

# CONSTRUCTING GALOIS REPRESENTATIONS WITH PRESCRIBED SATO–TATE DISTRIBUTION

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SATO–TATE DISTRIBUTION

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Abstract here.

## **BIOGRAPHICAL SKETCH**

Daniel Miller was born in St. Paul, Minnesota. He completed his Bachelor of Science at the University of Nebraska–Omaha, during which he attended Cornell’s Summer Mathematics Institute in 2011. He started his Ph.D. at Cornell planning on a career in academia, but halfway through had a change of heart, and will be joining Microsoft’s Analysis and Experimentation team as a Data Scientist after graduation.

This thesis is dedicated to my undergraduate thesis advisor, Griff Elder. He is the reason I considered a career in math, his infectious enthusiasm for number theory has inspired me more than I can say.

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# CHAPTER 1

## INTRODUCTION

### 1.1 Motivation from classical analytic number theory

We start with a problem near and dear to every number theorist's heart: counting prime numbers. As usual, let  $\pi(x)$  be the number of rational primes  $\leq x$  and  $\text{Li}(x) = \int_2^x \frac{dt}{\log t}$  be the logarithmic integral. For any  $x$ , we have the normalized empirical measure

$$P_x = \frac{1}{\pi(x)} \sum_{p \leq x} \delta_{p/x},$$

which is supported on the unit interval  $[0, 1]$ . The prime number theorem tells us that as  $x \rightarrow \infty$ , these empirical measures approach the “true” measure  $L_x = \frac{\text{Li}(tx)}{\text{Li}(x)} dt$ . This is proven via demonstrating analytic properties of the Riemann zeta function.

**Theorem 1.1.1.** *We have  $P_x \rightarrow L_x$  (in the weak sense) if and only if  $\zeta(s)$  admits a meromorphic continuation past  $\{\Re = 1\}$ , with at most a simple pole at  $s = 1$ .*

Since  $\zeta(s)$  has the desired properties, the prime number theorem is true. It is natural to try to quantify the rate of converge of  $P_x$  to  $L_x$ . One easy way to do this is via the discrepancy

$$D(P_x, L_x) = \sup_{t \in [0, 1]} |P_x[0, t] - L_x[0, t]| = \sup_{t \in [0, 1]} \left| \frac{\pi(tx)}{\pi(x)} - \frac{\int_2^{tx} \frac{ds}{\log s}}{\int_2^x \frac{ds}{\log s}} \right|.$$

Numerical experiments suggest that  $D(P_x, L_x) \ll x^{-\frac{1}{2}+\epsilon}$ , and in fact we have the following result.

**Theorem 1.1.2.** *We have  $D(P_x, L_x) \ll x^{-\frac{1}{2}+\epsilon}$  if and only if the Riemann Hypothesis holds.*

Of course here, neither side is known for certain to be true! This discussion finds a natural generalization in Artin  $L$ -functions.

Let  $K/\mathbf{Q}$  be a finite Galois extension with group  $G = \text{Gal}(K/\mathbf{Q})$ . For any irreducible representation  $\rho: G \rightarrow \text{GL}_d(\mathbf{C})$ , there is a corresponding  $L$ -function defined as

$$L(\rho, s) = \prod_p \frac{1}{\det(1 - \rho(\text{fr}_p)p^{-s})},$$

where we tacitly omit those primes  $p$  at which  $\rho$  is ramified. Given a cutoff  $x$ , there is a natural empirical measure

$$P_x = \frac{1}{\pi(x)} \sum_{p \leq x} \delta_{\text{fr}_p},$$

where  $\text{fr}_p$  is a conjugacy class in  $G$ . Let

$$D(P_x) = \sup_{c \in G^\natural} \left| P_x(c) - \frac{1}{\#G} \right|,$$

where  $G^\natural$  is the set of conjugacy classes in  $G$ .

**Theorem 1.1.3.** *We have convergence  $P_x \rightarrow L$  if and only if  $L(\rho, s)$  admits analytic continuation past  $\{\Re = 1\}$  for all nontrivial  $\rho$ .*

Both sides of this equivalence are true, and known as the Chebotarev density theorem. Moreover, there is a version of the strong Prime Number Theorem in this context.

**Theorem 1.1.4.** *We have  $D(P_x) \ll x^{-\frac{1}{2}+\epsilon}$  if and only if  $L(\rho, s)$  satisfies the Riemann Hypothesis for all nontrivial  $\rho$ .*

## 1.2 Discrepancy and Riemann Hypothesis for elliptic curves

Let's start with something basic, an elliptic curve  $E/\mathbf{Q}$ . For any prime  $l$ , we have the Tate module of  $E$ , written  $T_l E$ . This is a rank-2  $\mathbf{Z}_l$ -module with continuous

$G_{\mathbf{Q}}$ -action, so it induces a continuous representation

$$\rho_{E,l}: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{Z}_l).$$

It is known (citation?) that the quantities  $a_p(E) = \mathrm{tr} \rho_l(\mathrm{fr}_p)$  lie in  $\mathbf{Z}$  and satisfy the Hasse bound

$$|a_p(E)| \leq 2\sqrt{p}.$$

Thus we can define, for each prime  $p$ , the corresponding Satake parameter for  $E$ .

$$\theta_p(E) = \cos^{-1} \left( \frac{a_p(E)}{2\sqrt{p}} \right) \in [0, \pi).$$

The Satake parameters are packaged into an  $L$ -function as follows:

$$L^{\mathrm{an}}(E, s) = \prod_p \frac{1}{(1 - e^{i\theta_p(E)} p^{-s})(1 - e^{-i\theta_p(E)} p^{-s})}.$$

More generally we have, for each  $k \geq 1$ , the  $k$ -th symmetric power  $L$ -function

$$L^{\mathrm{an}}(\mathrm{sym}^k E, s) = \prod_p \prod_{j=0}^k \frac{1}{1 - e^{i(k-2j)\theta_p(E)} p^{-s}}.$$

Numerical experiments suggest that the Satake parameters are distributed with respect to the Sato–Tate distribution  $\mathrm{ST} = \frac{2}{\pi} \sin^2 \theta \, d\theta$ . The “goodness of fit” of the Satake parameters to the Sato–Tate distribution is quantified by the *discrepancy*:

$$D^*(\{\theta_p(E)\}_{p \leq X}, \mathrm{ST}) = \sup_{x \in [0, \pi]} \left| \frac{\#\{p \leq X : \theta_p(E) \in [0, x]\}}{\pi(X)} - \int_0^x d\mathrm{ST} \right|.$$

The decay of the discrepancy is closely related to the analytic properties of the  $L(\mathrm{sym}^k E, s)$ . First, here is the famous Sato–Tate conjecture (now a theorem) in the language we have defined.

**Theorem 1.2.1** (Sato–Tate conjecture).  $D^*(\{\theta_p(E)\}_{p \leq X}, \mathrm{ST}) \rightarrow 0$ .

**Theorem 1.2.2.** *The Sato–Tate conjecture for  $E$  holds if and only if each of the functions  $L(\mathrm{sym}^k E, s)$  have analytic continuation past  $\Re s = 1$ .*

The stunning recent proof of the Sato–Tate conjecture (citation) in fact showed that the functions  $L(\mathrm{sym}^k E, s)$  were potentially automorphic, which gives analytic continuation.

There is an analogy between the above equivalence and classical analytic number theory. Let  $K/\mathbf{Q}$  be a finite Galois extension, and  $\rho: \mathrm{Gal}(K/\mathbf{Q}) \rightarrow \mathrm{GL}_n(\mathbf{C})$  an irreducible representation. Recall the Artin  $L$ -function is

$$L(\rho, s) = \prod_p \frac{1}{1 - \mathrm{tr} \rho(\mathrm{fr}_p) p^{-s}}.$$

Let  $\mathrm{Gal}(K/\mathbf{Q})^\natural$  be the set of conjugacy classes in  $\mathrm{Gal}(K/\mathbf{Q})$ . The analogue of discrepancy here is:

$$D(\{\mathrm{fr}_p\}_{p \leq X}) = \sup_{c \in \mathrm{Gal}(K/\mathbf{Q})^\natural} \left| \frac{\#\{p \leq X : \rho(\mathrm{fr}_p) \in c\}}{\pi(X)} - \frac{1}{\#\mathrm{Gal}(K/\mathbf{Q})^\natural} \right|.$$

**Theorem 1.2.3.** *The “discrepancy”  $D(\{\mathrm{fr}_p\}_{p \leq X}) \rightarrow 0$  if and only if  $L(\rho, s)$  has analytic continuation past  $\Re s = 1$  for all non-trivial irreducible representations  $\rho$  of  $\mathrm{Gal}(K/\mathbf{Q})$ .*

In the case of Artin  $L$ -functions, we know moreover that

**Theorem 1.2.4.** *The “discrepancy” satisfies the bound  $D(\{\mathrm{fr}_p\}_{p \leq X}) \ll X^{-1/2+\epsilon}$  if and only if  $L(\rho, s)$  satisfies the Riemann Hypothesis for all non-trivial irreducible representation  $\rho$  of  $\mathrm{Gal}(K/\mathbf{Q})$ .*

In this context, the “Riemann Hypothesis” for  $L(\rho, s)$  means exactly that  $\log L(\rho, s)$  has analytic continuation to  $\Re s = 1/2$ .

The connection between the Riemann Hypothesis and “strong Sato–Tate” generalizes to elliptic curves and more general motives. For the moment, we stick to elliptic curves. In this case, “strong Sato–Tate” was conjectured by Akiyama–Tanigawa. More precisely,

Conjecture:

Let  $E/\mathbf{Q}$  be a non-CM elliptic curve. Then  $D^*(\{\theta_p(E)\}_{p \leq X}, \text{ST}) \ll X^{-1/2+\epsilon}$ .

Moreover, one side of the equivalence “Riemann Hypothesis  $\Leftrightarrow$  strong Sato–Tate” is known.

**Theorem 1.2.5.** *Let  $E/\mathbf{Q}$  be an elliptic curve. If the Akiyama–Tanigawa conjecture for  $E$  holds, then all  $L(\text{sym}^k E, s)$  satisfy the Riemann Hypothesis.*

It is natural to assume that the converse to this theorem holds. However (and that is the main point of this thesis) it does not! In this thesis, I construct a range of counterexamples to the implication “strong Sato–Tate implies Riemann,” and explore why the two are equivalent for Artin  $L$ -functions.

I also provide computational evidence for the Akiyama–Tanigawa conjecture (for elliptic curves and also generic abelian 2-folds).

Similar work: [Pan11].

To-do: conjectural framework? Can I find the rank from  $\{\text{sign}(a_p)\}$ ?

See [Maz08] for a nice discussion.

## CHAPTER 2

### DISCREPANCY

#### 2.1 Definitions and first results

The discrepancy (also known as the Kolmogorov–Smirnov statistic) is a way of measuring how closely sample data fits a predicted distribution. It has many applications in computer science and statistics, but here we will focus on only the basic known properties, as well as how discrepancy changes when sequences are tweaked and/or combined.

Discrepancy will be defined for measures on the  $d$ -dimensional half-open box  $[0, \infty)^d$ . For vectors  $x, y \in [0, \infty)^d$ , we say  $x < y$  if  $x_1 < y_1, \dots, x_d < y_d$ , and in that case write  $[x, y)$  for the half-open box  $[x_1, y_1) \times \dots \times [x_d, y_d)$ .

