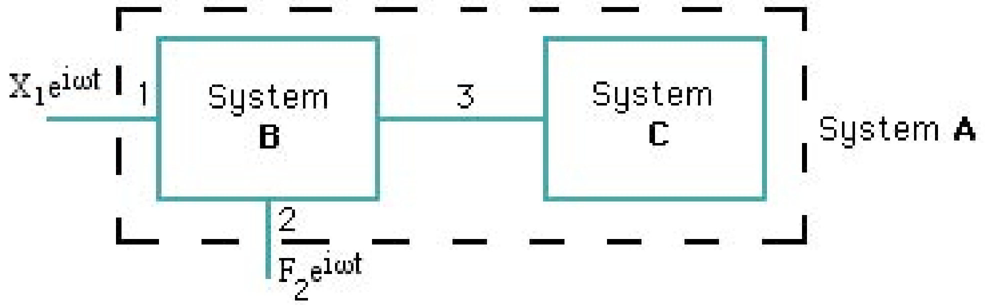
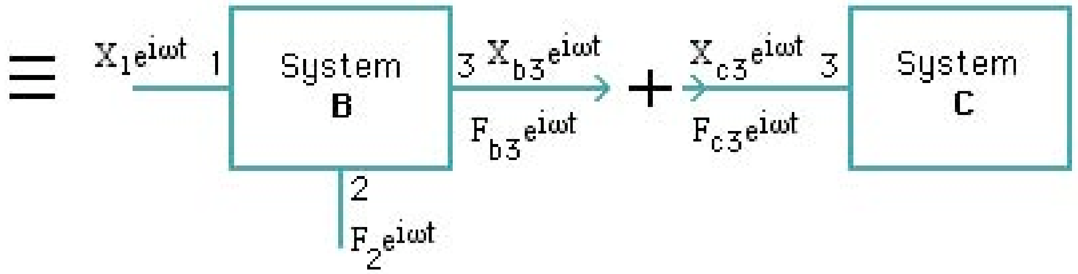
APPENDIX 1: Addition of a system at a remote co-ordinate

When the main system **B** is a real machine it is rarely possible to attach another system **C** at the point on **B** where the exciting force is applied. Also the relevant displacement will often be at the same position but in a different direction. It follows that the cross receptance b_{12} is the required receptance where 1 and 2 refer to the same position but a different direction. Also 1 and 2 may be defined as a relative displacement and a relative force (ie. equal and opposite forces) without changing the theory of system addition.

It is thus necessary to derive the equation for α_{12} for a combined system **A** when the additional system **C** has been attached away from the cutting position, ie. at some co-ordinate 3. This situation is shown in Figure 3 with the associated separation into individual systems to enable the analysis to be carried out. The forces and displacements introduced at 3 are such that the separated systems behave in the same way as when joined.





For system **C** by definition

$$X_{c3} = \gamma_{33}F_{c3} \quad \text{.......................................} \quad (9.27)$$

For system **B** since two forces are applied the displacement at any co-ordinate will be the sum of the displacements caused by each force. ie. superposition is assumed to apply.

Thus

$$X_{b3} = \beta_{33}F_{b3} + \beta_{32}F_2 \quad \text{.......................................} \quad (9.28)$$

and

$$X_1 = \beta_{12}F_2 + \beta_{13}F_{b3} \quad \text{.......................................} \quad (9.29)$$

Now for the systems to be identical when separated as when joined

$$X_{b3} = X_{c3} \quad \text{...} \quad (9.30)$$

and since there is no external force at 3.

$$F_{b3} + F_{c3} = 0 \quad \text{...} \quad (9.31)$$

Substitute $F_{c3} = -F_{b3}$ from (9.31) and for $\mathbf{X}_{b3}$ from (9.28) and $\mathbf{X}_{c3}$ from (9.27) in equation (9.30).

