{C}_\Lambda$ , let  $\mathrm{Ex}_{S_0}$  be the category of square-zero thickenings of  $S_0$ . An object of  $\mathrm{Ex}_{S_0}$  is a closed embedding  $S_0 \hookrightarrow S$  whose ideal of definition has square zero. Should be “exponential exact sequence”

$$0 \longrightarrow \mathfrak{g}(I) \longrightarrow \mathcal{G}(S) \longrightarrow \mathcal{G}(S_0) \longrightarrow 1$$

This gives us a class  $\exp \in H^2(\mathcal{G}(S_0), \mathfrak{g}(I))$ . For  $\rho_0: \Gamma \rightarrow \mathcal{G}(S_0)$ , the obstruction class is  $o(\rho_0, I) = \rho_0^*(\exp) \in H^2(\Gamma, \mathfrak{g}(I))$ . It’s easy to check that  $o(\rho_0, I) = 0$  if and only if  $\rho_0$  lifts to  $\rho$ . So obstruction theory naturally for  $\mathrm{Rep}^\square(\Gamma, \mathcal{G})$ .

[Use [Wei94, 6.6.4]. Given setting as above,  $\rho_0^*(\exp)$  is the pullback by  $\rho_0$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{g}(I) & \longrightarrow & \mathcal{G}(S) \times_{\mathcal{G}(S_0)} \Gamma & \longrightarrow & \Gamma \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \rho_0 \\ 0 & \longrightarrow & \mathfrak{g}(I) & \longrightarrow & \mathcal{G}(S) & \longrightarrow & \mathcal{G}(S_0) \longrightarrow 1 \end{array}$$

Computing explicitly, we see the result. ]

**Proposition 5.4.1.** *Let  $f: G \rightarrow H$  be a morphism of profinite groups. Suppose  $M$  is a discrete  $H$ -module and  $c \in H^2(H, M)$  corresponds to the extension*

$$0 \longrightarrow M \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 1.$$

Then  $f^*c = 0$  in  $H^2(G, M)$  if and only if there is a map  $\tilde{f}: G \rightarrow \tilde{H}$  making the following diagram commute:

$$\begin{array}{ccc} & & \tilde{H} \\ & \nearrow \tilde{f} & \downarrow \\ G & & H. \\ & \searrow f & \end{array}$$

*Proof.* By [Wei94, 6.6.4], the class  $f^*c$  corresponds to the pullback diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & G \times_H \tilde{H} & \longrightarrow & G \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow f \\ 0 & \longrightarrow & M & \longrightarrow & \tilde{H} & \longrightarrow & H \longrightarrow 1. \end{array}$$

Writing explicitly what it means for  $G \times_H \tilde{H} \rightarrow G$  to split yields the result.  $\square$

## 5.5 Deformation theory over $\mathbb{Q}$

Todo: summarize [Pan11].

## CHAPTER 6

### CONSTRUCTING GALOIS REPRESENTATIONS

#### 6.1 Main idea

Basic idea is as follows. Start with  $\rho_1: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{F}_l)$ . At each stage, we have  $\rho_n: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{Z}/l^n)$ . At that stage, we're allowed to choose the (integral) characteristic polynomial for Frobenius at an arbitrarily large set of primes  $R_n$ . Then some (but density zero) extra primes ramify, and then we get  $\rho_{n+1}$  that agrees with our choices for  $R_1 \cup \dots \cup R_n$ . Then we have to choose characteristic polynomials for  $R_{n+1}$ , but these are already determined modulo  $l^{n+1}$ .

Basic idea is, we can choose the  $R_n$  to be so huge that the primes involved are way way bigger than  $l^n$ . For example, we could have  $R_n = \{p \leq l^n\}$ . Thus the set of possible  $a_p$ 's is very big (big enough) so that we can get discrepancy to behave as we like (both decaying slowly and decaying quickly) to any measure  $\mu$  such that  $\mathrm{ST}/\mu$  is bounded away from zero. (By this, we mean: if  $\mathrm{ST} = f \cdot \lambda$  and  $\mu = g \cdot \lambda$ , where  $\lambda$  is Lebesgue, then  $f/g$  is bounded away from zero.)

We loosely summarize [Pan11], changing the notation to make things clearer.

Start with a prime  $l$  and a representation  $\bar{\rho}: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{Z}/l)$ . For any set  $S$  of primes outside which  $\bar{\rho}$  is unramified, let  $\mathcal{X}_{\bar{\rho},S}$  be the space of deformations of  $\bar{\rho}$  that are also unramified outside  $S$ . If  $\chi: G_{\mathbf{Q},S} \rightarrow \mathbf{Z}_l^\times$  is a character (perhaps a power of the cyclotomic character) let  $\mathcal{X}_{\bar{\rho},S}^\chi$  be the subspace of  $\mathcal{X}_{\bar{\rho},S}$  with determinant  $\chi$ .

It is known that the obstruction to lifting an element of  $\mathcal{X}_{\bar{\rho},S}^\chi(\mathbf{Z}/l^n)$  to  $\mathcal{X}_{\bar{\rho},S}^\chi(\mathbf{Z}/l^{n+1})$  lies in  $H^2(G_{\mathbf{Q},S}, \mathrm{Ad}^0 \bar{\rho})$ . If the obstruction vanishes, lifts are a torsor under a natural action of  $H^1(G_{\mathbf{Q},S}, \mathrm{Ad}^0 \bar{\rho})$ .