**Definition 2.1.1.** *Let  $\mu, \nu$  be probability measures on  $[0, \infty)^d$ . The discrepancy of  $\mu$  with respect to  $\nu$  is*

$$D(\mu, \nu) = \sup_{x < y} |\mu[x, y) - \nu[x, y)|,$$

where  $x < y$  range over  $[0, \infty)^d$ .

The star discrepancy of  $\mu$  with respect to  $\nu$  is

$$D^*(\mu, \nu) = \sup_{0 < y} |\mu[0, y) - \nu[0, y)|,$$

where  $y$  ranges over  $[0, \infty)^d$ .

**Lemma 2.1.2.** *Let  $\mu, \nu$  be Borel measures on  $\mathbf{R}^d$ . Then*

$$D^*(\mu, \nu) \leq D(\mu, \nu) \leq 2^d D^*(\mu, \nu).$$

*Proof.* The first inequality holds because the supremum defining the discrepancy is taken over a larger set than that defining star discrepancy. To prove the second inequality, let  $x < y$  be in  $[0, \infty)^d$ . For  $S \subset \{1, \dots, d\}$ , let

$$I_S = \{t \in [0, y) : t_i < x_i \text{ for all } i \in S\}.$$

The inclusion-exclusion principle for measures tells us that:

$$\mu[x, y) = \sum_{S \subset \{1, \dots, d\}} (-1)^{\#S} \mu(I_S),$$

and similarly for  $\nu$ . Since each of the  $I_S$  are “half-open boxes” we know that  $|\mu(I_S) - \nu(I_S)| \leq D^*(\mu, \nu)$ . It follows that

$$|\mu[x, y) - \nu[x, y)| \leq \sum_{S \subset \{1, \dots, d\}} |\mu(I_S) - \nu(I_S)| \leq 2^d D^*(\mu, \nu).$$

□

We are usually interested in comparing empirical measures and their conjectured distribution. Namely, let  $\mathbf{x} = \{x_p\}$  be a sequence in  $[0, \infty)^d$  indexed by the prime numbers, and  $\mu$  a Borel measure on  $[0, \infty)^d$ . For any real number  $N \geq 2$ , we write  $\mathbf{x}^N$  for the empirical measure given by

$$\mathbf{x}^N(S) = \frac{1}{\pi(N)} \sum_{p \leq N} \delta_{x_p}(S) = \frac{\#\{p \leq N : x_p \in S\}}{\pi(N)}.$$

Also, we write  $\mathbf{x}_{\geq N}$  for the truncated sequence  $(x_p)_{p \geq N}$ , and similarly for  $\mathbf{x}_{\leq N}$ , etc.

In this context,

$$D^*(\mathbf{x}^N, \nu) = \sup_{y \in [0, \infty)^d} \left| \frac{\#\{p \leq N : x_p \in [0, y)\}}{\pi(N)} - \int_{[0, y)} d\nu \right|.$$

If the measure  $\nu$  is only defined on a subset of  $[0, \infty)^d$ , we will tacitly extend it by zero. Moreover, if the sequence  $\mathbf{x}$  actually lies in a torus  $(\mathbf{R}/a\mathbf{Z})^d$ , we identify

that torus with the  $[0, a)^d \subset [0, \infty)^d$ . If  $\nu$  is the Lebesgue measure (on  $[0, \infty)^d$ ) or the normalized Haar measure on the torus, we write  $D^*(\mathbf{x}^N)$  in place of  $D^*(\mathbf{x}^N, \nu)$ .

Sometimes the sequence  $\mathbf{x}$  will not be indexed by the prime numbers, but rather by some other discrete subset of  $\mathbf{R}^+$ . In that case we will still use the notations  $\mathbf{x}^N$ ,  $\mathbf{x}_{\geq N}$ , etc., keeping in mind that  $\pi(N)$  is replaced by  $\#\{\text{indices} \leq N\}$ .

Todo: give some basic examples of equidistributed sequences, talk about equidistribution, almost-everywhere continuous functions. Prove basic facts about van der Corput sequence for arbitrary measures.

## 2.2 The Koksma–Hlawka inequality

Basically just summarize the paper [Ö99].

**Theorem 2.2.1** (Koksma–Hlawka). *Todo.*

## 2.3 Comparing sequences

**Lemma 2.3.1.** *Let  $\mathbf{x}$  and  $\mathbf{y}$  be sequences in  $[0, \infty)$ . Suppose  $\nu = f \cdot \lambda$  for  $f$  a bounded continuous function and  $\lambda$  the Lebesgue measure. Then*

$$|D^*(\mathbf{x}^N, \nu) - D^*(\mathbf{y}^N, \nu)| \leq \|f\|_\infty \epsilon + D^*(\mathbf{x}^N, \nu) + \frac{\#\{p \leq N : \|x_p - y_p\|_\infty \geq \epsilon\}}{\pi(N)}.$$

*Proof.* Let  $\epsilon > 0$  and  $t \in [0, \infty)$  be arbitrary. For all  $p \leq N$  such that  $y_p < t$ , either  $x_p < t + \epsilon$  or  $\|x_p - y_p\|_\infty \geq \epsilon$ . It follows that

$$\mathbf{y}^N[0, t) \leq \mathbf{x}^N[0, t + \epsilon) + \frac{\#\{p \leq N : \|x_p - y_p\|_\infty \geq \epsilon\}}{\pi(N)}.$$

Moreover, we trivially have

$$|\mathbf{x}^N[0, t + \epsilon) - \nu[0, t + \epsilon)| \leq D^*(\mathbf{x}^N, \nu).$$



Putting these together, we get:

$$\begin{aligned}
\mathbf{y}^N[0, t) - \nu[0, t) &\leq \mathbf{x}^N[0, t + \epsilon) - \nu[0, t) + \frac{\#\{p \leq N : \|x_p - y_p\|_\infty \geq \epsilon\}}{\pi(N)} \\
&\leq \nu[t, t + \epsilon) + D^*(\mathbf{x}^N, \nu) + \frac{\#\{p \leq N : \|x_p - y_p\|_\infty \geq \epsilon\}}{\pi(N)} \\
&\leq \|f\|_\infty \epsilon + D^*(\mathbf{x}^N, \nu) + \frac{\#\{p \leq N : \|x_p - y_p\|_\infty \geq \epsilon\}}{\pi(N)}
\end{aligned}$$

as desired.  $\square$

**Lemma 2.3.2.** *Let  $\sigma$  be an isometry of  $\mathbf{R}$ , and  $\mathbf{x}$  a sequence in  $[0, \infty)$  such that  $\sigma(\mathbf{x})$  is also in  $[0, \infty)$ . Let  $\nu$  be an absolutely continuous measure on  $[0, \infty)$  such that  $\sigma_*\nu$  is also supported on  $[0, \infty)$ . Then*

$$|D(\mathbf{x}^N, \nu) - D(\sigma_*\mathbf{x}^N, \sigma_*\nu)| \leq \frac{2}{\pi(N)}.$$

*Proof.* Every isometry of  $\mathbf{R}$  is a combination of translations and reflections. The statement is clear with translations (the two discrepancies are equal). So, suppose  $\sigma(t) = a - t$  for some  $a > 0$ . Since  $\nu$  is absolutely continuous,  $\nu\{t\} = 0$  for all  $t \geq 0$ . In particular,  $\nu[s, t) = \nu(s, t]$ . In contrast,  $\mathbf{x}^N\{t\} \leq \pi(N)^{-1}$ . For any interval  $[s, t)$  in  $[0, \infty)$ , we know that

$$|\mathbf{x}^N[s, t) - \mathbf{x}^N(s, t]| \leq \frac{2}{\pi(N)},$$

hence

$$|\mathbf{x}^N[s, t) - \nu[s, t) - (\sigma_*\mathbf{x}^N)[a - t, a - s) - (\sigma_*\nu)[a - t, a - s)| \leq \frac{2}{\pi(N)}.$$

This proves the result.  $\square$

## 2.4 Combining sequences

**Definition 2.4.1.** Let  $\mathbf{x}$  and  $\mathbf{y}$  be sequences in  $[0, \infty)^d$ . We write  $\mathbf{x} \wr \mathbf{y}$  for the interleaved sequence

$$(x_2, y_2, x_3, y_3, x_5, y_5, \dots, x_p, y_p, \dots).$$

For the interleaved sequence  $\mathbf{x} \wr \mathbf{y}$ , we write  $(\mathbf{x} \wr \mathbf{y})^N$  for the empirical measure

$$(\mathbf{x} \wr \mathbf{y})^N = \frac{1}{2\pi(N)} \sum_{p \leq N} \delta_{x_p} + \delta_{y_p}.$$

**Theorem 2.4.2.** Let  $I$  and  $J$  be disjoint open boxes in  $[0, \infty)^d$ , and let  $\mu, \nu$  be absolutely continuous probability measures on  $I$  and  $J$ , respectively. Let  $\mathbf{x}$  be a sequence in  $I$  and  $\mathbf{y}$  be a sequence in  $J$ . Then

$$\max\{D(\mathbf{x}^N, \mu), D(\mathbf{y}^N, \nu)\} \leq D((\mathbf{x} \wr \mathbf{y})^N, \mu + \nu) \leq D(\mathbf{x}^N, \mu) + D(\mathbf{y}^N, \nu)$$

*Proof.* Any half-open box in  $[0, \infty)^d$  can be split by a coordinate hyperplane into two disjoint half-open boxes  $[a, b) \sqcup [s, t)$ , each of which intersects at most one of  $I$  and  $J$ . We may assume that  $[a, b) \cap J = \emptyset$  and  $[s, t) \cap I = \emptyset$ . Then

$$\begin{aligned} |(\mathbf{x} \wr \mathbf{y})^N([a, b) \sqcup [s, t)) - (\mu + \nu)([a, b) \sqcup [s, t))| &\leq |\mathbf{x}^N[a, b) - \mu[a, b)| + |\mathbf{y}^N[s, t) - \nu[s, t)| \\ &\leq D(\mathbf{x}^N, \mu) + D(\mathbf{y}^N, \nu). \end{aligned}$$

This yields the second inequality in the statement of the theorem. To see the first, assume that the maximum discrepancy is  $D(\mathbf{x}^N, \mu)$ , and let  $[s, t)$  be a half-open box such that  $|\mathbf{x}^N[s, t) - \mu[s, t)|$  is within an arbitrary  $\epsilon$  of  $D(\mathbf{x}^N, \mu)$ . We can assume that  $[s, t)$  does not intersect  $J$ , and thus

$$|(\mathbf{x} \wr \mathbf{y})^N[s, t) - (\mu + \nu)[s, t)| = |\mathbf{x}^N[s, t) - \mu[s, t)|,$$

which yields the result. □

# CHAPTER 3

## STRANGE DIRICHLET SERIES

### 3.1 Definitions

We start by considering a very general class of Dirichlet series. In fact, they are all Dirichlet series that admit a product formula with degree-1 factors, but in this thesis they will be called strange Dirichlet series. The motivating example was suggested by Ramakrishna. Let  $E/\mathbf{Q}$  be an elliptic curve and let

$$L_{\text{sgn}}(E, s) = \prod_p \frac{1}{1 - \text{sgn}(a_p)p^{-s}}.$$

How much can we say about the behavior of  $L_{\text{sgn}}(E, s)$ ? For example, does it “know” the rank of  $E$ ?

