CHAPTER 9: Receptances.
A systems approach.

As engineering systems have become more and more complex, so have the methods which have been developed for their analysis. This has resulted in the availability of mathematical models which may very accurately represent the real system. These allow engineers to examine possible problems at the drawing board stage or even existing problems with a working system. However, because the modelling is itself so complex, it is as difficult to comprehend the model as the original problem. Various parameters may be varied and the effect on the whole system determined but with so many parameters, which often interact, the task becomes daunting. When models were simpler, there was at least the advantage that the major parameters were isolated and it was easier to find possible ways of solving problems. Nevertheless the disadvantage with the simple model was always the concern about the accuracy with which the real system was modelled. It therefore appears that an insuperable problem exists. Either the model is too simple but may be comprehensible, or the model accurately represents the system but is incomprehensible as far as finding a solution to a problem.

In many branches of engineering the way forward has been to adopt a systems approach. Thus a complex system is broken down into a number of sub-systems which may be modelled separately. The problem then becomes one of the interaction of the sub-systems. The effect on the total system of modifications to a single sub-system are more easily comprehended. Such a method may be developed for the general area of vibration. It is intended to introduce the relevant concepts which allow even complex problems to be amenable to solution using the systems approach. The concepts will be illustrated by simple examples and then applied to more complex systems.

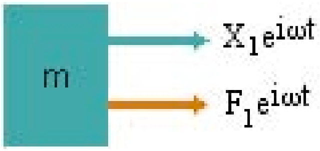
9.1 Definition of receptance

The systems approach uses receptances - the receptance α_{12} is defined as,

$$\alpha_{12} = \frac{X_1 e^{i\omega t}}{F_2 e^{i\omega t}} = \frac{X_1}{F_2} \quad \text{...} \quad (9.1)$$

Where X_1 is the response of a system at a position and in a direction defined by the subscript 1 and is often complex and indicates a phase with respect to the exciting force $F_2 e^{i\omega t}$ which is applied to the system at a position and in a direction defined by the subscript 2. α_{12} is termed the cross receptance, whereas α_{11} is called a direct receptance as then the force and displacement are at the same position and in the same direction. It is useful to consider some simple systems and derive their receptances. These receptances can then be used to build more complex systems.

9.1.1 Rigid mass



Let the force be applied through the centre of mass.

Thus from Newton II

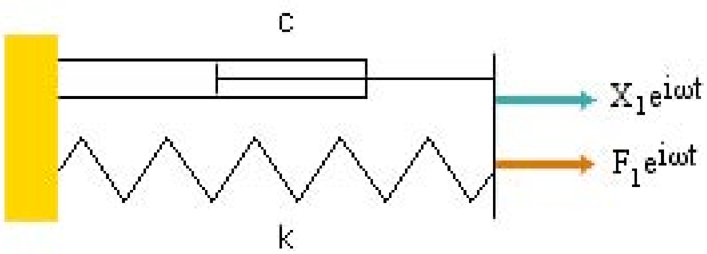
$$F_1e^{i\omega t} = mx_1''$$

Since $x_1 = X_1e^{i\omega t}$ we obtain $F_1e^{i\omega t} = -m\omega^2X_1e^{i\omega t}$

Therefore

$$\alpha_{11} = \frac{X_1}{F_1} = -\frac{1}{m\omega^2} \quad \text{...} \quad (9.2)$$

9.1.2 Spring and viscous damper in parallel



$$F_1e^{i\omega t} = kx_1 + cx_1'$$

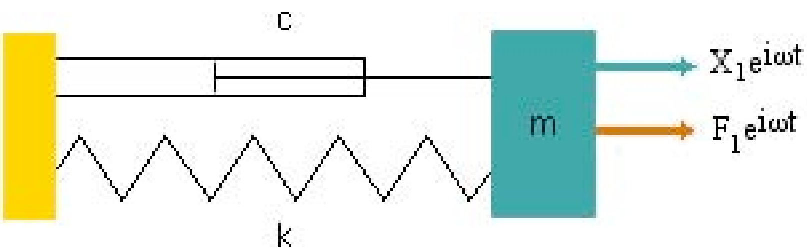
substituting $x_1 = X_1e^{i\omega t}$

$$F_1e^{i\omega t} = kX_1e^{i\omega t} + i\omega cX_1e^{i\omega t}$$

Therefore

$$\alpha_{11} = \frac{X_1}{F_1} = \frac{1}{k + i\omega c} \quad \text{..} \quad (9.3)$$

9.1.3 Spring/mass system with viscous damping



$$F_1e^{i\omega t} = kx_1 + cx_1' + mx_1''$$

substituting for $x_1 = X_1e^{i\omega t}$

$$F_1e^{i\omega t} = kX_1e^{i\omega t} + i\omega cX_1e^{i\omega t} - m\omega^2X_1e^{i\omega t}$$

therefore

$$\alpha_{11} = \frac{X_1}{F_1} = \frac{1}{k - m\omega^2 + i\omega c} \quad \text{..} \quad (9.4)$$

The main point to note is that the receptances vary with frequency and also are complex numbers which indicates a phase between X_1 and F_1.

Thus in general $\alpha_{11} = a + ib = \sqrt{(a^2 + b^2)}e^{i\theta}$ where $\tan\theta = b/a$.

The main use of receptances is in predicting the response of complex systems by adding together simpler sub-systems.

9.2 Addition of Two systems

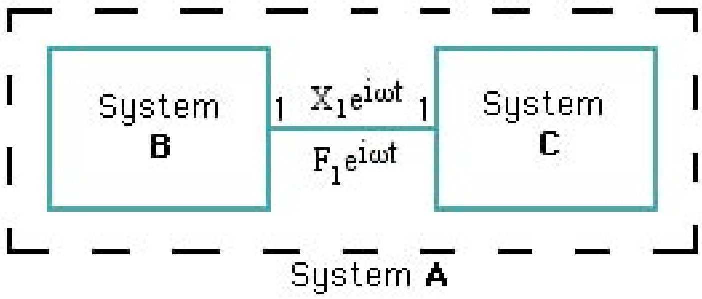


Figure 9.1 The addition of two sub-systems

Consider the original system to be **B** in Figure 1 and let the additional system **C** be added at co-ordinate 1. Note the line joining the coordinates 1 on systems **B** and **C** is drawn for convenience here and in subsequent analyses. In practice this line has zero length as the two systems are joined at a common coordinate 1. Now, for this first example, let the excitation and response be measured at co-ordinate 1 so that it will be necessary to find the direct receptance α_{11} of the combined system **A** in terms of the direct receptances of the two sub systems β_{11} and γ_{11}.

The procedure is to separate the combined system **A** into its component systems, ie. system **B** and **C** and introduce forces at the coordinate 1 on each subsisted so that the separate systems are behaving in an identical manner to that when they are joined. This separation is shown in Figure 9.2.

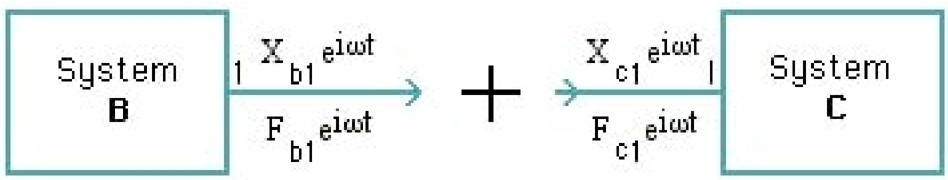


Figure 9.2. Sub-systems separated

It is helpful to introduce an additional subscript (ie. b or c) to indicate which system is being considered.

Consider system **B** By definition $\beta_{11} = \dfrac{X_{b1}}{F_{b1}}$ .... (9.5)

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and for system **C** By definition $$\gamma_{11} = \frac{X_{c1}}{F_{c1}} \qquad (9.6)$$

Now for the situation shown in Figure 9.2 to be identical to that shown in Figure 9.1 then all the displacements are identical,

$$X_{b1} = X_{c1} = X_1 \qquad (9.7)$$

and the sum of the forces on the sub systems is equal to the force applied to combined system,

$$F_{b1} + F_{c1} = F_1 \qquad (9.8)$$

ie. the force F_1 is shared between the two systems.

It remains to determine X_1/F_1 , ie. α_{11}, from equations (9.5)-(9.8)

From (9.5) $F_{b1} = \dfrac{X_{b1}}{\beta_{11}}$ and substituting from (9.7) $F_{b1} = \dfrac{X_1}{\beta_{11}}$

From (9.6) $F_{c1} = \dfrac{X_{c1}}{\gamma_{11}}$ and substituting from (9.7) $F_{c1} = \dfrac{X_1}{\gamma_{11}}$

Substituting in the force equation (9.8) gives

$$\frac{X_1}{\beta_{11}} + \frac{X_1}{\gamma_{11}} = F_1$$

$$\frac{F_1}{X_1} = \frac{1}{\alpha_{11}} = \frac{1}{\beta_{11}} + \frac{1}{\gamma_{11}} \qquad (9.9)$$

This result appears very simple but it is important to remember that β_{11}, γ_{11} and α_{11} may be complex quantities because of phase and that for conventional systems they will all be functions of frequency. The addition has thus to be made at each frequency of interest.

As a simple example of the addition of two systems consider the simple spring/mass system with damping which was examined previously. This may be considered as the addition of a mass to a spring and dashpot as shown in figure 9.3.

Since the mass is rigid the force has the same effect whether it is applied at the join or on the free side of the mass. The receptances of the sub-systems have been derived.

