CHAPTER 3: Beam on springs.

We have considered one and two degree-of-freedom vibration. It was found that a one degree-of-freedom system had one natural frequency and that a two degree-of-freedom system had two natural frequencies and two associated mode shapes. The examples considered were the axial vibration of systems including springs, masses and viscous dampers. Before considering more than two degrees-of-freedom it is appropriate to demonstrate that the degrees-of-freedom need not be restricted to the axial positions of a mass. Consider a long slender (but rigid) beam supported on springs at each end (figure 3.1).

3.1 Transient vibration

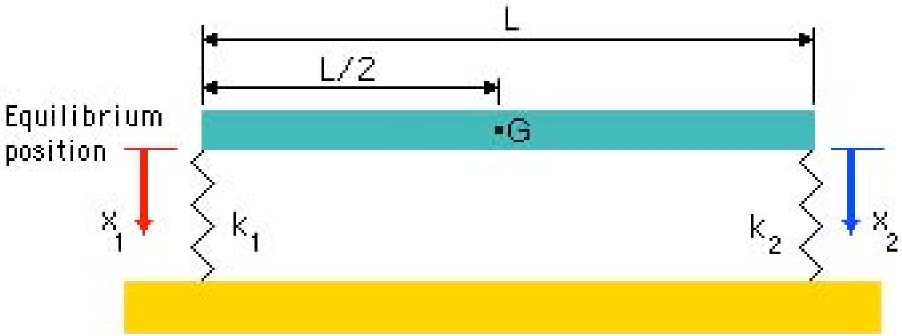

Figure 3.1 Beam on springs.

The static deflections will be omitted. We will consider all deflections relative to the equilibrium position. We have a choice over the two coordinates to use to describe the degrees-of-freedom. We could use the rotation of the beam and the vertical motion of the centre of gravity. However, we will use the vertical deflections at each end of the beam. Thus the two coordinates are x_1 and x_2. Consider the free-body diagram of the beam as shown in figure 3.2.

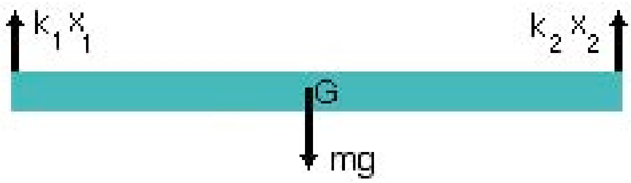

Figure 3.2 Free body diagram of beam.

As we omit the static deflections then the mg term need not be included as the static deflection produces a resultant force that is equal and opposite to mg. Considering the deflections from the equilibrium position.

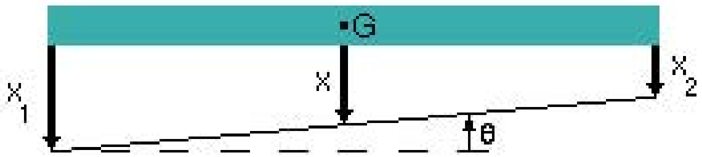

Figure 3.3 Deflections of the beam

Thus x(t) the position of the centre of mass is given by,

$$x = \left(x_1 + x_2\right) / 2 \quad \ldots\ldots \quad (3.1)$$

and for small angles,

$$\theta = \left(x_1 - x_2\right) / L \quad \ldots\ldots \quad (3.2)$$

Now to the kinematics. The linear motion of the centre of mass gives,

$$mx'' = -k_1 x_1 - k_2 x_2 \quad \ldots\ldots \quad (3.3)$$

and the angular motion about the centre of mass gives

$$\frac{1}{12} mL^2 \theta'' = -k_1 x_1 L / 2 + k_2 x_2 L / 2 \quad \ldots\ldots \quad (3.4)$$

Substituting for x and θ from equations (3.1) and (3.2),

$$m(x_1'' + x_2'') = 2\left(-k_1 x_1 - k_2 x_2\right) \quad \ldots\ldots \quad (3.5)$$

$$m(x_1'' - x_2'') = 6\left(-k_1 x_1 + k_2 x_2\right) \quad \ldots\ldots \quad (3.6)$$

