KOLMOGOROV–SMIRNOV STATISTICS AND THE ANALYTIC PROPERTIES OF DIRICHLET SERIES ASSOCIATED TO ELLIPTIC CURVES

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Abstract here.

BIOGRAPHICAL SKETCH

Brief biographical sketch.

Dedication here.

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CHAPTER 1
INTRODUCTION

Let's start with something basic, an elliptic curve $E_{/\mathbf{Q}}$. For any prime l, we have the Tate module of E, written $\mathrm{T}_l E$. This is a rank-2 $\mathbf{Z}_l$-module with continuous $G_{\mathbf{Q}}$-action, so it induces a continuous representation

$$\rho_{E,l} \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l).$$

It is known (citation?) that the quantities $a_p(E) = \operatorname{tr} \rho_l(\mathrm{fr}_p)$ lie in $\mathbf{Z}$ and satisfy the Hasse bound

$$|a_p(E)| \leqslant 2\sqrt{p}.$$

Thus we can define, for each prime p, the corresponding Satake parameter for E.

$$\theta_p(E) = \cos^{-1}\left(\frac{a_p(E)}{2\sqrt{p}}\right) \in [0, \pi).$$

The Satake parameters are packaged into an L-function as follows:

$$L^{\mathrm{an}}(E, s) = \prod_p \frac{1}{(1 - e^{i\theta_p(E)}p^{-s})(1 - e^{-i\theta_p(E)}p^{-s})}.$$

More generally we have, for each $k \geqslant 1$, the k-th symmetric power L-function

$$L^{\mathrm{an}}(\mathrm{sym}^k E, s) = \prod_p \prod_{j=0}^{k} \frac{1}{1 - e^{i(k-2j)\theta_p(E)}p^{-s}}.$$

Numerical experiments suggest that the Satake parameters are distributed with respect to the Sato–Tate distribution $\mathrm{ST} = \frac{2}{\pi}\sin^2\theta \,\mathrm{d}\theta$. The "goodness of fit" of the Satake parameters to the Sato–Tate distribution is quantified by the *discrepancy*:

$$\mathrm{D}^\star(\{\theta_p(E)\}_{p \leqslant X}, \mathrm{ST}) = \sup_{x \in [0,\pi]} \left| \frac{\#\{p \leqslant X : \theta_p(E) \in [0, x)\}}{\pi(X)} - \int_0^x \mathrm{dST} \right|.$$

The decay of the discrepancy is closely related to the analytic properties of the $L(\mathrm{sym}^k E, s)$. First, here is the famous Sato–Tate conjecture (now a theorem) in the language we have defined.

Theorem 1.0.1 (Sato–Tate conjecture)**.** $\mathrm{D}^\star(\{\theta_p(E)\}_{p \leqslant X}, \mathrm{ST}) \to 0$.

Theorem 1.0.2. *The Sato–Tate conjecture for E holds if and only if each of the functions $L(\mathrm{sym}^k E, s)$ have analytic continuation past $\Re s = 1$.*

The stunning recent proof of the Sato–Tate conjecture (citation) in fact showed that the functions $L(\mathrm{sym}^k E, s)$ were potentially automorphic, which gives analytic continuation.

There is an analogy between the above equivalence and classical analytic number theory. Let $K/\mathbf{Q}$ be a finite Galois extension, and $\rho \colon \mathrm{Gal}(K/\mathbf{Q}) \to \mathrm{GL}_n(\mathbf{C})$ an irreducible representation. Recall the Artin L-function is

$$L(\rho, s) = \prod_p \frac{1}{1 - \operatorname{tr} \rho(\mathrm{fr}_p)p^{-s}}.$$

Let $\mathrm{Gal}(K/\mathbf{Q})^\natural$ be the set of conjugacy classes in $\mathrm{Gal}(K/\mathbf{Q})$. The analogue of discrepancy here is:

$$\mathrm{D}(\{\mathrm{fr}_p\}_{p \leqslant X}) = \sup_{c \in \mathrm{Gal}(K/\mathbf{Q})^\natural} \left| \frac{\#\{p \leqslant X : \rho(\mathrm{fr}_p) \in c\}}{\pi(X)} - \frac{1}{\#\,\mathrm{Gal}(K/\mathbf{Q})^\natural} \right|.$$

Theorem 1.0.3. *The "discrepancy" $\mathrm{D}(\{\mathrm{fr}_p\}_{p \leqslant X}) \to 0$ if and only if $L(\rho, s)$ has analytic continuation past $\Re s = 1$ for all non-trivial irreducible representations ρ of $\mathrm{Gal}(K/\mathbf{Q})$.*

In the case of Artin L-functions, we know moreover that

Theorem 1.0.4. *The "discrepancy" satisfies the bound* $\mathrm{D}(\{\mathrm{fr}_p\}_{p\leqslant X}) \ll X^{-1/2+\epsilon}$ *if and only if* $L(\rho, s)$ *satisfies the Riemann Hypothesis for all non-trivial irreducible representation* ρ *of* $\mathrm{Gal}(K/\mathbf{Q})$.

In this context, the "Riemann Hypothesis" for $L(\rho, s)$ means exactly that $\log L(\rho, s)$ has analytic continuation to $\Re s = 1/2$.

The connection between the Riemann Hypothesis and "strong Sato–Tate" generalizes to elliptic curves and more general motives. For the moment, we stick to elliptic curves. In this case, "strong Sato–Tate" was conjectured by Akiyama–Tanigawa. More precisely,

Conjecture:

Let $E_{/\mathbf{Q}}$ be a non-CM elliptic curve. Then $\mathrm{D}^\star(\{\theta_p(E)\}_{p\leqslant X}, \mathrm{ST}) \ll X^{-1/2+\epsilon}$.