Thus $$\beta_{33}F_{b3} + \beta_{32}F_2 = -\gamma_{33}F_{b3}$$

therefore $$F_{b3} = \frac{\beta_{32}F_2}{\beta_{33} + \gamma_{33}}$$

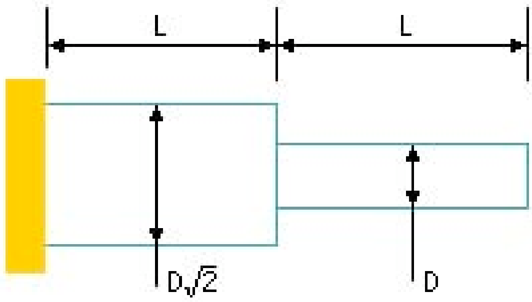
and substituting for F_{b3} in (9.29) gives $X_1 = \beta_{12}F_2 + \dfrac{\beta_{13}\beta_{32}F_2}{\beta_{33} + \gamma_{33}}$

therefore

$$\frac{X_1}{F_2} = \alpha_{12} = \beta_{12} + \frac{\beta_{13}\beta_{32}}{\beta_{33} + \gamma_{33}} \quad \text{..} \quad (9.32)$$

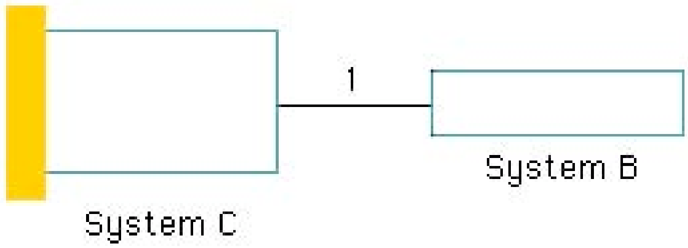
It is possible to measure β_{12}, β_{13}, β_{32} and β_{33} as functions of frequency and predict the effect on α_{12} of adding a particular system **C**.

APPENDIX 2: Axial Vibration of a stepped bar



Find the first natural frequency of the stepped bar shown. Each section has the same length L but the large bar has a diameter √2 times the smaller bar.

First divide in to two sub-systems,



The natural frequency equation is given by,

$$\beta_{11} + \gamma_{11} = 0$$

where

$$\gamma_{11} = \frac{\sin\lambda_c L}{A_c E_c \lambda_c \cos\lambda_c L}$$

and

$$\beta_{11} = -\frac{\cos\lambda_b L}{A_b E_b \lambda_b \sin\lambda_b L}$$

If the bars are made from the same material

$$E_b = E_c = E \text{ and also } \rho_b = \rho_c = \rho$$

$$\text{so that } \lambda_b = \lambda_c = \lambda = \omega\sqrt{\frac{\rho}{E}}$$

However because of the different diameters $A_b \neq A_c$.

$$A_c = \frac{\pi D_c^2}{4} = \frac{\pi 2 D_b^2}{4} = 2A_b$$

thus

$$\gamma_{11} = \frac{\sin\lambda L}{2A_b E\lambda\cos\lambda L}$$

and

$$\beta_{11} = -\frac{\cos\lambda L}{AE_b\lambda\sin\lambda L}$$

The natural frequency equation is thus

$$\frac{\sin\lambda L}{2A_bE\lambda\cos\lambda L} - \frac{\cos\lambda L}{A_bE\lambda\sin\lambda L} = 0$$

so that

$$\frac{\sin\lambda L}{2\cos\lambda L} = \frac{\cos\lambda L}{\sin\lambda L} \quad \text{and hence}$$

$$\tan^2 \lambda L = 2$$

and

$$\tan \lambda L = \pm\sqrt{2}$$

Note the negative sign is important when higher natural frequencies are to be found. For the first natural frequency,

$$\tan \lambda L = +\sqrt{2}$$

$$\lambda L = 0.9553$$

$$\text{since } \lambda^2 = \omega^2\rho / E$$

$$\omega_{n1} = \frac{0.9553}{L}\sqrt{\frac{E}{\rho}}$$

and for the second natural frequency

$$\tan \lambda L = -\sqrt{2}$$

$$\lambda L = 2.186$$

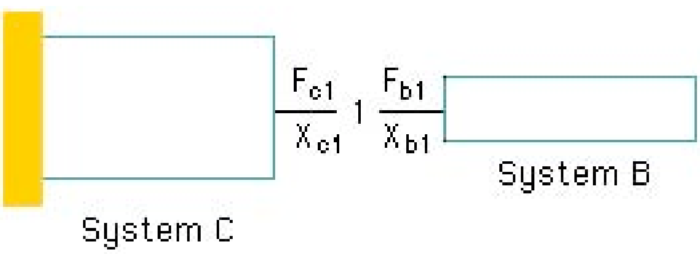
$$\text{since } \lambda^2 = \omega^2\rho / E$$