Thus for the mass $$\gamma_{11} = -\frac{1}{m\omega^2}$$

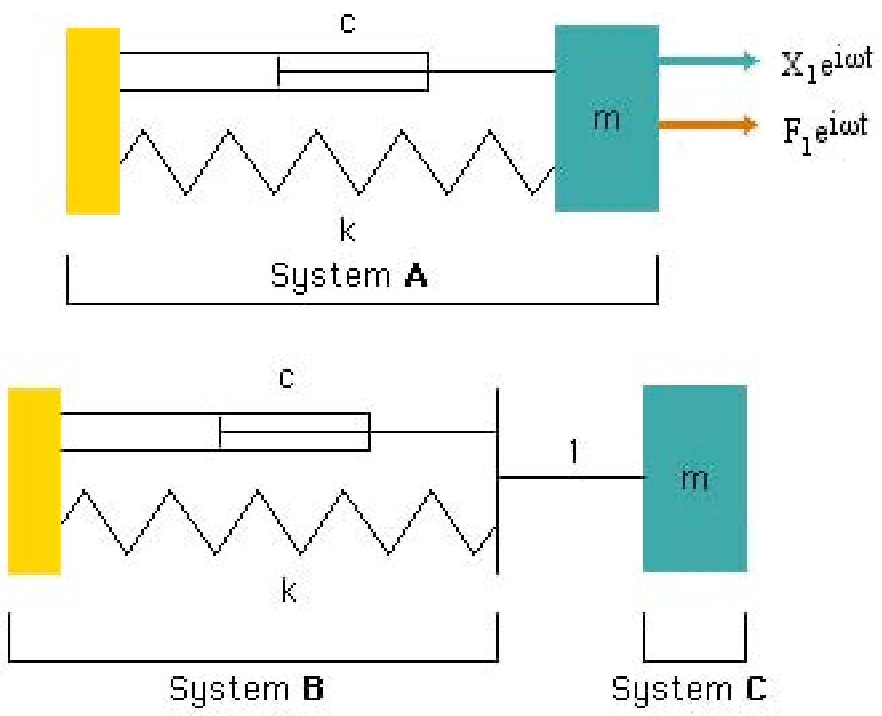


Figure 9.3 Separation of system into known components

and for the spring and damper
$$\beta_{11} = \frac{1}{k + i\omega c}$$

The receptance of the combined system is therefore given by

$$\frac{1}{\alpha_{11}} = \frac{1}{\beta_{11}} + \frac{1}{\gamma_{11}} = k + i\omega c - m\omega^2$$

so that

$$\alpha_{11} = \frac{1}{k - m\omega^2 + i\omega c}$$

which is the response of a spring/mass system with damping which was found previously.

9.3 Addition of two systems in series

It is possible to derive receptance addition formulae which allow complex systems to be built. In this section we will consider a simple addition in series but addition at a remote coordinate is useful in practice and Appendix 1 gives the derivation of the equations for that case. However, consider first addition in series.

Consider a system **C** to which is added a system **B** at coordinate 2 as shown in figure 9.4. We will find the receptance of the combined system **A** at coordinate 1. We first divide the combined system into its component systems and introduce forces at the join so that the vibration remains unchanged from when joined.

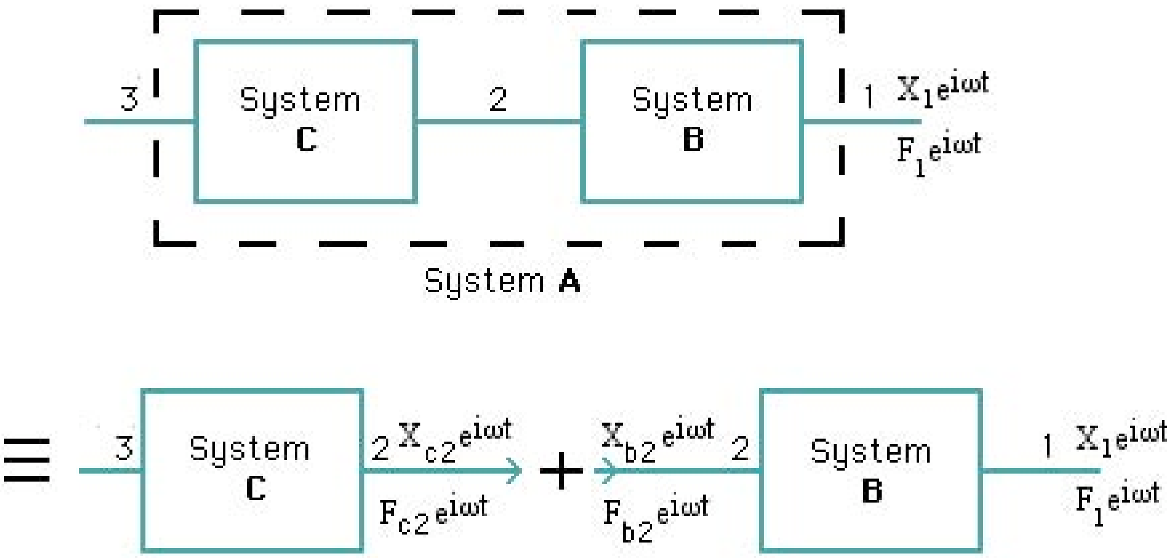


Figure 9.4 Addition of systems in series

Thus for system **C** by definition $$X_{c2} = \gamma_{22}F_{c2} \quad \text{...} \quad (9.10)$$

For system **B** since two forces are applied the displacement at any co-ordinate will be the sum of the displacements caused by each force. ie. superposition is assumed to apply.

Thus $$X_{b2} = \beta_{22}F_{b2} + \beta_{21}F_1 \quad \text{.......................................} \quad (9.11)$$

and $$X_1 = \beta_{11}F_1 + \beta_{12}F_{b2} \quad \text{...} \quad (9.12)$$

Now for the systems to be identical when separated as when joined

$$X_{b2} = X_{c2} \quad \text{...} \quad (9.13)$$

and since there is no external force at 2.

$$F_{b2} + F_{c2} = 0 \quad \text{..} \quad (9.14)$$

Substitute $F_{c2} = -F_{b2}$ from (9.14) and for X_{b2} from (9.11) and X_{c2} from (9.10) in equation (9.13).

Thus $$\beta_{22}F_{b2} + \beta_{21}F_1 = -\gamma_{22}F_{b2}$$

therefore $$F_{b2} = \frac{-\beta_{21}F_1}{\beta_{22} + \gamma_{22}} \quad \text{...} \quad (9.15)$$

and substituting for F_{b2} in (9.12) gives $X_1 = \beta_{11}F_1 - \dfrac{\beta_{12}\beta_{21}F_1}{\beta_{22} + \gamma_{22}}$

For linear conservative systems, Maxwell's Reciprocal Theorem holds and $\beta_{12} = \beta_{21}$ therefore

$$\frac{X_1}{F_1} = \alpha_{11} = \beta_{11} - \frac{\beta_{12}^2}{\beta_{22} + \gamma_{22}} \quad \text{..} \quad (9.16)$$

It is also possible to find other receptances of the combined system. thus from (9.11)

$$X_2 = X_{b2} = \beta_{23}F_{b2} + \beta_{21}F_1 = -\frac{\beta_{21}\beta_{23}F_1}{\beta_{22} + \gamma_{22}} + \beta_{21}F_1$$

$$\therefore \frac{X_2}{F_1} = \alpha_{21} = \beta_{21} - \frac{\beta_{21}\beta_{23}}{\beta_{22} + \gamma_{22}} \quad \text{..} \quad (9.17)$$

Also for system C

$$X_2 = X_{c3} = \gamma_{32}F_{c2} = -\gamma_{32}F_{b2} = \frac{\beta_{21}\gamma_{32}F_1}{\beta_{22} + \gamma_{22}}$$

$$\therefore \frac{X_3}{F_1} = \alpha_{31} = \frac{\beta_{21}\gamma_{32}}{\beta_{22} + \gamma_{22}} \quad \text{..} \quad (9.18)$$

9.4 Natural Frequency Calculations

In an undamped system the value of the receptance will tend to infinity as the excitation frequency tends to a natural frequency. For the addition of two systems in series it was found, equations (9.16) - (9.18), that

$$\alpha_{11} = \beta_{11} - \frac{\beta_{12}^2}{\beta_{22} + \gamma_{22}}; \quad \alpha_{21} = \beta_{21} - \frac{\beta_{21}\beta_{23}}{\beta_{22} + \gamma_{22}}; \quad \alpha_{31} = \frac{\beta_{21}\gamma_{32}}{\beta_{22} + \gamma_{22}}$$

Thus the natural frequencies occur when $\beta_{22} + \gamma_{22} = 0$. Also from the earlier example when a system was added at the excitation point it was found, equation (9.9), that

$$\frac{1}{\alpha_{11}} = \frac{1}{\beta_{11}} + \frac{1}{\gamma_{11}}$$

so that

$$\alpha_{11} = \frac{\beta_{11}\gamma_{11}}{\beta_{11} + \gamma_{11}}$$

and the natural frequencies occur when $\beta_{11} + \gamma_{11} = 0$.

In both the cases considered the natural frequencies are given when the sum of the direct receptances, at the point where the systems are joined, is zero. This confirms the fact that natural frequencies are governed by the geometry of a structure and not by the excitation point or the point where the response is measured.

Most real engineering structures have very light damping so that the natural frequencies may be determined by finding the frequencies when the real parts of β_{11} and γ_{11} add to zero. In practice, measured and/or calculated values may be used. It is convenient to plot β_{11} and $-\gamma_{11}$ against frequency and find the intercepts. Four examples will be given to illustrate this method of finding natural frequencies. We will use combinations of systems for which we already know the receptances.