<u>3.1.1 Natural frequencies and mode shapes.</u>

To find the natural frequencies we will use the approach that assumes sinusoidal motion such that

$$x_1 = X_1 \sin wt \quad \text{and} \quad x_2 = X_2 \sin wt$$

Substituting in equations (3.7) and (3.8) gives,

$$-m\omega^2 (X_1 + X_2) = 2\left(-k_1 X_1 - k_2 X_2\right)$$

$$\therefore X_1(2k_1 - m\omega^2) + X_2(2k_2 - m\omega^2) = 0 \quad \ldots\ldots \quad (3.7)$$

and

$$-m\omega^2 (X_1 - X_2) = 6\left(-k_1 X_1 + k_2 X_2\right)$$

$$\therefore X_1(6k_1 - m\omega^2) - X_2(6k_2 - m\omega^2) = 0 \quad \ldots\ldots \quad (3.8)$$

From (3.7)

$$X_2 = -\frac{(2k_1 - m\omega^2)}{(2k_2 - m\omega^2)} X_1 \quad \ldots\ldots \quad (3.9)$$

and substituting in (3.8)

$$\therefore X_1\left((6k_1 - m\omega^2) + \frac{(2k_1 - m\omega^2)(6k_2 - m\omega^2)}{(2k_2 - m\omega^2)}\right) = 0$$

$$\therefore X_1\left(\frac{24k_1k_2 - 6k_1m\omega^2 - 2k_2m\omega^2 + m^2\omega^4 - 6k_1m\omega^2 - 6k_2m\omega^2 + m^2\omega^4}{(2k_2 - m\omega^2)}\right) = 0$$

It follows that either $X_1 = 0$, and no motion exists, or motion of the assumed form may occur if,

$$m^2\omega^4 - 4m\omega^2\left(k_1 + k_2\right) + 12k_1k_2 = 0 \quad \ldots\ldots \quad (3.10)$$

As expected for a two degree-of-freedom system this gives two natural frequencies. Consider the special case of k_1=k and k_2=2k when (3.10) becomes,

$$m^2\omega^4 - 12mk\omega^2 + 24k^2 = 0$$

Solving this equation gives the two natural frequencies as,

$$\omega_{n1} = 1.592\sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_{n2} = 3.076\sqrt{\frac{k}{m}}$$

For each natural frequency the associated mode shapes X_1/X_2 are found by considering equations (3.9) with k_1=k and k_2=2k

$$\frac{X_1}{X_2} = -\frac{(4k - m\omega^2)}{(2k - m\omega^2)}$$

for $\omega_{n1} = 1.592\sqrt{\frac{k}{m}}$

$$\frac{X_1}{X_2} = -\frac{k(4 - 2.536)}{k(2 - 2.536)} = \frac{1.464}{0.536} = 2.73$$

and for $\omega_{n2} = 3.076\sqrt{\frac{k}{m}}$

$$\frac{X_1}{X_2} = -\frac{k(4 - 9.464)}{k(2 - 9.464)} = -\frac{5.464}{7.464} = -0.73$$

The natural frequencies and mode shapes for a range of k_1 and k_2 may be examined.

3.1.2 Transient motion

After much maths equations (3.5) and (3.6) may be solved for a given set of initial conditions. A typical response is shown in figure 3.4.

