Physically-based Modelling

Dynamics and Numerical Integration

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Dynamics

- Dynamics
  - force-based understanding of motions
- Important formula
  - $\mathrm{f} = m\mathrm{a}$
- Rigid body

$$\tau = \mathbf{I}\dot{\omega}$$

Integration of Equations of Motion

- Newton's 2nd law of motion
  - f = ma
- In other words,

$$\mathbf{f} = m\frac{d\mathbf{v}}{dt}$$

- dv = ?

$$\frac{\mathbf{f}dt}{m} = d\mathbf{v}$$

$$\int_{t_1}^{t_2} \frac{\mathbf{f}dt}{m} = \int_{\mathbf{v}_1}^{\mathbf{v}_2} d\mathbf{v}$$

if force is constant

$$\frac{\mathbf{f}}{m}(t_2 - t_1) = \mathbf{v}_2 - \mathbf{v}_1$$

$$\frac{\mathbf{f}}{m}\Delta t = \Delta\mathbf{v}$$

Initial Condition Problem

- Initial conditions
  - x(t), v(t)
  - location and velocity at current time t
- Problem
  - predict x(t+dt) and v(t+dt)
  - predict the location and velocity after a small time step dt
- Repeat with Initial conditions = x(t+dt), v(t+dt)

Velocity update

- Initial condition of velocity
  - v1 = v(t)
- Velocity to be found
  - v2 = v(t+dt)
- Velocity at the next frame

$$\mathbf{v}(t+\Delta t) = \mathbf{v}(t) + \frac{\mathbf{f}(t)}{m}\Delta t$$

  - or

$$\mathbf{v}(t+\Delta t) = \mathbf{v}(t) + \mathbf{a}(t)\Delta t$$

$$\frac{\mathbf{f}}{m}(t_2 - t_1) = \mathbf{v}_2 - \mathbf{v}_1$$

$$\frac{\mathbf{f}}{m}\Delta t = \Delta\mathbf{v}$$

Position Update

- Velocity and Position

$$\mathbf{v} = d\mathbf{s}/dt \qquad\qquad \mathbf{v}dt = d\mathbf{s}$$

- Numerical Integration

$$\mathbf{v}(t + \Delta t)\Delta t = \Delta\mathbf{s}$$

- If force (or acceleration) is known at time *t*
  - We can "approximately" predict velocity and location at time *t*+*dt*

Realtime Simulation

- compute force: f
- compute acceleration: a = f/m
- predict the velocity after time step
  - Euler integration

$$\mathbf{v}(t+\Delta t) = \mathbf{a}\Delta t$$

- predict the location after time step
  - Euler integration

$$\mathbf{x}(t+\Delta t) = \mathbf{v}(t+\Delta t)\Delta t$$

Simulation Code - main.cpp

```
void keyboardFunction(unsigned char key, int x, int y) {
    if (key == 27) exit(0);
    switch (key) {
        case 's':
            if(!myWatch.bRunning()) { Simulator->start(); myWatch.start(); }
            else { myWatch.stop(); Simulator->stop(); }
            break;
        case 'p':
            myWatch.pause(); Simulator->pause();
            break;
        case 'r':
            myWatch.resume(); Simulator->resume();
        default:
            break;
    }
}
void displayFunction(void) {
    …
    // check DT (in microsecond) from StopWatch and store it to "deltaTime" (in seconds)
    deltaTime = myWatch.checkAndComputeDT() / 1000000.0;
    currentTime = myWatch.getTotalElapsedTime() / 1000000.0;
    Simulator->actions(deltaTime, currentTime);
    glutSwapBuffers();
}
```