Moreover, one side of the equivalence "Riemann Hypothesis $\Leftrightarrow$ strong Sato–Tate" is known.

Theorem 1.0.5. *Let* $E_{/\mathbf{Q}}$ *be an elliptic curve. If the Akiyama–Tanigawa conjecture for* E *holds, then all* $L(\mathrm{sym}^k E, s)$ *satisfy the Riemann Hypothesis.*

It is natural to assume that the converse to this theorem holds. However (and that is the main point of this thesis) it does not! In this thesis, I construct a range of counterexamples to the implication "strong Sato–Tate implies Riemann," and explore why the two are equivalent for Artin L-functions.

I also provide computational evidence for the Akiyama–Tanigawa conjecture (for elliptic curves and also generic abelian 2-folds).

CHAPTER 2
DISCREPANCY

2.1 Definitions and first results

The discrepancy (also known as the Kolmogorov–Smirnov statistic) is a way of measuring how closely sample data fits a predicted distribution. It has many applications in computer science and statistics, but here we will focus on only the basic known properties, as well as how discrepancy changes when sequences are tweaked and/or combined.

Discrepancy will be defined for measures on the d-dimensional half-open box $[0,\infty)^d$. For vectors $x, y \in [0,\infty)^d$, we say $x < y$ if $x_1 < y_1, \ldots, x_d < y_d$, and in that case write $[x,y)$ for the half-open box $[x_1, y_1) \times \cdots \times [x_d, y_d)$.

Definition 2.1.1. *Let μ, ν be probability measures on $[0,\infty)^d$. The* discrepancy *of μ with respect to ν is*

$$\mathrm{D}(\mu,\nu) = \sup_{x<y} |\mu[x,y) - \nu[x,y)|,$$

where $x < y$ range over $[0,\infty)^d$.

The star discrepancy *of μ with respect to ν is*

$$\mathrm{D}^\star(\mu,\nu) = \sup_{0<y} |\mu[0,y) - \nu[0,y)|,$$

where y ranges over $[0,\infty)^d$.

Lemma 2.1.2. *Let μ, ν be Borel measures on $\mathbf{R}^d$. Then*

$$\mathrm{D}^\star(\mu,\nu) \leqslant \mathrm{D}(\mu,\nu) \leqslant 2^d\, \mathrm{D}^\star(\mu,\nu).$$

Proof. The first inequality holds because the supremum defining the discrepancy is taken over a larger set than that defining star discrepancy. To prove the second inequality, let $x < y$ be in $[0,\infty)^d$. For $S \subset \{1,\ldots,d\}$, let

$$I_S = \{t \in [0,y) : t_i < x_i \text{ for all } i \in S\}.$$

The inclusion-exclusion principle for measures tells us that:

$$\mu[x,y) = \sum_{S \subset \{1,\ldots,d\}} (-1)^{\#S} \mu(I_S),$$

and similarly for ν. Since each of the I_S are "half-open boxes" we know that $|\mu(I_S) - \nu(I_S)| \leqslant \mathrm{D}^\star(\mu,\nu)$. It follows that

$$|\mu[x,y) - \nu[x,y)| \leqslant \sum_{S \subset \{1,\ldots,d\}} |\mu(I_S) - \nu(I_S)| \leqslant 2^d\, \mathrm{D}^\star(\mu,\nu).$$

□

We are usually interested in comparing empirical measures and their conjectured distribution. Namely, let $\boldsymbol{x} = \{x_p\}$ be a sequence in $[0,\infty)^d$ indexed by the prime numbers, and μ a Borel measure on $[0\infty)^d$. For any real number $N \geqslant 2$, we write $\boldsymbol{x}^N$ for the empirical measure given by

$$\boldsymbol{x}^N(S) = \frac{1}{\pi(N)} \sum_{p \leqslant N} \delta_{x_p}(S) = \frac{\#\{p \leqslant N : x_p \in S\}}{\pi(N)}.$$

Also, we write $\boldsymbol{x}_{\geqslant N}$ for the truncated sequence $(x_p)_{p\geqslant N}$, and similarly for $\boldsymbol{x}_{\leqslant N}$, etc. In this context,

$$\mathrm{D}^\star(\boldsymbol{x}^N,\nu)=\sup_{y\in[0,\infty)^d}\left|\frac{\#\{p\leqslant N:x_p\in[0,y)\}}{\pi(N)}-\int_{[0,y)}\mathrm{d}\nu\right|.$$

If the measure ν is only defined on a subset of $[0,\infty)^d$, we will tacitly extend it by zero. Moreover, if the sequence $\boldsymbol{x}$ actually lies in a torus $(\mathbf{R}/a\mathbf{Z})^d$, we identify that torus with the $[0,a)^d\subset[0,\infty)^d$. If ν is the Lebesgue measure (on $[0,\infty)^d$) or the normalized Haar measure on the torus, we write $\mathrm{D}^\star(\boldsymbol{x}^N)$ in place of $\mathrm{D}^\star(\boldsymbol{x}^N,\nu)$.

Sometimes the sequence $\boldsymbol{x}$ will not be indexed by the prime numbers, but rather by some other discrete subset of $\mathbf{R}^+$. In that case we will still use the notations $\boldsymbol{x}^N$, $\boldsymbol{x}_{\geqslant N}$, etc., keeping in mind that $\pi(N)$ is replaced by $\#\{\text{indices }\leqslant N\}$.

2.2 The Koksma–Hlawka inequality

Basically just summarize the paper [Ö99].

