

KOLMOGOROV–SMIRNOV STATISTICS AND THE
ANALYTIC PROPERTIES OF DIRICHLET SERIES
ASSOCIATED TO ELLIPTIC CURVES

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Abstract here.

BIOGRAPHICAL SKETCH

Brief biographical sketch.

Dedication here.

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TABLE OF CONTENTS

1	Foundations	1
1.1	Category of test objects	1
1.2	Quotients in the flat topology	2
1.3	Groupoids and quotient stacks	4
1.4	Deformations of group representations	4
1.5	Tangent spaces and obstruction theory	5
2	?	7

CHAPTER 1 FOUNDATIONS

1.1 Category of test objects

The following is an exposition and explication of the theory outlined in [SGA 3₁, VII_B, §0–1]. In particular, we will heavily use the notions of a pseudocompact ring, pseudocompact modules, etc. Let Λ be a pseudocompact ring. Write \mathbf{C}_Λ for the opposite of the category of Λ -algebras which have finite length as Λ -modules. Given such a Λ -algebra A , write $X = \mathrm{Spf}(A)$ for the corresponding object of \mathbf{C}_Λ , and we put $A = \mathcal{O}(X)$.

Lemma 1.1.1. *Let Λ be a pseudocompact ring, \mathbf{C}_Λ as above. Then \mathbf{C}_Λ is closed under finite limits and colimits.*

Lemma 1.1.2. *Let Λ be a pseudocompact local ring. Then Λ is henselian, in any of the following senses:*

1. *d*

Proof. [EGA 4₄, 18.5.?] □

Following Grothendieck, if \mathcal{C} is an arbitrary category, we write $\widehat{\mathcal{C}} = \mathrm{hom}(\mathcal{C}^\circ, \mathbf{Set})$ for the category of contravariant functors $\mathcal{C} \rightarrow \mathbf{Set}$. We regard \mathcal{C} as a full subcategory of $\widehat{\mathcal{C}}$ via the Yoneda embedding, so for $X, Y \in \mathcal{C}$, we write $X(Y) = \mathrm{hom}_{\mathcal{C}}(Y, X)$. With this notation, the Yoneda Lemma states that $\mathrm{hom}_{\widehat{\mathcal{C}}}(X, P) = P(X)$ for all $X \in \mathcal{C}$.

Lemma 1.1.3. *Let $\mathcal{X} \in \widehat{\mathbf{C}_\Lambda}$. Then \mathcal{X} is left exact if and only if there exists a filtered system $\{X_i\}_{i \in I}$ in \mathbf{C}_Λ together with a natural isomorphism $\mathcal{X}(\cdot) \simeq \varinjlim X_i(\cdot)$. Write $\mathrm{Ind}(\mathbf{C}_\Lambda)$ for the category of such functors. Then $\mathrm{Ind}(\mathbf{C}_\Lambda)$ is closed under colimits, and the Yoneda embedding $\mathbf{C}_\Lambda \hookrightarrow \mathrm{Ind}(\mathbf{C}_\Lambda)$ preserves filtered colimits.*

Proof. This follows from the results of [KS06, 6.1]. □

If R is a pseudocompact Λ -algebra, write $\mathrm{Spf}(R)$ for the object of $\widehat{\mathbf{C}_\Lambda}$ defined by $\mathrm{Spf}(R)(A) = \mathrm{hom}_{\mathrm{cts}/\Lambda}(R, A)$, the set of continuous Λ -algebra homomorphisms.

Lemma 1.1.4. *The functor Spf induces an (anti-)equivalence between the category of pseudocompact Λ -algebras and $\mathrm{Ind}(\mathbf{C}_\Lambda)$.*

Proof. This is [SGA 3₁, VII_B 0.4.2 Prop.]. □

So $\mathrm{Ind}(\mathbf{C}_\Lambda)$ is the category of pro-representable functors on finite length Λ -algebras. *Warning:* in many papers, for example the foundational [Maz97], one reserves the term *pro-representable* for functors of the form $\mathrm{Spf}(R)$, where R is *noetherian*. We do not make this restriction.