**Definition 3.1.1.** Let  $\mathbf{z} = (z_2, z_3, z_5, \dots)$  be a sequence of complex numbers indexed by the primes. The associated strange Dirichlet series is

$$L(\mathbf{z}, s) = \prod_p \frac{1}{1 - z_p p^{-s}}.$$

If  $z_p$  is only defined for all but finitely many primes, then we tacitly set  $\mathbf{z}_p = 0$  for all primes for which  $z_p$  is not defined.

**Lemma 3.1.2.** Let  $\mathbf{z}$  be a sequence with  $\|\mathbf{z}\|_\infty \leq 1$ . Then  $L(\mathbf{z}, s)$  defines a holomorphic function on the region  $\{\Re s > 1\}$ . Moreover, on that region,

$$\log L(\mathbf{z}, s) = \sum_{p^r} \frac{z_p^n}{np^{ns}}.$$

*Proof.* Expanding the product for  $L(\mathbf{z}, s)$  formally, we have

$$L(\mathbf{z}, s) = \sum_{n \geq 1} \frac{\prod_p z_p^{v_p(n)}}{n^s}.$$

An easy comparison with the Riemann zeta function tells us that this sum is holomorphic on  $\{\Re s > 1\}$ . By [Apo76, Th. 11.7], the product formula holds in the same region. The formula for  $\log L(\mathbf{z}, s)$  comes from [Apo76, 11.9 Ex.2].  $\square$

**Lemma 3.1.3** (Abel summation). *Let  $\mathbf{z} = (z_2, z_3, z_5, \dots)$  be a sequence of complex numbers,  $f$  a smooth complex-valued function on  $\mathbf{R}$ . Then*

$$\sum_{p \leq N} f(p) z_p = f(N) \sum_{p \leq N} z_p - \int_2^N f'(x) \sum_{p \leq x} z_p \, dx.$$

*Proof.* Simply note that if  $p_1, \dots, p_n$  is an enumeration of the primes  $\leq N$ , we have

$$\begin{aligned} \int_2^N f'(x) \sum_{p \leq x} z_p \, dx &= \sum_{p \leq N} z_p \int_{p_n}^N f' + \sum_{i=1}^{n-1} \sum_{p \leq p_{i+1}} z_p \int_{p_i}^{p_{i+1}} f' \\ &= (f(N) - f(p_n)) \sum_{p \leq N} z_p + \sum_{i=1}^{n-1} (f(p_{i+1}) - f(p_i)) \sum_{p \leq p_{i+1}} z_p \\ &= f(N) \sum_{p \leq N} z_p - \sum_{p \leq N} f(p) z_p, \end{aligned}$$

as desired.  $\square$

**Theorem 3.1.4.** *Assume  $|\sum_{p \leq x} z_p| \ll x^{\alpha+\epsilon}$  for some  $\alpha \in [\frac{1}{2}, 1]$ . Then the series for  $\log L(\mathbf{z}, s)$  converges to a holomorphic function on the region  $\{\Re s > \alpha\}$ .*

*Proof.* Formally split the sum for  $\log L(\mathbf{z}, s)$  into two pieces:

$$\log L(\mathbf{z}, s) = \sum_p \frac{z_p}{p^s} + \sum_p \sum_{r \geq 2} \frac{z_p^r}{r p^{rs}}.$$

For each  $p$ , we have

$$\left| \sum_{r \geq 2} \frac{z_p^r}{r p^{rs}} \right| \leq \sum_{r \geq 2} p^{-r \Re s} = p^{-2 \Re s} \frac{1}{1 - p^{-\Re s}}.$$

Elementary analysis gives

$$1 \leq \frac{1}{1 - p^{-\Re s}} \leq 2 + 2\sqrt{2},$$

so the second piece of  $\log L(\mathbf{z}, s)$  converges absolutely when  $\Re s > \frac{1}{2}$ . We could simply cite [Ten95, II.1 Th. 10]; instead we prove directly that  $\sum_p \frac{z_p}{p^s}$  converges absolutely to a holomorphic function on the region  $\{\Re s > \alpha\}$ .

By Lemma 3.1.3 with  $f(x) = x^{-s}$ , we have

$$\begin{aligned} \sum_{p \leq N} \frac{z_p}{p^s} &= N^{-s} \sum_{p \leq N} z_p + s \int_2^N \sum_{p \leq x} z_p \frac{dx}{x^{s+1}} \\ &\ll N^{-\Re s + \alpha + \epsilon} + s \int_2^N x^{\alpha + \epsilon} \frac{dx}{x^{s+1}}. \end{aligned}$$

Since  $\alpha - \Re s < 0$ , the first term is bounded. Since  $s + 1 - \alpha > 1$  and  $\epsilon$  is arbitrary, the integral converges absolutely, and the proof is complete.  $\square$

**Theorem 3.1.5.** *Let  $\mathbf{z} = (z_2, z_3, \dots)$  be a sequence with  $\|\mathbf{z}\|_\infty \leq 1$ , and assume  $\log L(\mathbf{z}, s)$  has analytic continuation to  $\{\Re s > \alpha\}$  for some  $\alpha \in [\frac{1}{2}, 1]$ , and that for  $\sigma > \alpha$ , we have  $|\log L(\mathbf{z}, \sigma + it)| \ll |t|^{1-\epsilon}$  (implied constant independent of  $\sigma$ .) Then  $|\sum_{p \leq N} z_p| \ll N^{\alpha + \epsilon}$ .*

*Proof.* Recall that we can write

$$\log L(\mathbf{z}, s) = \sum_p \frac{z_p}{p^s} + \sum_p \sum_{r \geq 2} \frac{z_p^r}{r p^{rs}} = \sum_p \frac{z_p}{p^s} + O(\zeta(2\Re s)).$$

Thus, for any  $\epsilon > 0$ , analytic continuation and the bound on  $|\log L(\mathbf{z}, \sigma + it)|$  implies the same analytic continuation and bound for  $\sum \frac{z_p}{p^s}$  on  $\{\Re s > \alpha + \epsilon\}$ .

For any  $T > 0$ , let  $\gamma_T = \gamma_{1,T} + \gamma_{2,T} + \gamma_{3,T} + \gamma_{4,T}$  be the following contour:

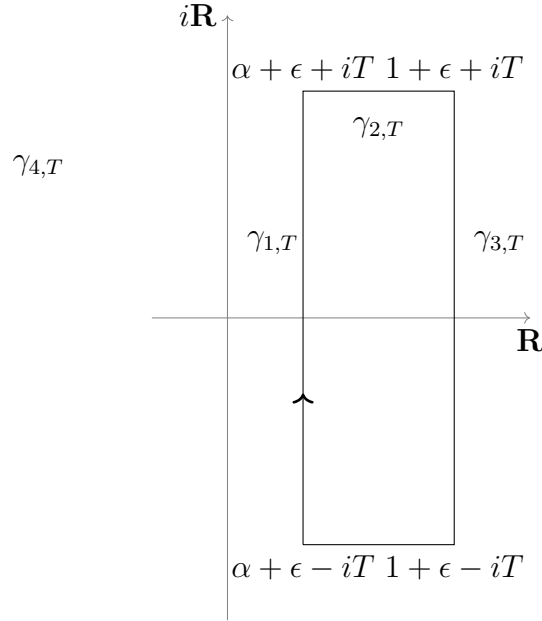
$$\gamma_{1,T}(t) = (\alpha + \epsilon) + it \quad t \in [-T, T]$$

$$\gamma_{2,T}(t) = t + iT \quad t \in [\alpha + \epsilon, 1 + \epsilon]$$

$$\gamma_{3,T}(t) = (1 + \epsilon) + it \quad t \in [T, -T]$$

$$\gamma_{4,T}(t) = t - iT \quad t \in [1 + \epsilon, \alpha + \epsilon].$$

Graphically, the contour looks like this:



By Perron's formula [Apo76, Th. 11.18],

$$\lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{-\gamma_{3,T}} \sum_p \frac{z_p}{p^s} N^z \frac{dz}{z} = \frac{1}{2} \sum_{p \leq N} z_p.$$

for  $N \in \mathbf{Z}$ , and the same without the  $\frac{1}{2}$  on the right-hand side when  $N \notin \mathbf{Z}$ .

Let  $h(s)$  be the analytic continuation of  $\sum z_p p^{-s}$  to  $\{\Re s > \alpha\}$ . Since  $\int_{\gamma_T} h(s) \frac{ds}{s} = 0$ , we obtain

$$\left| \sum_{p \leq N} z_p \right| \ll \lim_{T \rightarrow \infty} \left( \left| \int_{\gamma_{1,T}} h(s) N^s \frac{ds}{s} \right| + \left| \int_{\gamma_{2,T}} h(s) N^s \frac{ds}{s} \right| + \left| \int_{\gamma_{4,T}} h(s) N^s \frac{ds}{s} \right| \right).$$

We know that  $|h(\sigma + it)| \ll |t|^{1-\epsilon}$ , so we can bound

$$\left| \int_{\gamma_{2,T}} h(s) N^s \frac{ds}{s} \right| = \left| \int_{\alpha+\epsilon}^{1+\epsilon} \frac{h(t + iT) N^{t+iT}}{t + iT} dt \right| \ll \frac{N^{1+\alpha}}{T^\epsilon},$$

and similarly for  $\gamma_{4,T}$ . Finally, note that

$$\left| \int_{\gamma_{1,T}} h(s) N^s \frac{ds}{s} \right| \ll \int_{-T}^T |t|^{1-\epsilon} \frac{N^{\alpha+\epsilon}}{(\alpha + \epsilon)^2 + t^2} dt \ll N^{\alpha+\epsilon}.$$

Letting  $T \rightarrow \infty$  we obtain the desired result.  $\square$

In this thesis, we are interested in the following sort of strange Dirichlet series. Let  $X$  be a space,  $f: X \rightarrow \mathbf{C}$  a function with  $\|f\|_\infty \leq 1$ , and  $\mathbf{x} = (x_2, x_3, \dots)$  a sequence in  $X$ . Write

$$L_f(\mathbf{x}, s) = \prod_p \frac{1}{1 - f(x_p)p^{-s}},$$

for the associated strange Dirichlet series.

### **3.2 Relation to automorphic and motivic $L$ -functions**

### **3.3 The Riemann Hypothesis**

### **3.4 Discrepancy of sequences and the Riemann Hypothesis**

### **3.5 Strange Dirichlet series over function fields**

# CHAPTER 4

## IRRATIONALITY EXPONENTS

### 4.1 Definitions and first results

We follow the notation of [Lau09]. Let  $x = (x_1, \dots, x_d) \in \mathbf{R}^d$  be such that the  $x_i$  are  $\mathbf{Q}$ -linearly independent.

**Definition 4.1.1.** Let  $\omega_0(x)$  (resp.  $\omega_{d-1}(x)$ ) be the supremum of the set of real numbers  $\omega$  for which there exist infinitely many  $m = (m_0, \dots, m_d) \in \mathbf{Z}^{r+1}$  such that

$$\begin{aligned} \max\{|m_0 x_i - m_i|\} &\leq \|m\|_\infty^{-\omega} \quad (\text{resp.} \\ |m_0 + m_1 x_1 + \dots + m_r x_r| &\leq \|m\|_\infty^{-\omega}). \end{aligned}$$

These two quantities are related by Khintchine's Transference Principle, namely

$$\frac{\omega_{d-1}(x)}{(d-1)\omega_{d-1}(x) + d} \leq \omega(x) \leq \frac{\omega_{d-1}(x) - d + 1}{d}.$$

Moreover, these inequalities are sharp in a very strong sense.

