Лекция 11. ур. Фоккера Планка.

Уравнение Фоккера-Планка

- Пусть $x(t)$ $-$ Марковский случайный процесс.
- Определим $\delta x(t)$=$x(t+\delta t)-x(t)$.
- Постулируем для малых δt:

1) $\langle \delta x(t) \rangle$=$F_1(x(t))\ \delta t$

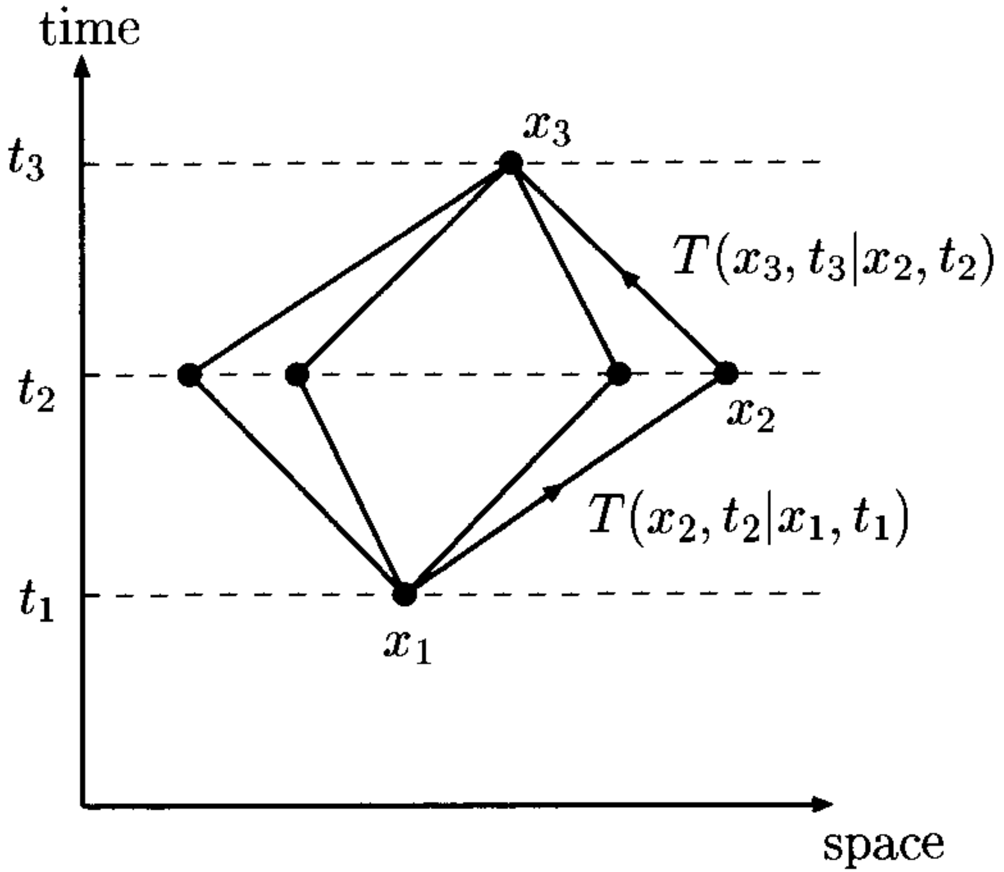
2) $\langle (\delta x(t))^2 \rangle$=$F_2(x(t))\ \delta t$

3) $\langle (\delta x(t))^n \rangle$=$O((\delta t\ )^2)$, n>2.

уравнения Чепмена-Колмогорова

$$T(x_2,t_2 \mid x_0,t_0) = \int dx_1 T(x_2,t_2 \mid x_1,t_1) T(x_1,t_1 \mid x_0,t_0)$$

$$T(x,t+\delta t \mid x_0,t_0) = \int dx' T(x,t+\delta t \mid x',t) T(x',t \mid x_0,t_0)$$



Детерминистический процесс

$$\frac{dx}{dt} = g(x(t)), \qquad (1)$$

$$T(x,t\,|\,x',t') = \delta(x - \Phi_{t-t'}(x'))$$

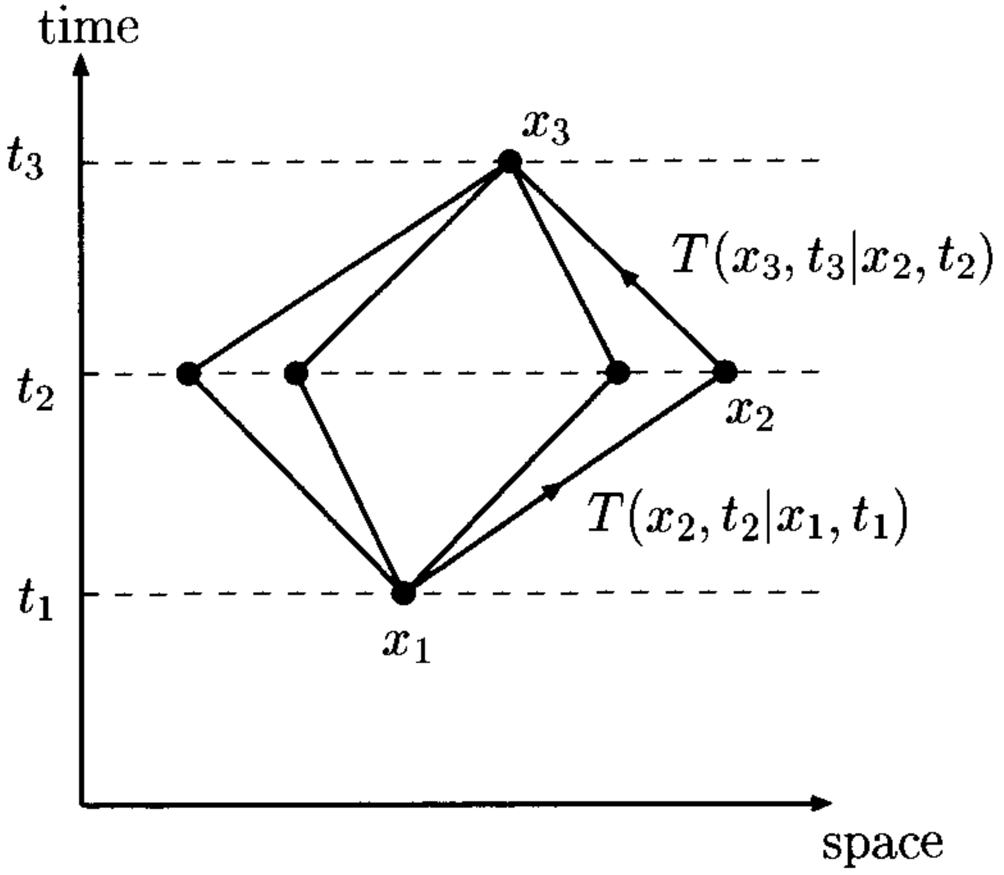
$\Phi_{t-t'}(x')$ – решение ур. (1) с начальным условием: $x(t = t') = x'$.

Стохастическое уравннеие

$$\frac{dx}{dt} = g(x(t)) + \xi(t), \qquad (1)$$

$$T(x,t\,|\,x',t') = \left\langle \delta(x - \Phi_{t-t'}(x')) \right\rangle_{\xi}$$

$\Phi_{t-t'}(x')$ – решение стохастического ур. (1)
с начальным условием: $x(t = t') = x'$.



$$T(x,t\mid x',t')=\left\langle \delta(x-\Phi_{t-t'}(x'))\right\rangle_{\xi}$$

$\Phi_{t-t'}(x')$ – решение стохастического ур. (1)
с начальным условием: $x(t = t') = x'$.

$$\frac{dx}{dt} = g(x(t)) + \xi(t), \qquad (1)$$

$$T(x,t \mid x',t') = \left\langle \delta(x - \Phi_{t-t'}(x')) \right\rangle_{\xi}$$

$\Phi_{t-t'}(x')$ – решение стохастического ур. (1)
с начальным условием: $x(t = t') = x'$.