Lemma 1.1.5. *The category $\mathrm{Ind}(\mathbf{C}_\Lambda)$ is an exponential ideal in $\widehat{\mathbf{C}_\Lambda}$.*

Proof. By this we mean the following. Let $\mathcal{X} \in \mathrm{Ind}(\mathbf{C}_\Lambda)$, $P \in \widehat{\mathbf{C}_\Lambda}$. Then the functor \mathcal{X}^P defined by

$$\mathcal{X}^P(S) = \mathrm{hom}_{\widehat{\mathbf{C}_\Lambda/S}}(P/S, \mathcal{X}/S)$$

is also in $\mathrm{Ind}(\mathbf{C}_\Lambda)$. Given the characterization of $\mathrm{Ind}(\mathbf{C}_\Lambda)$ as left exact functors, this is easy to prove, see e.g. [Joh02, 4.2.3]. □

If \mathcal{C} is a category, we write $\mathbf{Gp}(\mathcal{C})$ for the category of group objects in \mathcal{C} .

Corollary 1.1.6. *Let $\Gamma \in \mathbf{Gp}(\widehat{\mathbf{C}}_\Lambda)$ and $\mathcal{G} \in \mathbf{Gp}(\mathbf{Ind}(\mathbf{C}_\Lambda))$, then the functor $[\Gamma, \mathcal{G}]$ defined by*

$$[\Gamma, \mathcal{G}](S) = \mathrm{hom}_{\mathbf{Gp}/S}(\Gamma/S, \mathcal{G}/S)$$

is in $\mathbf{Ind}(\mathbf{C}_\Lambda)$. In particular, if Γ is a profinite group, then the functor

$$[\Gamma, \mathcal{G}](S) = \mathrm{hom}_{\mathbf{cts}/\mathbf{Gp}}(\Gamma, \mathcal{G}(S))$$

is in $\mathbf{Ind}(\mathbf{C}_\Lambda)$.

Proof. The first claim follows easily from 1.1.5. Just note that $[\Gamma, \mathcal{G}]$ is the equalizer:

$$[\Gamma, \mathcal{G}] \longrightarrow \mathcal{G}^\Gamma \xrightarrow[m_{\mathcal{G}*}]{m_\Gamma^*} \mathcal{G}^{\Gamma \times \Gamma},$$

that is, those $f: \Gamma \rightarrow \mathcal{G}$ such that $f \circ m_\Gamma = m_{\mathcal{G}} \circ (f \times f)$. The latter claim is just a special case. \square

1.2 Quotients in the flat topology

If Λ is a pseudocompact ring, the category $\mathbf{Ind}(\mathbf{C}_\Lambda)$ has nice “geometric” properties. However, for operations like taking quotients, we will embed it into the larger category $\mathbf{Sh}_\mathfrak{fl}(\mathbf{C}_\Lambda)$ of flat sheaves. We call a collection $\{U_i \rightarrow X\}$ of morphisms in \mathbf{C}_Λ a *flat cover* if each ring map $\mathcal{O}(X) \rightarrow \mathcal{O}(U_i)$ is flat, and moreover $\mathcal{O}(X) \rightarrow \prod \mathcal{O}(U_i)$ is faithfully flat. By [SGA 3₁, IV 6.3.1], this is a subcanonical Grothendieck topology on \mathbf{C}_Λ . We call it the *flat topology*, even though finite presentation comes for free because all the rings are finite length.