Simulation Code - Simulator.cpp

```
#include "Simulator.h"
#include <stdlib.h>
#include <stdio.h>
// Constructor
CSimulator::CSimulator(): bRunning(false) {  }
CSimulator::~CSimulator() { }

void CSimulator::actions(double dt, double currentTime) {
    if(bRunning) {
        doBeforeSimulation(dt, currentTime);
        doSimulation(dt, currentTime);
        doAfterSimulation(dt, currentTime);
    }
    visualize();
}
// Control Event Handlers
void CSimulator::start() {
    this->init();
    bRunning = true;
}
void CSimulator::stop() {
    bRunning = false;
    this->clean();
}
void CSimulator::pause() {}
void CSimulator::resume() {}
```

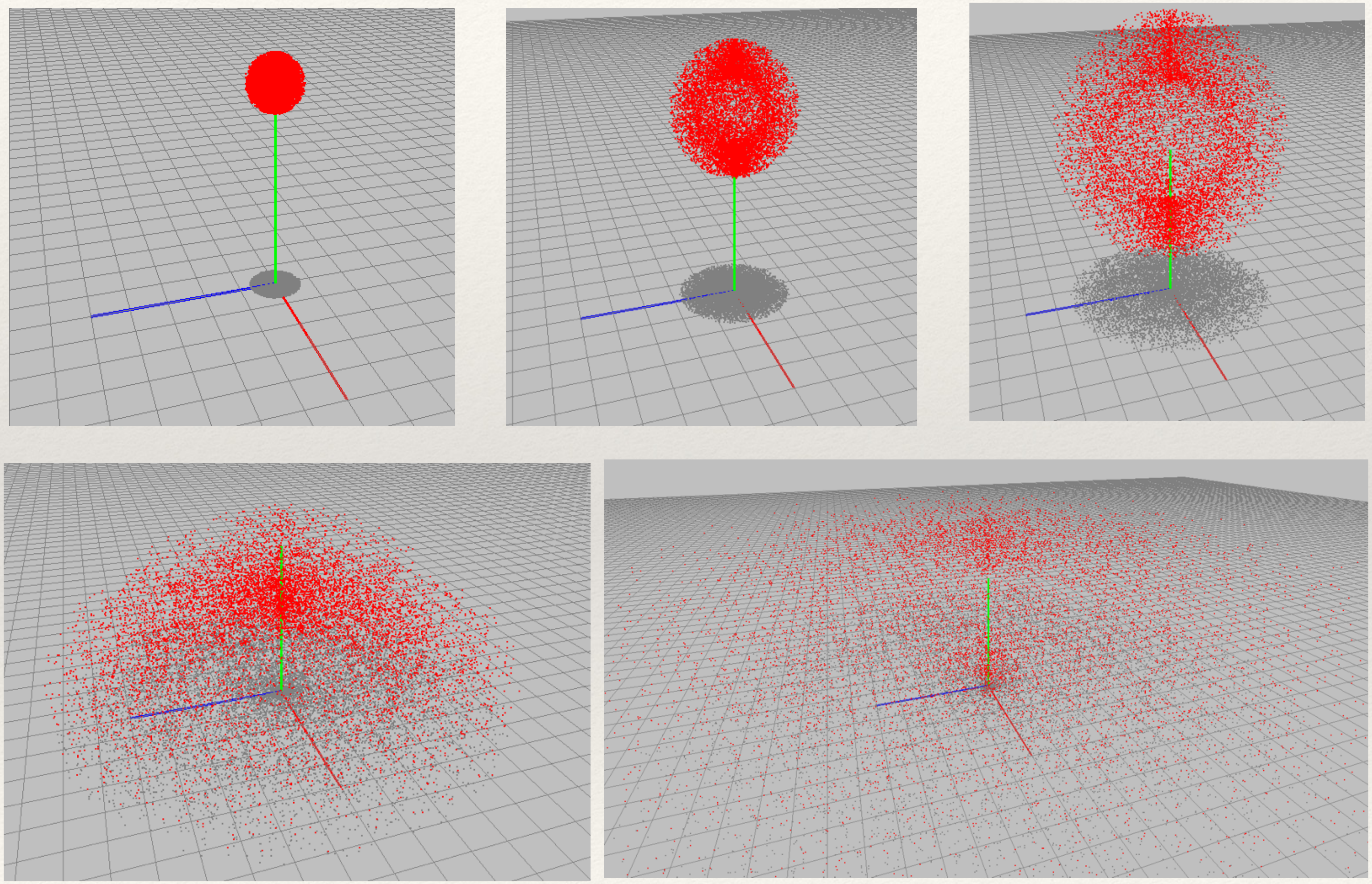
Simulation Code - DynamicSimulator.cpp

```
#include "DynamicSimulator.h"

CDynamicSimulator::CDynamicSimulator() : CSimulator() {}

void CDynamicSimulator::init() {
   for(int i=0;i<NUMPARTS;i++)particle[i].randomInit();
}

void CDynamicSimulator::clean() {}

void CDynamicSimulator::doBeforeSimulation(double dt, double currentTime) {}

void CDynamicSimulator::doSimulation(double dt, double currentTime) {
    for(int i=0;i<NUMPARTS;i++)
        particle[i].simulate(dt, currentTime);

}
void CDynamicSimulator::doAfterSimulation(double dt, double currentTime) {}

void CDynamicSimulator::visualize(void) {
    for(int i=0;i<NUMPARTS;i++) {
        particle[i].drawWithGL(POINT_DRAW);
    }
}
```

