KOLMOGOROV–SMIRNOV STATISTICS AND THE ANALYTIC PROPERTIES OF DIRICHLET SERIES ASSOCIATED TO ELLIPTIC CURVES

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BIOGRAPHICAL SKETCH

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4.1 Category of test objects

The following is an exposition and explication of the theory outlined in [SGA 3_1, $\mathrm{VII_B}$, §0–1]. In particular, we will heavily use the notions of a pseudocompact ring, pseudocompact modules, etc. Let Λ be a pseudocompact ring. Write C_Λ for the opposite of the category of Λ-algebras which have finite length as Λ-modules. Given such a Λ-algebra A, write $X = \operatorname{Spf}(A)$ for the corresponding object of C_Λ, and we put $A = \mathscr{O}(X)$.

Lemma 4.1.1. *Let Λ be a pseudocompact ring, C_Λ as above. Then C_Λ is closed under finite limits and colimits.*

Lemma 4.1.2. *Let Λ be a pseudocompact local ring. Then Λ is henselian, in any of the following senses:*

1. *d*

Proof. [EGA 4_4, 18.5.?] □

Following Grothendieck, if $\mathcal{C}$ is an arbitrary category, we write $\widehat{\mathcal{C}} = \hom(\mathcal{C}^\circ, \mathsf{Set})$ for the category of contravariant functors $\mathcal{C} \to \mathsf{Set}$. We regard $\mathcal{C}$ as a full subcategory of $\widehat{\mathcal{C}}$ via the Yoneda embedding, so for $X, Y \in \mathcal{C}$, we write $X(Y) = \hom_{\mathcal{C}}(Y, X)$. With this notation, the Yoneda Lemma states that $\hom_{\widehat{\mathcal{C}}}(X, P) = P(X)$ for all $X \in \mathcal{C}$.

Lemma 4.1.3. *Let $\mathcal{X} \in \widehat{\mathsf{C}_\Lambda}$. Then $\mathcal{X}$ is left exact if and only if there exists a filtered system $\{X_i\}_{i\in I}$ in $\mathcal{C}_\Lambda$ together with a natural isomorphism $\mathcal{X}(\cdot) \simeq \varinjlim X_i(\cdot)$. Write $\mathsf{Ind}(\mathsf{C}_\Lambda)$ for the category of such functors. Then $\mathsf{Ind}(\mathsf{C}_\Lambda)$ is closed under colimits, and the Yoneda embedding $\mathsf{C}_\Lambda \hookrightarrow \mathsf{Ind}(\mathsf{C}_\Lambda)$ preserves filtered colimits.*

Proof. This follows from the results of [KS06, 6.1]. □

If R is a pseudocompact Λ-algebra, write $\operatorname{Spf}(R)$ for the object of $\widehat{\mathsf{C}_\Lambda}$ defined by $\operatorname{Spf}(R)(A) = \hom_{\mathrm{cts}/\Lambda}(R, A)$, the set of continuous Λ-algebra homomorphisms.

Lemma 4.1.4. *The funtor* Spf *induces an (anti-)equivalence between the category of pseudocompact Λ-algebras and $\mathsf{Ind}(\mathsf{C}_\Lambda)$.*

Proof. This is [SGA 3_1, $\mathrm{VII_B}$ 0.4.2 Prop.]. □

So $\mathsf{Ind}(\mathsf{C}_\Lambda)$ is the category of pro-representable functors on finite length Λ-algebras. *Warning*: in many papers, for example the foundational [Maz97], one reserves the term *pro-representable* for functors of the form $\operatorname{Spf}(R)$, where R is *noetherian.* We do not make this restriction.

