CHAPTER 6: Modal analysis.

The modal analysis approach is the basis of many advanced computer programs for analysing vibration. In this chapter we will develop this method using some simple two degree of freedom examples. As matrix methods will be used systems with greater numbers of degrees of freedom can be treated in the same way. However the maths becomes extensive and computers are needed to produce solutions.

The concept of modal analysis is used to describe the vibration of a complex system by superposing modes. Each mode has a natural frequency and an associated mode shape. Each mode is considered to behave in the same manner as a single (one degree-of-freedom system. The number of modes will equal the number of degrees-of-freedom of the system. The modal characteristics can be thought of as building blocks for the calculation of the free vibration and forced vibration response of the system. As an example consider a two degree-of-freedom system.

6.1 Modes of transverse vibration of masses attached by tensioned strings.

Figure 6.1 shows two identical masses attached by tensioned strings. The equilibrium distance between the masses and that between the mass and wall are equal. Modal analysis will be used to study the transverse vibration of the masses. We first need to determine the mode characteristics, ie, the natural frequencies and mode shapes.

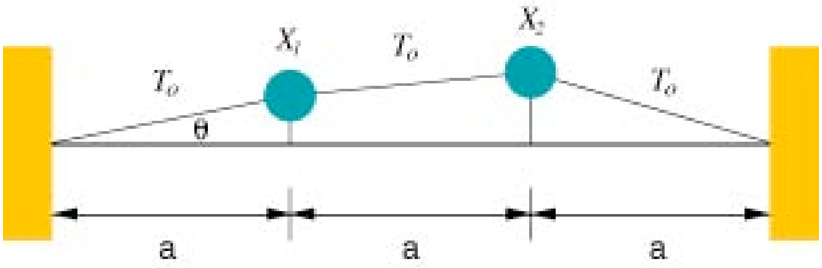

Figure 6.1 Transverse vibration of two masses connected by strings in tension.

The forces acting on the masses are caused by the transverse components of the tension in the strings (rubber bends). For small displacements, the component from the first string on the first mass is,

$$T_1 = -T_o \sin\theta \text{, where } \sin\theta = \frac{x_1}{\sqrt{a^2 + x_1^2}} \text{. When } |x_1| << a \text{, } \sin\theta \cong \frac{x_1}{a}.$$

$$\text{Therefore } T_1 \cong -\frac{T_o}{a} x_1 = -Kx_1 \text{ where } K = \frac{T_0}{a} .$$

Similarly the transverse force on the first mass from the middle string is given by,

$$T_2 \cong K(x_2 - x_1)$$

The transverse force on the second mass from the middle string,

$$T_2 \cong -K(x_2 - x_1)$$

and from the third string $T_3 \cong -Kx_2$

Thus the equations of motion for the two masses are,

$$Mx_1'' = K(x_2 - x_1) - Kx_1 \tag{6.1}$$

$$Mx_2'' = -K(x_2 - x_1) - Kx_2 \tag{6.2}$$

Assuming $x_1 = Ae^{\lambda t}$ and $x_2 = Be^{\lambda t}$, (6.1) and (6.2) become:

$$(\lambda^2 M + 2K)A - KB = 0 \tag{6.3}$$

$$-KA + (\lambda^2 M + 2K)B = 0 \tag{6.4}$$

In matrix form, we have

$$\begin{bmatrix} -\lambda^2 M + 2K & -K \\ -K & -\lambda^2 M + 2K \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0 \tag{6.5}$$

For a non-trivial solution, the determinant of the coefficient matrix in (6.5) must be zero

$$\det \begin{bmatrix} -\lambda^2 M + 2K & -K \\ -K & -\lambda^2 M + 2K \end{bmatrix} = 0 \tag{6.6}$$

That is

$$(\lambda^2 M + 2K)^2 = K^2 \tag{6.7}$$

Thus we may obtain the eigenvalues as

$$\lambda^2 = -\frac{K}{M} \quad \text{and} \quad \lambda^2 = -\frac{3K}{M}$$

so that we have four eigenvalues

$$\lambda_1 = i\sqrt{\frac{K}{M}} \quad \text{and} \quad \lambda_2 = -i\sqrt{\frac{K}{M}} \tag{6.8}$$

$$\lambda_3 = i\sqrt{\frac{3K}{M}} \quad \text{and} \quad \lambda_4 = -i\sqrt{\frac{3K}{M}} \tag{6.9}$$

The eigenvalues are related to the natural frequencies of the modes by,

$$\omega_i = |\mathrm{Im}(\lambda_i)|$$

Thus the first two eigenvalues give the first natural frequency as

$$\omega_1 = \sqrt{\frac{K}{M}} \tag{6.10}$$

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and the last two eigenvalues give the second natural frequency as

$$\omega_2 = \sqrt{\frac{3K}{M}} \quad \text{...} \quad (6.11)$$

For each eigenvalue, the solutions for the values of A and B can be obtained from (6.5) as corresponding eigenvectors. For example for $\lambda = \lambda_1$ and $\lambda = \lambda_2$; $A = B$ and for $\lambda = \lambda_3$ and $\lambda = \lambda_4$; $A = -B$

We then have the solutions corresponding to

$\lambda_1 = i\omega_1$

$$\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} A_{11} \\ B_{11} \end{bmatrix} e^{+i\omega_1 t} \text{, where } A_{11} = B_{11}$$

$\lambda_2 = -i\omega_1$

$$\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} A_{12} \\ B_{12} \end{bmatrix} e^{-i\omega_1 t} \text{, where } A_{12} = B_{12}$$

$\lambda_3 = i\omega_2$

$$\begin{bmatrix} x_{13} \\ x_{23} \end{bmatrix} = \begin{bmatrix} A_{21} \\ B_{21} \end{bmatrix} e^{+i\omega_2 t} \text{, where } A_{21} = -B_{21}$$

$\lambda_4 = -i\omega_2$

$$\begin{bmatrix} x_{14} \\ x_{24} \end{bmatrix} = \begin{bmatrix} A_{22} \\ B_{22} \end{bmatrix} e^{-i\omega_2 t} \text{, where } A_{22} = -B_{22}.$$

We now define the first two solutions $A_{11}e^{+i\omega_1 t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A_{12}e^{-i\omega_1 t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as the first mode of the two degree-of-freedom system with a natural frequency $\omega_1 = \sqrt{\frac{K}{M}}$. The shape of this first mode is represented by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which indicates that both masses move with the same amplitude and phase.

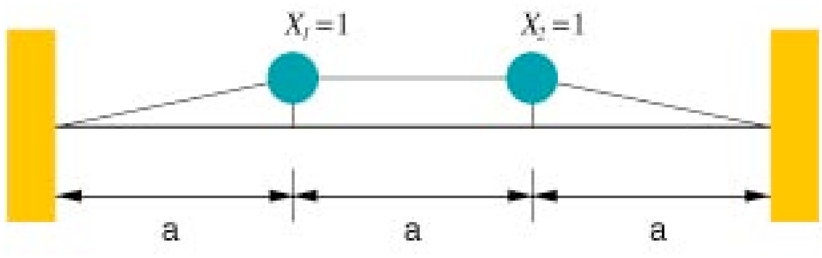
The second group of solutions is defined as the second mode $A_{21}e^{i\omega_2 t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $A_{22}e^{-i\omega_2 t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with a natural frequency $\omega_2 = \sqrt{\frac{3K}{M}}$. The corresponding mode shape is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ indicating that the two masses move in the same amplitude but opposite phase. The two mode shapes of the system are shown in figure 6.2.

