Optimization methods.
Seminar 10. Optimality conditions.

Alexandr Katrutsa

Moscow Institute of Physics and Technology
Department of Control and Applied Mathematics

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Remainder

- Conjugate function
- Young-Fenchel inequality
- Examples

Motivation

Question # 0

When does solution of the optimization problem exist?

Question # 1

How to check that a point is a solution of the optimization problem?

Question # 2

What conditions can give solution of the optimization problem?

Existence of the solution

Weierstrass theorem

Let $X \subset R^n$ be a compact set and $f : X \to \mathbb{R}$ be a continuous function on its domain. Then the global minimizer of f exists.

This theorem guarantees existence of the solution for most meaningful optimization problems

Optimality conditions

Definition

Optimality condition is a statement that gives necessary and/or sufficient condition for point to be local minimizer

Class of problems:

- General minimization problem
- Unconstrained minimization problem
- Equality constrained minimization problems
- Equality and inequality constrained minimization problem

General minimization problem

Problem

$$f(x) \to \min_{x \in X}$$

Optimality criterion

Assume $f(x)$ has domain $\mathrm{dom}\, f = X \subset \mathbb{R}^n$. Then

1. if x^* is a local minimizer of $f(x)$, then $0 \in \partial_X f(x^*)$
2. if subdifferential $\partial_X f(x^*)$ exists in some point $x^* \in X$ and $0 \in \partial_X f(x^*)$, then x^* is a local minimizer of $f(x)$.

What drawbacks have this criterion?

Examples

- $\mathbf{x}^\top \mathbf{x} + \alpha \|\mathbf{x} - \mathbf{c}\|_2 \to \min\limits_{\mathbf{x} \in \mathbb{R}^n}, \; \alpha > 0$
- $\mathbf{x}^\top \mathbf{x} + \alpha \|\mathbf{c}^\top \mathbf{x} - b\|_2 \to \min\limits_{\mathbf{x} \in \mathbb{R}^n}, \; \alpha > 0$
- Constraint on feasible set

$$
\begin{aligned}
& (x+2)^2 + |y+3| \to \min_{(x,y) \in \mathbb{R}^2} \\
\text{s.t. } & 8 + 2x - y \leq 0
\end{aligned}
$$

Unconstrained minimization problem

Problem: $f(x) \to \min\limits_{x \in \mathbb{R}^n}$.

Optimality criterion for convex functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then x^* is a solution of the unconstrained minimization problem iff $0 \in \partial f(x^*)$.

Corollary

If $f(x)$ is convex and differentiable, then x^* is a solution of the problem iff $\nabla f(x^*) = 0$.

Sufficient condition for non-convex functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable and x^* such that $\nabla f(x^*) = 0$. If $\nabla^2 f(x^*) \succ 0$, then x^* is a strict local minimizer.

Examples

- $x_1 e^{x_1} - (1 + e^{x_1}) \cos x_2 \to \min$
- Rosenbrock function:
  $(1 - x_1)^2 + \alpha \sum\limits_{i=2}^{n} (x_i - x_{i-1}^2)^2 \to \min, \; \alpha > 0$
- $x_1^2 + x_2^2 - x_1 x_2 + e^{x_1 + x_2} \to \min$

Minimization problem

$$f(x) \to \min_{x \in \mathbb{R}^n}$$
$$\text{s.t. } g_i(x) = 0, \; i = 1, \ldots, m$$

Lagrangian

$$L(x, \boldsymbol{\lambda}) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

Optimality condition (sufficient)

Let $f(x)$ and $g_i(x)$ be twice differentiable functions in x^* and continuously differentiable in some neighbourhood of x^*. Assume that $\nabla_x L(x^*, \boldsymbol{\lambda}) = 0$. The if $\mathbf{h}^\top \nabla^2 L(x^*, \boldsymbol{\lambda})\mathbf{h} > 0$, where $\mathbf{h} \in T(\mathbf{x}^* | G)$ — tangent cone, then x^* — point of the local minimum.