Lemma 4.1.5. *The category $\mathsf{Ind}(\mathsf{C}_\Lambda)$ is an exponential ideal in $\widehat{\mathsf{C}_\Lambda}$.*

Proof. By this we mean the following. Let $\mathcal{X} \in \mathsf{Ind}(\mathsf{C}_\Lambda)$, $P \in \widehat{\mathsf{C}_\Lambda}$. Then the functor $\mathcal{X}^P$ defined by

$$\mathcal{X}^P(S) = \hom_{\widehat{\mathsf{C}_\Lambda}_{/S}}(P_{/S}, \mathcal{X}_{/S})$$

is also in $\mathsf{Ind}(\mathsf{C}_\Lambda)$. Given the characterization of $\mathsf{Ind}(\mathsf{C}_\Lambda)$ as left exact functors, this is easy to prove, see e.g. [Joh02, 4.2.3]. □

If $\mathcal{C}$ is a category, we write $\mathsf{Gp}(\mathcal{C})$ for the category of group objects in $\mathcal{C}$.

Corollary 4.1.6. *Let $\Gamma \in \mathsf{Gp}(\widehat{\mathsf{C}_\Lambda})$ and $\mathcal{G} \in \mathsf{Gp}(\mathsf{Ind}(\mathsf{C}_\Lambda))$, then the functor $[\Gamma, \mathcal{G}]$ defined by*

$$[\Gamma, \mathcal{G}](S) = \hom_{\mathsf{Gp}/S}(\Gamma_{/S}, \mathcal{G}_{/S})$$

is in $\mathsf{Ind}(\mathsf{C}_\Lambda)$. In particular, if Γ is a profinite group, then the functor

$$[\Gamma, \mathcal{G}](S) = \hom_{\mathrm{cts}/\,\mathsf{Gp}}(\Gamma, \mathcal{G}(S))$$

is in $\mathsf{Ind}(\mathsf{C}_\Lambda)$.

Proof. The first claim follows easily from 4.1.5. Just note that $[\Gamma, \mathcal{G}]$ is the equalizer:

$$[\Gamma, \mathcal{G}] \longrightarrow \mathcal{G}^\Gamma \underset{m_{\mathcal{G}*}}{\overset{m_\Gamma^*}{\rightrightarrows}} \mathcal{G}^{\Gamma\times\Gamma},$$

that is, those $f\colon \Gamma \to \mathcal{G}$ such that $f \circ m_\Gamma = m_{\mathcal{G}} \circ (f \times f)$. The latter claim is just a special case. □

4.2 Quotients in the flat topology

If Λ is a pseudocompact ring, the category $\mathsf{Ind}(\mathsf{C}_\Lambda)$ has nice "geometric" properties. However, for operations like taking quotients, we will embed it into the larger category $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$ of flat sheaves. We call a collection $\{U_i \to X\}$ of morphisms in C_Λ a *flat cover* if each ring map $\mathscr{O}(X) \to \mathscr{O}(U_i)$ is flat, and moreover $\mathscr{O}(X) \to \prod \mathscr{O}(U_i)$ is faithfully flat. By [SGA 3_1, IV 6.3.1], this is a subcanonical Grothendieck topology on C_Λ. We call it the *flat topology*, even though finite presentation comes for free because all the rings are finite length.

Lemma 4.2.1. *Let $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$ be the category of sheaves (of sets) on C_Λ with respect to the flat topology. Then a presheaf $P \in \widehat{\mathsf{C}_\Lambda}$ lies in $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$ if and only if $P(\coprod U_i) = \prod P(U_i)$ and moreover, whenever $U \to X$ is a flat cover where $\mathscr{O}(U)$ and $\mathscr{O}(X)$ are local rings, the sequence*

$$P(X) \longrightarrow P(U) \rightrightarrows P(U \times_X U).$$

is exact. Moreover, $\mathsf{Ind}(\mathsf{C}_\Lambda) \subset \mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$.