Simulation Code - Particle.cpp

```
void CParticle::simulate(double dt, double et) {
    // Update velocity: Euler Integration of acceleration
    vel = vel + dt*gravity;
    // Update position: Euler Integration of velocity
    loc = loc + dt*vel;

    // collision handling
    if(loc[1]<0) {
        loc.set(loc[0], -0.9*loc[1], loc[2]);
        if(vel[1]<0) vel.set(vel[0], -0.9*vel[1], vel[2]);
    }
    setPosition(loc[0], loc[1], loc[2]);
}
```

Result



Forces

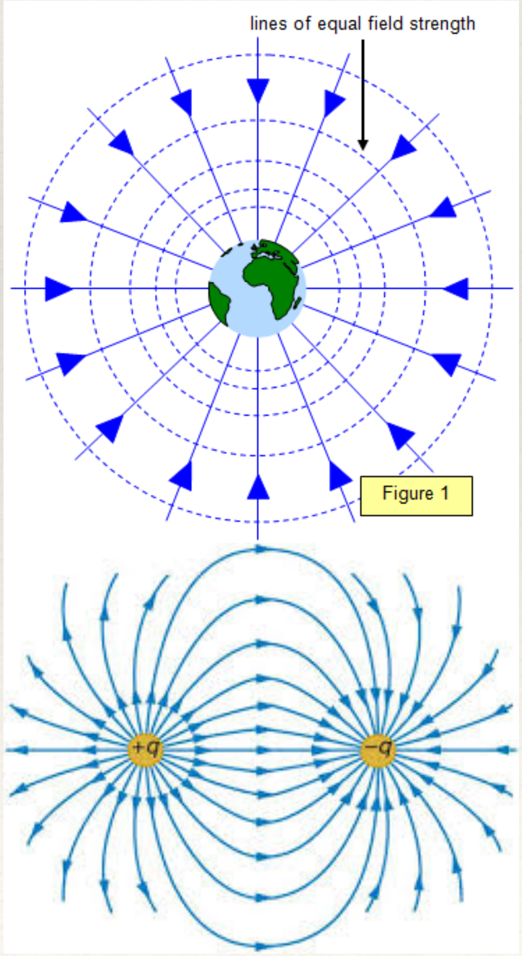
- Forces
  - cause motion
  - very important in dynamics
- Various force models

Concepts

- Force field : example - gravitational force
- Friction: a contact force that resists motion
- Fluid dynamic drag: resistance to objects moving in fluid
- Pressure: force per area
- Buoyancy: force pushing objects "upwards" when immersed in a fluid
- Springs and dampers: elastically ties objects
- Force applies torque to an object, and make it "rotate."