$\delta x(t) = x(t + \delta t) - x(t)$,
Динамика согласно уравнению (1).

$$\Phi_{\delta t}(x') = x' + \delta x(t)$$

$$\boxed{T(x, t + \delta t \mid x', t) = \left\langle \delta(x - \delta x(t) - x') \right\rangle.}$$

$$\frac{dx}{dt} = g(x(t)) + \xi(t),$$

$$\boxed{T(x, t+\delta t \mid x', t) = \langle \delta(x - \delta x(t) - x') \rangle.}$$

$$\boxed{\begin{aligned}
&T(x, t+\delta t \mid x', t) = \langle \delta(x - \delta x(t) - x') \rangle = \\
&= \left\{ 1 + \langle \delta x(t) \rangle \frac{d}{dx'} + \frac{1}{2} \left\langle [\delta x(t)]^2 \right\rangle \frac{d^2}{dx'^2} + \ldots \right\} \delta(x - x') = \\
&= \delta(x - x') + \delta t F_1(\mathrm{x}') \frac{d}{dx'} \delta(x - x') + \frac{1}{2} \delta t F_2(\mathrm{x}') \frac{d^2}{dx'^2} \delta(x - x') + O\left((\delta t)^2\right).
\end{aligned}}$$

$$T(x,t+\delta t\,|\,x',t)=\delta(x-x')+\delta t F_1(\mathrm{x}')\frac{d}{dx'}\delta(x-x')+\frac{1}{2}\delta t F_2(\mathrm{x}')\frac{d^2}{dx'^2}\delta(x-x')+O\left((\delta t)^2\right).$$

$$T(x,t+\delta t\,|\,x_0,t_0)=\int dx'\,T(x,t+\delta t\,|\,x',t)\,T(x',t\,|\,x_0,t_0)$$

$$T(x,t+\delta t\,|\,x_0,t_0)=T(x,t\,|\,x_0,t_0)-\delta t\frac{d}{dx}\left[F_1(x)T(x,t\,|\,x_0,t_0)\right]+\frac{1}{2}\delta t\frac{d^2}{dx^2}\left[F_2(x)T(x,t\,|\,x_0,t_0)\right].$$

$$\frac{\partial}{\partial t}T(x,t\,|\,x_0,t_0)=\frac{T(x,t+\delta t\,|\,x_0,t_0)-T(x,t\,|\,x_0,t_0)}{\delta t}=$$

$$=-\frac{d}{dx}\left[F_1(x)T(x,t\,|\,x_0,t_0)\right]+\frac{1}{2}\frac{d^2}{dx^2}\left[F_2(x)T(x,t\,|\,x_0,t_0)\right].$$

Итак, Уравнение Фоккера-Планка

$$\frac{\partial}{\partial t}T(x,t\,|\,x_0,t_0)=-\frac{d}{dx}\left[F_1(x)T(x,t\,|\,x_0,t_0)\right]+\frac{1}{2}\frac{d^2}{dx^2}\left[F_2(x)T(x,t\,|\,x_0,t_0)\right].$$

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3) $\langle (\delta x(t))^n \rangle$=$O((\delta t)^2)$, n>2.

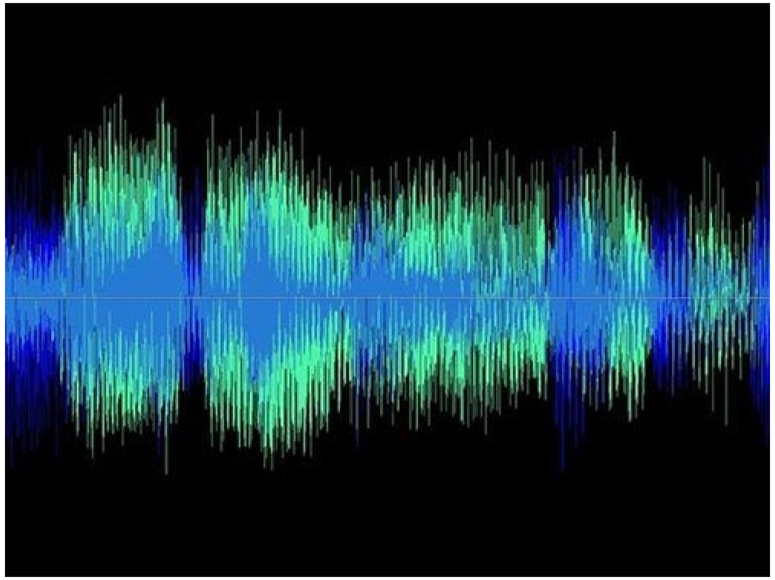
Уравнение Ланжевена

$$\frac{dx}{dt} = g(x(t)) + \xi(t),$$

$$\boxed{T(x, t+\delta t \mid x', t) = \left\langle \delta(x - \delta x(t) - x') \right\rangle.}$$

$$\delta x(t) = x(t+\delta t) - x(t) \approx g(x(t))\delta t + \int_t^{t+\delta t} \xi(t')dt',$$

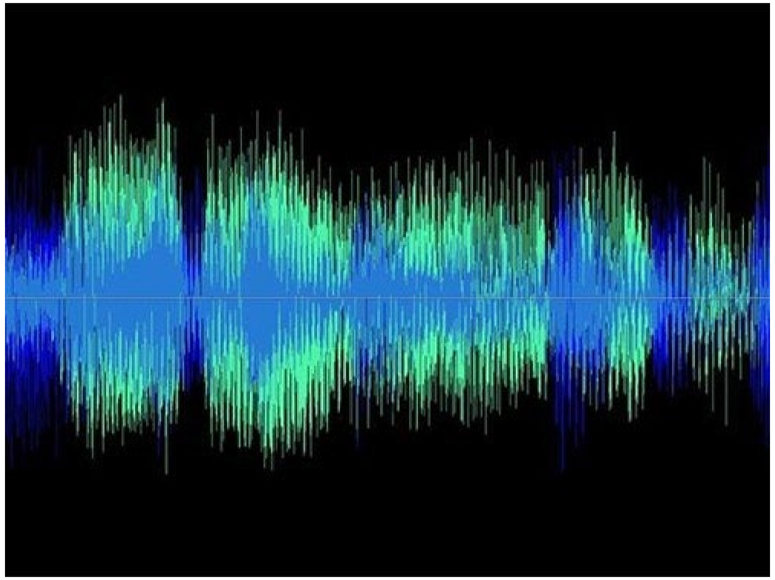
$$\delta x(t) = x(t+\delta t) - x(t) \approx g(x(t))\delta t + \int_{t}^{t+\delta t} \xi(t')dt',$$



$$F_1(x) = \langle \delta x(t) \rangle \approx g(x(t))\delta t + \int_{t}^{t+\delta t} \langle \xi(t') \rangle dt' = g(x(t))\delta t.$$

$$\delta x(t) = x(t+\delta t) - x(t) \approx g(x(t))\delta t + \int_t^{t+\delta t} \xi(t')dt',$$

$$\langle \xi(t_1)\xi(t_2)\rangle = D\delta(t_1 - t_2)$$



$$F_2(x) = \left\langle \left[\delta x(t)\right]^2 \right\rangle \approx \left\langle \left[g(x(t))\delta t + \int_t^{t+\delta t} \xi(t')dt' \right]^2 \right\rangle =$$

$$= \int_t^{t+\delta t} \left\langle \xi(t_1)\xi(t_2) \right\rangle dt_1 dt_2 + \mathrm{O}\left((\delta t)^2\right) = D\delta t + \mathrm{O}\left((\delta t)^2\right).$$

Подведем итоги.

Из ур. Ланжевена можно получить ур. Фоккера Планка:

$$\frac{\partial}{\partial t}T(x,t\,|\,x_0,t_0) = -\frac{d}{dx}\left[g(x)T(x,t\,|\,x_0,t_0)\right] + \frac{1}{2}\frac{d^2}{dx^2}\left[D(x)T(x,t\,|\,x_0,t_0)\right].$$

$$\frac{dx_i}{dt} = g_i(x(t)) + \xi_i(t),$$

$$\frac{\partial}{\partial t}T(x,t\,|\,x_0,t_0) = -\frac{d}{dx_i}\left[g_i(x)T(x,t\,|\,x_0,t_0)\right] + \frac{1}{2}\frac{d^2}{dx_i dx_j}\left[D_{ij}(x)T(x,t\,|\,x_0,t_0)\right].$$

Гауссовский шум: $\langle \xi_i(t_1)\xi_j(t_2)\rangle = D_{ij}\delta(t-t')$

Примеры, иллюстрирующие связь ур. Ланжевена и ур. Фоккера-Планка

Примеры. Движение в “вязкой” среде

$$\frac{d}{dt}\begin{pmatrix} r \\ p \end{pmatrix} = \begin{pmatrix} p/m \\ -\gamma p - \nabla U \end{pmatrix} + \begin{pmatrix} \xi_r(t) \\ \xi_p(t) \end{pmatrix},$$

$$\left\langle \xi_p(t)\xi_p(t') \right\rangle = D\delta(\mathrm{t} - \mathrm{t'}), \qquad \xi_r(t) \equiv 0.$$

$$\frac{\partial}{\partial t}\mathrm{f}(r,p,t) = -\frac{d}{dx_i}\left[g_i\,\mathrm{f}(r,p,t)\right] + \frac{1}{2}\frac{d^2}{dx_i dx_j}\left[D_{ij}\,\mathrm{f}(r,p,t)\right], \quad x = \begin{pmatrix} r \\ p \end{pmatrix}.$$

$$\boxed{\frac{\partial}{\partial t}\mathrm{f}(r,p,t) = -\frac{d}{dr}\left[\frac{p}{m}\mathrm{f}(r,p,t)\right] + \frac{d}{dp}\left[\left(\gamma p + \nabla U\right)\mathrm{f}(r,p,t)\right] + \frac{1}{2}\frac{d^2}{dp^2}\left[Df(r,p,t)\right].}$$

$$\boxed{\frac{\partial}{\partial t}\mathrm{f}(r,p,t)=-\frac{\partial}{\partial r}\left[\frac{p}{m}\mathrm{f}(r,p,t)\right]+\frac{\partial}{\partial p}\left[\left(\gamma p+\nabla U\right)\mathrm{f}(r,p,t)\right]+\frac{1}{2}\frac{\partial^2}{\partial p^2}\left[Df(r,p,t)\right].}$$

$$\boxed{\begin{aligned}&\frac{\partial}{\partial t}f(r,p,t)=-\frac{d}{dr}j_r-\frac{d}{dp}j_p,\\&j_r=\frac{p}{m}f(r,p,t),\\&j_p=-\left(\gamma p+\nabla U\right)f(r,p,t)-\frac{1}{2}D\frac{\partial}{\partial p}f(r,p,t).\end{aligned}}$$

Могут ли случайные силы появиться в кинетическом уравнении?