2.3 Comparing sequences

Lemma 2.3.1. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be sequences in $[0,\infty)$. Suppose $\nu=f\cdot\lambda$ for f a bounded continuous function and λ the Lebesgue measure. Then*

$$\left|\mathrm{D}^\star(\boldsymbol{x}^N,\nu)-\mathrm{D}^\star(\boldsymbol{y}^N,\nu)\right|\leqslant\|f\|_\infty\epsilon+\mathrm{D}^\star(\boldsymbol{x}^N,\nu)+\frac{\#\{p\leqslant N:\|x_p-y_p\|_\infty\geqslant\epsilon\}}{\pi(N)}.$$

Proof. Let $\epsilon>0$ and $t\in[0,\infty)$ be arbitrary. For all $p\leqslant N$ such that $y_p<t$, either $x_p<t+\epsilon$ or $\|x_p-y_p\|_\infty\geqslant\epsilon$. It follows that

$$\boldsymbol{y}^N[0,t)\leqslant\boldsymbol{x}^N[0,t+\epsilon)+\frac{\#\{p\leqslant N:\|x_p-y_p\|_\infty\geqslant\epsilon\}}{\pi(N)}.$$

Moreover, we trivially have

$$\left|\boldsymbol{x}^N[0,t+\epsilon)-\nu[0,t+\epsilon)\right|\leqslant\mathrm{D}^\star(\boldsymbol{x}^N,\nu).$$

Putting these together, we get:

$$\begin{aligned}\boldsymbol{y}^N[0,t)-\nu[0,t)&\leqslant\boldsymbol{x}^N[0,t+\epsilon)-\nu[0,t)+\frac{\#\{p\leqslant N:\|x_p-y_p\|_\infty\geqslant\epsilon\}}{\pi(N)}\\&\leqslant\nu[t,t+\epsilon)+\mathrm{D}^\star(\boldsymbol{x}^N,\nu)+\frac{\#\{p\leqslant N:\|x_p-y_p\|_\infty\geqslant\epsilon\}}{\pi(N)}\\&\leqslant\|f\|_\infty\epsilon+\mathrm{D}^\star(\boldsymbol{x}^N,\nu)+\frac{\#\{p\leqslant N:\|x_p-y_p\|_\infty\geqslant\epsilon\}}{\pi(N)}\end{aligned}$$

as desired. □

Lemma 2.3.2. *Let σ be an isometry of $\mathbf{R}$, and $\boldsymbol{x}$ a sequence in $[0,\infty)$ such that $\sigma(\boldsymbol{x})$ is also in $[0,\infty)$. Let ν be an absolutely continuous measure on $[0,\infty)$ such that $\sigma_*\nu$ is also supported on $[0,\infty)$. Then*

$$\left|\mathrm{D}(\boldsymbol{x}^N,\nu)-\mathrm{D}(\sigma_*\boldsymbol{x}^N,\sigma_*\nu)\right|\leqslant\frac{2}{\pi(N)}.$$

Proof. Every isometry of $\mathbf{R}$ is a combination of translations and reflections. The statement is clear with translations (the two discrepancies are equal). So, suppose $\sigma(t) = a - t$ for some $a > 0$. Since ν is absolutely continuous, $\nu\{t\} = 0$ for all $t \geqslant 0$. In particular, $\nu[s,t) = \nu(s,t]$. In contrast, $\boldsymbol{x}^N\{t\} \leqslant \pi(N)^{-1}$. For any interval $[s,t)$ in $[0,\infty)$, we know that

$$\left|\boldsymbol{x}^N[s,t) - \boldsymbol{x}^N(s,t]\right| \leqslant \frac{2}{\pi(N)},$$

hence

$$\left|\boldsymbol{x}^N[s,t) - \nu[s,t) - (\sigma_*\boldsymbol{x}^N)[a-t,a-s) - (\sigma_*\nu)[a-t,a-s)\right| \leqslant \frac{2}{\pi(N)}.$$

This proves the result. □

2.4 Combining sequences

Definition 2.4.1. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be sequences in $[0,\infty)^d$. We write $\boldsymbol{x} \wr \boldsymbol{y}$ for the interleaved sequence*

$$(x_2, y_2, x_3, y_3, x_5, y_5, \dots, x_p, y_p, \dots).$$

For the interleaved sequence $\boldsymbol{x} \wr \boldsymbol{y}$, we write $(\boldsymbol{x} \wr \boldsymbol{y})^N$ for the empirical measure

$$(\boldsymbol{x} \wr \boldsymbol{y})^N = \frac{1}{2\pi(N)} \sum_{p \leqslant N} \delta_{x_p} + \delta_{y_p}.$$

Theorem 2.4.2. *Let I and J be disjoint open boxes in $[0,\infty)^d$, and let μ, ν be absolutely continuous probability measures on I and J, respectively. Let $\boldsymbol{x}$ be a sequence in I and $\boldsymbol{y}$ be a sequence in J. Then*

$$\max\{\mathrm{D}(\boldsymbol{x}^N, \mu), \mathrm{D}(\boldsymbol{y}^N, \nu)\} \leqslant \mathrm{D}((\boldsymbol{x} \wr \boldsymbol{y})^N, \mu + \nu) \leqslant \mathrm{D}(\boldsymbol{x}^N, \mu) + \mathrm{D}(\boldsymbol{y}^N, \nu)$$

Proof. Any half-open box in $[0,\infty)^d$ can be split by a coordinate hyperplane into two disjoint half-open boxes $[a,b) \sqcup [s,t)$, each of which intersects at most one of I and J. We may assume that $[a,b) \cap J = \varnothing$ and $[s,t) \cap I = \varnothing$. Then

$$\begin{aligned}\left|(\boldsymbol{x} \wr \boldsymbol{y})^N([a,b) \sqcup [s,t)) - (\mu+\nu)([a,b) \sqcup [s,t))\right| &\leqslant |\boldsymbol{x}^N[a,b) - \mu[a,b)| + |\boldsymbol{y}^N[s,t) - \nu[s,t)| \\ &\leqslant \mathrm{D}(\boldsymbol{x}^N, \mu) + \mathrm{D}(\boldsymbol{y}^N, \nu).\end{aligned}$$

