CHAPTER 5: Multi degree-of-freedom vibration.

Initially we will just increase the number of degrees-of-freedom to three. Thereafter we will increase the numbers of degrees-of-freedom. It will soon become apparent that the maths will be overwhelming for systems with many degrees-of-freedom. For this reason computers are used extensively in vibration analysis. However it is important to have a good understanding of vibrating systems even when they have many degrees-of-freedom. The results from solving the equations of motion will be given but the mathematical detail will be omitted. In the next chapter more detail will be given of the mathematical solutions of the equations. At this stage it is the object to aim for understanding of the results rather than the means used to obtain them.

5.1 Three degrees-of-freedom

There are numerous examples that involve 3 degrees-of-freedom. The one chosen is an extension of the beam problem considered in chapter 3. Consider the beam mounted on springs at each end and an additional spring/mass system mounted on the beam as shown in figure 5.1.

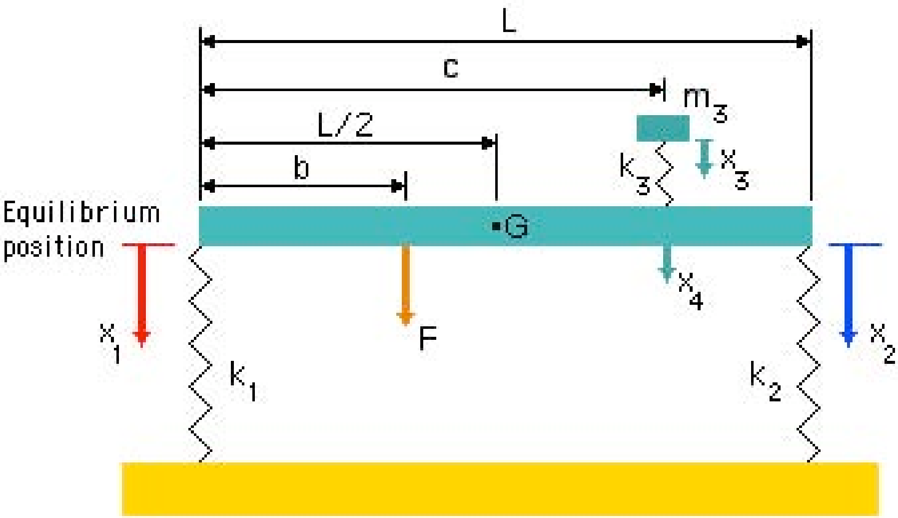


Figure 5.1 System with 3 degrees of freedom

We need to introduce another variable x_3 to locate the position of the mass m_3. The coordinate x_4 is introduced to aid in the deriving of the equations of motion but can be found from x_1 and x_2 so does not imply a fourth degree-of-freedom. From geometry,

$$x_4 = x_1 + \frac{c}{L}\left(x_2 - x_1\right) = x_1\left(1 - \frac{c}{L}\right) + \frac{c}{L}x_2 \quad \text{.............................} \quad (5.1)$$

The position of the centre of mass (x) is given by,

$$x = \left(x_1 + x_2\right)/2 \quad \text{...} \quad (5.2)$$

and for small angles,

$$\theta = \left(x_1 - x_2\right)/L \quad \text{..} \quad (5.3)$$

Free body diagrams ignoring gravity forces and working relative to the static equilibrium position are shown in figure 5.2.

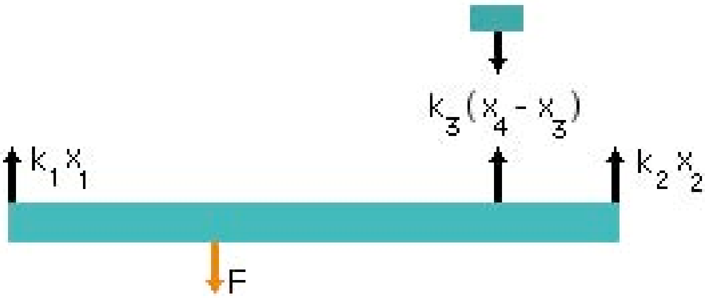


Figure 5.2 Free body diagrams.

Using Newton's second law for the linear motion of the centre of mass gives

$$mx'' = -k_1x_1 - k_2x_2 - k_3(x_4 - x_3) + F \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5.4)$$

and the angular motion about the centre of mass gives

$$\frac{1}{12}mL^2\theta'' = -k_1x_1L/2 + k_2x_2L/2 + k_3(x_4 - x_3)(c - L/2) + F(L/2 - b) \quad \ldots\ldots\ldots \quad (5.5)$$

and for the mass m_3,

$$m_3x_3'' = k_3(x_4 - x_3) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5.6)$$