Вернемся к задаче:

$$\frac{d}{dt}\begin{pmatrix} r \\ p \end{pmatrix} = \begin{pmatrix} p/m \\ -\gamma p - \nabla U \end{pmatrix} + \begin{pmatrix} \xi_r(t) \\ \xi_p(t) \end{pmatrix},$$

$$\left\langle \xi_p(t)\xi_p(t') \right\rangle = D\delta(\mathrm{t}-\mathrm{t}'), \qquad \xi_r(t) \equiv 0.$$

Мы формально имеем дело с обыкновенным диф. уравнением, решения которого, как мы давно знаем, определяют марковский случайный процесс.

Вспомним былые лекции… (лекцию 5)

Детерминистический процесс

$$\frac{dx}{dt} = g(x(t)), x \in R^N$$

$$T(x,t \mid x',t') = \delta(x - \Phi_{t-t'}(x'))$$

$\Phi_{t-t'}(x')$ – решение ур. (1) с начальным условием: $x(t = t') = x'$.

$$\boxed{\frac{\partial}{\partial t} p(x,t) = -\hat{\mathrm{L}}(x,t) \cdot p(x,t).}$$

$$\boxed{\hat{L} = \frac{d}{dx} g(x)} \Rightarrow \boxed{\frac{\partial}{\partial t} p(x,t) + \frac{d}{dx}\left(g(x) p(x,t)\right) = 0}$$

$$\boxed{\frac{\partial}{\partial t} p(x,t) + \mathrm{div}\left(j(x,t)\right) = 0}$$

$$\boxed{j(x,t) = g(x) p(x,t)}$$

Пример: Второй Закон Ньютона

$$\dot{r} = \frac{\partial H}{\partial p},$$
$$\dot{p} = -\frac{\partial H}{\partial r}.$$

$$x = q = (r, p),$$
$$g(x) = \left(\frac{\partial H}{\partial p}, \quad -\frac{\partial H}{\partial r} \right) = \left(\mathrm{v}(p), \quad F(r) \right)$$

Плотность вероятности обозначим p(x,t)=f(r,p,t). Получаем:

$$\frac{\partial}{\partial t} \mathrm{f}(r, p, t) + \frac{\partial}{\partial r}\left(\mathrm{v}(p)\, \mathrm{f}(r, p, t) \right) + \frac{\partial}{\partial p}\left(\mathrm{F}(r)\, \mathrm{f}(r, p, t) \right) = 0.$$

$$\frac{dx}{dt} = g(x(t)), x \in R^N$$

$$\frac{\partial}{\partial t} p(x,t) + \frac{d}{dx}\left(g(x)p(x,t)\right) = 0$$

$$\frac{d}{dt}\begin{pmatrix} r \\ p \end{pmatrix} = \begin{pmatrix} \mathrm{v}(p) \\ -\gamma p - \nabla U \end{pmatrix} + \begin{pmatrix} \xi_r(t) \\ \xi_p(t) \end{pmatrix} = \mathbf{g}(r,p),$$

$$\left\langle \xi_p(t)\xi_p(t')\right\rangle = D\delta(\mathrm{t}-\mathrm{t}'), \quad \xi_r(t) \equiv 0.$$

$$\frac{\partial}{\partial t}\mathrm{f}(r,p,t) + \frac{\partial}{\partial r}\left(\mathrm{v}(p)\,\mathrm{f}(r,p,t)\right) + \frac{\partial}{\partial p}\left(\left(-\gamma p - \nabla U + \xi_p(t)\right)\mathrm{f}(r,p,t)\right) = 0.$$

$$\frac{\partial}{\partial t}\mathrm{f}(r,p,t)+\frac{\partial}{\partial r}\left(\mathrm{v}(p)\,\mathrm{f}(r,p,t)\right)+\frac{\partial}{\partial p}\left(\left(-\gamma p-\nabla U+\xi_p(t)\right)\mathrm{f}(r,p,t)\right)=0.$$

$$\Downarrow$$

$$\frac{\partial}{\partial t}f(r,p,t)=\hat{\Omega}f=-\left[\frac{\partial}{\partial r}\left(\mathrm{v}(p)\,\mathrm{f}(r,p,t)\right)+\frac{\partial}{\partial p}\left(\left(-\gamma p-\nabla U+\xi_p(t)\right)\mathrm{f}(r,p,t)\right)\right].$$

$$\Downarrow$$

$$f(r,p,t+dt)=\left[1+\int_t^{t+dt}\hat{\Omega}(t_1)dt_1+\int_t^{t+dt}\hat{\Omega}(t_1)dt_1\int_t^{t_1}\hat{\Omega}(t_2)dt_2+\ldots\right]f(r,p,t).$$

$$\frac{\partial}{\partial t} f(r,p,t) = \hat{\Omega} f$$

$$\int_t^{t+dt} \frac{\partial}{\partial t} f(r,p,t')dt' = \int_t^{t+dt} \hat{\Omega} f\,dt'$$

$$f(r,p,t+dt) = f(r,p,t) + \int_t^{t+dt} \hat{\Omega}(t') f\,dt',$$

$$f^{(0)}(r,p,t+dt) \approx f(r,p,t)$$

$$f^{(1)}(r,p,t+dt) \approx f(r,p,t) + \int_t^{t+dt} \hat{\Omega}(t')dt' f(r,p,t),$$

$$f^{(2)}(r,p,t+dt) \approx f(r,p,t) + \int_t^{t+dt} \hat{\Omega}(t') f^{(1)}(r,p,t')dt' =$$

$$= f(r,p,t) + \int_t^{t+dt} \hat{\Omega}(t')dt' f(r,p,t) + \int_t^{t+dt} \hat{\Omega}(t') \int_t^{t'} \hat{\Omega}(t'')dt''dt' f(r,p,t)$$

$$\frac{\partial}{\partial t} f(r,p,t) = \hat{\Omega} f = -\left[\frac{\partial}{\partial r}\Big(\mathrm{v}(p)\,\mathrm{f}(r,p,t)\Big) + \frac{\partial}{\partial p}\Big(\big(-\gamma p - \nabla U + \xi_p(t)\big)\,\mathrm{f}(r,p,t)\Big) \right].$$

$$\left\langle f(r,p,t+dt) \right\rangle_\xi = \left\langle \left[1 + \int_t^{t+dt} \hat{\Omega}(t_1)dt_1 + \int_t^{t+dt} \hat{\Omega}(t_1)dt_1 \int_t^{t_1} \hat{\Omega}(t_2)dt_2 + \ldots \right] f(r,p,t) \right\rangle_\xi .$$

Случайная сила δ-коррелирована во времени! Усредним по флуктуациям случайной силы не везде… А только при $t \in [t, t + dt]$.

$$\left\langle f(r,p,t+dt) \right\rangle_\xi = \left\langle \left[1 + \int_t^{t+dt} \hat{\Omega}(t_1)dt_1 + \int_t^{t+dt} \hat{\Omega}(t_1)dt_1 \int_t^{t_1} \hat{\Omega}(t_2)dt_2 + \ldots \right] \right\rangle_\xi f(r,p,t).$$

$$\left\langle f(r,p,t+dt)\right\rangle_\xi = \left\langle \left[1+\int_t^{t+dt}\hat{\Omega}(t_1)dt_1+\int_t^{t+dt}\hat{\Omega}(t_1)dt_1\int_t^{t_1}\hat{\Omega}(t_2)dt_2+\dots\right]\right\rangle_\xi f(r,p,t).$$

$$\Downarrow$$

$$\left\langle f(r,p,t+dt)\right\rangle_\xi = \left[1+\int_t^{t+dt}\left\langle\hat{\Omega}(t_1)\right\rangle dt_1+\left\langle\int_t^{t+dt}\hat{\Omega}(t_1)dt_1\int_t^{t_1}\hat{\Omega}(t_2)dt_2\right\rangle+\dots\right] f(r,p,t).$$

$$\frac{\partial}{\partial t} f(r,p,t) = \hat{\Omega} f = -\left[\frac{\partial}{\partial r}\Big(\mathrm{v}(p)\,\mathrm{f}(r,p,t)\Big) + \frac{\partial}{\partial p}\Big(\big(-\gamma p - \nabla U + \xi_p(t)\big)\mathrm{f}(r,p,t)\Big) \right].$$

$$\left\langle f(r,p,t+dt) \right\rangle_{\xi} = \left[1 + \int_t^{t+dt} \left\langle \hat{\Omega}(t_1) \right\rangle dt_1 + \left\langle \int_t^{t+dt} \hat{\Omega}(t_1) dt_1 \int_t^{t_1} \hat{\Omega}(t_2) dt_2 \right\rangle + \ldots \right] f(r,p,t).$$

$$\int_t^{t+dt} \left\langle \hat{\Omega}(t_1) \right\rangle dt_1 f(r,p,t) = -dt \left[\frac{\partial}{\partial r}\Big(\mathrm{v}(p)\,\mathrm{f}(r,p,t)\Big) + \frac{\partial}{\partial p}\Big(\big(-\gamma p - \nabla U\big)\mathrm{f}(r,p,t)\Big) \right].$$

$$\left\langle \int_t^{t+dt} \hat{\Omega}(t_1) dt_1 \int_t^{t_1} \hat{\Omega}(t_2) dt_2 \right\rangle f(r,p,t) = \left\langle \int_t^{t+dt} dt_1 \int_t^{t_1} D\delta(t_1 - t_2) dt_2 \right\rangle \frac{\partial^2}{\partial p^2} f(r,p,t) = \frac{Ddt}{2} \frac{\partial^2}{\partial p^2} f(r,p,t).$$

$$f(r,p,t) \to \left\langle f(r,p,t) \right\rangle_{\xi}$$

Усреднили по флуктуациям в «прошлом».