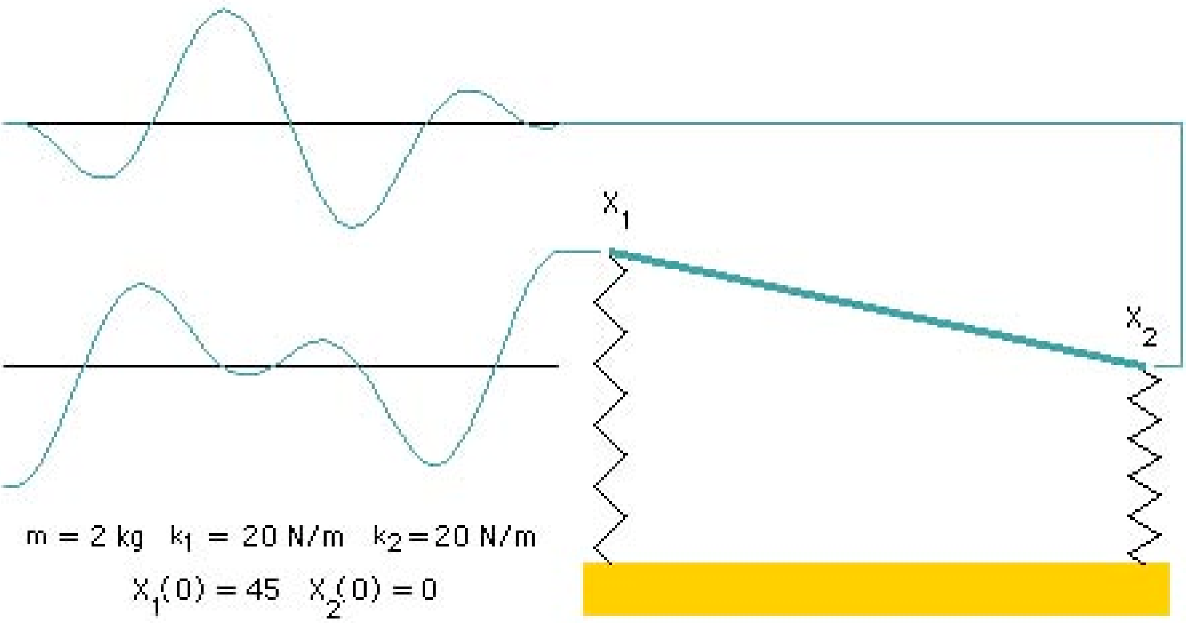

Figure 3.4 Transient motion

Different initial conditions and parameter values give different transients. The effect on the motion may be examined by running the program.

For the undamped case the transient motion is found to be the superposition of the two modes of vibration. It is possible to see the modal contributions by running the program

3.2 Forced vibration

Consider a force F(t) applied a distance b from the left hand end as shown in figure 3.5.

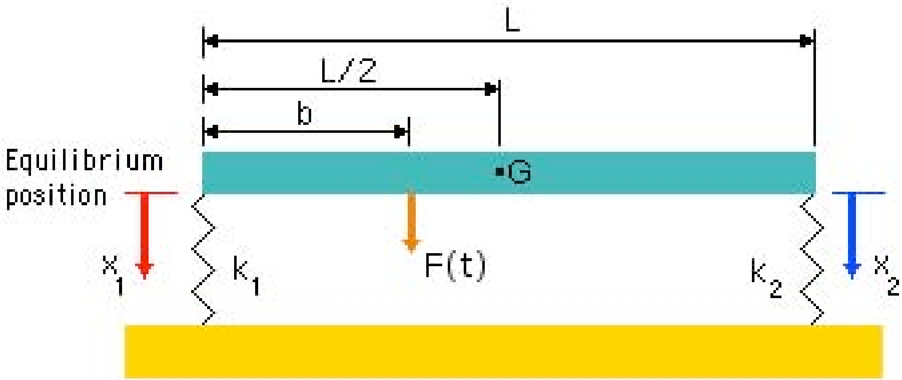

Figure 3.5 Excitation by an external force.

To obtain the equations of motion a free body diagram needs to be drawn as in figure 3.6

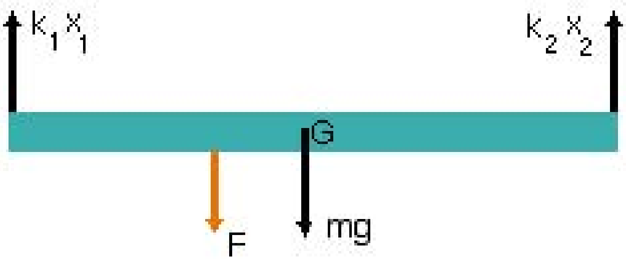

Figure 3.6 Free body diagram

Again as we omit static deflections then the mg term need not be included. Considering the deflections from the equilibrium position as before,

$$x = \left(x_1 + x_2\right)/2 \quad \text{..} \quad (3.1)$$

and for small angles,

$$\theta = \left(x_1 - x_2\right)/L \quad \text{..} \quad (3.2)$$

Now to the kinematics.

