Articulation Points

Let $G = (V, E)$ be a connected, undirected graph.
An *articulation point* (also *cut vertex*) of G
is a vertex whose removal disconnects G.

We can find all articulation points in a graph using depth-first search,
so let T be a depth-first search tree.

1. The root of T is an articulation point iff ...
   it has two or more children.

   Proof: If the root has less than two children, then deleting the root does not disconnect T so the root is not an articulation point.

   If the root has two or more children, then because there are no cross edges (G is undirected), every path from one child of the root to another contains the root. Therefore, the root is an articulation point. □

2. A non-root vertex v of T is an articulation point iff ...
   it has at least one child w and there is no back edge from a descendant of w to a proper ancestor of v.

   Proof: If there is no back edge from a descendant of w to a proper ancestor of v, then, because there are no cross edges, every path from w to the parent of v contains v.

   Suppose that there is a back edge from a descendant of each child of v to a proper ancestor of v ... (an exercise for the reader). □

The first item is easy to check, but how do we solve the second?
Consider the following function:

$$low[v] = \min\{d[v]\} \cup \{d[x] : (u, x) \text{ a back edge from some descendant } u \text{ of } v\}.$$

If we compute $low[v]$ for each vertex v, then v is an articulation point iff ...
$low[w] \geq d[v]$ for some child w of v in T.

Proof: We prove that there is a back edge from a descendant of a child w of v to a proper ancestor of v iff $low[w] \geq d[v]$.

If there is a back edge from a descendant to a proper ancestor x of v,
then $low[v] \leq d[x]$.
Since x is a proper ancestor of v, then $d[x] < d[v]$ so $low[v] < d[v]$.

If $low[w] \leq d[v]$, then there is a back edge from some descendant u of v to a vertex x with $d[x] < d[v]$.
Thus, x was discovered before v is x is a proper ancestor of v. □

We can compute $low[v]$ using dynamic programming (yes, that again!).
Let v be a vertex with children $w_1, w_2, \ldots, w_c$. Then, we have:

$$low[v] = \min\{d[v]\} \cup \{d[w] : (v, w) \text{ is a back edge}\} \cup \{low[w_i] : 1 \leq i \leq c\}.$$

The running time is ... $O(|E|)$.