$$\frac{\left\langle f(r,p,t+dt)\right\rangle_{\xi} - \left\langle f(r,p,t)\right\rangle_{\xi}}{dt} = \frac{1}{dt}\left[\int_{t}^{t+dt}\left\langle \hat{\Omega}(t_1)\right\rangle dt_1 + \left\langle \int_{t}^{t+dt}\hat{\Omega}(t_1)dt_1\int_{t}^{t_1}\hat{\Omega}(t_2)dt_2 \right\rangle + \ldots\right]\left\langle f(r,p,t)\right\rangle_{\xi} =$$

$$= -\left[\frac{\partial}{\partial r}\Big(\mathrm{v}(p)\left\langle f(r,p,t)\right\rangle_{\xi}\Big) + \frac{\partial}{\partial p}\Big(\left(-\gamma p - \nabla U\right)\left\langle f(r,p,t)\right\rangle_{\xi}\Big)\right] + \frac{D}{2}\frac{\partial^2}{\partial p^2}\left\langle f(r,p,t)\right\rangle_{\xi}.$$

Подведем итоги:

$$\frac{\partial\left\langle f(r,p,t)\right\rangle_{\xi}}{\partial t}+\frac{\partial}{\partial r}\left(\mathrm{v}(p)\left\langle f(r,p,t)\right\rangle_{\xi}\right)+\frac{\partial}{\partial p}\left(\left(-\gamma p-\nabla U\right)\left\langle f(r,p,t)\right\rangle_{\xi}\right)-\frac{D}{2}\frac{\partial^{2}}{\partial p^{2}}\left\langle f(r,p,t)\right\rangle_{\xi}=0.$$

$\left\langle f(r,p,t)\right\rangle_{\xi}$ – "угрубленная" по времени.

$\left\langle f(r,p,t)\right\rangle_{\xi}$ НЕ МЕНЯЕТСЯ на

масштабах порядка dt (корреляционное время случайной силы) !!!

УРАВНЕНИЯ ФОККЕРА-ПЛАНКА -- ПЕРЕХОД К БОЛЕЕ ГРУБЫМ МАСШТАБАМ!

Был вопрос, можно ли учесть силу трения в кинетическом уравнении.
Ответ мы получили,
дописывание силы трения к внешней силе требует добавления нетривиального диффузионного члена в кинетическое уравнение.

$$\boxed{\frac{\partial}{\partial t} f(r,p,t) = -\frac{\partial}{\partial r}\left[\frac{p}{m} f(r,p,t)\right] + \frac{\partial}{\partial p}\left[\left(\gamma p + \nabla U\right) f(r,p,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial p^2}\left[D f(r,p,t)\right].}$$

$$\Downarrow$$

$$\boxed{\frac{\partial}{\partial t} f(r,p,t) + \frac{\partial}{\partial r}\left[\mathrm{v}_{\mathrm{p}} f(r,p,t)\right] + \frac{\partial}{\partial p}\left[\left(-\gamma p - \nabla U\right) f(r,p,t)\right] = \frac{1}{2}\frac{\partial^2}{\partial p^2}\left[D f(r,p,t)\right].}$$

Соотношение Эйнштейна

$$\frac{\partial}{\partial t} f(r,p,t) = -\frac{d}{dr}\left[\frac{p}{m} f(r,p,t)\right] + \frac{d}{dp}\left[\left(\gamma p + \nabla U\right) f(r,p,t)\right] + \frac{1}{2}\frac{d^2}{dp^2}\left[D f(r,p,t)\right].$$

$$f_0(r,p) \propto \exp\left(-\frac{\varepsilon_p + U}{T}\right).$$

$$j_r = \frac{p}{m} f_0(r,p), \quad j_p = \left[-\left(\gamma p + \nabla U\right) + \frac{Dp}{2mT}\right] f_0(r,p).$$

$$0 = \frac{d}{dp}\left[\left(-\gamma p + \frac{Dp}{2mT}\right) f_0(r,p)\right].$$

$\Rightarrow$

$$\boxed{D = 2mT\gamma}$$

Как перейти к диффузии в реальном пространстве?
Уравнение Смолуховского.

$$\boxed{\frac{\partial}{\partial t}\mathrm{n}(r,t) = -\nabla j.}$$

Первый способ получения ур. Смолуховского

$$m\frac{d^2}{dt^2}r + m\gamma\frac{d}{dt}r = -\nabla U + \xi_p(t), \quad \left\langle \xi_p(t)\xi_p(t')\right\rangle = D\delta(\mathrm{t}-\mathrm{t'}).$$

Сила трения больше, чем инерция…

$$\frac{d}{dt}r = -\frac{1}{m\gamma}\nabla U + \frac{1}{m\gamma}\xi_p(t), \quad \left\langle \xi_p(t)\xi_p(t')\right\rangle = D\delta(\mathrm{t}-\mathrm{t'}).$$

$$\frac{d}{dt}r = -\frac{1}{m\gamma}\nabla U + \frac{1}{m\gamma}\xi_p(t), \quad \left\langle \xi_p(t)\xi_p(t')\right\rangle = D\delta(\mathrm{t}-\mathrm{t}').$$

$$\mathrm{n}(\mathrm{r},\mathrm{t}) = \int_p \mathrm{f}(r,p,t)$$

$$\frac{\partial}{\partial t}\mathrm{n}(\mathrm{r},\mathrm{t}) + \frac{\partial}{\partial r}\left[\left(-\frac{1}{m\gamma}\nabla U + \frac{1}{m\gamma}\xi_p(t)\right)\mathrm{n}(\mathrm{r},\mathrm{t})\right] = 0$$

$$\frac{\partial}{\partial t}\mathrm{n}(\mathrm{r},\mathrm{t}) = \hat{\Omega}\,\mathrm{n}(\mathrm{r},\mathrm{t}) = -\frac{\partial}{\partial r}\left[\left(-\frac{1}{m\gamma}\nabla U + \frac{1}{m\gamma}\xi_p(t)\right)\mathrm{n}(\mathrm{r},\mathrm{t})\right]$$

$$\frac{\partial}{\partial t}\mathrm{n(r,t)}=\hat{\Omega}\,\mathrm{n(r,t)}=-\frac{\partial}{\partial r}\left[\left(-\frac{1}{m\gamma}\nabla U+\frac{1}{m\gamma}\xi_p(t)\right)\mathrm{n(r,t)}\right]$$

$$\left\langle n(r,t+dt)\right\rangle_{\xi(t+0)}=\left[1+\int_t^{t+dt}\left\langle\hat{\Omega}(t_1)\right\rangle dt_1+\left\langle\int_t^{t+dt}\hat{\Omega}(t_1)dt_1\int_t^{t_1}\hat{\Omega}(t_2)dt_2\right\rangle+\ldots\right]\left\langle n(r,t)\right\rangle_{\xi(t-0)}.$$

$$\int_t^{t+dt}\left\langle\hat{\Omega}(t_1)\right\rangle dt_1 n(r,t)=-dt\frac{\partial}{\partial r}\left(-\frac{1}{m\gamma}\nabla U n(r,t)\right).$$

$$\frac{\partial}{\partial t}\mathrm{n(r,t)} = \hat{\Omega}\,\mathrm{n(r,t)} = -\frac{\partial}{\partial r}\left[\left(-\frac{1}{m\gamma}\nabla U + \frac{1}{m\gamma}\xi_p(t)\right)\mathrm{n(r,t)}\right]$$

$$\left\langle n(r,t+dt)\right\rangle_{\xi(t+0)} = \left[1+\int_t^{t+dt}\left\langle\hat{\Omega}(t_1)\right\rangle dt_1 + \left\langle\int_t^{t+dt}\hat{\Omega}(t_1)dt_1\int_t^{t_1}\hat{\Omega}(t_2)dt_2\right\rangle + \ldots\right]\left\langle n(r,t)\right\rangle_{\xi(t-0)}.$$

$$\left\langle\int_t^{t+dt}\hat{\Omega}(t_1)dt_1\int_t^{t_1}\hat{\Omega}(t_2)dt_2\right\rangle n(r,t) = \frac{1}{(m\gamma)^2}\left\langle\int_t^{t+dt}dt_1\int_t^{t_1}D\delta(t_1-t_2)dt_2\right\rangle\frac{\partial^2}{\partial r^2}n(r,t) = \frac{Ddt}{2}\frac{1}{(m\gamma)^2}\frac{\partial^2}{\partial r^2}n(r,t).$$

$$\left\langle n(r,t+dt)\right\rangle_{\xi(t+0)}=\left[1+\int_t^{t+dt}\left\langle\hat{\Omega}(t_1)\right\rangle dt_1+\left\langle\int_t^{t+dt}\hat{\Omega}(t_1)dt_1\int_t^{t_1}\hat{\Omega}(t_2)dt_2\right\rangle+\ldots\right]\left\langle n(r,t)\right\rangle_{\xi(t-0)}.$$

$$\int_t^{t+dt}\left\langle\hat{\Omega}(t_1)\right\rangle dt_1\left\langle n(r,t)\right\rangle=-dt\frac{\partial}{\partial r}\left(-\frac{1}{m\gamma}\nabla U\left\langle n(r,t)\right\rangle\right).$$

$$\left\langle\int_t^{t+dt}\hat{\Omega}(t_1)dt_1\int_t^{t_1}\hat{\Omega}(t_2)dt_2\right\rangle\left\langle n(r,t)\right\rangle=\frac{1}{(m\gamma)^2}\left\langle\int_t^{t+dt}dt_1\int_t^{t_1}D\delta(t_1-t_2)dt_2\right\rangle\frac{\partial^2}{\partial r^2}\left\langle n(r,t)\right\rangle=\frac{Ddt}{2}\frac{1}{(m\gamma)^2}\frac{\partial^2}{\partial r^2}\left\langle n(r,t)\right\rangle.$$

