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Abstract

1 Algorithm for Gibbs Sampler

Algorithm 1 Gibbs sampler

Require: Force observations $\mathbf{F}$, Visual observations $\mathbf{X}$

$\mathbf{Z} \leftarrow P(\mathbf{Z})$ {Initialise feature ownership matrix $\mathbf{Z}$ from the prior}
$\mathbf{A} \leftarrow P(\mathbf{A})$ {Initialise force basis matrix $\mathbf{A}$ from the prior}
$\mathbf{Y} \leftarrow P(\mathbf{Y})$ {Initialise image filter basis matrix $\mathbf{Y}$ from the prior}
while gathering posterior samples **do**
  $\mathbf{A} \leftarrow p(\mathbf{A}|\mathbf{F}, \mathbf{Z})$ {Resample $\mathbf{A}$ according to Eq. 1}
  for all $y_{k,t} \in \mathbf{Y}$ **do**
    $y_{k,t} \leftarrow P(y_{kt}|\mathbf{Z}, \mathbf{X}, \mathbf{Y}_{-k,t})$ {Resample each element of $\mathbf{Y}$ according to Eq. 2}
  end for
  for $i = 1$ to N **do**
    for $k = 1$ to K **do**
      $z_{i,k} \leftarrow P(z_{i,k}|\mathbf{f}_{i,:}, \mathbf{x}_{i,:}, \mathbf{A}, \mathbf{Y}, \mathbf{Z}_{-i,k})$ {Resample existing $z_{i,k}$ according to Eq. 5}
    end for
    $K_i^{new} \leftarrow P(K_i^{new}|\mathbf{x}_{i,:}, \mathbf{f}_{i,:}, \mathbf{z}_{i,:}, \mathbf{Y}, \mathbf{A})$ {Sample k_i^{new} according to Eq. 9}
    if $k_i^{new} > 0$ **then**
      $\mathbf{Z}_{new} \leftarrow FIX$ {Get new columns for $\mathbf{Z}$, each with a 1 at row i}
      $\mathbf{A}_{new} \leftarrow p(\mathbf{A}_{new}|\mathbf{F}, \mathbf{Z}_{new}, \mathbf{Z}, \mathbf{A})$ {Get new rows for $\mathbf{A}$, sampled from Eq. 13}
      $\mathbf{Z} \leftarrow [\mathbf{Z}, \mathbf{Z}_{new}]$ {Expand $\mathbf{Z}$ with the new columns}
      $\mathbf{A} \leftarrow [\mathbf{A}; \mathbf{A}_{new}]$ {Expand $\mathbf{A}$ with the new rows}
      $\mathbf{Y}_{new} \leftarrow 0_{k_i^{new} \times T}$ {Get new empty rows for $\mathbf{Y}$}
      $\mathbf{Y} \leftarrow [\mathbf{Y}; \mathbf{Y}_{new}]$ {Expand $\mathbf{Y}$ with the new rows}
      for all $y_{k,t} \in \mathbf{Y}_{K+1:K+k_i^{new},:}$ **do**
        $y_{k,t} \leftarrow P(y_{kt}|\mathbf{Z}, \mathbf{X}, \mathbf{Y}_{-k,t})$ {Resample the new empty parts of $\mathbf{Y}$ according to Eq. 2}
      end for
      $K \leftarrow K + K_i^{new}$
    end if
  end for
end while

2 Prior, Likelihood, and Posterior Terms

2.1 Posterior $p(\mathbf{A}|\mathbf{F}, \mathbf{Z})$

Given the observed forces $\mathbf{F}$, and the binary feature ownership matrix $\mathbf{Z}$, the posterior $\mathbf{A}$ for existing features is sampled from from the following Gaussian

$$p(\mathbf{A}|\mathbf{F}, \mathbf{Z}) \sim \mathcal{N}\left(\left(\mathbf{Z}^T\mathbf{Z} + \frac{\sigma_n^2}{\sigma_A^2}\mathbf{I}\right)^{-1} \mathbf{Z}^T\mathbf{F}, \sigma_n^2 \left(\mathbf{Z}^T\mathbf{Z} + \frac{\sigma_n^2}{\sigma_A^2}\mathbf{I}\right)^{-1}\right) \tag{1}$$

