Physically-based Modelling

Mass, Force, and Time

Young-Min Kang
Tongmyong University

Mass

- Mass
  - Resistance to being accelerated by force
  - accumulated density (kg/m^3)
    - integration of density

$$m = \int \rho dV$$

Centre of Mass

- theory

$$x_c = \frac{\int x_0 dm}{m}$$

$$y_c = \frac{\int y_0 dm}{m}$$

$$z_c = \frac{\int z_0 dm}{m}$$

$$c_g = \frac{\sum (c_{g_i} m_i)}{m}$$

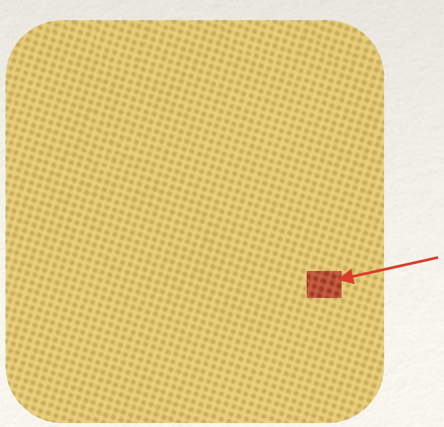
- computation

$$x_c = \frac{\sum x_i m_i}{\sum m_i}$$

$$y_c = \frac{\sum y_i m_i}{\sum m_i}$$

$$z_c = \frac{\sum z_i m_i}{\sum m_i}$$

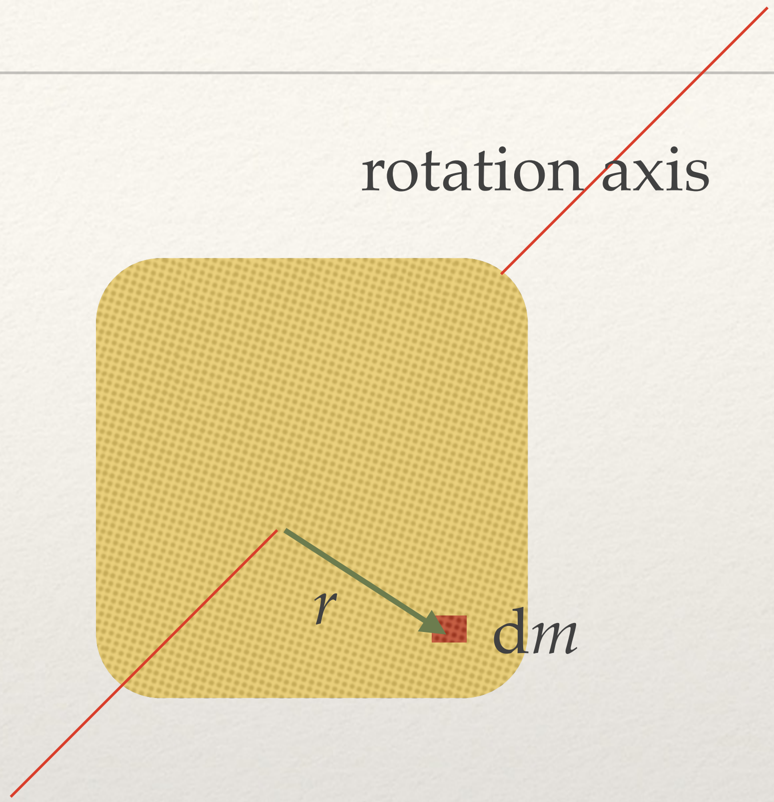
V



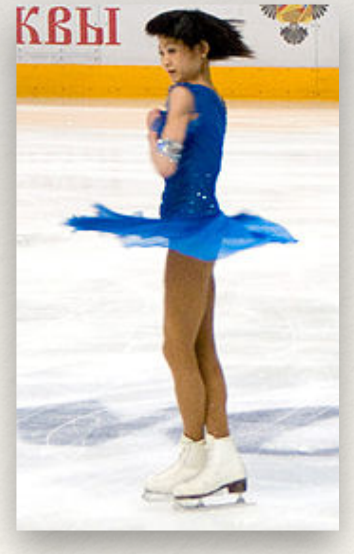
dm at (x_0, y_0, z_0)