Given  $\bar{\rho}$ , there is a set of primes  $p$  called *nice primes*, which are those  $p$  for

which:

1.  $\bar{\rho}$  is unramified at  $p$ .
2. The ratio of eigenvalues of  $\bar{\rho}(\text{fr}_p)$  is  $p$ .

If  $p$  is a nice prime, there is a subspace  $\mathcal{C}_p$  of deformations of  $\bar{\rho}|_{G_{\mathbf{Q}_p}}$  consisting of *nice deformations*. (Not too sure about that.)

At the very least,  $p$  is nice iff  $\text{Ad}^0 \bar{\rho} \simeq \mathbf{F}_l \oplus \mathbf{F}_l(1) \oplus \mathbf{F}_l(-1)$ . (Maybe this is a nicer way to think about it.)

Then put  $N_p = H^1(G_{\mathbf{Q}_p}, \mathbf{F}_l(1)) \subset H^1(G_{\mathbf{Q}_p}, \text{Ad}^0 \bar{\rho})$ . Let  $C_p$  be deformations of  $\bar{\rho}$  which are fixed by  $N_p$ . (What does this mean?)

## 6.2 Summary

Here we summarize the results of [KLR05; Pan11], adapting them as needed. Let  $\bar{\rho}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Z}/l)$  be a continuous, absolutely irreducible representation. By Serre's conjecture,  $\bar{\rho}$  is modular.

By [KLR05, Lem. 6], there exists a finite set  $S$  of primes, containing all primes at which  $\bar{\rho}$  ramifies, such that  $\text{III}_S^1(\text{Ad}^0 \bar{\rho}) = \text{III}_S^2(\text{Ad}^0 \bar{\rho}) = 0$ .

Since  $\bar{\rho}$  is modular, it admits a lift to  $\mathbf{Z}/l^2$ . Call it  $\rho_2$ . Let  $S_2 = S$ .

For primes  $p \in S_2$ , choose an arbitrary weight-2 lift  $\rho_p: G_{\mathbf{Q}_p} \rightarrow \text{GL}_2(\mathbf{Z}_l)$ .

Let  $R_2 = \{p \notin S_2 : p \leq r_2\}$ , for arbitrary choice of  $r_2$  (ideally very large compared to  $l$ ).

For primes  $p \in R_2$ , choose an arbitrary unramified weight-2 lift  $\rho_p: G_{\mathbf{Q}_p} \rightarrow \text{GL}_2(\mathbf{Z}_l)$ . This is determined by its characteristic polynomial, which we assume to be in  $\mathbf{Z}[t]$ .

By [KLR05, Lem. 8], there is a finite set  $Q_2$  of primes such that the following maps is an isomorphism:

$$H^1(G_{S_2 \cup Q_2}, \text{Ad}^0 \bar{\rho}) \rightarrow \prod_{p \in S_2} H^1(G_{\mathbf{Q}_p}, \text{Ad}^0 \bar{\rho}) \times \prod_{p \in R_2} H_{\text{nr}}^1(G_{\mathbf{Q}_p}, \text{Ad}^0 \bar{\rho}).$$

Since  $\rho_1$  has no obstructions to lifting, we know that  $H^1(G_{S_2 \cup Q_2}, \text{Ad}^0 \bar{\rho})$  acts simply transitively on the set of lifts of  $\rho_1$  to  $\mathbf{Z}/l^2$ . By the above isomorphism, there exists  $f_2 \in H^1(G_{S_2 \cup Q_2}, \text{Ad}^0 \bar{\rho})$  such that  $f_2 \cdot \rho_2|_{G_{\mathbf{Q}_p}} \equiv \rho_p \pmod{l^2}$  for all  $p \in S_2 \cup R_2$ .

Now clearly,  $f_2 \cdot \rho_2$  can be lifted to  $\mathbf{Z}/l^3$  when restricted to all primes in  $S_2 \cup R_2$ . However, it may not necessarily lift for primes in  $Q_2$ . By [Pan11, Lem. 3.10], there is a set  $A_2$ , of at most two primes, and  $g_2 \in H^1(G_{S_2 \cup Q_2 \cup A_2}, \text{Ad}^0 \bar{\rho})$  such that  $g_2 \cdot f_2 \cdot \rho_2$  still agrees with our chosen lifts on  $S_2 \cup R_2$ , and moreover has no local obstructions to lifting in  $Q_2 \cup A_2$ .

Put  $S_3 = S_2 \cup Q_2 \cup A_2$ , and let  $R_3 = R_2 \cup \{p \notin S_3 : p \leq r_3\}$ , where  $r_3 \gg r_2$ . Choose arbitrary lifts of  $g_2 f_2 \rho_2$  restricted to primes in  $S_3 \setminus S_2$  and  $R_3 \setminus R_2$ .

Generally, suppose we have a representation  $\rho_n$  to  $\mathbf{Z}/l^n$ , which agrees with our choices of lifts for all primes in  $S_n$  and  $R_n$ , and we choose lifts of  $\rho_n$  to  $\mathbf{Z}_l$  for primes in  $R_{n+1} \setminus R_n$  and  $S_{n+1} \setminus S_n$ . There will be a set  $Q_{n+1}^*$  such that if we allow ramification at  $Q_{n+1}^*$ , we can ensure that there is a lift of  $\rho_n$  to  $\mathbf{Z}/l^{n+1}$  which agrees with our choices of lifts for primes in  $S_{n+1}$  and  $R_{n+1}$ .