Уравнение Смолуховского:

$$\boxed{\frac{\partial}{\partial t}\left\langle \mathrm{n}(\mathrm{r},\mathrm{t})\right\rangle=\frac{\partial}{\partial r}\left[\left(\frac{1}{m\gamma}\nabla U+\frac{D}{2(m\gamma)^2}\frac{\partial}{\partial r}\right)\left\langle n(r,t)\right\rangle\right].}$$

$$\frac{\partial}{\partial t}\langle \mathrm{n(r,t)}\rangle = \frac{\partial}{\partial r}\left[\left(\frac{1}{m\gamma}\nabla U + \frac{D}{2(m\gamma)^2}\frac{\partial}{\partial r}\right)\langle n(r,t)\rangle\right].$$

Проверим соотношение Эйнштейна для ур. Смолуховского.

В равновесии – распределение Гиббса:

$$n(r) = n_0(r) \propto \exp(-U/T).$$

$$j = -\left(\frac{1}{m\gamma}\nabla U + \frac{D}{2(m\gamma)^2}\frac{\partial}{\partial r}\right)n_0(r,t) = 0,$$

$$j = -\left(\frac{1}{m\gamma}\nabla U - \frac{D}{2(m\gamma)^2 T}\frac{\partial}{\partial r}U\right)n_0(r,t) = 0.$$

$$\boxed{\frac{\partial}{\partial t}\left\langle \mathrm{n}(\mathrm{r},\mathrm{t})\right\rangle = \frac{\partial}{\partial r}\left[\left(\frac{1}{m\gamma}\nabla U + \frac{D}{2\left(m\gamma\right)^2}\frac{\partial}{\partial r}\right)\left\langle n(r,t)\right\rangle\right].}$$

Проверим соотношение Эйнштейна для ур. Смолуховского.

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$$\boxed{n(r) = n_0(r) \propto \exp(-U/T).}$$

$$j = -\left(\frac{1}{m\gamma}\nabla U + \frac{D}{2\left(m\gamma\right)^2}\frac{\partial}{\partial r}\right)n_0(r,t) = -\left(\frac{1}{m\gamma}\nabla U - \frac{D}{2\left(m\gamma\right)^2 T}\nabla U\right)n_0(r,t) = 0,$$

$D = 2m\gamma T$ – Соотношение Эйнштейна. Мы его уже получали...

Подведем Итоги:

$$\frac{d}{dt}r = -\frac{1}{m\gamma}\nabla U + \frac{1}{m\gamma}\xi_p(t), \quad \left\langle \xi_p(t)\xi_p(t')\right\rangle = D\delta(\mathrm{t}-\mathrm{t'}).$$

$$\frac{\partial}{\partial t}\left\langle \mathrm{n(r,t)}\right\rangle = \frac{\partial}{\partial r}\left[\left(\frac{1}{m\gamma}\nabla U + \frac{D}{2\left(m\gamma\right)^2}\frac{\partial}{\partial r}\right)\left\langle n(r,t)\right\rangle\right]$$

ур. Смолуховского.

$D = 2m\gamma T$ – Соотношение Эйнштейна. Мы его уже получали...

«Dynamics in condensed phases : relaxation, transfer, and reactions in condensed molecular systems»
Abraham Nitzan, Oxford University Press 2006, см. стр. 301.

Ур. Смолуховского.
Второй способ вывода, исходя из

$$\boxed{\frac{\partial}{\partial t}\mathrm{f}(r,p,t)=-\frac{d}{dr}\left[\frac{p}{m}\mathrm{f}(r,p,t)\right]+\frac{d}{dp}\left[\left(\gamma p+\nabla U\right)\mathrm{f}(r,p,t)\right]+\frac{1}{2}\frac{d^2}{dp^2}\left[Df(r,p,t)\right].}$$

$$\frac{\partial}{\partial t}\mathrm{f}(r,p,t) = -\frac{d}{dr}\left[\frac{p}{m}\mathrm{f}(r,p,t)\right] + \frac{d}{dp}\left[\left(\gamma p + \nabla U\right)\mathrm{f}(r,p,t)\right] + \frac{1}{2}\frac{d^2}{dp^2}\left[Df(r,p,t)\right].$$

$$\mathrm{n(r,t)} = \int_p \mathrm{f}(r,p,t)$$

$$\frac{\partial}{\partial t}\mathrm{n}(r,t) = -\frac{d}{dr}\int_p \left[\frac{p}{m}\mathrm{f}(r,p,t)\right] + \int_p \frac{d}{dp}\left[\left(\gamma p + \nabla U\right)\mathrm{f}(r,p,t)\right] + \frac{1}{2}\int_p \frac{d^2}{dp^2}\left[Df(r,p,t)\right].$$

То, что хочется получить:

$$\frac{\partial}{\partial t}\mathrm{n}(r,t) = -\nabla j.$$

$$\frac{\partial}{\partial t}\mathrm{f}(r,p,t)=-\frac{d}{dr}\left[\frac{p}{m}\mathrm{f}(r,p,t)\right]+\frac{d}{dp}\left[\left(\gamma p+\nabla U\right)\mathrm{f}(r,p,t)\right]+\frac{1}{2}\frac{d^2}{dp^2}\left[Df(r,p,t)\right].$$

$$\frac{\partial}{\partial t}j_\alpha=-\frac{d}{dr_\beta}\int_p\left[\frac{p_\alpha p_\beta}{m^2}\mathrm{f}(r,p,t)\right]+\int_p\frac{p_\alpha}{m}\frac{d}{dp_\beta}\left[\left(\gamma p_\beta+\nabla U\right)\mathrm{f}(r,p,t)\right]+\frac{1}{2}D\int_p\frac{p_\alpha}{m}\frac{d^2}{dp^2}f(r,p,t),$$

$$\int_p\left[\frac{p_\alpha p_\beta}{m^2}\mathrm{f}(r,p,t)\right]\approx\delta_{\alpha\beta}\frac{1}{3}\int_p\left[\mathrm{v}^2\mathrm{f}(r,p,t)\right]=\delta_{\alpha\beta}n(r,t)\frac{1}{3}T/m,$$

$$\int_p\frac{p_\alpha}{m}\frac{d}{dp_\beta}\left[\gamma p_\beta\,\mathrm{f}(r,p,t)\right]=-\frac{\gamma}{m}\int_p\left[p_\alpha\,\mathrm{f}(r,p,t)\right]=-\gamma j_\alpha,$$

$$\int_p\frac{p_\alpha}{m}\frac{d}{dp_\beta}\left[\nabla U\,\mathrm{f}(r,p,t)\right]=-\frac{\nabla U}{m}\int_p\left[\mathrm{f}(r,p,t)\right]=-\frac{n\nabla U}{m},$$

$$\int_p\frac{p_\alpha}{m}\frac{d^2}{dp^2}f(r,p,t)=0.$$

$$\frac{\partial}{\partial t} j_\alpha = -\frac{d}{dr_\beta}\int_p \left[\frac{p_\alpha p_\beta}{m^2} f(r,p,t)\right] + \int_p \frac{p_\alpha}{m}\frac{d}{dp_\beta}\left[\left(\gamma p_\beta + \nabla U\right) f(r,p,t)\right] + \frac{1}{2} D \int_p \frac{p_\alpha}{m}\frac{d^2}{dp^2} f(r,p,t),$$

$$\int_p \left[\frac{p_\alpha p_\beta}{m^2} f(r,p,t)\right] \approx \delta_{\alpha\beta}\frac{1}{3}\int_p \left[v^2 f(r,p,t)\right] = \delta_{\alpha\beta} n(r,t)\frac{1}{3} T/m,$$

$$\int_p \frac{p_\alpha}{m}\frac{d}{dp_\beta}\left[\gamma p_\beta f(r,p,t)\right] = -\frac{\gamma}{m}\int_p \left[p_\alpha f(r,p,t)\right] = -\gamma j_\alpha,$$

$$\int_p \frac{p_\alpha}{m}\frac{d}{dp_\beta}\left[\nabla U f(r,p,t)\right] = -\frac{\nabla U}{m}\int_p \left[f(r,p,t)\right] = -\frac{n\nabla U}{m},$$

$$\int_p \frac{p_\alpha}{m}\frac{d^2}{dp^2} f(r,p,t) = 0.$$

$$\boxed{\frac{\partial}{\partial t} j_\alpha = -\gamma j_\alpha + \frac{\partial}{\partial r_\alpha}\left(-\frac{n}{m}(U+T)\right) = -\gamma j_\alpha + B_\alpha,} \Rightarrow \boxed{j_\alpha \approx B_\alpha / \gamma.}$$

$$\boxed{\frac{\partial}{\partial t}\mathrm{n}(r,t) = -\nabla j.}$$

$$\boxed{j_\alpha \approx \frac{\partial}{\partial r_\alpha}\left(-\frac{n}{m}(U+T)\right)/\gamma.}$$

$$\boxed{\begin{aligned} &\frac{\partial}{\partial t}n(r,t) = D_r\nabla\left(\nabla n + \frac{n}{T}\nabla U\right), \\ &D_r = \frac{T}{m\gamma}. \end{aligned}}$$

$$\boxed{\frac{\partial}{\partial t}\mathrm{n}(r,t) = -\nabla j.}$$

$$\boxed{j_\alpha \approx \frac{\partial}{\partial r_\alpha}\left(-\frac{n}{m}(U+T)\right)/\gamma.}$$

$$\Rightarrow$$

$$\boxed{\begin{gathered}\frac{\partial}{\partial t}n(r,t) = D_r\nabla\left(\nabla n + \frac{n}{T}\nabla U\right),\\ D_r = \frac{T}{m\gamma}.\end{gathered}}$$