If in equations (5.4) - (5.6) we substitute for x from (5.2), θ from(5.3) and x_4 from (5.1) we obtain 3 equations with just the three degrees-of-freedom x_1, x_2 and x_3. Thus after much maths,

$$m(x_1'' + x_2'') = 2\left(x_1\left(-k_1 - k_3\left(1 - \frac{c}{L}\right)\right) + x_2\left(-k_2 - k_3\frac{c}{L}\right) + k_3x_3 + F\right) \quad \ldots\ldots\ldots \quad (5.7)$$

and

$$m(x_1'' - x_2'') =$$
$$12\left(x_1\left(-\frac{k_1}{2} + k_3\left(1 - \frac{c}{L}\right)\left(\frac{c}{L} - \frac{1}{2}\right)\right) + x_2\left(\frac{k_2}{2} + k_3\frac{c}{L}\left(\frac{c}{L} - \frac{1}{2}\right)\right) + x_3\left(-k_3\left(\frac{c}{L} - \frac{1}{2}\right)\right) + F\left(\frac{1}{2} - \frac{b}{L}\right)\right)$$
$$\ldots\ldots \quad (5.8)$$

Adding (5.7) and (5.8)

$$mx_1'' =$$
$$\left(x_1\left(-4k_1 - k_3\left(1 - \frac{c}{L}\right)\left(4 - 6\frac{c}{L}\right)\right) + x_2\left(2k_2 - k_3\frac{c}{L}\left(4 - 6\frac{c}{L}\right)\right) + x_3k_3\left(4 - 6\frac{c}{L}\right) + F\left(4 - \frac{6b}{L}\right)\right)$$
$$\ldots\ldots \quad (5.9)$$

Subtracting (5.7) and (5.8)

$$mx_2'' =$$

$$\left(x_1\left(2k_1 + k_3\left(1-\frac{c}{L}\right)\left(2-6\frac{c}{L}\right)\right) + x_2\left(-4k_2 + k_3\frac{c}{L}\left(2-6\frac{c}{L}\right)\right) - x_3k_3\left(2-6\frac{c}{L}\right) - F\left(2-\frac{6b}{L}\right)\right)$$

..... (5.10)

and $$m_3x_3'' = k_3\left(x_1\left(1-\frac{c}{L}\right) + \frac{c}{L}x_2 - x_3\right) \quad \text{.......................................} \quad (5.11)$$

Equations (5.9) - (5.11) may be written in matrix form

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix} + \begin{bmatrix} 4k_1 + k_3\left(1-\frac{c}{L}\right)\left(4-6\frac{c}{L}\right) & \left(-2k_2 + k_3\frac{c}{L}\left(4-6\frac{c}{L}\right)\right) & -k_3\left(4-6\frac{c}{L}\right) \\ -2k_1 - k_3\left(1-\frac{c}{L}\right)\left(2-6\frac{c}{L}\right) & 4k_2 - k_3\frac{c}{L}\left(2-6\frac{c}{L}\right) & -k_3\left(2-6\frac{c}{L}\right) \\ -k_3\left(1-\frac{c}{L}\right) & -k_3\frac{c}{L} & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} F\left(4-\frac{6b}{L}\right) \\ -F\left(2-\frac{6b}{L}\right) \\ 0 \end{bmatrix}$$

.. (5.12)

However the stiffness matrix is not symmetrical. If we had used the Lagrange equations the derivation would have been as follows,

$$T = \frac{1}{2}mx_G^2 + \frac{1}{2}I_G\theta^2 + \frac{1}{2}m_3x_3^2$$

$$I_G = \frac{mL^2}{12} \text{ (long slender beam)}$$

$$\text{and } V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 + \frac{1}{2}k_3(x_3 - x_4)^2$$

We need x_1, x_2 and x_4 in terms of x and θ.

From geometry $x = (x_1 + x_2)/2$ and $\theta = (x_1 - x_2)/L$ so that ,

$$x_1 = x + \frac{\theta L}{2} \text{ and } x_2 = x - \frac{\theta L}{2}$$

and

$$x_4 = x - \theta\left(c - \frac{L}{2}\right)$$

$$\text{so that } V = \frac{1}{2}k_1\left(x + \frac{\theta L}{2}\right)^2 + \frac{1}{2}k_2\left(x - \frac{\theta L}{2}\right)^2 + \frac{1}{2}k_3\left(x_3 - x + \theta\left(c - \frac{L}{2}\right)\right)^2$$

for the $q_i = x$

$$\frac{\partial V}{\partial x}=k_1\left(x+\frac{\theta L}{2}\right)+k_2\left(x-\frac{\theta L}{2}\right)-k_3\left(x_3-x+\theta\left(c-\frac{L}{2}\right)\right)$$

$$\text{and}\quad\left(\frac{\partial T}{\partial x}\right)=m_1 x\quad\text{so that}\quad\frac{d}{dt^2}\left(\frac{\partial T}{\partial x}\right)=m_1 x''$$

$$\therefore m_1 x''+k_1\left(x+\frac{\theta L}{2}\right)+k_2\left(x-\frac{\theta L}{2}\right)-k_3\left(x_3-x+\theta\left(c-\frac{L}{2}\right)\right)=0$$