Finally, the free vibration solution of (6.1) and (6.2) is the superposition of all the possible solutions:

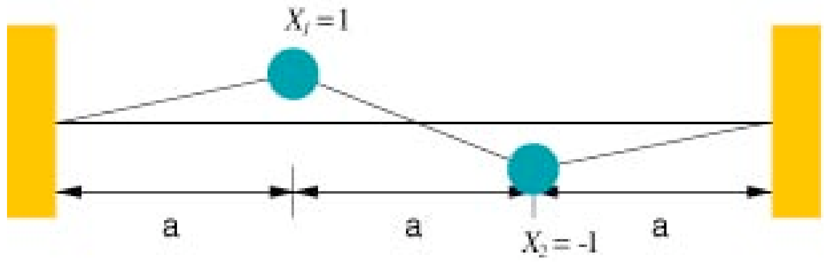
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = [A_{11}e^{+i\omega_1 t} + A_{12}e^{-i\omega_1 t}]\begin{bmatrix} 1 \\ 1 \end{bmatrix} + [A_{21}e^{+i\omega_2 t} + A_{22}e^{-i\omega_2 t}]\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{.....................} \quad (6.12)$$

where the four coefficients are determined by the four initial conditions given by,

$$x_1(0) = x_{10}\,,\ x_1'(0) = v_{10}\,, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6.13)$$
$$x_2(0) = x_{20}\,,\ x_2'(0) = v_{20}\,, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6.14)$$



First mode.



Second mode

Figure 6.2 Mode shapes.

Equation (6.12) can be converted to a form involving sines and cosines as follows.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \left[\frac{(A_{11}+A_{12})}{2}\left(e^{+i\omega_1 t}+e^{-i\omega_1 t}\right)+\frac{(A_{11}-A_{12})}{2}\left(e^{+i\omega_1 t}-e^{-i\omega_1 t}\right)\right]\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$+\left[\frac{(A_{21}+A_{22})}{2}\left(e^{+i\omega_2 t}+e^{-i\omega_2 t}\right)+\frac{(A_{21}-A_{22})}{2}\left(e^{+i\omega_2 t}-e^{-i\omega_2 t}\right)\right]\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so that

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \left[a_{11}\cos\omega_1 t + b_{11}\sin\omega_1 t\right]\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left[a_{12}\cos\omega_1 t + b_{12}\sin\omega_1 t\right]\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

6.2 Matrix expression of system equations

The above analysis can be simplified using matrix representations of the system displacements and equation of motion. Equations (6.5) and (6.6) written in matrix form are,

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} + \begin{bmatrix} 2K & -K \\ -K & 2K \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 0 \quad \ldots\ldots\ldots\ldots \quad (6.15)$$

We define a displacement vector

$$\{x(t)\} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6.16)$$

and mass and stiffness matrices

$$[M] = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}, [K] = \begin{bmatrix} 2K & -K \\ -K & 2K \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6.17)$$

Then the equation of motion is expressed as,

$$[M]\{x''\} + [K]\{x\} = \{0\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6.18)$$

It is necessary to note that the method presented below is not limited to the vibration system described in figure 6.1. As long as the equations of motion of the system can be written in the form of equation (6.18) and the mass and stiffness matrices satisfy some conditions, the system response can then be solved by using the method.

We can assume the following general solutions for (6.18):

$$\{x\} = \{u\}e^{\lambda t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6.19)$$

Substituting (6.19) into (6.18) we obtain we obtain the characteristic or eigenvalue equation of the system:

$$([K] + \lambda^2[M])\{u\} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6.20)$$

and the solution of

$$\det([K] + \lambda^2[M]) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6.21)$$

gives rise to four eigenvalues λ_i, i=1, 2, 3, 4 and their corresponding eigenvectors $\{u_i\}$. The first two eigenvalue are related to the first natural frequency by $\omega_i = |\mathrm{Im}(\lambda_i)|$ and the last two eigenvalues are related in the same way to the second natural frequency.

Equation (6.19) can be converted into a standard eigenvalue equation if $[M]$ is not singular:

$$([M]^{-1}[K] - \lambda^2[I])\{u\} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.22)$$

where $[M]^{-1}$ is the inverse of the mass matrix and $[I]$ is the identity matrix.

The eigenvectors of the system have the orthogonal property if the system mass and stiffness matrices are symmetrical so that

$$[M]^T = [M] \text{ and } [K]^T = [K] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6.23)$$

where the superscript T represents a matrix transpose. For example, these requirements are satisfied for the matrices in (6.17). This orthogonal property of eigenvectors can be illustrated by the following consideration. For the i^{th} eigenvalue $\lambda = \lambda_i$, equation (6.19) may be expressed as

$$[K]\{u_i\} = \lambda_i^2[M]\{u_i\} \quad \text{..} \quad (6.24)$$

and for the j^{th} eigenvalue, we have

$$([K]\{u_j\})^T = (\lambda_j^2[M]\{u_j\})^T \quad \text{...} \quad (6.25)$$

Multiplying $\{u_j\}^T$ on both sides of (6.24), and $\{u_i\}$ on the both sides of (6.25), we obtain respectively:

$$\{u_j\}^T[K]\{u_i\} = \lambda_i^2\{u_j\}^T[M]\{u_i\} \quad \text{....................................} \quad (6.26)$$

and

$$\{u_j\}^T[K]^T\{u_i\} = \lambda_j^2\{u_j\}^T[M]^T\{u_i\} \quad \text{..................................} \quad (6.27)$$

Subtracting (6.26) - (6.27) gives

$$(\lambda_i^2 - \lambda_j^2)\{u_j\}^T[M]\{u_i\} = 0 \quad \text{...} \quad (6.28)$$

where we have used the symmetrical properties of the system matrices. If $\lambda_i \neq \lambda_j$ for $i \neq j$, we obtain the orthogonal relationship of the eigenvectors as,

$$\{u_j\}^T[M]\{u_i\} = \begin{cases} 0 & i \neq j \\ \{u_i\}^T[M]\{u_i\} = M_i & i = j \end{cases} \quad \text{...............................} \quad (6.29)$$

where

$$M_i = \{u_i\}^T[M]\{u_i\} \quad \text{..} \quad (6.30)$$

is defined as the modal mass.

6.3 Steady state response

The steady state response of the system vibration refers to the system displacements $\{x\}$ due to a steady state force excitation. Free vibration may coexist with the steady state response initially, but it will die out due to the system damping present in all real systems.

When external harmonic forces are applied to the masses, the matrix equation becomes:

$$[M]\{\ddot{x}\} + [K]\{x\} = \{F\}e^{i\omega t} \quad \text{....................................} \quad (6.31)$$

where $\{F\} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ is the force vector.