This yields the second inequality in the statement of the theorem. To see the first, assume that the maximum discrepancy is $\mathrm{D}(\boldsymbol{x}^N, \mu)$, and let $[s,t)$ be a half-open box such that $|\boldsymbol{x}^N[s,t) - \mu[s,t)|$ is within an arbitrary ϵ of $\mathrm{D}(\boldsymbol{x}^N, \mu)$. We can assume that $[s,t)$ does not intersect J, and thus

$$\left|(\boldsymbol{x} \wr \boldsymbol{y})^N[s,t) - (\mu+\nu)[s,t)\right| = |\boldsymbol{x}^N[s,t) - \mu[s,t)|,$$

which yields the result. □

CHAPTER 3
STRANGE DIRICHLET SERIES

3.1 Definitions

strange Dirichlet series for a series of complex numbers
. . . for a function and a sequence in the domain space

3.2 Relation to automorphic and motivic L-functions

3.3 The Riemann Hypothesis

3.4 Discrepancy of sequences and the Riemann Hypothesis

CHAPTER 4

IRRATIONALITY EXPONENTS

CHAPTER 5
DEFORMATION THEORY

5.1 Category of test objects

The following is an exposition and explication of the theory outlined in [SGA 3_1, VII_B, §0–1]. In particular, we will heavily use the notions of a pseudocompact ring, pseudocompact modules, etc. Let Λ be a pseudocompact ring. Write C_Λ for the opposite of the category of Λ-algebras which have finite length as Λ-modules. Given such a Λ-algebra A, write $X = \mathrm{Spf}(A)$ for the corresponding object of C_Λ, and we put $A = \mathscr{O}(X)$.

Lemma 5.1.1. *Let Λ be a pseudocompact ring, C_Λ as above. Then C_Λ is closed under finite limits and colimits.*

Lemma 5.1.2. *Let Λ be a pseudocompact local ring. Then Λ is henselian, in any of the following senses:*

1. *d*

Proof. [EGA 4_4, 18.5.?] □

Following Grothendieck, if $\mathcal{C}$ is an arbitrary category, we write $\widehat{\mathcal{C}} = \hom(\mathcal{C}^\circ, \mathsf{Set})$ for the category of contravariant functors $\mathcal{C} \to \mathsf{Set}$. We regard $\mathcal{C}$ as a full subcategory of $\widehat{\mathcal{C}}$ via the Yoneda embedding, so for $X, Y \in \mathcal{C}$, we write $X(Y) = \hom_{\mathcal{C}}(Y, X)$. With this notation, the Yoneda Lemma states that $\hom_{\widehat{\mathcal{C}}}(X, P) = P(X)$ for all $X \in \mathcal{C}$.

Lemma 5.1.3. *Let $\mathcal{X} \in \widehat{\mathsf{C}_\Lambda}$. Then $\mathcal{X}$ is left exact if and only if there exists a filtered system $\{X_i\}_{i\in I}$ in $\mathcal{C}_\Lambda$ together with a natural isomorphism $\mathcal{X}(\cdot) \simeq \varinjlim X_i(\cdot)$. Write $\mathsf{Ind}(\mathsf{C}_\Lambda)$ for the category of such functors. Then $\mathsf{Ind}(\mathsf{C}_\Lambda)$ is closed under colimits, and the Yoneda embedding $\mathsf{C}_\Lambda \hookrightarrow \mathsf{Ind}(\mathsf{C}_\Lambda)$ preserves filtered colimits.*

Proof. This follows from the results of [KS06, 6.1]. □

If R is a pseudocompact Λ-algebra, write $\mathrm{Spf}(R)$ for the object of $\widehat{\mathsf{C}_\Lambda}$ defined by $\mathrm{Spf}(R)(A) = \hom_{\mathrm{cts}/\Lambda}(R, A)$, the set of continuous Λ-algebra homomorphisms.

Lemma 5.1.4. *The funtor* Spf *induces an (anti-)equivalence between the category of pseudocompact Λ-algebras and $\mathsf{Ind}(\mathsf{C}_\Lambda)$.*

Proof. This is [SGA 3_1, VII_B 0.4.2 Prop.]. □

So $\mathsf{Ind}(\mathsf{C}_\Lambda)$ is the category of pro-representable functors on finite length Λ-algebras. *Warning*: in many papers, for example the foundational [Maz97], one reserves the term *pro-representable* for functors of the form $\mathrm{Spf}(R)$, where R is *noetherian*. We do not make this restriction.

Lemma 5.1.5. *The category $\mathsf{Ind}(\mathsf{C}_\Lambda)$ is an exponential ideal in $\widehat{\mathsf{C}_\Lambda}$.*

Proof. By this we mean the following. Let $\mathcal{X} \in \mathsf{Ind}(\mathsf{C}_\Lambda)$, $P \in \widehat{\mathsf{C}_\Lambda}$. Then the functor $\mathcal{X}^P$ defined by

$$\mathcal{X}^P(S) = \hom_{\widehat{\mathsf{C}_{\Lambda/S}}}(P_{/S}, \mathcal{X}_{/S})$$

is also in $\mathsf{Ind}(\mathsf{C}_\Lambda)$. Given the characterization of $\mathsf{Ind}(\mathsf{C}_\Lambda)$ as left exact functors, this is easy to prove, see e.g. [Joh02, 4.2.3]. □

If $\mathcal{C}$ is a category, we write $\mathsf{Gp}(\mathcal{C})$ for the category of group objects in $\mathcal{C}$.