for the $q_i=\theta$

$$\frac{\partial V}{\partial \theta}=k_1\frac{L}{2}\left(x+\frac{\theta L}{2}\right)-k_2\frac{L}{2}\left(x-\frac{\theta L}{2}\right)+k_3\left(c-\frac{L}{2}\right)\left(x_3-x+\theta\left(c-\frac{L}{2}\right)\right)$$

$$\text{and}\quad\left(\frac{\partial T}{\partial \theta}\right)=I_G\theta\quad\text{so that}\quad\frac{d}{dt^2}\left(\frac{\partial T}{\partial \theta}\right)=I_G\theta''$$

$$I_G\theta''+k_1\frac{L}{2}\left(x+\frac{\theta L}{2}\right)-k_2\frac{L}{2}\left(x-\frac{\theta L}{2}\right)+k_3\left(c-\frac{L}{2}\right)\left(x_3-x+\theta\left(c-\frac{L}{2}\right)\right)=0$$

for the $q_i=x_3$

$$\frac{\partial V}{\partial x_3}=k_3\left(x_3-x+\theta\left(c-\frac{L}{2}\right)\right)$$

$$\text{and}\quad\left(\frac{\partial T}{\partial x_3}\right)=m_3 x_3\quad\text{so that}\quad\frac{d}{dt^2}\left(\frac{\partial T}{\partial x_3}\right)=m_3 x_3''$$

$$m_3 x_3''+k_3\left(x_3-x+\theta\left(c-\frac{L}{2}\right)\right)=0$$

In matrix form

$$\begin{bmatrix} m_1 & 0 & 0\\ 0 & I_G & 0\\ 0 & 0 & m_3\end{bmatrix}\begin{bmatrix} x''\\ \theta''\\ x_3''\end{bmatrix}$$

$$+\begin{bmatrix} k_1+k_2+k_3 & (k_1-k_2)L/2-k_3\left(c-\frac{L}{2}\right) & -k_3\\ (k_1-k_2)L/2-k_3\left(c-\frac{L}{2}\right) & (k_1+k_2)L^2/4+k_3\left(c-\frac{L}{2}\right)^2 & k_3\left(c-\frac{L}{2}\right)\\ -k_3 & k_3\left(c-\frac{L}{2}\right) & k_3\end{bmatrix}\begin{bmatrix} x\\ \theta\\ x_3\end{bmatrix}=0$$

This may be abbreviated to

$$[\mathbf{M}][\ddot{\mathbf{X}}]+[\mathbf{K}][\mathbf{X}]=0 \quad\text{...}\quad (5.13)$$

and in this case both the mass and stiffness matrices are symmetrical. Using the method described in chapter 4 we can find the eigen values and vectors and hence the natural frequencies and mode shapes.

5.1.1 Natural frequencies

If we put $\lambda = \omega^2$ and $[\mathbf{X}] = \{u\}$ we obtain the characteristic or eigen value equation of the system:

$$[[\mathbf{K}] - \lambda[\mathbf{M}]]\{u\} = 0 \quad \ldots\ldots\ldots\ldots \quad (4.17)$$

For non-zero solutions for $\{u\}$

$$\det[[\mathbf{K}] - \lambda[\mathbf{M}]] = 0 \quad \ldots\ldots\ldots\ldots \quad (4.18)$$

Making the substitutions,

$$\det\left[\begin{bmatrix} k_1+k_2+k_3 & (k_1-k_2)L/2-k_3\left(c-\frac{L}{2}\right) & -k_3 \\ (k_1-k_2)L/2-k_3\left(c-\frac{L}{2}\right) & (k_1+k_2)L^2/4+k_3\left(c-\frac{L}{2}\right)^2 & k_3\left(c-\frac{L}{2}\right) \\ -k_3 & k_3\left(c-\frac{L}{2}\right) & k_3 \end{bmatrix} - \lambda\begin{bmatrix} m_1 & 0 & 0 \\ 0 & I_G & 0 \\ 0 & 0 & m_3 \end{bmatrix}\right] = 0$$

This is a cubic equation in λ so that there will be three natural frequencies. When each of these values is substituted in (4.17) the associated eigen vector $\{u\}$ is found and this is the mode shape. For ease of comprehension it is convenient to represent the mode shapes in terms of the coordinates $x_1 = x + \frac{\theta L}{2}$ and $x_2 = x - \frac{\theta L}{2}$ and x_3 as shown in figure 5.3.