The linear motion of the centre of mass gives,

$$mx'' = -k_1x_1 - k_2x_2 + F(t) \quad \ldots\ldots \quad (3.11)$$

and the angular motion about the centre of mass gives

$$\frac{1}{12}mL^2\theta'' = -k_1x_1L/2 + k_2x_2L/2 + F(t)(L/2 - b) \quad \ldots\ldots \quad (3.12)$$

Substituting for x and q from equations (3.1) and (3.2).

$$x_1'' + x_2'' = 2(-k_1x_1 - k_2x_2 + F(t))/m \quad \ldots\ldots \quad (3.13)$$

$$x_1'' - x_2'' = 12(-k_1x_1/2 + k_2x_2/2 + F(t)(0.5 - b/L))/m \quad \ldots\ldots \quad (3.14)$$

Adding (3.13) and (3.14)

$$x_1'' = (-4k_1x_1 + 2k_2x_2 + F(t)(4 - 6b/L))/m \quad \ldots\ldots \quad (3.15)$$

Subtracting (3.13) and (3.14)

$$x_2'' = (+2k_1x_1 - 4k_2x_2 - F(t)(2 - 6b/L))/m \quad \ldots\ldots \quad (3.16)$$

Equations (3.15) and (3.16) can be solved in various ways. They have been presented in the form above as then a numerical method such as Runge Kutta may be used.

3.2.1 Sinusoidal excitation

If the force Fsinωt then as found previously for other examples there will be a start up transient. The maths is significant. A typical start up transient is shown in figure 3.7.

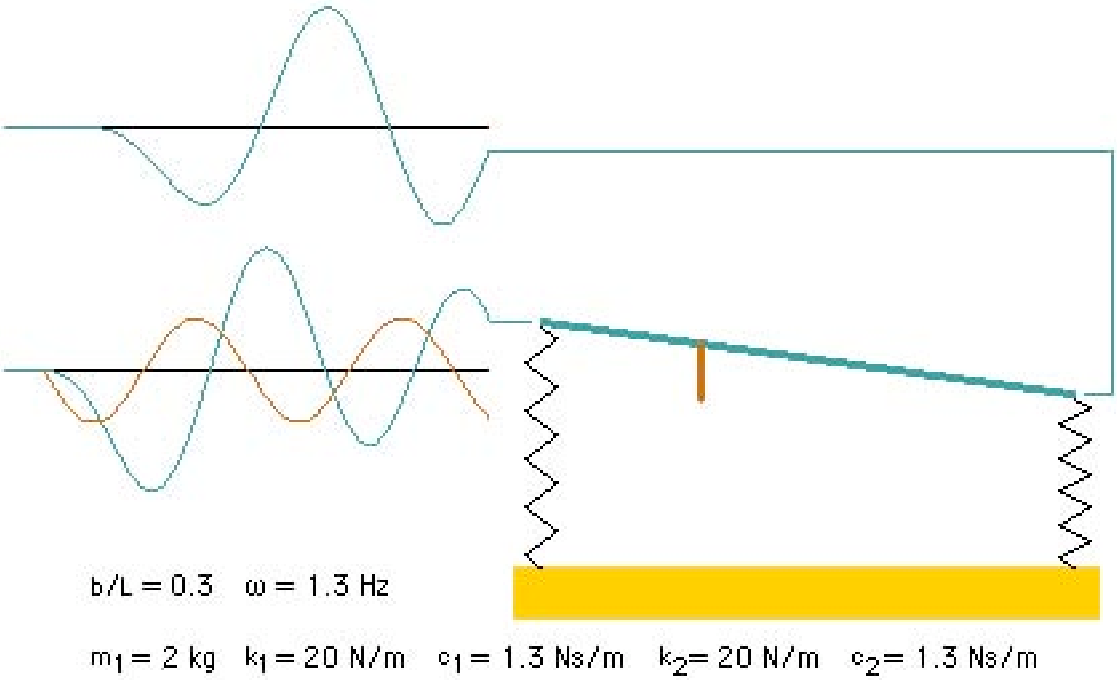

Figure 3.7 Start up transient with sinusoidal force excitation.

The motion depends on the system parameters and may be investigated by running the program.

