COUNTEREXAMPLES RELATED TO THE STRONG SATO–TATE CONJECTURE

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Let $E_{/\mathbf{Q}}$ be an elliptic curve. The Sato–Tate conjecture (now a theorem) tells us that the angles $\theta_p = \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right)$ are equidistributed in $[0,\pi]$ with respect to the measure $\frac{2}{\pi}\sin^2\theta$ if E is not CM, and uniformly distributed if E has CM over $\mathbf{Q}$. Call ST the measure in question. Akiyama and Tanigawa conjecture that in fact, the discrepancy

$$D_N = \sup_{x\in[0,\pi]} \left| \frac{1}{\pi(N)} \sum_{p\leqslant N} 1_{[0,x]}(\theta_p) - \int 1_{[0,x]}\,\mathrm{d}\mu \right|$$

decays as $D_N \ll N^{-\frac{1}{2}+\epsilon}$, as is suggested by computational evidence and certain reasonable heuristics. When E is non-CM, this implies the Riemann Hypothesis for all $L(\operatorname{sym}^k E, s)$. It is natural to assume that the converse holds, as is suggested by analogy with the Riemann Hypothesis for Artin L-functions. We show that when E has CM over $\mathbf{Q}$, there is no reason to believe that the converse holds, as there are "fake Satake parameters" yielding L-functions which satisfy the Generalized Riemann Hypothesis, but for which the discrepancy decays like $N^{-\epsilon}$.

We also show that there are Galois representations $\rho\colon \operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathbf{Z}_l)$, ramified at an arbitrarily thin set of primes, whose Satake parameters can be made to converge at any specified rate to any reasonable measure on $[0,\pi]$.

BIOGRAPHICAL SKETCH

Daniel Miller was born in St. Paul, Minnesota. He completed his Bachelor of Science at the University of Nebraska–Omaha, during which he attended Cornell's Summer Mathematics Institute in 2011. He started his Ph.D. at Cornell in 2012 planning on a career in academia, but halfway through had a change of heart, and will be joining Microsoft's Analysis and Experimentation team as a Data Scientist after graduation. He is happily married to Ivy Lai Miller, and enjoys hiking, exploring new food, and martial arts.

This thesis is dedicated to my undergraduate adviser, Griff Elder. He is the reason I considered a career in mathematics in the first place, and his infectious enthusiasm for number theory has inspired me more than I can say.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Motivation from classical analytic number theory

Start with a problem central to the history of number theory—counting prime numbers. As usual, let $\pi(x)$ be the number of rational primes $\leqslant x$ and $\operatorname{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}$ be the logarithmic integral. For any $x \geqslant 2$, there is a (normalized) empirical measure capturing the distribution of those primes $\leqslant x$:

$$P_x = \frac{1}{\pi(x)} \sum_{p \leqslant x} \delta_{p/x},$$

which is supported on the unit interval $[0,1]$. The prime number theorem tells us that as $x \to \infty$, these empirical measures approach the "true" measure $L_x = \frac{\operatorname{Li}(tx)}{\operatorname{Li}(x)}\,\mathrm{d}t$. The standare approach to proving the prime number theorem is by showing that the Riemann ζ-function has meromorphic continuation past $\Re = 1$.

Theorem 1.1.1. *The function $\zeta(s)$ admits a meromorphic continuation past $\Re = 1$ with at most a simple pole at $s = 1$, if and only if $P_x \to L_x$ in the weak sense.*

Since $\zeta(s)$ does have such a meromorphic continuation, the prime number theorem holds. It is natural to try to quantify the rate of converge of P_x to L_x. One way to do this is via the (star) discrepancy

$$\mathrm{D}^\star(P_x, L_x) = \sup_{t \in [0,1]} |P_x[0,t] - L_x[0,t]| = \sup_{t \in [0,1]} \left| \frac{\pi(tx)}{\pi(x)} - \frac{\int_2^{tx} \frac{\mathrm{d}s}{\log s}}{\int_2^x \frac{\mathrm{d}s}{\log s}} \right|.$$

Numerical experiments suggest that $\mathrm{D}^\star(P_x, L_x) \ll x^{-\frac{1}{2}+\epsilon}$, and in fact we have the following result.

Theorem 1.1.2. *The Riemann Hypothesis is true if and only if* $\mathrm{D}^\star(P_x, L_x) \ll x^{-\frac{1}{2}+\epsilon}$.

Of course, neither side of this equivalence is known for certain to be true!

The above discussion finds a natural generalization in Artin L-functions. Let $K/\mathbf{Q}$ be a finite Galois extension with group $G = \mathrm{Gal}(K/\mathbf{Q})$. For any irreducible representation $\rho\colon G \to \mathrm{GL}_d(\mathbf{C})$, there is a corresponding L-function defined as

$$L(\rho, s) = \prod_p \frac{1}{\det(1 - \rho(\mathrm{fr}_p)p^{-s})},$$

where here (and for the remainder of this thesis) we tacitly omit from the product those primes at which ρ is ramified. Given a cutoff x, there is a natural empirical measure

$$P_x = \frac{1}{\pi(x)} \sum_{p \leqslant x} \delta_{\mathrm{fr}_p},$$

where fr_p is the p-th Frobenius conjugacy class in G. Let

$$\mathrm{D}(P_x) = \sup_{S \subset G^\natural} \left| P_x(S) - \frac{\#S}{\#G^\natural} \right|,$$

where $G^\natural$ is the set of conjugacy classes in G.

Theorem 1.1.3. *The measure* P_x *converge weakly to the uniform measure on* $G^\natural$ *if and only if the function* $L(\rho, s)$ *admits analytic continuation past* $\Re = 1$ *for all nontrivial* ρ.

Both sides of this equivalence are true, and known as the Chebotarev density theorem. Moreover, there is a version of the strong Prime Number Theorem in this context.

Theorem 1.1.4. *The bound* $\mathrm{D}(P_x) \ll x^{-\frac{1}{2}+\epsilon}$ *holds if and only if each* $L(\rho, s)$*,* ρ *nontrivial, satisfies the Riemann Hypothesis.*

This whole discussion generalizes to a more complicated set of Galois representations—those arising from elliptic curves.

1.2 Discrepancy and Riemann Hypothesis for elliptic curves

Let $E_{/\mathbf{Q}}$ be an elliptic curve. For any prime l, there is an l-adic Galois representation $\mathrm{T}_l E$ associated to E, known as the Tate module. This is a rank-2 $\mathbf{Z}_l$-module with continuous $G_{\mathbf{Q}}$-action, so it induces a continuous representation $\rho_{E,l}\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$. It is known [Sil09, Th. V.1.1] that the quantities $a_p(E) = \operatorname{tr} \rho_l(\mathrm{fr}_p)$ lie in $\mathbf{Z}$ and satisfy the Hasse bound $|a_p(E)| \leqslant 2\sqrt{p}$. Thus we can define, for each unramified prime p, the corresponding Satake parameter for E:

$$\theta_p(E) = \cos^{-1}\left(\frac{a_p(E)}{2\sqrt{p}}\right) \in [0, \pi).$$

The Satake parameters are packaged into an L-function as follows:

$$L(E, s) = \prod_p \frac{1}{(1 - e^{i\theta_p(E)} p^{-s})(1 - e^{-i\theta_p(E)} p^{-s})} = \prod_p \frac{1}{1 - \det \begin{pmatrix} e^{i\theta_p} & \\ & e^{-i\theta_p} \end{pmatrix} p^{-s}}.$$

More generally we have, for each irreducible representation of $\mathrm{SU}(2)$, which will be sym^k for some $k \geqslant 1$, the k-th symmetric power L-function

$$L(\mathrm{sym}^k E, s) = \prod_p \prod_{j=0}^{k} \frac{1}{1 - e^{i(k-2j)\theta_p(E)} p^{-s}} = \prod_p \frac{1}{1 - \det \mathrm{sym}^k \begin{pmatrix} e^{i\theta_p} & \\ & e^{-i\theta_p} \end{pmatrix} p^{-s}}.$$

Numerical experiments suggest that the Satake parameters are distributed with respect to the Sato–Tate distribution $\mathrm{ST} = \frac{2}{\pi}\sin^2\theta\,\mathrm{d}\theta$. Indeed, for any cutoff x, let P_x be the empirical measure

$$P_x = \frac{1}{\pi(x)}\sum_{p\leqslant x}\delta_{\theta_p}.$$

The convergence of the P_x to the Sato–Tate measure is closely related to the analytic properties of the $L(\operatorname{sym}^k E, s)$. First, here is the famous Sato–Tate Conjecture (now a theorem) in our notation.

Theorem 1.2.1 (Sato–Tate conjecture)**.** *If E is non-CM, the measures P_x converge weakly to* ST.

Theorem 1.2.2. *Let Sato–Tate conjecture holds for (a non-CM) E if and only if each of the functions $L(\operatorname{sym}^k E, s)$ have analytic continuation past $\Re = 1$.*

The stunning recent proof of the Sato–Tate conjecture [CHT08; Tay08; HSBT10] in fact showed that the functions $L(\operatorname{sym}^k E, s)$ were potentially automorphic, which gives analytic continuation.

The "usual" Riemann Hypothesis, and its generalization to Artin L-functions, have a natural generalization to elliptic curves. In this context, the discrepancy of the set $\{\theta_p\}_{p\leqslant x}$ is

$$\mathrm{D}\left(\{\theta_p\}_{p\leqslant x}, \mathrm{ST}\right) = \sup_{t\in[0,\pi]}\left|P_x[0,t] - \mathrm{ST}[0,t]\right|.$$

The following conjecture is first made in [AT99]: for $E_{/\mathbf{Q}}$ a non-CM elliptic curve, the bound $\mathrm{D}\left(\{\theta_p\}_{p\leqslant x}, \mathrm{ST}\right) \ll x^{-\frac{1}{2}+\epsilon}$ holds. The authors go on to prove what is essentially the following theorem (fully fleshed out in [Maz08]).

Theorem 1.2.3. *If* $\mathrm{D}\left(\{\theta_p\}_{p\leqslant x}, \mathrm{ST}\right) \ll x^{-\frac{1}{2}+\epsilon}$*, then all the functions* $L(\operatorname{sym}^k E, s)$ *satisfy the Riemann Hypothesis.*

This discussion also makes sense when E has complex multiplication (say, defined over $\mathbf{Q}$), but the Sato–Tate measure is instead the Haar measure on SO(2), i.e. the uniform measure on $[0,\pi]$. Instead of symmetric power L-functions, one takes L-functions for each representation of SO(2).

It is natural to assume that the converse to the implication "Strong Sato–Tate implies General Riemann Hypothesis" holds. David Zywina first suggested to the author that it might not. In this thesis, we construct a range of counterexamples to the implication "Strong Sato–Tate implies Riemann Hypothesis." We also construct a broader conjectural framework generalizing Akiyama–Tanigawa's conjecture to more general motives. Moreover, we generalize the results of [Pan11] to show that there can be no purely Galois-theoretic proof of the Sato–Tate conjecture, for the are Galois representations with arbitrary Sato–Tate distributions! We also show that some of the results of [Sar07] about sums of the form $\sum_{p\leqslant x} \frac{a_p}{\sqrt{p}}$ cannot be generalized to general Galois representations.

1.3 Notational conventions

Throughout, whenever l is mentioned it is a rational prime $\geqslant 7$.

The symbol $f = \Omega(g)$ (in the convention of Hardy–Littlewood) means the negation of $f = O(g)$.

The symbol $f = \Theta(g)$ means there exist non-zero constants $0 < C_1 < C_2$ such that $C_1 g \leqslant f \leqslant C_2 f$.

If μ is a measure on $\mathbf{R}$, then $\mu[a, b] = \mu([a, b])$ and similarly for $[a, b)$, $(a, b])$, etc.

If μ is a measure on $\mathbf{R}$, then $\mathrm{cdf}_\mu(x) = \mu[-\infty, x]$.

CHAPTER 2

DISCREPANCY

2.1 Equidistribution

Discrepancy (also known as the Kolmogorov–Smirnov statistic) is a way of measuring how closely sample data fits a predicted distribution. It has many applications in computer science and statistics, but here we will focus on only the basic properties, such as how discrepancy changes when sequences are "tweaked" and combined.

First, recall that the discrepancy is a way of sharpening the "soft" convergence results of, say [Ser89, A.1]. Let X be a compact topological space, $\boldsymbol{x} = (x_2, x_3, x_5, \dots)$ a sequence of points in X indexed by the rational primes.

Definition 2.1.1. *Let μ be a continuous probability measure on X. The sequence $\boldsymbol{x}$ is* equidistributed *with respect to μ if for all $f \in C(X)$, we have*

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p\leqslant N} f(x_p) \to \int f \, \mathrm{d}\mu.$$

In other words, $\boldsymbol{x}$ is μ-equidistributed if the empirical measures $P_N = \frac{1}{\pi(N)} \sum_{p\leqslant N} \delta_{x_p}$ converge to μ in the weak topology. It is easy to see that $\boldsymbol{x}$ is μ-equidistributed if and only if $\left|\sum_{p\leqslant N} f(x_p)\right| = o(N)$ for all continuous f having $\int f \, \mathrm{d}\mu = 0$. In fact, one can restrict to a set of f which generate a dense subpace of $C(X)^{\mu=0}$.

In the discussion in [Ser89, A.1], X is the space of conjugacy classes in a compact Lie group, and f is allowed to range over the characters of irreducible, nontrivial representations of the group. We will see that the entire discussion can be generalized to a much broader class of *strange Dirichlet series*, which are of the form

$$L_f(\boldsymbol{x}, s) = \prod_p \frac{1}{1 - f(x_p)p^{-s}}.$$

In fact, we can consider functions f which are only only continuous almost everywhere.

Theorem 2.1.2. *Let X be a compact separable metric space with no isolated points. Let μ be a Borel measure on X and let $f \colon X \to \mathbf{C}$ be bounded and measurable. Then f is continuous almost everywhere if and only if*

$$\lim_{N \to \infty} \frac{1}{\pi(N)} \sum_{p \leqslant N} f(x_p) = \int f \, \mathrm{d}\mu$$

for all μ-equidistributed sequences $\boldsymbol{x}$.

Proof. This follows immediately from the proof of [Maz95, Th. 1] □

2.2 Definitions and first results

We will define discrepancy for measures on the d-dimensional half-open box $[0, \infty)^d$. For vectors $x, y \in [0, \infty)^d$, we say $x < y$ if $x_1 < y_1$,. . . ,$x_d < y_d$, and in that case write $[x, y)$ for the half-open box $[x_1, y_1) \times \cdots \times [x_d, y_d)$.

Definition 2.2.1. *Let μ, ν be probability measures on $[0, \infty)^d$. The* discrepancy *of μ with respect to ν is*

$$\mathrm{D}(\mu, \nu) = \sup_{x<y} \left| \mu[x, y) - \nu[x, y) \right|,$$

where $x < y$ range over $[0, \infty)^d$.

The star discrepancy *of μ with respect to ν is*

$$\mathrm{D}^\star(\mu,\nu) = \sup_{0<y} |\mu[0,y) - \nu[0,y)|\,,$$

where y ranges over $[0,\infty)^d$.

Lemma 2.2.2. *Let μ, ν be Borel measures on $\mathbf{R}^d$. Then*

$$\mathrm{D}^\star(\mu,\nu) \leqslant \mathrm{D}(\mu,\nu) \leqslant 2^d\,\mathrm{D}^\star(\mu,\nu).$$

Proof. The first inequality holds because the supremum defining the discrepancy is taken over a larger set than that defining star discrepancy. To prove the second inequality, let $x < y$ be in $[0,\infty)^d$. For $S \subset \{1,\dots,d\}$, let

$$I_S = \{t \in [0,y) : t_i < x_i \text{ for all } i \in S\}.$$

Inclusion-exclusion principle tells us that: $\mu[x,y) = \sum_{S\subset\{1,\dots,d\}} (-1)^{\#S} \mu(I_S)$, and similarly for ν. Since each of the I_S are "half-open boxes" we know that $|\mu(I_S) - \nu(I_S)| \leqslant \mathrm{D}^\star(\mu,\nu)$. It follows that

$$|\mu[x,y) - \nu[x,y)| \leqslant \sum_{S\subset\{1,\dots,d\}} |\mu(I_S) - \nu(I_S)| \leqslant 2^d\,\mathrm{D}^\star(\mu,\nu).$$

For a discussion and related context, see [KN74, Ch. 2 Ex. 1.2]. □

Since we are only interested in the asymptotics of discrepancy, we will gloss over the distinction between discrepancy and star discrepancy, using whichever makes a proof easier to follow.