2.2 Posterior $P(y_{k,t}|\mathbf{Z}, \mathbf{X}, \mathbf{Y}_{-k,t})$

When performing updates to $\mathbf{y}_{:,t}$, the t-th column of the image filter basis matrix $\mathbf{Y}$, this impacts the t-th column of the reconstructed images. Accordingly, the posterior from proportional is

$$\begin{aligned} &P(y_{k,t} = a|\mathbf{Z}, \mathbf{X}, \mathbf{Y}_{-k,t}) \\ &\quad \propto P(y_{k,t} = a) P(\mathbf{x}_{:,t}|\mathbf{Z}, \mathbf{y}_{:,t})|_{y_{k,t}=a} \end{aligned} \tag{2}$$

The prior is defined using the hyperparameter ϕ as

$$P(y_{k,t} = a) = \phi^a (1-\phi)^{1-a} \tag{3}$$

And the likelihood is:

$$P(\mathbf{x}_{:,t}|\mathbf{Z}, \mathbf{y}_{:,t}) = \prod_{i=1}^{N} \left[1 - (1-\lambda)^{\mathbf{z}_{i,:}\mathbf{y}_{:,t}}(1-\epsilon)\right]^{x_{i,t}} \times \left[(1-\lambda)^{\mathbf{z}_{i,:}\mathbf{y}_{:,t}}(1-\epsilon)\right]^{1-x_{i,t}} \tag{4}$$

$|_{y_{k,t}=a}$ denotes carrying out the calculation with $y_{k,t}$ temporarily set to a.

2.3 Posterior $P(z_{i,k}|\mathbf{f}_{i,:},\mathbf{x}_{i,:},\mathbf{A},\mathbf{Y},\mathbf{Z}_{-i,k})$

As our force and image observations are conditionally independent given $\mathbf{Z}$, our posterior proportional is

$$\begin{aligned} &P(z_{i,k}=a|\mathbf{f}_{i,:},\mathbf{x}_{i,:},\mathbf{A},\mathbf{Y},\mathbf{Z}_{-i,k}) \\ &\quad \propto P(z_{i,k}=a|\mathbf{Z}_{-i,k})P(\mathbf{x}_{i,:}|\mathbf{Y},\mathbf{z}_{i,:})p(\mathbf{f}_{i,:}|\mathbf{A},\mathbf{z}_{i,:})|_{z_{i,k}=a} \end{aligned} \tag{5}$$

The prior $P(z_{i,k}=a|\mathbf{Z}_{-i,k})$ is defined with $m_{-i,k}=\sum_{j\neq i}^{N} z_{j,k}$ as

$$P(z_{i,k}=a|\mathbf{Z}_{-i,k}) = (\frac{m_{-i,k}}{N})^a(1-\frac{m_{-i,k}}{N})^{1-a} \tag{6}$$

The likelihood term for the visual observation is $P(\mathbf{x}_{i,:}|\mathbf{Y},\mathbf{z}_{i,:})$ is

$$P(\mathbf{x}_{i,:}|\mathbf{Y},\mathbf{z}_{i,:}) = \prod_{t=1}^{T}\left[1-(1-\lambda)^{\mathbf{z}_{i,:}\mathbf{y}_{:,t}}(1-\epsilon)\right]^{x_{i,t}} \times \left[(1-\lambda)^{\mathbf{z}_{i,:}\mathbf{y}_{:,t}}(1-\epsilon)\right]^{1-x_{i,t}} \tag{7}$$

and the likelihood term for the force observation is $p(\mathbf{f}_{i,:}|\mathbf{A},\mathbf{z}_{i,:})$ is simply

$$p(\mathbf{f}_{i,:}|\mathbf{A},\mathbf{z}_{i,:}) = \frac{1}{(2\pi\sigma_n^2)^{D/2}}\exp\left(-\frac{1}{2\sigma_n^2}(\mathbf{f}_{i,:}-\mathbf{z}_{i,:}\mathbf{A})^T(\mathbf{f}_{i,:}-\mathbf{z}_{i,:}\mathbf{A})\right) \tag{8}$$

As previously, The symbol $|_{z_{i,k}=a}$ denotes carrying out the calculation with $z_{i,k}$ temporarily set to a.