For the particular case of steady state sinusoidal excitation,

$$x_1 = X_1 e^{i\omega t} \text{ and } x_2 = X_2 e^{i\omega t} \text{ and } F = Fe^{i\omega t}$$

so that (3.15) becomes

$$-\omega^2 X_1 = \left(-4k_1 X_1 + 2k_2 X_2 + F(4 - 6b / L)\right) / m$$

and rearranging

$$\left(4k_1 - m\omega^2\right) X_1 - 2k_2 X_2 = F(4 - 6b / L) \quad \text{..............................} \quad (3.17)$$

and (3.16) becomes

$$-\omega^2 X_2 = \left(+2k_1 X_1 - 4k_2 X_2 - F(2 - 6b / L)\right) / m$$

and rearranging

$$-2k_1 X_1 + \left(4k_2 - m\omega^2\right) X_2 = -F(2 - 6b / L) \quad \text{.........................} \quad (3.18)$$

(3.17) and (3.18) may be re-written

$$aX_1 + bX_2 = c \quad \text{and} \quad dX_1 + eX_2 = f$$

then

$$X_1 = \frac{ec - bf}{ae - bd} = \frac{\left(4k_2 - m\omega^2\right)(4 - 6b / L) - 2k_2(2 - 6b / L)}{\left(4k_1 - m\omega^2\right)\left(4k_2 - m\omega^2\right) - 4k_2 k_1} F$$

$$\frac{X_1}{F} = \frac{12k_2(1 - b / L) - m\omega^2(4 - 6b / L)}{\left(4k_1 - m\omega^2\right)\left(4k_2 - m\omega^2\right) - 4k_2 k_1} \quad \text{..............................} \quad (3.19)$$

and

$$X_2 = \frac{af - cd}{ae - bd} = \frac{-\left(4k_1 - m\omega^2\right)(2 - 6b / L) + 2k_1(4 - 6b / L)}{\left(4k_1 - m\omega^2\right)\left(4k_2 - m\omega^2\right) - 4k_1 k_2} F$$

$$\frac{X_2}{F} = \frac{12k_1 b / L + m\omega^2(2 - 6b / L)}{\left(4k_1 - m\omega^2\right)\left(4k_2 - m\omega^2\right) - 4k_1 k_2} \quad \text{.............................} \quad (3.20)$$

Some typical responses are shown in figure 3.8.