We are usually interested in comparing empirical measures and their conjectured distribution. Namely, let $\boldsymbol{x} = (x_2, x_3, x_5, \dots)$ be a sequence in $[0,\infty)^d$ indexed by the rational primes, and μ a probability measure on $[0,\infty)^d$. For any real number $N \geqslant 2$, we write $\boldsymbol{x}^N$ for the empirical measure given by

$$\boldsymbol{x}^N(S) = \frac{1}{\pi(N)} \sum_{p \leqslant N} \delta_{x_p}(S) = \frac{\#\{p \leqslant N : x_p \in S\}}{\pi(N)}.$$

Also, we write $\boldsymbol{x}_{\geqslant N}$ for the truncated sequence $(x_p)_{p \geqslant N}$, and similarly for $\boldsymbol{x}_{\leqslant N}$, etc. In this context,

$$\mathrm{D}^\star(\boldsymbol{x}^N, \nu) = \sup_{y \in [0,\infty)^d} \left| \frac{\#\{p \leqslant N : x_p \in [0, y)\}}{\pi(N)} - \int_{[0,y)} \mathrm{d}\nu \right|.$$

If the measure ν is only defined on a subset of $[0,\infty)^d$, we will tacitly extend it by zero. Moreover, if the sequence $\boldsymbol{x}$ actually lies in a torus $(\mathbf{R}/a\mathbf{Z})^d$, we identify that torus with $[0, a)^d \subset [0,\infty)^d$. If ν is normalized Haar measure on the torus, we write $\mathrm{D}^\star(\boldsymbol{x}^N)$ in place of $\mathrm{D}^\star(\boldsymbol{x}^N, \nu)$.

If the sequence $\boldsymbol{x}$ lies in the space $G^\natural$ of conjugacy classes in a compact Lie group G, choose a maximal torus $T \subset G$, and recall that $G^\natural = T/W$, where W is the Weyl group of T. There is a "half-open box" in $\mathfrak{t} = \mathrm{Lie}(T)$ which maps bijectively to T under the exponential map. Choose a "half-open" polyhedral set Q that maps bijectively to T/W. Then $Q \subset \mathfrak{t}$ and, if we choose a basis for $\mathfrak{t}$ mapping to zero in T, then it makes sense to talk about the discrepancy of a sequence in $G^\natural$ with respect to the Haar measure. The paper [Ros13] has a different definition of discrepancy which only works for semisimple simply-connected groups, but also proves an Erdös–Turán inequality in that context. It is likely that a reasonable application of isotropic discrepancy would render these definitions equivalent, at least for asymptotic purposes, but as the two definitions coincide for SU(2), we do not explore this further.

Sometimes the sequence $\boldsymbol{x}$ will not be indexed by the prime numbers, but rather by some other discrete subset of $\mathbf{R}^+$. In that case we will still use the notations $\boldsymbol{x}^N$, $\boldsymbol{x}_{\geqslant N}$, etc., keeping in mind that $\pi(N)$ is replaced by $\#\{\text{indices } \leqslant N\}$.

2.3 Statistical heuristics

Replace the Satake parameters θ_p of an elliptic curve with a sequence $\{\theta_p\}$ of iid random variables with common distribution $\mu = \frac{2}{\pi}\sin^2\theta\,\mathrm{d}\theta$ supported on $[0,\pi]$. Then the discrepancy (known as the Kolmogorov–Smirnov statistic in this context) is the random variable

$$D_N = \sup_{x\in[0,\pi]} \left| \frac{1}{\pi(N)} \sum_{p\leqslant N} 1_{[0,x]} \circ \theta_p - \int 1_{[0,x]}\,\mathrm{d}\mu \right|.$$

Kolmogorov and Smirnov proved that the inside of the absolute value converges to zero. The Glivenko–Cantelli Theorem says that $D_N \to 0$ almost surely, and even better that the normalized discrepancy $\sqrt{\pi(N)}D_N$ approaches a limiting distribution K (supremum of the Brownian Bridge) which does not depend on μ. The rate of convergence of $\sqrt{\pi(N)}D_N$ to that distribution is quantified by the Dvoretzky–Kiefer–Wolfowitz inequality, which tells us that

$$\mathrm{P}\left(\sqrt{\pi(N)}D_N > z\right) \leqslant 2e^{-2z^2}.$$

These theorems suggest that for $E_{/\mathbf{Q}}$ an elliptic curve, the "true" discrepancy $\mathrm{D}(\boldsymbol{\theta}^N, \mathrm{ST})$ should decay like $\pi(N)^{-\frac{1}{2}}$, or at least $N^{-\frac{1}{2}+\epsilon}$. Ideally, the normalized discrepancy $\sqrt{\pi(N)}\,\mathrm{D}^\star(\boldsymbol{\theta}^N, \mathrm{ST})$ would also be equidistributed, but sadly numerical experiments suggest that this is not the case.

2.4 Examples

One of the first examples of equidistributed sequences is the set of translates of an irrational number modulo one.

Theorem 2.4.1. *Let $a \in \mathbf{R}$ be irrational. Then the sequence $\boldsymbol{x} = (a \mod 1, 2a \mod 1, 3a \mod 1, \dots)$ is equidistributed in $[0,1]$.*

Proof. This follows from the more precise results of Chapter 4. □

Sequences of this form will have discrepancy that decays like $N^{-\alpha\pm\epsilon}$, for $\alpha \in (0,1/2)$. It can be useful to have a sequence whose discrepancy decays faster. The best known rate of decay is achieved by the following sequence.

Definition 2.4.2. *The* van der Corput sequence *is given by $\boldsymbol{v} = \left(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \dots\right)$. More precisely, write n in base 2 as $n = \sum a_i 2^i$. Then $v_n = \sum a_i 2^{-(i+1)}$.*

The van der Corput sequence has generalizations to other bases and higher dimensions. It is well-known for being “very equidistributed”—i.e., its discrepancy has extremely fast convergence to zero.

Lemma 2.4.3. *Let $\boldsymbol{v} = \{v_n\}$ be the van der Corput sequence. Then $\mathrm{D}(\boldsymbol{v}^N) \leqslant \frac{\log(N+1)}{N\log 2}$.*

Proof. This is [KN74, Ch. 2 Th. 3.5]. In particular, we will use often that $\mathrm{D}(\boldsymbol{v}^N) \ll \frac{\log N}{N}$. □

The van der Corput sequence is uniformly distributed, but there is a convenient trick to construct sequences equidistributed with respect to more general measures.

Definition 2.4.4. *Let μ be a probability measure on $[a,b]$. We call μ* good *if* cdf_μ *is continuous, strictly increasing, and sends $a \mapsto 0$.*

Note that if μ is a good measure, then cdf_μ is an order isomorphism from $[a,b]$ to $[0,1]$.

Theorem 2.4.5. *Let μ be a good measure on a closed interval. Then there exists a sequence $\boldsymbol{x} = (x_1, x_2, \dots)$ such that $\mathrm{D}(\boldsymbol{x}^N, \mu) \ll \frac{\log(N)}{N}$.*

Proof. Since μ is good, cdf_μ is an order isomorphism. Then Lemma 2.6.3 tells us that for $\boldsymbol{v}$ the van der Corput sequence on $[0,1]$, we have $\mathrm{D}(\mathrm{cdf}_\mu^{-1}(\boldsymbol{v})^N, \mu) = \mathrm{D}(\boldsymbol{v}^N, \mu)$, which gives us the desired result with $\boldsymbol{x} = \mathrm{cdf}_\mu^{-1}(\boldsymbol{v})$. □

Theorem 2.4.6. *Let μ be a good measure. Fix $\alpha \in (0,1)$. Then there exists a sequence $\boldsymbol{x} = (x_1, x_2, \dots)$ such that $\mathrm{D}^\star(\boldsymbol{x}^N, \mu) = \Theta(N^{-\alpha})$.*

Proof. If $\boldsymbol{x}_{\leqslant N}$ is a sequence of length N, let $\boldsymbol{x}_{\leqslant N} : a^M$ be the sequence $(x_1, \dots, x_N, a, \dots, a)$ (M copies of a). Then

$$\mathrm{D}^\star(\boldsymbol{x}^N : a^M, \mu) \geqslant \left| \frac{\#\{n \leqslant N+M : x_n = a\}}{N+M} - \mu\{a\} \right| \geqslant \frac{M}{N+M}.$$

On the other hand,

$$\begin{aligned} |\mathrm{cdf}_{N,M}(t) - \mathrm{cdf}_N(t)| &\leqslant \frac{\left|\#\{n \leqslant N : x_n \leqslant t\} + M - \frac{M+N}{N}\#\{n \leqslant N : x_n \leqslant t\}\right|}{M+N} \\ &\leqslant \frac{2M}{M+N}, \end{aligned}$$

which implies that $\mathrm{D}^\star\left(\boldsymbol{x}^N : a^M, \mu\right) \leqslant \mathrm{D}^\star\left(\boldsymbol{x}^N, \mu\right) + \frac{2M}{M+N}$. Let $\boldsymbol{v}$ be the μ-equidistributed van der Corput sequence of Theorem 2.4.5, possibly transformed linearly to lie in $[a,b]$. We know that $\mathrm{D}(\boldsymbol{v}^N, \mu) \ll \frac{\log N}{N}$, which converges to zero faster than $N^{-\alpha}$.

We construct the sequence $\boldsymbol{x}$ via the following recipe. Start with $(x_1 = v_1, x_2 = v_2, \dots)$ until, for some N_1, $\mathrm{D}^\star(\boldsymbol{x}^{N_1}, \mu) < N_1^{-\alpha}$. Then set $x_{N_1+1} = a$, $x_{N_1+2} = a$, $\dots$, until $\mathrm{D}^\star(\boldsymbol{x}^{N_1+M_1}, \mu) > (N_1 + M_1)^{-\alpha}$. Then set $x_{N_1+M_1+1} = v_{N_1+1}$, $x_{N_1+M_1+2} = v_{N_1+2}$, $\dots$, until once again $\mathrm{D}^\star(\boldsymbol{x}^{N_1+M_1+N_2}, \mu) < (N_1 + M_1 + N_2)^{-\alpha}$. Repeat indefinitely. We will show first, that the two steps are possible, and that nowhere does $\mathrm{D}^\star(\boldsymbol{x}^N, \mu)$ differ by too much from $N^{-\alpha}$.

Note that $\frac{M+1}{N+M+1} - \frac{M}{N+M} \leqslant N^{-1}$. This tells us that when we are adding a's at the end of $\boldsymbol{x}^N$, the discrepancy of $\boldsymbol{x}_{\leqslant N} : a_{\leqslant M}$ increases by at most N^{-1} at each step. So if $\mathrm{D}^\star(\boldsymbol{x}^N, \mu) < N^{-\alpha}$, we can ensure that $\mathrm{D}^\star(\boldsymbol{x}^N : a^M, \mu)$ is at most N^{-1} greater than $N^{-\alpha}$.

Moreover, we know that $\mathrm{D}^\star(\boldsymbol{x}^N : a, \mu)$ is at most $\frac{2}{N+1}$ away from $\mathrm{D}^\star(\boldsymbol{x}^N, \mu)$. So when adding van der Corput elements to the end of the sequence, its discrepancy cannot decay any faster than by $\frac{2}{N+1}$ per a added. This yields

$$\left|\mathrm{D}^\star(\boldsymbol{x}^N, \mu) - N^{-\alpha}\right| \ll N^{-1},$$

which is even stronger than we need. □

In the remainder of this thesis, we will refer to the sequences constructed in this theorem as "$N^{-\alpha}$-decay van der Corput sequences."

2.5 The Koksma–Hlawka inequality

Here we summarize the results of the paper [Ö99], generalizing them as needed for our context. Recall that a function f on $[0, \infty)^d$ is said to be of *bounded variation* if there is a finite Radon measure ν such that $f(x) - f(0) = \nu[0, x]$. In such a case we write $\mathrm{Var}(f) = |\nu|$. If the appropriate differentiability conditions are satisfied, then

$$\mathrm{Var}(f) = \int_{[0,\infty)^d} \left| \frac{\mathrm{d}^d f}{\mathrm{d}x_1 \dots \mathrm{d}x_d} \right| .$$

Theorem 2.5.1 (Koksma–Hlawka)**.** *Let μ be a probability measure on $[0, \infty)^d$, f a function of bounded variation. Then for any sequence $\boldsymbol{x} = (x_1, x_2, \dots)$ in $[0, \infty)^d$, we have*

$$\left| \frac{1}{N} \sum_{n \leqslant N} f(x_n) - \int f \, \mathrm{d}\mu \right| \leqslant \mathrm{Var}(f) \, \mathrm{D}(\boldsymbol{x}^N, \mu).$$

Proof. By our assumptions there is a Radon measure ν such that $f(y) - f(0) = \nu[0, y]$. What follows is essentially trivial, noting that $1_{[0,x]}(y) = 1_{[y,\infty)^d}(x)$.

$$\begin{aligned} \frac{1}{N} \sum_{n \leqslant N} f(x_n) - \int f \, \mathrm{d}\mu &= \frac{1}{N} \sum_{n \leqslant N} (f(x_n) - f(0)) - \int (f - f(0)) \, \mathrm{d}\mu \\ &= \frac{1}{N} \sum_{n \leqslant N} \int 1_{[y,\infty)^d}(x_n) \, \mathrm{d}\nu(y) - \int \int 1_{[0,y]} \, \mathrm{d}\nu \, \mathrm{d}\mu(y) \\ &= \int \frac{1}{N} \sum_{n \leqslant N} 1_{[y,\infty)^d}(x_n) - \int 1_{[y,\infty)^d} \, \mathrm{d}\mu \, \mathrm{d}\nu(y) \end{aligned}$$

It follows that

$$\left| \frac{1}{N} \sum_{n \leqslant N} f(x_n) - \int f \, \mathrm{d}\mu \right| \leqslant \sup_{y \in [0,\infty)} \left| \frac{1}{N} \sum_{n \leqslant N} 1_{[y,\infty)}(x_n) - \int 1_{[y,\infty)} \, \mathrm{d}\mu \right| \cdot |\nu|.$$

The supremum in question is clearly bounded above by $\mathrm{D}(\boldsymbol{x}^N, \mu)$, so the proof is complete. □

This theorem is proved in a somewhat restrictive setting, and can be generalized. For f a function on $\mathbf{R}^+$ that is bounded variation in the traditional sense (for example, piecewise continuous) and μ a continuous probability measure, the inequality

$$\left|\frac{1}{N}\sum_{n\leqslant N} f(x_n) - \int f\,\mathrm{d}\mu\right| \leqslant \mathrm{Var}(f)\,\mathrm{D}^\star(\boldsymbol{x}^N,\mu)$$

holds [KN74, Ch. 2, Th. 5.1]. In particular, when μ is the Sato–Tate measure and f is piecewise continuous, we can apply this inequality.