9.4.1 Clamped/free bar attached to a spring

For the first example consider the natural frequencies for axial vibration of a bar clamped at one end with a spring connected to earth added at the other end. Figure 9.5 shows the separation into two systems for which the receptances are known. In chapter 7 the end response of a clamped free bar was found and this is the receptance at the free end so that,

$$\beta_{11} = \frac{\sin \lambda L}{AE\lambda \cos \lambda L}$$

The receptance of a spring attached to earth is given by,

$$\gamma_{11} = \frac{1}{k}$$

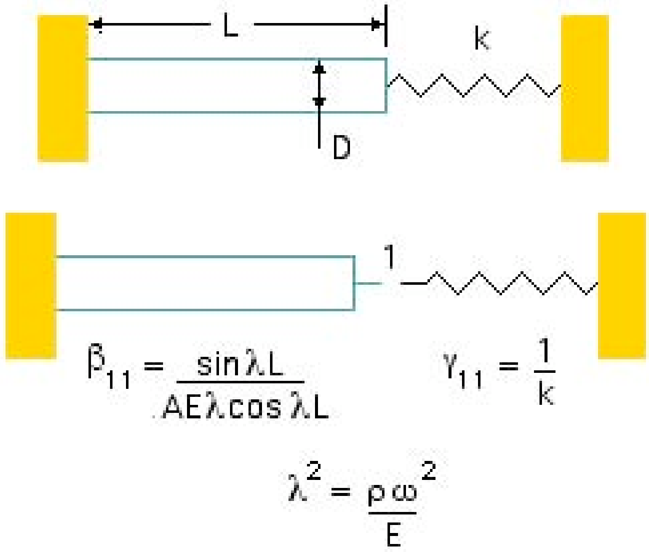


Figure 9.5 Component systems of a clamped/free bar attached to a spring.

If the bar is made of steel with L = 1.42m and D = 0.06m then the receptance β_{11} is as shown in red in Figure 9.6. If the stiffness is 10^9 N/m then $-\gamma_{11}$ is shown in blue and the intersections show 3 natural frequencies in the range 0 - 5000Hz.

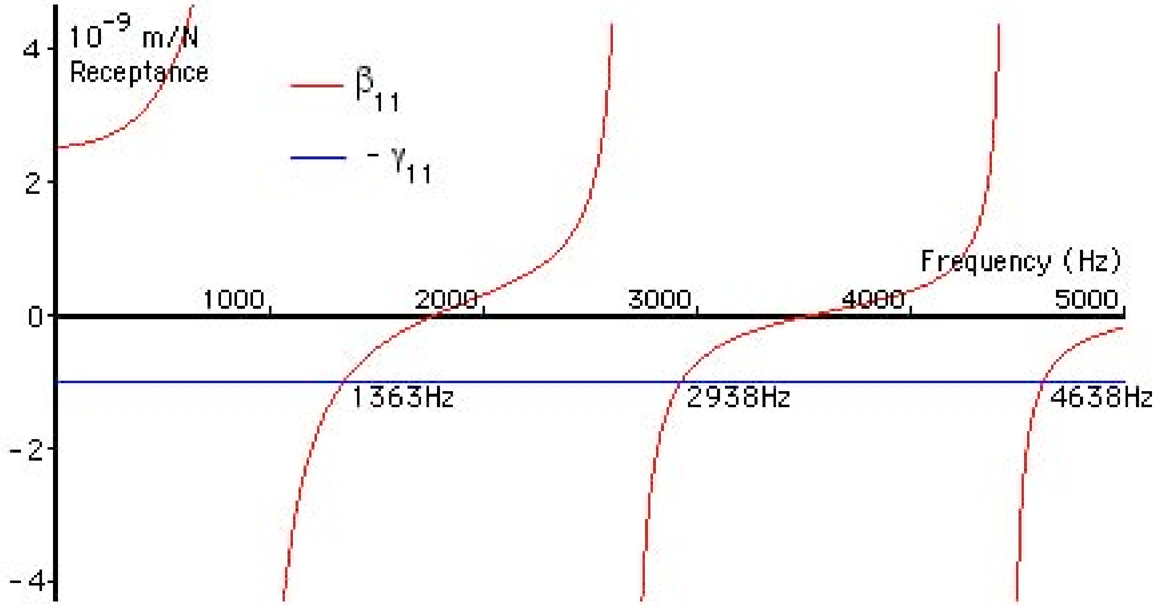


Figure 9.6 Determination of natural frequencies by finding when $\beta_{11} + \gamma_{11} = 0$ for a clamped/free bar attached to a spring.

<u>9.4.2 Clamped/free bar attached to a mass</u>

For the second example consider the axial natural frequencies for the same clamped/free bar with a mass added at the other end. Figure 9.7 shows the components and their receptances.

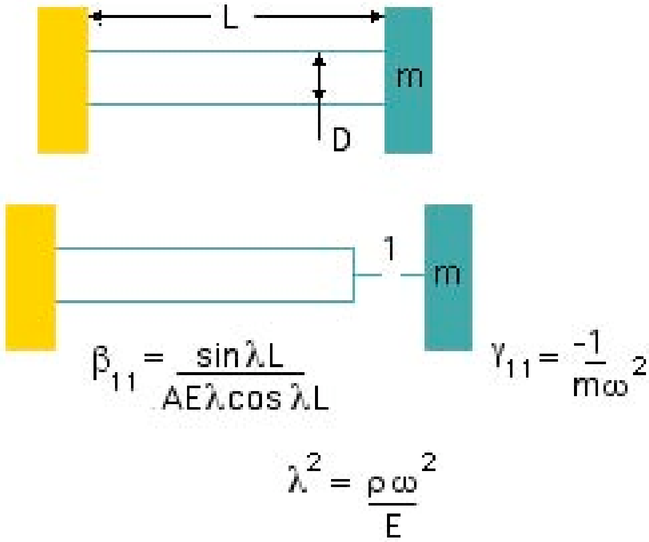


Figure 9.7 Component systems of a clamped/free bar attached to a mass.

The receptance of a mass has previously been derived. The receptance β_{11} of the bar is as shown in red in Figure 9.8. If the mass is 5kg then $-\gamma_{11}$ is shown in blue and the intersections show 3 natural frequencies in the range 0 - 5000Hz.

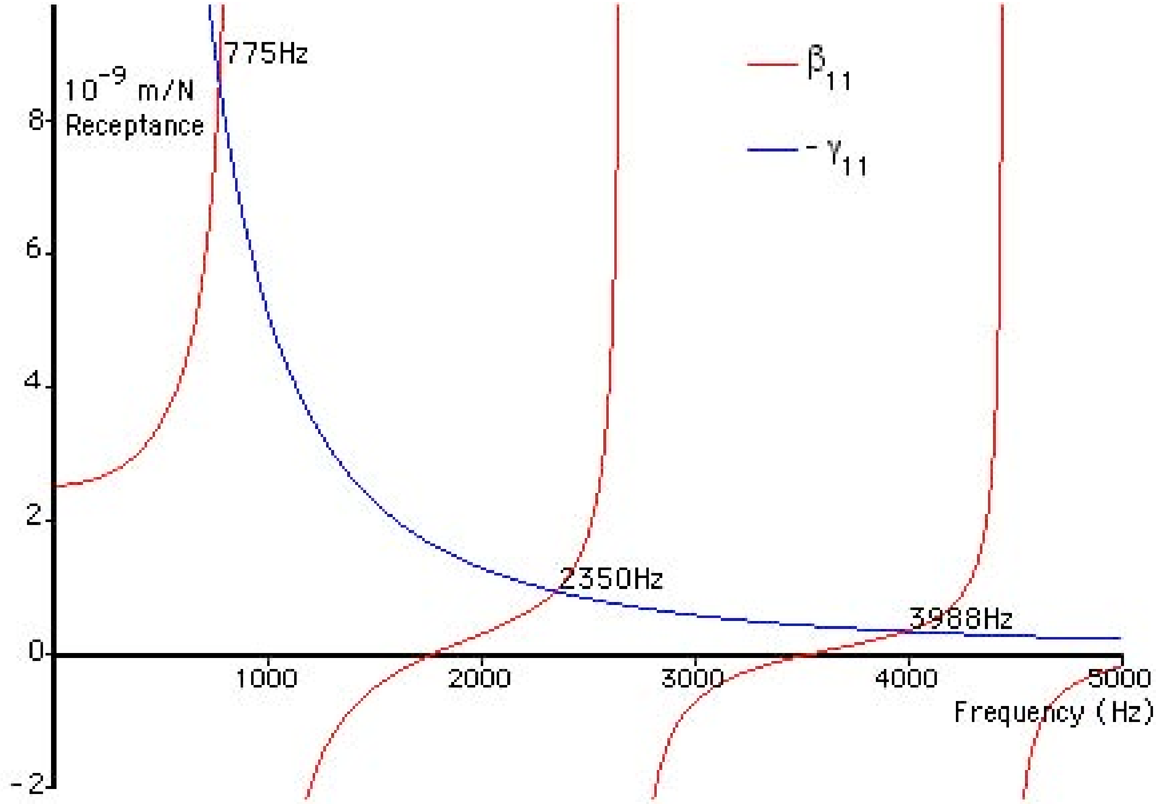
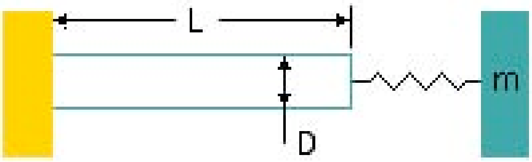


Figure 9.8 Determination of natural frequencies by finding when $\beta_{11} + \gamma_{11} = 0$ for a clamped/free bar attached to a mass.