### 6.3 More summary

We begin by bounding the size of some sets. In the discussion below, sets  $Q_{n+1}^*$  arise. By the proof of [KLR05, Lem. 8], we see that  $\#Q_{n+1}^* = h^1(G_{\mathbf{Q}, S_{n+1} \cup R_{n+1}}, \text{Ad}^0 \bar{\rho}^*)$ .



Adapting equation (9) of the same paper with  $M = \text{Ad}^0 \bar{\rho}$  and  $L_p = 0$ , we see that

$$h^1(G_{\mathbf{Q}, S_{n+1} \cup R_{n+1}}, \text{Ad}^0 \bar{\rho}^*) = h^0(\mathbf{Q}, \text{Ad} \bar{\rho}^*) + \sum_{p \in S_{n+1} \cup R_{n+1}} h^0(\mathbf{Q}_p, \text{Ad}^0 \bar{\rho}).$$

The  $h^0(\mathbf{Q}, \text{Ad}^0 \bar{\rho}) = 0$ , at least if  $\bar{\rho}$  surjects onto  $\text{GL}_2(\mathbf{Z}/l)$  (prove this!). Moreover, if  $p$  is a nice prime, then  $h^0(\mathbf{Q}_p, \text{Ad}^0 \bar{\rho}) = 1$ . It follows that

$$\#Q_{n+1}^* = \#S_{n+1} + \sum_{p \in R_{n+1}} h^0(\mathbf{Q}_p, \text{Ad}^0 \bar{\rho}).$$

Now,  $S_{n+1} = S_2 \cup_{m \leq n} Q_m^* \cup_{m \leq n} A_m$ , and  $\#A_m \leq 2$ . So

$$\#Q_{n+1}^* \leq \#S_2 + \sum_{m \leq n} \#Q_m^* + 2n + 3\#R_{n+1}.$$

(Not sure if this matters...)

Let  $\bar{\rho}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Z}/l)$  be a continuous, odd, absolutely irreducible representation. By results of Khare–Wintenberger,  $\bar{\rho}$  is modular. By [KLR05, Lem. 6], there exists a finite set  $S$  of primes, containing the primes at which  $\bar{\rho}$  ramifies, such that  $\text{III}_S^1(\text{Ad}^0 \bar{\rho}) = \text{III}_S^2(\text{Ad}^0 \bar{\rho}) = 0$ , and thus any obstructions to lifting can be detected locally.

Let  $S_2 = S$ , and for each  $p \in S_2$ , choose a lift  $\rho_p: G_{\mathbf{Q}_p} \rightarrow \text{GL}_2(\mathbf{Z}_l)$  of  $\bar{\rho}|_{G_{\mathbf{Q}_p}}$ . Choose  $r_2$ , and let  $R_2 = \{p \notin S_2 : p \leq r_2\}$ . Choose  $a_p \in \mathbf{Z}$  (hence a lift  $\rho_p: G_{\mathbf{Q}_p} \rightarrow \text{GL}_2(\mathbf{Z}_l)$  of  $\bar{\rho}|_{G_{\mathbf{Q}_p}}$ ) for all  $p \in R_2$ .

Let  $\rho_2^*$  be any lift of  $\bar{\rho}$  to  $G_{\mathbf{Q}, S_2} \rightarrow \text{GL}_2(\mathbf{Z}/l^2)$ . Such lifts exist because  $\bar{\rho}$  is modular. By [KLR05, Lem. 8], there is a finite set  $Q_2^*$  of primes such that the following maps is an isomorphism:

$$H^1(G_{\mathbf{Q}, S_2 \cup Q_2^*}, \text{Ad}^0 \bar{\rho}) \rightarrow \prod_{p \in S_2} H^1(G_{\mathbf{Q}_p}, \text{Ad}^0 \bar{\rho}) \times \prod_{p \in R_2} H_{\text{nr}}^1(G_{\mathbf{Q}_p}, \text{Ad}^0 \bar{\rho}).$$

Looking at the proof of [KLR05, Fact 5], we can ensure that all primes in  $Q_2^*$  are larger than an arbitrary fixed bound  $q_2$ . In fact, we can ensure that the primes in  $Q_2^*$  (and  $Q_2$ ) grow faster than any function we give.

## 6.4 A precise theorem

If  $\rho$  is a Galois representation, let  $\pi_{\text{nr}(\rho)}(x)$  be the number of  $p \leq x$  such that  $\rho$  is unramified at  $p$ .

**Theorem 6.4.1.** *Let  $\mathbf{x} = (x_1, x_2, \dots)$  be a sequence in  $[-1, 1]$ . Let  $l \geq 5$  be a rational prime, and let  $\bar{\rho}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{F}_l)$  be a continuous, absolutely irreducible representation that is odd of weight 2. Fix a function  $h: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  which decreases to zero, and a function  $b: \mathbf{N} \rightarrow \mathbf{N}$  which increases to infinity. Then there exists a continuous lift of  $\bar{\rho}$ , namely  $\rho: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Z}_l)$ , also of weight 2, such that*

1.  $\pi_{\text{nr}(\rho)}(x) \ll h(x)\pi(x)$ .
2. For each unramified prime  $p$ ,  $a_p = \text{tr } \rho(\text{fr}_p) \in \mathbf{Z}$ , satisfies  $|a_p| \leq 2\sqrt{p}$ .
3. If  $\{x_p\}$  is a renumeration of  $\mathbf{x}$  by unramified primes, then  $\left| \frac{a_p}{2\sqrt{p}} - x_p \right| \leq lb(p)p^{-1/2}$ .