Проверим соотношение Эйнштейна,

В равновесии:

$$\boxed{n(r) = n_0(r) \propto \exp(-U/T).}$$

$$\boxed{\nabla n_0 + \frac{n_0}{T}\nabla U = 0.}$$

Уравнение Фоккера-Планка и интеграл столкновений кинетического уравнения

Кинетическое уравнение (в общем виде):

Столкновительный член

$$\frac{\partial}{\partial t}p(x,t)+\frac{\partial}{\partial x}\big(g(x)p(x,t)\big)=\int \underbrace{dz\big(\mathrm{W}(x\,|\,z)p(\mathrm{z},t)}_{\text{приходный член}}-\underbrace{\mathrm{W}(z\,|\,x)p(x,t)}_{\text{уходный член}}\big).$$

Уравнения Фоккера-Планка

$$\boxed{\frac{\partial}{\partial t}p(x,t)+\frac{\partial}{\partial x}\Big(g(x)p(x,t)\Big)=\int dz\Big(\mathrm{W}(x\,|\,z)p(\mathrm{z},t)-\mathrm{W}(z\,|\,x)p(x,t)\Big).}$$

$$\boxed{W(x\,|\,x',t)=\Omega(x',y,t),\qquad y=x-x'.}$$

$$\boxed{\frac{\partial}{\partial t}p(x,t)+\frac{\partial}{\partial x}\Big(g(x)p(x,t)\Big)=\int dy\Big(\Omega(x-y,y,t)p(\mathrm{x}-\mathrm{y},t)-\Omega(x,y,t)p(x,t)\Big).}$$

$$\frac{\partial}{\partial t}p(x,t)+\frac{\partial}{\partial x_i}\left(g_i(x)p(x,t)\right)=\int dy\left(\Omega(x-y,y,t)p(\mathrm{x}-\mathrm{y},t)-\Omega(x,y,t)p(x,t)\right)=$$

$$=\int dy\left(\Omega(x,y,t)p(\mathrm{x},t)-\mathrm{y}_i\frac{\partial}{\partial x_i}\left(\Omega(x,y,t)p(\mathrm{x},t)\right)+\frac{1}{2}\mathrm{y}_i\,y_j\frac{\partial^2}{\partial x_i\partial x_j}\left(\Omega(x,y,t)p(\mathrm{x},t)\right)-\Omega(x,y,t)p(x,t)\right)=$$

$$=-\frac{\partial}{\partial x_i}\left(A_i(x,t)p(x,t)\right)+\frac{1}{2}\frac{\partial^2}{\partial x_i\partial x_j}\left(D_{ij}(x,t)p(x,t)\right).$$

$$A_i(x,t)=\int dy\,\mathrm{y}_i\,\Omega(x,y,t)=\int(x'-x)_i W(x'\,|\,x,t)dx',$$

$$D_{ij}(x,t)\int dy\,\mathrm{y}_i\mathrm{y}_j\,\Omega(x,y,t)=\int(x'-x)_i(x'-x)_j W(x'\,|\,x,t)dx'.$$

$$\frac{\partial}{\partial t}p(x,t)+\frac{\partial}{\partial x_i}J_i(x,t)=0.$$

$$J_i=G_i(x,t)p(x,t)-\frac{1}{2}\frac{\partial}{\partial x_j}\left(D_{ij}(x,t)p(x,t)\right).$$

$$G_i(x,t)=A_i(x,t)-g_i(x,t),$$

$$\frac{\partial}{\partial t} p(x,t) + \frac{\partial}{\partial x_i} J_i(x,t) = 0.$$

$$J_i = G_i(x,t) p(x,t) - \frac{1}{2}\frac{\partial}{\partial x_j}\Big(D_{ij}(x,t) p(x,t)\Big).$$

$$G_i(x,t) = A_i(x,t) - g_i(x,t),$$

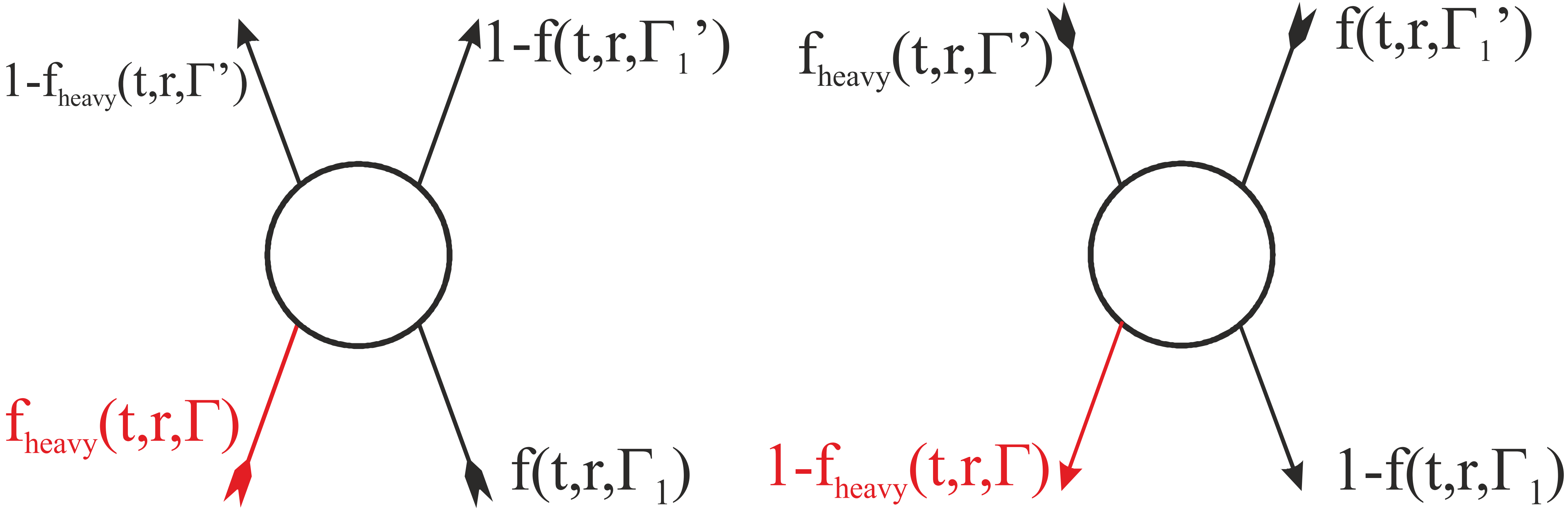
Данные уравнения типа Фоккера-Планка
можно получить из уравнений Ланжевена:

$$\frac{dx_i}{dt} = -G_i(x(t)) + \xi_i(t),$$

$$\langle \xi_i(t) \rangle = 0, \qquad \langle \xi_i(t) \xi_j(t') \rangle = \delta(t - t') D_{ij}.$$

Задание: тяжелая частица в легком газе

- Столкновения парные…
- Тяжелая частица при столкновении с легкой получает небольшое приращение импульса – условие применимости приближения Фоккера-Планка для интеграла столкновений.



$$\frac{\partial}{\partial t} f_h(r,\Gamma,t) + \mathrm{v}(\Gamma)\frac{\partial}{\partial r}\left(f_h(r,\Gamma,t)\right) + F(r,t)\frac{\partial}{\partial \Gamma}\left(f_h(r,\Gamma,t)\right) =$$

$$= \sum_{\Gamma_1,\Gamma',\Gamma_1'} w\left(\Gamma\Gamma_1 \,|\, \Gamma'\Gamma_1'\right) f_h(r,\Gamma',t) f(r,\Gamma_1',t)\left(1 - f(r,\Gamma_1,t)\right)\left(1 - f_h(r,\Gamma,t)\right) -$$

$$- \sum_{\Gamma_1,\Gamma',\Gamma_1'} w\left(\Gamma'\Gamma_1' \,|\, \Gamma\Gamma_1\right)\left(1 - f_h(r,\Gamma',t)\right)\left(1 - f(r,\Gamma_1',t)\right) f(r,\Gamma_1,t) f_h(r,\Gamma,t).$$