9.4.3 Clamped/free bar attached to a spring/mass

For the third example consider the axial natural frequencies for the same clamped/free bar with a spring/mass added at the other end.

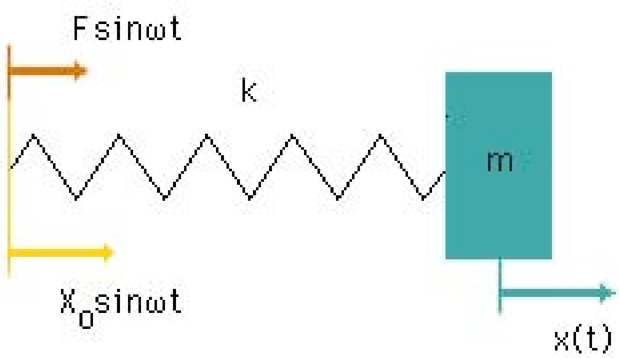


The receptance of the spring/mass system needs some thought. In chapter 2 when considering detuners we showed that for a spring mass system,

$$\frac{F}{X_0} = \frac{-km\omega^2}{(k - m\omega^2)}$$

so that rearranging we obtain

$$\gamma_{11} = \frac{X_0}{F} = \frac{-(k - m\omega^2)}{km\omega^2} = \frac{1}{k} - \frac{1}{m\omega^2}$$



The component systems and their receptances are thus as shown in figure 9.9.

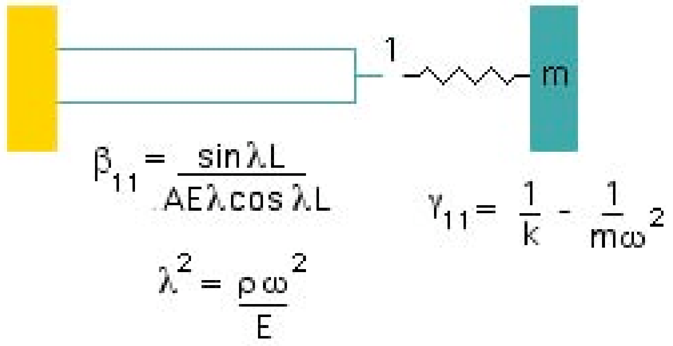


Figure 9.9 Component systems of a clamped/free bar attached to a spring/mass system.

The receptance β_{11} of the bar is as shown in red in Figure 9.10. If the mass is 5kg and the spring stiffness 10^8 N/m then $-\gamma_{11}$ is shown in blue and the intersections show 4 natural frequencies in the range 0 - 5000Hz.

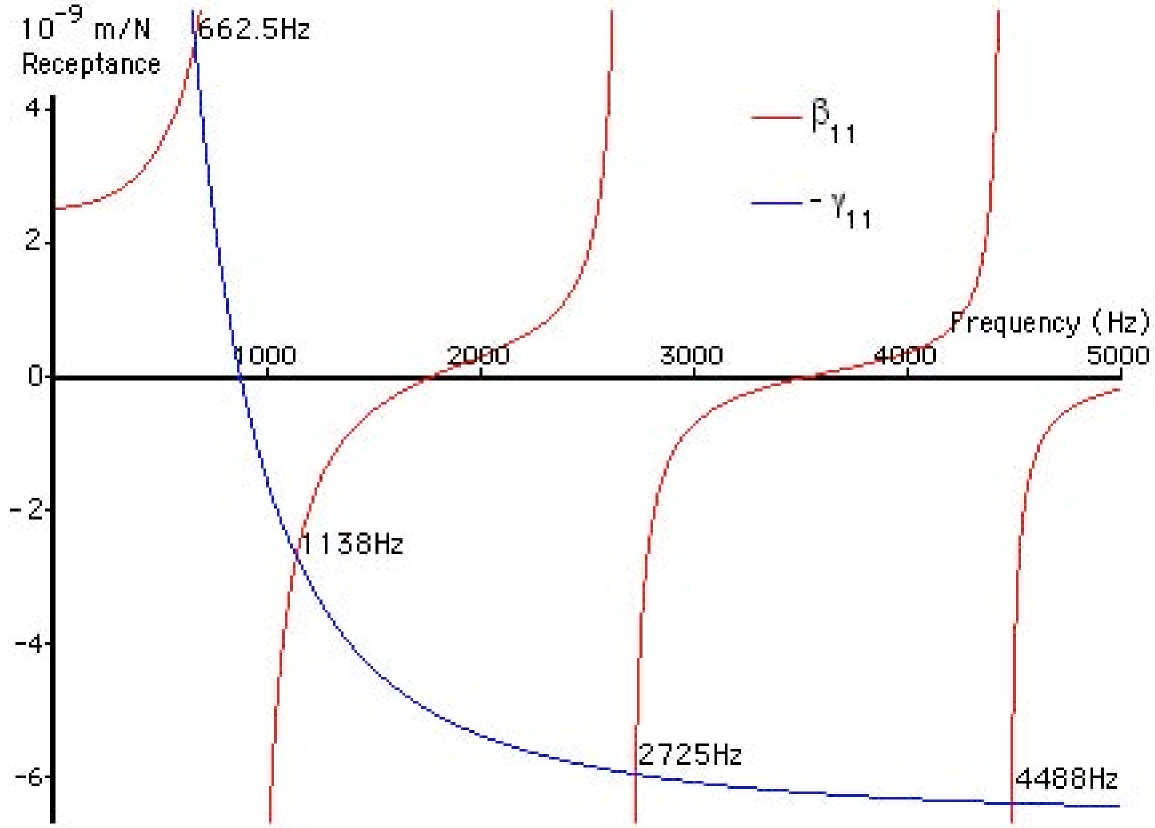


Figure 9.10 Determination of natural frequencies by finding when $\beta_{11} + \gamma_{11} = 0$ for a clamped/free bar attached to a spring/mass system.

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9.4.4 Clamped/free bar with a second bar attached

Finally consider the same clamped/free/bar with another bar attached to the free end. The component systems and their receptances are as shown in figure 9.11.

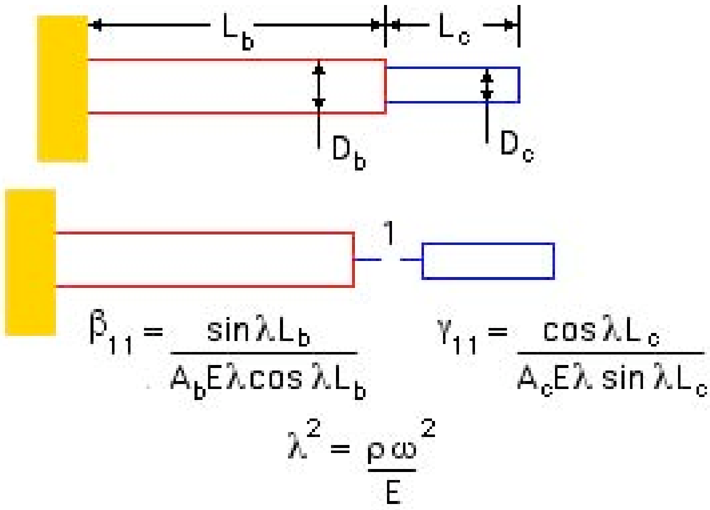

Figure 9.11 Component systems of a clamped/free bar attached to a second bar.

The receptance of a free/free bar has previously been derived. The receptance β_{11} of the clamped/free bar is as shown in red in Figure 9.12. If the second bar is also made of steel with L = 1.2m and D = 0.04m then $-\gamma_{11}$ is shown in blue and the intersections show 5 natural frequencies in the range 0 - 5000Hz.

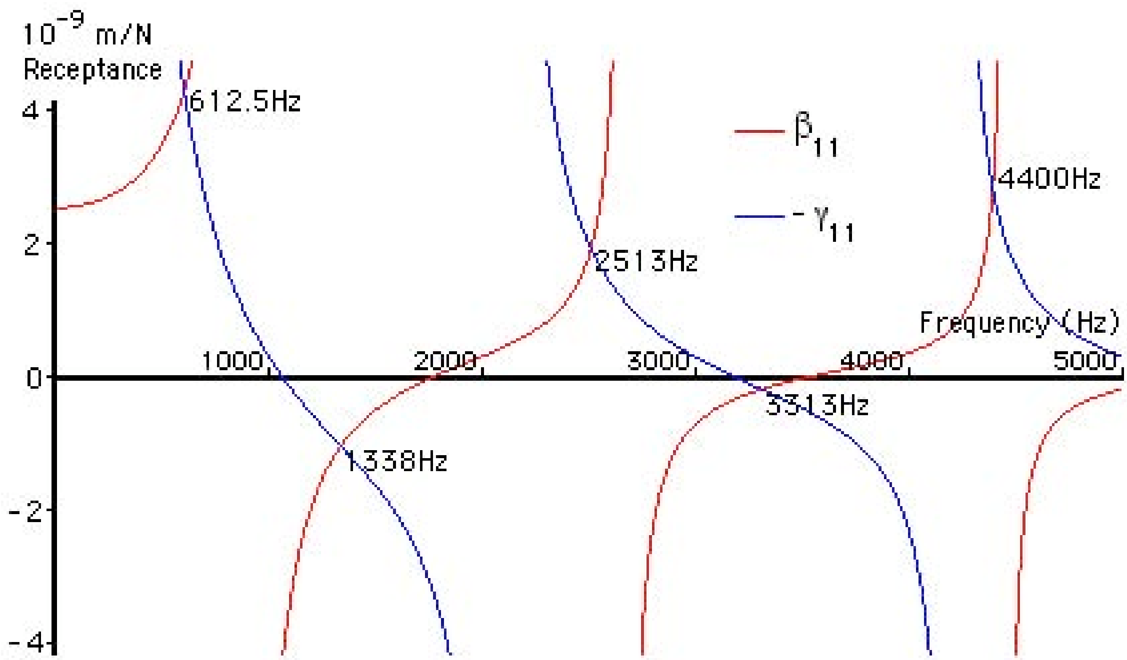

Figure 9.12 Determination of natural frequencies by finding when $\beta_{11} + \gamma_{11} = 0$
for a clamped/free bar attached to a free/free bar.