*Proof.* By [KLR05, Lem. 6], there exist a finite set  $S$ , containing the primes at which  $\bar{\rho}$  ramifies, such that  $\text{III}_S^1(\text{Ad}^0 \bar{\rho}) = \text{III}_S^2(\text{Ad}^0 \bar{\rho}) = 0$ . We will consider deformations of  $\bar{\rho}$  that are also of weight 2 and ramified at  $S$  (or a larger, but finite set). Thus, a deformation to  $\mathbf{Z}/l^n$  will admit a lift to  $\mathbf{Z}/l^{n+1}$  if and only if all its restrictions to  $G_{\mathbf{Q}_p}$  admit such lifts. As we progress, re-index  $\mathbf{x}$  so that its segment is indexed by unramified primes of  $\rho_n$ .

Let  $S_2$  be the union of  $S$  and those primes  $p$  for which there does *not* exist an  $a \in \mathbf{Z}$  with  $a \equiv \text{tr } \bar{\rho}(\text{fr}_p) \pmod{l}$  and  $|a| \leq 2\sqrt{p}$ . For the remainder of the proof, we modify the methods of [KLR05; Pan11]. For all primes  $p \in S_2$ , choose a ramified lift  $\rho_p$  of  $\bar{\rho}|_{G_{\mathbf{Q}_p}}$  to  $\mathbf{Z}_l$ . Let  $r_2$  be large enough so that  $r_2 < p$  implies  $l < b(p)$ . Let  $R_2 = \{p \notin S_2 : p \leq r_2\}$ , and choose  $a_p$  for  $p \in R_2$  so that  $|a_p| \leq 2\sqrt{p}$  and

$\left| \frac{a_p}{2\sqrt{p}} - x_p \right| < lp^{-1/2}$ . For  $p \in R_2$ , let  $\rho_p$  be the unique unramified representation  $G_{\mathbf{Q}_p} \rightarrow \mathrm{GL}_2(\mathbf{Z}_l)$  that has weight 2 and  $\mathrm{tr} \rho_p(\mathrm{fr}_p) = a_p$ .

By results of Fontaine and Wintenberger,  $\bar{\rho}$  is modular, so in particular it admits a lift  $\rho_2^*$  to  $\mathbf{Z}/l^2$ . By [KLR05, Lem. 8], there exists a finite set of nice primes  $Q_2^*$ , such that the map

$$H^1(\mathbf{Q}_{S_2 \cup Q_2^*}, \mathrm{Ad}^0 \bar{\rho}) \rightarrow \prod_{p \in S_2} H^1(\mathbf{Q}_p, \mathrm{Ad}^0 \bar{\rho}) \times \prod_{p \in R_2} H_{\mathrm{nr}}^1(\mathbf{Q}_p, \mathrm{Ad}^0 \bar{\rho})$$

is an isomorphism. Moreover, an examination of the proof of [KLR05, Fact 5] shows that we may ensure that  $\pi_{\mathrm{nr}(\rho_2^*)}(p) \leq h(p)\pi(p)$  for all  $p \in Q_2^*$ . Since the above map is an isomorphism, there exists  $f_2 \in H^1(\mathbf{Q}_{S_2 \cup Q_2^*}, \mathrm{Ad}^0 \bar{\rho})$  such that  $f_2 \cdot \bar{\rho} \equiv \rho_p \pmod{l^2}$  for all  $p \in S_2 \cup R_2$ . Clearly  $\rho_2^*$  can lift to  $\mathbf{Z}/l^3$  when restricted to all primes in  $S_2 \cup R_2$ , but it might be obstructed at some primes in  $Q_2^*$ . By [Pan11, Prop. 3.10], there exists a set  $A_2$  of nice primes, with  $\#A_2 \leq 2$  and with  $\pi_{\mathrm{nr}}(p) \leq h(p)\pi(p)$  for  $p \in A_2$ , together with  $g_2 \in H^1(\mathbf{Q}_{S_2 \cup Q_2^* \cup A_2}, \mathrm{Ad}^0 \bar{\rho})$ , such that  $(f_2 + g_2) \cdot \rho_2^*$  still agrees with  $\rho_p$  modulo  $l^2$  for all  $p \in S_2 \cup R_2$ , and  $(f_2 + g_2) \cdot \rho_2^*$  is unobstructed at all primes in  $Q_2 = Q_2^* \cup A_2$ .

Set  $\rho_2 = (f_2 + g_2) \cdot \rho_2^*$  and put  $S_3 = S_2 \cup Q_2$ . Then  $\rho_2$  admits a lift to  $\mathbf{Z}/l^3$ ; call it  $\rho_3^*$ . For all  $p \in S_3 \setminus S_2$ , choose a ramified, weight-2 lift  $\rho_p$  of  $\rho_2|_{G_{\mathbf{Q}_p}}$  to  $\mathbf{Z}_l$ . Let  $r_3$  be large enough so that  $r_3 < p$  implies  $l^2 < b(p)$ , and let  $R_3 = \{p \notin S_3 : p \leq r_3\}$ . For all  $p \in R_3 \setminus R_2$ , choose  $a_p$  such that  $\left| \frac{a_p}{2\sqrt{p}} - x_p \right| < l^2 p^{-1/2}$  and  $|a_p| \leq 2\sqrt{p}$  and  $a_p \equiv \mathrm{tr} \rho_2(\mathrm{fr}_p) \pmod{l^2}$ .