2.6 Comparing sequences

Lemma 2.6.1. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be sequences in $[0,\infty)$. Suppose μ is an absolutely continuous probability measure on $[0,\infty)$ with continuous bounded Radon–Nikodym derivative $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$, where λ is the Lebesgue measure. Then*

$$\left|\mathrm{D}^\star(\boldsymbol{x}^N,\nu) - \mathrm{D}^\star(\boldsymbol{y}^N,\nu)\right| \leqslant \left\|\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right\|_\infty \epsilon + \frac{\#\{n\leqslant N : |x_n-y_n|\geqslant\epsilon\}}{N}.$$

Proof. Let $\epsilon > 0$ and $t \in [0,\infty)$ be arbitrary. For all $n \leqslant N$ such that $y_n < t$, either $x_n < t+\epsilon$ or $|x_n - y_n| \geqslant \epsilon$. It follows that

$$\boldsymbol{y}^N[0,t) \leqslant \boldsymbol{x}^N[0,t+\epsilon) + \frac{\#\{n\leqslant N : |x_n-y_n|\geqslant\epsilon\}}{N}.$$

Moreover, we have $\left|\boldsymbol{x}^N[0,t+\epsilon) - \nu[0,t+\epsilon)\right| \leqslant \mathrm{D}^\star(\boldsymbol{x}^N,\nu)$. Putting these together, we get:

$$\begin{aligned}\boldsymbol{y}^N[0,t) - \nu[0,t) &\leqslant \boldsymbol{x}^N[0,t+\epsilon) - \nu[0,t) + \frac{\#\{n\leqslant N : |x_n-y_n|\geqslant\epsilon\}}{N}\\ &\leqslant \nu[t,t+\epsilon) + \mathrm{D}^\star(\boldsymbol{x}^N,\nu) + \frac{\#\{n\leqslant N : |x_n-y_n|\geqslant\epsilon\}}{N}\\ &\leqslant \left\|\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right\|_\infty \epsilon + \mathrm{D}^\star(\boldsymbol{x}^N,\nu) + \frac{\#\{n\leqslant N : |x_n-y_n|\geqslant\epsilon\}}{N}\end{aligned}$$

This tells us that

$$\mathrm{D}^\star(\boldsymbol{y}^N,\nu) \leqslant \left\|\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right\|_\infty \epsilon + \mathrm{D}^\star(\boldsymbol{x}^N,\nu) + \frac{\#\{n\leqslant N : |x_n-y_n|\geqslant\epsilon\}}{N}.$$

Reversing the roles of $\boldsymbol{x}$ and $\boldsymbol{y}$, we obtain the desired result. □

Lemma 2.6.2. *Let σ be an isometry of $\mathbf{R}$, and $\boldsymbol{x}$ a sequence in $[0,\infty)$ such that $\sigma(\boldsymbol{x})$ is also in $[0,\infty)$. Let ν be an absolutely continuous measure on $[0,\infty)$ such that $\sigma_*\nu$ is also supported on $[0,\infty)$. Then*

$$\left|\mathrm{D}(\boldsymbol{x}^N,\nu)-\mathrm{D}(\sigma_*\boldsymbol{x}^N,\sigma_*\nu)\right|\leqslant\frac{2}{\pi(N)}.$$

Proof. Every isometry of $\mathbf{R}$ is a combination of translations and reflections. The statement is clear with translations (the two discrepancies are equal). So, suppose $\sigma(t)=a-t$ for some $a>0$. Since ν is absolutely continuous, $\nu\{t\}=0$ for all $t\geqslant 0$. In particular, $\nu[s,t)=\nu(s,t]$. In contrast, $\boldsymbol{x}^N\{t\}\leqslant\pi(N)^{-1}$. For any interval $[s,t)$ in $[0,\infty)$, we know that

$$\left|\boldsymbol{x}^N[s,t)-\boldsymbol{x}^N(s,t]\right|\leqslant\frac{2}{\pi(N)},$$

hence

$$\left|\boldsymbol{x}^N[s,t)-\nu[s,t)-(\sigma_*\boldsymbol{x}^N)[a-t,a-s)-(\sigma_*\nu)[a-t,a-s)\right|\leqslant\frac{2}{\pi(N)}.$$

This proves the result. □

A trick we will use throughout this thesis involves comparing the discrepancy of a sequence with the discrepancy of a pushforward sequence, with respect to the pushforward measure.

Lemma 2.6.3. *Let f be an order isomorphism $f\colon[a,b]\to[c,d]$. If $\boldsymbol{x}$ is a sequence on $[a,b]$ and μ is a probability measure on $[a,b]$, then $\mathrm{D}(\boldsymbol{x}^N,\mu)=\mathrm{D}(f(\boldsymbol{x})^N,f_*\mu)$, and likewise for star discrepancy.*

Proof. This is a simple computation, which we only check for star discrepancy:

$$\begin{aligned}\mathrm{D}^\star(f(\boldsymbol{x})^N, f_*\mu) &= \sup_{t\in[c,d]}\left|\frac{\#\{n\leqslant N : f(x_n)\leqslant t\}}{N} - (f_*\mu)[a,t]\right| \\ &= \sup_{t\in[a,b]}\left|\frac{\#\{n\leqslant N : x_n\leqslant f^{-1}(t)\}}{N} - \mu[a,f^{-1}(t)]\right| \\ &= \mathrm{D}(\boldsymbol{x}^N,\mu).\end{aligned}$$

□

Lemma 2.6.4. *Let f be an order anti-automorphism $[a,b]\to[c,d]$. If $\boldsymbol{x}$ is a sequence on $[a,b]$ and μ is a probability measure on $[a,b]$, then $\mathrm{D}(\boldsymbol{x}^N,\mu)$?.*

Proof. Repeat the proof of Lemma 2.6.3, except we have $s\leqslant f(x_n)\leqslant t$ if and only if $f^{-1}(t)\leqslant f(x_n)\leqslant f^{-1}(s)$, and likewise $(f_*\mu)[s,t]=\mu[f^{-1}(t),f^{-1}(s)]$. □

2.7 Combining sequences

Definition 2.7.1. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be sequences in $[0,\infty)^d$. We write $\boldsymbol{x}\wr\boldsymbol{y}$ for the interleaved sequence*

$$(x_2, y_2, x_3, y_3, x_5, y_5, \ldots, x_p, y_p, \ldots).$$

For the interleaved sequence $\boldsymbol{x}\wr\boldsymbol{y}$, we write $(\boldsymbol{x}\wr\boldsymbol{y})^N$ for the empirical measure

$$(\boldsymbol{x}\wr\boldsymbol{y})^N = \frac{1}{2\pi(N)}\sum_{p\leqslant N}\delta_{x_p}+\delta_{y_p}.$$

Theorem 2.7.2. *Let I and J be disjoint open boxes in $[0,\infty)^d$, and let μ, ν be absolutely continuous probability measures on I and J, respectively. Let $\boldsymbol{x}$ be a sequence in I and $\boldsymbol{y}$ be a sequence in J. Then*

$$\max\{\mathrm{D}(\boldsymbol{x}^N,\mu),\mathrm{D}(\boldsymbol{y}^N,\nu)\}\leqslant\mathrm{D}((\boldsymbol{x}\wr\boldsymbol{y})^N,\mu+\nu)\leqslant\mathrm{D}(\boldsymbol{x}^N,\mu)+\mathrm{D}(\boldsymbol{y}^N,\nu)$$

Proof. Any half-open box in $[0,\infty)^d$ can be split by a coordinate hyperplane into two disjoint half-open boxes $[a,b)\sqcup[s,t)$, each of which intersects at most one of I and J. We may assume that $[a,b)\cap J=\varnothing$ and $[s,t)\cap I=\varnothing$. Then

$$\begin{aligned}\big|(\boldsymbol{x}\wr\boldsymbol{y})^N([a,b)\sqcup[s,t))-(\mu+\nu)([a,b)\sqcup[s,t))\big| &\leqslant |\boldsymbol{x}^N[a,b)-\mu[a,b)|+|\boldsymbol{y}^N[s,t)-\nu[s,t)|\\ &\leqslant \mathrm{D}(\boldsymbol{x}^N,\mu)+\mathrm{D}(\boldsymbol{y}^N,\nu).\end{aligned}$$

This yields the second inequality in the statement of the theorem. To see the first, assume that the maximum discrepancy is $\mathrm{D}(\boldsymbol{x}^N,\mu)$, and let $[s,t)$ be a half-open box such that $|\boldsymbol{x}^N[s,t)-\mu[s,t)|$ is within an arbitrary ϵ of $\mathrm{D}(\boldsymbol{x}^N,\mu)$. We can assume that $[s,t)$ does not intersect J, and thus

$$\big|(\boldsymbol{x}\wr\boldsymbol{y})^N[s,t)-(\mu+\nu)[s,t)\big|=|\boldsymbol{x}^N[s,t)-\mu[s,t)|,$$

which yields the result. □

CHAPTER 3

STRANGE DIRICHLET SERIES

3.1 Definitions

We start by considering a very general class of Dirichlet series. In fact, they are all Dirichlet series that admit a product formula with degree-1 factors, but in this thesis they will be called strange Dirichlet series. The motivating example was suggested to the author by Ravi Ramakrishna. Let $E_{/\mathbf{Q}}$ be an elliptic curve and let

$$L_{\text{sgn}}(E,s) = \prod_p \frac{1}{1-\operatorname{sgn}(a_p)p^{-s}}.$$

How much can we say about the behavior of $L_{\text{sgn}}(E,s)$? For example, does it admit analytic continuation past $\Re = 1$? Can the rank of E be found from $L_{\text{sgn}}(E,s)$?

Definition 3.1.1. *Let $\boldsymbol{z} = (z_2, z_3, z_5, \dots)$ be a sequence of complex numbers indexed by the primes. The associated* strange Dirichlet series *is*

$$L(\boldsymbol{z},s) = \prod_p \frac{1}{1-z_p p^{-s}}.$$

If z_p is only defined for all but finitely many primes, then we tacitly set $\boldsymbol{z}_p = 0$ for all primes for which z_p is not defined.

Lemma 3.1.2. *Let $\boldsymbol{z}$ be a sequence with $\|\boldsymbol{z}\|_\infty \leqslant 1$. Then $L(\boldsymbol{z},s)$ defines a holomorphic function on the region $\{\Re s > 1\}$. Moreover, on that region,*

$$\log L(\boldsymbol{z},s) = \sum_{p^r} \frac{z_p^n}{np^{ns}}.$$

Proof. Expanding the product for $L(\boldsymbol{z}, s)$ formally, we have

$$L(\boldsymbol{z}, s) = \sum_{n \geqslant 1} \frac{\prod_p z_p^{v_p(n)}}{n^s}.$$

An easy comparison with the Riemann zeta function tells us that this sum is holomorphic on $\{\Re s > 1\}$. By [Apo76, Th. 11.7], the product formula holds in the same region. The formula for $\log L(\boldsymbol{z}, s)$ comes from [Apo76, 11.9 Ex.2]. □

Lemma 3.1.3 (Abel summation). *Let $\boldsymbol{z} = (z_2, z_3, z_5, \dots)$ be a sequence of complex numbers, f a smooth complex-valued function on $\mathbf{R}$. Then*

$$\sum_{p \leqslant N} f(p) z_p = f(N) \sum_{p \leqslant N} z_p - \int_2^N f'(x) \sum_{p \leqslant x} z_p \, \mathrm{d}x.$$

Proof. Simply note that if $p_1, \dots, p_n$ is an enumeration of the primes $\leqslant N$, we have

$$\begin{aligned}
\int_2^N f'(x) \sum_{p \leqslant x} z_p \, \mathrm{d}x &= \sum_{p \leqslant N} z_p \int_{p_n}^N f' + \sum_{i=1}^{n-1} \sum_{p \leqslant p_{i+1}} z_p \int_{p_i}^{p_{i+1}} f' \\
&= (f(N) - f(p_n)) \sum_{p \leqslant N} z_p + \sum_{i=1}^{n-1} (f(p_{i+1}) - f(p_i)) \sum_{p \leqslant p_{i+1}} z_p \\
&= f(N) \sum_{p \leqslant N} z_p - \sum_{p \leqslant N} f(p) z_p,
\end{aligned}$$

as desired. □

Theorem 3.1.4. *Assume $|\sum_{p \leqslant x} z_p| \ll x^{\alpha+\epsilon}$ for some $\alpha \in [\frac{1}{2}, 1]$. Then the series for $\log L(\boldsymbol{z}, s)$ converges to a holomorphic function on the region $\{\Re s > \alpha\}$.*

Proof. Formally split the sum for $\log L(\boldsymbol{z}, s)$ into two pieces:

$$\log L(\boldsymbol{z}, s) = \sum_p \frac{z_p}{p^s} + \sum_p \sum_{r \geqslant 2} \frac{z_p^r}{r p^{rs}}.$$

For each p, we have

$$\left| \sum_{r \geqslant 2} \frac{z_p^r}{r p^{rs}} \right| \leqslant \sum_{r \geqslant 2} p^{-r \Re s} = p^{-2\Re s} \frac{1}{1 - p^{-\Re s}}.$$

Elementary analysis gives

$$1 \leqslant \frac{1}{1 - p^{-\Re s}} \leqslant 2 + 2\sqrt{2},$$

so the second piece of $\log L(\boldsymbol{z}, s)$ converges absolutely when $\Re s > \frac{1}{2}$. We could simply cite [Ten95, II.1 Th. 10] to finish the proof; instead we prove directly that $\sum \frac{z_p}{p^s}$ converges absolutely to a holomorphic function on the region $\{\Re s > \alpha\}$.

By Lemma 3.1.3 with $f(x) = x^{-s}$, we have

$$\begin{aligned} \sum_{p \leqslant N} \frac{z_p}{p^s} &= N^{-s} \sum_{p \leqslant N} z_p + s \int_2^N \sum_{p \leqslant x} z_p \frac{\mathrm{d}x}{x^{s+1}} \\ &\ll N^{-\Re s + \alpha + \epsilon} + s \int_2^N x^{\alpha+\epsilon} \frac{\mathrm{d}x}{x^{s+1}}. \end{aligned}$$

Since $\alpha - \Re s < 0$, the first term is bounded. Since $\Re s + 1 - \alpha > 1$ and ϵ is arbitrary, the integral converges absolutely, and the proof is complete. □

Theorem 3.1.5. *Let* $\boldsymbol{z} = (z_2, z_3, \dots)$ *be a sequence with* $\|\boldsymbol{z}\|_\infty \leqslant 1$*, and assume* $\log L(\boldsymbol{z}, s)$ *has analytic continuation to* $\{\Re s > \alpha\}$ *for some* $\alpha \in \left[\frac{1}{2}, 1\right]$*, and that for* $\sigma > \alpha$*, we have* $|\log L(\boldsymbol{z}, \sigma + it)| \ll |t|^{1-\epsilon}$ *(implied constant independent of* σ*.) Then* $|\sum_{p \leqslant N} z_p| \ll N^{\alpha+\epsilon}$*.*

Proof. Recall that we can write

$$\log L(\boldsymbol{z}, s) = \sum_p \frac{z_p}{p^s} + \sum_p \sum_{r \geqslant 2} \frac{z_p^r}{r p^{rs}} = \sum_p \frac{z_p}{p^s} + O(\zeta(2\Re s)).$$

Thus, for any $\epsilon > 0$, analytic continuation and the bound on $|\log L(\boldsymbol{z}, \sigma + it)|$ implies the same analytic continuation and bound for $\sum \frac{z_p}{p^s}$ on $\{\Re s > \alpha + \epsilon\}$.

For any $T > 0$, let $\gamma_T = \gamma_{1,T} + \gamma_{2,T} + \gamma_{3,T} + \gamma_{4,T}$ be the following contour:

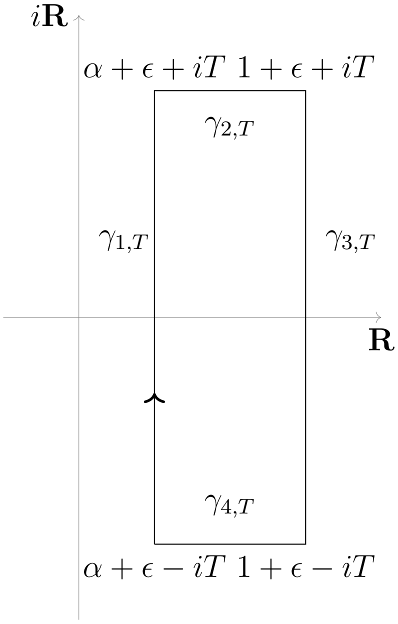
$$\gamma_{1,T}(t) = (\alpha + \epsilon) + it \qquad t \in [-T, T]$$

$$\gamma_{2,T}(t) = t + iT \qquad t \in [\alpha + \epsilon, 1 + \epsilon]$$

$$\gamma_{3,T}(t) = (1 + \epsilon) + it \qquad t \in [T, -T]$$

$$\gamma_{4,T}(t) = t - iT \qquad t \in [1 + \epsilon, \alpha + \epsilon].$$

As a picture, γ_T looks like this:



By Perron's formula [Apo76, Th. 11.18],

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{-\gamma_{3,T}} \sum_p \frac{z_p}{p^s} N^z \frac{\mathrm{d}z}{z} = \frac{1}{2} \sum_{p \leqslant N} z_p.$$

for $N \in \mathbf{Z}$, and the same without the $\frac{1}{2}$ on the right-hand side when $N \notin \mathbf{Z}$.