В больцмановском пределе

$$\frac{\partial}{\partial t} f_h(r,\Gamma,t) + \mathrm{v}(\Gamma)\frac{\partial}{\partial r}\left(f_h(r,\Gamma,t)\right) + F(r,t)\frac{\partial}{\partial \Gamma}\left(f_h(r,\Gamma,t)\right) =$$
$$= \sum_{\Gamma_1,\Gamma',\Gamma_1'} w\left(\Gamma\Gamma_1 \,|\, \Gamma'\Gamma_1'\right) f_h(r,\Gamma',t) f(r,\Gamma_1',t)\left(1 - f(r,\Gamma_1,t)\right)\left(1 - f_h(r,\Gamma,t)\right) -$$
$$- \sum_{\Gamma_1,\Gamma',\Gamma_1'} w\left(\Gamma'\Gamma_1' \,|\, \Gamma\Gamma_1\right)\left(1 - f_h(r,\Gamma',t)\right)\left(1 - f(r,\Gamma_1',t)\right) f(r,\Gamma_1,t) f_h(r,\Gamma,t).$$

$$\Downarrow$$

$$\frac{\partial}{\partial t} f_h(r,\Gamma,t) + \mathrm{v}(\Gamma)\frac{\partial}{\partial r}\left(f_h(r,\Gamma,t)\right) + F(r,t)\frac{\partial}{\partial \Gamma}\left(f_h(r,\Gamma,t)\right) =$$
$$= \sum_{\Gamma_1,\Gamma',\Gamma_1'} w\left(\Gamma\Gamma_1 \,|\, \Gamma'\Gamma_1'\right) f_h(r,\Gamma',t) f(r,\Gamma_1',t) - \sum_{\Gamma_1,\Gamma',\Gamma_1'} w\left(\Gamma'\Gamma_1' \,|\, \Gamma\Gamma_1\right) f(r,\Gamma_1,t) f_h(r,\Gamma,t).$$

$$\frac{\partial}{\partial t} f_h(r,\Gamma,t) + \mathrm{v}(\Gamma)\frac{\partial}{\partial r}\big(f_h(r,\Gamma,t)\big) + F(r,t)\frac{\partial}{\partial \Gamma}\big(f_h(r,\Gamma,t)\big) =$$
$$= \sum_{\Gamma_1,\Gamma',\Gamma_1'} w\big(\Gamma\Gamma_1 \,|\, \Gamma'\Gamma_1'\big) f_h(r,\Gamma',t) f(r,\Gamma_1',t) - \sum_{\Gamma_1,\Gamma',\Gamma_1'} w\big(\Gamma'\Gamma_1' \,|\, \Gamma\Gamma_1\big) f(r,\Gamma_1,t) f_h(r,\Gamma,t).$$

$$\frac{\partial}{\partial t} f_h(q,\Gamma,t) + \mathrm{v}(\Gamma)\frac{\partial}{\partial q}\big(f_h(q,\Gamma,t)\big) + F(q,t)\frac{\partial}{\partial \Gamma}\big(f_h(q,\Gamma,t)\big) =$$
$$= \sum_{\Gamma'} W\big(\Gamma \,|\, \Gamma'\big) f_h(q,\Gamma',t) - \sum_{\Gamma'_1} W\big(\Gamma' \,|\, \Gamma\big) f(q,\Gamma_1,t),$$
$$W\big(\Gamma \,|\, \Gamma'\big) = \sum_{\Gamma_1,\Gamma_1'} w\big(\Gamma\Gamma_1 \,|\, \Gamma'\Gamma_1'\big) f(q,\Gamma_1',t),$$
$$W\big(\Gamma' \,|\, \Gamma\big) = \sum_{\Gamma_1,\Gamma_1'} w\big(\Gamma'\Gamma_1' \,|\, \Gamma\Gamma_1\big) f(q,\Gamma_1,t).$$

Диффузия в импульсном пространстве – уравнение больцмана для тяжелых [heavy (h)] частиц

$$\frac{\partial}{\partial t} f_h(r,\Gamma,t) + v(\Gamma)\frac{\partial}{\partial x}\left(f_h(r,\Gamma,t)\right) + F(r,t)\frac{\partial}{\partial \Gamma}\left(f_h(r,\Gamma,t)\right) =$$

$$= \int \left(W(\Gamma\,|\,\Gamma') f_h(r,\Gamma',t) - W(\Gamma'\,|\,\Gamma) f_h(r,\Gamma,t)\right) d\Gamma' = \left(St\, f_h\right)_{in} - \left(St\, f_h\right)_{out} = I_{st}.$$

$$I_{st} = -\frac{\partial}{\partial p_i}\left(A_i(r,p,t) f_h(r,p,t)\right) + \frac{1}{2}\frac{\partial^2}{\partial p_i \partial p_j}\left(D_{ij}(r,p,t) f_h(r,p,t)\right),$$

$$A_i(r,p,t) = \int (p' - p)_i W(p'\,|\,p,t)\, d\Gamma'_p,$$

$$D_{ij}(r,p,t) = \int (p' - p)_i (p' - p)_j W(p'\,|\,p,t)\, d\Gamma'_p \approx$$

$$\approx \delta_{ij}\frac{1}{3}\int (p' - p)^2 W(p'\,|\,p,t)\, d\Gamma'_p = \delta_{ij} D.$$

$$\frac{\partial}{\partial t} f_h(r,\Gamma,t) + v(\Gamma)\frac{\partial}{\partial x}\left(f_h(r,\Gamma,t)\right) + F(r,t)\frac{\partial}{\partial \Gamma}\left(f_h(r,\Gamma,t)\right) =$$

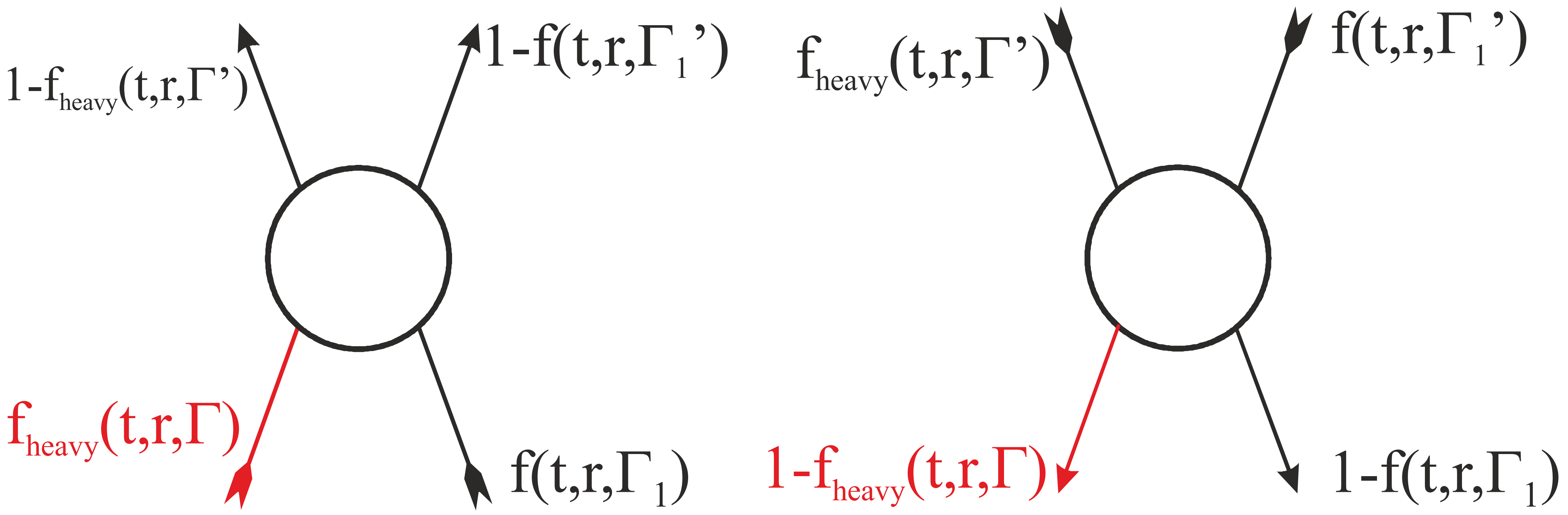
$$= -\frac{\partial}{\partial p_i}\left(A_i(r,p,t) f_h(r,p,t) - \frac{1}{2}\frac{\partial}{\partial p}\left(D(r,p,t) f_h(r,p,t)\right)\right).$$

В равновесии

$$f_0(r,p) \propto \exp\left(-\frac{\varepsilon_p}{T}\right). \quad \Rightarrow \quad A_i + \frac{1}{2MT} p_i D = 0.$$

$$\frac{d}{dt} f_h = \frac{1}{2} D \frac{\partial}{\partial p_i}\left(\frac{p_i}{MT} f_h(r,p,t) + \frac{\partial}{\partial p} f_h(r,p,t)\right).$$

$$D = \frac{1}{3}\int (p' - p)^2 W(p' \mid p, t)\, d\Gamma'_p = \frac{1}{3}\int (p' - p)^2 w\left(\Gamma'\Gamma'_1 \mid \Gamma\Gamma_1\right) f(p_1, t)\, d\Gamma'\, d\Gamma_1 d\Gamma'_1.$$



$$D = \frac{1}{3}\int (p'_1 - p_1)^2 w\left(\Gamma'\Gamma'_1 \mid \Gamma\Gamma_1\right) f(p_1, t)\, d\Gamma'\, d\Gamma_1 d\Gamma'_1.$$

$$D=\frac{1}{3}\int(p_1'-p_1)^2 w\left(\Gamma'\Gamma_1' \mid \Gamma\Gamma_1\right) f(p_1,t)d\Gamma' d\Gamma_1 d\Gamma_1' = \frac{1}{3}\int(p_1'-p_1)^2 n_{l.p.} v_1 d\sigma \approx$$
$$\approx \frac{1}{3}\int 2p_1^2(1-\cos\alpha) n_{l.p.} v_1 d\sigma = \frac{2}{3m} n_{l.p.} \left\langle p_1^3 \sigma_{transport} \right\rangle$$