Force field

- Force field
  - a vector field indicating the force exerted by on object on another
- Very good example
  - gravitational force field
  - electromagnetic force field



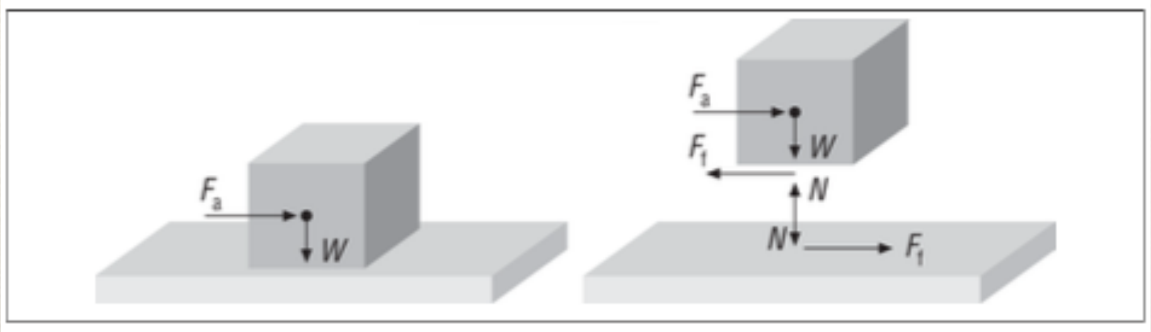
Gravitational force field

- universal gravitational force $$|\mathbf{f}_u| = Gm_1m_2/r^2$$
  - G: gravitation constance $6.673 \times 10^{-11}(N \cdot m^2)/kg^2$
  - r: distance between masses
  - m_{1,2}: mass of each objects
- gravity on Earth
  - mass of Earth: $5.98 \times 10^{24}kg$
  - radius of Earth: 6.38×10^6m
  - gravitational acceleration

$$\frac{Gm_{earth}}{r^2} \simeq (\frac{6.673 \times 5.98}{6.38^2}) \times 10m/s^2 \simeq 9.8034m/s^2$$

friction

- resistance force due to the contacting surfaces
    - contact force
    - normal force: N is important



- two friction
    - static friction: maximum friction

$$|\mathbf{f}_{max}| = \mu_s \mathbf{N}$$

    - kinetic friction

$$|\mathbf{f}_k| = \mu_k \mathbf{N}$$

Coefficients of friction

- Well known surfaces

Surface condition	M_s	M_u	% difference
Dry glass on glass	0.94	0.4	54%
Dry iron on iron	1.1	0.15	86%
Dry rubber on pavement	0.55	0.4	27%
Dry steel on steel	0.78	0.42	46%
Dry Teflon on Teflon	0.04	0.04	—
Dry wood on wood	0.38	0.2	47%
Ice on ice	0.1	0.03	70%
Oiled steel on steel	0.10	0.08	20%

Fluid dynamic drag

- similar to friction
  - friction is also important component of drag
  - but, it's not the only one
- viscous drag for "slow-moving" objects (laminar)
  - $f = -C v$
- for "fast-moving" objects (turbulence)
  - $f = -C v^2$

Pressure

- Pressure is not force
  - pressure = force per unit area
  - $F = PA$ (force = pressure x area)
  - $P = F/A$
- Pressure is important in simulating
  - boats, hovercrafts,…