As the response has the same time dependent term as that of the exciting force, the assumed solution may take the following form:

$$\{x\} = \sum_{i=1}^{2} q_i \{u_i\} e^{i\omega t} \quad \text{...} \quad (6.32)$$

Substituting equation (6.32) into equation (6.31), we have

$$\sum_{i=1}^{2} q_i (\omega_i^2 - \omega^2)[M]\{u_i\} = \{F\} \quad \text{..} \quad (6.33)$$

Multiplying $\{u_i\}^T$ on the both sides of Equation (6.33), and using the orthogonal properties of the eigenvectors, we obtain

$$q_i = \frac{\{u_i\}^T \{F\}}{M_i(\omega_i^2 - \omega^2)} \quad \text{..} \quad (6.34)$$

Therefore, the steady state response is

$$\{x\} = \sum_{i=1}^{2} \frac{\{u_i\}^T \{F\}\{u_i\}}{M_i(\omega_i^2 - \omega^2)} e^{i\omega t} \quad \text{...} \quad (6.35)$$

6.4 Free vibration

The free vibration results from the initial disturbance and/or initial conditions described by equations (6.13) and (6.14). These initial conditions may also be expressed as vectors:

$$\{x(0)\} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \text{ and } \{x'(0)\} = \begin{bmatrix} v_{10} \\ v_{20} \end{bmatrix} \quad \text{......................................} \quad (6.36)$$

The assumed vibration can be expressed in terms of eigenvectors:

$$\{x\} = \sum_{i=1}^{2} q_i(t)\{u_i\} \quad \text{...} \quad (6.37)$$

where $q_i(t)$ is the coefficient of the i^{th} mode. By submitting this assumed solution into the matrix equation, we have

$$\sum_{i=1}^{2} \{q_i''[M]\{u_i\} + q_i[K]\{u_i\}\} = \{0\} \quad \text{..................................} \quad (6.38)$$

Using the eigenvalue equation and $\lambda_i^2 = -\omega_i^2$

$$[K]\{u_i\} = \lambda_i^2[M]\{u_i\} = \omega_i^2[M]\{u_i\} \quad \text{.................................} \quad (6.39)$$

and the orthogonal property of the eigenvectors we obtain the following modal equations (one degree-of-freedom equivalent):

$$q_i'' + \omega_i^2 q_i = 0\,,\ i = 1,2 \quad \text{..} \quad (6.40)$$

with a general solution of

$$q_i(t) = A_{i1}e^{i\omega_i t} + A_{i2}e^{-i\omega_i t} \quad \text{.....................................} \quad (6.41)$$

The modal equation has the same form as that of a single degree-of-freedom system. It describes the time function of the modal components. As a result, the system displacement vector is

$$\{x\} = \sum_{i=1}^{2}(A_{i1}e^{i\omega_i t} + A_{i2}e^{-i\omega_i t})\{u_i\} \quad \text{..............................} \quad (6.42)$$

where A_{i1} and A_{i2} are determined from the initial conditions (6.36):

$$\{x(0)\} = \sum_{i=1}^{2}(A_{i1} + A_{i2})\{u_i\} \quad \text{.....................................} \quad (6.43)$$

$$\{\dot{x}(0)\} = \sum_{i=1}^{2} i\omega_i(A_{i1} - A_{i2})\{u_i\} \quad \text{.................................} \quad (6.44)$$

Multiplying by $\{u_j\}^T[M]$ on both sides of Equations (6.43) and (6.44), we have for $i = 1,2$

$$A_{i1} + A_{i2} = \frac{\{u_i\}^T[M]\{x(0)\}}{M_i} \quad \text{.....................................} \quad (6.45)$$

$$A_{i1} - A_{i2} = \frac{\{u_i\}^T[M]\{x'(0)\}}{i\omega_i M_i} \quad \text{.....................................} \quad (6.46)$$

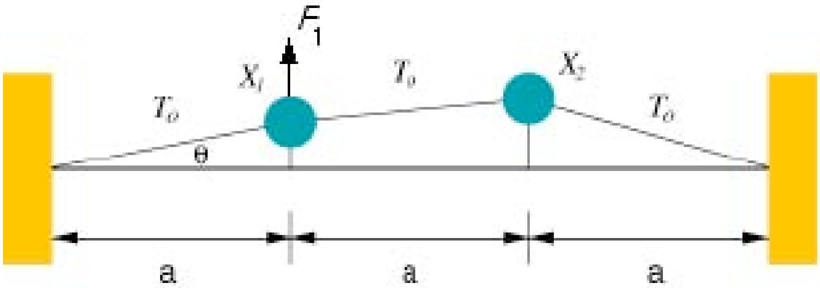
so that adding and subtracting (6.45) and (6.46) gives

$$A_{i1} = \frac{1}{2}\left[\frac{\{u_i\}^T[M]\{x(0)\}}{M_i} + \frac{\{u_i\}^T[M]\{x'(0)\}}{i\omega_i M_i}\right] \quad \text{..............................} \quad (6.47)$$

$$A_{i2} = \frac{1}{2}\left[\frac{\{u_i\}^T[M]\{x(0)\}}{M_i} - \frac{\{u_i\}^T[M]\{x'(0)\}}{i\omega_i M_i}\right] \quad \text{.............................} \quad (6.48)$$

The method of modal analysis developed in sections 6.3 and 6.4 will be applied to the vibration behaviour of the two degree-of-freedom system shown in figure 6.1.

6.5 Example: steady state response of a two degree-of-freedom mass-string system.



For this example let both masses be 1kg and the tension and length of string be such that $K = \frac{T_0}{a} = 1$ N/m. The system mass and stiffness matrices are then

$$[M] = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{.....................................} \quad (6.49)$$

$$[K] = \begin{bmatrix} 2K & -K \\ -K & 2K \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{..................................} \quad (6.50)$$

Let the force vector applied to the system be

$$\{F\} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad \text{...} \quad (6.51)$$

This allows us to subsequently obtain the response to F_1 alone by putting F_2 =0. However it will also allow the response resulting from F_2 alone to be found by putting F_1 = 0.

The displacement vector due to the steady state excitation of this force vector at a frequency ω can be calculated from (6.35). First the natural frequencies are obtained from the eigenvalue equation of the system:

$$([K] + \lambda^2[M])\{u\} = 0 \quad \text{..} \quad (6.20)$$

These are given by

$$\det([K] + \lambda^2[M]) = 0 \quad \text{..} \quad (6.21)$$

For complex systems a computer would be used to produce a solution but for this two degree-of-freedom system we may solve long hand as follows.

$$\det\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \lambda^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} 2+\lambda^2 & -1 \\ -1 & 2+\lambda^2 \end{vmatrix} = 0$$

$$(2+\lambda^2)^2 - (-1)^2 = 0$$
$$\lambda^2 + 4\lambda + 3 = 0$$

Thus

$$\lambda_1 = i1.0 \quad \lambda_2 = -i1.0 \quad \lambda_3 = i3.0 \quad \lambda_4 = -i3.0$$

so that

$$\omega_1 = |\mathrm{Im}(\lambda_1)| = 1.0 \text{ rad/s} \quad \ldots\ldots\ldots\ldots \quad (6.52)$$

$$\omega_2 = |\mathrm{Im}(\lambda_3)| = 3.0 \text{ rad/s} \quad \ldots\ldots\ldots\ldots \quad (6.53)$$

The corresponding eigenvectors are found from equation (6.20). Thus for $\omega_1 = 1.0$, $\lambda^2 = -\omega_1^2 = -1.0$ so that equation (6.20) gives

$$\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - 1.0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\{u\} = 0$$
$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\{u\} = 0$$

$$\{u_1\} = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix} \quad \ldots\ldots\ldots\ldots \quad (6.54)$$

(The eigenvectors have been normalized by making that the sum of the squares of the elements equal to one.)