Proof. The first claim is the content of [SGA 3_1, IV 6.3.1(ii)]. For the second, note that any $\mathcal{X} \in \mathsf{Ind}(\mathsf{C}_\Lambda)$ will, by 4.1.3, convert (arbitrary) colimits into limits. Thus $\mathcal{X}(\coprod U_i) = \prod \mathcal{X}(U_i)$. If $U \to X$ is a flat cover, then by (loc. cit.), $U \times_X U \rightrightarrows U \to X$ is a coequalizer diagram in C_Λ, hence $\mathcal{X}(X) \to \mathcal{X}(U) \rightrightarrows \mathcal{X}(U \times_X U)$ is an equalizer. □

Our main reason for introducing the category $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$ is that, as a (Grothendieck) topos, it is closed under arbitrary colimits. Recall that in an *equivalence relation* in $\widehat{\mathsf{C}_\Lambda}$ is a morphism $R \to X \times X$ such that, for all S, the map $R(S) \to X(S) \times X(S)$ is an injection whose image is an equivalence relation on $X(S)$. We define the quotient X/R to be the coequalizer

$$R \rightrightarrows X \longrightarrow X/R.$$

By Giraud's Theorem [MLM94, App.], for any $S \in \mathsf{C}_\Lambda$, the natural map $X(S)/R(S) \to (X/R)(S)$ is injective. It will not be surjective in general.

We let $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$ inherit definitions from C_Λ as follows. If P is a property of maps in C_Λ (for example, "flat," or "smooth,") and $f\colon X \to Y$ is a morphism in $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_\Lambda)$, we say that f has P if for all $S \in \mathsf{C}_\Lambda$ and $y \in Y(S)$, the pullback $X_S = X \times_Y S$ lies in C_Λ, and the pullback map $X_S \to S$ has property P. For example, if $X = \mathrm{Spf}(R')$ and $Y = \mathrm{Spf}(R)$, then $X \to Y$ has property P if and only if for all finite length A and continuous Λ-algebra maps $R \to A$, the induced map $A \to R' \otimes_R A$ has P.

Theorem 4.2.2. *Let $\mathcal{R} \to \mathcal{X} \times \mathcal{X}$ be an equivalence relation in $\mathsf{Ind}(\mathsf{C}_\Lambda)$ such that one of the maps $\mathcal{R} \to \mathcal{X}$ is flat. Then the quotient $\mathcal{X}/\mathcal{R}$ lies in $\mathsf{Ind}(\mathsf{C}_\Lambda)$, and $\mathcal{X} \to \mathcal{X}/\mathcal{R}$ is a flat cover.*

Proof. This is [SGA 3_1, $\mathrm{VII_B}$ 1.4]. □

By [Mat89, 29.7], if k is a field and R is a complete regular local k-algebra, then $R \simeq k[\![t_1, \dots, t_n]\!]$. In particular, R admits an augmentation $\epsilon\colon R \to k$. There is a general analogue of this result, but first we need a definition.

Definition 4.2.3. *A map $f\colon \mathcal{X} \to \mathcal{Y}$ in $\mathsf{Ind}(\mathsf{C}_\Lambda)$ is a* residual isomorphism *if for all $S = \mathrm{Spf}(k) \in \mathsf{C}_\Lambda$ where k is a field, the map $f\colon \mathcal{X}(S) \to \mathcal{Y}(S)$ is a bijection.*

Lemma 4.2.4. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a smooth map in $\mathsf{Ind}(\mathsf{C}_\Lambda)$ that is a residual isomorphism. Then f admits a section.*

Proof. By [SGA 3_1, $\mathrm{VII_B}$ 0.1.1], it suffices to prove the result when $\mathcal{X} = \mathrm{Spf}(R')$, $\mathcal{Y} = \mathrm{Spf}(R)$, for local Λ-algebras $R \to R'$ with the same residue field. Let $k = R/\mathfrak{m}_R \xrightarrow{\sim} R'/\mathfrak{m}_{R'}$ be their common residue field. From the diagram

$$\begin{tikzcd} R' \arrow[r, dotted] \arrow[dr, two heads] & R \arrow[d, two heads] \\ R \arrow[u] \arrow[r, two heads] \arrow[ur, equal] & k, \end{tikzcd}$$

the definition of (formal) smoothness, and a limiting argument involving the finite length quotients $R/\mathfrak{a}$, we obtain the result. □