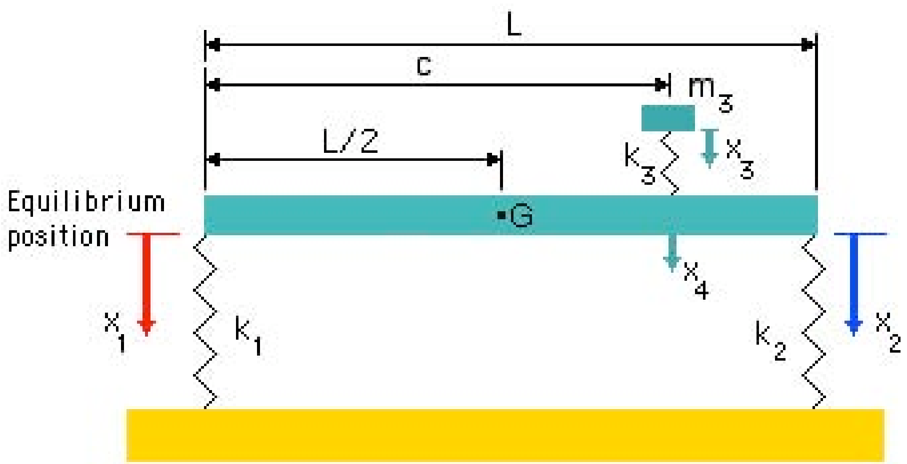
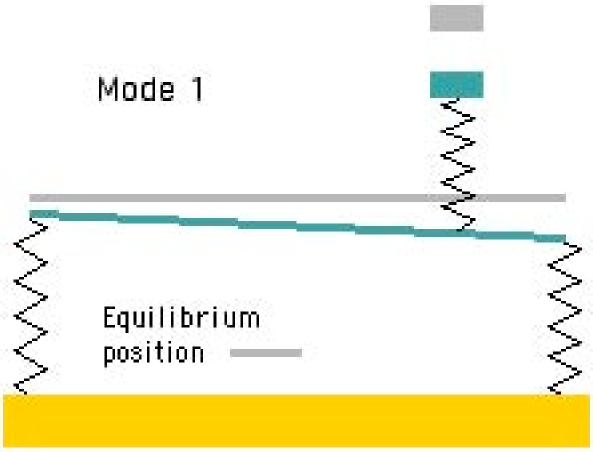
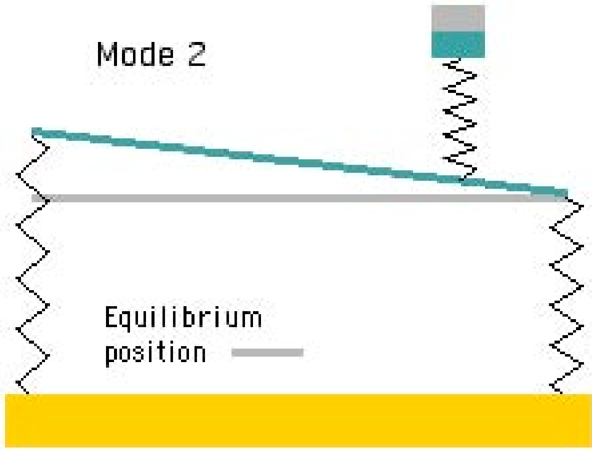


Figure 5.3 Coordinate definition

The amount of maths is significant so a program has been written to calculate the natural frequencies and the associated mode shapes. The results are shown in figure 5.4 for

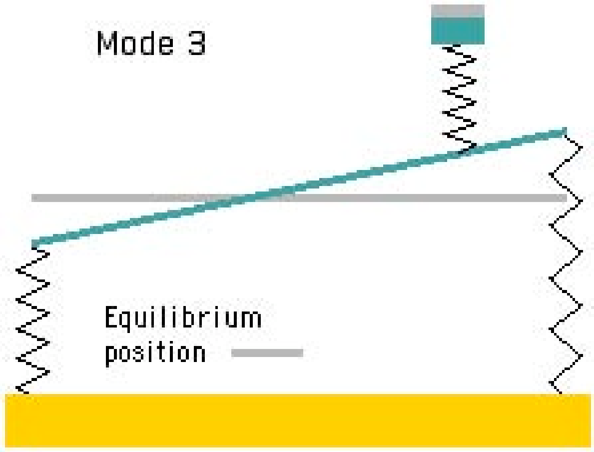
$$m_1 = 0.5\text{kg};\ k_1 = 10\text{N/m};\ k_2 = 10\text{N/m};\ m_3 = 0.2\text{kg};\ k_3 = 10\text{N/m};\ c/L = 0.8$$





$$\omega_{n1} = 0.4857 \text{ Hz} \qquad \{u_1\} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.2683 \\ 0.6008 \\ 1.0 \end{bmatrix}$$

$$\omega_{n2} = 0.9106 \text{ Hz} \qquad \{u_2\} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.0866 \\ -0.4227 \end{bmatrix}$$



It is helpful to see the modes animated.