Once again, there exists a finite set of primes  $Q_3$  and  $f_3 \in H^1(\mathbf{Q}_{S_3 \cup Q_3}, \mathrm{Ad}^0 \bar{\rho})$  such that  $f_3 \cdot \rho_3^* \equiv \rho_p \pmod{l^3}$  for all  $p \in S_3 \cup R_3$ , and  $\rho_3 = f_3 \cdot \rho_3^*$  is unobstructed for primes in  $Q_3$ . Moreover, we can ensure that  $\pi_{\mathrm{nr}}(p) \leq h(p)\pi(p)$  for all  $p \in Q_3$ . Set  $S_4 = S_3 \cup Q_3$  and continue in the same manner.

For the general inductive step, let  $\rho_{n+1}^*$  be a lift of  $\rho_n$  from  $\mathbf{Z}/l^n$  to  $\mathbf{Z}/l^{n+1}$ .

Let  $r_{n+1}$  be large enough so that  $r_{n+1} < p$  implies  $l^n < b(p)$ , and let  $R_{n+1} = \{p \notin S_{n+1} : p \leq r_{n+1}\}$ . For  $p \in S_{n+1} \setminus S_n$ , choose a ramified, weight-2 lift  $\rho_p$  of  $\rho_n|_{G_{\mathbf{Q}_p}}$  to  $\mathbf{Z}_l$ , and for  $p \in R_{n+1} \setminus R_n$ , choose  $a_p \in \mathbf{Z}$  satisfying the Hasse bound such that  $\left| \frac{a_p}{2\sqrt{p}} - x_p \right| < l^n p^{-1/2}$ . As before, there exists a finite set  $Q_{n+1}$  of nice primes, enforcing  $\pi_{\text{nr}} \leq h\pi$ , and  $f_{n+1} \in H^1(\mathbf{Q}_{S_{n+1} \cup Q_{n+1}}, \text{Ad}^0 \bar{\rho})$  such that  $\rho_{n+1} = f_{n+1} \cdot \rho_{n+1}^*$  agrees with  $\rho_p$  modulo  $l^{n+1}$  for all  $p \in S_{n+1} \cup R_{n+1}$ , and moreover  $\rho_{n+1}$  is unobstructed at  $Q_{n+1}$ . Set  $S_{n+2} = S_{n+1} \cup Q_{n+1}$ , and the inductive step is complete.  $\square$

## CHAPTER 7

### COUNTEREXAMPLE VIA DIOPHANTINE APPROXIMATION

#### 7.1 Supporting results

Give  $(\mathbf{R}/\mathbf{Z})^d$  the natural Haar measure normalized to have total mass one. Recall that for any  $f \in L^1((\mathbf{R}/\mathbf{Z})^d)$ , the Fourier coefficients of  $f$  are, for  $m \in \mathbf{Z}^d$

$$\widehat{f}(m) = \int_{(\mathbf{R}/\mathbf{Z})^d} e^{2\pi i \langle m, x \rangle} dx,$$

where  $\langle m, x \rangle = m_1 x_1 + \cdots + m_d x_d$  is the usual inner product.

**Theorem 7.1.1.** *Fix  $x \in (\mathbf{R}/\mathbf{Z})^d$  with  $\omega_{d-1}(x)$  finite. Then*

$$\left| \sum_{n \leq N} e^{2\pi i \langle m, nx \rangle} \right| \ll |m|^{\omega_{d-1}(x) + \epsilon}$$

as  $m$  ranges over  $\mathbf{Z}^r \setminus 0$ .

*Proof.* From Lemma 4.2.2 we know that

$$\left| \sum_{n \leq N} e^{2\pi i \langle m, nx \rangle} \right| \ll \frac{1}{d(\langle m, x \rangle, \mathbf{Z})},$$

and from Lemma 4.1.4, we know that  $d(\langle m, x \rangle, \mathbf{Z})^{-1} \ll |m|^{\omega_{d-1}(x) + \epsilon}$ . The result follows.  $\square$

**Theorem 7.1.2.** *Let  $x \in \mathbf{R}^d$  with  $\omega_{d-1}(x)$  finite. Then let  $f \in L^1((\mathbf{R}/\mathbf{Z})^d)$  with  $\widehat{f}(0) = 0$  and suppose the Fourier coefficients of  $f$  satisfy the bound  $|\widehat{f}(m)| \ll |m|^{-\frac{1}{d-1} - \omega_{d-1}(x) - \epsilon}$ . Then*

$$\left| \sum_{n \leq N} f(nx) \right| \ll 1.$$

*Proof.* Write  $f$  as a Fourier series:

$$f(x) = \sum_{m \in \mathbf{Z}^r} \widehat{f}(m) e^{2\pi i \langle m, x \rangle}.$$

Since  $\widehat{f}(0) = 0$ , we can compute:

$$\begin{aligned}
\left| \sum_{n \leq N} f(nx) \right| &= \left| \sum_{n \leq N} \sum_{m \in \mathbf{Z}^d \setminus 0} \widehat{f}(m) e^{2\pi i \langle m, x \rangle} \right| \\
&\leq \sum_{m \in \mathbf{Z}^d \setminus 0} |\widehat{f}(m)| \left| \sum_{n \leq N} e^{2\pi i n \langle m, x \rangle} \right| \\
&\ll \sum_{m \in \mathbf{Z}^d \setminus 0} |m|^{-\frac{1}{d-1} - \omega_{d-1}(x) - \epsilon} |m|^{\omega_{d-1}(x) + \epsilon/2} \\
&\ll \sum_{m \in \mathbf{Z}^d \setminus 0} |m|^{-\frac{1}{d-1} - \epsilon/2}.
\end{aligned}$$