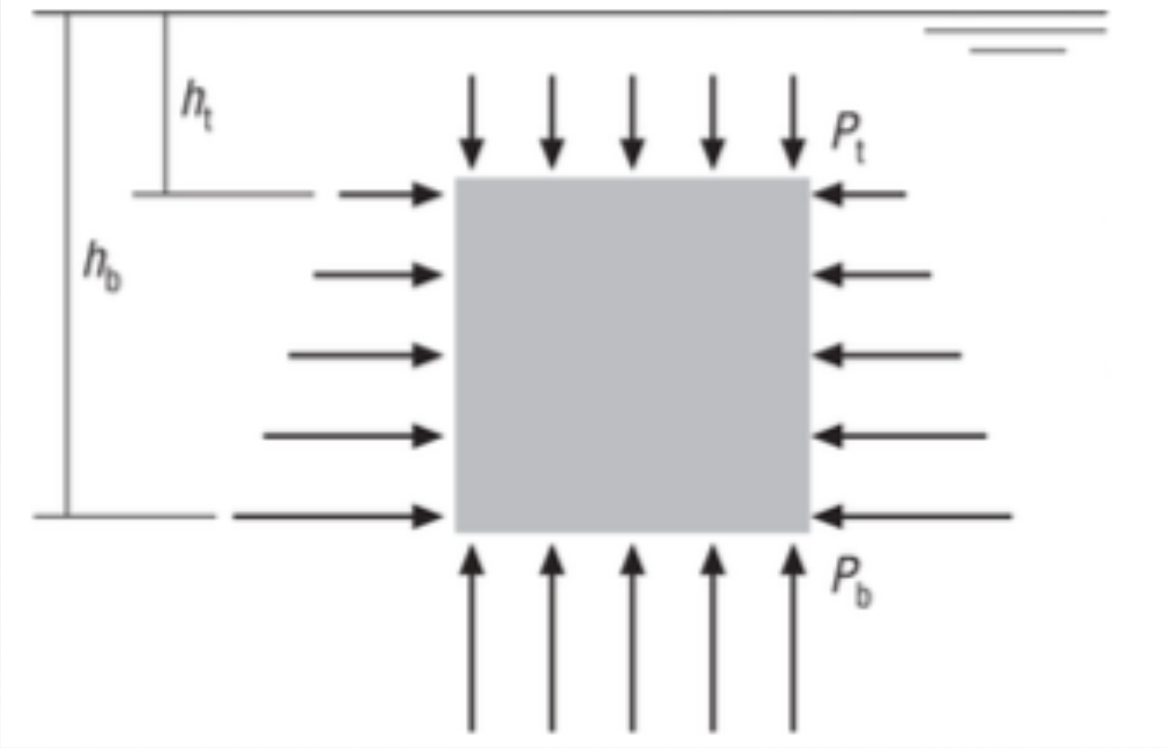
Buoyancy

- Different pressure in fluid
- Horizontal net force = 0
- Vertical net force = bottom force - top force
- F = PA
- Pressures: function(density, gravity)
  - on top surface

$$\mathbf{P}_t = \rho \mathbf{g} h_t$$

  - on bottom surface

$$\mathbf{P}_b = \rho \mathbf{g} h_b$$



Buoyancy

- forces $$\mathbf{f}_t = \mathbf{P}_t A_t = \rho \mathbf{g} h_t s^2$$
  $$\mathbf{f}_b = \mathbf{P}_b A_b = \rho \mathbf{g} h_b s^2$$
- difference

$$\begin{aligned}
\mathbf{f}_b - \mathbf{f}_t &= \rho \mathbf{g} h_b s^2 - \rho \mathbf{g} h_t s^2 \\
&= \rho \mathbf{g} (h_b - h_t) s^2 \\
&= -\rho \mathbf{g} s^3 \\
&= -\rho \mathbf{g} V \quad (V : volume)
\end{aligned}$$