Let $h(s)$ be the analytic continuation of $\sum z_p p^{-s}$ to $\{\Re s > \alpha\}$. Since $\int_{\gamma_T} h(s) \frac{\mathrm{d}s}{s} = 0$, we obtain

$$\left|\sum_{p\leqslant N} z_p\right| \ll \lim_{T\to\infty} \left(\left| \int_{\gamma_{1,T}} h(s) N^s \frac{\mathrm{d}s}{s} \right| + \left| \int_{\gamma_{2,T}} h(s) N^s \frac{\mathrm{d}s}{s} \right| + \left| \int_{\gamma_{4,T}} h(s) N^s \frac{\mathrm{d}s}{s} \right| \right).$$

We know that $|h(\sigma+it)| \ll |t|^{1-\epsilon}$, so we can bound

$$\left|\int_{\gamma_{2,T}} h(s) N^s \frac{\mathrm{d}s}{s}\right| = \left|\int_{\alpha+\epsilon}^{1+\epsilon} \frac{h(t+iT)N^{t+iT}}{t+iT}\,\mathrm{d}t\right| \ll \frac{N^{1+\alpha}}{T^\epsilon},$$

and similarly for $\gamma_{4,T}$. Finally, note that

$$\left|\int_{\gamma_{1,T}} h(s) N^s \frac{\mathrm{d}s}{s}\right| \ll \int_{-T}^{T} |t|^{1-\epsilon} \frac{N^{\alpha+\epsilon}}{(\alpha+\epsilon)^2+t^2}\,\mathrm{d}t \ll N^{\alpha+\epsilon}.$$

Letting $T \to \infty$ we obtain the desired result. □

Let X be a space, $f\colon X \to \mathbf{C}$ a function with $\|f\|_\infty \leqslant 1$, and $\boldsymbol{x} = (x_2, x_3, \dots)$ a sequence in X. Write

$$L_f(\boldsymbol{x}, s) = \prod_p \frac{1}{1 - f(x_p)p^{-s}},$$

for the associated strange Dirichlet series. In the remainder, we will exclusively focus on strange Dirichlet series of this form.

3.2 Relation to automorphic and motivic L-functions

Suppose G is a compact group, $G^\natural$ the space of conjugacy classes in G. If $\boldsymbol{x} = (x_2, x_3, x_5, \dots)$ is a sequence in $G^\natural$ and ρ is a finite-dimensional representation of G, put

$$L(\rho(\boldsymbol{x}), s) = \prod_p \frac{1}{\det(1 - \rho(x_p)p^{-s})}.$$

Clearly $L((\rho_1 \oplus \rho_2)(\boldsymbol{x}), s) = L(\rho_1(\boldsymbol{x}), s) L(\rho_2(\boldsymbol{x}), s)$. Now, let $T \subset G$ be a maximal torus, and recall that $T \twoheadrightarrow G^\natural$. The representation $\rho|_T$ decomposes as $\bigoplus \chi^{\oplus m_\chi}$, where χ ranges over characters of T and the entire expression is W-invariant. We may regard the x_p as lying in T/W, so we have

$$L(\rho(\boldsymbol{x}), s) = \prod_\chi L(\chi(\boldsymbol{x}), s)^{m_\chi}.$$

If the trivial representation appears in $\rho|_T$, this product formula will include a copy (possibly several) of $\zeta(s)$.

3.3 Discrepancy of sequences and the Riemann Hypothesis

Definition 3.3.1. *Let $L(s)$ be a Dirichlet series with full product formula. Then the Riemann Hypothesis for L holds if* $\log L$ *admits analytic continuation to* $\{\Re > 1/2\}$.

In effect, Theorem 3.1.5 says that under reasonable analytic hypotheses, the Riemann Hypothesis for $L(\boldsymbol{z}, s)$ implies the estimate $|\sum_{p\leqslant N} z_p| \ll N^{-\frac{1}{2}+\epsilon}$.

Theorem 3.3.2. *Let (X, μ) be a probability space in which discrepancy makes sense, and let $\boldsymbol{x} = (x_2, x_3, x_5, \dots)$ be a sequence in X with* $\mathrm{D}(\boldsymbol{x}^N, \mu) \ll N^{-\frac{1}{2}+\epsilon}$. *Then for any function f on X of bounded variation, the strange Dirichlet series $L_f(\boldsymbol{x}, s)$ satisfies the Riemann Hypothesis.*

Proof. The bound on discrepancy yields (by the Koksma–Hlawka inequality) the estimate

$$\left|\sum_{p\leqslant N} f(x_p)\right| \ll N^{-\frac{1}{2}+\epsilon}.$$

By Theorem 3.1.4, the Riemann Hypothesis holds for $L_f(\boldsymbol{x}, s)$. □

Let $F = \mathbf{F}_q(t)$ be a function field, $E_{/F}$ a generic elliptic curve. There is, for every prime $\mathfrak{p}$ of F, a Satake parameter $\theta_{\mathfrak{p}} \in [0, \pi]$, defined in the usual way. It is known [Kat88, Ch. 3] that

$$\left|\sum_{\mathrm{N}(\mathfrak{p})\leqslant x} U_k(\theta_{\mathfrak{p}})\right| \ll k\sqrt{x}.$$

This tells us that for any $f \in C(\mathrm{SU}(2))$ with $\sum_{k\geqslant 1} |\widehat{f}(\mathrm{sym}^k)| < \infty$, the strange Dirichlet series $L_f(\boldsymbol{\theta}, s)$ satisfies the Riemann Hypothesis.

For function fields, the best estimate on discrepancy is found in [Nie91], where it is shown that $D_N \ll N^{-1/4}$ by applying a generalization of the Koksma–Hlawka inequality to $\mathrm{SU}(2)^\natural$. Namely, for any odd r, we have

$$\mathrm{D}(\boldsymbol{\theta}^x, \mathrm{ST}) \ll \frac{1}{r} + \sum_{k=1}^{2r-1} \frac{1}{k} \left| \frac{1}{\pi_F(x)} \sum_{\mathrm{N}(\mathfrak{p})\leqslant x} U_k(\boldsymbol{\theta}_{\mathfrak{p}}) \right|.$$

Using the above estimate on $\sum U_k(\boldsymbol{\theta}_{\mathfrak{p}})$, he is able to derive $D_N \ll N^{-1/4}$. This fits the results of [BK15; RT16], both of which derive estimates of the form $D_N \ll N^{-1/4+\epsilon}$ under GRH + functional equation for the (non-CM) elliptic curve in question.

CHAPTER 4

IRRATIONALITY EXPONENTS

4.1 Definitions and first results

We follow the notation of [Lau09]. Let $x = (x_1, \dots, x_d) \in \mathbf{R}^d$ be such that the x_i are $\mathbf{Q}$-linearly independent.

Definition 4.1.1. *Let $\omega_0(x)$ (resp. $\omega_{d-1}(x)$) be the supremum of the set of real numbers ω for which there exist infinitely many $m = (m_0, \dots, m_d) \in \mathbf{Z}^{d+1}$ such that*

$$\max\{|m_0 x_i - m_i|\} \leqslant \|m\|_\infty^{-\omega} \qquad \textit{(resp.}$$

$$|m_0 + m_1 x_1 + \cdots + m_r x_r| \leqslant \|m\|_\infty^{-\omega}\textit{).}$$

These two quantities are related by Khintchine's Transference Principle, namely

$$\frac{\omega_{d-1}(x)}{(d-1)\omega_{d-1}(x) + d} \leqslant \omega_(x) \leqslant \frac{\omega_{d-1}(x) - d + 1}{d}.$$

Moreover, these inequalities are sharp in a very strong sense.

Theorem 4.1.2 (Jarník)**.** *Let $w \geqslant 1/d$. Then there exists $x \in \mathbf{R}^d$ such that $\omega_0(x) = w$ and $\omega_{d-1}(x) = dw + d - 1$.*

Theorem 4.1.3. *If $d = 1$, then $\omega_0(x) = \mu - 1$, where μ is the traditional irrationality measure of x.*

So Roth's Theorem tells us that for x an algebraic irrational, $\omega_0(x) = 1$.

Now given $x \in \mathbf{R}^d$, we write $d(x, \mathbf{Z}^d) = \min_{m \in \mathbf{Z}^d} |x - m|$, where $|\cdot|$ is any fixed norm on $\mathbf{R}^d$. Note that $d(x, \mathbf{Z}^d) = 0$ if and only if $x \in \mathbf{Z}^d$.

Lemma 4.1.4. *Let $x \in \mathbf{R}^d$ with $\|x\|_\infty \leqslant 1$ and $\omega_0(x)$ (resp. $\omega_{d-1}(x)$) finite. Then*

$$\frac{1}{d(nx, \mathbf{Z}^d)} \ll |n|^{\omega_0(x)+\epsilon} \qquad \textit{(resp.}$$
$$\frac{1}{d(\langle m, x\rangle, \mathbf{Z})} \ll |m|^{\omega_{d-1}(x)+\epsilon} \qquad \textit{for } m \in \mathbf{Z}^d\textit{).}$$

Proof. Let $\epsilon > 0$. Then there are only finitely many $n \in \mathbf{Z}$ (resp. $m \in \mathbf{Z}^d$) such that the inequalities in Definition 4.1.1 hold with $\omega_0(x) + \epsilon$ (resp. $\omega_{d-1}(x) + \epsilon$). In other words, there exist constants $C_0, C_{d-1} > 0$ such that

$$\max\{|m_0 x_i - m_i|\} \geqslant C_0 \|m\|_\infty^{-\omega_0(x)-\epsilon},$$
$$|m_0 + m_1 x_1 + \cdots + m_d x_d| \geqslant C_{d-1} \|m\|_\infty^{-\omega_{d-1}(x)-\epsilon}$$

for all $m \neq 0$.

Start with the first inequality in the statement of the result, where up to constant, we may assume that $|\cdot| = \|\cdot\|_\infty$ in the definition of $d(nx, \mathbf{Z}^d)$. Let $m = (m_1, \ldots, m_d)$ be the lattice point achieving the minimum $|nx - m|$. Then we know that

$$d(nx, \mathbf{Z}^d) \geqslant C_0 \|(m_1, \ldots, m_d)\|_\infty^{-\omega_0(x)-\epsilon}.$$

Moreover, since $|nx - m| < 1$, there exists a constant C_0' such that

$$d(nx, \mathbf{Z}^d) \geqslant C_0' |n|^{-\omega_0(x)-\epsilon}.$$

It follows that

$$\frac{1}{d(nx, \mathbf{Z}^d)} \ll |n|^{\omega_0(x)+\epsilon},$$

the implied constant depending on x, ϵ, and the choice of norm $|\cdot|$.

Now let's consider the second inequality in the statement of the result. Note that $d(m_1 x_1 + \cdots + m_d x_d, \mathbf{Z}) = |m_0 + m_1 x_1 + \cdots + m_d x_d$ for some m_0 with $|m_0| \leqslant$

$\|(m_1,\dots,m_d)\|_2\|x\|_2+1$. Thus $\|(m_1,\dots,m_d)\|_\infty \ll \|x\|_2\|(m_1,\dots,m_d)\|_2$, which gives us

$$d(m_1x_1+\cdots+m_dx_d,\mathbf{Z}) \geqslant C_{d-1}\|(m_1,\dots,m_d)\|_2^{-\omega_{d-1}(x)-\epsilon}.$$

This implies

$$\frac{1}{d(\langle m,x\rangle,\mathbf{Z})} \ll |m|^{\omega_{r-1}(x)+\epsilon},$$

the implied constant depending on x, ϵ, and the choice of $|\cdot|$. □

4.2 Irrationality exponents and discrepancy

Let $x\in\mathbf{R}^d$ with $x_1,\dots,x_d$ linearly independent over $\mathbf{Q}$. We wish to control the discrepancy of the sequence $\{x,2x,3x,\dots\}$ in $(\mathbf{R}/\mathbf{Z})^d$.

Theorem 4.2.1 (Erdös–Turán–Koksma)**.** *Let $\boldsymbol{x}$ be a sequence in $\mathbf{R}^d$ and h an arbitrary integer. Then*

$$\mathrm{D}(\boldsymbol{x}^N) \ll \frac{1}{h} + \sum_{0\leqslant\|m\|_\infty\leqslant h} \frac{1}{r(m)}\left|\frac{1}{N}\sum_{n\leqslant N} e^{2\pi i\langle m,x_n\rangle}\right|,$$

where the first sum ranges over $m\in\mathbf{Z}^d$, $r(m)=\prod\max\{1,|m_i|\}$, and the implied constant depends only on d.

Proof. This is [DT97, Th. 1.21]. □

Lemma 4.2.2. *Let $x\in\mathbf{R}$. Then*

$$\left|\sum_{n\leqslant N} e^{2\pi i n x}\right| \ll \frac{1}{d(x,\mathbf{Z})}.$$

Proof. We begin with an easy bound:

$$\left|\sum_{n\leqslant N} e^{2\pi i n x}\right| = \frac{|e^{2\pi i(N+1)x}-1|}{|e^{2\pi i x}-1|} \leqslant \frac{2}{|e^{2\pi i x}-1|}.$$

Since $|e^{2\pi i m x}-1| = \sqrt{2-2\cos(2\pi x)}$ and $\cos(2\theta) = 1-2\sin^2\theta$, we obtain

$$\left|\sum_{n\leqslant N} e^{2\pi i n x}\right| \leqslant \frac{1}{|\sin(\pi x)|}.$$

It is easy to check that $|\sin(\pi x)| \geqslant d(x, \mathbf{Z})$, whence the result. □

Corollary 4.2.3. *Let $x \in (\mathbf{R}/\mathbf{Z})^d$ with $(x_1, \ldots, x_d)$ linearly independent over $\mathbf{Q}$. Then for $\boldsymbol{x} = (x, 2x, 3x, \ldots)$, we have*

$$\mathrm{D}(\boldsymbol{x}^N) \ll \frac{1}{h} + \frac{1}{N} \sum_{0<\|m\|_\infty \leqslant h} \frac{1}{r(m)d(\langle m, x\rangle, \mathbf{Z})}$$

for any integer h, with the implied constant depending only on d.

Proof. Apply the Erdös–Turán–Koksma inequality and bound the exponential sums using Lemma 4.2.2. □

Theorem 4.2.4. *Let $\boldsymbol{x} = (x, 2x, 3x, \ldots)$ in $(\mathbf{R}/\mathbf{Z})^d$. Then*

$$\mathrm{D}(\boldsymbol{x}^N) \ll N^{-\frac{1}{\omega_{d-1}(x)+1}+\epsilon}.$$

Proof. Choose $\delta > 0$ such that $\frac{1}{\omega_{d-1}(x)+1+\delta} = \frac{1}{\omega_{d-1}(x)+1} - \epsilon$.

By Corollary 4.2.3, we know that

$$\mathrm{D}(\boldsymbol{x}^N) \ll \frac{1}{h} + \frac{1}{N} \sum_{0<\|m\|_\infty \leqslant h} \frac{1}{r(m)d(\langle m, x\rangle, \mathbf{Z})},$$

and by Lemma 4.1.4, we know that $d(\langle m, x\rangle, \mathbf{Z})^{-1} \ll |m|^{\omega_{d-1}(x)+\delta}$. It follows that

$$\mathrm{D}(\boldsymbol{x}^N) \ll \frac{1}{h} + \frac{1}{N} \sum_{0<\|m\|_\infty \leqslant h} \frac{|m|^{\omega_{d-1}(x)+\delta}}{r(m)}.$$

The only tricky part is bounding the sum.

$$\begin{aligned}\sum_{0<\|m\|_\infty\leqslant h}\frac{|m|_\infty^{\omega_{d-1}(x)+\delta}}{r(m)} &\ll \int_1^h\int_1^{t_d}\cdots\int_1^{t_2}\frac{t_d^{\omega_{d-1}(x)+\delta}}{t_1\ldots t_d}\,\mathrm{d}t_1\ldots\mathrm{d}t_d\\ &\ll \int_1^h t^{\omega_{d-1}(x)+\delta-1}\,\mathrm{d}t\prod_{j=1}^{d-1}\int_1^h\frac{\mathrm{d}t}{t}\\ &\ll (\log h)^{d-1}h^{\omega_{d-1}(x)+\delta}.\end{aligned}$$

It follows that

$$\mathrm{D}(\boldsymbol{x}^N)\ll\frac{1}{h}+\frac{1}{N}(\log h)^{d-1}h^{\omega_{d-1}(x)+\delta}.$$

Setting $h\approx N^{\frac{1}{1+\omega_{d-1}(x)+\delta}}$, we see that

$$D(\boldsymbol{x}^N)\ll N^{-\frac{1}{\omega_{d-1}(x)+1+\delta}}=N^{-\frac{1}{\omega_{d-1}(x)+1}+\epsilon}.$$

For a slightly different proof of a similar result (given as a sequence of exercises), see [KN74, Ch. 2, Ex. 3.15, 16, 17]. □

Theorem 4.2.5. *Let $x\in\mathbf{R}$ be such that $x_1,\ldots,x_d$ are linearly independent over $\mathbf{Q}$, and let $\boldsymbol{x}=(x,2x,3x,\ldots)$ in $(\mathbf{R}/\mathbf{Z})^d$. Then*

$$\mathrm{D}(\boldsymbol{x}^N)=\Omega\left(N^{-\frac{d}{\omega_0(x)}-\epsilon}\right).$$

Proof. Here $f=\Omega(g)$ in the sense of Hardy, namely that $\limsup\frac{f}{g}>0$. We follow the proof of [KN74, Ch. 2, Th. 3.3]. Given $\epsilon>0$, there exists $\delta>0$ such that $\frac{d}{\omega_0(x)-\delta}=\frac{d}{\omega_0(x)}+\epsilon$.