Moment of Inertia

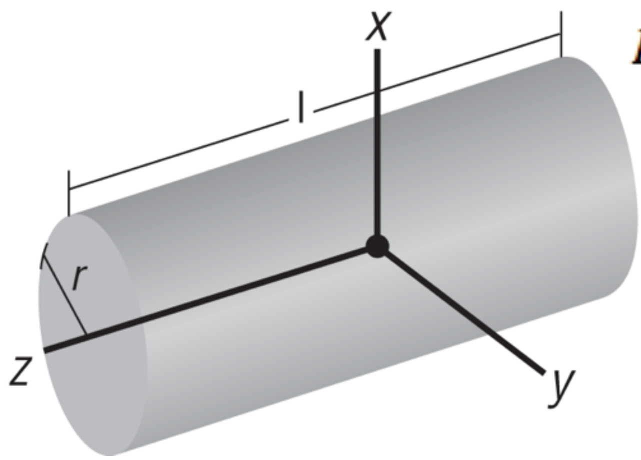
- angular mass
  - rotation axis is important
  - resistance to angular acceleration along the axis



$$I = \int_m r^2 \mathrm{d}m$$

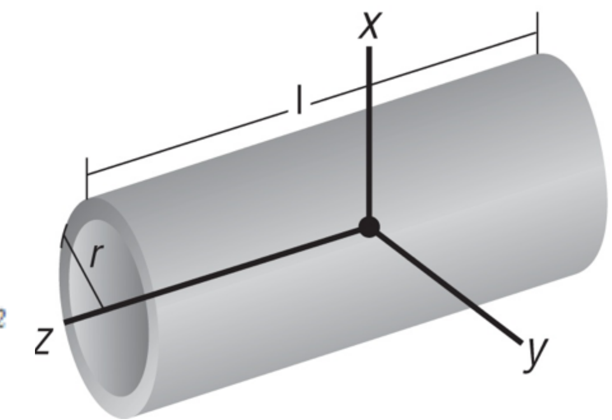


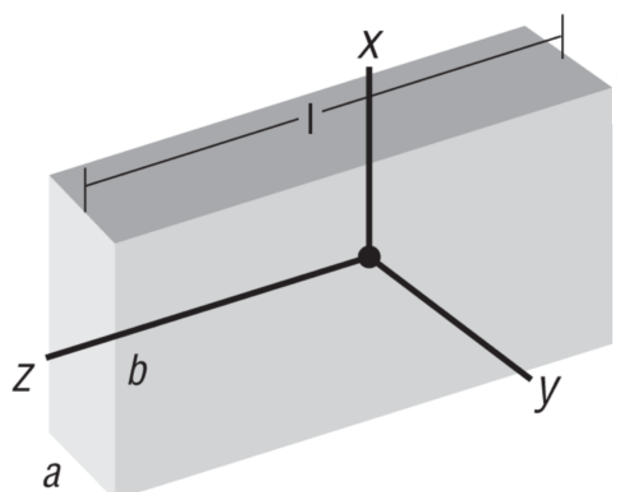
Examples



$$I_{xx} = I_{yy} = (1/4)\,mr^2 + (1/12)\,ml^2 ;\ I_{zz} = (1/2)\,mr^2$$

$$I_{xx} = I_{yy} = (1/4)\,mr^2 + (1/12)\,ml^2 ;\ I_{zz} = (1/2)\,mr^2$$

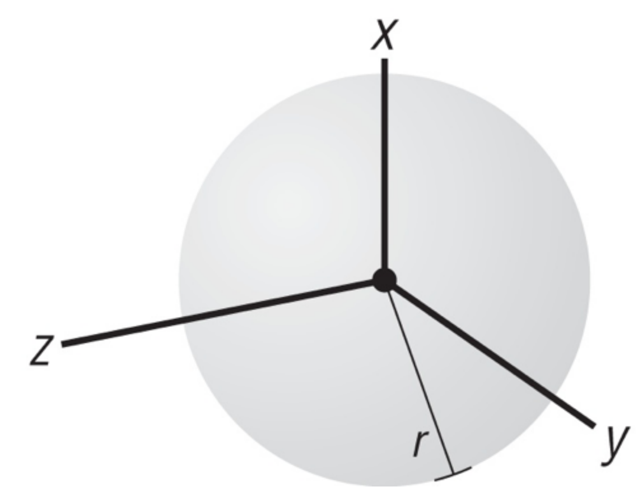




$$I_{xx} = I_{yy} = (1/4)\,mr^2 + (1/12)\,ml^2 ;\ I_{zz} = (1/2)\,mr^2$$

sphere $I_{xx} = I_{yy} = I_{zz} = (2/5)\,mr^2$

spherical shell $I_{xx} = I_{yy} = I_{zz} = (2/3)\,mr^2$



Newton's 2nd law of motion

- force = mass x acceleration

$$\mathbf{f} = m\mathbf{a}$$

- total force $\mathbf{f}_{net}$
  - sum of all the exerted forces $\mathbf{f}_{net} = \sum \mathbf{f}_i$