Corollary 4.2.5. *Let $\mathcal{R} \to \mathcal{X} \times \mathcal{X}$ be an equivalence relation satisfying the hypotheses of 4.2.2. Suppose further that*

1. *One of the maps $\mathcal{R} \to \mathcal{X}$ is smooth, and*

2. *The projection $\mathcal{X} \to \mathcal{X}/\mathcal{R}$ is a residual isomorphism.*

Then $\mathcal{X} \to \mathcal{X}/\mathcal{R}$ admits a section, so $\mathcal{X}(S)/\mathcal{R}(S) \xrightarrow{\sim} (\mathcal{X}/\mathcal{R})(S)$ for all $S \in \mathsf{C}_\Lambda$.

Proof. By 4.2.4, it suffices to prove that $\mathcal{X} \to \mathcal{X}/\mathcal{R}$ is smooth. By [EGA 4_4, 17.7.3(ii)], smoothness can be detected after flat descent. So base-change with respect to the projection $\mathcal{X} \to \mathcal{X}/\mathcal{R}$. In the following commutative diagram

$$\begin{tikzcd} \mathcal{R} \arrow[dr, equal] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \mathcal{X} \times_{\mathcal{X}/\mathcal{R}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ & \mathcal{X} \arrow[r] & \mathcal{X}/\mathcal{R} \end{tikzcd}$$

we can ensure the smoothness of $\mathcal{R} \to \mathcal{X}$ by our hypotheses. Since $\mathcal{X} \to \mathcal{X}/\mathcal{R}$ is smooth after flat base-change, the original map is smooth. □

Example 4.2.6. The hypothesis on residue fields in 4.2.5 is necessary. To see this, let $\Lambda = k$ be a field, $k \hookrightarrow K$ a finite Galois extension with Galois group G. Then $G \times \mathrm{Spf}(K) \rightrightarrows \mathrm{Spf}(K)$ has quotient $\mathrm{Spf}(k)$, but the map $\mathrm{Spf}(K)(S) \to \mathrm{Spf}(k)(S)$ is *not* surjective for all $S \in \mathsf{C}_k$, e.g. it is not for $S = \mathrm{Spf}(k)$.

Example 4.2.7. The hypothesis of smoothness in 4.2.5 is necessary. To see this, let k be a field of characteristic $p > 0$. Then the formal additive group $\widehat{\mathbf{G}}_{\mathrm{a}} = \mathrm{Spf}(k[\![t]\!])$ has a subgroup $\boldsymbol{\alpha}_p$ defined by

$$\boldsymbol{\alpha}_p(S) = \{s \in \mathscr{O}(S) \colon s^p = 0\}.$$

The quotient $\widehat{\mathbf{G}}_{\mathrm{a}}/\boldsymbol{\alpha}_p$ has as affine coordinate ring $k[\![t^p]\!]$. In particular, the following sequence is exact in the flat topology:

$$0 \longrightarrow \boldsymbol{\alpha}_p \longrightarrow \widehat{\mathbf{G}}_{\mathrm{a}} \xrightarrow{(\cdot)^p} \widehat{\mathbf{G}}_{\mathrm{a}} \longrightarrow 0.$$

It follows that $\boldsymbol{\alpha}_p \times \widehat{\mathbf{G}}_{\mathrm{a}} \rightrightarrows \widehat{\mathbf{G}}_{\mathrm{a}} \xrightarrow{(\cdot)^p} \widehat{\mathbf{G}}_{\mathrm{a}}$ is a coequalizer in $\mathsf{Sh}_{\mathrm{fl}}(\mathsf{C}_k)$ satisfying all the hypothese of 4.2.5 except smoothness. And indeed, as one sees by letting $S = \mathrm{Spf}(A)$ for any non-perfect k-algebra A, the map $(\cdot)^p \colon \widehat{\mathbf{G}}_{\mathrm{a}}(S) \to \widehat{\mathbf{G}}_{\mathrm{a}}(S)$ is *not* surjective for all S.