By the definition of $\omega_0(x)$, there exist infinitely many $(q,m_1,\ldots,m_d)$ with $q>0$ such that

$$\|qx-m\|_\infty\leqslant\|(q,m_1,\ldots,m_d)\|_\infty^{-\omega_0(x)+\delta/2}.$$

Since $\|(q, m_1, \ldots, m_d)\|_\infty \geqslant q$, we derive the stronger statement that for infinitely many $q \to \infty$, there exists $m = (m_1, \ldots, m_d) \in \mathbf{Z}^d$ such that $\|qx - m\|_\infty \leqslant q^{-\omega_0(x)+\delta/2}$ or, equivalently, $|x - \frac{m}{q}| \leqslant q^{-1-\omega_0(x)+\delta/2}$. Pick such a q, and let $N = \lfloor q^{\omega_0(x)-\delta} \rfloor$. Then for each $n \leqslant N$, we have $\|nx - \frac{n}{q}m\|_\infty \leqslant q^{-1-\delta/2}$. Thus, for each $n \leqslant N$, each nx is within $q^{-1-\delta/2}$ of the grid $\frac{1}{q}\mathbf{Z}^d \subset (\mathbf{R}/\mathbf{Z})^d$. Thus, they miss a box with side lengths $q^{-1} - 2q^{-1-\delta/2}$. For q sufficiently large, $q^{-1} - 2q^{-1-\delta/2} \geqslant 1/2q$, so the discrepancy of $\boldsymbol{x}^N$ is bounded below by $2^{-d}q^{-d}$. Since $q^{\omega_0(x)-\delta} \leqslant 2N$, the discrepancy at N is bounded below by

$$2^{-d}\left((2N)^{-\frac{1}{\omega_0(x)+\delta}}\right)^{-d} = 2^{-d-\frac{d}{\omega_0(x)+\delta}} N^{-\frac{d}{\omega_0(x)+\delta}} = 2^{-d\left(1+\frac{1}{\omega_0(x)}\right)-\epsilon} N^{-\frac{d}{\omega_0(x)}-\epsilon}.$$

□

CHAPTER 5

DEFORMATION THEORY

5.1 Category of test objects

This section summarizes the theory in [SGA 3_1, $\mathrm{VII}_{\mathrm{B}}$, §0–1], adapting it to the deformation theory of Galois representations. All rings are commutative with unit.

Definition 5.1.1. *Let Λ be a ring. A topological Λ-module M is* pseudocompact *if it is a filtered inverse limit of discrete finite-length Λ-modules. The ring Λ is pseudocompact if it is pseudocompact as a module over itself.*

Let Λ be a topological ring. Given a pseudocompact Λ-algebra A, write C_Λ for the opposite of the category of Λ-algebras which have finite length as Λ-modules. Given such a Λ-algebra A, write $X = \mathrm{Spf}(A)$ for the corresponding object of C_Λ, and we put $A = \mathscr{O}(X)$.

Lemma 5.1.2. *Let Λ be a pseudocompact ring, C_Λ as above. Then C_Λ is closed under finite limits and colimits.*

Proof. That C_Λ is closed under finite colimits follows from the fact that finite-length Λ-algebras are closed under finite limits (the underlying modules are closed under finite limits). Moreover, since the tensor product of finite length modules also has finite length, and quotients of length modules have finite length, C_Λ is closed under finite limits. □

Lemma 5.1.3. *Let Λ be a pseudocompact local ring. Then Λ is henselian, in any of the following senses:*

1. *Every finite Λ-algebra is a product of local Λ-algebras.*

2. *The first condition is satisfied for Λ-algebras of the form $\Lambda[t]/f$, where f is monic.*

3. *Let $\mathfrak{m}$ be the maximal ideal of Λ. Then $A \mapsto A/\mathfrak{m}$ is an equivalence of categories from finite étale Λ-algebras to finite étale $\Lambda/\mathfrak{m}$-algebras.*

Proof. The conditions are equivalent by [EGA 4_4, 18.5.11]. Recall that $\Lambda = \varprojlim \Lambda/\mathfrak{a}$, where $\mathfrak{a}$ ranges over closed ideals of finite index. Let A be a pseudo-compact Λ-algebra. For any ideal $\mathfrak{a} \subset \Lambda$, the ring $\Lambda/\mathfrak{a}$ is henselian by [EGA 4_4, 18.5.14], so $A/\mathfrak{a}$ is a product of local $\Lambda/\mathfrak{a}$-algebras. Moreover, by [EGA 4_4, 18.5.4], the map $A/\mathfrak{a} \to A/\mathfrak{m}$ is a bijection on idempotents. The inverse limit of these compatible systems of idempotents decompose A into a product of local Λ-algebras. □

Following Grothendieck, if $\mathcal{C}$ is an arbitrary category, we write $\widehat{\mathcal{C}} = \hom(\mathcal{C}^\circ, \mathsf{Set})$ for the category of contravariant functors $\mathcal{C} \to \mathsf{Set}$. We regard $\mathcal{C}$ as a full subcategory of $\widehat{\mathcal{C}}$ via the Yoneda embedding, so for $X, Y \in \mathcal{C}$, we write $X(Y) = \hom_{\mathcal{C}}(Y, X)$. With this notation, the Yoneda Lemma states that $\hom_{\widehat{\mathcal{C}}}(X, P) = P(X)$ for all $X \in \mathcal{C}$.

Lemma 5.1.4. *Let $\mathcal{X} \in \widehat{\mathsf{C}_\Lambda}$. Then $\mathcal{X}$ is left exact if and only if there exists a filtered system $\{X_i\}_{i \in I}$ in $\mathcal{C}_\Lambda$ together with a natural isomorphism $\mathcal{X}(\cdot) \simeq \varinjlim X_i(\cdot)$. Write $\mathsf{Ind}(\mathsf{C}_\Lambda)$ for the category of such functors. Then $\mathsf{Ind}(\mathsf{C}_\Lambda)$ is closed under colimits, and the Yoneda embedding $\mathsf{C}_\Lambda \hookrightarrow \mathsf{Ind}(\mathsf{C}_\Lambda)$ preserves filtered colimits.*

Proof. This follows from the results of [KS06, 6.1]. □

Lemma 5.1.5. *The functors* $\mathsf{C}_\Lambda \to \mathsf{Ind}(\mathsf{C}_\Lambda) \to \widehat{\mathsf{C}_\Lambda}$ *are left exact.*

Proof. This is [KS06, 6.1.17]. □

If R is a pseudocompact Λ-algebra, write $\mathrm{Spf}(R)$ for the object of $\widehat{\mathsf{C}_\Lambda}$ defined by $\mathrm{Spf}(R)(A) = \hom_{\mathrm{cts}/\Lambda}(R, A)$, the set of continuous Λ-algebra homomorphisms.

Lemma 5.1.6. *The funtor* Spf *induces an (anti-)equivalence between the category of pseudocompact* Λ*-algebras and* $\mathsf{Ind}(\mathsf{C}_\Lambda)$*.*

Proof. This is [SGA 3_1, $\mathrm{VII_B}$ 0.4.2 Prop.]. □

So $\mathsf{Ind}(\mathsf{C}_\Lambda)$ is the category of pro-representable functors on finite length Λ-algebras. *Warning*: in many papers, for example the foundational [Maz97], one reserves the term *pro-representable* for functors of the form $\mathrm{Spf}(R)$, where R is required to be noetherian. We do not make this restriction.

Lemma 5.1.7. *The category* $\mathsf{Ind}(\mathsf{C}_\Lambda)$ *is an exponential ideal in* $\widehat{\mathsf{C}_\Lambda}$*.*

Proof. By this we mean the following. Let $\mathcal{X} \in \mathsf{Ind}(\mathsf{C}_\Lambda)$, $P \in \widehat{\mathsf{C}_\Lambda}$. Then the functor $\mathcal{X}^P$ defined by

$$\mathcal{X}^P(S) = \hom_{\widehat{\mathsf{C}_\Lambda}_{/S}}(P_{/S}, \mathcal{X}_{/S})$$

is also in $\mathsf{Ind}(\mathsf{C}_\Lambda)$. Given the characterization of $\mathsf{Ind}(\mathsf{C}_\Lambda)$ as left exact functors, this is easy to prove, see e.g. [Joh02, 4.2.3]. □

If $\mathcal{C}$ is a category, we write $\mathsf{Gp}(\mathcal{C})$ for the category of group objects in $\mathcal{C}$.

Corollary 5.1.8. *Let* $\Gamma \in \mathsf{Gp}(\widehat{\mathsf{C}_\Lambda})$ *and* $\mathcal{G} \in \mathsf{Gp}(\mathsf{Ind}(\mathsf{C}_\Lambda))$*, then the functor* $[\Gamma, \mathcal{G}]$ *defined by*

$$[\Gamma, \mathcal{G}](S) = \hom_{\mathsf{Gp}/S}(\Gamma_{/S}, \mathcal{G}_{/S})$$

is in $\mathsf{Ind}(\mathsf{C}_\Lambda)$*. In particular, if* Γ *is a profinite group, then the functor*

$$[\Gamma, \mathcal{G}](S) = \hom_{\mathrm{cts}/\,\mathsf{Gp}}(\Gamma, \mathcal{G}(S))$$

is in $\mathsf{Ind}(\mathsf{C}_\Lambda)$*.*

Proof. The first claim follows easily from Lemma 5.1.7 and Lemma 5.1.5. Just note that $[\Gamma, \mathcal{G}]$ is the equalizer:

$$\begin{tikzcd}[\Gamma, \mathcal{G}] \arrow[r] & \mathcal{G}^{\Gamma} \arrow[r, shift left, "m_\Gamma^*"] \arrow[r, shift right, "m_{\mathcal{G}*}"'] & \mathcal{G}^{\Gamma\times\Gamma},\end{tikzcd}$$

that is, those $f\colon \Gamma \to \mathcal{G}$ such that $f \circ m_\Gamma = m_{\mathcal{G}} \circ (f \times f)$. The latter claim is just a special case. □

5.2 Quotients in the flat topology

If Λ is a pseudocompact ring, the category $\mathsf{Ind}(\mathsf{C}_\Lambda)$ has nice "geometric" properties. However, for operations like taking quotients, we will embed it into the larger category $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$ of flat sheaves. We call a collection $\{U_i \to X\}$ of morphisms in C_Λ a *flat cover* if each ring map $\mathscr{O}(X) \to \mathscr{O}(U_i)$ is flat, and moreover $\mathscr{O}(X) \to \prod \mathscr{O}(U_i)$ is faithfully flat. By [SGA 3_1, IV 6.3.1], this is a subcanonical Grothendieck topology on C_Λ. We call it the *flat topology*, even though finite presentation comes for free because all the rings are finite length.

Lemma 5.2.1. *Let $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$ be the category of sheaves (of sets) on C_Λ with respect to the flat topology. Then a presheaf $P \in \widehat{\mathsf{C}_\Lambda}$ lies in $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$ if and only if $P(\coprod U_i) = \prod P(U_i)$ and moreover, whenever $U \to X$ is a flat cover where $\mathscr{O}(U)$ and $\mathscr{O}(X)$ are local rings, the sequence*

$$P(X) \longrightarrow P(U) \rightrightarrows P(U \times_X U).$$

is exact. Moreover, $\mathsf{Ind}(\mathsf{C}_\Lambda) \subset \mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$.

Proof. The first claim is the content of [SGA 3_1, IV 6.3.1(ii)]. For the second, note that any $\mathcal{X} \in \mathsf{Ind}(\mathsf{C}_\Lambda)$ will, by 5.1.4, convert (arbitrary) colimits into limits. Thus $\mathcal{X}(\coprod U_i) = \prod \mathcal{X}(U_i)$. If $U \to X$ is a flat cover, then by (loc. cit.), $U \times_X U \rightrightarrows U \to X$ is a coequalizer diagram in C_Λ, hence $\mathcal{X}(X) \to \mathcal{X}(U) \rightrightarrows \mathcal{X}(U \times_X U)$ is an equalizer. □

Our main reason for introducing the category $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$ is that, as a (Grothendieck) topos, it is closed under arbitrary colimits. Recall that in an *equivalence relation* in $\widehat{\mathsf{C}_\Lambda}$ is a morphism $R \to X \times X$ such that, for all S, the map $R(S) \to X(S) \times X(S)$ is an injection whose image is an equivalence relation on $X(S)$. We define the quotient X/R to be the coequalizer

$$R \rightrightarrows X \longrightarrow X/R.$$

By Giraud's Theorem [MLM94, App.], for any $S \in \mathsf{C}_\Lambda$, the natural map $X(S)/R(S) \to (X/R)(S)$ is injective. It will not be surjective in general.

We let $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$ inherit definitions from C_Λ as follows. If P is a property of maps in C_Λ (for example, "flat," or "smooth,") and $f\colon X \to Y$ is a morphism in $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$, we say that f has P if for all $S \in \mathsf{C}_\Lambda$ and $y \in Y(S)$, the pullback $X_S = X \times_Y S$ lies in C_Λ, and the pullback map $X_S \to S$ has property P. For example, if $X = \mathrm{Spf}(R')$ and $Y = \mathrm{Spf}(R)$, then $X \to Y$ has property P if and only if for all finite length A and continuous Λ-algebra maps $R \to A$, the induced map $A \to R' \otimes_R A$ has P.

Theorem 5.2.2. *Let $\mathcal{R} \to \mathcal{X} \times \mathcal{X}$ be an equivalence relation in $\mathsf{Ind}(\mathsf{C}_\Lambda)$ such that one of the maps $\mathcal{R} \to \mathcal{X}$ is flat. Then the quotient $\mathcal{X}/\mathcal{R}$ lies in $\mathsf{Ind}(\mathsf{C}_\Lambda)$, and $\mathcal{X} \to \mathcal{X}/\mathcal{R}$ is a flat cover.*

Proof. This is [SGA 3_1, $\mathrm{VII_B}$ 1.4]. □

By [Mat89, 29.7], if k is a field and R is a complete regular local k-algebra, then $R \simeq k[\![t_1, \ldots, t_n]\!]$. In particular, R admits an augmentation $\epsilon\colon R \to k$. There is a general analogue of this result, but first we need a definition.

Definition 5.2.3. *A map $f\colon \mathcal{X} \to \mathcal{Y}$ in $\mathsf{Ind}(\mathsf{C}_\Lambda)$ is a* residual isomorphism *if for all $S = \mathrm{Spf}(k) \in \mathsf{C}_\Lambda$ where k is a field, the map $f\colon \mathcal{X}(S) \to \mathcal{Y}(S)$ is a bijection.*

Lemma 5.2.4. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a smooth map in $\mathsf{Ind}(\mathsf{C}_\Lambda)$ that is a residual isomorphism. Then f admits a section.*

Proof. By [SGA 3_1, $\mathrm{VII_B}$ 0.1.1], it suffices to prove the result when $\mathcal{X} = \mathrm{Spf}(R')$, $\mathcal{Y} = \mathrm{Spf}(R)$, for local Λ-algebras $R \to R'$ with the same residue field. Let $k = R/\mathfrak{m}_R \xrightarrow{\sim} R'/\mathfrak{m}_{R'}$ be their common residue field. From the diagram

$$\begin{tikzcd}
R' \arrow[r, dotted] \arrow[dr, two heads] & R \arrow[d, two heads] \\
R \arrow[u] \arrow[r, two heads] \arrow[ur, equal] & k,
\end{tikzcd}$$

the definition of (formal) smoothness, and a limiting argument involving the finite length quotients $R/\mathfrak{a}$, we obtain the result. $\square$

Corollary 5.2.5. *Let $\mathcal{R} \to \mathcal{X} \times \mathcal{X}$ be an equivalence relation satisfying the hypotheses of Theorem 5.2.2. Suppose further that*

1. *One of the maps $\mathcal{R} \to \mathcal{X}$ is smooth, and*

2. *The projection $\mathcal{X} \to \mathcal{X}/\mathcal{R}$ is a residual isomorphism.*

Then $\mathcal{X} \to \mathcal{X}/\mathcal{R}$ admits a section, so $\mathcal{X}(S)/\mathcal{R}(S) \xrightarrow{\sim} (\mathcal{X}/\mathcal{R})(S)$ for all $S \in \mathsf{C}_\Lambda$.