**Theorem 4.1.2** (Jarník). Let  $w \geq 1/d$ . Then there exists  $x \in \mathbf{R}^d$  such that  $\omega_0(x) = w$  and  $\omega_{d-1}(x) = dw + d - 1$ .

*Proof.* Do this. □

**Theorem 4.1.3.** When  $d = 1$ , relate  $\omega_0(x)$  to the irrationality measure.

*Proof.* Recall that the irrationality measure  $\mu(x)$  is the infimum of the set of positive reals  $\mu$  such that

$$0 < \left| x - \frac{p}{q} \right| < q^{-\mu}$$

has only finitely many solutions  $p/q$  with  $p, q$  integers. □



Mention Roth's theorem... generalize to higher dimension?

Now given  $x \in \mathbf{R}^d$ , we write  $d(x, \mathbf{Z}^d) = \min_{m \in \mathbf{Z}^d} |x - m|$ , where  $|\cdot|$  is any fixed norm on  $\mathbf{R}^d$ . Note that  $d(x, \mathbf{Z}^d) = 0$  if and only if  $x \in \mathbf{Z}^d$ .

**Lemma 4.1.4.** *Let  $x \in \mathbf{R}^d$  with  $\|x\|_\infty \leq 1$  and  $\omega_0(x)$  (resp.  $\omega_{d-1}(x)$ ) finite. Then*

$$\begin{aligned} \frac{1}{d(nx, \mathbf{Z}^d)} &\ll |n|^{\omega_0(x)+\epsilon} && (\text{resp.} \\ \frac{1}{d(\langle m, x \rangle, \mathbf{Z})} &\ll |m|^{\omega_{d-1}(x)+\epsilon} && \text{for } m \in \mathbf{Z}^d). \end{aligned}$$

*Proof.* Let  $\epsilon > 0$ . Then there are only finitely many  $n \in \mathbf{Z}$  (resp.  $m \in \mathbf{Z}^d$ ) such that the inequalities in Definition 4.1.1 hold with  $\omega_0(x) + \epsilon$  (resp.  $\omega_{d-1}(x) + \epsilon$ ). In other words, there exist constants  $C_0, C_{d-1} > 0$  such that

$$\begin{aligned} \max\{|m_0 x_i - m_i|\} &\geq C_0 \|m\|_\infty^{-\omega_0(x)-\epsilon}, \\ |m_0 + m_1 x_1 + \cdots + m_d x_d| &\geq C_{d-1} \|m\|_\infty^{-\omega_{d-1}(x)-\epsilon} \end{aligned}$$

for all  $m \neq 0$ .

Start with the first inequality in the statement of the result, where up to constant, we may assume that  $|\cdot| = \|\cdot\|_\infty$  in the definition of  $d(nx, \mathbf{Z}^d)$ . Let  $m = (m_1, \dots, m_d)$  be the lattice point achieving the minimum  $|nx - m|$ . Then we know that

$$d(nx, \mathbf{Z}^d) \geq C_0 \|(m_1, \dots, m_d)\|_\infty^{-\omega_0(x)-\epsilon}.$$

Moreover, since  $|nx - m| < 1$ , there exists a constant  $C'_0$  such that

$$d(nx, \mathbf{Z}^d) \geq C'_0 |n|^{-\omega_0(x)-\epsilon}.$$

It follows that

$$\frac{1}{d(nx, \mathbf{Z}^d)} \ll |n|^{\omega_0(x)+\epsilon},$$

the implied constant depending on  $x$ ,  $\epsilon$ , and the choice of norm  $|\cdot|$ .

Now let's consider the second inequality in the statement of the result. Note that  $d(m_1x_1 + \dots + m_dx_d, \mathbf{Z}) = |m_0 + m_1x_1 + \dots + m_dx_d|$  for some  $m_0$  with  $|m_0| \leq \|(m_1, \dots, m_d)\|_2 \|x\|_2 + 1$ . Thus  $\|(m_1, \dots, m_d)\|_\infty \ll \|x\|_2 \|(m_1, \dots, m_d)\|_2$ , which gives us

$$d(m_1x_1 + \dots + m_dx_d, \mathbf{Z}) \geq C_{d-1} \|(m_1, \dots, m_d)\|_2^{-\omega_{d-1}(x) - \epsilon}.$$

This implies

$$\frac{1}{d(\langle m, x \rangle, \mathbf{Z})} \ll |m|^{\omega_{r-1}(x) + \epsilon},$$

the implied constant depending on  $x$ ,  $\epsilon$ , and the choice of  $|\cdot|$ .  $\square$

## 4.2 Irrationality exponents and discrepancy

Let  $x \in \mathbf{R}^d$  with  $x_1, \dots, x_d$  linearly independent over  $\mathbf{Q}$ . We wish to control the discrepancy of the sequence  $\{x, 2x, 3x, \dots\}$  in  $(\mathbf{R}/\mathbf{Z})^d$ .

**Theorem 4.2.1** (Erdős–Turán–Koksma). *Let  $\mathbf{x}$  be a sequence in  $\mathbf{R}^d$  and  $h$  an arbitrary integer. Then*

$$D(\mathbf{x}^N) \ll \frac{1}{h} + \sum_{0 \leq \|m\|_\infty \leq h} \frac{1}{r(m)} \left| \frac{1}{N} \sum_{n \leq N} e^{2\pi i \langle m, x_n \rangle} \right|,$$

where the first sum ranges over  $m \in \mathbf{Z}^d$ ,  $r(m) = \prod \max\{1, |m_i|\}$ , and the implied constant depends only on  $d$ .

*Proof.* This is [DT97, Th. 1.21].  $\square$

**Lemma 4.2.2.** *Let  $x \in \mathbf{R}$ . Then*

$$\left| \sum_{n \leq N} e^{2\pi i n x} \right| \ll \frac{1}{d(x, \mathbf{Z})}.$$

*Proof.* We begin with an easy bound:

$$\left| \sum_{n \leq N} e^{2\pi i n x} \right| = \frac{|e^{2\pi i (N+1)x} - 1|}{|e^{2\pi i x} - 1|} \leq \frac{2}{|e^{2\pi i x} - 1|}.$$

Since  $|e^{2\pi i m x} - 1| = \sqrt{2 - 2\cos(2\pi x)}$  and  $\cos(2\theta) = 1 - 2\sin^2 \theta$ , we obtain

$$\left| \sum_{n \leq N} e^{2\pi i n x} \right| \leq \frac{1}{|\sin(\pi x)|}.$$

It is easy to check that  $|\sin(\pi x)| \geq d(x, \mathbf{Z})$ , whence the result.  $\square$

**Corollary 4.2.3.** *Let  $x \in (\mathbf{R}/\mathbf{Z})^d$  with  $(x_1, \dots, x_d)$  linearly independent over  $\mathbf{Q}$ .*

*Then for  $\mathbf{x} = (x, 2x, 3x, \dots)$ , we have*

$$D(\mathbf{x}^N) \ll \frac{1}{h} + \frac{1}{N} \sum_{0 < \|m\|_\infty \leq h} \frac{1}{r(m)d(\langle m, x \rangle, \mathbf{Z})}$$

*for any integer  $h$ , with the implied constant depending only on  $d$ .*

*Proof.* Apply the Erdős–Turán–Koksma inequality and bound the exponential sums using Lemma 4.2.2.  $\square$

**Theorem 4.2.4.** *Let  $\mathbf{x} = (x, 2x, 3x, \dots)$  in  $(\mathbf{R}/\mathbf{Z})^d$ . Then*

$$D(\mathbf{x}^N) \ll N^{-\frac{1}{\omega_{d-1}(x)+1} + \epsilon}.$$

*Proof.* Choose  $\delta > 0$  such that  $\frac{1}{\omega_{d-1}(x)+1+\delta} = \frac{1}{\omega_{d-1}(x)+1} - \epsilon$ .

By Corollary 4.2.3, we know that

$$D(\mathbf{x}^N) \ll \frac{1}{h} + \frac{1}{N} \sum_{0 < \|m\|_\infty \leq h} \frac{1}{r(m)d(\langle m, x \rangle, \mathbf{Z})},$$

and by Lemma 4.1.4, we know that  $d(\langle m, x \rangle, \mathbf{Z})^{-1} \ll |m|^{\omega_{d-1}(x)+\delta}$ . It follows that

$$D(\mathbf{x}^N) \ll \frac{1}{h} + \frac{1}{N} \sum_{0 < \|m\|_\infty \leq h} \frac{|m|^{\omega_{d-1}(x)+\delta}}{r(m)}.$$

The only tricky part is bounding the sum.

$$\begin{aligned}
\sum_{0 < \|m\|_\infty \leq h} \frac{|m|_\infty^{\omega_{d-1}(x)+\delta}}{r(m)} &\ll \int_1^h \int_1^{t_d} \cdots \int_1^{t_2} \frac{t_d^{\omega_{d-1}(x)+\delta}}{t_1 \cdots t_d} dt_1 \cdots dt_d \\
&\ll \int_1^h t^{\omega_{d-1}(x)+\delta-1} dt \prod_{j=1}^{d-1} \int_1^h \frac{dt}{t} \\
&\ll (\log h)^{d-1} h^{\omega_{d-1}(x)+\delta}.
\end{aligned}$$

It follows that

$$D(\mathbf{x}^N) \ll \frac{1}{h} + \frac{1}{N} (\log h)^{d-1} h^{\omega_{d-1}(x)+\delta}.$$

Setting  $h \approx N^{\frac{1}{1+\omega_{d-1}(x)+\delta}}$ , we see that

$$D(\mathbf{x}^N) \ll N^{-\frac{1}{\omega_{d-1}(x)+1+\delta}} = N^{-\frac{1}{\omega_{d-1}(x)+1}+\epsilon}.$$

For a slightly different proof of a similar result (given as a sequence of exercises), see [KN74, Ch. 2, Ex. 3.15, 16, 17].  $\square$

**Theorem 4.2.5.** *Let  $x \in \mathbf{R}$  be such that  $x_1, \dots, x_d$  are linearly independent over  $\mathbf{Q}$ , and let  $\mathbf{x} = (x, 2x, 3x, \dots)$  in  $(\mathbf{R}/\mathbf{Z})^d$ . Then*

$$D(\mathbf{x}^N) = \Omega \left( N^{-\frac{d}{\omega_0(x)}-\epsilon} \right).$$

*Proof.* Here  $f = \Omega(g)$  in the sense of Hardy, namely that  $\limsup \frac{f}{g} > 0$ . We follow the proof of [KN74, Ch. 2, Th. 3.3]. Given  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $\frac{d}{\omega_0(x)-\delta} = \frac{d}{\omega_0(x)} + \epsilon$ .