2.4 Posterior $P(K_i^{new}|\mathbf{x}_{i,:},\mathbf{f}_{i,:},\mathbf{z}_{i,:},\mathbf{Y},\mathbf{A})$

During the process of sampling, it might be necessary to increase the number of latent features. As new features is initially only active for the i-th row of $\mathbf{z}$, we only need to consider how this effects the i-th observation.

$$\begin{aligned} &P(K_i^{new}|\mathbf{x}_{i,:},\mathbf{f}_{i,:},\mathbf{z}_{i,:},\mathbf{Y},\mathbf{A}) \\ &\quad \propto P(k_i^{new})P(\mathbf{x}_{i,:}|\mathbf{z}_{i,:},\mathbf{Y},K_i^{new})p(\mathbf{f}_{i,:}|\mathbf{z}_{i,:},\mathbf{A},K_i^{new}) \end{aligned} \tag{9}$$

The prior $P(k_i^{new})$ is simply a Poisson:

$$P(k_i^{new}) = \text{Pois}\left(\frac{\alpha}{N}\right) \tag{10}$$

The likelihood term for the visual observations

$$\begin{aligned} &P(\mathbf{x}_{i,:}|\mathbf{z}_{i,:},\mathbf{Y},K_i^{new}) \\ &= \prod_{t=1}^{T}\left[1-(1-\epsilon)(1-\lambda)^{\mathbf{z}_{i,:}\mathbf{y}_{:,t}}(1-\lambda\phi)^{K_i^{new}}\right]^{x_{i,t}} \times \left[(1-\epsilon)(1-\lambda)^{\mathbf{z}_{i,:}\mathbf{y}_{:,t}}(1-\lambda\phi)^{K_i^{new}}\right]^{1-x_{i,t}} \end{aligned} \tag{11}$$

To derive the likelihood for the force observations,

$$\begin{aligned} &p(\mathbf{f}_{i,:}|\mathbf{z}_{i,:},\mathbf{A},K_i^{new}) = \int p(\mathbf{f}_{i,:}|\mathbf{z}_{i,:},\mathbf{A},\mathbf{A}_{new},K_i^{new})p(\mathbf{A}_{new})\mathrm{d}\mathbf{A}_{new} \\ &= \frac{1}{\sqrt{(2\pi)^D|1_{k_{new}\times k_{new}}+\frac{\sigma_n^2}{\sigma_A^2}\mathbf{I}|}}\exp\left(-\frac{1}{2}(\mathbf{f}_{i,:}-\mathbf{z}_{i,:}\mathbf{A})^T\left(1_{k_{new}\times k_{new}}+\frac{\sigma_n^2}{\sigma_A^2}\mathbf{I}\right)^{-1}(\mathbf{f}_{i,:}-\mathbf{z}_{i,:}\mathbf{A})\right) \end{aligned} \tag{12}$$

2.5 Posterior $p(\mathbf{A}_{new}|\mathbf{F},\mathbf{Z}_{new},\mathbf{Z},\mathbf{A})$

When k_{new} features were added to $\mathbf{Z}$ in the form of $\mathbf{Z}_{new}$, $\mathbf{A}$ must also be expanded with k_{new} rows.

$$\begin{aligned} &p(\mathbf{A}_{new}|\mathbf{F},\mathbf{Z}_{new},\mathbf{Z},\mathbf{A}) \\ &\quad \propto p(A_{new})p(\mathbf{F}|\mathbf{Z}_{new},\mathbf{Z},\mathbf{A},\mathbf{A}_{new}) \\ &\quad \sim \mathcal{N}\left(\left(1_{k_{new}\times k_{new}}+\frac{\sigma_n^2}{\sigma_A^2}\mathbf{I}\right)^{-1}\mathbf{Z}_{new}^T(\mathbf{F}-\mathbf{Z}\mathbf{A}),\sigma_n^2\left(1_{k_{new}\times k_{new}}+\frac{\sigma_n^2}{\sigma_A^2}\mathbf{I}\right)^{-1}\right) \end{aligned} \tag{13}$$