$$\omega_{n2} = \frac{2.186}{L}\sqrt{\frac{E}{\rho}}$$

<u>9.7.1 Classical Mode Shapes</u>

If the system has no damping then it will have undamped natural frequencies. If we wish to determine the deflected shapes (mode shapes) at these frequencies we cannot have an external excitation as at an undamped natural frequency the response will tend to infinity. Hence we apply no external excitation. If any point in the system is assumed to have an amplitude of unity then we find the forces at the joins, all of which will have zero external excitation.

Hence to find the associated mode shapes, divide the system into its sub-systems and assume an amplitude of unity at the join.



Thus we have $X_{b1} = X_{c1} = 1.0$ and we have already shown,

$$\beta_{11} = -\frac{\cos\lambda L}{A_b E\lambda \sin\lambda L}$$

Thus

$$\beta_{11} = \frac{X_{b1}}{F_{b1}} = \frac{1}{F_{b1}} = -\frac{\cos\lambda L}{A_b E\lambda \sin\lambda L}$$

$$\therefore F_{b1} = -\frac{A_b E\lambda \sin\lambda L}{\cos\lambda L}$$

Now consider the amplitudes along the bar B at some position x from the left hand end,

$$\beta_{x1} = \frac{X_{bx}}{F_{b1}} = -\frac{\cos\lambda(L - x)}{A_b E\lambda \sin\lambda L}$$

and substituting for F_{b1}

For the first mode,

$$\lambda L = 0.9553 \text{ and thus } X_{bx} = \frac{\cos 0.9553\left(1 - \frac{x}{L}\right)}{\cos 0.9553} = 1.733\cos 0.9553\left(1 - \frac{x}{L}\right)$$

and for the second mode

$$\lambda L = 2.186 \text{ and thus } X_{bx} = \frac{\cos 2.186\left(1 - \frac{x}{L}\right)}{\cos 2.186} = -1.733\cos 2.186\left(1 - \frac{x}{L}\right)$$

Now consider the sub system C. There is no external force at coordinate 1 so that

$$F_{c1} = -F_{b1} = \frac{A_b E\lambda \sin\lambda L}{\cos\lambda L}$$

Now consider the amplitudes along the bar C at some position y from the left hand end,
(Note: it is not wise to use x again for position as confusion may arise with system C)

$$\frac{X_{cy}}{F_{c1}} = \gamma_{11} = \frac{\sin\lambda y}{A_c E\lambda \cos\lambda L}$$

and substituting for F_{c1}

$$U_{cy} = \frac{\sin\lambda y}{A_c E\lambda \cos\lambda L}\frac{A_b E\lambda \sin\lambda L}{\cos\lambda L}$$

Since $A_c = 2A_b$

$$\theta_{cy} = \frac{\sin\lambda y \tan\lambda L}{2\cos\lambda L}$$

For the first mode,

$$\lambda L = 0.9553 \text{ and thus } X_{cy} = \frac{\sin 0.9553\frac{y}{L}\tan 0.9553}{2\cos 0.9553} = 1.225\sin 0.9553\frac{y}{L}$$

and for the second mode

$$\lambda L = 2.186 \text{ and thus } X_{cy} = \frac{\sin 2.186\frac{y}{L}\tan 2.186}{2\cos 2.186} = 1.226\sin 2.186\frac{y}{L}$$

To plot the mode shape take ten points on each bar.

