Optimization methods.
Seminar 8. Tangent and feasible direction cones and sharp extremum

Alexandr Katrutsa

Moscow Institute of Physics and Technology
Department of Control and Applied Mathematics

June 28, 2018

Reminder

- Subdifferential
- Conditional subdifferential
- Normal cone

Feasible direction cone

Definition

Feasible direction cone for a set $G \subset \mathbb{R}^n$ in a point $\mathbf{x}_0 \in G$ is a set $\Gamma(\mathbf{x}_0 | G) = \{\mathbf{s} \in \mathbb{R}^n | \mathbf{x}_0 + \alpha \mathbf{s} \in G, \; 0 \leq \alpha \leq \overline{\alpha}(\mathbf{s})\}$, where $\overline{\alpha}(\mathbf{s}) > 0$.

Definition for convex set

Feasible direction cone for a *convex* set $X \subset \mathbb{R}^n$ in a point$\mathbf{x}_0 \in X$ is a set $\Gamma(\mathbf{x}_0 | X) = \{\mathbf{s} \in \mathbb{R}^n | \mathbf{s} = \lambda(\mathbf{x} - \mathbf{x}_0), \; \lambda > 0, \forall \mathbf{x} \in X\}$.

How normal cone and feasible direction cone are related?

Example

Useful fact

Assume $G = \{\mathbf{x} \in \mathbb{R}^n | \varphi_i(\mathbf{x}) \leq 0,\ i = \overline{0, n-1};\ \varphi_i(\mathbf{x}) = \mathbf{a}_i^\mathsf{T}\mathbf{x} - b_i = 0,\ i = \overline{n, m}\}$. Then if $\varphi_i(\mathbf{x})$ is convex and set G is regular, then

$$\Gamma(\mathbf{x}_0 | G) = \{\mathbf{s} \in \mathbb{R}^n | \nabla\varphi_i(\mathbf{x}_0)^\mathsf{T}\mathbf{s} \leq 0, i \in I, \mathbf{a}_i^\mathsf{T}\mathbf{s} = 0, i = \overline{n, m}\}$$

and

$$\Gamma^*(\mathbf{x}_0 | G) = \left\{ \mathbf{p} \in \mathbb{R}^n \middle| \mathbf{p} = \sum_{i=n}^{m} \lambda_i \mathbf{a}_i - \sum_{i \in I} \mu_i \nabla\varphi_i(\mathbf{x}_0) \right\},$$

where $\lambda_i \in \mathbb{R}$, $\mu_i \geq 0$, $\mathbf{x}_0 \in G$ and $I = \{i : \varphi_i(\mathbf{x}_0) = 0,\ i = \overline{0, n-1}\}$.

Find $\Gamma(\mathbf{x}_0 | X)$ и $\Gamma^*(\mathbf{x}_0 | X)$ for the following sets:
$X = \{\mathbf{x} \in \mathbb{R}^2 | x_1^2 + 2x_2^2 \leq 3,\ x_1 + x_2 = 0\}$.

Tangent cone

Definition

Tangent cone to the set G in the point $\mathbf{x}_0 \in \overline{G}$ is the following set $T(\mathbf{x}_0|G) = \{\lambda\mathbf{z}|\lambda > 0,\ \exists\{\mathbf{x}_k\} \subset G,\ \mathbf{x}_k \to \mathbf{x}_0, \mathbf{x}_k \neq \mathbf{x}_0,\ \lim\limits_{k\to\infty} \frac{\mathbf{x}_k - \mathbf{x}_0}{\|\mathbf{x}_k - \mathbf{x}_0\|_2} = \mathbf{z}\}$

Remark

Tangent cone consists of all directions such that sequences from the set G converge to the point x_0 in this direction.

Lemma

If G is a convex set, then $T(\mathbf{x}_0|G) = \Gamma(\mathbf{x}_0|G)$.

Useful fact

Assume a set
$G = \{\mathbf{x} \in \mathbb{R}^n | \varphi_i(\mathbf{x}) \leq 0, i = \overline{0, n-1}\ \varphi_i(\mathbf{x}) = 0, i = \overline{n, m}\}$ is regular, then
$T(\mathbf{x}_0|G) = \{\mathbf{z} \in \mathbb{R}^n | \nabla\varphi_i^\mathsf{T}(\mathbf{x}_0)\mathbf{z} \leq 0, i \in I,\ \nabla\varphi_i^\mathsf{T}(\mathbf{x}_0)\mathbf{z} = 0, i = \overline{n, m}\}$
and

$$T^*(\mathbf{x}_0|G) = \left\{ \mathbf{p} \in \mathbb{R}^n \middle| \mathbf{p} = \sum_{i=n}^{m} \lambda_i \nabla\varphi_i(\mathbf{x}_0) - \sum_{i \in I} \mu_i \nabla\varphi_i(\mathbf{x}_0) \right\},$$

where $\mu_i \geq 0,\ \lambda_i \in \mathbb{R},\ I = \{i | \varphi_i(\mathbf{x}_0) = 0, i = \overline{0, n-1}\}$
Example: find $T(\mathbf{x}_0|G)$ and $T^*(\mathbf{x}_0|G)$ for a set
$G = \{\mathbf{x} \in \mathbb{R}^2 | x_1 + x_2 \leq 1,\ x_1^2 + 2x_2^2 = 1\}$

Sharp extremum

Definition

A point $\mathbf{x}^*$ is a point of sharp extremum of the function f on the set G, if there exists $\gamma > 0$ such that
$f(\mathbf{x}) - f(\mathbf{x}^*) \geq \gamma \|\mathbf{x} - \mathbf{x}^*\|_2, \; \forall x \in G$.

Lemma

Assume f is a differentiable function on $G \subset \mathbb{R}^n$. Then $\mathbf{x}^*$ is a point of sharp extremum of function f on the set G iff there exists $\alpha > 0$, such that
$\nabla f^\mathsf{T}(\mathbf{x}^*)\mathbf{z} \geq \alpha > 0, \; \mathbf{z} \in T(\mathbf{x}^* | G), \|\mathbf{z}\|_2 = 1$.

Examples

- $x_1^2 + x_2^2 \to \mathrm{extr}_G, \; G = \{(x_1, x_2) | x_1^2 + 2x_2^2 = 2, \; x_1 + x_2 \leq 1\}$
- $x_1 + 2x_2 \to \mathrm{extr}_G$

Recap

- Feasible direction cone
- Tangent cone
- Sharp extremum