By the definition of  $\omega_0(x)$ , there exist infinitely many  $(q, m_1, \dots, m_d)$  with  $q > 0$  such that

$$\|qx - m\|_\infty \leq \|(q, m_1, \dots, m_d)\|_\infty^{-\omega_0(x)+\delta/2}.$$

Since  $\|(q, m_1, \dots, m_d)\|_\infty \geq q$ , we derive the stronger statement that for infinitely many  $q \rightarrow \infty$ , there exists  $m = (m_1, \dots, m_d) \in \mathbf{Z}^d$  such that  $\|qx - m\|_\infty \leq$

$q^{-\omega_0(x)+\delta/2}$  or, equivalently,  $|x - \frac{m}{q}| \leq q^{-1-\omega_0(x)+\delta/2}$ . Pick such a  $q$ , and let  $N = \lfloor q^{\omega_0(x)-\delta} \rfloor$ . Then for each  $n \leq N$ , we have  $\|nx - \frac{n}{q}m\|_\infty \leq q^{-1-\delta/2}$ . Thus, for each  $n \leq N$ , each  $nx$  is within  $q^{-1-\delta/2}$  of the grid  $\frac{1}{q}\mathbf{Z}^d \subset (\mathbf{R}/\mathbf{Z})^d$ . Thus, they miss a box with side lengths  $q^{-1} - 2q^{-1-\delta/2}$ . For  $q$  sufficiently large,  $q^{-1} - 2q^{-1-\delta/2} \geq 1/2q$ , so the discrepancy of  $\mathbf{x}^N$  is bounded below by  $2^{-d}q^{-d}$ . Since  $q^{\omega_0(x)-\delta} \leq 2N$ , the discrepancy at  $N$  is bounded below by

$$2^{-d} \left( (2N)^{-\frac{1}{\omega_0(x)+\delta}} \right)^{-d} = 2^{-d-\frac{d}{\omega_0(x)+\delta}} N^{-\frac{d}{\omega_0(x)+\delta}} = 2^{-d(1+\frac{1}{\omega_0(x)})-\epsilon} N^{-\frac{d}{\omega_0(x)}-\epsilon}.$$

□

## CHAPTER 5

### DEFORMATION THEORY

#### 5.1 Category of test objects

The following is an exposition and explication of the theory outlined in [SGA 3<sub>1</sub>, VII<sub>B</sub>, §0–1]. In particular, we will heavily use the notions of a pseudocompact ring, pseudocompact modules, etc. Let  $\Lambda$  be a pseudocompact ring. Write  $\mathcal{C}_\Lambda$  for the opposite of the category of  $\Lambda$ -algebras which have finite length as  $\Lambda$ -modules. Given such a  $\Lambda$ -algebra  $A$ , write  $X = \mathrm{Spf}(A)$  for the corresponding object of  $\mathcal{C}_\Lambda$ , and we put  $A = \mathcal{O}(X)$ .

**Lemma 5.1.1.** *Let  $\Lambda$  be a pseudocompact ring,  $\mathcal{C}_\Lambda$  as above. Then  $\mathcal{C}_\Lambda$  is closed under finite limits and colimits.*

**Lemma 5.1.2.** *Let  $\Lambda$  be a pseudocompact local ring. Then  $\Lambda$  is henselian, in any of the following senses:*

1. *d*

*Proof.* [EGA 4<sub>4</sub>, 18.5.?] □

Following Grothendieck, if  $\mathcal{C}$  is an arbitrary category, we write  $\widehat{\mathcal{C}} = \mathrm{hom}(\mathcal{C}^\circ, \mathbf{Set})$  for the category of contravariant functors  $\mathcal{C} \rightarrow \mathbf{Set}$ . We regard  $\mathcal{C}$  as a full subcategory of  $\widehat{\mathcal{C}}$  via the Yoneda embedding, so for  $X, Y \in \mathcal{C}$ , we write  $X(Y) = \mathrm{hom}_{\mathcal{C}}(Y, X)$ . With this notation, the Yoneda Lemma states that  $\mathrm{hom}_{\widehat{\mathcal{C}}}(X, P) = P(X)$  for all  $X \in \mathcal{C}$ .

**Lemma 5.1.3.** *Let  $\mathcal{X} \in \widehat{\mathcal{C}_\Lambda}$ . Then  $\mathcal{X}$  is left exact if and only if there exists a filtered system  $\{X_i\}_{i \in I}$  in  $\mathcal{C}_\Lambda$  together with a natural isomorphism  $\mathcal{X}(\cdot) \simeq \varinjlim X_i(\cdot)$ . Write*

$\mathbf{Ind}(\mathbf{C}_\Lambda)$  for the category of such functors. Then  $\mathbf{Ind}(\mathbf{C}_\Lambda)$  is closed under colimits, and the Yoneda embedding  $\mathbf{C}_\Lambda \hookrightarrow \mathbf{Ind}(\mathbf{C}_\Lambda)$  preserves filtered colimits.

*Proof.* This follows from the results of [KS06, 6.1].  $\square$

If  $R$  is a pseudocompact  $\Lambda$ -algebra, write  $\mathrm{Spf}(R)$  for the object of  $\widehat{\mathbf{C}}_\Lambda$  defined by  $\mathrm{Spf}(R)(A) = \mathrm{hom}_{\mathrm{cts}/\Lambda}(R, A)$ , the set of continuous  $\Lambda$ -algebra homomorphisms.

**Lemma 5.1.4.** *The functor  $\mathrm{Spf}$  induces an (anti-)equivalence between the category of pseudocompact  $\Lambda$ -algebras and  $\mathbf{Ind}(\mathbf{C}_\Lambda)$ .*

*Proof.* This is [SGA 3<sub>I</sub>, VII<sub>B</sub> 0.4.2 Prop.].  $\square$

So  $\mathbf{Ind}(\mathbf{C}_\Lambda)$  is the category of pro-representable functors on finite length  $\Lambda$ -algebras. *Warning:* in many papers, for example the foundational [Maz97], one reserves the term *pro-representable* for functors of the form  $\mathrm{Spf}(R)$ , where  $R$  is *noetherian*. We do not make this restriction.

**Lemma 5.1.5.** *The category  $\mathbf{Ind}(\mathbf{C}_\Lambda)$  is an exponential ideal in  $\widehat{\mathbf{C}}_\Lambda$ .*

*Proof.* By this we mean the following. Let  $\mathcal{X} \in \mathbf{Ind}(\mathbf{C}_\Lambda)$ ,  $P \in \widehat{\mathbf{C}}_\Lambda$ . Then the functor  $\mathcal{X}^P$  defined by

$$\mathcal{X}^P(S) = \mathrm{hom}_{\widehat{\mathbf{C}}_{\Lambda/S}}(P/S, \mathcal{X}/S)$$

is also in  $\mathbf{Ind}(\mathbf{C}_\Lambda)$ . Given the characterization of  $\mathbf{Ind}(\mathbf{C}_\Lambda)$  as left exact functors, this is easy to prove, see e.g. [Joh02, 4.2.3].  $\square$

If  $\mathcal{C}$  is a category, we write  $\mathbf{Gp}(\mathcal{C})$  for the category of group objects in  $\mathcal{C}$ .

**Corollary 5.1.6.** *Let  $\Gamma \in \mathbf{Gp}(\widehat{\mathbf{C}}_\Lambda)$  and  $\mathcal{G} \in \mathbf{Gp}(\mathbf{Ind}(\mathbf{C}_\Lambda))$ , then the functor  $[\Gamma, \mathcal{G}]$  defined by*

$$[\Gamma, \mathcal{G}](S) = \mathrm{hom}_{\mathbf{Gp}/S}(\Gamma/S, \mathcal{G}/S)$$

is in  $\text{Ind}(\mathbf{C}_\Lambda)$ . In particular, if  $\Gamma$  is a profinite group, then the functor

$$[\Gamma, \mathcal{G}](S) = \text{hom}_{\text{cts}/\mathbf{G}_p}(\Gamma, \mathcal{G}(S))$$

is in  $\text{Ind}(\mathbf{C}_\Lambda)$ .

*Proof.* The first claim follows easily from 5.1.5. Just note that  $[\Gamma, \mathcal{G}]$  is the equalizer:

$$[\Gamma, \mathcal{G}] \longrightarrow \mathcal{G}^\Gamma \xrightleftharpoons[m_{\mathcal{G}*}]{m_\Gamma^*} \mathcal{G}^{\Gamma \times \Gamma},$$

that is, those  $f: \Gamma \rightarrow \mathcal{G}$  such that  $f \circ m_\Gamma = m_{\mathcal{G}} \circ (f \times f)$ . The latter claim is just a special case.  $\square$

## 5.2 Quotients in the flat topology

If  $\Lambda$  is a pseudocompact ring, the category  $\text{Ind}(\mathbf{C}_\Lambda)$  has nice “geometric” properties. However, for operations like taking quotients, we will embed it into the larger category  $\text{Sh}_{\text{fl}}(\mathbf{C}_\Lambda)$  of flat sheaves. We call a collection  $\{U_i \rightarrow X\}$  of morphisms in  $\mathbf{C}_\Lambda$  a *flat cover* if each ring map  $\mathcal{O}(X) \rightarrow \mathcal{O}(U_i)$  is flat, and moreover  $\mathcal{O}(X) \rightarrow \prod \mathcal{O}(U_i)$  is faithfully flat. By [SGA 3<sub>I</sub>, IV 6.3.1], this is a subcanonical Grothendieck topology on  $\mathbf{C}_\Lambda$ . We call it the *flat topology*, even though finite presentation comes for free because all the rings are finite length.

**Lemma 5.2.1.** *Let  $\text{Sh}_{\text{fl}}(\mathbf{C}_\Lambda)$  be the category of sheaves (of sets) on  $\mathbf{C}_\Lambda$  with respect to the flat topology. Then a presheaf  $P \in \widehat{\mathbf{C}_\Lambda}$  lies in  $\text{Sh}_{\text{fl}}(\mathbf{C}_\Lambda)$  if and only if  $P(\coprod U_i) = \prod P(U_i)$  and moreover, whenever  $U \rightarrow X$  is a flat cover where  $\mathcal{O}(U)$  and  $\mathcal{O}(X)$  are local rings, the sequence*

$$P(X) \longrightarrow P(U) \rightrightarrows P(U \times_X U).$$

*is exact. Moreover,  $\text{Ind}(\mathbf{C}_\Lambda) \subset \text{Sh}_{\text{fl}}(\mathbf{C}_\Lambda)$ .*



*Proof.* The first claim is the content of [SGA 3<sub>I</sub>, IV 6.3.1(ii)]. For the second, note that any  $\mathcal{X} \in \mathbf{Ind}(\mathbf{C}_\Lambda)$  will, by 5.1.3, convert (arbitrary) colimits into limits. Thus  $\mathcal{X}(\coprod U_i) = \prod \mathcal{X}(U_i)$ . If  $U \rightarrow X$  is a flat cover, then by (loc. cit.),  $U \times_X U \rightrightarrows U \rightarrow X$  is a coequalizer diagram in  $\mathbf{C}_\Lambda$ , hence  $\mathcal{X}(X) \rightarrow \mathcal{X}(U) \rightrightarrows \mathcal{X}(U \times_X U)$  is an equalizer.  $\square$

Our main reason for introducing the category  $\mathbf{Sh}_\#(\mathbf{C}_\Lambda)$  is that, as a (Grothendieck) topos, it is closed under arbitrary colimits. Recall that in an *equivalence relation* in  $\widehat{\mathbf{C}_\Lambda}$  is a morphism  $R \rightarrow X \times X$  such that, for all  $S$ , the map  $R(S) \rightarrow X(S) \times X(S)$  is an injection whose image is an equivalence relation on  $X(S)$ . We define the quotient  $X/R$  to be the coequalizer

$$R \rightrightarrows X \longrightarrow X/R.$$

By Giraud's Theorem [MLM94, App.], for any  $S \in \mathbf{C}_\Lambda$ , the natural map  $X(S)/R(S) \rightarrow (X/R)(S)$  is injective. It will not be surjective in general.