It is also possible to investigate the effects of changing the parameter values.

$$\omega_{n3} = 1.412 \text{ Hz} \qquad \{u_2\} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.699 \\ 1.0 \\ -0.2247 \end{bmatrix}$$

Figure 5.4 Natural frequencies and mode shapes

<u>5.1.2 Forced vibration</u>

Consider again the equations of motion

$$mx_1'' = \left(x_1\left(-4k_1 - k_3\left(1 - \frac{c}{L}\right)\left(4 - 6\frac{c}{L}\right)\right) + x_2\left(2k_2 - k_3\frac{c}{L}\left(4 - 6\frac{c}{L}\right)\right) + x_3k_3\left(4 - 6\frac{c}{L}\right) + F\left(4 - \frac{6b}{L}\right)\right) \quad \ldots\ldots \quad (5.9)$$

$$mx_2'' = \left(x_1\left(2k_1 + k_3\left(1 - \frac{c}{L}\right)\left(2 - 6\frac{c}{L}\right)\right) + x_2\left(-4k_2 + k_3\frac{c}{L}\left(2 - 6\frac{c}{L}\right)\right) - x_3k_3\left(2 - 6\frac{c}{L}\right) - F\left(2 - \frac{6b}{L}\right)\right) \quad \ldots\ldots \quad (5.10)$$

and $$m_3x_3'' = k_3\left(x_1\left(1 - \frac{c}{L}\right) + \frac{c}{L}x_2 - x_3\right) \quad \ldots\ldots \quad (5.11)$$

For steady state vibration the exiting force is F$\sin\omega t$ and $x_1=X_1\sin\omega t$, $x_2=X_2\sin\omega t$ and $x_3=X_3\sin\omega t$. This is because there is no damping. Substituting in (5.9), (5.10) and (5.11)

$$X_1\left(4k_1+k_3\left(1-\frac{c}{L}\right)\left(4-6\frac{c}{L}\right)-m\omega^2\right)+X_2\left(-2k_2+k_3\frac{c}{L}\left(4-6\frac{c}{L}\right)\right)+X_3k_3\left(-4+6\frac{c}{L}\right)=F\left(4-\frac{6b}{L}\right)$$

$$X_1\left(-2k_1-k_3\left(1-\frac{c}{L}\right)\left(2-6\frac{c}{L}\right)\right)+X_2\left(4k_2-k_3\frac{c}{L}\left(2-6\frac{c}{L}\right)-m\omega^2\right)+X_3k_3\left(2-6\frac{c}{L}\right)=-F\left(2-\frac{6b}{L}\right)$$

and

$$-k_3X_1\left(1-\frac{c}{L}\right)-k_3\frac{c}{L}X_2+X_3\left(k_3-m_3\omega^2\right)=0$$

and writing these in matrix form

$$\begin{bmatrix} 4k_1+k_3\left(1-\frac{c}{L}\right)\left(4-6\frac{c}{L}\right)-m\omega^2 & -2k_2+k_3\frac{c}{L}\left(4-6\frac{c}{L}\right) & k_3\left(-4+6\frac{c}{L}\right) \\ -2k_1-k_3\left(1-\frac{c}{L}\right)\left(2-6\frac{c}{L}\right) & 4k_2-k_3\frac{c}{L}\left(2-6\frac{c}{L}\right)-m\omega^2 & k_3\left(2-6\frac{c}{L}\right) \\ -k_3\left(1-\frac{c}{L}\right) & -k_3\frac{c}{L} & k_3-m_3\omega^2 \end{bmatrix}\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}=\begin{bmatrix}F\left(4-\frac{6b}{L}\right)\\-F\left(2-\frac{6b}{L}\right)\\0\end{bmatrix}$$

....................5.14)

after much maths it is possible to obtain the responses $\frac{X_1}{F}$, $\frac{X_2}{F}$ and $\frac{X_3}{F}$

and hence the response X/F at any position Xpos along the beam may be determined

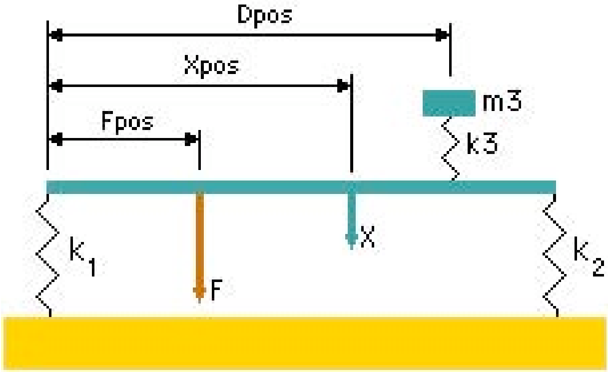


Figure 5.5 Excitation and response positions.