Possible options

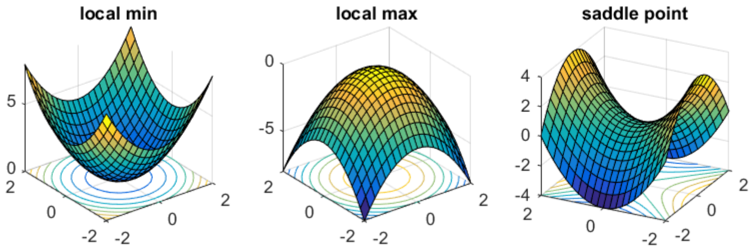

Figure is from
http://www.offconvex.org/2016/03/22/saddlepoints/

Examples

- $\sum\limits_{i=1}^{n} \alpha_i x_i^4 \to \mathrm{extr}_{\mathbf{x} \in G}, \; G = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{c}^\mathsf{T}\mathbf{x} = 1\},$
  $\alpha_i > 0, \; c_i > 0$
- $x_1 + 4x_2 + 9x_3 \to \mathrm{extr}_{\mathbf{x} \in G}, \; G = \left\{ \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 1 \right\}$

Equality and inequality constrained minimization problem

Minimization problem

$$\begin{gathered}\min_{x \in \mathbb{R}^n} f(x) \\ \text{s.t. } g_i(x) = 0, \; i = 1, \dots, m \\ h_j(x) \leq 0, \; j = 1, \dots, p\end{gathered}$$

Lagrangian

$$L(x, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{p} \mu_j h_j(x)$$

Optimality conditions

Necessary condition (Karush - Kuhn - Tucker)

Пусть x^* решение задачи математического программирования, и функции f, h_j, g_i дифференцирумы. Тогда найдутся такие $\boldsymbol{\mu}^*$ и $\boldsymbol{\lambda}^*$, что выполнены следующие условия:

- $g_i(x^*) = 0,\ i = 1, \ldots, m$
- $h_j(x^*) \leq 0,\ j = 1, \ldots, p$
- $\mu_j^* \geq 0,\ j = 1, \ldots, p$
- $\mu_j^* h_j(x^*) = 0,\ j = 1, \ldots, p$
- $\nabla_x L(x^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = 0$

If the minimization problem is convex, then the necessary optimality condition is also sufficient.

Optimality conditions (cont'd)

If the problem in non-convex, then

First order sufficient condition

If a stationary point $(x^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ gives number of active constraints $|J|$ such that $n = m + |J|$ and $\mu_j > 0$, $j \in J$, then this point is local minimizer.

Second order sufficient condition

If the number of active constraints is less than the dimension of the, then x^* is a local minimizer if the following conditions hold

$$\mathbf{z}^\mathsf{T} \nabla^2_{xx} L(x^*) \mathbf{z} > 0$$

for

- $\mathbf{z} \neq 0$ и $\nabla g_i^\mathsf{T}(x^*) \mathbf{z} = 0$
- $j \in J$ and $\mu_j > 0$, $\nabla h_j^\mathsf{T}(x^*) \mathbf{z} = 0$
- $j \in J$ and $\mu_j = 0$, $\nabla h_j^\mathsf{T}(x^*) \mathbf{z} \leq 0$

Examples

- Example 1

$$\begin{aligned} &\operatorname{extr}(x_1 - 3)(x_2 - 2) \\ \text{s.t. } & x_1 + 2x_2 = 4 \\ & x_1^2 + x_2^2 \le 5 \\ & x_1 \ge 0, \ x_2 \ge 0 \end{aligned}$$

- Example 2

$$\begin{aligned} &\operatorname{extr} \sum_{i=1}^{n} \frac{c_i}{x_i} \\ \text{s.t. } & \sum_{i=1}^{n} a_i x_i \le b \\ & x_i > 0, \ b > 0, \ c_i > 0, \ a_i > 0 \end{aligned}$$

- Example 3

$$\begin{aligned} &\operatorname{extr}(x_1 x_3 - 2x_2) \\ \text{s.t. } & 2x_1 - x_2 - 3x_3 \le 10 \\ & 3x_1 + 2x_2 + x_3 = 6 \\ & x_2 \ge 0 \end{aligned}$$

Recap

- Existence of the solution of the minimization problem
- Optimality conditions for
  - general minimization problem
  - unconstrained minimization problem
  - equality constrained minimization problem
  - equality and inequality constrained minimization problem