Lemma 1.2.1. *Let $\mathbf{Sh}_\mathfrak{fl}(\mathbf{C}_\Lambda)$ be the category of sheaves (of sets) on \mathbf{C}_Λ with respect to the flat topology. Then a presheaf $P \in \widehat{\mathbf{C}}_\Lambda$ lies in $\mathbf{Sh}_\mathfrak{fl}(\mathbf{C}_\Lambda)$ if and only if $P(\coprod U_i) = \prod P(U_i)$ and moreover, whenever $U \rightarrow X$ is a flat cover where $\mathcal{O}(U)$ and $\mathcal{O}(X)$ are local rings, the sequence*

$$P(X) \longrightarrow P(U) \rightrightarrows P(U \times_X U).$$

is exact. Moreover, $\mathbf{Ind}(\mathbf{C}_\Lambda) \subset \mathbf{Sh}_\mathfrak{fl}(\mathbf{C}_\Lambda)$.

Proof. The first claim is the content of [SGA 3₁, IV 6.3.1(ii)]. For the second, note that any $\mathcal{X} \in \mathbf{Ind}(\mathbf{C}_\Lambda)$ will, by 1.1.3, convert (arbitrary) colimits into limits. Thus $\mathcal{X}(\coprod U_i) = \prod \mathcal{X}(U_i)$. If $U \rightarrow X$ is a flat cover, then by (loc. cit.), $U \times_X U \rightrightarrows U \rightarrow X$ is a coequalizer diagram in \mathbf{C}_Λ , hence $\mathcal{X}(X) \rightarrow \mathcal{X}(U) \rightrightarrows \mathcal{X}(U \times_X U)$ is an equalizer. \square

Our main reason for introducing the category $\mathbf{Sh}_\mathfrak{fl}(\mathbf{C}_\Lambda)$ is that, as a (Grothendieck) topos, it is closed under arbitrary colimits. Recall that in an *equivalence relation* in $\widehat{\mathbf{C}}_\Lambda$ is a morphism $R \rightarrow X \times X$ such that, for all S , the map $R(S) \rightarrow X(S) \times X(S)$ is an injection whose image is an equivalence relation on $X(S)$. We define the quotient X/R to be the coequalizer

$$R \rightrightarrows X \longrightarrow X/R.$$

By Giraud’s Theorem [MLM94, App.], for any $S \in \mathbf{C}_\Lambda$, the natural map $X(S)/R(S) \rightarrow (X/R)(S)$ is injective. It will not be surjective in general.

We let $\mathbf{Sh}_\mathfrak{fl}(\mathbf{C}_\Lambda)$ inherit definitions from \mathbf{C}_Λ as follows. If P is a property of maps in \mathbf{C}_Λ (for example, “flat,” or “smooth,”) and $f: X \rightarrow Y$ is a morphism in $\mathbf{Sh}_\mathfrak{fl}(\mathbf{C}_\Lambda)$, we say that f has P if for all $S \in \mathbf{C}_\Lambda$ and $y \in Y(S)$, the pullback $X_S = X \times_Y S$ lies in \mathbf{C}_Λ , and the pullback map $X_S \rightarrow S$ has property P . For example, if $X = \mathrm{Spf}(R')$ and $Y = \mathrm{Spf}(R)$, then $X \rightarrow Y$ has property P if and only if for all finite length A and continuous Λ -algebra maps $R \rightarrow A$, the induced map $A \rightarrow R' \otimes_R A$ has P .

Theorem 1.2.2. *Let $\mathcal{R} \rightarrow \mathcal{X} \times \mathcal{X}$ be an equivalence relation in $\mathrm{Ind}(\mathbf{C}_\Lambda)$ such that one of the maps $\mathcal{R} \rightarrow \mathcal{X}$ is flat. Then the quotient \mathcal{X}/\mathcal{R} lies in $\mathrm{Ind}(\mathbf{C}_\Lambda)$, and $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$ is a flat cover.*

Proof. This is [SGA 3_I, VII_B 1.4]. \square

By [Mat89, 29.7], if k is a field and R is a complete regular local k -algebra, then $R \simeq k[[t_1, \dots, t_n]]$. In particular, R admits an augmentation $\epsilon: R \rightarrow k$. There is a general analogue of this result, but first we need a definition.