For bar C

$$\text{mode 1 } X_{cy} = 1.225\sin 0.9553\frac{y}{L} \qquad \text{mode 2 } X_{cy} = 1.226\sin 2.186\frac{y}{L}$$

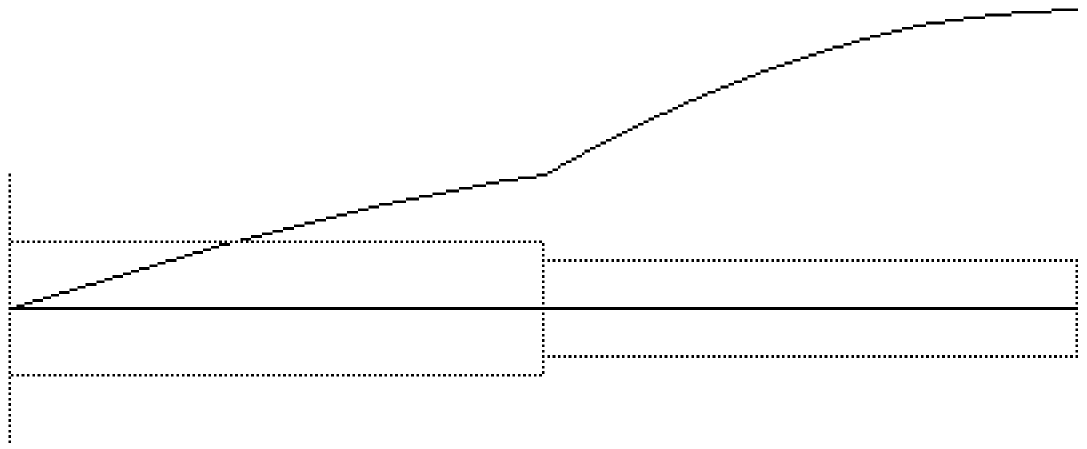
y/L=0	$U_{cy} = 0$	$U_{cy} = 0$
y/L=0.1	$U_{cy} = 0.1169$	$U_{cy} = 0.2659$
y/L=0.2	$U_{cy} = 0.2326$	$U_{cy} = 0.5191$
y/L=0.3	$U_{cy} = 0.3463$	$U_{cy} = 7476$
y/L=0.4	$U_{cy} = 0.4568$	$U_{cy} = 0.9405$
y/L=0.5	$U_{cy} = 0.5631$	$U_{cy} = 1.0887$
y/L=0.6	$U_{cy} = 0.6643$	$U_{cy} = 1.1851$
y/L=0.7	$U_{cy} = 0.7595$	$U_{cy} = 1.2250$
y/L=0.8	$U_{cy} = 0.8480$	$U_{cy} = 1.2066$
y/L=0.9	$U_{cy} = 0.9282$	$U_{cy} = 1.1308$
y/L=1.0	$U_{cy} = 1.0$	$U_{cy} = 1.0$

For bar B

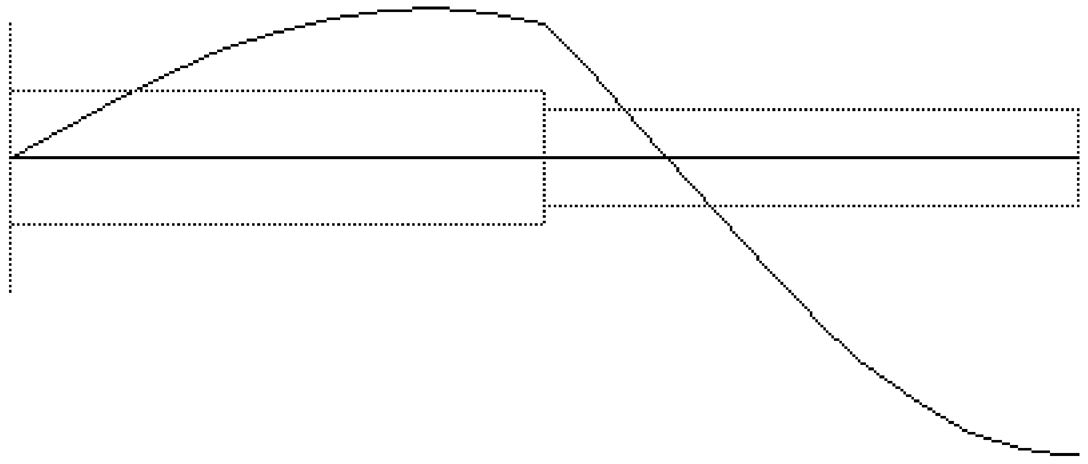
$$\text{mode 1 } X_{bx} = 1.733\cos 0.9553\left(1-\frac{x}{L}\right) \qquad \text{mode 2 } X_{bx} = -1.733\cos 2.186\left(1-\frac{x}{L}\right)$$

