Optimization Methods.
Seminar 4. Conjugate sets. Farkas' lemma

Alexandr Katrutsa

Moscow Institute of Physics and Technology,
Department of Control and Applied Mathematics

October 1, 2017

Reminder

- Interior and relative interior of convex set
- Projection onto set
- Separation of convex sets
- Support hyperplane

Conjugate set

Conjugate set

Let X^* be a conjugate (dual) set to the set $X \subseteq \mathbb{R}^n$ such that

$$X^* = \{\mathbf{p} \in \mathbb{R}^n | \langle \mathbf{p}, \mathbf{x} \rangle \geq -1, \ \forall \mathbf{x} \in X\}.$$

Conjugate cone

If $X \subseteq \mathbb{R}^n$ is a cone, then

$$X^* = \{\mathbf{p} \in \mathbb{R}^n | \langle \mathbf{p}, \mathbf{x} \rangle \geq 0, \ \forall \mathbf{x} \in X\}.$$

Conjugate subspace

If X is a linear subspace of $\mathbb{R}^n$, then

$$X^* = \{\mathbf{p} \in \mathbb{R}^n | \langle \mathbf{p}, \mathbf{x} \rangle = 0, \ \forall \mathbf{x} \in X\}.$$

Claims about conjugate sets

Theorem

Let X be an arbitrary subset of $\mathbb{R}^n$. Then

$$X^{**} = \overline{conv\left(X \cup \{0\}\right)}.$$

Theorem

Let X be a closed convex set with zero. Then $X^{**} = X$.

Theorem

If $X_1 \subset X_2$, then $X_2^* \subset X_1^*$.

Examples

Find conjugate sets for the following sets:

1. Nonnegative orthant: $\mathbb{R}^n_+$
2. Cone of positive semidefinite matrices: $\mathbf{S}^n_+$
3. $\{(x_1, x_2) | |x_1| \leq x_2\}$
4. $\{\mathbf{x} \in \mathbb{R}^n | \|x\| \leq r\}$
5. $\{(\mathbf{x}, t) \in \mathbb{R}^{n+1} | \|x\| \leq t\}$

Farkas' lemma

Lemma (Farkas)

Assume $\mathbf{A} \in \mathbb{R}^{m \times n}$ *и* $\mathbf{b} \in \mathbb{R}^m$*. Then exactly one of the following system is feasible:*

$$1)\ \mathbf{A}\mathbf{x} = \mathbf{b},\ \ \mathbf{x} \geq 0$$

$$2)\ \mathbf{p}^\mathsf{T}\mathbf{A} \geq 0,\ \ \langle \mathbf{p}, \mathbf{b} \rangle < 0$$

Important corollary

Assume $\mathbf{A} \in \mathbb{R}^{m \times n}$ и $\mathbf{b} \in \mathbb{R}^m$. Then exactly one of the following systems is feasible:

$$1)\ \mathbf{A}\mathbf{x} \leq \mathbf{b}$$

$$2)\ \mathbf{p}^\mathsf{T}\mathbf{A} = 0,\ \ \langle \mathbf{p}, \mathbf{b} \rangle < 0,\ \ \mathbf{p} \geq 0$$

Application

If the feasible set in linear programming problem is nonempty and objective function is bounded below, then the problem is feasible.

Geometric interpretation

Farkas' lemma from geometric perspective

- $\mathbf{A}\mathbf{x} = \mathbf{b}$ with $\mathbf{x} \geq 0$ means that $\mathbf{b}$ lies in cone generated by the columns of matrix $\mathbf{A}$
- $\mathbf{p}^\top \mathbf{A} \geq 0, \ \langle \mathbf{p}, \mathbf{b} \rangle < 0$ means that there exists separation hyperplane between vector $\mathbf{b}$ and cone generated by the columns of matrix $\mathbf{A}$

Recap

- Conjugate sets
- Properties of conjugate sets
- Farkas' lemma