Definition 1.2.3. *A map $f: \mathcal{X} \rightarrow \mathcal{Y}$ in $\mathrm{Ind}(\mathbf{C}_\Lambda)$ is a residual isomorphism if for all $S = \mathrm{Spf}(k) \in \mathbf{C}_\Lambda$ where k is a field, the map $f: \mathcal{X}(S) \rightarrow \mathcal{Y}(S)$ is a bijection.*

Lemma 1.2.4. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a smooth map in $\mathrm{Ind}(\mathbf{C}_\Lambda)$ that is a residual isomorphism. Then f admits a section.*

Proof. By [SGA 3_I, VII_B 0.1.1], it suffices to prove the result when $\mathcal{X} = \mathrm{Spf}(R')$, $\mathcal{Y} = \mathrm{Spf}(R)$, for local Λ -algebras $R \rightarrow R'$ with the same residue field. Let $k = R/\mathfrak{m}_R \xrightarrow{\sim} R'/\mathfrak{m}_{R'}$ be their common residue field. From the diagram

$$\begin{array}{ccc} R' & \cdots \cdots \rightarrow & R \\ \uparrow & \searrow & \downarrow \\ R & \longrightarrow & k, \end{array}$$

the definition of (formal) smoothness, and a limiting argument involving the finite length quotients R/\mathfrak{a} , we obtain the result. \square

Corollary 1.2.5. *Let $\mathcal{R} \rightarrow \mathcal{X} \times \mathcal{X}$ be an equivalence relation satisfying the hypotheses of 1.2.2. Suppose further that*

1. *One of the maps $\mathcal{R} \rightarrow \mathcal{X}$ is smooth, and*
2. *The projection $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$ is a residual isomorphism.*

Then $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$ admits a section, so $\mathcal{X}(S)/\mathcal{R}(S) \xrightarrow{\sim} (\mathcal{X}/\mathcal{R})(S)$ for all $S \in \mathbf{C}_\Lambda$.

Proof. By 1.2.4, it suffices to prove that $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$ is smooth. By [EGA 4₄, 17.7.3(ii)], smoothness can be detected after flat descent. So base-change with respect to the projection $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$. In the following commutative diagram

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{\quad} & \mathcal{X} \\ \parallel & \searrow & \downarrow \\ \mathcal{X} \times_{\mathcal{X}/\mathcal{R}} \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X}/\mathcal{R} \end{array}$$

we can ensure the smoothness of $\mathcal{R} \rightarrow \mathcal{X}$ by our hypotheses. Since $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$ is smooth after flat base-change, the original map is smooth. \square

Example 1.2.6. The hypothesis on residue fields in 1.2.5 is necessary. To see this, let $\Lambda = k$ be a field, $k \hookrightarrow K$ a finite Galois extension with Galois group G . Then $G \times \mathrm{Spf}(K) \rightrightarrows \mathrm{Spf}(K)$ has quotient $\mathrm{Spf}(k)$, but the map $\mathrm{Spf}(K)(S) \rightarrow \mathrm{Spf}(k)(S)$ is *not* surjective for all $S \in \mathbf{C}_k$, e.g. it is not for $S = \mathrm{Spf}(k)$.

Example 1.2.7. The hypothesis of smoothness in 1.2.5 is necessary. To see this, let k be a field of characteristic $p > 0$. Then the formal additive group $\widehat{\mathbf{G}}_a = \mathrm{Spf}(k[[t]])$ has a subgroup α_p defined by

$$\alpha_p(S) = \{s \in \mathcal{O}(S) : s^p = 0\}.$$

The quotient $\widehat{\mathbf{G}}_a/\alpha_p$ has as affine coordinate ring $k[[t^p]]$. In particular, the following sequence is exact in the flat topology:

$$0 \longrightarrow \alpha_p \longrightarrow \widehat{\mathbf{G}}_a \xrightarrow{(\cdot)^p} \widehat{\mathbf{G}}_a \longrightarrow 0.$$