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9.5 Building complex systems

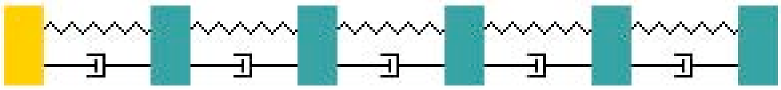
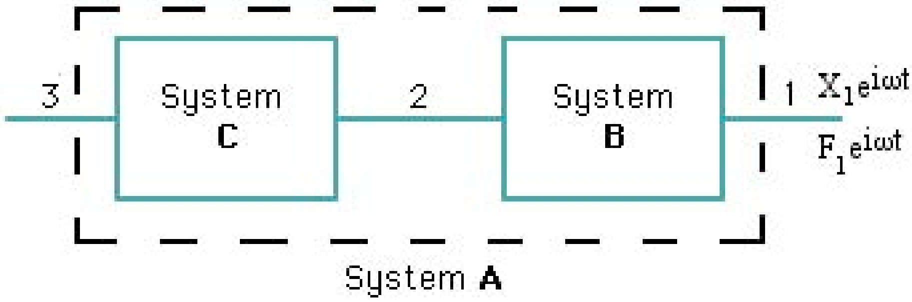


Figure 9.13 Axial system with five degrees-of-freedom

Consider the axial system shown which has five degrees of freedom. We could write down the 5 equations of motion and after a considerable amount of maths determine the steady state response X/F. Alternatively we can apply the series addition of two systems four times. We have shown for series addition,

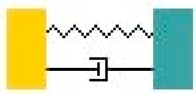


$$\alpha_{11} = \beta_{11} - \frac{\beta_{12}^2}{\beta_{22} + \gamma_{22}} \quad \text{...} \quad (9.16)$$

$$\alpha_{21} = \beta_{21} - \frac{\beta_{21}\beta_{23}}{\beta_{22} + \gamma_{22}} \quad \text{...} \quad (9.17)$$

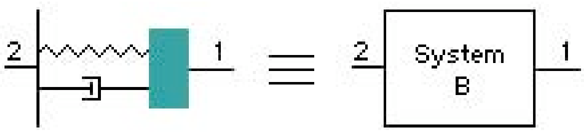
$$\alpha_{31} = \frac{\beta_{21}\gamma_{32}}{\beta_{22} + \gamma_{22}} \quad \text{...} \quad (9.18)$$

The first system is as shown and we have found the receptance for the system to be,

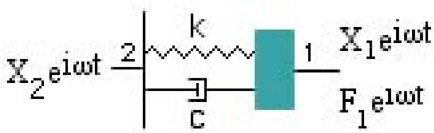


$$\gamma_{11} = \frac{1}{k - m\omega^2 + i\omega c} \quad \text{.................................} \quad (9.4)$$

Now we need to determine the receptances of the system shown below.



We need to find β_{11}, β_{12}, β_{22}. Consider first an oscillating force applied at coordinate 1.



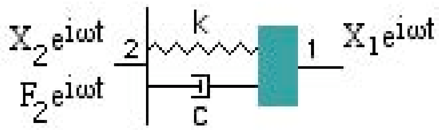
The spring and damper are considered massless so that this is the same as the excitation of a mass. Thus using equation (9.2)

$$\beta_{11} = \frac{X_1}{F_1} = -\frac{1}{m\omega^2}$$

Also since there is no force on the massless spring/damper there is no deflection across them and $X_2 = X_1$ so that

$$\beta_{21} = \frac{X_2}{F_1} = \frac{X_1}{F_1} = -\frac{1}{m\omega^2}$$

Now consider an exciting force at coordinate 2



The spring/damper are massless so that there can be no resultant force on them. Thus the force $F_2e^{i\omega t}$ also acts on the mass (so as to give an equal and opposite force on the spring/ damper). Thus

$$\beta_{12} = \frac{X_1}{F_2} = -\frac{1}{m\omega^2} \quad \text{and note } \beta_{12} = \beta_{21}$$

Now consider the deflection across the spring/damper

$$F_2e^{i\omega t} = k(x_2 - x_1) + c(x_2' - x_1')$$

and substituting $x_1 = X_1e^{i\omega t}$ and $x_2 = X_2e^{i\omega t}$ gives

$$F_2e^{i\omega t} = k(X_2 - X_1)e^{i\omega t} + i\omega c(X_2 - X_1)e^{i\omega t}$$

$$\text{so that } \frac{X_2 - X_1}{F_2} = \frac{1}{k + i\omega c}$$

$$\text{and therefore } \frac{X_2}{F_2} = \frac{1}{k + i\omega c} + \frac{X_1}{F_2}$$

$$\text{and hence } \frac{X_2}{F_2} = \beta_{22} = \frac{1}{k + i\omega c} - \frac{1}{m\omega^2}$$

We are now in a position to consider the system of Figure 9.13 which may be constructed by a succession of series additions as shown in Figure 9.14. After each addition we have a new system C to which another system B is added. Finally we achieve the complete system and have the receptance at the right hand end.

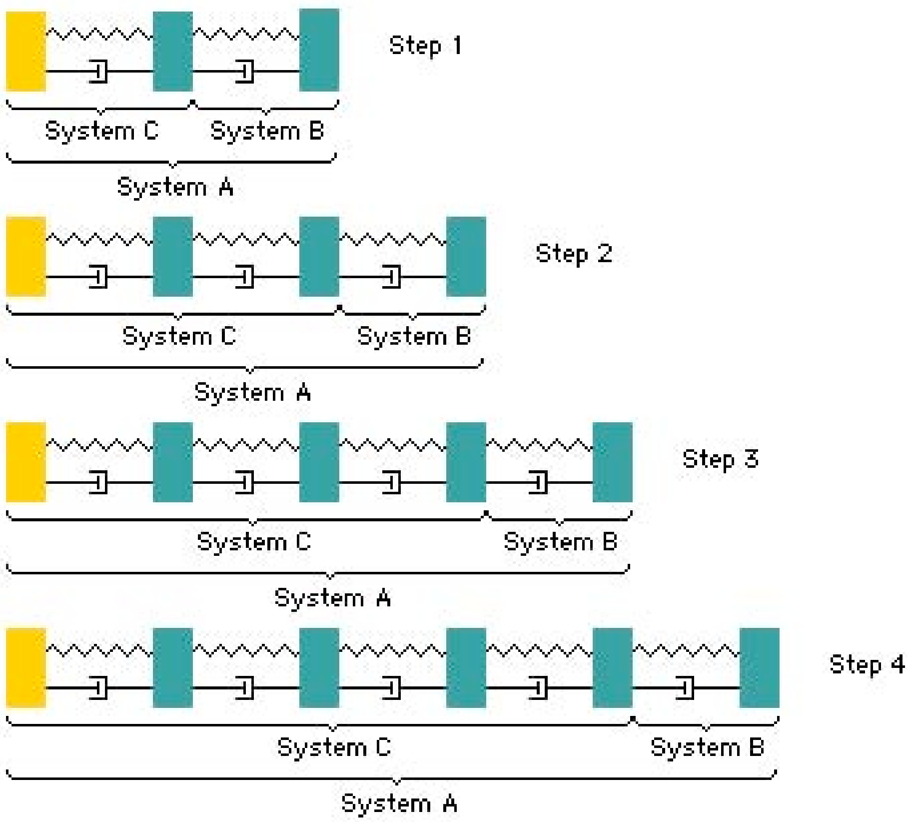


Figure 9.14 Successive series additions.

Step 1 involves a series addition of System **C** and system **B** to form a new system **A**. This system now becomes the new system **C** for the next series addition in step 2. This is repeated until the complete system is constructed. The response of such a 5 degree-of-freedom system is shown in Figure 9.15.