and for $\omega_2 = 3.0$, $\lambda^2 = -\omega_2^2 = -3.0$ equation (6.20) gives

$$\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - 3.0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\{u\} = 0$$
$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\{u\} = 0$$

$$\{u_2\} = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix} \quad \ldots\ldots\ldots\ldots \quad (6.55)$$

It has been shown that

$$\{x\} = \sum_{i=1}^{2} \frac{\{u_i\}^T\{F\}\{u_i\}}{M_i(\omega_i^2 - \omega^2)} e^{i\omega t} \quad \ldots\ldots\ldots\ldots \quad (6.35)$$

where

$$M_i = \{u_i\}^T[M]\{u_i\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6.30)$$

Using the values in equations (6.52 – 6.55)

$$M_1 = \{u_1\}^T[M]\{u_1\} = \begin{bmatrix}0.7071\\0.7071\end{bmatrix}^T \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \begin{bmatrix}0.7071\\0.7071\end{bmatrix} = \begin{bmatrix}0.7071 & 0.7071\end{bmatrix} \begin{bmatrix}0.7071\\0.7071\end{bmatrix} = 1.0$$

$$M_2 = \{u_2\}^T[M]\{u_2\} == \begin{bmatrix}-0.7071\\0.7071\end{bmatrix}^T \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \begin{bmatrix}-0.7071\\0.7071\end{bmatrix} = \begin{bmatrix}-0.7071 & 0.7071\end{bmatrix} \begin{bmatrix}-0.7071\\0.7071\end{bmatrix} = 1.0$$

Now substituting in equation (6.35)

$$\{x\} = \sum_{i=1}^{2} \frac{\{u_i\}^T\{F\}\{u_i\}}{M_i(\omega_i^2 - \omega^2)} e^{i\omega t}$$

$$\{x\} = \frac{\{u_1\}^T\{F\}\{u_1\}}{M_1(\omega_1^2 - \omega^2)} e^{i\omega t} + \frac{\{u_2\}^T\{F\}\{u_2\}}{M_2(\omega_2^2 - \omega^2)} e^{i\omega t}$$

and using the values obtained above

$$\{x\} = \frac{\begin{bmatrix}0.7071\\0.7071\end{bmatrix}^T \begin{bmatrix}F_1\\F_2\end{bmatrix} \begin{bmatrix}0.7071\\0.7071\end{bmatrix}}{1.0(\omega_1^2 - \omega^2)} e^{i\omega t} + \frac{\begin{bmatrix}-0.7071\\0.7071\end{bmatrix}^T \begin{bmatrix}F_1\\F_2\end{bmatrix} \begin{bmatrix}-0.7071\\0.7071\end{bmatrix}}{1.0(\omega_2^2 - \omega^2)} e^{i\omega t}$$

$$\{x\} = \frac{\begin{bmatrix}0.5F_1 + 0.5F_2\\0.5F_1 + 0.5F_2\end{bmatrix}}{(\omega_1^2 - \omega^2)} e^{i\omega t} + \frac{\begin{bmatrix}0.5F_1 - 0.5F_2\\-0.5F_1 + 0.5F_2\end{bmatrix}}{(\omega_2^2 - \omega^2)} e^{i\omega t}$$

Thus for $F_2 = 0$

$$\{x\} = \frac{\begin{bmatrix}0.5F_1\\0.5F_1\end{bmatrix}}{(\omega_1^2 - \omega^2)} e^{i\omega t} + \frac{\begin{bmatrix}0.5F_1\\-0.5F_1\end{bmatrix}}{(\omega_2^2 - \omega^2)} e^{i\omega t}$$

Since

$$\{x\} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}X_1\\X_2\end{bmatrix} e^{i\omega t}$$

it follows

$$\begin{bmatrix}X_1\\X_2\end{bmatrix} = \frac{\begin{bmatrix}0.5F_1\\0.5F_1\end{bmatrix}}{(\omega_1^2 - \omega^2)} + \frac{\begin{bmatrix}0.5F_1\\-0.5F_1\end{bmatrix}}{(\omega_2^2 - \omega^2)}$$

and hence

$$\frac{X_1}{F_1} = \frac{0.5}{(\omega_1^2 - \omega^2)} + \frac{0.5}{(\omega_2^2 - \omega^2)} \quad \text{and} \quad \frac{X_2}{F_1} = \frac{0.5}{(\omega_1^2 - \omega^2)} - \frac{0.5}{(\omega_2^2 - \omega^2)}$$

These responses are shown in figure 6.3.

If the excitation was on mass 2 then $F_1 = 0$ and

$$\{x\} = \frac{\begin{bmatrix} 0.5F_2 \\ 0.5F_2 \end{bmatrix}}{(\omega_1^2 - \omega^2)} e^{i\omega t} + \frac{\begin{bmatrix} -0.5F_2 \\ 0.5F_2 \end{bmatrix}}{(\omega_2^2 - \omega^2)} e^{i\omega t}$$

and hence

$$\frac{X_1}{F_2} = \frac{0.5}{(\omega_1^2 - \omega^2)} - \frac{0.5}{(\omega_2^2 - \omega^2)} \quad \text{and} \quad \frac{X_2}{F_2} = \frac{0.5}{(\omega_1^2 - \omega^2)} + \frac{0.5}{(\omega_2^2 - \omega^2)}$$

which is to be expected from symmetry as both masses are the same.

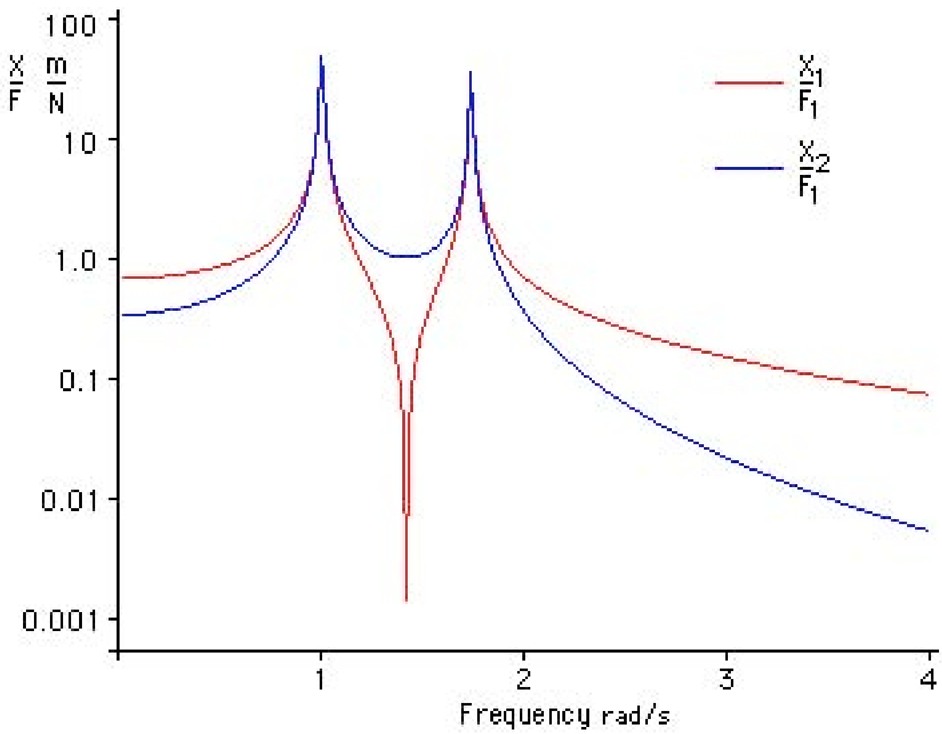


Figure 6.3 Response resulting from excitation on mass 1.