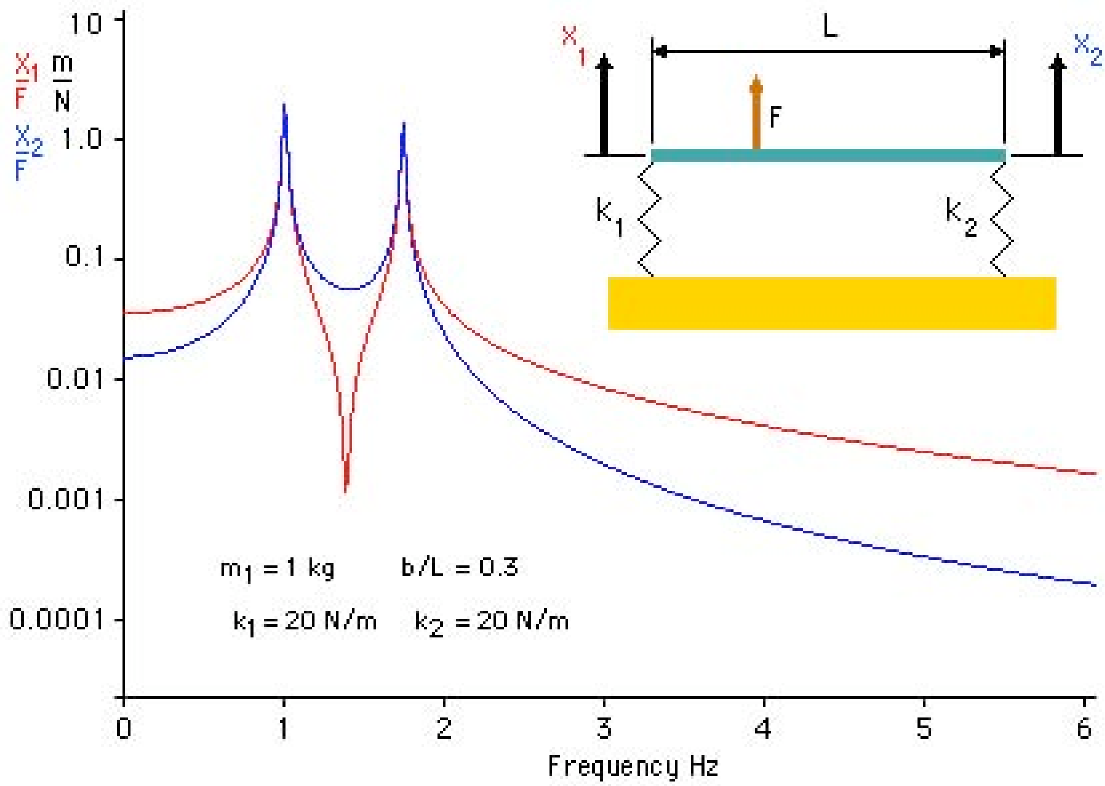

Figure 3.8 End responses for force excitation.

Note

The natural frequencies are given when the denominator of equations (3.19) and (3.20) is zero (and resonance occurs). Thus when

$$m^2\omega^4 - 4m\omega^2(k_1 + k_2) + 12k_1k_2 = 0$$

This is identical to the natural frequency equation obtained previously - equation (3.10). The associated mode shapes can be found from equation (3.17) with F=0,

$$\frac{X_1}{X_2} = \frac{2k_2}{4k_1 - m\omega^2} \qquad \text{(3.21)}$$

3.3 The effect of damping.

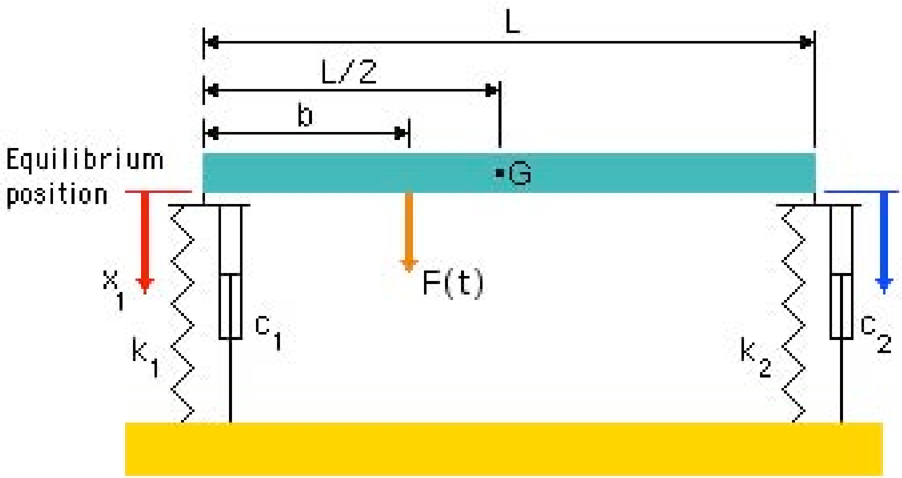

Figure 3.9 Beam supported on springs and viscous dampers.

When viscous damping is included in parallel with the springs, wherever there was a kx term we will now have $kx + cx'$ so that equations (3.11)and (3.12) become

$$mx'' = -k_1x_1 - c_1x_1' - k_2x_2 - c_2x_2' + F(t) \quad \text{..............................} \quad (3.22)$$

$$\frac{1}{12}mL^2\theta'' = (-k_1x_1 - c_1x_1')L/2 + (k_2x_2 + c_2x_2')L/2 + F(L/2 - b) \quad \text{..........} \quad (3.23)$$

Substituting for x and θ from equations (3.1) and (3.2).

$$x_1'' + x_2'' = 2(-k_1x_1 - c_1x_1' - k_2x_2 - c_2x_2' + F)/m \quad \text{..........................} \quad (3.24)$$

$$x_1'' - x_2'' = 12((-k_1x_1 - c_1x_1')/2 + (k_2x_2 + c_2x_2')/2 + F(0.5 - b/L))/m \quad \text{.........} \quad (3.25)$$

Adding (3.24) and (3.25)

$$x_1'' = (-4k_1x_1 - 4c_1x_1' + 2k_2x_2 + 2c_2x_2' + F(4 - 6b/L))/m \quad \text{...................} \quad (3.26)$$