Corollary 5.1.6. *Let $\Gamma \in \mathsf{Gp}(\widehat{\mathsf{C}_\Lambda})$ and $\mathcal{G} \in \mathsf{Gp}(\mathsf{Ind}(\mathsf{C}_\Lambda))$, then the functor $[\Gamma, \mathcal{G}]$ defined by*

$$[\Gamma, \mathcal{G}](S) = \hom_{\mathsf{Gp}/S}(\Gamma_{/S}, \mathcal{G}_{/S})$$

is in $\mathsf{Ind}(\mathsf{C}_\Lambda)$. In particular, if Γ is a profinite group, then the functor

$$[\Gamma, \mathcal{G}](S) = \hom_{\mathrm{cts}/\,\mathsf{Gp}}(\Gamma, \mathcal{G}(S))$$

is in $\mathsf{Ind}(\mathsf{C}_\Lambda)$.

Proof. The first claim follows easily from 5.1.5. Just note that $[\Gamma, \mathcal{G}]$ is the equalizer:

$$[\Gamma, \mathcal{G}] \longrightarrow \mathcal{G}^\Gamma \underset{m_{\mathcal{G}*}}{\overset{m_\Gamma^*}{\rightrightarrows}} \mathcal{G}^{\Gamma\times\Gamma},$$

that is, those $f\colon \Gamma \to \mathcal{G}$ such that $f \circ m_\Gamma = m_\mathcal{G} \circ (f \times f)$. The latter claim is just a special case. □

5.2 Quotients in the flat topology

If Λ is a pseudocompact ring, the category $\mathsf{Ind}(\mathsf{C}_\Lambda)$ has nice "geometric" properties. However, for operations like taking quotients, we will embed it into the larger category $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$ of flat sheaves. We call a collection $\{U_i \to X\}$ of morphisms in C_Λ a *flat cover* if each ring map $\mathscr{O}(X) \to \mathscr{O}(U_i)$ is flat, and moreover $\mathscr{O}(X) \to \prod \mathscr{O}(U_i)$ is faithfully flat. By [SGA 3_1, IV 6.3.1], this is a subcanonical Grothendieck topology on C_Λ. We call it the *flat topology*, even though finite presentation comes for free because all the rings are finite length.

Lemma 5.2.1. *Let $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$ be the category of sheaves (of sets) on C_Λ with respect to the flat topology. Then a presheaf $P \in \widehat{\mathsf{C}_\Lambda}$ lies in $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$ if and only if $P(\coprod U_i) = \prod P(U_i)$ and moreover, whenever $U \to X$ is a flat cover where $\mathscr{O}(U)$ and $\mathscr{O}(X)$ are local rings, the sequence*

$$P(X) \longrightarrow P(U) \rightrightarrows P(U \times_X U).$$

is exact. Moreover, $\mathsf{Ind}(\mathsf{C}_\Lambda) \subset \mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$.

Proof. The first claim is the content of [SGA 3_1, IV 6.3.1(ii)]. For the second, note that any $\mathcal{X} \in \mathsf{Ind}(\mathsf{C}_\Lambda)$ will, by 5.1.3, convert (arbitrary) colimits into limits. Thus $\mathcal{X}(\coprod U_i) = \prod \mathcal{X}(U_i)$. If $U \to X$ is a flat cover, then by (loc. cit.), $U \times_X U \rightrightarrows U \to X$ is a coequalizer diagram in C_Λ, hence $\mathcal{X}(X) \to \mathcal{X}(U) \rightrightarrows \mathcal{X}(U \times_X U)$ is an equalizer. □

Our main reason for introducing the category $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$ is that, as a (Grothendieck) topos, it is closed under arbitrary colimits. Recall that in an *equivalence relation* in $\widehat{\mathsf{C}_\Lambda}$ is a morphism $R \to X \times X$ such that, for all S, the map $R(S) \to X(S) \times X(S)$ is an injection whose image is an equivalence relation on $X(S)$. We define the quotient X/R to be the coequalizer

$$R \rightrightarrows X \longrightarrow X/R.$$

By Giraud's Theorem [MLM94, App.], for any $S \in \mathsf{C}_\Lambda$, the natural map $X(S)/R(S) \to (X/R)(S)$ is injective. It will not be surjective in general.

We let $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$ inherit definitions from C_Λ as follows. If P is a property of maps in C_Λ (for example, "flat," or "smooth,") and $f\colon X \to Y$ is a morphism in $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$, we say that f has P if for all $S \in \mathsf{C}_\Lambda$ and $y \in Y(S)$, the pullback $X_S = X \times_Y S$ lies in C_Λ, and the pullback map $X_S \to S$ has property P. For example, if $X = \mathrm{Spf}(R')$ and $Y = \mathrm{Spf}(R)$, then $X \to Y$ has property P if and only if for all finite length A and continuous Λ-algebra maps $R \to A$, the induced map $A \to R' \otimes_R A$ has P.

Theorem 5.2.2. *Let $\mathcal{R} \to \mathcal{X} \times \mathcal{X}$ be an equivalence relation in $\mathsf{Ind}(\mathsf{C}_\Lambda)$ such that one of the maps $\mathcal{R} \to \mathcal{X}$ is flat. Then the quotient $\mathcal{X}/\mathcal{R}$ lies in $\mathsf{Ind}(\mathsf{C}_\Lambda)$, and $\mathcal{X} \to \mathcal{X}/\mathcal{R}$ is a flat cover.*

Proof. This is [SGA 3_1, VII_B 1.4]. □

By [Mat89, 29.7], if k is a field and R is a complete regular local k-algebra, then $R \simeq k[\![t_1, \ldots, t_n]\!]$. In particular, R admits an augmentation $\epsilon\colon R \to k$. There is a general analogue of this result, but first we need a definition.