Proof. By 5.2.4, it suffices to prove that $\mathcal{X} \to \mathcal{X}/\mathcal{R}$ is smooth. By [EGA 4_4, 17.7.3(ii)], smoothness can be detected after flat descent. So base-change with respect to the projection $\mathcal{X} \to \mathcal{X}/\mathcal{R}$. In the following commutative diagram

$$\begin{tikzcd}
\mathcal{R} \arrow[dr, equal] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathcal{X} \times_{\mathcal{X}/\mathcal{R}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
& \mathcal{X} \arrow[r] & \mathcal{X}/\mathcal{R}
\end{tikzcd}$$

we can ensure the smoothness of $\mathcal{R} \to \mathcal{X}$ by our hypotheses. Since $\mathcal{X} \to \mathcal{X}/\mathcal{R}$ is smooth after flat base-change, the original map is smooth. $\square$

Example 5.2.6. The hypothesis on residue fields in 5.2.5 is necessary. To see this, let $\Lambda = k$ be a field, $k \hookrightarrow K$ a finite Galois extension with Galois group G. Then $G \times \mathrm{Spf}(K) \rightrightarrows \mathrm{Spf}(K)$ has quotient $\mathrm{Spf}(k)$, but the map $\mathrm{Spf}(K)(S) \to \mathrm{Spf}(k)(S)$ is *not* surjective for all $S \in \mathsf{C}_k$, e.g. it is not for $S = \mathrm{Spf}(k)$.

Example 5.2.7. The hypothesis of smoothness in Theorem 5.2.5 is necessary. To see this, let k be a field of characteristic $p > 0$. Then the formal additive group

$\widehat{\mathbf{G}_{\mathrm{a}}} = \operatorname{Spf}(k[\![t]\!])$ has a subgroup $\boldsymbol{\alpha}_p$ defined by

$$\boldsymbol{\alpha}_p(S) = \{s \in \mathscr{O}(S) \colon s^p = 0\}.$$

The quotient $\widehat{\mathbf{G}_{\mathrm{a}}}/\boldsymbol{\alpha}_p$ has as affine coordinate ring $k[\![t^p]\!]$. In particular, the following sequence is exact in the flat topology:

$$0 \longrightarrow \boldsymbol{\alpha}_p \longrightarrow \widehat{\mathbf{G}_{\mathrm{a}}} \xrightarrow{(\cdot)^p} \widehat{\mathbf{G}_{\mathrm{a}}} \longrightarrow 0.$$

It follows that $\boldsymbol{\alpha}_p \times \widehat{\mathbf{G}_{\mathrm{a}}} \rightrightarrows \widehat{\mathbf{G}_{\mathrm{a}}} \xrightarrow{(\cdot)^p} \widehat{\mathbf{G}_{\mathrm{a}}}$ is a coequalizer in $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_k)$ satisfying all the hypothese of 5.2.5 except smoothness. And indeed, as one sees by letting $S = \operatorname{Spf}(A)$ for any non-perfect k-algebra A, the map $(\cdot)^p \colon \widehat{\mathbf{G}_{\mathrm{a}}}(S) \to \widehat{\mathbf{G}_{\mathrm{a}}}(S)$ is *not* surjective for all S.

5.3 Deformations of group representations

Here we elaborate on (and correct some mistakes in) the arguments in [BÏ3, §2.1].

Let $\Gamma \in \mathsf{Gp}(\widehat{\mathsf{C}_\Lambda})$ and $G_{/\Lambda}$ be a smooth group scheme of finite type. Write $\widehat{G}$ for the group object in $\mathsf{Ind}(\mathsf{C}_\Lambda)$ given by $\widehat{G}(\operatorname{Spf} A) = G(\operatorname{Spec} A)$. By 5.1.8, the functor

$$\operatorname{Rep}^\square(\Gamma, \widehat{G})(S) = \hom_{\mathsf{Gp}/S}(\Gamma_S, \widehat{G}_S) = \hom_{\mathsf{Gp}}(\Gamma(S), G(S))$$

is in $\mathsf{Ind}(\mathsf{C}_\Lambda)$. We would like to define an ind-scheme $\operatorname{Rep}(\Gamma, \mathcal{G})$ as "$\operatorname{Rep}^\square(\Gamma, \mathcal{G})$ modulo conjugation," but this requires some care. The conjugation action of $\mathcal{G}$ on $\operatorname{Rep}^\square(\Gamma, \mathcal{G})$ will have fixed points, so the quotient will be badly behaved. We loosely follow [Til96, Ch. 2–3].

Assume Λ is local, with maximal ideal $\mathfrak{m}$ and residue field k. Fix $\bar\rho \in \operatorname{Rep}^\square(\Gamma, \widehat{G})(k)$, i.e. a residual representation $\bar\rho \colon \Gamma \to G(k)$. Let $\operatorname{Rep}^\square(\Gamma, \widehat{G})_{\bar\rho}$ be the connected component of $\bar\rho$ in $\operatorname{Rep}^\square(\Gamma, \widehat{G})$. Assume that G and $\mathrm{Z}(G)$ are smooth; then the quotient $\widehat{G}^{\circ,\mathrm{ad}} = \widehat{G}^\circ / \mathrm{Z}(\widehat{G}^\circ)$ is also smooth, where $(-)^\circ$ denotes "connected component of identity." Since $\mathrm{Z}(\widehat{G}^\circ)$ is also smooth, the quotient sheaf $\widehat{G}^{\circ,\mathrm{ad}}$ is the same as the quotient presheaf.

Theorem 5.3.1. *Suppose $(\Lambda, \mathfrak{m}, k)$ is local. If $\mathcal{X}, \mathcal{Y} \in \mathsf{Ind}(\mathsf{C}_\Lambda)$ are connected and $\mathcal{X}(k) \neq \varnothing$, then $\mathcal{X} \times_\Lambda \mathcal{Y}$ is connected.*

Proof. We are reduced to proving the following result from commutative algebra: if R, S are local pro-artinian Λ-algebras and R has residue field $\mathbf{k}$, then $R \widehat{\otimes}_\Lambda S$ is local. Since $R \widehat{\otimes}_\Lambda S = \varprojlim (R/\mathfrak{r}) \otimes_\Lambda (S/\mathfrak{s})$, $\mathfrak{r}$ (resp. $\mathfrak{s}$) ranges over all open ideals in R (resp. S), we may assume that both R and S are artinian. The rings R and S are henselian, so $R \otimes S$ is local if and only if $(R/\mathfrak{m}_R) \otimes (S/\mathfrak{m}_S) = S/\mathfrak{m}_S$ is local, which it is. □

We conclude that the action of $\widehat{G}^{\circ,\mathrm{ad}}$ on $\operatorname{Rep}^\square(\Gamma, \widehat{G})$ preserves $\operatorname{Rep}^\square(\Gamma, \mathcal{G})_{\bar\rho}$. Thus we may put

$$\operatorname{Rep}(\Gamma, G)_{\bar\rho} = \operatorname{Rep}^\square(\Gamma, \widehat{G})_{\bar\rho} / \widehat{G}^{\circ,\mathrm{ad}}.$$

Obviously, that quotient will only be well-behaved if $\widehat{G}^{\circ,\mathrm{ad}}$ acts faithfully on $\operatorname{Rep}^\square(\Gamma, \widehat{G})$. Whether or not it does is governed by the action of Γ on $\mathfrak{g}(k)$. Let $\mathfrak{z} = \operatorname{Lie}(\mathrm{Z}(G))$.

Theorem 5.3.2. *Let $\Gamma, G, \bar\rho$ be as above. Assume $\mathrm{H}^0(\Gamma, \mathfrak{g}(k)) = \mathfrak{z}(k)$. Then $\operatorname{Rep}(\Gamma, G)_{\bar\rho}$ exists and, for A a local Λ-algebra, consists of $\widehat{G}^\circ(A)$-conjugacy classes of maps $\Gamma \to G(A)$ that reduce to $\bar\rho$ modulo $\mathfrak{m}_A$.*

Proof. We begin by proving that if $\mathrm{H}^0(\Gamma, \mathfrak{g}(k)) = \mathfrak{z}(k)$, then $\widehat{G}^{\circ,\mathrm{ad}}$ acts faithfully on $\mathrm{Rep}^{\square}(\Gamma, \widehat{G})$. First, note that if the given $\mathrm{H}^0 = \mathfrak{z}$, then $\mathrm{H}^0(\Gamma, \mathfrak{g}^{\mathrm{ad}}(k)) = 0$, where $\mathfrak{g}^{\mathrm{ad}} = \mathrm{Lie}(\widehat{G}^{\circ,\mathrm{ad}}) = \mathfrak{g}/\mathfrak{z}$. The action of $\widehat{G}^{\circ,\mathrm{ad}}$ is faithful if and only if, whenever A is a local Artinian Λ-algebra and $\rho\colon \Gamma \to G(A)$ agrees with $\bar{\rho}$ modulo $\mathfrak{m}_A$, and for $g \in G(A)$ with $g \equiv 1$ modulo $\mathfrak{m}_A$, then if $g\rho g^{-1} = \rho$, we have $g = 1$.

Let $I \subset A$ be a square-zero ideal that is one-dimensional over k. By induction, we may assume that in fact $g \equiv 1$ modulo I. There is the exponential short exact sequence

$$0 \longrightarrow \mathfrak{g}^{\mathrm{ad}}(I) \xrightarrow{\exp} \widehat{G}^{\circ,\mathrm{ad}}(A) \longrightarrow \widehat{G}^{\circ,\mathrm{ad}}(A/I) \longrightarrow 1$$

Since $g \equiv 1$ modulo I, we can write $g = \exp(X)$ for some $X \in \mathfrak{g}^{\mathrm{ad}}(I) \simeq \mathfrak{g}^{\mathrm{ad}}(k)$. Our assumption $g\rho g^{-1}$ tells us that for all $\gamma \in \Gamma$, $\exp(X)\rho(\gamma)\exp(-X) = \rho(\gamma)$, i.e. $[\exp(X), \rho(\gamma)] = 0$. From this, it follows that X is fixed by all $\mathrm{Ad}\,\rho(\gamma)$ acting on $\mathfrak{g}^{\mathrm{ad}}(I)$, so $X = 0$, i.e. $g = 1$.

Since the action of $\widehat{G}^{\circ,\mathrm{ad}}$ on $\mathrm{Rep}^{\square}(\Gamma, \widehat{G})_{\bar{\rho}}$ is faithful, we can apply Theorem 5.2.5 to see that $\mathrm{Rep}(\Gamma, G)_{\bar{\rho}}$ is the presheaf quotient, which is exactly what is described in the statement of the theorem. □

5.4 Tangent spaces and obstruction theory

A routine argument shows that if $I \subset A$ is a square-zero ideal, and $\rho\colon \Gamma \to G(A/I)$ admits some lift to $G(A)$, then the set of such lifts form a $\mathrm{H}^1(\Gamma, \mathfrak{g}(I))$-torsor. We show that obstruction theory also works in this more general context.

For $S_0 \in \mathsf{C}_\Lambda$, let Ex_{S_0} be the category of square-zero thickenings of S_0. An object of Ex_{S_0} is a closed embedding $S_0 \hookrightarrow S$ whose ideal of definition has square zero. For any such object, there is an "exponential exact sequence"

$$0 \longrightarrow \mathfrak{g}(I) \longrightarrow G(S) \longrightarrow G(S_0) \longrightarrow 1$$

This gives us a class $\exp \in \mathrm{H}^2(G(S_0), \mathfrak{g}(I))$. For $\rho_0 \colon \Gamma \to G(S_0)$, the obstruction class is $o(\rho_0, I) = \rho_0^*(\exp) \in \mathrm{H}^2(\Gamma, \mathfrak{g}(I))$. It's easy to check that $o(\rho_0, I) = 0$ if and only if ρ_0 lifts to ρ. More formally, we use [Wei94, 6.6.4]. Given setting as above, $\rho_0^*(\exp)$ is the pullback by ρ_0:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{g}(I) \arrow[r] \arrow[d, equal] & \mathcal{G}(S) \times_{\mathcal{G}(S_0)} \Gamma \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d, "\rho_0"] & 1 \\
0 \arrow[r] & \mathfrak{g}(I) \arrow[r] & \mathcal{G}(S) \arrow[r] & \mathcal{G}(S_0) \arrow[r] & 1
\end{tikzcd}$$

Computing explicitly, we see the result.

Proposition 5.4.1. *Let $f \colon G \to H$ be a morphism of profinite groups. Suppose M is a discrete H-module and $c \in \mathrm{H}^2(H, M)$ corresponds to the extension*

$$0 \longrightarrow M \longrightarrow \widetilde{H} \longrightarrow H \longrightarrow 1.$$

Then $f^*c = 0$ in $\mathrm{H}^2(G, M)$ if and only if there is a map $\widetilde{f} \colon G \to \widetilde{H}$ making the following diagram commute:

$$\begin{tikzcd}
 & \widetilde{H} \arrow[dd] \\
G \arrow[ur, "\widetilde{f}"] \arrow[dr, "f"'] & \\
 & H.
\end{tikzcd}$$

Proof. By [Wei94, 6.6.4], the class f^*c corresponds to the pullback diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & G \times_H \widetilde{H} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "f"] & 1 \\
0 \arrow[r] & M \arrow[r] & \widetilde{H} \arrow[r] & H \arrow[r] & 1.
\end{tikzcd}$$

Writing explicitly what it means for $G \times_H \widetilde{H} \to G$ to split yields the result. □

Corollary 5.4.2. *Let $\rho \in \mathrm{Rep}(\Gamma, G)_{\bar{\rho}}(A/I)$, where I is a square-zero ideal. Then ρ lifts to A if and only if $\rho^*(\exp_I) = 0$ in the group $\mathrm{H}^2(\Gamma, \mathfrak{g}(I))$.*

CHAPTER 6

CONSTRUCTING GALOIS REPRESENTATIONS

6.1 Notation and necessary results

In this chapter we loosely summarize, and adapt as needed, the results of [KLR05; Pan11]. Throughout, if F is a field, M a G_F-module, we write $\mathrm{H}^i(F, M)$ in place of $\mathrm{H}^1(G_F, M)$. All Galois representations will be to $\mathrm{GL}_2(\mathbf{Z}/l^n)$ or $\mathrm{GL}_2(\mathbf{Z}_l)$ for l a (fixed) rational prime, and all deformations will have fixed determinant, so we only consider the cohomology of $\mathrm{Ad}^0\,\bar{\rho}$, the induced representation on trace-zero matrices by conjugation.

If S is a set of rational primes, $\mathbf{Q}_S$ denotes the largest extension of $\mathbf{Q}$ unramified outside S. So $\mathrm{H}^i(\mathbf{Q}_S, -)$ is what is usually written as $\mathrm{H}^1(G_{\mathbf{Q},S}, -)$. If M is a $G_{\mathbf{Q}}$-module and S a finite set of primes, write

$$Ш^i_S(M) = \ker\left(\mathrm{H}^i(\mathbf{Q}_S, M) \to \prod_{p\in S} \mathrm{H}^i(\mathbf{Q}_p, M)\right).$$

If l is a rational prime and S a finite set of primes containing l, then for any $\mathbf{F}_l[G_{\mathbf{Q}_S}]$-module M, write $M^\vee = \hom_{\mathbf{F}_l}(M, \mathbf{F}_l)$ with the obvious $G_{\mathbf{Q}_S}$-action, and write $M^* = M^\vee(1)$ for the Cartier dual. By [NSW08, Th. 8.6.7], there is an isomorphism $Ш^1_S(M^*) = Ш^2_S(M)^\vee$.

Definition 6.1.1. *A* good residual representation *is an odd, absolutely irreducible, weight-*2 *representation* $\bar{\rho}\colon G_{\mathbf{Q}_S} \to \mathrm{GL}_2(\mathbf{F}_l)$*, where* $l \geqslant 7$ *is a rational prime.*

Roughly, “good residual representations” have enough properties that we can prove quite a lot about their lifts. By results of Khare–Wintenberger, we know that good residual representations have characteristic-zero lifts. Even better, they admit $\mathbf{Z}_l$-lifts.