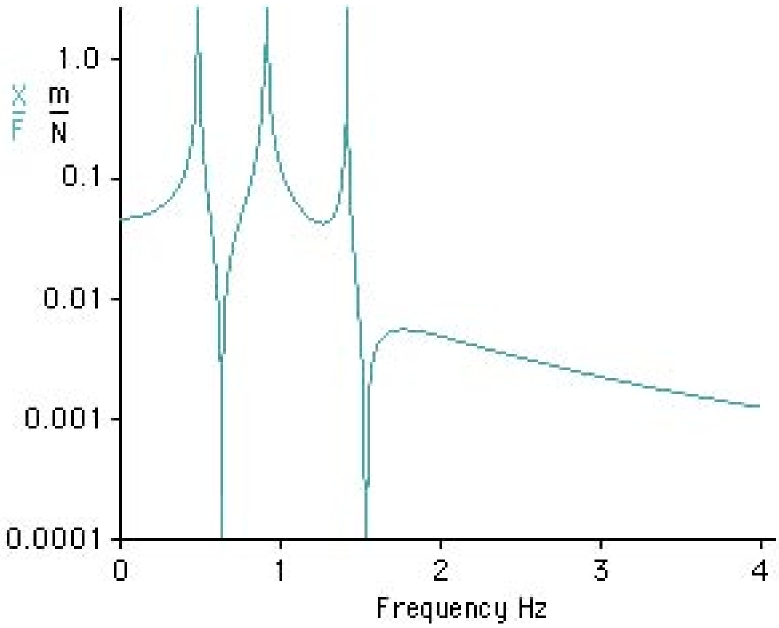


Figure 5.6 Response of beam

The response shown in figure 5.6 is for the same values as used previously with the excitation (Fpos) and response position (Xpos) as shown,

$$\text{Fpos} = 0.3L;\ \text{Xpos} = 0.6L;\ \text{Dpos} = 0.8L;$$

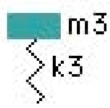
Note that there are infinite resonances at the natural frequencies because there is no damping.

To examine the effect of varying the parameters run the program

Note that the resonant frequencies do not change with Fpos or Xpos.

To get a better understanding of this system an animation program has been written.

This shows the system vibrating and allows the excitation frequency to be set by dragging the blue line on the graph (see figure 5.7). As shown this is set to a minimum on the graph. At this frequency the beam has no vibration amplitude for the Xpos value. The beam is vibrating as if there was a pivot at Xpos. It is of interest to investigate when the additional spring/mass acts as a detuner.



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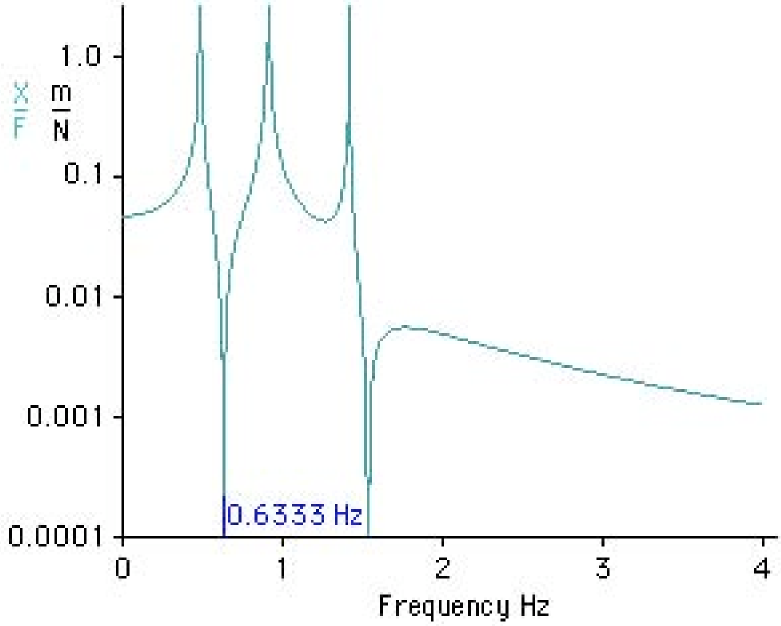


Figure 5.7 Excitation frequency variation.

The detuned frequency is given by $\omega = \sqrt{\dfrac{k_3}{m_3}} = \sqrt{\dfrac{10}{0.5}} = 4.472\text{rad / s} = 0.7118\text{Hz}$. At this frequency the beam will have no amplitude of vibration where the detuner is attached. This is because the detuner exhibits infinite dynamic stiffness at this frequency. If Xpos = Dpos then the response will be zero at this frequency. However note that the response graph is drawn at discrete frequencies and the frequency for zero response will be close to 0.7118 Hz but not exactly this value.

If Xpos and Dpos are kept the same but varied it will be found that the response is always zero at 0.7118 Hz.

5.1.3 Effects of damping

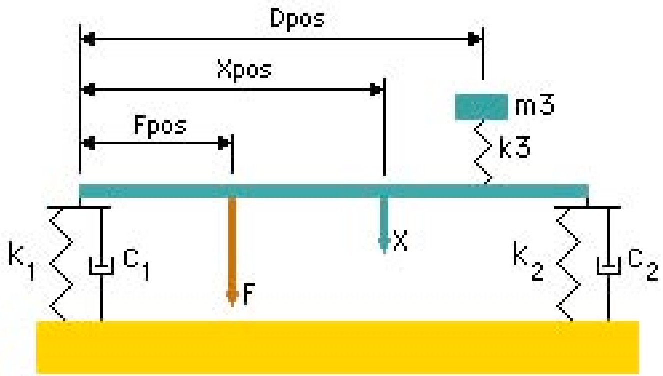


Figure 5.6 Three degree of freedom system with damping.