Definition 5.2.3. *A map $f\colon \mathcal{X} \to \mathcal{Y}$ in $\mathsf{Ind}(\mathsf{C}_\Lambda)$ is a* residual isomorphism *if for all $S = \mathrm{Spf}(k) \in \mathsf{C}_\Lambda$ where k is a field, the map $f\colon \mathcal{X}(S) \to \mathcal{Y}(S)$ is a bijection.*

Lemma 5.2.4. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a smooth map in $\mathsf{Ind}(\mathsf{C}_\Lambda)$ that is a residual isomorphism. Then f admits a section.*

Proof. By [SGA 3_1, VII_B 0.1.1], it suffices to prove the result when $\mathcal{X} = \mathrm{Spf}(R')$, $\mathcal{Y} = \mathrm{Spf}(R)$, for local Λ-algebras $R \to R'$ with the same residue field. Let $k = R/\mathfrak{m}_R \xrightarrow{\sim} R'/\mathfrak{m}_{R'}$ be their common residue field. From the diagram

$$\begin{array}{ccc} R' & \dashrightarrow & R \\ \uparrow & \searrow & \downarrow \\ R & \twoheadrightarrow & k, \end{array}$$

the definition of (formal) smoothness, and a limiting argument involving the finite length quotients $R/\mathfrak{a}$, we obtain the result. □

Corollary 5.2.5. *Let $\mathcal{R} \to \mathcal{X} \times \mathcal{X}$ be an equivalence relation satisfying the hypotheses of 5.2.2. Suppose further that*

1. *One of the maps $\mathcal{R} \to \mathcal{X}$ is smooth, and*
2. *The projection $\mathcal{X} \to \mathcal{X}/\mathcal{R}$ is a residual isomorphism.*

Then $\mathcal{X} \to \mathcal{X}/\mathcal{R}$ admits a section, so $\mathcal{X}(S)/\mathcal{R}(S) \xrightarrow{\sim} (\mathcal{X}/\mathcal{R})(S)$ for all $S \in \mathsf{C}_\Lambda$.

Proof. By 5.2.4, it suffices to prove that $\mathcal{X} \to \mathcal{X}/\mathcal{R}$ is smooth. By [EGA 4_4, 17.7.3(ii)], smoothness can be detected after flat descent. So base-change with respect to the projection $\mathcal{X} \to \mathcal{X}/\mathcal{R}$. In the following commutative diagram

$$\begin{array}{ccc} \mathcal{R} & & \\ \| & \searrow & \\ & \mathcal{X} \times_{\mathcal{X}/\mathcal{R}} \mathcal{X} & \longrightarrow \mathcal{X} \\ & \downarrow & \downarrow \\ & \mathcal{X} & \longrightarrow \mathcal{X}/\mathcal{R} \end{array}$$

we can ensure the smoothness of $\mathcal{R} \to \mathcal{X}$ by our hypotheses. Since $\mathcal{X} \to \mathcal{X}/\mathcal{R}$ is smooth after flat base-change, the original map is smooth. □

Example 5.2.6. The hypothesis on residue fields in 5.2.5 is necessary. To see this, let $\Lambda = k$ be a field, $k \hookrightarrow K$ a finite Galois extension with Galois group G. Then $G \times \mathrm{Spf}(K) \rightrightarrows \mathrm{Spf}(K)$ has quotient $\mathrm{Spf}(k)$, but the map $\mathrm{Spf}(K)(S) \to \mathrm{Spf}(k)(S)$ is *not* surjective for all $S \in \mathsf{C}_k$, e.g. it is not for $S = \mathrm{Spf}(k)$.

Example 5.2.7. The hypothesis of smoothness in 5.2.5 is necessary. To see this, let k be a field of characteristic $p > 0$. Then the formal additive group $\widehat{\mathbf{G}_{\mathrm{a}}} = \mathrm{Spf}(k[\![t]\!])$ has a subgroup $\boldsymbol{\alpha}_p$ defined by

$$\boldsymbol{\alpha}_p(S) = \{s \in \mathscr{O}(S) \colon s^p = 0\}.$$

The quotient $\widehat{\mathbf{G}_{\mathrm{a}}}/\boldsymbol{\alpha}_p$ has as affine coordinate ring $k[\![t^p]\!]$. In particular, the following sequence is exact in the flat topology:

$$0 \longrightarrow \boldsymbol{\alpha}_p \longrightarrow \widehat{\mathbf{G}_{\mathrm{a}}} \xrightarrow{(\cdot)^p} \widehat{\mathbf{G}_{\mathrm{a}}} \longrightarrow 0.$$

It follows that $\boldsymbol{\alpha}_p \times \widehat{\mathbf{G}_{\mathrm{a}}} \rightrightarrows \widehat{\mathbf{G}_{\mathrm{a}}} \xrightarrow{(\cdot)^p} \widehat{\mathbf{G}_{\mathrm{a}}}$ is a coequalizer in $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_k)$ satisfying all the hypothese of 5.2.5 except smoothness. And indeed, as one sees by letting $S = \mathrm{Spf}(A)$ for any non-perfect k-algebra A, the map $(\cdot)^p \colon \widehat{\mathbf{G}_{\mathrm{a}}}(S) \to \widehat{\mathbf{G}_{\mathrm{a}}}(S)$ is *not* surjective for all S.