The sum converges since the exponent is less than  $-\frac{1}{d-1}$ , and it doesn't depend on  $N$ , hence the result.  $\square$

## 7.2 Pathological Satake parameters

Let  $p_1 = 2, p_2 = 3, p_3 = 5, \dots$  be an enumeration of the prime numbers. Let  $y \in \mathbf{R}^d$  with  $y_1, \dots, y_d$  linearly independent over  $\mathbf{Q}$ . The associated sequence of “fake Satake parameters” is

$$\mathbf{x} = (y, 2y, 3y, 4y, \dots),$$

where we put  $x_{p_n} = ny \pmod{\mathbf{Z}^d}$ . By Theorem 4.1.2, we can arrange for  $\omega_0(y) = w$  and  $\omega_{d-1}(y) = dw + d - 1$ .

**Theorem 7.2.1.** *The sequence  $\mathbf{x}$  is equidistributed in  $(\mathbf{R}/\mathbf{Z})^d$ , with discrepancy decaying as*

$$D(\mathbf{x}^N) \ll N^{-\frac{1}{dw+d} + \epsilon}$$

and for which

$$D(\mathbf{x}^N) = \Omega\left(N^{-\frac{d}{w} - \epsilon}\right).$$

However, for any  $f \in C^\infty((\mathbf{R}/\mathbf{Z})^d)$  with  $\widehat{f}(0) = 0$ , the strange Dirichlet series  $L_f(\mathbf{x}, s)$  satisfies the Riemann Hypothesis.

### 7.3 Some remarks on isotropic discrepancy

d

## CHAPTER 8

### DIRECT COUNTEREXAMPLE

#### 8.1 Main ideas

This chapter has two parts. First, for any reasonable measure  $\mu$  on  $[0, \pi]$  invariant under the same “flip” automorphism as the Sato–Tate measure, there is a sequence  $\{a_p\}$  of integers satisfying the Hasse bound  $|a_p| \leq 2\sqrt{p}$ , such that for  $\theta_p = \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right)$ , the discrepancy  $D(\{\theta_p\}_{p \leq x}, \mu)$  behaves like  $x^{-\alpha}$  for predetermined  $\alpha \in (0, 1/2]$ , while for any smooth  $f$  satisfying  $f(\pi - \theta) = -f(\theta)$  (and hence  $\int f d\mu = 0$ ), the strange Dirichlet series  $L_f(\{\theta_p\}, s)$  satisfies the Riemann Hypothesis.

In the second part of this chapter, we associate (infinitely ramified) Galois representations to the fake Satake parameters above, using techniques from [Pan11; KLR05].

#### 8.2 Construction

**Theorem 8.2.1.** *Let  $\mu$  be a probability measure on  $[0, \pi]$  such that  $\cos_* \mu$  is good, and fix  $\alpha \in (0, 1/2)$ . Then there exists a sequence of integers  $a_p \in \mathbf{Z}$  with  $|a_p| \leq 2\sqrt{p}$ , such that if we set  $\theta_p = \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right)$ , then  $D^*(\boldsymbol{\theta}^N, \mu) = \Theta(\pi(N)^{-\alpha})$ .*

*Proof.* Apply Theorem 2.3.6 to find a sequence  $\mathbf{x}$  such that  $D(\mathbf{x}^N, \cos_* \mu) = \Theta(\pi(N)^{-\alpha})$ . For each prime  $p$ , there exists an integer  $a_p$  such that  $|a_p| \leq 2\sqrt{p}$  and  $\left|\frac{a_p}{2\sqrt{p}} - x_p\right| \leq p^{-1/2}$ . Let  $y_p = \frac{a_p}{2\sqrt{p}}$ . Now apply Lemma 2.5.1 with  $\epsilon = N^{-1/2}$ . We obtain

$$|D(\mathbf{x}^N, \cos_* \mu) - D(\mathbf{y}^N, \cos_* \mu)| \ll N^{-1/2} + \frac{\pi(N^{1/2})}{\pi(N)},$$



which tells us that  $D(\mathbf{y}^N, \cos_* \mu) = \Theta(\pi(N)^{-\alpha})$ . Now let  $\boldsymbol{\theta} = \cos^{-1}(\mathbf{y})$ . Apply Lemma 2.5.4 to  $\boldsymbol{\theta} = \cos^{-1}(\mathbf{y})$ , and we see that  $D(\boldsymbol{\theta}^N, \mu) = \Theta(\pi(N)^{-\alpha})$ .  $\square$

We can improve this example by controlling the behavior of sums of the form  $\sum_{p \leq N} f(\theta_p)$ , at least for “odd”  $f$ . Let  $\sigma$  be the involution of  $[0, \pi]$  given by  $\sigma(\theta) = \pi - \theta$ . Note that  $\sigma_* \text{ST} = \text{ST}$ . Moreover, note that for any  $f$  with  $f \circ \sigma = -f$ , then  $\int f \, d\text{ST} = 0$ .