With damping as shown above we retain the detuner as it has no damping. The transient response involves a decaying vibration and is a major maths exercise to determine. For the steady state solution for sinusoidal excitation the previous undamped solution (equation

(5.14)) is easily modified. When viscous damping is included in parallel with the springs, wherever there was a kx term we will now have $kx + cx'$ so that equation (5.14) becomes,

$$\begin{bmatrix} 4(k_1+i\omega c_1)+k_3\left(1-\frac{c}{L}\right)\left(4-6\frac{c}{L}\right)-m\omega^2 & -2(k_2+i\omega c_2)+k_3\frac{c}{L}\left(4-6\frac{c}{L}\right) & k_3\left(-4+6\frac{c}{L}\right) \\ -2(k_1+i\omega c_1)-k_3\left(1-\frac{c}{L}\right)\left(2-6\frac{c}{L}\right) & 4(k_2+i\omega c_2)-k_3\frac{c}{L}\left(2-6\frac{c}{L}\right)-m\omega^2 & k_3\left(2-6\frac{c}{L}\right) \\ -k_3\left(1-\frac{c}{L}\right) & -k_3\frac{c}{L} & k_3-m_3\omega^2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$= \begin{bmatrix} F\left(4-\frac{6b}{L}\right) \\ -F\left(2-\frac{6b}{L}\right) \\ 0 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5.15)$$

after even more maths it is possible to obtain the responses $\frac{X_1}{F}$, $\frac{X_2}{F}$ and $\frac{X_3}{F}$ (these will be complex, ie having amplitude and phase).

The response X/F at any position Xpos along the beam may be determined. The response shown is for

$m_1 = 1.0\text{kg};\ k_1 = 10\text{N/m};\ k_2 = 10\text{N/m};$
$c_1 = c_2 = 0.3\text{Ns/m};$
$\text{Fpos} = 0.3L;\ \text{Xpos} = 0.6L;\ \text{Dpos} = 0.8L;$
$m_3 = 0.5\text{kg};\ k_3 = 10\text{N/m};$

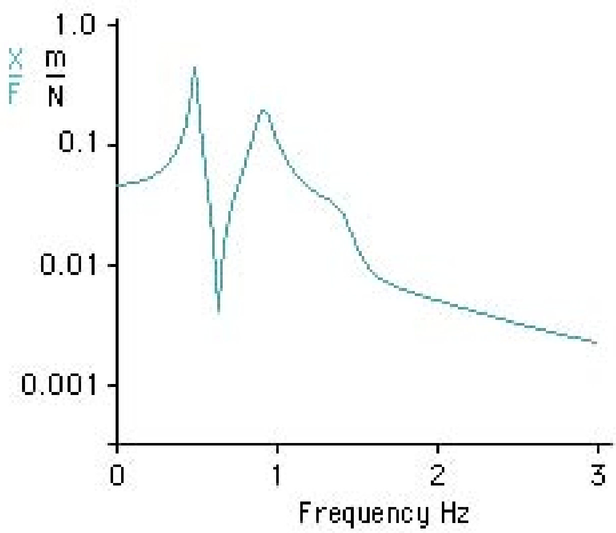


Figure 5.7 Response of damped system.

It is of interest to note that the highest natural frequency is barely visible. The effect of damping may be examined using the program and varying c_1 and c_2.

There is still a detuner with a detuned frequency $\omega = \sqrt{\frac{k_3}{m_3}} = \sqrt{\frac{10}{0.5}} = 4.472\text{rad/s} = 0.7118\text{Hz}$.

At this frequency the beam will have no amplitude of vibration where the detuner is attached. The animation program will allow this to be checked.

As for the undamped system, if Xpos = Dpos then the response will be zero at this frequency. If Xpos and Dpos are kept the same but varied it will be found that the response is always zero at 0.7118 Hz.

5.2 Five degrees-of-freedom

5.2.1 Natural frequencies

To illustrate how the maths compounds and why computers are needed we will consider an axial system with five degrees-of-freedom as shown in figure 5.7.

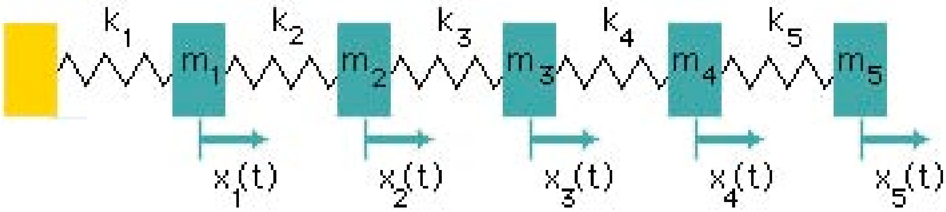


Figure 5.7 System with five degrees of freedom.