x/L=0 $U_{bx} = 1.0$		$U_{bx} = 1.0$
x/L=0.1	$U_{bx} = 1.1310$	$U_{bx} = 0.6694$
x/L=0.2	$U_{bx} = 1.2510$	$U_{bx} = 0.3069$
x/L=0.3	$U_{bx} = 1.3598$	$U_{bx} = -0.0703$
x/L=0.4	$U_{bx} = 1.4560$	$U_{bx} = -0.4442$
x/L=0.5	$U_{bx} = 1.5390$	$U_{bx} = -0.7969$
x/L=0.6	$U_{bx} = 1.6080$	$U_{bx} = -1.1117$
x/L=0.7	$U_{bx} = 1.6623$	$U_{bx} = -1.3735$
x/L=0.8	$U_{bx} = 1.7015$	$U_{bx} = -1.5700$
x/L=0.9	$U_{bx} = 1.7251$	$U_{bx} = -1.6918$
x/L=1.0	$U_{bx} = 1.7330$	$U_{bx} = -1.7330$

We may present these mode shapes in graphical form

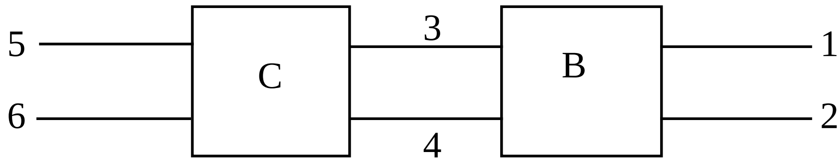


Mode 1



Mode 2.

APPENDIX 3: Receptance Addition with two coupling coordinates



For system B

$$U_1 = \beta_{11}F_1 + \beta_{12}F_2 + \beta_{13}F_{b3} + \beta_{14}F_{b4} \quad \ldots\ldots \quad (9.33)$$

$$U_2 = \beta_{12}F_1 + \beta_{22}F_2 + \beta_{23}F_{b3} + \beta_{24}F_{b4} \quad \ldots\ldots \quad (9.34)$$

$$U_{b3} = \beta_{13}F_1 + \beta_{23}F_2 + \beta_{33}F_{b3} + \beta_{34}F_{b4} \quad \ldots\ldots \quad (9.35)$$

$$U_{b4} = \beta_{14}F_1 + \beta_{24}F_2 + \beta_{34}F_{b3} + \beta_{44}F_{b4} \quad \ldots\ldots \quad (9.36)$$

For system C

$$U_{c3} = \gamma_{33}F_{c3} + \gamma_{34}F_{c4} + \gamma_{35}F_5 + \gamma_{36}F_6 \quad \ldots\ldots \quad (9.37)$$

$$U_{c4} = \gamma_{34}F_{c3} + \gamma_{44}F_{c4} + \gamma_{45}F_5 + \gamma_{46}F_6 \quad \ldots\ldots \quad (9.38)$$

$$U_5 = \gamma_{35}F_{c3} + \gamma_{45}F_{c4} + \gamma_{55}F_5 + \gamma_{56}F_6 \quad \ldots\ldots \quad (9.39)$$

$$U_6 = \gamma_{36}F_{c3} + \gamma_{46}F_{c4} + \gamma_{56}F_5 + \gamma_{66}F_6 \quad \ldots\ldots \quad (9.40)$$