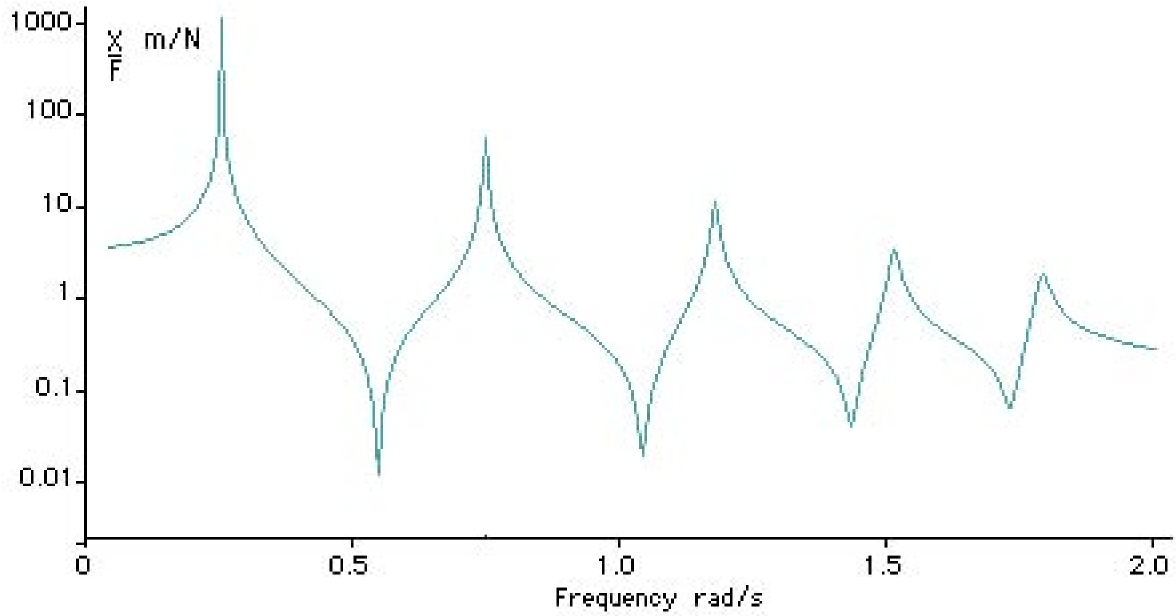


Figure 9.15 Receptance of system shown in Figure 9.9.

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It is possible to determine the response of any mass resulting from excitation on any mass. This is effected in the same way as finding the "mode" or deflected shapes.

9.6 Prediction of the system "Mode" shapes

The mode shape should really be called the deflected shape when exciting at a particular frequency. When the system is damped the mode shapes will not consist solely of positive or negative real displacements, there will be phase variations through the structure. In practice, it is not generally possible to excite a single mode. We may therefore be interested in the actual response and deflected shape at any frequency and that will include all the mode contributions.

The method we will use will be to give the last sub-system added a unit displacement at the end. Consider a typical system made up of several sub-systems (as shown in the Figure 9.16). For the sake of this example let us assume that there is an external force (F_2) applied at coordinate 2.

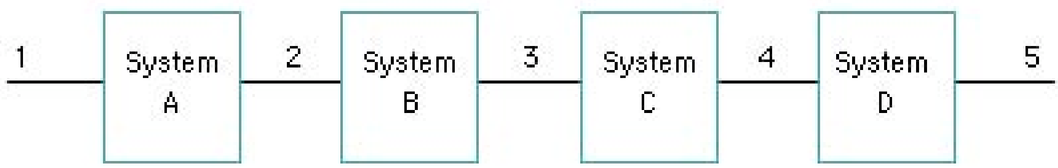


Figure 9.16 A complex system comprising four sub systems.

We set the amplitude at coordinate 5 on sub-system D to 1.0, ie. $X_{d5} = 1.0$. Now since we know all the receptances for each system it follows that we would know the cross-receptance for the whole system that connects 2 and 5. We would thus be able to calculate the value of F_2 to give $X_{d5} = 1.0$.

For system D we may write down the equation,

$$X_{d5} = \delta_{45} F_{d4} \quad \text{...} \quad (9.19)$$

There is no term involving F_5 because this is a free end and so $F_5 = 0$. As we know X_{d5} and δ_{45} we may easily determine F_{d4}. We may also calculate X_{d4} from

$$X_{d4} = \delta_{44} F_{d4} \quad \text{..} \quad (9.20)$$

Now consider sub-system C, $F_{c4} = -F_{d4}$ as there is no external force and $X_{c4} = X_{d4}$ from compatibility at the join. Thus,

$$X_{c4} = (X_{d4}) = \gamma_{34} F_{c3} + \gamma_{44} F_{c4} \quad \text{......................................} \quad (9.21)$$

In equation (9.21) the only unknown is F_{c3} so this may be determined. Also

$$X_{c3} = \gamma_{33}F_{c3} + \gamma_{34}F_{c4} \quad \text{...} \quad (9.22)$$

and so X_{c3} is now known.

Now consider sub-system B, $F_{b3} = -F_{c3}$ as there is no external force and $X_{b3} = X_{c3}$ from compatibility at the join. Thus,

$$X_{b3} = (X_{c3}) = \beta_{23}F_{b2} + \beta_{33}F_{b3} \quad \text{..} \quad (9.23)$$

In equation (9.23) the only unknown is F_{b2} so this may be determined. Also

$$X_{b2} = \beta_{23}F_{b2} + \beta_{33}F_{b3} \quad \text{...} \quad (9.24)$$

so that U_{b2} may be calculated.

Now consider sub-system A, $F_{a2} = F_2 - F_{b2}$ as there is an external force F_2 (which is known) and $X_{a2} = X_{b2}$ from compatibility at the join. Thus,

$$X_{a2} = (X_{b2}) = \alpha_{12}F_{a1} + \alpha_{22}F_{a2} \quad \text{...} \quad (9.25)$$

In equation (9.25) the only unknown is F_{a1} so this may be determined.

$$\text{also} \quad X_{a1} = \alpha_{12}F_{a2} \quad \text{...} \quad (9.26)$$

There is no force F_1 and so U_{a1} is now known.

This method lends itself to programming as the process is largely repetitive. At the end of the process, the forces at each end, for any sub-system, are known. Thus if the internal receptances are known it is possible to calculate any intermediate displacement that is required. This was the basis of the axial multi degree-of-freedom program considered in an earlier chapter.

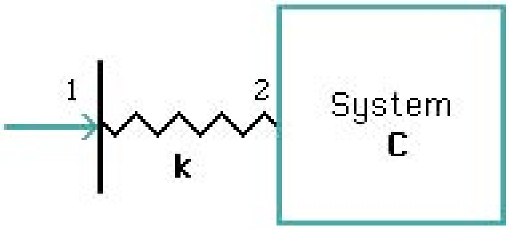
If at the start of the process only the direct receptance at the point of excitation is known for the whole system, it is then appropriate to make the displacement unity at the excitation point and work out (one direction at a time) from that point. The procedure is identical to that described above.

An example that does not require a computer is given in Appendix 2.

9.7 Solving vibration problems

9.7.1 Changing natural frequencies

The question that still needs an answer is, 'What can be done if the natural frequencies are close to any excitation frequencies? A possible solution in this case is to use a flexible coupling between parts of a system. To enable this approach to be understood it is necessary to consider the excitation of a system through a spring of negligible mass.

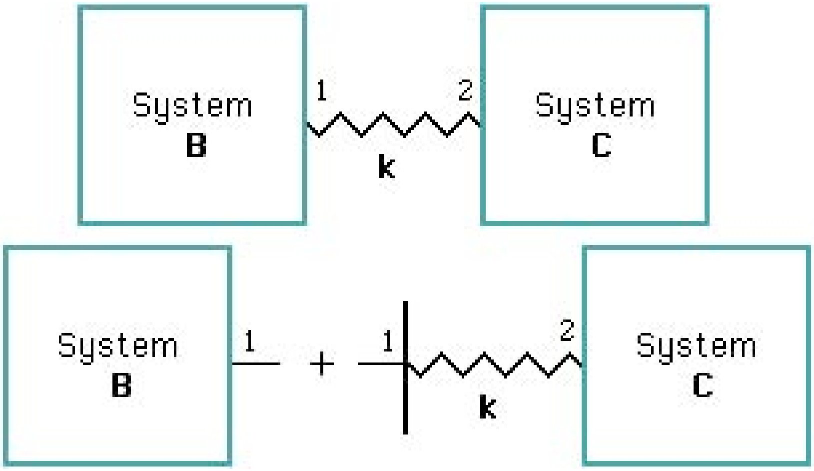


Consider an excitation force $F_1e^{i\omega t}$ at co-ordinate 1 with resulting displacements $X_1e^{i\omega t}$ at 1 and $X_2e^{i\omega t}$ at 2. Since the spring has negligible mass the force acting on system **C** will also be $F_1e^{i\omega t}$ so that $\gamma_{22} = X_{c2} / F_1$ and considering the spring X_1-X_{c2} =F_1/k. Thus eliminating X_{c2} from these two equations gives,

$$\frac{X_1}{F_1} = \alpha_{11} = \frac{1}{k} + \gamma_{11}$$

If system **C** is a mass then, $\gamma_{11} = \dfrac{-1}{m\omega^2}$ and so as found previously $\dfrac{X_1}{F_1} = \alpha_{11} = \dfrac{1}{k} - \dfrac{1}{m\omega^2}$

It follows that when two systems are joined through a flexible coupling that the natural frequency equation becomes,



$$\beta_{11} + \frac{1}{k} + \gamma_{11}$$

If the means adopted previously of finding the natural frequencies is used then a plot of β_{11} against frequency should have a plot of $-(1/k + \gamma_{11})$ superimposed. As the term -1/k is independent of frequency this simply moves the original -γ_{11} plot in the negative y-direction. It is therefore a simple matter to observe the change in resulting natural frequencies arising from a variation in k. For example, see figure 9.17. which shows the way in which the natural frequencies vary for a rigid coupling and for three values of k. The system considered is a clamped free bar (L =1.0m, D = 0.1m) with a spring (k) and mass (m = 20kg) at one end.