**Theorem 8.2.2.** *Let  $\mu$  be a probability measure on  $[0, \pi]$  such that  $\sigma_* \mu = \mu$  and  $\mu|_{[0, \pi/2]}$  is good. Fix  $\alpha \in (0, 1/2)$ . Then there exists a sequence of integers  $a_p$  with  $|a_p| \leq 2\sqrt{p}$  such that for  $\theta_p = \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right)$ , we have  $D(\boldsymbol{\theta}^N, \mu) = \Theta(\pi(N)^{-\alpha})$ , and moreover  $\left|\sum_{p \leq N} f(\theta_p)\right| \ll N^{-1/2+\epsilon}$  whenever  $f \circ \sigma = -f$ , and  $f$  is the restriction to  $[0, \pi]$  of a smooth periodic function on  $[-\pi, \pi]$  satisfying  $f(-\theta) = f(\theta)$ .*

*Proof.* The basic idea is as follows. Enumerate the primes  $p_1 = 2, p_2 = 3, p_3 = 5$ , and divide them into the “odd indexed primes” and the “even indexed primes.” For  $n$  odd, choose  $a_{p_n}$  so that  $\theta_{p_n} \in [0, \pi/2)$  are equidistributed with respect to  $\mu|_{[0, \pi/2]}$  with desired rate of convergence. Then choose, for  $n$  odd,  $a_{p_{n+1}}$  so that  $\theta_{p_{n+1}} \in [\pi/2, \pi]$  is very close to  $\pi - \theta_{p_n}$ . We can ensure that the discrepancy of the combined sequence decays at the correct rate. Moreover, for functions with  $f(\pi - \theta) = -f(\theta)$ , sums like  $\sum_{p \leq N} f(\theta_p)$  have a bunch of terms looking like

$$f(\theta_{p_n}) + f(\theta_{p_{n+1}}) \approx f(\theta) + f(\pi - \theta) \approx f(\theta) - f(\theta) \approx 0.$$

We proceed to do this rigorously.

Let  $\mathbf{x} = (x_1, x_2, \dots)$  be the sequence of Theorem 2.3.6 for  $\cos_* \mu|_{[0, \pi/2]}$  and  $\alpha$ . This is supported on  $[0, 1]$ . Choose  $a_{p_{2n-1}}$  so that  $\left|\frac{a_{p_{2n-1}}}{2\sqrt{p_{2n-1}}} - x_n\right| \leq p^{-1/2}$  and also...

Let  $p < q$  be successive primes. Suppose we have already chosen  $a_p < 0$ . Then we can choose  $a_q > 0$  to guarantee that

$$\left| \frac{a_p}{2\sqrt{p}} + \frac{a_q}{2\sqrt{q}} \right| \leq \frac{1}{\sqrt{q}}.$$

□

### 8.3 Associated Galois representation

Fix, for the remainder of this section, a continuous representation

$$\bar{\rho}_l: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{F}_l).$$

For each  $p$  at which  $\bar{\rho}_l$  is unramified, we write

$$\Theta_p(\bar{\rho}_l) = \left\{ \cos^{-1} \left( \frac{a}{2\sqrt{p}} \right) : a \in \mathbf{Z}, |a| \leq 2\sqrt{p}, \text{ and } a \equiv \mathrm{tr} \bar{\rho}_l(\mathrm{fr}_p) \pmod{l} \right\}.$$

For the finitely many primes  $p$  for which  $\Theta_p(\bar{\rho}_l)$  is empty, redefine  $\Theta_p(\bar{\rho}_l)$  to include some elements for which  $|a| > 2\sqrt{p}$ . We have a sequence of  $\Theta_p(\bar{\rho}_l)$  for which at most finitely many do not satisfy the Hasse bound.

**Theorem 8.3.1.** *There exists a choice of  $\theta_p \in \Theta_p(\bar{\rho}_l)$  for odd-indexed primes  $\{2, 5, 11, \dots\}$  such that*

1.  $\theta_p \in [0, \pi/2)$  for all but finitely many  $p$ .
2.  $D \left( \boldsymbol{\theta}_{\mathrm{odd}}^N, \mathrm{ST}|_{[0, \pi/2)} \right) \rightarrow 0$ , but is not  $\ll N^{-\epsilon}$  for any  $\epsilon > 0$ .

*Proof.* This is intuitively obvious, but a bit tricky to prove rigorously.

Two key ideas:

1. If we're given a "bad" finite distribution  $\nu$ , we can choose "good"  $\theta_p$ 's to make the combined distribution close enough (discrepancy-wise) to ST.

2. If we're given a “good” finite distribution  $\nu$ , we can choose “bad”  $\theta_p \sim \pi/2$  to make the combined distribution far away (discrepancy-wise) from ST.  $\square$

Claim: let  $\mu, \nu$  be two absolutely continuous distributions. Suppose there is a sequence  $\{T_p\}$  of  $\mu$ -distributed sets, such that  $D(T_p, \mu) \ll p^{-1/2}$ . Suppose moreover that  $\mu/\nu$  is bounded away from zero (at the pdf side). Then we can choose  $t_p \in T_p$  so that  $\{t_p\}$  is  $\nu$ -equidistributed with good discrepancy.