6.6 Damped vibration using modal analysis

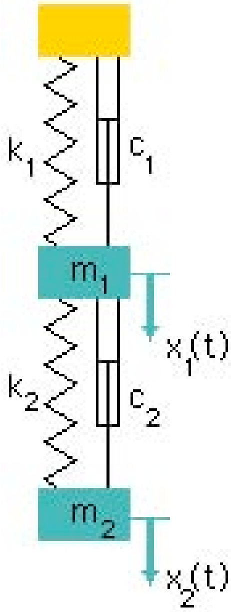
We have shown modes to be orthogonal for undamped vibrating systems. It is now appropriate to consider the effects of damping. For modal analysis, the system eigenvalue equation needs to be transformed into the standard form.

$$([A] + \lambda[B])\{u\} = 0 \quad \text{...} \quad (6.56)$$

We shall investigate the damped axial system considered in chapter 2 so that comparisons can be made with the exact solution. The equations of motion are,

$$mx_1'' = -k_1x_1 + k_2(x_2 - x_1) - c_1x_1' + c_2(x_2' - x_1')$$
$$mx_2'' = -k_2(x_2 - x_1) - c_2(x_2' - x_1')$$

These equations may be written in matrix form as,



$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6.57)$$

In order to have this equation in the form of equation (6.56) above we introduce a velocity vector

$$\{x'\} = \{y\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.58)$$

Multiplying (6.57) by $[M]^{-1}$

$$\{x''\} + [M]^{-1}[C]\{x'\} + [M]^{-1}[K]\{x\} = 0$$

and substituting from (6.58)

$$\{y'\} + [M]^{-1}[C]\{y\} + [M]^{-1}[K]\{x\} = 0 \quad \ldots\ldots\ldots\ldots\ldots (6.59)$$

we may write equations (6.58) and (6.59) as a single matrix equation

$$\begin{Bmatrix} \{x'\} \\ \{y'\} \end{Bmatrix} + \begin{bmatrix} [0] & -[I] \\ [M]^{-1}[K] & [M]^{-1}[C] \end{bmatrix} \begin{Bmatrix} \{x\} \\ \{y\} \end{Bmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6.60)$$

Assuming

$$\begin{Bmatrix} \{x\} \\ \{y\} \end{Bmatrix} = \begin{Bmatrix} \{\tilde{u}\} \\ \{\tilde{v}\} \end{Bmatrix} e^{\lambda t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.61)$$

we obtain the following standard eigenvalue equation:

$$\left(\begin{bmatrix} [0] & -[I] \\ [M]^{-1}[K] & [M]^{-1}[C] \end{bmatrix} + \lambda[I] \right) \begin{Bmatrix} \{\tilde{u}\} \\ \{\tilde{v}\} \end{Bmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6.62)$$

In equation (6.62), the matrix $\begin{bmatrix} [0] & -[I] \\ [M]^{-1}[K] & [M]^{-1}[C] \end{bmatrix}$ is not symmetrical. Therefore the eigenvalues are complex and their eigenvectors are not orthogonal. The imaginary and real parts of each complex eigenvalue are related to the damped natural frequency and modal damping ratio of the corresponding damped mode. The solution of l results in a pair of complex conjugates,

$$\lambda_i^{(1,2)} = -\xi_i\omega_i \pm i\omega_i\sqrt{1-\xi_i^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.63)$$

Its corresponding eigenvector $\begin{Bmatrix} \{\tilde{u}\} \\ \{\tilde{v}\} \end{Bmatrix}$ represents the mode shape.

As an example take $m_1 = m_2 = 1\,\text{kg}$; $k_1 = k_2 = 1\,\text{N/m}$; $c_1 = c_2 = 0.08\,\text{Ns/m}$. The four eigenvalues are found to be,

$$\lambda_1 = -0.0153 + i0.6178 \qquad \lambda_2 = -0.0153 - i0.6178$$
$$\lambda_3 = -0.1047 + i1.6146 \qquad \lambda_3 = -0.1047 + i1.6146$$

The first two eigenvalues belong to the first mode of the system, and the last two eigenvalues to the second mode.

Thus for the first mode,

$$\omega_1 = \omega_{n1}\sqrt{1-\xi_1^2} = 0.6178\ \text{rad/s} \ \text{ and } \ \xi_1\omega_1 = 0.01528\ \text{rad/s}$$

and hence from these values $\omega_{n1} = 0.618\ \text{rad/s}$ and $\xi_1 = 0.02472$

and for the second mode,

$$\omega_2 = \omega_{n2}\sqrt{1-\xi_2^2} = 1.615\ \text{rad/s} \ \text{ and } \ \xi_2\omega_2 = 0.1047\ \text{rad/s}$$

and hence from these values $\omega_{n2} = 1.618\ \text{rad/s}$ and $\xi_2 = 0.06472$

These values of the undamped natural frequencies compare well with the values calculated from an undamped model.

$$\omega_{n1} = 0.618\ \text{rad/s} \ \text{ and } \ \omega_{n2} = 1.618\ \text{rad/s}$$

Using the program the system parameters may be changed and the effect on the eigenvalues and undamped natural frequencies investigated. It will be found that for large values of damping the agreement is not so good.

A good estimate of the response of the damped vibration system is to use the modal characteristics of the corresponding undamped system and introduce the modal damping obtained from equation (6.63) into the modal equation. For this case, the orthogonal properties of the eigenvectors of the undamped system can be used to determine an estimate of the displacement vector using

$$\{x(t)\} = \sum_{i=1}^{2} q_i(t)\{u_i\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6.64)$$

Substituting (6.64) into (6.57), and using the orthogonal properties of the eigenvectors, we obtain the damped modal equations:

$$\ddot{q}_i + 2\xi_i\omega_i\dot{q}_i + \omega_i^2 q_i = \frac{\{u_i\}^T\{f\}}{M_i} \quad \text{for } i = 1,2 \quad \text{......................} (6.65)$$

where ξ_i is the damping ratio of the i^{th} mode obtained from (6.63).

Therefore, for steady state vibration $\{f\} = \{F\}e^{j\omega t}$, the displacement vector of the damped system is obtained as

$$\{x(t)\} = \sum_{i=1}^{2} \frac{\{u_i\}^T\{F\}\{u_i\}}{M_i(\omega_i^2 + j2\xi_i\omega_i\omega - \omega^2)} e^{j\omega t} \quad \text{.........................} \quad (6.66)$$

Equation (6.66) is used for the steady state displacement response. The modal damping terms are obtained from the eigenvalues of the damped system:

$$\xi_1\omega_1 = 0.0153 \text{ (and hence } \xi_1 = 0.02472)$$

$$\xi_2\omega_2 = 0.1047 \text{ (and hence } \xi_2 = 0.06472)$$

Figure 6.4 shows the forced displacement response of the damped system obtained using modal analysis. The results are compared with an exact solution.