Subtracting (3.24) and (3.25)

$$x_2'' = (2k_1x_1 + 2c_1x_1' - 4k_2x_2 - 4c_2x_2' - F(2 - 6b/L))/m \quad \text{.....................} \quad (3.27)$$

Equations (3.26) and (3.27) can be solved in various ways. For transients Runge Kutta may be used. A typical transient response is shown in figure 3.10

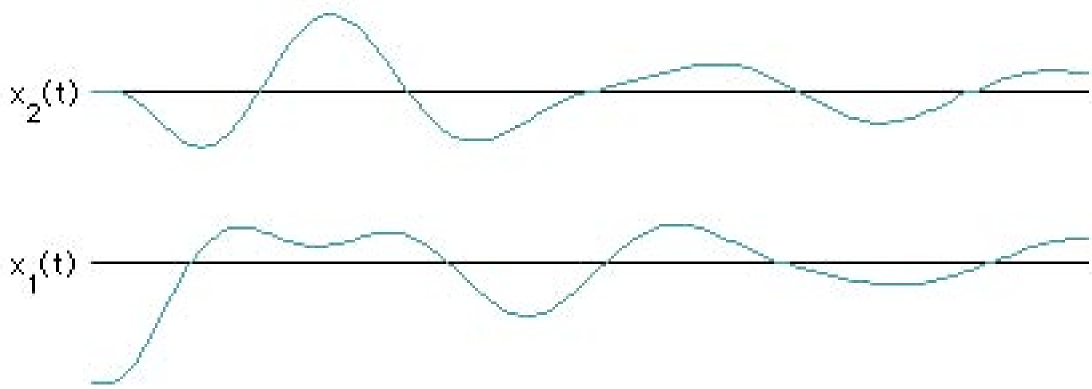

Figure 3.10 Transient response of beam on damped supports.

Note that the dampers are omitted from the animation for reasons of clarity.

Except for particular values of the damping coefficients it is not possible to separate the motion into modes that may be superimposed.

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For the particular case of steady state sinusoidal excitation,

$$x_1 = X_1 e^{i\omega t} \text{ and } x_2 = X_2 e^{i\omega t} \text{ and } F = F e^{i\omega t}$$

so that (3.26) becomes

$$-\omega^2 X_1 = \left(-4k_1 X_1 - 4i\omega c_1 X_1 + 2k_2 X_2 + 2i\omega c_2 X_2 + F(4 - 6b / L)\right) / m$$

and rearranging

$$\left(4k_1 - m\omega^2 + 4i\omega c_1\right) X_1 - \left(2k_2 + 2i\omega c_2\right) X_2 = F(4 - 6b / L) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.28)$$

and (3.27) becomes

$$-\omega^2 X_2 = \left(+2k_1 X_1 + i\omega c_1 X - 4k_2 X_2 - 4i\omega c_2 X_2 - F(2 - 6b / L)\right) / m$$

and rearranging

$$-\left(2k_1 + 2i\omega c_1\right) X_1 + \left(4k_2 - m\omega^2 + 4i\omega c_2\right) X_2 = -F(2 - 6b / L) \quad \ldots\ldots\ldots\ldots\ldots \quad (3.29)$$

(3.28) and (3.29) may be re-written

$$aX_1 + bX_2 = c \quad \text{and} \quad dX_1 + eX_2 = f$$

then

$$X_1 = \frac{ec - bf}{ae - bd} = \frac{\left(4k_2 - m\omega^2 + 4i\omega c_2\right)(4 - 6b / L) - \left(2k_2 + 2i\omega c_2\right)(2 - 6b / L)}{\left(4k_1 - m\omega^2 + 4i\omega c_1\right)\left(4k_2 - m\omega^2 + 4i\omega c_2\right) - 4\left(k_2 + i\omega c_2\right)\left(k_1 + i\omega c_1\right)} F$$

$$\frac{X_1}{F} = \frac{12k_2(1 - b / L) - m\omega^2(4 - 6b / L) + 12i\omega c_2(1 - b / L)}{\left(4k_1 - m\omega^2\right)\left(4k_2 - m\omega^2\right) - 4k_2 k_1 - 12\omega^2 c_1 c_2 + 4\omega i\left(c_1(3k_2 - m\omega^2) + c_2(3k_1 - m\omega^2)\right)}$$