We let  $\mathbf{Sh}_\#(\mathbf{C}_\Lambda)$  inherit definitions from  $\mathbf{C}_\Lambda$  as follows. If  $P$  is a property of maps in  $\mathbf{C}_\Lambda$  (for example, “flat,” or “smooth,”) and  $f: X \rightarrow Y$  is a morphism in  $\mathbf{Sh}_\#(\mathbf{C}_\Lambda)$ , we say that  $f$  has  $P$  if for all  $S \in \mathbf{C}_\Lambda$  and  $y \in Y(S)$ , the pullback  $X_S = X \times_Y S$  lies in  $\mathbf{C}_\Lambda$ , and the pullback map  $X_S \rightarrow S$  has property  $P$ . For example, if  $X = \mathrm{Spf}(R')$  and  $Y = \mathrm{Spf}(R)$ , then  $X \rightarrow Y$  has property  $P$  if and only if for all finite length  $A$  and continuous  $\Lambda$ -algebra maps  $R \rightarrow A$ , the induced map  $A \rightarrow R' \otimes_R A$  has  $P$ .

**Theorem 5.2.2.** *Let  $\mathcal{R} \rightarrow \mathcal{X} \times \mathcal{X}$  be an equivalence relation in  $\mathbf{Ind}(\mathbf{C}_\Lambda)$  such that one of the maps  $\mathcal{R} \rightarrow \mathcal{X}$  is flat. Then the quotient  $\mathcal{X}/\mathcal{R}$  lies in  $\mathbf{Ind}(\mathbf{C}_\Lambda)$ , and  $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$  is a flat cover.*

*Proof.* This is [SGA 3<sub>I</sub>, VII<sub>B</sub> 1.4].  $\square$

By [Mat89, 29.7], if  $k$  is a field and  $R$  is a complete regular local  $k$ -algebra, then  $R \simeq k[[t_1, \dots, t_n]]$ . In particular,  $R$  admits an augmentation  $\epsilon: R \rightarrow k$ . There is a general analogue of this result, but first we need a definition.

**Definition 5.2.3.** *A map  $f: \mathcal{X} \rightarrow \mathcal{Y}$  in  $\text{Ind}(\mathbf{C}_\Lambda)$  is a residual isomorphism if for all  $S = \text{Spf}(k) \in \mathbf{C}_\Lambda$  where  $k$  is a field, the map  $f: \mathcal{X}(S) \rightarrow \mathcal{Y}(S)$  is a bijection.*

**Lemma 5.2.4.** *Let  $f: \mathcal{X} \rightarrow \mathcal{Y}$  be a smooth map in  $\text{Ind}(\mathbf{C}_\Lambda)$  that is a residual isomorphism. Then  $f$  admits a section.*

*Proof.* By [SGA 3<sub>I</sub>, VII<sub>B</sub> 0.1.1], it suffices to prove the result when  $\mathcal{X} = \text{Spf}(R')$ ,  $\mathcal{Y} = \text{Spf}(R)$ , for local  $\Lambda$ -algebras  $R \rightarrow R'$  with the same residue field. Let  $k = R/\mathfrak{m}_R \xrightarrow{\sim} R'/\mathfrak{m}_{R'}$  be their common residue field. From the diagram

$$\begin{array}{ccc} R' & \cdots\cdots\cdots & R \\ \uparrow & \searrow & \downarrow \\ R & \longrightarrow & k, \end{array}$$

the definition of (formal) smoothness, and a limiting argument involving the finite length quotients  $R/\mathfrak{a}$ , we obtain the result.  $\square$

**Corollary 5.2.5.** *Let  $\mathcal{R} \rightarrow \mathcal{X} \times \mathcal{X}$  be an equivalence relation satisfying the hypotheses of 5.2.2. Suppose further that*

1. *One of the maps  $\mathcal{R} \rightarrow \mathcal{X}$  is smooth, and*
2. *The projection  $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$  is a residual isomorphism.*

*Then  $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$  admits a section, so  $\mathcal{X}(S)/\mathcal{R}(S) \xrightarrow{\sim} (\mathcal{X}/\mathcal{R})(S)$  for all  $S \in \mathbf{C}_\Lambda$ .*

*Proof.* By 5.2.4, it suffices to prove that  $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$  is smooth. By [EGA 4<sub>4</sub>, 17.7.3(ii)], smoothness can be detected after flat descent. So base-change with

respect to the projection  $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$ . In the following commutative diagram

$$\begin{array}{ccc}
 \mathcal{R} & \xrightarrow{\quad} & \mathcal{X} \\
 \parallel & \searrow & \downarrow \\
 \mathcal{X} \times_{\mathcal{X}/\mathcal{R}} \mathcal{X} & \longrightarrow & \mathcal{X} \\
 \downarrow & & \downarrow \\
 \mathcal{X} & \longrightarrow & \mathcal{X}/\mathcal{R}
 \end{array}$$

we can ensure the smoothness of  $\mathcal{R} \rightarrow \mathcal{X}$  by our hypotheses. Since  $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{R}$  is smooth after flat base-change, the original map is smooth.  $\square$

**Example 5.2.6.** The hypothesis on residue fields in 5.2.5 is necessary. To see this, let  $\Lambda = k$  be a field,  $k \hookrightarrow K$  a finite Galois extension with Galois group  $G$ . Then  $G \times \mathrm{Spf}(K) \rightrightarrows \mathrm{Spf}(K)$  has quotient  $\mathrm{Spf}(k)$ , but the map  $\mathrm{Spf}(K)(S) \rightarrow \mathrm{Spf}(k)(S)$  is *not* surjective for all  $S \in \mathbf{C}_k$ , e.g. it is not for  $S = \mathrm{Spf}(k)$ .

**Example 5.2.7.** The hypothesis of smoothness in 5.2.5 is necessary. To see this, let  $k$  be a field of characteristic  $p > 0$ . Then the formal additive group  $\widehat{\mathbf{G}}_{\mathbf{a}} = \mathrm{Spf}(k[[t]])$  has a subgroup  $\alpha_p$  defined by

$$\alpha_p(S) = \{s \in \mathcal{O}(S) : s^p = 0\}.$$

The quotient  $\widehat{\mathbf{G}}_{\mathbf{a}}/\alpha_p$  has as affine coordinate ring  $k[[t^p]]$ . In particular, the following sequence is exact in the flat topology:

$$0 \longrightarrow \alpha_p \longrightarrow \widehat{\mathbf{G}}_{\mathbf{a}} \xrightarrow{(\cdot)^p} \widehat{\mathbf{G}}_{\mathbf{a}} \longrightarrow 0.$$

It follows that  $\alpha_p \times \widehat{\mathbf{G}}_{\mathbf{a}} \rightrightarrows \widehat{\mathbf{G}}_{\mathbf{a}} \xrightarrow{(\cdot)^p} \widehat{\mathbf{G}}_{\mathbf{a}}$  is a coequalizer in  $\mathbf{Sh}_{\mathbf{fl}}(\mathbf{C}_k)$  satisfying all the hypotheses of 5.2.5 except smoothness. And indeed, as one sees by letting  $S = \mathrm{Spf}(A)$  for any non-perfect  $k$ -algebra  $A$ , the map  $(\cdot)^p : \widehat{\mathbf{G}}_{\mathbf{a}}(S) \rightarrow \widehat{\mathbf{G}}_{\mathbf{a}}(S)$  is *not* surjective for all  $S$ .

### 5.3 Groupoids and quotient stacks

**Lemma 5.3.1.** *Let  $\mathcal{G} \in \text{Ind}(\mathbf{C}_\Lambda)$  be a smooth connected group. Then every  $\mathcal{G}$ -torsor is trivial.*

*Proof.* Let  $\mathcal{P} \rightarrow \mathcal{B}$  be a  $\mathcal{G}$ -torsor in  $\text{Ind}(\mathbf{C}_\Lambda)$ . That is,  $\mathcal{P}$  has an action of  $\mathcal{G}_S$  for which  $\mathcal{P} \times_{\mathcal{B}} \mathcal{P} \simeq \mathcal{G} \times \mathcal{P}$  as  $\mathcal{G}$ -spaces. [...not done...]  $\square$

**Theorem 5.3.2.** *Let  $\mathcal{G}$  be a smooth connected group in  $\text{Ind}(\mathbf{C}_\Lambda)$ , and  $\mathcal{X} \in \text{Ind}(\mathbf{C}_\Lambda)$  a  $\mathcal{G}$ -object. Then the quotient stack  $[\mathcal{X}/\mathcal{G}](S)$  has as objects  $\mathcal{X}(S)/\mathcal{G}(S)$ , but with extra automorphisms?*

*Proof.* Use triviality of torsors.  $\square$

### 5.4 Deformations of group representations

Let  $\Gamma \in \text{Gp}(\widehat{\mathbf{C}_\Lambda})$  and  $\mathcal{G} \in \text{Ind}(\mathbf{C}_\Lambda)$ . By 5.1.6, the functor

$$\text{Rep}^\square(\Gamma, \mathcal{G})(S) = \text{hom}_{\mathbf{Gp}/S}(\Gamma_S, \mathcal{G}_S)$$

is in  $\text{Ind}(\mathbf{C}_\Lambda)$ . We would like to define an ind-scheme  $\text{Rep}(\Gamma, \mathcal{G})$  as “ $\text{Rep}^\square(\Gamma, \mathcal{G})$  modulo conjugation,” but this requires some care. The conjugation action of  $\mathcal{G}$  on  $\text{Rep}^\square(\Gamma, \mathcal{G})$  will have fixed points, so the quotient will be badly behaved. We loosely follow [Til96].

Assume  $\Lambda$  is local, with maximal ideal  $\mathfrak{m}$  and residue field  $\mathbf{k}$ . Fix  $\bar{\rho} \in \text{Rep}^\square(\Gamma, \mathcal{G})(\mathbf{k})$ , i.e. a residual representation  $\bar{\rho}: \Gamma \rightarrow \mathcal{G}(\mathbf{k})$ . Let  $\text{Rep}^\square(\Gamma, \mathcal{G})_{\bar{\rho}}$  be the connected component of  $\bar{\rho}$  in  $\text{Rep}^\square(\Gamma, \mathcal{G})$ . Assume that  $\mathcal{G}$  and  $Z(\mathcal{G})$  are smooth; then the quotient  $\mathcal{G}^{\text{ad}} = \mathcal{G}/Z(\mathcal{G})$  is also smooth. Let  $\mathcal{G}^{\text{ad}, \circ}$  be the connected component of 1 in  $\mathcal{G}^{\text{ad}}$ .