It follows that $\alpha_p \times \widehat{\mathbf{G}}_a \rightrightarrows \widehat{\mathbf{G}}_a \xrightarrow{(\cdot)^p} \widehat{\mathbf{G}}_a$ is a coequalizer in $\mathrm{Sh}_{\mathrm{fl}}(\mathbf{C}_k)$ satisfying all the hypotheses of 1.2.5 except smoothness. And indeed, as one sees by letting $S = \mathrm{Spf}(A)$ for any non-perfect k -algebra A , the map $(\cdot)^p : \widehat{\mathbf{G}}_a(S) \rightarrow \widehat{\mathbf{G}}_a(S)$ is *not* surjective for all S .

1.3 Groupoids and quotient stacks

Lemma 1.3.1. *Let $\mathcal{G} \in \mathrm{Ind}(\mathbf{C}_\Lambda)$ be a smooth connected group. Then every \mathcal{G} -torsor is trivial.*

Proof. Let $\mathcal{P} \rightarrow \mathcal{B}$ be a \mathcal{G} -torsor in $\mathrm{Ind}(\mathbf{C}_\Lambda)$. That is, \mathcal{P} has an action of \mathcal{G}_S for which $\mathcal{P} \times_{\mathcal{B}} \mathcal{P} \simeq \mathcal{G} \times \mathcal{P}$ as \mathcal{G} -spaces. [...not done...] \square

Theorem 1.3.2. *Let \mathcal{G} be a smooth connected group in $\mathrm{Ind}(\mathbf{C}_\Lambda)$, and $\mathcal{X} \in \mathrm{Ind}(\mathbf{C}_\Lambda)$ a \mathcal{G} -object. Then the quotient stack $[\mathcal{X}/\mathcal{G}](S)$ has as objects $\mathcal{X}(S)/\mathcal{G}(S)$, but with extra automorphisms?*

Proof. Use triviality of torsors. \square

1.4 Deformations of group representations

Let $\Gamma \in \mathrm{Gp}(\widehat{\mathbf{C}}_\Lambda)$ and $\mathcal{G} \in \mathrm{Ind}(\mathbf{C}_\Lambda)$. By 1.1.6, the functor

$$\mathrm{Rep}^\square(\Gamma, \mathcal{G})(S) = \mathrm{hom}_{\mathrm{Gp}/S}(\Gamma_S, \mathcal{G}_S)$$

is in $\mathrm{Ind}(\mathbf{C}_\Lambda)$. We would like to define an ind-scheme $\mathrm{Rep}(\Gamma, \mathcal{G})$ as “ $\mathrm{Rep}^\square(\Gamma, \mathcal{G})$ modulo conjugation,” but this requires some care. The conjugation action of \mathcal{G} on $\mathrm{Rep}^\square(\Gamma, \mathcal{G})$ will have fixed points, so the quotient will be badly behaved. We loosely follow [Til96].

Assume Λ is local, with maximal ideal \mathfrak{m} and residue field \mathbf{k} . Fix $\bar{\rho} \in \mathrm{Rep}^\square(\Gamma, \mathcal{G})(\mathbf{k})$, i.e. a residual representation $\bar{\rho} : \Gamma \rightarrow \mathcal{G}(\mathbf{k})$. Let $\mathrm{Rep}^\square(\Gamma, \mathcal{G})_{\bar{\rho}}$ be the connected component of $\bar{\rho}$ in $\mathrm{Rep}^\square(\Gamma, \mathcal{G})$. Assume that \mathcal{G} and $Z(\mathcal{G})$ are smooth; then the quotient $\mathcal{G}^{\mathrm{ad}} = \mathcal{G}/Z(\mathcal{G})$ is also smooth. Let $\mathcal{G}^{\mathrm{ad}, \circ}$ be the connected component of 1 in $\mathcal{G}^{\mathrm{ad}}$.