Theorem 6.1.2. *Let $\bar{\rho}\colon G_{\mathbf{Q}_S} \to \mathrm{GL}_2(\mathbf{F}_l)$ be a good residual representation. Then there exists a weight-2 lift of $\bar{\rho}$ to $\mathbf{Z}_l$.*

Proof. This is [Ram02, Th. 1], taking into account that the paper in question allows for arbitrary fixed determinants. □

Definition 6.1.3. *Let $\bar{\rho}\colon G_{\mathbf{Q}_S} \to \mathrm{GL}_2(\mathbf{F}_l)$ be a good residual representation. A prime $p \not\equiv \pm 1 \pmod{l}$ is* nice *if $\mathrm{Ad}^0\,\bar{\rho} \simeq \mathbf{F}_l \oplus \mathbf{F}_l(1) \oplus \mathbf{F}_l(-1)$, i.e. if the eigenvalues of $\bar{\rho}(\mathrm{fr}_p)$ have ratio p.*

Theorem 6.1.4. *Let $\bar{\rho}$ be a good residual representation and p a nice prime. Then any deformation of $\bar{\rho}|_{G_{\mathbf{Q}_p}}$ is induced by $G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\mathbf{Z}_l[\![a,b]\!]/\langle ab\rangle)$, sending*

$$\mathrm{fr}_p \mapsto \begin{pmatrix} p(1+a) & \\ & (1+a)^{-1} \end{pmatrix} \qquad \tau_p \mapsto \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix},$$

where $\tau_p \in G_{\mathbf{Q}_p}$ is a generator for tame inertia.

Proof. This is mentioned in KLR, find the real proof. □

We close this section by introducing some new terminology and notation to condense the lifting process used in [KLR05].

Fix a good residual representation $\bar{\rho}$. We will consider weight-2 deformations of $\bar{\rho}$ to $\mathbf{Z}/l^n$ and $\mathbf{Z}_l$. Call such a deformation a "lift of $\bar{\rho}$ to $\mathbf{Z}/l^n$ (resp. $\mathbf{Z}_l$)." We will often restrict the local behavior of such lifts, i.e. the restrictions of a lift to $G_{\mathbf{Q}_p}$ for p in some set of primes. The necessary constraints are captured in the following definition.

Definition 6.1.5. *Let $\bar{\rho}$ be a good residual representation, $h\colon \mathbf{R}^+ \to \mathbf{R}^+$ a function decreasing to zero. An h-*bounded lifting datum *is a tuple $(\rho_n, R, U, \{\rho_p\}_{p \in R \cup U})$, where*

1. *$\rho_n\colon G_{\mathbf{Q}_R} \to \mathrm{GL}_2(\mathbf{Z}/l^n)$ is a lift of $\bar{\rho}$.*
2. *R and U are finite sets of primes, R containing l and all primes at which ρ_n ramifies.*
3. *$\pi_R(x) \leqslant h(x)\pi(x)$ for all x.*
4. *$\text{Ш}^1_R(\mathrm{Ad}^0\,\bar{\rho}) = \text{Ш}^2_R(\mathrm{Ad}^0\,\bar{\rho}) = 0$.*
5. *For all $p \in R \cup U$, $\rho_p \equiv \rho_n|_{G_{\mathbf{Q}_p}} \pmod{l^n}$.*
6. *For all $p \in R$, ρ_p is ramified.*
7. *ρ_n admits a lift to $\mathbf{Z}/l^{n+1}$.*

If $(\rho_n, R, U, \{\rho_p\})$ is an h-bounded lifting datum, we call another h-bounded lifting datum $(\rho_{n+1}, R', U', \{\rho_p\})$ a *lift of* $(\rho_n, R, U, \{\rho_p\})$ if $U \subset U'$, $R \subset R'$, and for all $p \in R \cup U$, the two possible "ρ_p" agree.

Theorem 6.1.6. *Let $\bar{\rho}$ be a good residual representation, $h\colon \mathbf{R}^+ \to \mathbf{R}^+$ decreasing to zero. If $(\rho_n, R, U, \{\rho_p\})$ is an h-bounded lifting datum, $U' \supset U$ is a finite set of primes disjoint from R, and $\{\rho_p\}_{p \in U'}$ extends $\{\rho_p\}_{p \in U}$, then there exists an h-bounded lift $(\rho_{n+1}, R', U', \{\rho_p\})$ of $(\rho_n, R, U, \{\rho_p\})$.*

Proof. Note that we do not bound the size of $R' \smallsetminus R$. It is possible that this can be done, using unpublished results of Ramakrishna, but that is not necessary for the results that follow.

By [KLR05, Lem. 8], there exists a finite set N of what they call *nice primes*, such that the map

$$\mathrm{H}^1(\mathbf{Q}_{R\cup N}, \operatorname{Ad}^0\bar\rho) \to \prod_{p\in R} \mathrm{H}^1(\mathbf{Q}_p, \operatorname{Ad}^0\bar\rho) \times \prod_{p\in U'} \mathrm{H}^1_{\mathrm{nr}}(\mathbf{Q}_p, \operatorname{Ad}^0\bar\rho) \tag{6.1}$$

is an isomorphism. In fact, $\# N = h^1(\mathbf{Q}_{R\cup N}, \operatorname{Ad}^0\bar\rho^*)$, and the primes in N are chosen, one at a time, from Chebotarev sets. This means we can force them to be large enough to ensure that the bound $\pi_{R\cup N}(x) \leqslant h(x)\pi(x)$ continues to hold.

By our hypothesis, ρ_n admits a lift to $\mathbf{Z}/l^{n+1}$; call one such lift ρ^*. For each $p \in R\cup U'$, $\mathrm{H}^1(\mathbf{Q}_p, \operatorname{Ad}^0\bar\rho)$ acts simply transitively on lifts of $\rho_n|_{G_{\mathbf{Q}_p}}$ to $\mathbf{Z}/l^{n+1}$. In particular, there are cohomology classes $f_p \in \mathrm{H}^1(\mathbf{Q}_p, \operatorname{Ad}^0\bar\rho)$ such that $f_p\cdot\rho^* \equiv \rho_p \pmod{l^{n+1}}$ for all $p\in R\cup U'$. Moreover, for all $p\in U'$, the class f_p is unramified. Since the map in (6.1) is an isomorphism, there exists $f \in \mathrm{H}^1(\mathbf{Q}_{R\cup N}, \operatorname{Ad}^0\bar\rho)$ such that $f\cdot\rho^*|_{G_{\mathbf{Q}_p}} \equiv \rho_p \pmod{l^{n+1}}$ for all $p\in R\cup U'$.

Clearly $f\cdot\rho^*|_{G_{\mathbf{Q}_p}}$ admits a lift to $\mathbf{Z}_l$ for all $p\in R\cup U'$, but it does not necessarily admit such a lift for $p\in N$. By repeated applications of [Pan11, Prop. 3.10], there exists a set $N' \supset N$, with $\# N' \leqslant 2\# N$, of nice primes and $g\in \mathrm{H}^1(\mathbf{Q}_{R\cup N'}, \operatorname{Ad}^0\bar\rho)$ such that $(g+f)\cdot\rho^*$ still agrees with ρ_p for $p\in R\cup U'$, and $(g+f)\cdot\rho^*$ is nice for all $p\in N'$. As above, the primes in N' are chosen one at a time from Chebotarev sets, so we can continue to ensure the bound $\pi_{R\cup N'}(x)\leqslant h(x)\pi(x)$. Let $\rho_{n+1} = (g+f)\cdot\rho^*$. Let $R' = R\cup N'$. For each $p\in R'\smallsetminus R$, choose a ramified lift ρ_p of $\rho_{n+1}|_{G_{\mathbf{Q}_p}}$ to $\mathbf{Z}_l$.

Since $\rho_{n+1}|_{G_{\mathbf{Q}_p}}$ admits a lift to $\mathbf{Z}/l^{n+2}$ (in fact, it admits a lift to $\mathbf{Z}_l$) for each p, and $\text{Ш}^2_{R'}(\operatorname{Ad}^0\bar\rho) = 0$, the deformation ρ_{n+1} admits a lift to $\mathbf{Z}/l^{n+2}$. Thus $(\rho_{n+1}, R', U', \{\rho_p\})$ is the desired lift of $(\rho_n, R, U, \{\rho_p\})$. □

6.2 Galois representations with specified Satake parameters

Fix a good residual representation $\bar{\rho}$. We consider weight-2 deformations of $\bar{\rho}$. The final deformation, $\rho\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$, will be constructed as the inverse limit of a compatible collection of lifts $\rho_n\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}/l^n)$. At any given stage, we will be concerned with making sure that a) there exists a lift to the next stage, and b) there is a lift with the necessary properties. Fix a sequence $\boldsymbol{x} = (x_1, x_2, \dots)$ in $[-1, 1]$. The set of unramified primes of ρ is not determined at the beginning, but at each stage there will be a large finite set U of primes which we know will remain unramified. Re-indexing $\boldsymbol{x}$ by these unramified primes, we will construct ρ so that for all unramified primes p, $\operatorname{tr}\rho(\mathrm{fr}_p) \in \mathbf{Z}$, satisfies the Hasse bound, and has $\operatorname{tr}\rho(\mathrm{fr}_p) \approx x_p$. Moreover, we can ensure that the set of ramified primes has density zero in a very strong sense (controlled by a parameter function h) and that our trace of Frobenii are very close to specified values (the "closeness" again controlled by a parameter function b).

Given any deformation ρ, write $\pi_{\mathrm{ram}(\rho)}(x)$ for the function which counts ρ_n-ramified primes $\leqslant x$.

Theorem 6.2.1. *Let l, $\bar{\rho}$, $\boldsymbol{x}$ be as above. Fix functions $h\colon \mathbf{R}^+ \to \mathbf{R}^+$ (resp. $b\colon \mathbf{R}^+ \to \mathbf{R}_{\geqslant 1}$) which decrease to zero (resp. increase to infinity). Then there exists a weight-2 deformation ρ of $\bar{\rho}$, such that*

1. *$\pi_{\mathrm{ram}(\rho)}(x) \ll h(x)\pi(x)$.*

2. *For each unramified prime p, $a_p = \operatorname{tr}\rho(\mathrm{fr}_p) \in \mathbf{Z}$ and satisfies the Hasse*

bound.

3. *For each unramified prime* p, $\left|\frac{a_p}{2\sqrt{p}} - x_p\right| \leqslant \frac{lb(p)}{2\sqrt{p}}$.

Proof. Begin with $\rho_1 = \bar{\rho}$. By [KLR05, Lem. 6], there exists a finite set R, containing the set of primes at which $\bar{\rho}$ ramifies, such that $Ш^1_R(\operatorname{Ad}^0 \bar{\rho}) = Ш^2_R(\operatorname{Ad}^0 \bar{\rho}) = 0$. Let R_2 be the union of R and all primes p with $\frac{l}{2\sqrt{p}} > 2$. For all $p \notin R_2$ and any $a \in \mathbf{F}_l$, there exists $a_p \in \mathbf{Z}$ satisfying the Hasse bound with $a_p \equiv a \pmod{l}$. In fact, given any $x_p \in [-1, 1]$, there exists $a_p \in \mathbf{Z}$ satisfying the Hasse bound such that $\left|\frac{a_p}{2\sqrt{p}} - x_p\right| \leqslant \frac{l}{2\sqrt{p}}$. Choose, for all primes $p \in R_2$, a ramified lift ρ_p of $\rho_1|_{G_{\mathbf{Q}_p}}$. Let U_2 be the set of primes not in R_2 such that $\frac{l^2}{2\sqrt{p}} > \min\left(2, \frac{lb(p)}{2\sqrt{p}}\right)$. For each $p \in U_2$, there exists $a_p \in \mathbf{Z}$, satisfying the Hasse bound, such that

$$\left|\frac{a_p}{2\sqrt{p}} - x_p\right| \leqslant \frac{l}{2\sqrt{p}} \leqslant \frac{lb(p)}{2\sqrt{p}},$$

and moreover $a_p \equiv \operatorname{tr} \bar{\rho}(\mathrm{fr}_p) \pmod{l}$. For each $p \in U_2$, let ρ_p be an unramified lift of $\bar{\rho}|_{G_{\mathbf{Q}_p}}$ with $a_p \equiv \operatorname{tr} \rho_p(\mathrm{fr}_p) \pmod{l}$. It may not be that $\pi_{R_2}(x) \leqslant h(x)\pi(x)$ for all x, but there is a scalar multiple h^* of h so that $\pi_{R_2}(x) \leqslant h^*(x)\pi(x)$ for all x.

We have constructed our first h^*-bounded lifting datum $(\rho_1, R_2, U_2, \{\rho_p\})$. We proceed to construct $\rho = \varprojlim \rho_n$ inductively, by constructing a new h^*-bounded lifting datum for each n. We ensure that U_n contains all primes for which $\frac{l^n}{2\sqrt{p}} > \min\left(2, \frac{lb(p)}{2\sqrt{p}}\right)$, so there are always integral a_p satisfying the Hasse bound which satisfy any mod-l^n constraint, and that can always choose these a_p so as to preserve statement 2 in the theorem.

The base case is already complete, so suppose we are given $(\rho_n, R_n, U_n, \{\rho_p\})$. We may assume that U_n contains all primes for which $\frac{l^n}{2\sqrt{p}} > \min\left(2, \frac{lb(p)}{2\sqrt{p}}\right)$. Let

U_{n+1} be the set of all primes not in R_n such that $\frac{l^{n+1}}{2\sqrt{p}} > \min\left(2, \frac{lb(p)}{2\sqrt{p}}\right)$. For each $p \in U_{n+1} \setminus U_n$, there is an integer a_p, satisfying the Hasse bound, such that $a_p \equiv \rho_n(\mathrm{fr}_p) \pmod{l^n}$, and moreover $\left|\frac{a_p}{2\sqrt{p}} - x_p\right| \leqslant \frac{lb(p)}{2\sqrt{p}}$. For such p, let ρ_p be an unramified lift of $\rho_n|_{G_{\mathbf{Q}_p}}$ such that $a_p \equiv \operatorname{tr}\rho_n(\mathrm{fr}_p) \pmod{l^n}$. By Theorem 6.1.6, there exists an h^*-bounded lifting datum $(\rho_{n+1}, R_{n+1}, U_{n+1}, \{\rho_p\})$ extending and lifting $(\rho_n, R_n, U_n, \{\rho_p\})$. This completes the inductive step. □

CHAPTER 7

COUNTEREXAMPLE VIA DIOPHANTINE APPROXIMATION

7.1 Supporting results

Give $(\mathbf{R}/\mathbf{Z})^d$ the natural Haar measure normalized to have total mass one. Recall that for any $f \in L^1((\mathbf{R}/\mathbf{Z})^d)$, the Fourier coefficients of f are, for $m \in \mathbf{Z}^d$

$$\widehat{f}(m) = \int_{(\mathbf{R}/\mathbf{Z})^d} e^{2\pi i \langle m,x\rangle}\,\mathrm{d}x,$$

where $\langle m, x\rangle = m_1x_1 + \cdots + m_dx_d$ is the usual inner product.

Theorem 7.1.1. *Fix $x \in (\mathbf{R}/\mathbf{Z})^d$ with $\omega_{d-1}(x)$ finite. Then*

$$\left|\sum_{n\leqslant N} e^{2\pi i\langle m,nx\rangle}\right| \ll |m|^{\omega_{d-1}(x)+\epsilon}$$

as m ranges over $\mathbf{Z}^r \smallsetminus 0$.

Proof. From Lemma 4.2.2 we know that

$$\left|\sum_{n\leqslant N} e^{2\pi i\langle m,nx\rangle}\right| \ll \frac{1}{d(\langle m,x\rangle,\mathbf{Z})},$$

and from Lemma 4.1.4, we know that $d(\langle m,x\rangle,\mathbf{Z})^{-1} \ll |m|^{\omega_{d-1}(x)+\epsilon}$. The result follows. □

Theorem 7.1.2. *Let $x \in \mathbf{R}^d$ with $\omega_{d-1}(x)$ finite. Then let $f \in L^1((\mathbf{R}/\mathbf{Z})^d)$ with $\widehat{f}(0) = 0$ and suppose the Fourier coefficients of f satisfy the bound $|\widehat{f}(m)| \ll |m|^{-\frac{1}{d-1}-\omega_{d-1}(x)-\epsilon}$. Then*

$$\left|\sum_{n\leqslant N} f(nx)\right| \ll 1.$$

Proof. Write f as a Fourier series:

$$f(x) = \sum_{m \in \mathbf{Z}^r} \widehat{f}(m) e^{2\pi p i \langle m, x \rangle}.$$

Since $\widehat{f}(0) = 0$, we can compute:

$$\begin{aligned} \left| \sum_{n \leqslant N} f(nx) \right| &= \left| \sum_{n \leqslant N} \sum_{m \in \mathbf{Z}^d \smallsetminus 0} \widehat{f}(m) e^{2\pi i \langle m, x \rangle} \right| \\ &\leqslant \sum_{m \in \mathbf{Z}^d \smallsetminus 0} |\widehat{f}(m)| \left| \sum_{n \leqslant N} e^{2\pi i n \langle m, x \rangle} \right| \\ &\ll \sum_{m \in \mathbf{Z}^d \smallsetminus 0} |m|^{-\frac{1}{d-1} - \omega_{d-1}(x) - \epsilon} |m|^{\omega_{d-1}(x) + \epsilon/2} \\ &\ll \sum_{m \in \mathbf{Z}^d \smallsetminus 0} |m|^{-\frac{1}{d-1} - \epsilon/2}. \end{aligned}$$

The sum converges since the exponent is less than $-\frac{1}{d-1}$, and it doesn't depend on N, hence the result. □

7.2 Pathological Satake parameters

Let $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ be an enumeration of the prime numbers. Let $y \in \mathbf{R}^d$ with $y_1, \ldots, y_d$ linearly independent over $\mathbf{Q}$. The associated sequence of "fake Satake parameters" is

$$\boldsymbol{x} = (y, 2y, 3y, 4y, \ldots),$$

where we put $x_{p_n} = ny \mod \mathbf{Z}^d$. By Theorem 4.1.2, we can arrange for $\omega_0(y) = w$ and $\omega_{d-1}(y) = dw + d - 1$.