Let  $\mu$  be an absolutely continuous measure on  $[0, \pi]$  such that the pushforward  $\cos_* \mu$  is bounded (this is true for the Sato–Tate measure). Fix a prime  $l \geq 5$  and a constant  $\alpha \in (0, 1/2]$ . We want to construct a weight-2 Galois representation  $\rho_l: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{Z}_l)$ , ramified at a density zero set of primes, such that

1. If  $\rho_l$  is unramified at  $p$ , then  $a_p = \mathrm{tr} \rho_l(\mathrm{fr}_p) \in \mathbf{Z}$  and satisfies the Hasse bound  $|a_p| \leq 2\sqrt{p}$ .
2. If we write  $\theta_p = \cos^{-1}(a_p/2\sqrt{p})$  for the Satake parameters at unramified primes, then  $D(\boldsymbol{\theta}^N, \mu) \ll N^{-\alpha+\epsilon}$  and  $D(\boldsymbol{\theta}^N, \mu) = \Omega(N^{-\alpha-\epsilon})$ .

Recall the van der Corput sequence  $\{x_p\}$  satisfies  $D(\mathbf{x}^N) \ll N^{-1+\epsilon}$ . Let  $\nu = \cos_* \mu$ ; this is an absolutely continuous measure supported on  $[-1, 1]$ . By transforming the van der Corput sequence by a continuous map, we may assume that in fact  $D(\mathbf{x}^N, \nu) \ll N^{-1+\epsilon}$ . In fact, by alternating between “van der Corput elements” and “bad elements” we can ensure that not only does  $D(\mathbf{x}^N, \nu) \ll N^{-\alpha+\epsilon}$ , but also  $D(\mathbf{x}^N, \nu) = \Omega(N^{-\alpha-\epsilon})$ .

We start by choosing a modular mod- $l$  representation  $\rho_1: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{Z}/l)$ , which is ramified at a finite set of primes  $S_1$ . Let  $R_1 = \{p \leq r_1 : p \notin S_1\}$ . For  $p \in R_1$ , we can choose  $a_p \in \mathbf{Z}$  subject only to the condition  $a_p \equiv \mathrm{tr} \rho_1(\mathrm{fr}_p) \pmod{l}$ .

For any  $p \in R_1$ , the set

$$T_p(l) = \left\{ \frac{a}{2\sqrt{p}} : |a| \leq 2\sqrt{p} \text{ and } a \equiv \text{tr } \rho_1(\text{fr}_p) \pmod{l} \right\}$$

has an element within  $lp^{-1/2}$  of any element of  $[-1, 1]$ . Choose  $a_p \in T_p(l)$  so that

$$\left| \frac{a_p}{2\sqrt{p}} - x_p \right| \leq lp^{-1/2}. \text{ It follows that for } p \in R_1, \text{ we have}$$

$$|D(\{a_p/2\sqrt{p}\}_{p \leq N}, \nu) - D(\mathbf{x}^N, \nu)| \ll lN^{-1/2}$$

We get a lift of  $\rho_1$  to  $\rho_2: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Z}/l^2)$  respecting our choices of the  $a_p$  for  $p \in R_1$ , which is ramified at one (perhaps two) extra primes.

What happens next is in stages. We'll already have a mod- $l^{n+1}$  representation  $\rho_{n+1}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Z}/l^n)$ , together with choices of  $a_p$  for  $p \in R_1 \cup \dots \cup R_n$  that ensure  $|D(\{a_p/2\sqrt{p}\}_{p \leq N}, \nu) - D(\mathbf{x}^N, \nu)| \ll ?$

The main question is: how do we choose  $r_1$ , and the later  $r_n$ ? We ensure that a) the set  $T_p(l^n)$  are non-empty, and that b)  $l^n < \log(r_n)$ . This gives us that for  $N \leq r_n$ , we have

$$|D(\{a_p/2\sqrt{p}\}_{p \leq N}, \nu) - D(\mathbf{x}^N, \nu)| \ll N^{-\frac{1}{2} + \epsilon}.$$

Todo: can I make  $\sum a_p = ?$  anything from  $-\infty$  to  $\infty$ ?

What if I make a fake modular form with these “bad” Satake parameters?

What can I say about it?

## 8.4 Informal approach

This discussion is inspired by [Pan11]. Throughout, all Galois representations have weight 2, i.e. determinant is the cyclotomic power.

Fix a prime  $l \geq 5$  and a (modular) representation  $\rho_1: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Z}/l)$ . We claim that there is a finite set  $S$  such that  $\text{III}_S^1(\text{Ad}^0 \rho_1) = \text{III}_S^2(\text{Ad}^0 \rho_1) = 0$ . Moreover, all the local deformation spaces are smooth? (Why?)

Set  $S = S_2$ . Choose lifts  $\rho_p: G_{\mathbf{Q}_p} \rightarrow \mathrm{GL}_2(\mathbf{Z}_l)$  of  $\rho_1|_{G_{\mathbf{Q}_p}}$  for all  $p \in S_2$ . We can ensure that the  $\rho_p$  are ramified. (Can we also ensure that their characteristic polynomials are well behaved? By [KR01], these characteristic polynomials are well-defined for all but finitely many primes.) Now let  $R_2 = \{p \notin S_2 : p \leq r_2\}$ , where  $r_2$  is a yet unspecified large constant (say  $l^{100}$ ). Choose  $a_p$  for all  $p \in R_2$ . By [Pan11, Lem. 5.1], there is a set  $Q_2$  (bound the size of  $Q_2$ !)

CHAPTER 9

COMPUTATIONAL EVIDENCE FOR THE  
AKIYAMA–TANIGAWA CONJECTURE

## CHAPTER 10

### CONCLUDING REMARKS AND FUTURE DIRECTIONS

Todo: future direction, discrete dense subgroups of  $SU(2)$  and other compact, semisimple groups.

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