Optimization Methods. Seminar 2. Convex sets.

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Reminder

- Objective of the Optimization Methods course
- General optimization problem statement
- Examples of optimization problems:
    - linear programming
    - least squares problem
    - convex optimization problems
- Why convex optimization problems are good?

Affine sets

Affine set

Let A be affine set if for any x_1, $x_2 \in A$ and $\theta \in \mathbb{R}$ point $\theta x_1 + (1-\theta)x_2 \in A$.

Examples: $\mathbb{R}^n$, hyperplane, single point.

Affine combination of points

Assume $x_1, \dots, x_k \in G$, then point $\theta_1 x_1 + \ldots + \theta_k x_k$ is called affine combination of the points $x_1, \dots, x_k$ if $\sum\limits_{i=1}^{k} \theta_i = 1$.

Affine hull of set

A set $\left\{ \sum\limits_{i=1}^{k} \theta_i x_i \mid x_i \in G, \sum\limits_{i=1}^{k} \theta_i = 1 \right\}$ is called affine hull of set G and is denoted by $\mathbf{aff}(\mathrm{G})$.

Claims

Claim 1

A set G is affine if and only if it contains all affine combinations of points from G.

Claim 2

A set G is affine if and only if it can be represented in the form $G = \{\mathbf{x} | \mathbf{A}\mathbf{x} = \mathbf{b}\}$.

Convex set

Convex set

Let C be a convex set if

$$\forall x_1,\ x_2 \in C, \theta \in [0,1] \to \theta x_1 + (1-\theta) x_2 \in C.$$

$\emptyset$ and $\{x_0\}$ are also convex by definition.

Examples: $\mathbb{R}^n$, affine set, half-open segment, segment.

Convex combination of points

Assume $x_1, \ldots, x_k \in G$, then point $\theta_1 x_1 + \ldots + \theta_k x_k$ is called convex combination of points $x_1, \ldots, x_k$ if $\sum\limits_{i=1}^{k} \theta_i = 1,\ \theta_i \geq 0$.

Convex hull

A set $\left\{ \sum\limits_{i=1}^{k} \theta_i x_i \mid x_i \in G, \sum\limits_{i=1}^{k} \theta_i = 1, \theta_i \geq 0 \right\}$ is called convex combination of a set G and is denoted by $\mathbf{conv}(\mathrm{G})$.

Operations that preserve convexity

- Intersection of any number (finite or infinite) of convex sets is convex set.
- Image of convex set under affine function is convex function
- Linear combination of convex sets is convex
- Cartesian product of convex sets is convex
- Perspective function

$$f : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}^n \quad f(\mathbf{x}) = \begin{bmatrix} x_1/x_{n+1} \\ \vdots \\ x_n/x_{n+1} \end{bmatrix}$$

maps convex set to another convex set

Examples

Check if the following sets are affine and/or convex:

1. Half-space: $\{\mathbf{x} | \mathbf{a}^\intercal \mathbf{x} \leq c\}$
2. Polyhedron: $\{\mathbf{x} | \mathbf{A}\mathbf{x} \preceq \mathbf{b},\ \mathbf{C}\mathbf{x} = 0\}$
3. A ball induced by norm in $\mathbb{R}^n$:
   $B(r, x_c) = \{x \mid \|x - x_c\| \leq r\}$
4. Ellipsoid: $\mathcal{E}(x_c, \mathbf{P}, r) = \{x \mid (x - x_c)^\intercal \mathbf{P}^{-1}(x - x_c) \leq r\}$
5. A set of symmetric and positive-definite matrices:
   $\mathbf{S}^n_+ = \{\mathbf{X} \in \mathbb{R}^{n \times n} \mid \mathbf{X}^\intercal = \mathbf{X},\ \mathbf{X} \succeq 0\}$
6. $\{\mathbf{X} \in \mathbb{R}^{n \times n} \mid \mathrm{Tr}(\mathbf{X}) = const\}$
7. Hyberbolic set: $\{\mathbf{x} \in \mathbb{R}^n_+ \mid \prod\limits_{i=1}^{n} x_i \geq 1\}$

Cone

Cone (convex)

Let a set C be a cone (convex cone), if

$$\forall x \in C, \theta \geq 0 \to \theta x \in C$$

$$(\forall x_1, x_2 \in C, \theta_1, \theta_2 \geq 0 \to \theta_1 x_1 + \theta_2 x_2 \in C)$$

Examples: $\mathbb{R}^n$, affine set with 0, half-open segment.

Conical (non-negative) combination of points

Assume $x_1, \ldots, x_k \in G$, then point $\theta_1 x_1 + \ldots + \theta_k x_k$ is called conic (non-negative) combination of points $x_1, \ldots, x_k$ if $\theta_i \geq 0$.

Conical hull

A set $\left\{ \sum\limits_{i=1}^{k} \theta_i x_i \mid x_i \in G, \theta_i \geq 0 \right\}$ is called conical hull of a set G and is denoted by **cone**(G).

Examples

1. $\mathbf{S}^n_+$
2. Normal cone: $\{(\mathbf{x}, t) \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\| \leq t\}$
   In case of ℓ_2 norm it is called second-order cone or Lorentz cone
3. Some special cases

Recap

- Affine set
- Convex set
- Cone
- Methods to check properties of given set