Как найти подвижность???

$$\frac{\partial}{\partial t} f_h(r,\mathrm{p},t) + \mathrm{v}(\Gamma)\frac{\partial}{\partial x}\left(f_h(r,\mathrm{p},t)\right) + F(r,t)\frac{\partial}{\partial p}\left(f_h(r,\mathrm{p},t)\right) =$$
$$= \frac{1}{2} D \frac{\partial}{\partial p_i}\left(\frac{p_i}{MT} f_h(r,p,t) + \frac{\partial}{\partial p} f_h(r,p,t)\right).$$

Подвижность

$$\boxed{\begin{aligned}&\frac{\partial}{\partial t} f_h(r,\mathrm{p},t) + v(\Gamma)\frac{\partial}{\partial x}\left(f_h(r,\mathrm{p},t)\right) + F(r,t)\frac{\partial}{\partial p}\left(f_h(r,\mathrm{p},t)\right) =\\ &= \frac{1}{2} D \frac{\partial}{\partial p_i}\left(\frac{p_i}{MT} f_h(r,p,t) + \frac{\partial}{\partial p} f_h(r,p,t)\right).\end{aligned}}$$

$$\boxed{\left\langle \mathbf{v} \right\rangle = \int \mathbf{v}_p \delta f_h(r,p,t)\, d\Gamma = b\mathbf{F}.}$$

$$\frac{\partial}{\partial t} f_h(r,\mathrm{p},t) + v(\Gamma)\frac{\partial}{\partial x}\left(f_h(r,\mathrm{p},t)\right) + F(r,t)\frac{\partial}{\partial p}\left(f_h(r,\mathrm{p},t)\right) =$$

$$= \frac{1}{2} D \frac{\partial}{\partial p_i}\left(\frac{p_i}{MT} f_h(r,p,t) + \frac{\partial}{\partial p} f_h(r,p,t)\right).$$

$$F\frac{\partial}{\partial p}\left(f_h^{(0)}\right) = \frac{1}{2} D \frac{\partial}{\partial p_i}\left(\frac{p_i}{MT}\delta f_h + \frac{\partial}{\partial p}\delta f_h\right).$$

$$F f_h^{(0)} = \frac{1}{2} D \left(\frac{p_i}{MT}\delta f_h + \frac{\partial}{\partial p}\delta f_h\right).$$

$$\boxed{F_i f_h^{(0)} = \frac{1}{2} D \left(\frac{p_i}{MT} \delta f_h + \frac{\partial}{\partial p_i} \delta f_h \right).}$$

$$\boxed{\delta f_h = f_h^{(0)} \frac{2\mathbf{F} \cdot \mathbf{p}}{D}.}$$

$$\boxed{\begin{aligned} &\langle \mathbf{v} \rangle = \int \mathbf{v}_p \delta f_h(r, p, t) d\Gamma = \int \mathbf{v}_p f_h^{(0)} \frac{2\mathbf{F} \cdot \mathbf{p}}{D} d\Gamma = \frac{2\mathbf{F}M}{3D} \int \mathbf{v}^2 f_h^{(0)} d\Gamma = \frac{2\mathbf{F}T}{3D} n_h, \\ &b = \frac{2n_h T}{3D}. \end{aligned}}$$

Диффузионный марковский процесс

Диффузионный пропагатор

$$T(x,t \mid x_0, t_0) = 0, \quad t < t_0.$$

$$T(x,t \mid x_0, t_0) \propto \theta(t - t_0)$$

Уравнение Фоккера-Планка

$$\frac{\partial}{\partial t}T(x,t\,|\,x_0,t_0) = -\frac{d}{dx}\left[F_1(x)T(x,t\,|\,x_0,t_0)\right] + \frac{1}{2}\frac{d^2}{dx^2}\left[F_2(x)T(x,t\,|\,x_0,t_0)\right].$$

- Пусть $x(t)$ – Марковский случайный процесс.
- Определим $\delta x(t) = x(t+\delta t) - x(t)$.
- Постулируем для малых δt:

1) $\langle \delta x(t) \rangle = F_1(x(t))\,\delta t$
2) $\langle (\delta x(t))^2 \rangle = F_2(x(t))\,\delta t$
3) $\langle (\delta x(t))^n \rangle = O((\delta t)^2)$, n>2.

$$\frac{\partial}{\partial t}T(x,t\,|\,x_0,t_0) = -\frac{d}{dx}\left[F_1(x)T(x,t\,|\,x_0,t_0)\right] + \frac{1}{2}\frac{d^2}{dx^2}\left[F_2(x)T(x,t\,|\,x_0,t_0)\right] + \delta(x-x_0)\delta(t-t_0).$$

$$\frac{\partial}{\partial t}T(x,t\,|\,x_0,t_0) = -\frac{d}{dx}\left[g(x)T(x,t\,|\,x_0,t_0)\right] + \frac{1}{2}\frac{d^2}{dx^2}\left[D(x)T(x,t\,|\,x_0,t_0)\right].$$

$$\frac{dx_i}{dt} = g_i(x(t)) + \xi_i(t),$$

$$\frac{\partial}{\partial t}T(x,t\,|\,x_0,t_0) = -\frac{d}{dx_i}\left[g_i(x)T(x,t\,|\,x_0,t_0)\right] + \frac{1}{2}\frac{d^2}{dx_i dx_j}\left[D_{ij}(x)T(x,t\,|\,x_0,t_0)\right].$$

Гауссовский шум: $\langle \xi_i(t_1)\xi_j(t_2)\rangle = D_{ij}\delta(t-t')$

Диффузионный процесс

$$\frac{\partial}{\partial t}p(x,\,t) + \frac{\partial}{\partial x_i}J_i(x,\,t) = 0$$

$$J_i = G_i(x,t)\,p(x,t) - \frac{1}{2}\frac{\partial}{\partial x_j}\left(D_{ij}(x,t)\,p(x,t)\right).$$

$$G_i(x,t) = A_i(x,t) - g_i(x,t),$$

$$G(x,t) = 0,$$

$$D = 1.$$

$$\frac{\partial}{\partial t}p(x,t) - \frac{1}{2}\frac{\partial^2}{\partial x^2}p(x,t) = 0,$$

$$\frac{\partial}{\partial t}T(x,t\,|\,x',t') - \frac{1}{2}\frac{\partial^2}{\partial x^2}T(x,t\,|\,x',t') = \delta(x-x')\delta(t-t').$$

$$T(x,\,t|x',\,t') = \frac{1}{\sqrt{2\pi(t-t')}}\exp\left(-\frac{(x-x')^2}{2(t-t')}\right)$$

$$p(x,\,t=0) = \delta(x)$$

Диффузионный процесс

$$\frac{dx_i}{dt} = \xi_i(t), \qquad \langle \xi_i(t_1)\xi_j(t_2)\rangle = \delta(t - t')$$

$$\boxed{\begin{aligned} G(x,t) &= 0, \\ D &= 1. \end{aligned}}$$

$$\boxed{\begin{aligned} &\frac{\partial}{\partial t} p(x,t) - \frac{1}{2}\frac{\partial^2}{\partial x^2} p(x,t) = 0, \\ &\frac{\partial}{\partial t} T(x,t\,|\,x',t') - \frac{1}{2}\frac{\partial^2}{\partial x^2} T(x,t\,|\,x',t') = \delta(x-x')\delta(t-t'). \end{aligned}}$$

$$T(x,\, t|x',\, t') = \frac{1}{\sqrt{2\pi(t-t')}} \exp\left(-\frac{(x-x')^2}{2(t-t')}\right)$$

$$p(x,\, t=0) = \delta(x)$$

Диффузионный процесс

$$\frac{\partial}{\partial t}p(x,\, t) + \frac{\partial}{\partial x_i}J_i(x,\, t) = 0$$

$$J_i = G_i(x,t)p(x,t) - \frac{1}{2}\frac{\partial}{\partial x_j}\left(D_{ij}(x,t)p(x,t)\right).$$

$$G_i(x,t) = A_i(x,t) - g_i(x,t),$$

$$G(x,t) = k,$$
$$D = 1.$$

$$\frac{\partial}{\partial t}p(x,t) + \frac{\partial}{\partial x}kp(x,t) - \frac{1}{2}\frac{\partial^2}{\partial x^2}p(x,t) = 0,$$

$$\frac{\partial}{\partial t}\mathrm{T}(x,t\,|\,x',t') + \frac{\partial}{\partial x}k\,\mathrm{T}(x,t\,|\,x',t') - \frac{1}{2}\frac{\partial^2}{\partial x^2}\mathrm{T}(x,t\,|\,x',t') = \delta(x-x').$$

$$T(x,\, t|x',\, t') = \sqrt{\frac{k}{\pi D\left[1-e^{-2k(t-t')}\right]}}\exp\left(-\frac{k(x-e^{-k(t-t')}x')^2}{D\left[1-e^{-2k(t-t')}\right]}\right)$$