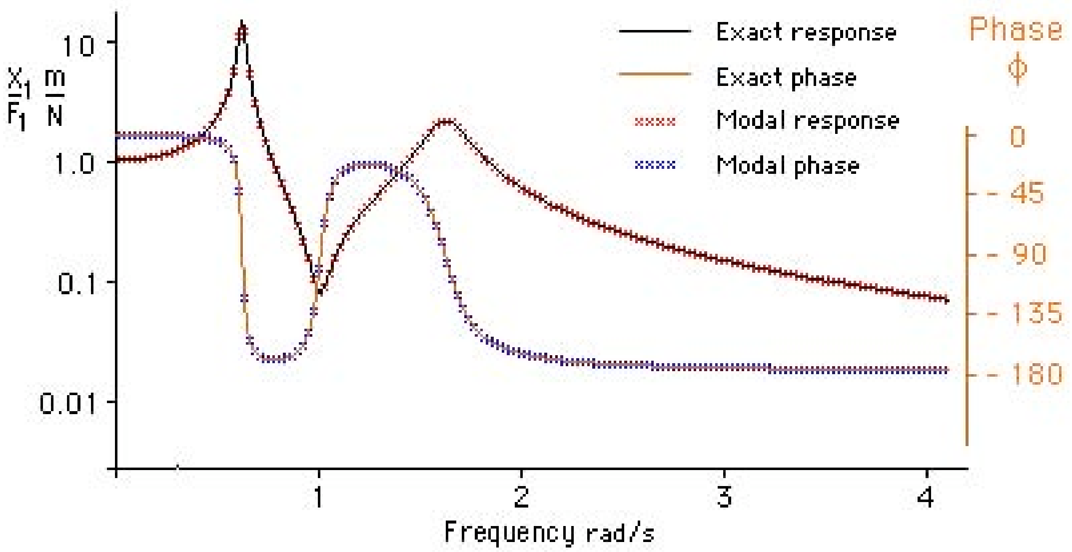


Figure 6,4 Comparison of modal analysis response with the exact solution

If the program is used it will be found that the agreement is not always as good, especially if one of the damping values is increased significantly.

6.7 Modal analysis of real systems

In practice it is often the case that the location and magnitude of the damping in a structure is not well defined. As a result it is not possible to have a model of an engineering structure with the damping represented accurately. It is a common practice to conduct an undamped analysis as the mass and stiffness matrices, that are much more amenable to modeling accurately. Then a modal damping is assumed for each mode. The values of such damping are either known from experience or can be estimated from a measured response on a real structure. As an example of how this is achieved we will consider a two degree-of-freedom system with viscous damping.

An exact solution will be calculated and plotted. Then the modal damping for each mode will be determined using the "Q" factor for each resonance. In chapter 1 the dynamic magnification factor (Q) was introduced for a one degree-of-freedom system.

$$\frac{1}{Q} = \frac{\Delta\omega}{\omega_r} = 2\xi \qquad (1.32)$$

where ω_r is the resonant frequency and $\Delta\omega$ is the frequency bandwidth at the 3dB point, ie, at the maximum response/$\sqrt{2}$. Thus from a resonance it is possible to estimate the damping for the mode. It is found for a one degree-of-freedom system that this method is very accurate for low (and realistic) levels of damping. If the program is run it will become apparent when the estimate of damping becomes inaccurate.

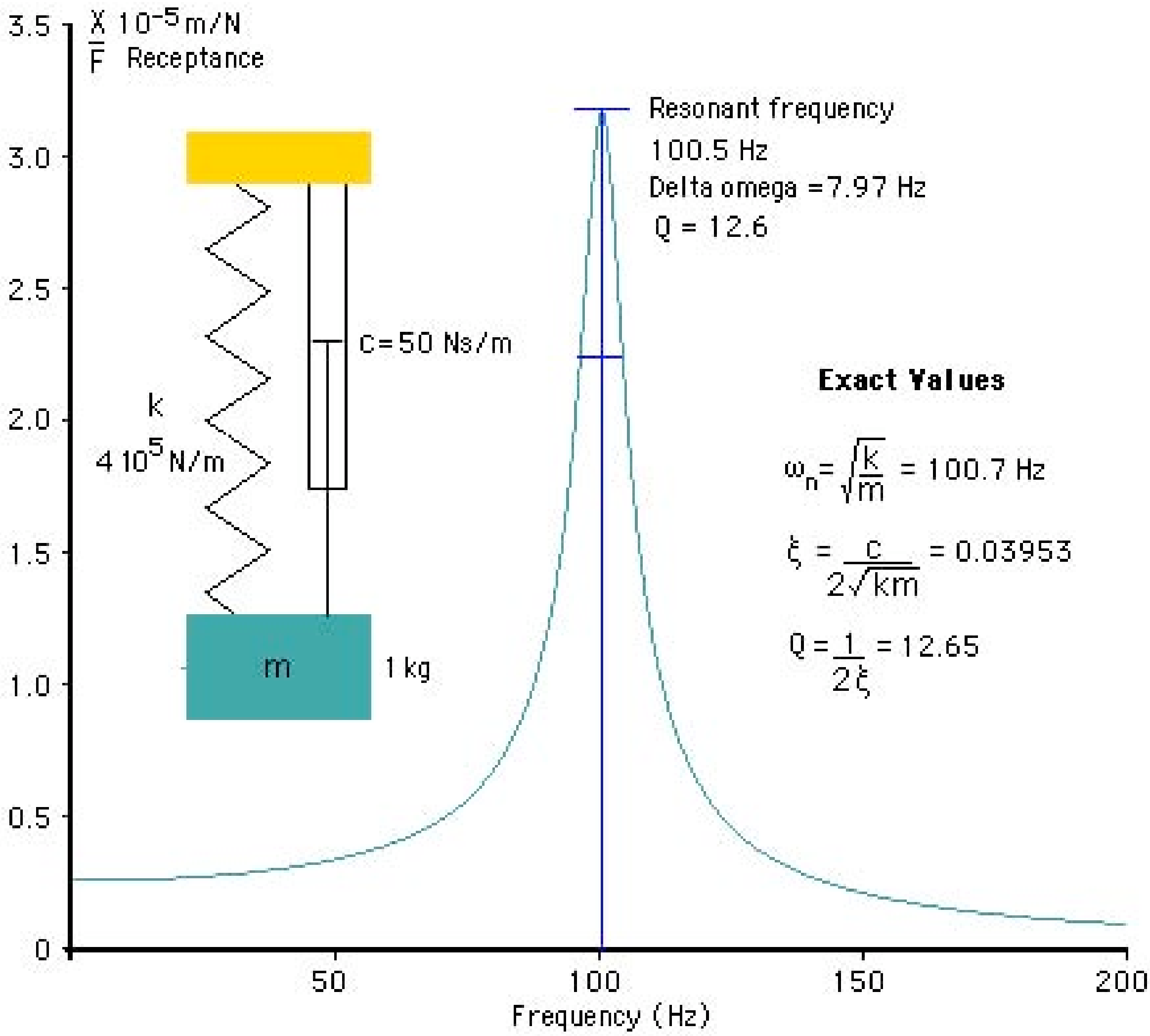


Figure 6.5 Q factor estimate from resonant curve

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Consider first the two degree-of-freedom system with viscous damping shown in figure 6.6. It is possible to calculate the exact response for x1/F1. It is also possible to calculate the undamped natural frequencies and mode shapes. If a Q factor is estimated for each mode from the exact response then it is possible to determine Xiw1 for each mode. An estimate of the response can then be made using equation (6.67) ie, by modal superposition.