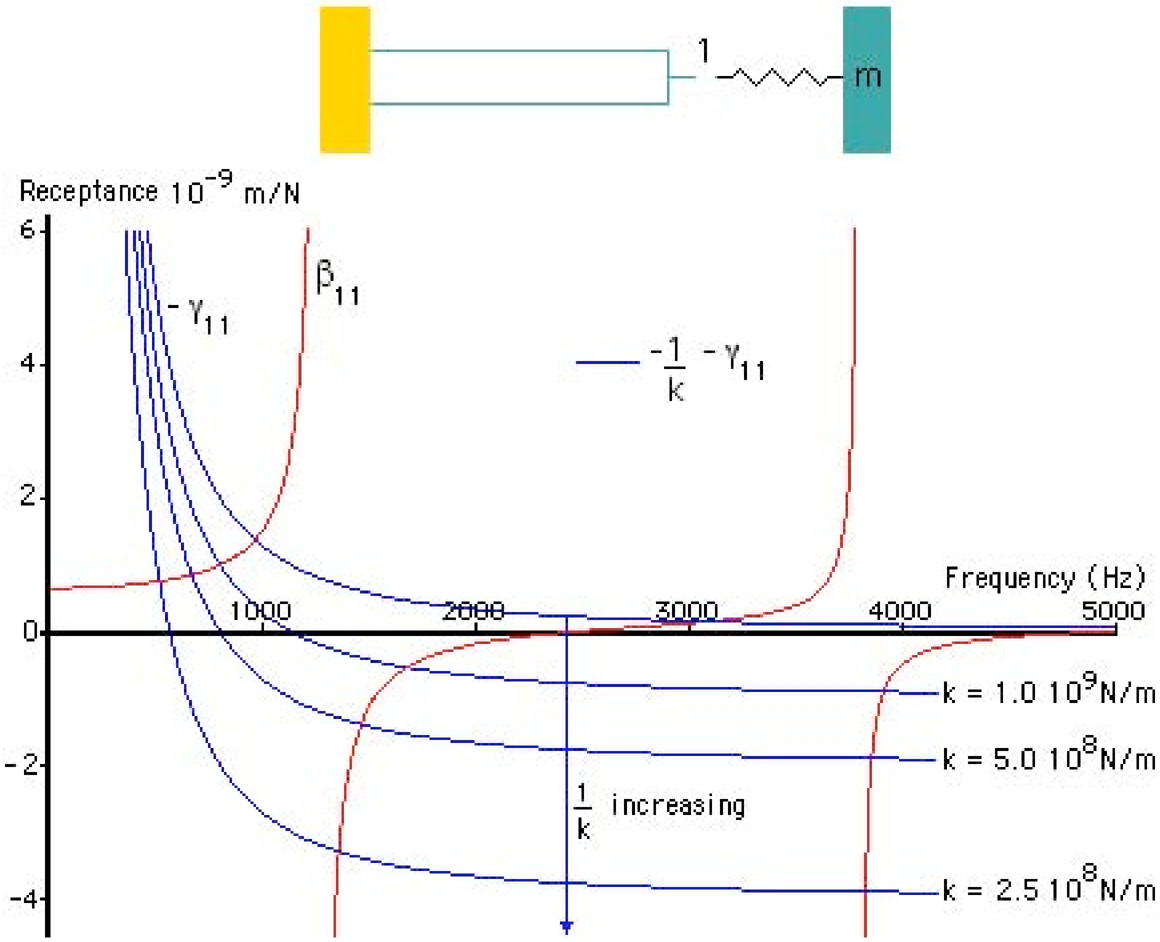


Figure 9.17 Effect of varying coupling stiffness.

The change in the natural frequencies is shown in Table 9.1.

Stiffness (N/m)	First natural frequency (Hz)	First natural frequency (Hz)	First natural frequency (Hz)
No coupling	975	3088	5425
k =1.0x10^9	800	1688	3913
k = 5.0x10^8	675	1475	3863
k = 2.5x10^8	525	1363	3838

Table 9.1 Variation of natural frequencies with coupling stiffness

It is of practical importance to note that the change in natural frequency is different for each mode. Also the change is not proportional to the change in stiffness. However it is always the case that as the stiffness of the coupling is reduced so are the natural frequencies.

9.7.2 Vibration absorbers: Damping a clamped/free bar.

A clamped/free bar is an approximate representation of many real engineering structures that are susceptible to transverse vibration. For example a boring bar used in machining operations and also a wind turbine tower. Such clamped/free bars often have little damping and hence a large resonant response. It has been shown previously that a vibration absorber is a useful way of reducing resonant amplitudes.

Vibration absorber at the end of a bar

Consider a uniform circular clamped/free bar as shown in figure 9.18. The vibration absorber consists of a spring, mass and hysteretic damper.

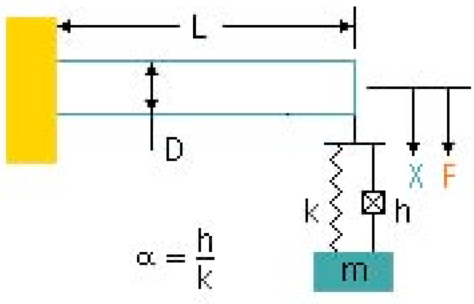


Figure 9.18 Clamped/free bar with a vibration absorber added.

For this example the receptance at the end with the absorber added is given by,

$$\frac{1}{\alpha_{11}} = \frac{1}{\beta_{11}} + \frac{1}{\gamma_{11}}$$

where β_{11}, the tip receptance of the bar, is given by,

$$\beta_{11} = -\frac{(\cos\lambda L \sinh\lambda L - \sin\lambda L \cosh\lambda L)}{EI\lambda^3(\cos\lambda L \cosh\lambda L + 1)}$$

$$\text{where } \lambda^4 = \frac{\rho A \omega^2}{EI}$$

and the absorber receptance is given by,

$$\gamma_{11} = \frac{1}{k(1 + i\alpha)} - \frac{1}{m\omega^2}$$

A computer program is required to optimise the absorber for any given bar. In practice the spring and damping are supplied by a rubber element and alpha is fixed. Thus for a given absorber mass (the larger the better) there is only the parameter k to optimise. For a bar of length L = 1.2m, diameter D = 0.15m and for rubber with α = 0.25 and an absorber mass of 2kg, the optimum response is shown in figure 9.19.

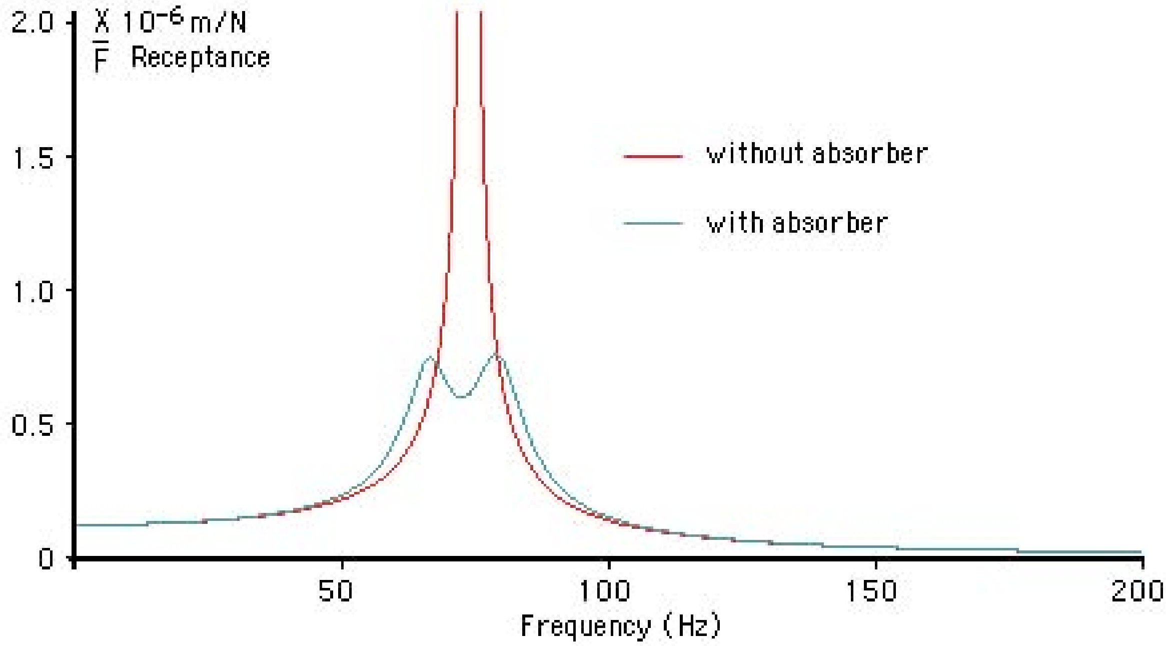


Figure 9.19 Bar response with and without absorber (k =3.93 10^5 N/m)

Program allowing parameter variation

Optimising program

It should be noted that the bar without absorber has an infinite response at resonance as no damping has been included when determining the transverse receptances of bars. In practice such bars have little damping so that the improvement that may be achieved by using an absorber is significant.

Using a second bar.

If a bar is rotating there is a practical problem with a vibration absorber in that it may become unbalanced and actually cause vibration. It has been found useful to use a second bar. This may be mounted within the original bar if it is a tube. For ease of visualisation we will consider to clamped/free bars of the same length mounted in parallel with a rubber insert place between them at the free end (see figure 9.20). We will again use an inset with hysteretic damping as in the vibration absorber.