5.3 Groupoids and quotient stacks

Lemma 5.3.1. *Let $\mathcal{G} \in \mathsf{Ind}(\mathsf{C}_\Lambda)$ be a smooth connected group. Then every $\mathcal{G}$-torsor is trivial.*

Proof. Let $\mathcal{P} \to \mathcal{B}$ be a $\mathcal{G}$-torsor in $\mathsf{Ind}(\mathsf{C}_\Lambda)$. That is, $\mathcal{P}$ has an action of $\mathcal{G}_{\mathcal{S}}$ for which $\mathcal{P} \times_{\mathcal{B}} \mathcal{P} \simeq \mathcal{G} \times \mathcal{P}$ as $\mathcal{G}$-spaces. [... not done...] □

Theorem 5.3.2. *Let $\mathcal{G}$ be a smooth connected group in $\mathsf{Ind}(\mathsf{C}_\Lambda)$, and $\mathcal{X} \in \mathsf{Ind}(\mathsf{C}_\Lambda)$ a $\mathcal{G}$-object. Then the quotient stack $[\mathcal{X}/\mathcal{G}](S)$ has as objects $\mathcal{X}(S)/\mathcal{G}(S)$, but with extra automorphisms?*

Proof. Use triviality of torsors. □

5.4 Deformations of group representations

Let $\Gamma \in \mathsf{Gp}(\widehat{\mathsf{C}_\Lambda})$ and $\mathcal{G} \in \mathsf{Ind}(\mathsf{C}_\Lambda)$. By 5.1.6, the functor

$$\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})(S) = \hom_{\mathsf{Gp}/S}(\Gamma_S, \mathcal{G}_S)$$

is in $\mathsf{Ind}(\mathsf{C}_\Lambda)$. We would like to define an ind-scheme $\mathrm{Rep}(\Gamma, \mathcal{G})$ as "$\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})$ modulo conjugation," but this requires some care. The conjugation action of $\mathcal{G}$ on $\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})$ will have fixed points, so the quotient will be badly behaved. We loosely follow [Til96].

Assume Λ is local, with maximal ideal $\mathfrak{m}$ and residue field $\mathbf{k}$. Fix $\bar{\rho} \in \mathrm{Rep}^{\square}(\Gamma, \mathcal{G})(\mathbf{k})$, i.e. a residual representation $\bar{\rho} \colon \Gamma \to \mathcal{G}(\mathbf{k})$. Let $\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})_{\bar{\rho}}$ be the connected component of $\bar{\rho}$ in $\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})$. Assume that $\mathcal{G}$ and $\mathrm{Z}(\mathcal{G})$ are smooth; then the quotient $\mathcal{G}^{\mathrm{ad}} = \mathcal{G}/\mathrm{Z}(\mathcal{G})$ is also smooth. Let $\mathcal{G}^{\mathrm{ad},\circ}$ be the connected component of 1 in $\mathcal{G}^{\mathrm{ad}}$.

Theorem 5.4.1. *Suppose $(\Lambda, \mathfrak{m}, \mathbf{k})$ is local. If $\mathcal{X}, \mathcal{Y} \in \mathsf{Ind}(\mathsf{C}_\Lambda)$ are connected and $\mathcal{X}(\mathbf{k}) \neq \varnothing$, then $\mathcal{X} \times_\Lambda \mathcal{Y}$ is connected.*

Proof. We are reduced to proving the following result from commutative algebra: if R, S are local pro-artinian Λ-algebras and R has residue field $\mathbf{k}$, then $R \widehat{\otimes}_\Lambda S$ is local. Since $R \widehat{\otimes}_\Lambda S = \varprojlim (R/\mathfrak{r}) \otimes_\Lambda (S/\mathfrak{s})$, $\mathfrak{r}$ (resp. $\mathfrak{s}$) ranges over all open ideals in R (resp. S), we may assume that both R and S are artinian. The rings R and S are henselian, so $R \otimes S$ is local if and only if $(R/\mathfrak{m}_R) \otimes (S/\mathfrak{m}_S) = S/\mathfrak{m}_S$ is local, which it is. □

We conclude that the action of $\mathcal{G}^{\mathrm{ad},\circ}$ on $\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})$ preserves $\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})_{\bar{\rho}}$. Thus we may put

$$\mathrm{Rep}(\Gamma, \mathcal{G})_{\bar{\rho}} = [\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})_{\bar{\rho}}/\mathcal{G}^{\mathrm{ad},\circ}].$$

If $\mathcal{G}^{\mathrm{ad},\circ}$ acts faithfully on $\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})_{\bar{\rho}}$, then we recover the classical notion of the deformation functor.

Theorem 5.4.2. *Let Γ be a profinite group, $\bar{\rho}\colon \Gamma \to \mathcal{G}(\mathbf{k})$ a representation with $\mathrm{H}^0(\Gamma, \operatorname{Ad}\bar{\rho}) = 0$. Then $\operatorname{Rep}(\Gamma, \mathcal{G})_{\bar{\rho}}$ exists and is what you expect.*

Proof. Need assumptions on $\mathrm{Z}(\mathcal{G})$, $\mathcal{G}$ should be smooth.