At boundary, for compatability,

$$U_3 = U_{b3} = U_{c3} \quad \ldots\ldots \quad (9.41)$$

$$U_4 = U_{b4} = U_{c4} \quad \ldots\ldots \quad (9.42)$$

At boundary, for force equivalence,

$$F_{b3} + F_{c3} = F_3 \quad \ldots\ldots \quad (9.43)$$

$$F_{b4} + F_{c4} = F_4 \quad \ldots\ldots \quad (9.44)$$

Substituting in (9.41) and (19.42) from (9.35), (9.36), (9.37) and (9.38),

$$\beta_{13}F_1 + \beta_{23}F_2 + \beta_{33}F_{b3} + \beta_{34}F_{b4} = \gamma_{33}F_{c3} + \gamma_{34}F_{c4} + \gamma_{35}F_5 + \gamma_{36}F_6 \quad \ldots\ldots \quad (9.45)$$

$$\beta_{14}F_1 + \beta_{24}F_2 + \beta_{34}F_{b3} + \beta_{44}F_{b4} = \gamma_{34}F_{c3} + \gamma_{44}F_{c4} + \gamma_{45}F_5 + \gamma_{46}F_6 \quad \ldots\ldots \quad (9.46)$$

Substituting for F_{c3} and F_{c4} from (9.43) and (9.344) in (9.45)and (9.46)

$$\beta_{13}F_1 + \beta_{23}F_2 + \beta_{33}F_{b3} + \beta_{34}F_{b4} = \gamma_{33}(F_3 - F_{b3}) + \gamma_{34}(F_4 - F_{b4}) + \gamma_{35}F_5 + \gamma_{36}F_6$$

$$\beta_{14}F_1 + \beta_{24}F_2 + \beta_{34}F_{b3} + \beta_{44}F_{b4} = \gamma_{34}(F_3 - F_{b3}) + \gamma_{44}(F_4 - F_{b4}) + \gamma_{45}F_5 + \gamma_{46}F_6$$

rearranging,

$$(\beta_{33} + \gamma_{33})F_{b3} + (\beta_{34} + \gamma_{34})F_{b4} = -\beta_{13}F_1 - \beta_{23}F_2 + \gamma_{33}F_3 + \gamma_{34}F_4 + \gamma_{35}F_5 + \gamma_{36}F_6$$

$$(\beta_{34} + \gamma_{34})F_{b3} + (\beta_{44} + \gamma_{44})F_{b4} = -\beta_{14}F_1 - \beta_{24}F_2 + \gamma_{34}F_3 + \gamma_{44}F_4 + \gamma_{45}F_5 + \gamma_{46}F_6$$

and substituting,

$$P = -\beta_{13}F_1 - \beta_{23}F_2 + \gamma_{33}F_3 + \gamma_{34}F_4 + \gamma_{35}F_5 + \gamma_{36}F_6 \quad \ldots\ldots \quad (9.47)$$

$$Q = -\beta_{14}F_1 - \beta_{24}F_2 + \gamma_{34}F_3 + \gamma_{44}F_4 + \gamma_{45}F_5 + \gamma_{46}F_6 \quad \ldots\ldots \quad (9.48)$$

gives,

$$(\beta_{33} + \gamma_{33})F_{b3} + (\beta_{34} + \gamma_{34})F_{b4} = P$$

$$(\beta_{34}+\gamma_{34})F_{b3}+(\beta_{44}+\gamma_{44})F_{b4}=Q$$

Solving for F_{b3} and F_{b4} gives,

$$F_{b3}=\frac{P(\beta_{44}+\gamma_{44})-Q(\beta_{34}+\gamma_{34})}{(\beta_{33}+\gamma_{33})(\beta_{44}+\gamma_{44})-(\beta_{34}+\gamma_{34})^2}$$

and

$$F_{b4}=\frac{Q(\beta_{33}+\gamma_{33})-P(\beta_{34}+\gamma_{34})}{(\beta_{33}+\gamma_{33})(\beta_{44}+\gamma_{44})-(\beta_{34}+\gamma_{34})^2}$$

and substituting,

$$\Delta=(\beta_{33}+\gamma_{33})(\beta_{44}+\gamma_{44})-(\beta_{34}+\gamma_{34})^2 \qquad \text{......} \qquad (9.49)$$

yields,

$$F_{b3}=\frac{P(\beta_{44}+\gamma_{44})-Q(\beta_{34}+\gamma_{34})}{\Delta} \qquad \text{......} \qquad (9.50)$$

$$F_{b4}=\frac{Q(\beta_{33}+\gamma_{33})-P(\beta_{34}+\gamma_{34})}{\Delta} \qquad \text{......} \qquad (9.51)$$