$$\sum f_x = ma_x$$
$$\sum f_y = ma_y$$
$$\sum f_z = ma_z$$

  - acceleration is determined by the total force

$$\mathbf{a} = \frac{\mathbf{f}_{net}}{m}$$

Linear Momentum and its Derivative

- Linear Momentum: **G**
  - mass x velocity
  - $\mathbf{G} = m\mathbf{v}$
- Derivative of Linear Momentum with respect to time

$$\frac{d\mathbf{G}}{dt} = \frac{dm\mathbf{v}}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}$$

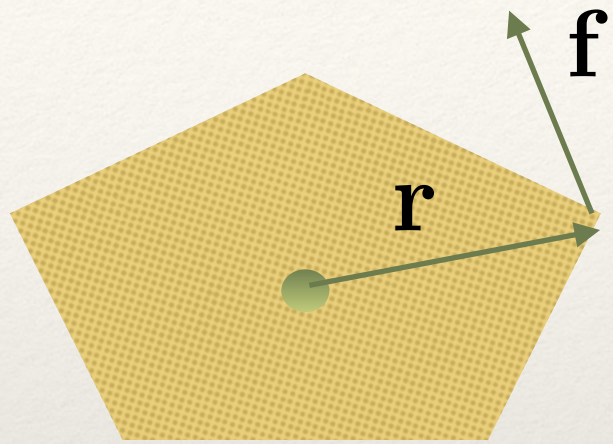
  - therefore,

$$\frac{d\mathbf{G}}{dt} = \sum \mathbf{f}$$

Rotational Motion

- Torque: τ
    - "force" in rotation

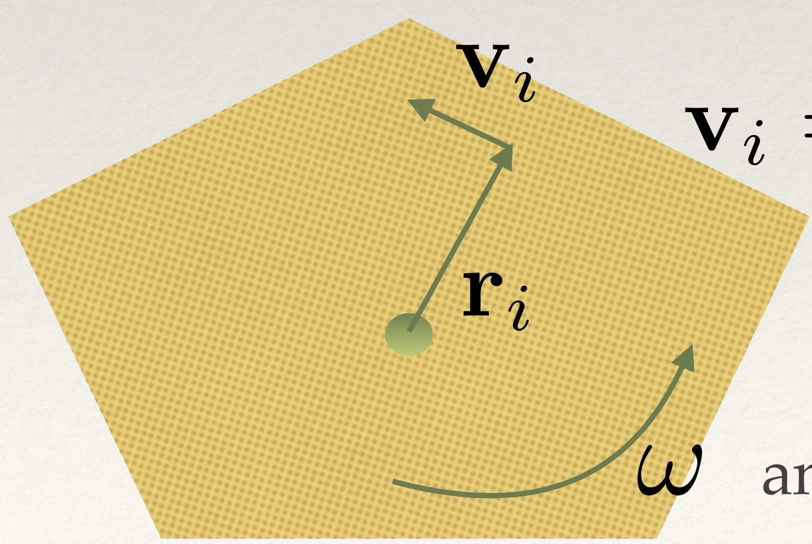
$$\tau = \mathbf{r} \times \mathbf{f}$$



- angular momentum
    - sum of moments of momentum of all particles

$$\mathbf{H} = \sum \mathbf{r}_i \times m_i \mathbf{v}_i$$

$$\mathbf{H} = \sum \mathbf{r}_i \times m_i (\omega \times \mathbf{r}_i)$$



$$\mathbf{v}_i = \omega \times \mathbf{r}_i$$

angular velocity

Angular Momentum = A. Mass x A. Velocity

- Linar Momentum: **G**
  - $\mathbf{G} = m\mathbf{v}$
- Angular Momentum

$$\mathbf{H} = \sum \mathbf{r}_i \times m_i(\omega \times \mathbf{r}_i)$$

$$\begin{aligned}\mathbf{H} &= \int \omega \mathbf{r}^2 dm \\ &= \omega \int \mathbf{r}^2 dm \\ &= \omega \mathbf{I} = \mathbf{I}\omega\end{aligned}$$

Inertia Tensor
(Angular Mass)