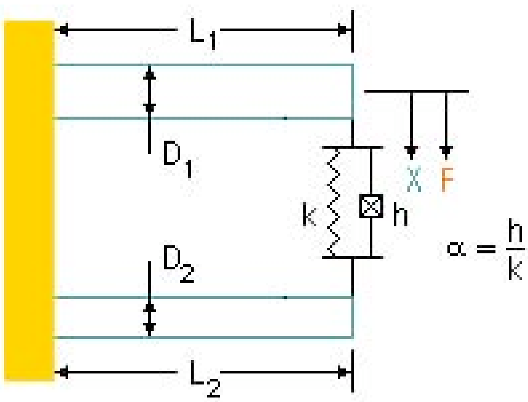


Figure 9.20 Parallel bars with a damping insert

Before presenting the analysis it is informative to consider why the use of a second bar may help damp the first bar. When a vibration absorber was used the damping insert was connected to a mass and this is replaced by a clamped free bar. If the receptances of the mass and bar are plotted on the same graph (figure 9.21) then a striking feature is observed. Above the first natural frequency of the bar both receptances are negative. In effect the bar behaves as a large mass and thus in this range could be used as a vibration absorber mass.

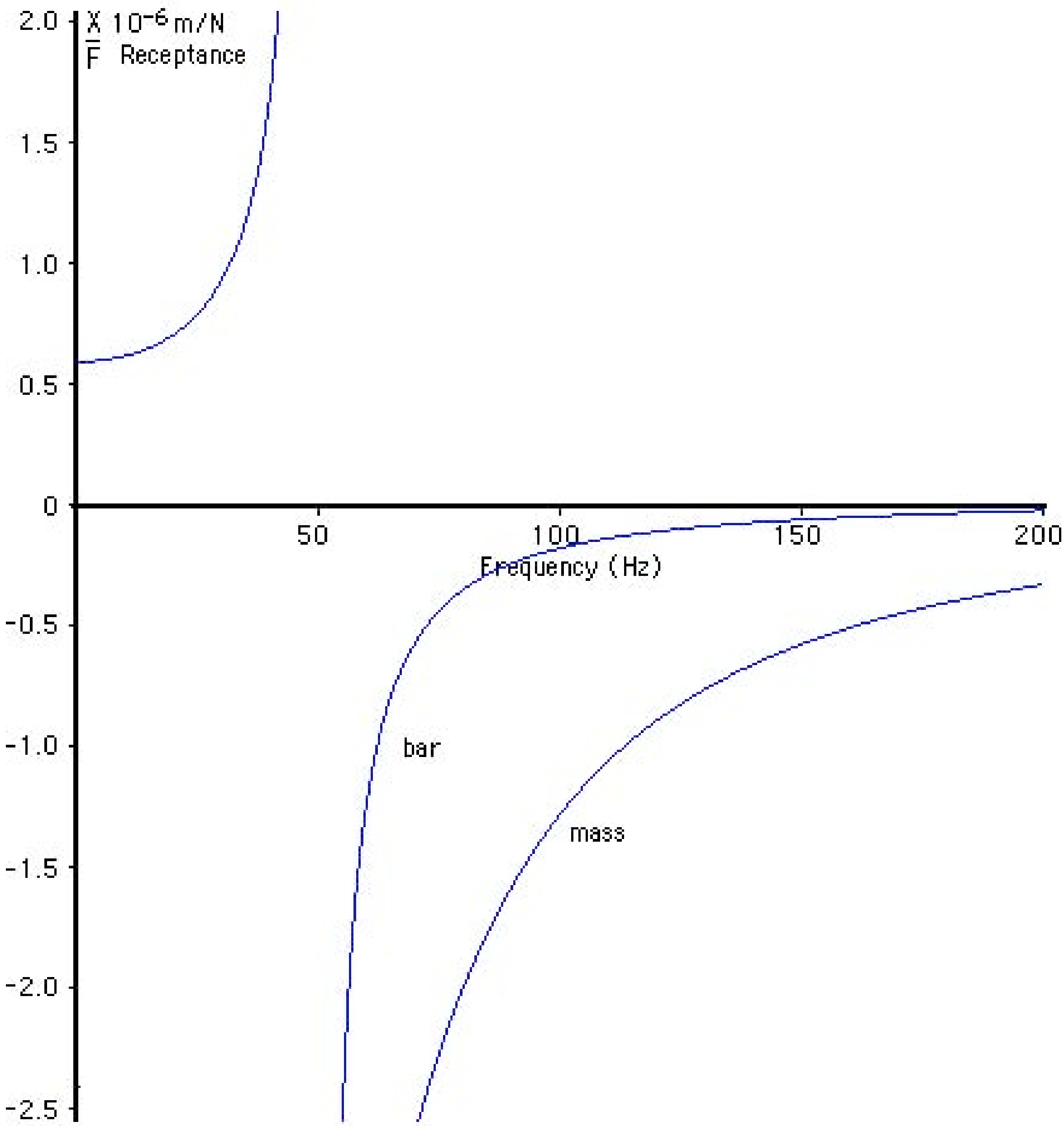


Figure 9.21 Comparison of the receptances of a mass and a bar.

It is necessary to choose the diameter of the "absorber" bar so that in the frequency range that includes the resonance of the main bar the secondary bar is behaving as a mass. This in effect means that the natural frequency of the "absorber" bar has to be lower than that of the main bar. (In passing it should be noted that we could use the region beyond the second natural frequency of the "absorber" bar. Also it might be possible to damp more than the first resonance of the main bar using the absorber bar)

The receptance analysis follows that of a bar with a vibration absorber, except that the mass receptance is replace by that of a clamped free bar. So that

$$\gamma_{22} = -\frac{(\cos\lambda L \sinh\lambda L - \sin\lambda L \cosh\lambda L)}{EI\lambda^3(\cos\lambda L \cosh\lambda L + 1)}$$

and

$$\gamma_{11} = \frac{1}{k(1 + i\alpha)} + \gamma_{22}$$

and finally

$$\frac{1}{\alpha_{11}} = \frac{1}{\beta_{11}} + \frac{1}{\gamma_{11}}$$

An improved response is shown in figure 9.22. This is for the same main bar as used for the vibration absorber example, L = 1.2m and D = 0.15m. The magnitude of the receptance of the "absorber bar" (diameter 0.1m) is shown. The damping ratio α was 0,4 and for the results shown $k = 1.38\ 10^6$ N/m. This stiffness is much greater than that for the vibration absorber as the effective mass of the "absorber" bar is much greater than the 2kg mass used for the vibration absorber. Figure 9.21 shows the relevant receptances.

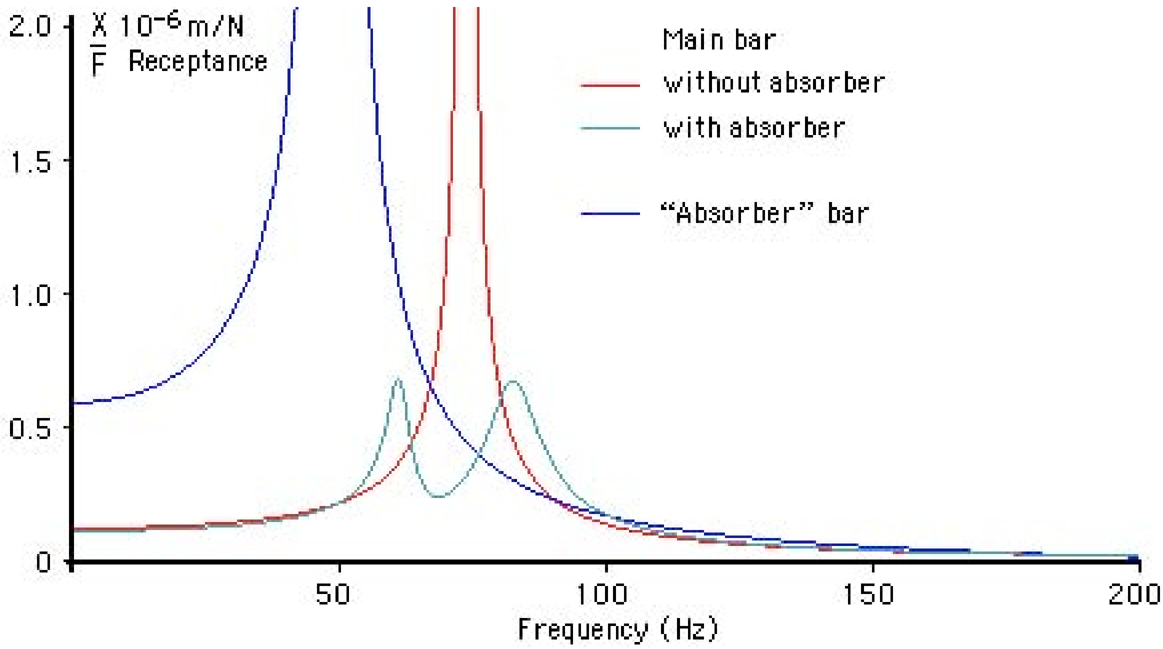


Figure 9.22 Response using an "absorber" bar and damping insert.

Program allowing parameter variation

Optimisation for a selected absorber bar diameter

9.8 Complex systems

If two bars are added together and we are interested in either the axial or torsional vibration receptances then there is only one coordinate that has to be used to join the bars. However if the transverse vibration is of interest then the coupling involves two coordinates, the transverse displacement/shear force and the slope/bending moment. The receptances for the addition under these conditions are derived in Appendix 3. The maths again becomes extensive. However, the systems approach can be used with a matrix formulation and this lends itself to programming on a computer.

9.9 Conclusions

The systems approach using receptances has been described. This method has the advantage of generating complex systems by the addition of much simpler known systems. The use of a vibration absorber system has again been shown to be useful.

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