Need $\mathrm{Z}(\mathcal{G}) = \ker(\mathcal{G} \to \mathrm{GL}(\mathfrak{g}))$ in connected case. This should use $\mathfrak{g} = \operatorname{Lie}(\operatorname{Aut}\mathcal{G})$, via deviations in [SGA 3_1]. □

[. . . local conditions]

5.5 Tangent spaces and obstruction theory

For $S_0 \in \mathsf{C}_\Lambda$, let Ex_{S_0} be the category of square-zero thickenings of S_0. An object of Ex_{S_0} is a closed embedding $S_0 \hookrightarrow S$ whose ideal of definition has square zero. Should be "exponential exact sequence"

$$0 \longrightarrow \mathfrak{g}(I) \longrightarrow \mathcal{G}(S) \longrightarrow \mathcal{G}(S_0) \longrightarrow 1$$

This gives us a class $\exp \in \mathrm{H}^2(\mathcal{G}(S_0), \mathfrak{g}(I))$. For $\rho_0\colon \Gamma \to \mathcal{G}(S_0)$, the obstruction class is $o(\rho_0, I) = \rho_0^*(\exp) \in \mathrm{H}^2(\Gamma, \mathfrak{g}(I))$. It's easy to check that $o(\rho_0, I) = 0$ if and only if ρ_0 lifts to ρ. So obstruction theory naturally for $\operatorname{Rep}^{\square}(\Gamma, \mathcal{G})$.

[Use [Wei94, 6.6.4]. Given setting as above, $\rho_0^*(\exp)$ is the pullback by ρ_0:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{g}(I) & \longrightarrow & \mathcal{G}(S) \times_{\mathcal{G}(S_0)} \Gamma & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \rho_0} & & \\
0 & \longrightarrow & \mathfrak{g}(I) & \longrightarrow & \mathcal{G}(S) & \longrightarrow & \mathcal{G}(S_0) & \longrightarrow & 1
\end{array}$$

Computing explicitly, we see the result.]

Proposition 5.5.1. *Let $f\colon G \to H$ be a morphism of profinite groups. Suppose M is a discrete H-module and $c \in \mathrm{H}^2(H, M)$ corresponds to the extension*

$$0 \longrightarrow M \longrightarrow \widetilde{H} \longrightarrow H \longrightarrow 1.$$

Then $f^*c = 0$ in $\mathrm{H}^2(G, M)$ if and only if there is a map $\widetilde{f}\colon G \to \widetilde{H}$ making the following diagram commute:

$$\begin{array}{ccc}
 & & \widetilde{H} \\
 & \overset{\widetilde{f}}{\nearrow} & \downarrow \\
G & & \\
 & \underset{f}{\searrow} & \downarrow \\
 & & H.
\end{array}$$

Proof. By [Wei94, 6.6.4], the class f^*c corresponds to the pullback diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & G \times_H \widetilde{H} & \longrightarrow & G & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & M & \longrightarrow & \widetilde{H} & \longrightarrow & H & \longrightarrow & 1.
\end{array}$$

Writing explicitly what it means for $G \times_H \widetilde{H} \to G$ to split yields the result. □

Let $\mathcal{X} \in \mathsf{Ind}(\mathsf{C}_{/\Lambda})$ be smooth, and $\mathrm{L}_{\mathcal{X}/\Lambda} \simeq \Omega^1_{\mathcal{X}/\Lambda}[0]$ be its cotangent complex. Fix $x_0 \in \mathcal{X}(S_0)$. From the chain $S_0 \xrightarrow{x_0} \mathcal{X} \to \operatorname{Spf}(\Lambda)$, we get a distinguished triangle [Ill71, II 2.1.5.6]

$$x_0^*\mathrm{L}_{\mathcal{X}/\Lambda} \longrightarrow \mathrm{L}_{S_0/\Lambda} \longrightarrow \mathrm{L}_{S_0/\mathcal{X}} \longrightarrow .$$

If I is a coherent sheaf on S_0, we get a long exact sequence:

$$\mathrm{Ext}^0(\mathrm{L}_{S_0/\Lambda}, M) \longrightarrow \mathrm{Ext}^0(x_0^*\mathrm{L}_{\mathcal{X}/\Lambda}, M) \longrightarrow \mathrm{Ext}^1(\mathrm{L}_{S_0/\mathcal{X}}, M) \longrightarrow \mathrm{Ext}^1(\mathrm{L}_{S_0/\Lambda}, M) \longrightarrow \mathrm{Ext}^1(x_0^*\mathrm{L}_{\mathcal{X}/\Lambda}, M)$$

If $\mathcal{X}_{/\Lambda}$ is smooth, then $\mathrm{Ext}^1(x_0^*\mathrm{L}_{\mathcal{X}/\Lambda}, M) = 0$ and $\mathrm{L}_{\mathcal{X}/\Lambda} = \Omega^1_{\mathcal{X}/\Lambda}$. This gives us an exact sequence

$$\mathrm{Ext}^0(\mathrm{L}_{S_0/\Lambda}, M) \longrightarrow \hom(\Omega^1_{\mathcal{X}/\Lambda}, M) \longrightarrow \mathrm{Ext}^1(\mathrm{L}_{S_0/\mathcal{X}}, M) \longrightarrow \mathrm{Ext}^1(\mathrm{L}_{S_0/\Lambda}, M) \longrightarrow 0.$$

The result [Ill71, III 2.1.7] tells us that the choice of $S \in \mathsf{Ex}_{S_0}(M)$ gives us an element of $\mathrm{Ext}^1(\mathrm{L}_{S_0/\Lambda}, M)$. Its fiber admits an action of $\hom(\Omega^1_{\mathcal{X}/\Lambda}, M)$. The only thing remaining is: we need $\mathrm{Ext}^0(\mathrm{L}_{S_0/\Lambda}, M) = 0$, which doesn't hold in complete generality.

CHAPTER 6

CONSTRUCTING GALOIS REPRESENTATIONS

CHAPTER 7

FIRST COUNTEREXAMPLE

CHAPTER 8

SECOND COUNTEREXAMPLE

CHAPTER 9

COMPUTATIONAL EVIDENCE FOR THE AKIYAMA–TANIGAWA CONJECTURE

CHAPTER 10

CONCLUDING REMARKS AND FUTURE DIRECTIONS

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