Angular Velocity

Derivative of Angular Momentum

- d**G** / dt = total force
- d**H** / dt = total torque

$$\frac{d\mathbf{H}}{dt} = \frac{d\mathbf{I}\omega}{dt} = \mathbf{I}\frac{d\omega}{dt} = \mathbf{I}\alpha$$

$$\longrightarrow \quad \sum \tau = \mathbf{I}\alpha \quad \longrightarrow \quad \alpha = \mathbf{I}^{-1} \sum \tau$$

Tensor

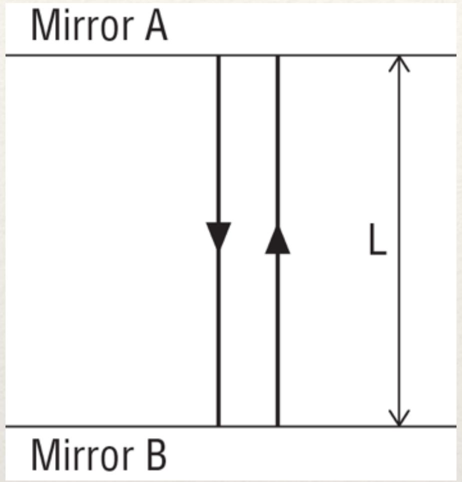
- Tensor
  - mathematical expression that has magnitude and direction
  - its magnitude may not be unique depending on the direction
  - typically used to represent properties of materials where these properties have different magnitudes in different directions.
- Isotropic vs. Anisotropic
  - isotropic: properties are the same in all direction
  - anisotropic: properties vary depending on direction
- Moment of Inertia
  - inertia tensor (in 3D)
  - nine components to fully describe it for any arbitrary rotation. (3x3 matrix)
  - property of the body that varies with the axis of rotation.

Time

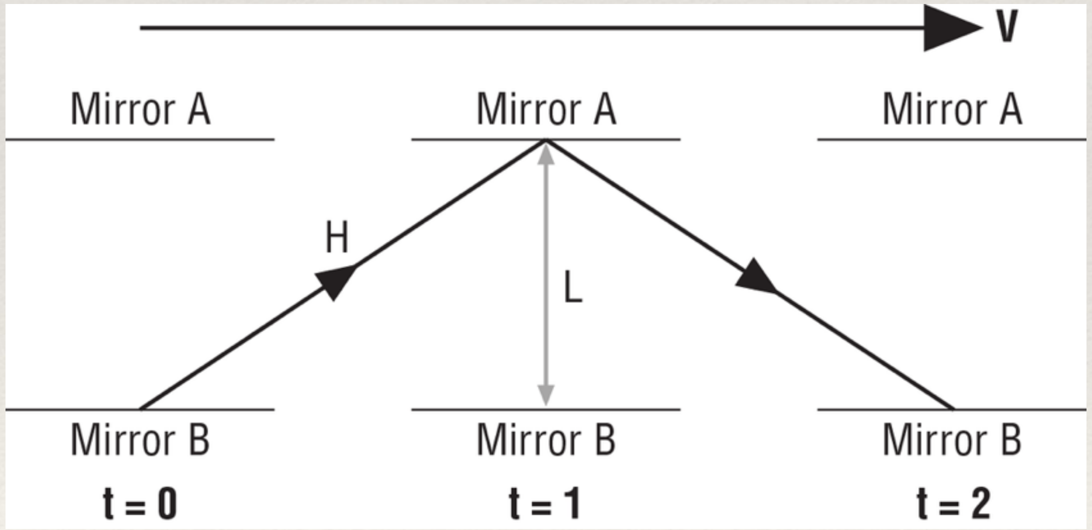
- Classic Physics
  - Time is “constant”
- Modern Physics
  - Time is “variable”
  - “Speed of Light” is constant: c
  - c = 299,792,458 m/s

Relativistic Time - Just for fun

We are not likely to use this relativistic time for this course…



$$c = 2L/\Delta t$$



mirrors are travelling in a spaceship

in the spaceship

$$c = 2L/\Delta t_s$$

on earth

$$c = 2H/\Delta t_e$$

if c is constant
time should be DIFFERENT

Time Dilation

- Simple geometry

$$H^2 = L^2 + \left(\frac{\mathbf{v}\Delta t_e}{2}\right)^2 \Rightarrow \left(\frac{c\Delta t_e}{2}\right)^2 = L^2 + \left(\frac{\mathbf{v}\Delta t_e}{2}\right)^2$$