**Theorem 5.4.1.** *Suppose  $(\Lambda, \mathfrak{m}, \mathbf{k})$  is local. If  $\mathcal{X}, \mathcal{Y} \in \text{Ind}(\mathbf{C}_\Lambda)$  are connected and  $\mathcal{X}(\mathbf{k}) \neq \emptyset$ , then  $\mathcal{X} \times_\Lambda \mathcal{Y}$  is connected.*

*Proof.* We are reduced to proving the following result from commutative algebra: if  $R, S$  are local pro-artinian  $\Lambda$ -algebras and  $R$  has residue field  $\mathbf{k}$ , then  $R\widehat{\otimes}_\Lambda S$  is local. Since  $R\widehat{\otimes}_\Lambda S = \varprojlim (R/\mathfrak{r}) \otimes_\Lambda (S/\mathfrak{s})$ ,  $\mathfrak{r}$  (resp.  $\mathfrak{s}$ ) ranges over all open ideals in  $R$  (resp.  $S$ ), we may assume that both  $R$  and  $S$  are artinian. The rings  $R$  and  $S$  are henselian, so  $R \otimes S$  is local if and only if  $(R/\mathfrak{m}_R) \otimes (S/\mathfrak{m}_S) = S/\mathfrak{m}_S$  is local, which it is.  $\square$

We conclude that the action of  $\mathcal{G}^{\text{ad}, \circ}$  on  $\text{Rep}^\square(\Gamma, \mathcal{G})$  preserves  $\text{Rep}^\square(\Gamma, \mathcal{G})_{\bar{\rho}}$ . Thus we may put

$$\text{Rep}(\Gamma, \mathcal{G})_{\bar{\rho}} = [\text{Rep}^\square(\Gamma, \mathcal{G})_{\bar{\rho}} / \mathcal{G}^{\text{ad}, \circ}].$$

If  $\mathcal{G}^{\text{ad}, \circ}$  acts faithfully on  $\text{Rep}^\square(\Gamma, \mathcal{G})_{\bar{\rho}}$ , then we recover the classical notion of the deformation functor.

**Theorem 5.4.2.** *Let  $\Gamma$  be a profinite group,  $\bar{\rho}: \Gamma \rightarrow \mathcal{G}(\mathbf{k})$  a representation with  $H^0(\Gamma, \text{Ad } \bar{\rho}) = 0$ . Then  $\text{Rep}(\Gamma, \mathcal{G})_{\bar{\rho}}$  exists and is what you expect.*

*Proof.* Need assumptions on  $Z(\mathcal{G})$ ,  $\mathcal{G}$  should be smooth.

Need  $Z(\mathcal{G}) = \ker(\mathcal{G} \rightarrow \text{GL}(\mathfrak{g}))$  in connected case. This should use  $\mathfrak{g} = \text{Lie}(\text{Aut } \mathcal{G})$ , via deviations in [SGA 3<sub>1</sub>].  $\square$

[...local conditions]

## 5.5 Tangent spaces and obstruction theory

For  $S_0 \in \mathbf{C}_\Lambda$ , let  $\text{Ex}_{S_0}$  be the category of square-zero thickenings of  $S_0$ . An object of  $\text{Ex}_{S_0}$  is a closed embedding  $S_0 \hookrightarrow S$  whose ideal of definition has square zero. Should be “exponential exact sequence”

$$0 \longrightarrow \mathfrak{g}(I) \longrightarrow \mathcal{G}(S) \longrightarrow \mathcal{G}(S_0) \longrightarrow 1$$

This gives us a class  $\exp \in H^2(\mathcal{G}(S_0), \mathfrak{g}(I))$ . For  $\rho_0: \Gamma \rightarrow \mathcal{G}(S_0)$ , the obstruction class is  $o(\rho_0, I) = \rho_0^*(\exp) \in H^2(\Gamma, \mathfrak{g}(I))$ . It's easy to check that  $o(\rho_0, I) = 0$  if and only if  $\rho_0$  lifts to  $\rho$ . So obstruction theory naturally for  $\text{Rep}^\square(\Gamma, \mathcal{G})$ .

[Use [Wei94, 6.6.4]. Given setting as above,  $\rho_0^*(\exp)$  is the pullback by  $\rho_0$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{g}(I) & \longrightarrow & \mathcal{G}(S) \times_{\mathcal{G}(S_0)} \Gamma & \longrightarrow & \Gamma \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \rho_0 \\ 0 & \longrightarrow & \mathfrak{g}(I) & \longrightarrow & \mathcal{G}(S) & \longrightarrow & \mathcal{G}(S_0) \longrightarrow 1 \end{array}$$

Computing explicitly, we see the result. ]

**Proposition 5.5.1.** *Let  $f: G \rightarrow H$  be a morphism of profinite groups. Suppose  $M$  is a discrete  $H$ -module and  $c \in H^2(H, M)$  corresponds to the extension*

$$0 \longrightarrow M \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 1.$$

*Then  $f^*c = 0$  in  $H^2(G, M)$  if and only if there is a map  $\tilde{f}: G \rightarrow \tilde{H}$  making the following diagram commute:*

$$\begin{array}{ccc} & & \tilde{H} \\ & \nearrow \tilde{f} & \downarrow \\ G & & H \\ & \searrow f & \\ & & \end{array}$$

*Proof.* By [Wei94, 6.6.4], the class  $f^*c$  corresponds to the pullback diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & G \times_H \tilde{H} & \longrightarrow & G \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow f \\ 0 & \longrightarrow & M & \longrightarrow & \tilde{H} & \longrightarrow & H \longrightarrow 1. \end{array}$$

Writing explicitly what it means for  $G \times_H \tilde{H} \rightarrow G$  to split yields the result.  $\square$

Let  $\mathcal{X} \in \text{Ind}(\mathbf{C}/\Lambda)$  be smooth, and  $L_{\mathcal{X}/\Lambda} \simeq \Omega_{\mathcal{X}/\Lambda}^1[0]$  be its cotangent complex. Fix  $x_0 \in \mathcal{X}(S_0)$ . From the chain  $S_0 \xrightarrow{x_0} \mathcal{X} \rightarrow \text{Spf}(\Lambda)$ , we get a distinguished

triangle [Ill71, II 2.1.5.6]

$$x_0^* L_{\mathcal{X}/\Lambda} \longrightarrow L_{S_0/\Lambda} \longrightarrow L_{S_0/\mathcal{X}} \longrightarrow .$$

If  $I$  is a coherent sheaf on  $S_0$ , we get a long exact sequence:

$$\mathrm{Ext}^0(L_{S_0/\Lambda}, M) \longrightarrow \mathrm{Ext}^0(x_0^* L_{\mathcal{X}/\Lambda}, M) \longrightarrow \mathrm{Ext}^1(L_{S_0/\mathcal{X}}, M) \longrightarrow \mathrm{Ext}^1(L_{S_0/\Lambda}, M) \longrightarrow \mathrm{Ext}^1(x_0^* L_{\mathcal{X}/\Lambda}, M) \longrightarrow \dots$$

If  $\mathcal{X}/\Lambda$  is smooth, then  $\mathrm{Ext}^1(x_0^* L_{\mathcal{X}/\Lambda}, M) = 0$  and  $L_{\mathcal{X}/\Lambda} = \Omega_{\mathcal{X}/\Lambda}^1$ . This gives us an exact sequence

$$\mathrm{Ext}^0(L_{S_0/\Lambda}, M) \longrightarrow \mathrm{hom}(\Omega_{\mathcal{X}/\Lambda}^1, M) \longrightarrow \mathrm{Ext}^1(L_{S_0/\mathcal{X}}, M) \longrightarrow \mathrm{Ext}^1(L_{S_0/\Lambda}, M) \longrightarrow 0.$$

The result [Ill71, III 2.1.7] tells us that the choice of  $S \in \mathbf{Ex}_{S_0}(M)$  gives us an element of  $\mathrm{Ext}^1(L_{S_0/\Lambda}, M)$ . Its fiber admits an action of  $\mathrm{hom}(\Omega_{\mathcal{X}/\Lambda}^1, M)$ . The only thing remaining is: we need  $\mathrm{Ext}^0(L_{S_0/\Lambda}, M) = 0$ , which doesn't hold in complete generality.

## CHAPTER 6

### CONSTRUCTING GALOIS REPRESENTATIONS

#### 6.1 Main idea

Basic idea is as follows. Start with  $\rho_1: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{F}_l)$ . At each stage, we have  $\rho_n: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{Z}/l^n)$ . At that stage, we're allowed to choose the (integral) characteristic polynomial for Frobenius at an arbitrarily large set of primes  $R_n$ . Then some (but density zero) extra primes ramify, and then we get  $\rho_{n+1}$  that agrees with our choices for  $R_1 \cup \dots \cup R_n$ . Then we have to choose characteristic polynomials for  $R_{n+1}$ , but these are already determined modulo  $l^{n+1}$ .

Basic idea is, we can choose the  $R_n$  to be so huge that the primes involved are way way bigger than  $l^n$ . For example, we could have  $R_n = \{p \leq l^n\}$ . Thus the set of possible  $a_p$ 's is very big (big enough) so that we can get discrepancy to behave as we like (both decaying slowly and decaying quickly) to any measure  $\mu$  such that  $\mathrm{ST}/\mu$  is bounded away from zero. (By this, we mean: if  $\mathrm{ST} = f \cdot \lambda$  and  $\mu = g \cdot \lambda$ , where  $\lambda$  is Lebesgue, then  $f/g$  is bounded away from zero.)



## CHAPTER 7

### COUNTEREXAMPLE VIA DIOPHANTINE APPROXIMATION

#### 7.1 Supporting results

Give  $(\mathbf{R}/\mathbf{Z})^d$  the natural Haar measure normalized to have total mass one. Recall that for any  $f \in L^1((\mathbf{R}/\mathbf{Z})^d)$ , the Fourier coefficients of  $f$  are, for  $m \in \mathbf{Z}^d$

$$\widehat{f}(m) = \int_{(\mathbf{R}/\mathbf{Z})^d} e^{2\pi i \langle m, x \rangle} dx,$$

where  $\langle m, x \rangle = m_1 x_1 + \cdots + m_d x_d$  is the usual inner product.

**Theorem 7.1.1.** *Fix  $x \in (\mathbf{R}/\mathbf{Z})^d$  with  $\omega_{d-1}(x)$  finite. Then*

$$\left| \sum_{n \leq N} e^{2\pi i \langle m, nx \rangle} \right| \ll |m|^{\omega_{d-1}(x) + \epsilon}$$

as  $m$  ranges over  $\mathbf{Z}^r \setminus 0$ .

*Proof.* From Lemma 4.2.2 we know that

$$\left| \sum_{n \leq N} e^{2\pi i \langle m, nx \rangle} \right| \ll \frac{1}{d(\langle m, x \rangle, \mathbf{Z})},$$

and from Lemma 4.1.4, we know that  $d(\langle m, x \rangle, \mathbf{Z})^{-1} \ll |m|^{\omega_{d-1}(x) + \epsilon}$ . The result follows.  $\square$

**Theorem 7.1.2.** *Let  $x \in \mathbf{R}^d$  with  $\omega_{d-1}(x)$  finite. Then let  $f \in L^1((\mathbf{R}/\mathbf{Z})^d)$  with  $\widehat{f}(0) = 0$  and suppose the Fourier coefficients of  $f$  satisfy the bound  $|\widehat{f}(m)| \ll |m|^{-\frac{1}{d-1} - \omega_{d-1}(x) - \epsilon}$ . Then*

$$\left| \sum_{n \leq N} f(nx) \right| \ll 1.$$

*Proof.* Write  $f$  as a Fourier series:

$$f(x) = \sum_{m \in \mathbf{Z}^r} \widehat{f}(m) e^{2\pi i \langle m, x \rangle}.$$

Since  $\widehat{f}(0) = 0$ , we can compute:

$$\begin{aligned}
\left| \sum_{n \leq N} f(nx) \right| &= \left| \sum_{n \leq N} \sum_{m \in \mathbf{Z}^d \setminus 0} \widehat{f}(m) e^{2\pi i \langle m, x \rangle} \right| \\
&\leq \sum_{m \in \mathbf{Z}^d \setminus 0} |\widehat{f}(m)| \left| \sum_{n \leq N} e^{2\pi i n \langle m, x \rangle} \right| \\
&\ll \sum_{m \in \mathbf{Z}^d \setminus 0} |m|^{-\frac{1}{d-1} - \omega_{d-1}(x) - \epsilon} |m|^{\omega_{d-1}(x) + \epsilon/2} \\
&\ll \sum_{m \in \mathbf{Z}^d \setminus 0} |m|^{-\frac{1}{d-1} - \epsilon/2}.
\end{aligned}$$

The sum converges since the exponent is less than  $-\frac{1}{d-1}$ , and it doesn't depend on  $N$ , hence the result.  $\square$

## 7.2 Pathological Satake parameters

Let  $p_1 = 2, p_2 = 3, p_3 = 5, \dots$  be an enumeration of the prime numbers. Let  $y \in \mathbf{R}^d$  with  $y_1, \dots, y_d$  linearly independent over  $\mathbf{Q}$ . The associated sequence of “fake Satake parameters” is

$$\mathbf{x} = (y, 2y, 3y, 4y, \dots),$$

where we put  $x_{p_n} = ny \pmod{\mathbf{Z}^d}$ . By Theorem 4.1.2, we can arrange for  $\omega_0(y) = w$  and  $\omega_{d-1}(y) = dw + d - 1$ .