Theorem 1.4.1. *Suppose $(\Lambda, \mathfrak{m}, \mathbf{k})$ is local. If $\mathcal{X}, \mathcal{Y} \in \mathrm{Ind}(\mathbf{C}_\Lambda)$ are connected and $\mathcal{X}(\mathbf{k}) \neq \emptyset$, then $\mathcal{X} \times_\Lambda \mathcal{Y}$ is connected.*

Proof. We are reduced to proving the following result from commutative algebra: if R, S are local pro-artinian Λ -algebras and R has residue field \mathbf{k} , then $R \widehat{\otimes}_\Lambda S$ is local. Since $R \widehat{\otimes}_\Lambda S = \varprojlim (R/\mathfrak{r}) \otimes_\Lambda (S/\mathfrak{s})$, \mathfrak{r} (resp. \mathfrak{s}) ranges over all open ideals in R (resp. S), we may assume that both R and S are artinian. The rings R and S are henselian, so $R \otimes S$ is local if and only if $(R/\mathfrak{m}_R) \otimes (S/\mathfrak{m}_S) = S/\mathfrak{m}_S$ is local, which it is. \square

We conclude that the action of $\mathcal{G}^{\mathrm{ad}, \circ}$ on $\mathrm{Rep}^\square(\Gamma, \mathcal{G})$ preserves $\mathrm{Rep}^\square(\Gamma, \mathcal{G})_{\bar{\rho}}$. Thus we may put

$$\mathrm{Rep}(\Gamma, \mathcal{G})_{\bar{\rho}} = [\mathrm{Rep}^\square(\Gamma, \mathcal{G})_{\bar{\rho}}/\mathcal{G}^{\mathrm{ad}, \circ}].$$

If $\mathcal{G}^{\mathrm{ad}, \circ}$ acts faithfully on $\mathrm{Rep}^\square(\Gamma, \mathcal{G})_{\bar{\rho}}$, then we recover the classical notion of the deformation functor.

Theorem 1.4.2. *Let Γ be a profinite group, $\bar{\rho}: \Gamma \rightarrow \mathcal{G}(\mathbf{k})$ a representation with $H^0(\Gamma, \text{Ad } \bar{\rho}) = 0$. Then $\text{Rep}(\Gamma, \mathcal{G})_{\bar{\rho}}$ exists and is what you expect.*

Proof. Need assumptions on $Z(\mathcal{G})$, \mathcal{G} should be smooth.

Need $Z(\mathcal{G}) = \ker(\mathcal{G} \rightarrow \text{GL}(\mathfrak{g}))$ in connected case. This should use $\mathfrak{g} = \text{Lie}(\text{Aut } \mathcal{G})$, via deviations in [SGA 3₁]. \square

[... local conditions]

1.5 Tangent spaces and obstruction theory

For $S_0 \in \mathbf{C}_\Lambda$, let Ex_{S_0} be the category of square-zero thickenings of S_0 . An object of Ex_{S_0} is a closed embedding $S_0 \hookrightarrow S$ whose ideal of definition has square zero. Should be “exponential exact sequence”

$$0 \longrightarrow \mathfrak{g}(I) \longrightarrow \mathcal{G}(S) \longrightarrow \mathcal{G}(S_0) \longrightarrow 1$$

This gives us a class $\exp \in H^2(\mathcal{G}(S_0), \mathfrak{g}(I))$. For $\rho_0: \Gamma \rightarrow \mathcal{G}(S_0)$, the obstruction class is $o(\rho_0, I) = \rho_0^*(\exp) \in H^2(\Gamma, \mathfrak{g}(I))$. It's easy to check that $o(\rho_0, I) = 0$ if and only if ρ_0 lifts to ρ . So obstruction theory naturally for $\text{Rep}^\square(\Gamma, \mathcal{G})$.

[Use [Wei94, 6.6.4]. Given setting as above, $\rho_0^*(\exp)$ is the pullback by ρ_0 :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{g}(I) & \longrightarrow & \mathcal{G}(S) \times_{\mathcal{G}(S_0)} \Gamma & \longrightarrow & \Gamma \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \rho_0 \\ 0 & \longrightarrow & \mathfrak{g}(I) & \longrightarrow & \mathcal{G}(S) & \longrightarrow & \mathcal{G}(S_0) \longrightarrow 1 \end{array}$$

Computing explicitly, we see the result.]