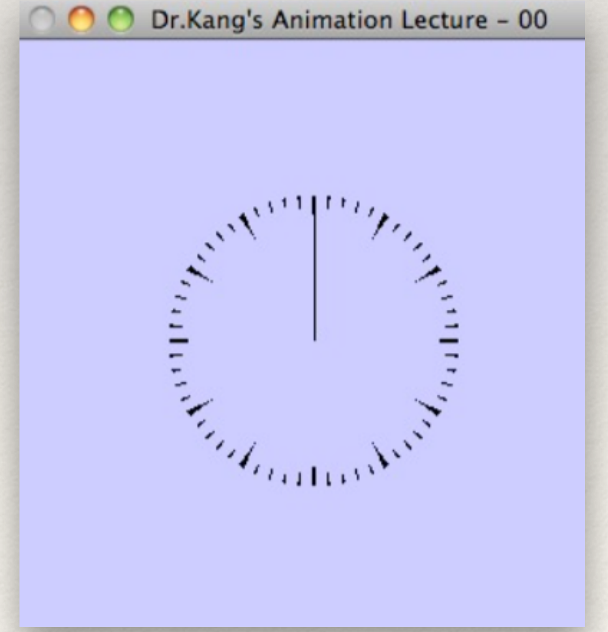
- time dilation

$$\boxed{\begin{aligned} 4L^2 &= c^2\Delta t_e^2 - \mathbf{v}^2\Delta t_e^2 \\ 4L^2 &= (c^2 - \mathbf{v}^2)\Delta t_e^2 \\ \frac{4L^2}{c^2} &= (1 - \frac{\mathbf{v}^2}{c^2})\Delta t_e^2 \end{aligned}} \Rightarrow \boxed{\begin{aligned} \frac{2L}{c} &= \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}\Delta t_e \\ \Delta t &= \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}\Delta t_e \end{aligned}}$$

$$\Delta t_e = \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}\Delta t$$

Time and Animation

- Animation
  - change over time
- Computer Animation
  - computing the physical state in accordance with “time”
- We must measure the time
  - Stop watch is needed
  - Let’s make our own stop watch for animation



Stop Watch (header)

```
#ifndef _STOPWATCH_YMKANG_H
#define _STOPWATCH_YMKANG_H

#ifdef WIN32   // Windows system specific
#include <windows.h>
#else          // Unix based system specific
#include <sys/time.h>
#endif

class StopWatch {
#ifdef WIN32
    LARGE_INTEGER frequency;                    // ticks per second
    LARGE_INTEGER startCount;                   //
    LARGE_INTEGER endCount;                     //
#else
    timeval startCount;                         //
    timeval endCount;                           //
#endif
    double startTimeInMicroSec;
    double endTimeInMicroSec;
public:
    StopWatch();
    void start();               // start StopWatch and record time to "startCount"
    void stop();               // stop StopWatch and record time to "endCount"
    double getElapsedTime();    // return the elapsed time at the last stop since the last start (microsec)
};

#endif
```

Stop Watch (implementation)

```
/*
 *  StopWatch.cpp
 *  Young-Min Kang
 *  Tongmyong University
 *
 */

#include "StopWatch.h"

StopWatch::StopWatch() {
#ifdef WIN32
    QueryPerformanceFrequency(&frequency);
    startCount.QuadPart = 0;
    endCount.QuadPart = 0;
#else
    startCount.tv_sec = startCount.tv_usec = 0;
    endCount.tv_sec = endCount.tv_usec = 0;
#endif
    startTimeInMicroSec = endTimeInMicroSec = 0.0;
}

void StopWatch::start() {
#ifdef WIN32
    QueryPerformanceCounter(&startCount);
#else
    gettimeofday(&startCount, NULL);
#endif
}

void StopWatch::stop() {
#ifdef WIN32
    QueryPerformanceCounter(&endCount);
#else
    gettimeofday(&endCount, NULL);
#endif
}

double StopWatch::getElapsedTime(){
#ifdef WIN32
    startTimeInMicroSec = startCount.QuadPart * (1000000.0 / frequency.QuadPart);
    endTimeInMicroSec = endCount.QuadPart * (1000000.0 / frequency.QuadPart);
#else
    startTimeInMicroSec = (startCount.tv_sec * 1000000.0) + startCount.tv_usec;
    endTimeInMicroSec = (endCount.tv_sec * 1000000.0) + endCount.tv_usec;
#endif
    return endTimeInMicroSec - startTimeInMicroSec;
}
```

Visualise Your Stop Watch

- Implement your stop watch