4.3 Groupoids and quotient stacks

Lemma 4.3.1. *Let $\mathcal{G} \in \mathsf{Ind}(\mathsf{C}_\Lambda)$ be a smooth connected group. Then every $\mathcal{G}$-torsor is trivial.*

Proof. Let $\mathcal{P} \to \mathcal{B}$ be a $\mathcal{G}$-torsor in $\mathsf{Ind}(\mathsf{C}_\Lambda)$. That is, $\mathcal{P}$ has an action of $\mathcal{G}_{\mathcal{S}}$ for which $\mathcal{P} \times_{\mathcal{B}} \mathcal{P} \simeq \mathcal{G} \times \mathcal{P}$ as $\mathcal{G}$-spaces. [...not done...] $\square$

Theorem 4.3.2. *Let $\mathcal{G}$ be a smooth connected group in $\mathsf{Ind}(\mathsf{C}_\Lambda)$, and $\mathcal{X} \in \mathsf{Ind}(\mathsf{C}_\Lambda)$ a $\mathcal{G}$-object. Then the quotient stack $[\mathcal{X}/\mathcal{G}](S)$ has as objects $\mathcal{X}(S)/\mathcal{G}(S)$, but with extra automorphisms?*

Proof. Use triviality of torsors. $\square$

4.4 Deformations of group representations

Let $\Gamma \in \mathsf{Gp}(\widehat{\mathsf{C}_\Lambda})$ and $\mathcal{G} \in \mathsf{Ind}(\mathsf{C}_\Lambda)$. By 4.1.6, the functor

$$\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})(S) = \hom_{\mathsf{Gp}/S}(\Gamma_S, \mathcal{G}_S)$$

is in $\mathsf{Ind}(\mathsf{C}_\Lambda)$. We would like to define an ind-scheme $\mathrm{Rep}(\Gamma, \mathcal{G})$ as "$\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})$ modulo conjugation," but this requires some care. The conjugation action of $\mathcal{G}$ on $\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})$ will have fixed points, so the quotient will be badly behaved. We loosely follow [Til96].

Assume Λ is local, with maximal ideal $\mathfrak{m}$ and residue field $\mathbf{k}$. Fix $\bar\rho \in \mathrm{Rep}^{\square}(\Gamma, \mathcal{G})(\mathbf{k})$, i.e. a residual representation $\bar\rho \colon \Gamma \to \mathcal{G}(\mathbf{k})$. Let $\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})_{\bar\rho}$ be the connected component of $\bar\rho$ in $\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})$. Assume that $\mathcal{G}$ and $\mathrm{Z}(\mathcal{G})$ are smooth; then the quotient $\mathcal{G}^{\mathrm{ad}} = \mathcal{G}/\mathrm{Z}(\mathcal{G})$ is also smooth. Let $\mathcal{G}^{\mathrm{ad},\circ}$ be the connected component of 1 in $\mathcal{G}^{\mathrm{ad}}$.

Theorem 4.4.1. *Suppose $(\Lambda, \mathfrak{m}, \mathbf{k})$ is local. If $\mathcal{X}, \mathcal{Y} \in \mathsf{Ind}(\mathsf{C}_\Lambda)$ are connected and $\mathcal{X}(\mathbf{k}) \neq \varnothing$, then $\mathcal{X} \times_\Lambda \mathcal{Y}$ is connected.*