Proposition 1.5.1. *Let $f: G \rightarrow H$ be a morphism of profinite groups. Suppose M is a discrete H -module and $c \in H^2(H, M)$ corresponds to the extension*

$$0 \longrightarrow M \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 1.$$

*Then $f^*c = 0$ in $H^2(G, M)$ if and only if there is a map $\tilde{f}: G \rightarrow \tilde{H}$ making the following diagram commute:*

$$\begin{array}{ccc} & & \tilde{H} \\ & \nearrow \tilde{f} & \downarrow \\ G & & H \\ & \searrow f & \end{array}$$

Proof. By [Wei94, 6.6.4], the class f^*c corresponds to the pullback diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & G \times_H \tilde{H} & \longrightarrow & G \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow f \\ 0 & \longrightarrow & M & \longrightarrow & \tilde{H} & \longrightarrow & H \longrightarrow 1. \end{array}$$

Writing explicitly what it means for $G \times_H \tilde{H} \rightarrow G$ to split yields the result. \square

Let $\mathcal{X} \in \text{Ind}(\mathbf{C}/\Lambda)$ be smooth, and $L_{\mathcal{X}/\Lambda} \simeq \Omega_{\mathcal{X}/\Lambda}^1[0]$ be its cotangent complex. Fix $x_0 \in \mathcal{X}(S_0)$. From the chain $S_0 \xrightarrow{x_0} \mathcal{X} \rightarrow \text{Spf}(\Lambda)$, we get a distinguished triangle [Ill71, II 2.1.5.6]

$$x_0^* L_{\mathcal{X}/\Lambda} \longrightarrow L_{S_0/\Lambda} \longrightarrow L_{S_0/\mathcal{X}} \longrightarrow .$$

If I is a coherent sheaf on S_0 , we get a long exact sequence:

$$\mathrm{Ext}^0(\mathrm{L}_{S_0/\Lambda}, M) \rightarrow \mathrm{Ext}^0(x_0^* \mathrm{L}_{\mathcal{X}/\Lambda}, M) \rightarrow \mathrm{Ext}^1(\mathrm{L}_{S_0/\mathcal{X}}, M) \rightarrow \mathrm{Ext}^1(\mathrm{L}_{S_0/\Lambda}, M) \rightarrow \mathrm{Ext}^1(x_0^* \mathrm{L}_{\mathcal{X}/\Lambda}, M)$$

If \mathcal{X}/Λ is smooth, then $\mathrm{Ext}^1(x_0^* \mathrm{L}_{\mathcal{X}/\Lambda}, M) = 0$ and $\mathrm{L}_{\mathcal{X}/\Lambda} = \Omega_{\mathcal{X}/\Lambda}^1$. This gives us an exact sequence

$$\mathrm{Ext}^0(\mathrm{L}_{S_0/\Lambda}, M) \rightarrow \mathrm{hom}(\Omega_{\mathcal{X}/\Lambda}^1, M) \rightarrow \mathrm{Ext}^1(\mathrm{L}_{S_0/\mathcal{X}}, M) \rightarrow \mathrm{Ext}^1(\mathrm{L}_{S_0/\Lambda}, M) \rightarrow 0.$$

The result [Ill71, III 2.1.7] tells us that the choice of $S \in \mathrm{Ex}_{S_0}(M)$ gives us an element of $\mathrm{Ext}^1(\mathrm{L}_{S_0/\Lambda}, M)$. Its fiber admits an action of $\mathrm{hom}(\Omega_{\mathcal{X}/\Lambda}^1, M)$. The only thing remaining is: we need $\mathrm{Ext}^0(\mathrm{L}_{S_0/\Lambda}, M) = 0$, which doesn't hold in complete generality.

CHAPTER 2
?

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