**Theorem 7.2.1.** *The sequence  $\mathbf{x}$  is equidistributed in  $(\mathbf{R}/\mathbf{Z})^d$ , with discrepancy decaying as*

$$D(\mathbf{x}^N) \ll N^{-\frac{1}{dw+d} + \epsilon}$$

and for which

$$D(\mathbf{x}^N) = \Omega\left(N^{-\frac{d}{w} - \epsilon}\right).$$

However, for any  $f \in C^\infty((\mathbf{R}/\mathbf{Z})^d)$  with  $\widehat{f}(0) = 0$ , the strange Dirichlet series  $L_f(\mathbf{x}, s)$  satisfies the Riemann Hypothesis.

### 7.3 Some remarks on isotropic discrepancy

d

## CHAPTER 8

### DIRECT COUNTEREXAMPLE

#### 8.1 Main ideas

The goal for this chapter is to construct a Galois representation  $\rho_l: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{Z}_l)$ , ramified at a density-zero set of primes, such that

1. We have  $a_p = \mathrm{tr} \rho(\mathrm{fr}_p) \in \mathbf{Z}$  whenever  $\rho_l$  is unramified at  $p$ .
2.  $|a_p| \leq 2\sqrt{p}$ . ( $\rho_l$  satisfies the Hasse bound.)
3. For  $\theta_p = \cos^{-1} a_p/2\sqrt{p}$ , the discrepancy  $D^*(\boldsymbol{\theta}^N, \mathrm{ST}) \rightarrow 0$ , but slower than any  $N^{-\epsilon}$ .
4. For any smooth  $f \in C^\infty(\mathbf{R}/2\pi\mathbf{Z})$  such that  $f(\pi - \theta) = -f(\theta)$ , the associated  $L$ -function  $L_f(\boldsymbol{\theta}, s)$  satisfies the Riemann Hypothesis.

These points together imply that  $\rho_l$  satisfies the Sato–Tate conjecture, and  $L(\mathrm{sym}^k \rho_l, s)$  satisfy the Riemann Hypothesis for odd  $k$ , but that  $\rho_l$  does not satisfy the strong Sato–Tate conjecture.

#### 8.2 Construction

Fix, for the remainder of this section, a continuous representation

$$\bar{\rho}_l: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{F}_l).$$

For each  $p$  at which  $\bar{\rho}_l$  is unramified, we write

$$\Theta_p(\bar{\rho}_l) = \left\{ \cos^{-1} \left( \frac{a}{2\sqrt{p}} \right) : a \in \mathbf{Z}, |a| \leq 2\sqrt{p}, \text{ and } a \equiv \mathrm{tr} \bar{\rho}_l(\mathrm{fr}_p) \pmod{l} \right\}.$$

For the finitely many primes  $p$  for which  $\Theta_p(\bar{\rho}_l)$  is empty, redefine  $\Theta_p(\bar{\rho}_l)$  to include some elements for which  $|a| > 2\sqrt{p}$ . We have a sequence of  $\Theta_p(\bar{\rho}_l)$  for which at most finitely many do not satisfy the Hasse bound.

**Theorem 8.2.1.** *There exists a choice of  $\theta_p \in \Theta_p(\bar{\rho}_l)$  for odd-indexed primes  $\{2, 5, 11, \dots\}$  such that*

1.  $\theta_p \in [0, \pi/2)$  for all but finitely many  $p$ .
2.  $D\left(\theta_{\text{odd}}^N, \text{ST}|_{[0, \pi/2)}\right) \rightarrow 0$ , but is not  $\ll N^{-\epsilon}$  for any  $\epsilon > 0$ .

*Proof.* This is intuitively obvious, but a bit tricky to prove rigorously.

Two key ideas:

1. If we're given a "bad" finite distribution  $\nu$ , we can choose "good"  $\theta_p$ 's to make the combined distribution close enough (discrepancy-wise) to ST.
2. If we're given a "good" finite distribution  $\nu$ , we can choose "bad"  $\theta_p \sim \pi/2$  to make the combined distribution far away (discrepancy-wise) from ST.  $\square$

Claim: let  $\mu, \nu$  be two absolutely continuous distributions. Suppose there is a sequence  $\{T_p\}$  of  $\mu$ -distributed sets, such that  $D(T_p, \mu) \ll p^{-1/2}$ . Suppose moreover that  $\mu/\nu$  is bounded away from zero (at the pdf side). Then we can choose  $t_p \in T_p$  so that  $\{t_p\}$  is  $\nu$ -equidistributed with good discrepancy.

### 8.3 Associated Galois representation

Let  $\mu$  be an absolutely continuous measure on  $[0, \pi]$  such that the pushforward  $\cos_* \mu$  is bounded (this is true for the Sato–Tate measure). Fix a prime  $l \geq 5$  and a constant  $\alpha \in (0, 1/2]$ . We want to construct a weight-2 Galois representation  $\rho_l: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Z}_l)$ , ramified at a density zero set of primes, such that

1. If  $\rho_l$  is unramified at  $p$ , then  $a_p = \text{tr } \rho_l(\text{fr}_p) \in \mathbf{Z}$  and satisfies the Hasse bound  $|a_p| \leq 2\sqrt{p}$ .
2. If we write  $\theta_p = \cos^{-1}(a_p/2\sqrt{p})$  for the Satake parameters at unramified primes, then  $D(\boldsymbol{\theta}^N, \mu) \ll N^{-\alpha+\epsilon}$  and  $D(\boldsymbol{\theta}^N, \mu) = \Omega(N^{-\alpha-\epsilon})$ .

Recall the van der Corput sequence  $\{x_p\}$  satisfies  $D(\boldsymbol{x}^N) \ll N^{-1+\epsilon}$ . Let  $\nu = \cos_* \mu$ ; this is an absolutely continuous measure supported on  $[-1, 1]$ . By transforming the van der Corput sequence by a continuous map, we may assume that in fact  $D(\boldsymbol{x}^N, \nu) \ll N^{-1+\epsilon}$ . In fact, by alternating between “van der Corput elements” and “bad elements” we can ensure that not only does  $D(\boldsymbol{x}^N, \nu) \ll N^{-\alpha+\epsilon}$ , but also  $D(\boldsymbol{x}^N, \nu) = \Omega(N^{-\alpha-\epsilon})$ .

We start by choosing a modular mod- $l$  representation  $\rho_1: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Z}/l)$ , which is ramified at a finite set of primes  $S_1$ . Let  $R_1 = \{p \leq r_1 : p \notin S_1\}$ . For  $p \in R_1$ , we can choose  $a_p \in \mathbf{Z}$  subject only to the condition  $a_p \equiv \text{tr } \rho_1(\text{fr}_p) \pmod{l}$ . For any  $p \in R_1$ , the set

$$T_p(l) = \left\{ \frac{a}{2\sqrt{p}} : |a| \leq 2\sqrt{p} \text{ and } a \equiv \text{tr } \rho_1(\text{fr}_p) \pmod{l} \right\}$$

has an element within  $lp^{-1/2}$  of any element of  $[-1, 1]$ . Choose  $a_p \in T_p(l)$  so that  $\left| \frac{a_p}{2\sqrt{p}} - x_p \right| \leq lp^{-1/2}$ . It follows that for  $p \in R_1$ , we have

$$|D(\{a_p/2\sqrt{p}\}_{p \leq N}, \nu) - D(\boldsymbol{x}^N, \nu)| \ll lN^{-1/2}$$

We get a lift of  $\rho_1$  to  $\rho_2: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Z}/l^2)$  respecting our choices of the  $a_p$  for  $p \in R_1$ , which is ramified at one (perhaps two) extra primes.

What happens next is in stages. We'll already have a mod- $l^{n+1}$  representation  $\rho_{n+1}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Z}/l^n)$ , together with choices of  $a_p$  for  $p \in R_1 \cup \dots \cup R_n$  that ensure  $|D(\{a_p/2\sqrt{p}\}_{p \leq N}, \nu) - D(\boldsymbol{x}^N, \nu)| \ll ?$

The main question is: how do we choose  $r_1$ , and the later  $r_n$ ? We ensure that  
a) the set  $T_p(l^n)$  are non-empty, and that b)  $l^n < \log(r_n)$ . This gives us that for  
 $N \leq r_n$ , we have

$$|D(\{a_p/2\sqrt{p}\}_{p \leq N}, \nu) - D(\mathbf{x}^N, \nu)| \ll N^{-\frac{1}{2}+\epsilon}.$$

Todo: can I make  $\sum a_p = ?$  anything from  $-\infty$  to  $\infty$ ?

What if I make a fake modular form with these “bad” Satake parameters?

What can I say about it?

## 8.4 Informal approach

This discussion is inspired by [Pan11]. Throughout, all Galois representations have weight 2, i.e. determinant is the cyclotomic power.

Fix a prime  $l \geq 5$  and a (modular) representation  $\rho_1: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{Z}/l)$ . We claim that there is a finite set  $S$  such that  $\mathrm{III}_S^1(\mathrm{Ad}^0 \rho_1) = \mathrm{III}_S^2(\mathrm{Ad}^0 \rho_1) = 0$ . Moreover, all the local deformation spaces are smooth? (Why?)

Set  $S = S_2$ . Choose lifts  $\rho_p: G_{\mathbf{Q}_p} \rightarrow \mathrm{GL}_2(\mathbf{Z}_l)$  of  $\rho_1|_{G_{\mathbf{Q}_p}}$  for all  $p \in S_2$ . We can ensure that the  $\rho_p$  are ramified. (Can we also ensure that their characteristic polynomials are well behaved? By [KR01], these characteristic polynomials are well-defined for all but finitely many primes.) Now let  $R_2 = \{p \notin S_2 : p \leq r_2\}$ , where  $r_2$  is a yet unspecified large constant (say  $l^{100}$ ). Choose  $a_p$  for all  $p \in R_2$ . By [Pan11, Lem. 5.1], there is a set  $Q_2$  (bound the size of  $Q_2$ !)

CHAPTER 9

COMPUTATIONAL EVIDENCE FOR THE  
AKIYAMA–TANIGAWA CONJECTURE



## CHAPTER 10

### CONCLUDING REMARKS AND FUTURE DIRECTIONS

Todo: future direction, discrete dense subgroups of  $SU(2)$  and other compact, semisimple groups.

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