Proof. We are reduced to proving the following result from commutative algebra: if R, S are local pro-artinian Λ-algebras and R has residue field $\mathbf{k}$, then $R \widehat{\otimes}_\Lambda S$ is local. Since $R \widehat{\otimes}_\Lambda S = \varprojlim (R/\mathfrak{r}) \otimes_\Lambda (S/\mathfrak{s})$, $\mathfrak{r}$ (resp. $\mathfrak{s}$) ranges over all open ideals in R (resp. S), we may assume that both R and S are artinian. The rings R and S are henselian, so $R \otimes S$ is local if and only if $(R/\mathfrak{m}_R) \otimes (S/\mathfrak{m}_S) = S/\mathfrak{m}_S$ is local, which it is. $\square$

We conclude that the action of $\mathcal{G}^{\mathrm{ad},\circ}$ on $\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})$ preserves $\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})_{\bar\rho}$. Thus we may put

$$\mathrm{Rep}(\Gamma, \mathcal{G})_{\bar\rho} = [\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})_{\bar\rho}/\mathcal{G}^{\mathrm{ad},\circ}].$$

If $\mathcal{G}^{\mathrm{ad},\circ}$ acts faithfully on $\mathrm{Rep}^{\square}(\Gamma, \mathcal{G})_{\bar\rho}$, then we recover the classical notion of the deformation functor.

Theorem 4.4.2. *Let Γ be a profinite group, $\bar{\rho}\colon \Gamma \to \mathcal{G}(\mathbf{k})$ a representation with $\mathrm{H}^0(\Gamma, \operatorname{Ad}\bar{\rho}) = 0$. Then $\operatorname{Rep}(\Gamma, \mathcal{G})_{\bar{\rho}}$ exists and is what you expect.*

Proof. Need assumptions on $\mathrm{Z}(\mathcal{G})$, $\mathcal{G}$ should be smooth.

Need $\mathrm{Z}(\mathcal{G}) = \ker(\mathcal{G} \to \mathrm{GL}(\mathfrak{g}))$ in connected case. This should use $\mathfrak{g} = \operatorname{Lie}(\operatorname{Aut}\mathcal{G})$, via deviations in [SGA 3_1]. □

[...local conditions]

4.5 Tangent spaces and obstruction theory

For $S_0 \in \mathsf{C}_\Lambda$, let Ex_{S_0} be the category of square-zero thickenings of S_0. An object of Ex_{S_0} is a closed embedding $S_0 \hookrightarrow S$ whose ideal of definition has square zero. Should be "exponential exact sequence"

$$0 \longrightarrow \mathfrak{g}(I) \longrightarrow \mathcal{G}(S) \longrightarrow \mathcal{G}(S_0) \longrightarrow 1$$

This gives us a class $\exp \in \mathrm{H}^2(\mathcal{G}(S_0), \mathfrak{g}(I))$. For $\rho_0\colon \Gamma \to \mathcal{G}(S_0)$, the obstruction class is $o(\rho_0, I) = \rho_0^*(\exp) \in \mathrm{H}^2(\Gamma, \mathfrak{g}(I))$. It's easy to check that $o(\rho_0, I) = 0$ if and only if ρ_0 lifts to ρ. So obstruction theory naturally for $\operatorname{Rep}^\square(\Gamma, \mathcal{G})$.

[Use [Wei94, 6.6.4]. Given setting as above, $\rho_0^*(\exp)$ is the pullback by ρ_0:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{g}(I) \arrow[r] \arrow[d, equal] & \mathcal{G}(S) \times_{\mathcal{G}(S_0)} \Gamma \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d, "\rho_0"] & 1 \\
0 \arrow[r] & \mathfrak{g}(I) \arrow[r] & \mathcal{G}(S) \arrow[r] & \mathcal{G}(S_0) \arrow[r] & 1
\end{tikzcd}$$

Computing explicitly, we see the result.]