All six responses U_1, U_2, U_3, U_4, U_5 and U_6 may be obtained using equations (9.33) to (9.51),

$$U_1=\beta_{11}F_1+\beta_{12}F_2+\beta_{13}F_{b3}+\beta_{14}F_{b4} \qquad \text{......} \qquad (9.33)$$

$$U_2=\beta_{12}F_1+\beta_{22}F_2+\beta_{23}F_{b3}+\beta_{24}F_{b4} \qquad \text{......} \qquad (9.34)$$

$$U_3=\beta_{13}F_1+\beta_{23}F_2+\beta_{33}F_{b3}+\beta_{34}F_{b4} \qquad \text{......} \qquad (9.35)$$

$$U_4=\beta_{14}F_1+\beta_{24}F_2+\beta_{34}F_{b3}+\beta_{44}F_{b4} \qquad \text{......} \qquad (9.36)$$

$$U_5=\gamma_{35}(F_3-F_{b3})+\gamma_{45}(F_4-F_{b4})+\gamma_{55}F_5+\gamma_{56}F_6 \qquad \text{......} \qquad (9.39)$$

$$U_6=\gamma_{36}(F_3-F_{b3})+\gamma_{46}(F_4-F_{b4})+\gamma_{56}F_5+\gamma_{66}F_6 \qquad \text{......} \qquad (9.40)$$

For any particular receptance we require the response to just one force. Thus for example, α_{13} is, by definition, given by

$$\alpha_{13}=\frac{U_1}{F_3} \text{ with all other forces than } F_3 \text{ zero.}$$

therefore

$$U_1=\beta_{13}\left[\frac{P(\beta_{44}+\gamma_{44})-Q(\beta_{34}+\gamma_{34})}{\Delta}\right]+\beta_{14}\left[\frac{Q(\beta_{33}+\gamma_{33})-P(\beta_{34}+\gamma_{34})}{\Delta}\right]$$

and P and Q reduce to,

$$P=\gamma_{33}F_3 \text{ and } Q=\gamma_{34}F_3$$

so that

$$U_1=\beta_{13}\left[\frac{\gamma_{33}F_3(\beta_{44}+\gamma_{44})-\gamma_{34}F_3(\beta_{34}+\gamma_{34})}{\Delta}\right]+\beta_{14}\left[\frac{\gamma_{34}F_3(\beta_{33}+\gamma_{33})-\gamma_{33}F_3(\beta_{34}+\gamma_{34})}{\Delta}\right]$$

therefore

$$\alpha_{13}=\frac{U_1}{F_3}=\beta_{13}\left[\frac{\gamma_{33}(\beta_{44}+\gamma_{44})-\gamma_{34}(\beta_{34}+\gamma_{34})}{\Delta}\right]+\beta_{14}\left[\frac{\gamma_{34}(\beta_{33}+\gamma_{33})-\gamma_{33}(\beta_{34}+\gamma_{34})}{\Delta}\right]$$

or rearranging,

$$\alpha_{13} = \frac{U_1}{F_3} = \frac{\beta_{13}\left[\gamma_{33}(\beta_{44}+\gamma_{44})-\gamma_{34}(\beta_{34}+\gamma_{34})\right]+\beta_{14}\left[\gamma_{34}(\beta_{33}+\gamma_{33})-\gamma_{33}(\beta_{34}+\gamma_{34})\right]}{\Delta}$$

The same result could have been obtained using,

$$\alpha_{13} = \frac{U_3}{F_1} \text{ with all other forces than } F_1 \text{ zero.}$$

This only applies for linear conservative systems when Maxwell's reciprocal Theorem holds.

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