Spring force

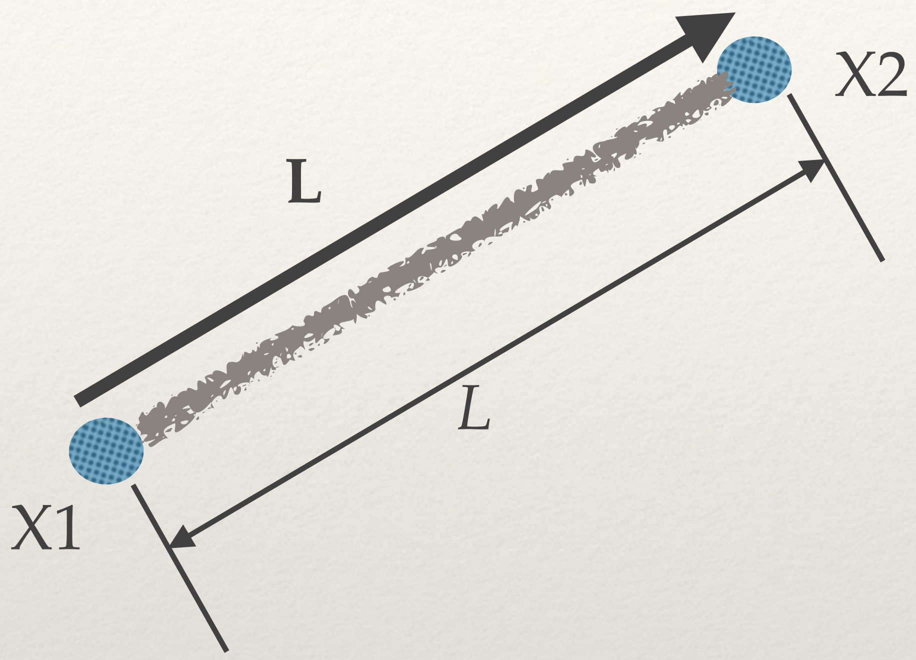
- Hooke's law
  - force needed to extend or compress a spring by a distance x is proportional to that distance
    - f = -k x
    - k: spring constant
- rest length of spring:
- current length of spring:
  - force magnitude: $|\mathbf{f}| = k_s(L - r)$
  - force direction: obj1 and obj2 are linked and located at x1 and x2.
    - $\dfrac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|}$ and $-\dfrac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|}$

Damper

- Springs do not oscillate forever
  - energy dissipation
  - simple model
    - damping force

$$\mathbf{f}_d = k_d(\mathbf{v}_1 - \mathbf{v}_2)$$

Spring and Damper

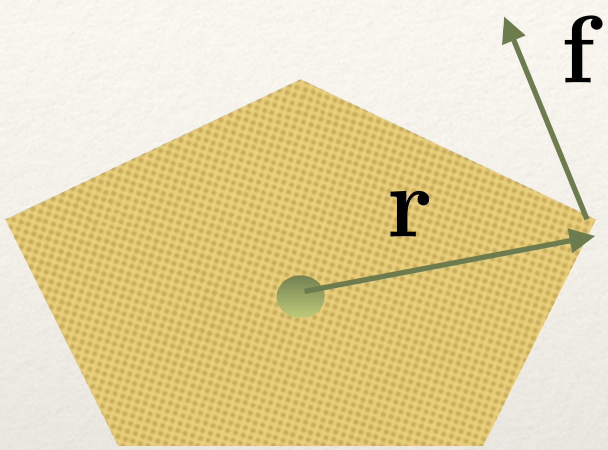


$$\mathbf{f}_1 = -(k_s(L - r) + k_d(\mathbf{v}_1 - \mathbf{v}_2) \cdot \frac{\mathbf{L}}{L})\frac{\mathbf{L}}{L}$$

$$\mathbf{f}_2 = -\mathbf{f}_1$$

Force and Torque

- Force
  - causes linear acceleration
- Torque
  - causes angular acceleration
- Torque: τ
  - a vector
    - magnitude
      - how quickly the angular velocity is changed
        - $|\mathrm{r} \times \mathrm{f}|$
    - direction
      - rotational axis = $(\mathrm{r} \times \mathrm{f}) / |\mathrm{r} \times \mathrm{f}|$



$$\tau = \mathbf{r} \times \mathbf{f}$$