Theorem 7.2.1. *The sequence $\boldsymbol{x}$ is equidistributed in $(\mathbf{R}/\mathbf{Z})^d$, with discrepancy decaying as*

$$\mathrm{D}(\boldsymbol{x}^N) \ll N^{-\frac{1}{dw+d} + \epsilon}$$

and for which

$$\mathrm{D}(\boldsymbol{x}^N) = \Omega\left(N^{-\frac{d}{w}-\epsilon}\right).$$

However, for any $f \in C^\infty((\mathbf{R}/\mathbf{Z})^d)$ *with* $\widehat{f}(0) = 0$*, the strange Dirichlet series* $L_f(\boldsymbol{x}, s)$ *satisfies the Riemann Hypothesis.*

Now, as the sequences in this theorem are uniformly distributed, not equidistributed with respect to the Haar measure on $\mathrm{SU}(2)$ or any other semisimple group, this example may not say too much about non-CM elliptic curves. However, let $E_1, \ldots, E_n$ be pairwise non-isogenous elliptic curves with complex multiplication defined over $\mathbf{Q}$. Then the Sato–Tate group of the abelian variety $A = E_1 \times \cdots \times E_n$ is $\mathrm{ST}(A) = \mathrm{SO}(2)^n = (S^1)^n$. The theorem above shows that the truth of the Generalized Riemann Hypothesis for A says nothing about the rate of decay of the discrepancy for the Satake parameters of A.

CHAPTER 8

DIRECT COUNTEREXAMPLE

8.1 Main ideas

For $k \geqslant 1$, let

$$U_k(\theta) = \operatorname{tr} \operatorname{sym}^k \left(\begin{smallmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{smallmatrix}\right) = \frac{\sin((k+1)\theta)}{\sin\theta}.$$

Then $U_k(\cos^{-1} t)$ is the k-th Chebyshev polynomial of the 2nd kind. Moreover, $\{U_k\}$ forms an orthonormal basis for $L^2([0,\pi], \mathrm{ST}) = L^2(\mathrm{SU}(2)^\natural)$.

This chapter has two parts. First, for any reasonable measure μ on $[0,\pi]$ invariant under the same "flip" automorphism as the Sato–Tate measure, there is a sequence $\{a_p\}$ of integers satisfying the Hasse bound $|a_p| \leqslant 2\sqrt{p}$, such that for $\theta_p = \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right)$, the discrepancy $\mathrm{D}(\boldsymbol{\theta}^N, \mu)$ behaves like $\pi(N)^{-\alpha}$ for predetermined $\alpha \in (0, 1/2]$, while for any odd k, the strange Dirichlet series $L_{U_k}(\boldsymbol{\theta}, s)$, which we will write $L(\operatorname{sym}^k \boldsymbol{\theta}, s)$, satisfies the Riemann Hypothesis.

In the second part of this chapter, we associate (infinitely ramified) Galois representations to the fake Satake parameters above, using the techniques in Chapter 6.

8.2 Construction

Definition 8.2.1. *Let* $\mu = f(t)\,\mathrm{d}t$ *be a good measure on* $[0,\pi]$*. If* $f(t) \ll \sin(t)$*, then* μ *is a* Sato–Tate compatible measure.

The key facts about Sato–Tate compatible measures are that $\cos_* \mu$ satisfies the hypotheses of Theorem 2.4.6, so there are "$N^{-\alpha}$-decaying van der Corput sequences" for $\cos_* \mu$, and also that since $\cos\colon [0,\pi] \to [-1,1]$ is an order anti-isomorphism, we know that for any sequence $\boldsymbol{x}$ on $[-1,1]$, there is equality $\mathrm{D}(\boldsymbol{x}^N, \cos_* \mu) = \mathrm{D}(\cos^{-1} \boldsymbol{x}^N, \mu)$.

Theorem 8.2.2. *Let μ be a Sato–Tate compatible measure, and fix $\alpha \in (0,1/2)$. Then there exists a sequence of integers a_p satisfying the Hasse bound, such that if we set $\theta_p = \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right)$, then $\mathrm{D}^\star(\boldsymbol{\theta}^N, \mu) = \Theta(\pi(N)^{-\alpha})$.*

Proof. Apply Theorem 2.4.6 to find a sequence $\boldsymbol{x}$ such that $\mathrm{D}(\boldsymbol{x}^N, \cos_* \mu) = \Theta(\pi(N)^{-\alpha})$. For each prime p, there exists an integer a_p such that $|a_p| \leqslant 2\sqrt{p}$ and $\left|\frac{a_p}{2\sqrt{p}} - x_p\right| \leqslant p^{-1/2}$. Let $y_p = \frac{a_p}{2\sqrt{p}}$. Now apply Lemma 2.6.1 with $\epsilon = N^{-1/2}$. We obtain

$$\left|\mathrm{D}(\boldsymbol{x}^N, \cos_* \mu) - \mathrm{D}(\boldsymbol{y}^N, \cos_* \mu)\right| \ll N^{-1/2} + \frac{\pi(N^{1/2})}{\pi(N)},$$

which tells us that $\mathrm{D}(\boldsymbol{y}^N, \cos_* \mu) = \Theta(\pi(N)^{-\alpha})$. Now let $\boldsymbol{\theta} = \cos^{-1}(\boldsymbol{y})$. Apply Lemma 2.6.4 to $\boldsymbol{\theta} = \cos^{-1}(\boldsymbol{y})$, and we see that $\mathrm{D}(\boldsymbol{\theta}^N, \mu) = \Theta(\pi(N)^{-\alpha})$. □

We can improve this example by controlling the behavior of sums of the form $\sum_{p \leqslant N} U_k(\theta_p)$ for odd k. Let σ be the involution of $[0,\pi]$ given by $\sigma(\theta) = \pi - \theta$. Note that $\sigma_* \mathrm{ST} = \mathrm{ST}$. Moreover, note that for any odd k, $U_k \circ \sigma = -U_k$, so $\int U_k \, \mathrm{dST} = 0$. (Of course, $\int U_k = 0$ for the reason that U_k is the trace of a nontrivial unitary representation, but we will directly exploit the "oddness" of U_k in what follows.)

Theorem 8.2.3. *Let μ be a σ-invariant Sato–Tate compatible measure. Fix $\alpha \in$*

$(0,1/2)$. *Then there is a sequence of integers a_p, satisfying the Hasse bound, such that for $\theta_p = \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right)$, we have*

1. $\mathrm{D}(\boldsymbol{\theta}^N, \mu) = \Theta(\pi(N)^{-\alpha})$.

2. *For all odd k,* $\left|\sum_{k\leqslant N} U_k(\theta_p)\right| \ll \pi(N)^{1/2}$.

Proof. The basic ideas is as follows. Enumerate the primes

$$p_1 = 2, q_1 = 3, p_2 = 5, q_2 = 7, p_3 = 11, q_3 = 13, \dots.$$

Consider the measure $\mu|_{[0,\pi/2)}$. An argument nearly identical to the proof of Theorem 8.2.2 shows that we can choose a_{p_i} satisfying the Hasse bound so that

$$\mathrm{D}\left(\{\theta_{p_i}\}_{i\leqslant N}, \mu|_{[0,\pi/2)}\right) = \Theta(N^{-\alpha}).$$

We can also choose the $a_{q_i} \in [\pi/2, \pi]$ so that

$$\left|\frac{a_{p_i}}{2\sqrt{p_i}} + \frac{a_{q_i}}{2\sqrt{q_i}}\right| \ll \frac{1}{\sqrt{p_i}}.$$

If $\boldsymbol{x}$ is the sequence of the $\frac{a_{p_i}}{2\sqrt{p_i}}$ and $\boldsymbol{y}$ is the similar sequence with the q_i-s, then Lemma 2.6.2, Lemma 2.6.1, and Theorem 2.7.2 tell us that $\mathrm{D}((\boldsymbol{x} \wr \boldsymbol{y})^N, \mu) = \Theta(N^{-\alpha})$.

Moreover, $U_k(\cos^{-1} t)$ is an odd polynomial in t, so if $|x_i - (-y_i)| \ll p_i^{-1/2}$, then $|U_k(\theta_{p_i}) + U_k(\theta_{q_i})| \ll p_i^{-1/2}$. We can then bound

$$\left|\sum_{i\leqslant N} U_k(\theta_{p_i}) + U_k(\theta_{q_i})\right| \ll \sum_{p\leqslant N} p^{-1/2} \ll \pi(N)^{1/2}.$$

□

8.3 Associated Galois representation

Now we combine the results of the last section and Chapter 6 to obtain a "beefed-up" version of Theorem 8.2.3.

Theorem 8.3.1. *Let μ be a Sato–Tate compatible σ-invariant measure on $[0,\pi]$. Fix $\alpha \in (0,1/2)$ and a good residual representation $\rho\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_l)$. Then there exists a weight-2 lift $\rho\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$ of $\bar{\rho}$ such that*

1. $\pi_{\mathrm{ram}(\rho)}(x) \ll e^{-x}\pi(x)$.
2. *For each unramified prime p, $a_p = \operatorname{tr}\rho(\mathrm{fr}_p) \in \mathbf{Z}$ and satisfies the Hasse bound.*
3. *If, for unramified p we set $\theta_p = \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right)$, then $\mathrm{D}(\boldsymbol{\theta}^N, \mu) = \Theta(\pi(N)^{-\alpha})$.*
4. *For each odd k, the function $L(\operatorname{sym}^k \rho, s)$ satisfies the Riemann Hypothesis.*

Proof. Let $\boldsymbol{x}$ be an $N^{-\alpha}$-decay van der Corput sequence for $\cos_* \mu|_{[0,\pi/2)}$. Let $\boldsymbol{y} = -\boldsymbol{x}$. Then $\mathrm{D}((\boldsymbol{x} \wr \boldsymbol{y})^N, \cos_* \mu) = \Theta(N^{-\alpha})$. Set $h(x) = e^{-x}$ and $b(x) = \log(x)$. By Theorem 6.2.1, there is a $\rho\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$ lifting $\bar{\rho}$ such that parts 1 and 2 of the theorem hold. The discrepancy estimate comes from Lemma 2.6.2, Lemma 2.6.1, and Theorem 2.7.2 as above, while the Riemann Hypothesis for odd symmetric powers follows from the proof of Theorem 8.2.3. □

CHAPTER 9

CONCLUDING REMARKS AND FUTURE DIRECTIONS

9.1 Fake modular forms

The Galois representations of Theorem 8.3.1 have "fake modular forms" associated to them. Namely, there is a representation of $\mathrm{GL}_2(\mathbf{A})$ with the specified Satake parameters at each prime (for now, set $\theta_p = 0$ at ramified primes). It is natural to ask if these "fake modular forms" have any interesting properties. For example, we know that all their odd symmetric powers satisfy the Riemann Hypothesis. The author is unaware of any further results (say about analytic continuation or functional equation) concerning these fake modular forms.

9.2 Dense free subgroups of compact semisimple groups

Let G be a compact semisimple Lie group, for example $\mathrm{SU}(2)$. By [BG03], G contains a dense free subgroup $\Gamma = \langle \gamma_1, \gamma_2 \rangle$. We will now follow the argument of [AK63] to hint at how Γ may yield equidistributed sequences with "bad" discrepancy and small character sums.

Given an integer N, let B_N be the "closed ball of size N" in Γ, that is the set of products $\gamma_{\sigma(1)} \dots \gamma_{\sigma(n)}$, where $n \leqslant N$ and $\sigma\colon \{1, \dots, n\} \to \{1, 2\}$ is a function. We will write $\sigma\colon [n] \to [2]$ in this case. Given an irreducible unitary representation $\rho \in \widehat{G}$, we wish to control the behavior of $\sum_{\gamma \in B_N} \operatorname{tr} \rho(\gamma)$, ideally to show an estimate of the form

$$\left|\sum_{\gamma\in B_N} \operatorname{tr}\rho(\gamma)\right| \ll (\#B_N)^{\frac{1}{2}+\epsilon}.$$

In fact, $\#B_N = \sum_{n=0}^N 2^n = 2^{N+1}-1$. We can encode these sums in terms of convolutions of a measure as follows. Let μ be the measure $\delta_{\gamma_1^{-1}}+\delta_{\gamma_2^{-1}}$ on G. If ρ is any unitary representation (not necessarily irreducible or even finite-dimensional) then μ acts on ρ via $\rho(\mu)\int \rho\,\mathrm{d}\mu$. So, if $\rho = L^2(G)$ via the left regular representation, then $(\mu\cdot f)(x) = f(\gamma_1 x)+f(\gamma_2 x)$, while if $\rho\in\widehat{G}$ and $v\in\rho$, then $\mu\cdot v = \rho(\gamma_1)v+\rho(\gamma_2)v$. Note that

$$\mu^{*n} = \sum_{\sigma\colon [n]\to[2]} \delta_{\gamma_{\sigma(1)}\cdots\gamma_{\sigma(n)}}.$$

This tells us that $\sum_{\gamma\in B_N} f(\gamma) = \sum_{n\leqslant N}\mu^{*n}(f)$. So we really only need to study how μ and its powers act on the functions $\operatorname{tr}\rho$, $\rho\in\widehat{G}$.

First note that $\operatorname{tr}\rho$ generates a subrepresentation of $L^2(G)$ which is isomorphic to ρ. On that representation, we claim that μ is invertible, hence $\sum_{n=0}^N \mu^{*n} = (\mu^{*(N+1)}-1)(\mu-1)^{-1}$.It follows that $\|\sum_{n=0}^N \mu^{*n}\| \leqslant \frac{\|\mu\|^{N+1}}{\|\mu-1\|}$,

Note that $\|\mu\|^{N+1}\leqslant 2^{(N+1)\alpha}$ if and only if $\|\mu\|\leqslant 2^\alpha$. In other words, to get the Riemann Hypothesis for L-functions coming from Γ, we need $\|\mu\|\leqslant\sqrt{2}$. If $v\in\rho$ has norm 1, then

$$\begin{aligned}\|\rho(\mu)v\|^2 &= \langle \rho(\gamma_1^{-1})v+\rho(\gamma_2^{-1})v, \rho(\gamma_1^{-1})v+\rho(\gamma_2^{-1})v\rangle\\ &= 2\|v\|^2 + 2\Re\langle\rho(\gamma_2\gamma_1^{-1})v,v\rangle.\end{aligned}$$

So, we want $\Re\langle\rho(\gamma_2\gamma_1^{-1})v,v\rangle\leqslant 0$ for all irreducible ρ. Sadly, even for $\mathrm{SU}(2)$, this is not possible.

Write $\gamma=\gamma_2\gamma_1^{-1}$, then the identity $\langle\rho(\gamma)\rho(\delta)v,\rho(\delta)v\rangle = \langle\rho(\delta^{-1}\gamma\delta)v,v\rangle$ tells us that we can restrict our search to γ of the form $\left(\begin{smallmatrix} a & \\ & \bar{a} \end{smallmatrix}\right)$ with $|a| = 1$. Now

$$\langle \left(\begin{smallmatrix} a & \\ & \bar{a} \end{smallmatrix}\right) \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right), \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \rangle = \Re(a),$$

which appears to be promising. But a similar computation with sym^2 shows that one can always get $\langle \mathrm{sym}^2 \gamma v, v \rangle = 1$, so the above approach fails.

There may be alternative ways of bounding the sums $\sum \mu^{*n}(\operatorname{tr} \rho)$, but we do not investigate them here.

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