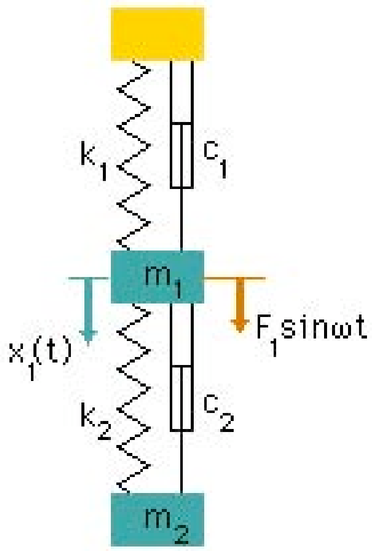

Figure 6.6

As an example and using more realistic values for k1 and k2 the accuracy of such modelling can be seen in figure 6.7. It is evident that good agreement can be achieved and this is generally the case if the damping is low as it is in most engineering structures.

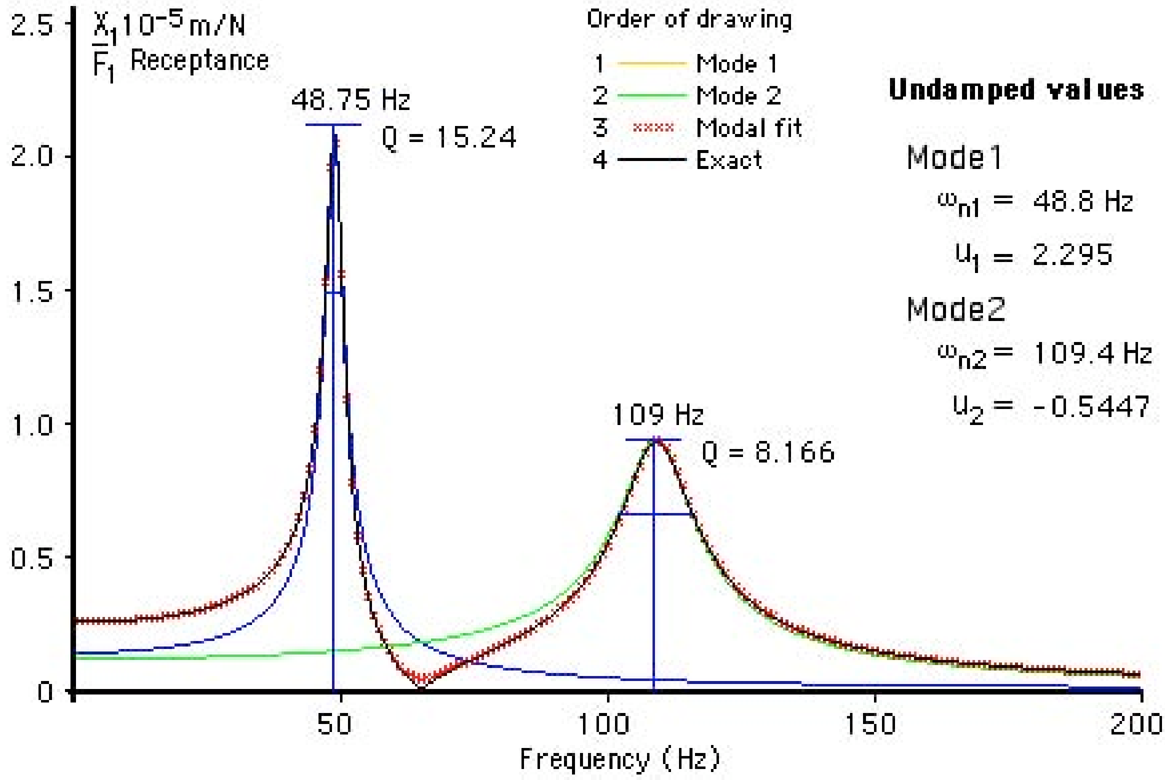

Figure 6.7 Response using modal analysis and damping determined from Q factors.

6.8 Transient Vibration

The transient vibration of a damped system may also be considered by modal superposition. For free vibration there is no excitation so that,

$$[M]\{x''\} + [C]\{x'\} + [K]\{x\} = \{0\} \quad \text{.........................} (6.67)$$

if the initial conditions are:

$$\{x(0)\} = \{x_0\} \text{ and } \{x'(0)\} = \{v_0\} \qquad (6.68)$$

then by describing the transient displacement vector by the system of undamped modes:

$$\{x\} = \sum_{i=1}^{2} q_i(t)\{u_i\} \qquad (6.69)$$

substituting from (6.69) in (6.67) gives rise to a damped modal equation

$$\sum_{i=1}^{2} [q_i''[M]\{u_i\} + q_i'[C]\{u_i\} + q_i[K]\{u_i\}] = \{0\} \qquad (6.70)$$

From the eigenvalue equation,

$$[K]\{u_i\} = \lambda_i[M]\{u_i\} = \omega_i^2[M]\{u_i\} \qquad (6.\ 41)$$

substituting in (6.70)

$$\sum_{i=1}^{2} [q_i''[M]\{u_i\} + q_i'[C]\{u_i\} + q_i\omega_i^2[M]\{u_i\}] = \{0\}$$

so that, if the orthogonal property of the eigenvectors holds,

$$q_i'' + 2\xi_i\omega_i q_i' + \omega_i^2 q_i = 0, \qquad i = 1,2 \qquad (6.71)$$

where the modal damping terms $2\xi_i\omega_i$ are obtained from the eigenvalues of the eigenvalue equation of the damped system. It should be noted that the use of the modal damping in the modal analysis of the transient vibration is an approximation. In this approximation, we have assumed that:

$$\{u_j\}^T[C]\{u_i\} = \begin{cases} 2\xi_i\omega_i M_i & i = j \\ 0 & i \neq j \end{cases} \qquad (6.72)$$

Equation (6.72) is exact only for the special case where $[C]$ is a linear combination of the $[M]$ and $[K]$ matrices.

Using the result obtained from the vibration analysis of a one degree-of-freedom system, we have,

$$q_i(t) = e^{-\xi_i\omega_i t}(A_{i1}e^{i\omega_i\sqrt{1-\xi_i^2}\,t} + A_{i2}e^{-i\omega_i\sqrt{1-\xi_i^2}\,t}), \quad i = 1,2 \qquad (6.73)$$

The coefficients are obtained from the initial conditions. Using the orthogonal properties of the eigenvectors, we have

$$\frac{\{u_i\}^T[M]\{x_0\}}{M_i} = A_{i1} + A_{i2} \qquad (6.74)$$

and

$$\frac{\{u_i\}^T[M]\{v_0\}}{M_i} = (-\xi_i\omega_i + i\omega_i\sqrt{1-\xi_i^2})A_{i1} + (-\xi_i\omega_i - i\omega_i\sqrt{1-\xi_i^2})A_{i2} \quad \text{..................} (6.75)$$

which give

$$A_{i1} = \frac{v_{0i} + (\xi_i\omega_i + i\omega_i\sqrt{1-\xi_i^2})x_{0i}}{i2\omega_i\sqrt{1-\xi_i^2}} \quad \text{..............................} (6.76)$$

and

$$A_{i2} = -\frac{v_{0i} + (\xi_i\omega_i - i\omega_i\sqrt{1-\xi_i^2})x_{0i}}{i2\omega_i\sqrt{1-\xi_i^2}} \quad \text{..............................} (6.77)$$

where

$$x_{0i} = \frac{\{u_i\}^T[M]\{x_0\}}{M_i} \quad \text{..............................} (6.78)$$

and

$$v_{0i} = \frac{\{u_i\}^T[M]\{v_0\}}{M_i} \quad \text{..............................} (6.79)$$