Proposition 4.5.1. *Let $f\colon G \to H$ be a morphism of profinite groups. Suppose M is a discrete H-module and $c \in \mathrm{H}^2(H, M)$ corresponds to the extension*

$$0 \longrightarrow M \longrightarrow \widetilde{H} \longrightarrow H \longrightarrow 1.$$

Then $f^*c = 0$ in $\mathrm{H}^2(G, M)$ if and only if there is a map $\widetilde{f}\colon G \to \widetilde{H}$ making the following diagram commute:

$$\begin{tikzcd}
 & \widetilde{H} \arrow[d] \\
G \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & H.
\end{tikzcd}$$

Proof. By [Wei94, 6.6.4], the class f^*c corresponds to the pullback diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & G \times_H \widetilde{H} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "f"] & 1 \\
0 \arrow[r] & M \arrow[r] & \widetilde{H} \arrow[r] & H \arrow[r] & 1.
\end{tikzcd}$$

Writing explicitly what it means for $G \times_H \widetilde{H} \to G$ to split yields the result. □

Let $\mathcal{X} \in \mathsf{Ind}(\mathsf{C}_{/\Lambda})$ be smooth, and $\mathrm{L}_{\mathcal{X}/\Lambda} \simeq \Omega^1_{\mathcal{X}/\Lambda}[0]$ be its cotangent complex. Fix $x_0 \in \mathcal{X}(S_0)$. From the chain $S_0 \xrightarrow{x_0} \mathcal{X} \to \operatorname{Spf}(\Lambda)$, we get a distinguished triangle [Ill71, II 2.1.5.6]

$$x_0^*\mathrm{L}_{\mathcal{X}/\Lambda} \longrightarrow \mathrm{L}_{S_0/\Lambda} \longrightarrow \mathrm{L}_{S_0/\mathcal{X}} \longrightarrow .$$

If I is a coherent sheaf on S_0, we get a long exact sequence:

$$\mathrm{Ext}^0(\mathrm{L}_{S_0/\Lambda}, M) \longrightarrow \mathrm{Ext}^0(x_0^*\mathrm{L}_{\mathcal{X}/\Lambda}, M) \longrightarrow \mathrm{Ext}^1(\mathrm{L}_{S_0/\mathcal{X}}, M) \longrightarrow \mathrm{Ext}^1(\mathrm{L}_{S_0/\Lambda}, M) \longrightarrow \mathrm{Ext}^1(x_0^*\mathrm{L}_{\mathcal{X}/\Lambda}, M)$$

If $\mathcal{X}_{/\Lambda}$ is smooth, then $\mathrm{Ext}^1(x_0^*\mathrm{L}_{\mathcal{X}/\Lambda}, M) = 0$ and $\mathrm{L}_{\mathcal{X}/\Lambda} = \Omega^1_{\mathcal{X}/\Lambda}$. This gives us an exact sequence

$$\mathrm{Ext}^0(\mathrm{L}_{S_0/\Lambda}, M) \longrightarrow \hom(\Omega^1_{\mathcal{X}/\Lambda}, M) \longrightarrow \mathrm{Ext}^1(\mathrm{L}_{S_0/\mathcal{X}}, M) \longrightarrow \mathrm{Ext}^1(\mathrm{L}_{S_0/\Lambda}, M) \longrightarrow 0.$$

The result [Ill71, III 2.1.7] tells us that the choice of $S \in \mathsf{Ex}_{S_0}(M)$ gives us an element of $\mathrm{Ext}^1(\mathrm{L}_{S_0/\Lambda}, M)$. Its fiber admits an action of $\hom(\Omega^1_{\mathcal{X}/\Lambda}, M)$. The only thing remaining is: we need $\mathrm{Ext}^0(\mathrm{L}_{S_0/\Lambda}, M) = 0$, which doesn't hold in complete generality.

CHAPTER 5

CONSTRUCTING GALOIS REPRESENTATIONS WITH SPECIFIED PROPERTIES

CHAPTER 6
FIRST COUNTEREXAMPLE

CHAPTER 7
SECOND COUNTEREXAMPLE

CHAPTER 8
CONCLUDING REMARKS AND FUTURE DIRECTIONS

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