..... (3.30)

and

$$X_2 = \frac{af - cd}{ae - bd} = \frac{-\left(4k_1 - m\omega^2 + 4i\omega c_1\right)(2 - 6b / L) + (2k_1 + 2i\omega c_1)(4 - 6b / L)}{\left(4k_1 - m\omega^2 + 4i\omega c_1\right)\left(4k_2 - m\omega^2 + 4i\omega c_2\right) - 4\left(k_2 + i\omega c_2\right)\left(k_1 + i\omega c_1\right)} F$$

$$\frac{X_2}{F} = \frac{12k_1 b / L + m\omega^2(2 - 6b / L) + 12i\omega c_1 b / L}{\left(4k_1 - m\omega^2\right)\left(4k_2 - m\omega^2\right) - 4k_2 k_1 - 12\omega^2 c_1 c_2 + 4\omega i\left(c_1(3k_2 - m\omega^2) + c_2(3k_1 - m\omega^2)\right)}$$

..... (3.31)

The responses have amplitude and phase. A computer program was written to allow the amplitude of the response at each end of the beam to be calculated.

Typical responses are shown in figure 3.11.

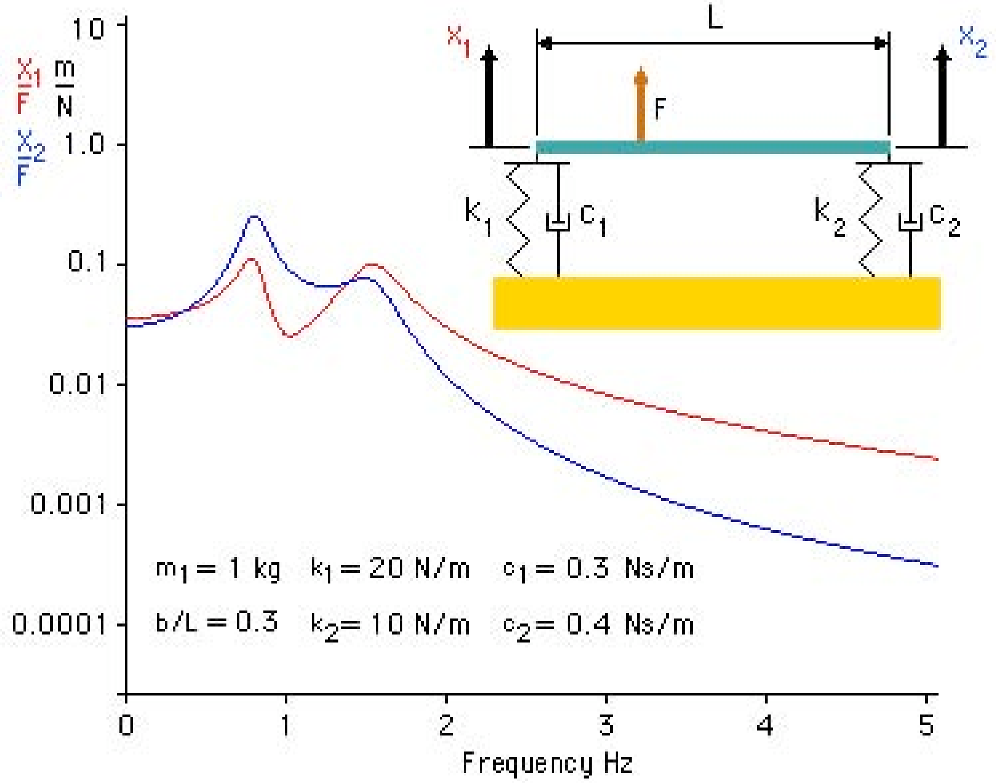

Figure 3.11 End responses for force excitation.

3.4 Conclusions.

It should be noted that as for the axial two degree-of-freedom system there are two resonances close to the undamped natural frequencies. In practice these resonances are important and one of the major goals of vibration analysis is to determine the natural frequencies. The next chapter considers some of the ways that are used to find these natural frequencies for complex systems. However to ease the maths the methods will be illustrated using the two degree-of-freedom systems already considered.

Next chapter

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