Substituting (6.73) in (6.69), the system transient response is expressed as,

$$\{x\} = \sum_{i=1}^{2} e^{-\xi_i\omega_i t}(A_{i1}e^{i\omega_i\sqrt{1-\xi_i^2}t} + A_{i2}e^{-i\omega_i\sqrt{1-\xi_i^2}t})\{u_i\}$$

$$= \sum_{i=1}^{2} A_{mi}\cos(\omega_{di}t - \varphi_i)\{u_i\} \quad \text{..............................} (6.80)$$

where

$$A_{mi} = \sqrt{x_{0i}^2 + \left(\frac{v_{0i} + \xi_i\omega_i x_{0i}}{\omega_{di}}\right)^2} e^{-\xi_i\omega_i t} \quad \text{..............................} (6.81)$$

and

$$\varphi_i = \tan^{-1}\frac{v_{0i} + \xi_i\omega_i x_{0i}}{\omega_{di}x_{0i}} \quad \text{..............................} (6.82)$$

and the damped natural frequency is

$$\omega_{di} = \omega_i\sqrt{1-\xi_i^2} \quad \text{..............................} (6.83)$$

It is shown that the transient response of the system due to initial conditions is contributed by the superposition of two decaying modes at their own damped natural frequencies and modal damping ratios. Figure 6.7 shows the free vibration of the damped system considered previously for which $m_1 = m_2 = 1$ kg; $k_1 = k_2 = 1$ N/m; $c_1 = c_2 = 0.08$ Ns/m. The initial conditions of the two masses are:

$$\{x_0\} = \begin{bmatrix} 0.001 \\ 0 \end{bmatrix}, \{v_0\} = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}.$$

Note that the modal contributions to X_2 are not shown as they are simply scaled versions of those for X_1. The ratio of X_1 to X_2 for the first mode being the first mode shape and similarly for the second mode.

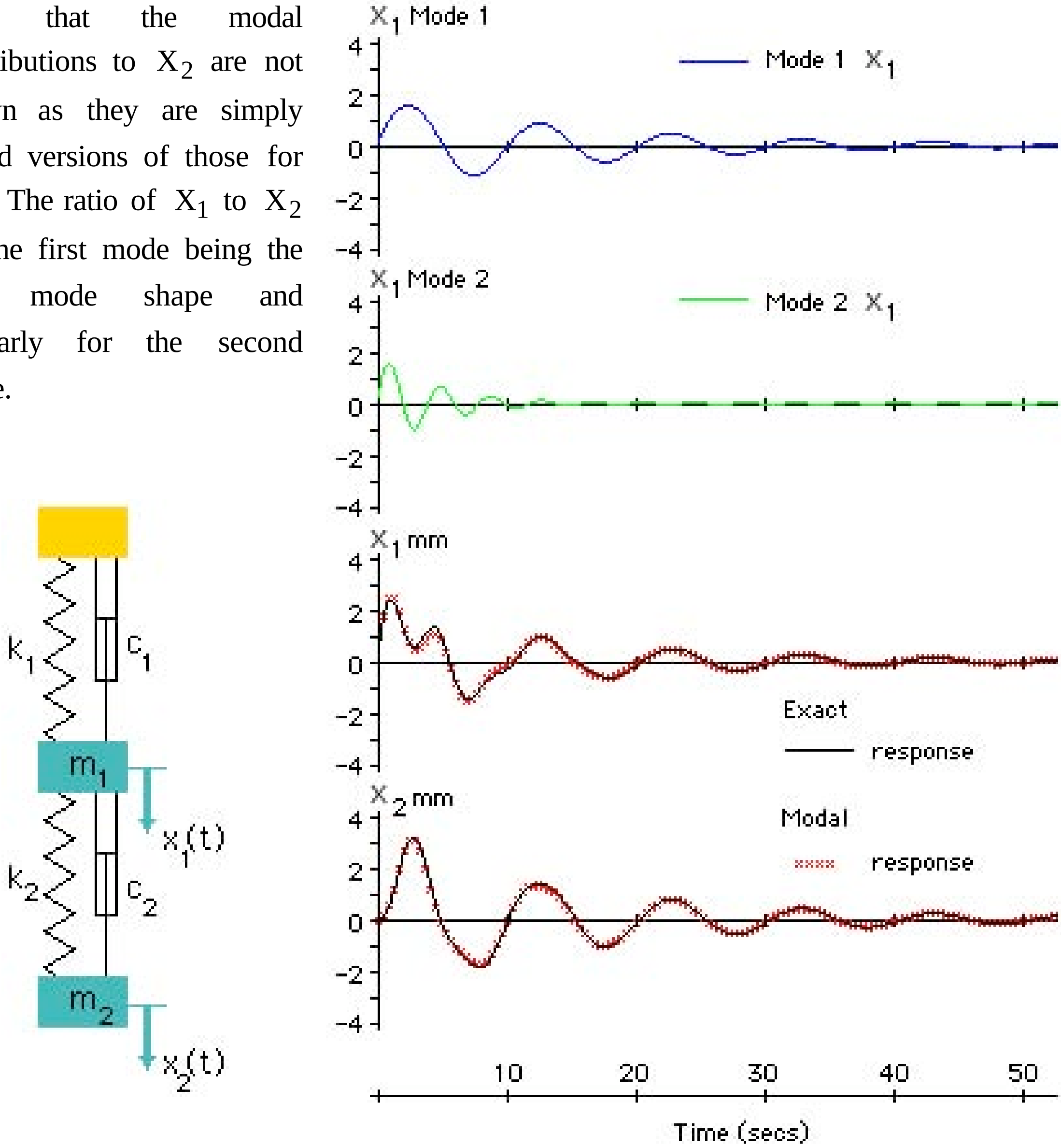

Figure 6.7 Displacement response of a two degree-of-freedom system due to initial conditions.

The program may be used to investigate different systems and the effect of different initial conditions on the transient response. In general one mode will have a larger damping ratio and dies out more rapidly. The displacements are then dominated by the mode with smaller damping ratio.

It will be found that as with steady state excitation the modal analysis method is very accurate except when the levels of damping are large.

6.9 Conclusions

All the examples in this chapter have been of systems with two degrees-of-freedom. At times the maths was extensive and computer programs needed to find solutions. However, as the equations were presented in matrix form they still hold for systems with more degrees-of-freedom. For example the mass matrix $[M]$ was a 2 x 2 diagonal matrix but could easily have been a 10 x 10 diagonal matrix for a system with 10 degrees-of-freedom. When computers are used the matrix equations are the same as presented in this chapter but just of higher order.

The modal analysis approach that has presented has been shown to be accurate when the levels of damping are small, which is the case in most real structures. Thus undamped models are used to find the natural frequencies and mode shapes and then measured (or known from experience) modal damping is applied and the modes summed.

We have so far considered only discrete systems made up of discrete masses, springs and dampers. Many real systems are better modeled as continuous systems and that is the topic of the next chapter.

Next chapter

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