Using Newton's second law we obtain five equations of motion.

$$\begin{aligned}
m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1) \\
m_2 x_2'' &= -k_2(x_2 - x_1) + k_3(x_3 - x_2) \\
m_3 x_3'' &= -k_3(x_3 - x_2) + k_4(x_4 - x_3) \\
m_4 x_4'' &= -k_4(x_4 - x_3) + k_5(x_5 - x_4) \\
m_5 x_5'' &= -k_5(x_5 - x_4)
\end{aligned}$$

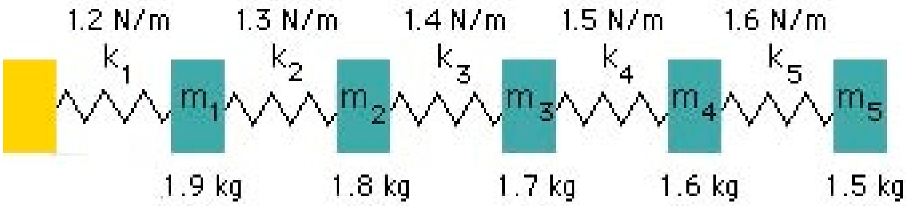
and if we rearrange and write these in matrix form we have,

$$\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 \\ 0 & 0 & 0 & 0 & m_4 \end{bmatrix} \begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \\ x_4'' \\ x_5'' \end{bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 & 0 & 0 & 0 \\ -k_2 & (k_2+k_3) & -k_3 & 0 & 0 \\ 0 & -k_3 & (k_3+k_4) & -k_4 & 0 \\ 0 & 0 & -k_4 & (k_4+k_5) & -k_5 \\ 0 & 0 & 0 & -k_5 & k_5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0 \quad \text{.....(5.16)}$$

This may be abbreviated to the standard form

$$[\mathbf{M}][\ddot{\mathbf{X}}] + [\mathbf{K}][\mathbf{X}] = 0 \quad \text{...} \quad (5.13)$$

As an example consider



The mass matrix $[\mathbf{M}]$ is then $\begin{bmatrix} 1.9 & 0 & 0 & 0 & 0 \\ 0 & 1.8 & 0 & 0 & 0 \\ 0 & 0 & 1.7 & 0 & 0 \\ 0 & 0 & 0 & 1.6 & 0 \\ 0 & 0 & 0 & 0 & 1.5 \end{bmatrix}$ kg

and the stiffness matrix $[\mathbf{K}]$ is then $\begin{bmatrix} 25 & -13 & 0 & 0 & 0 \\ -13 & 27 & -14 & 0 & 0 \\ 0 & -14 & 29 & -15 & 0 \\ 0 & 0 & -15 & 31 & -16 \\ 0 & 0 & 0 & -16 & 16 \end{bmatrix}$ N/m

Using the eigen value approach and a computer program we find the five modes as shown in Figure 5.8.

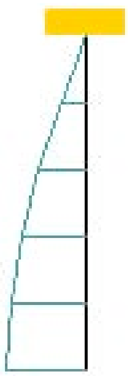
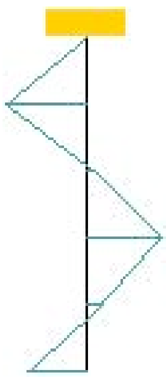
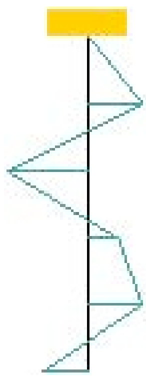
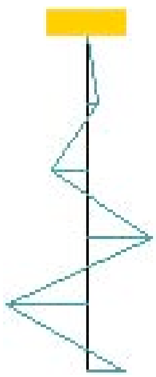
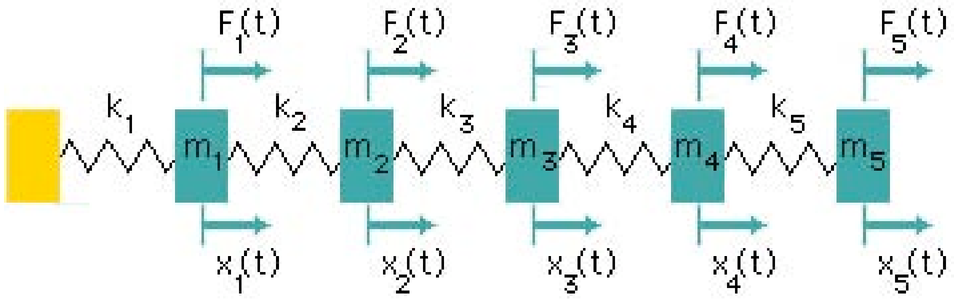
Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
$\lambda_1 = 0.0652 \frac{\text{rad}^2}{\text{sec}^2}$	$\lambda_2 = 0.5619 \frac{\text{rad}^2}{\text{sec}^2}$	$\lambda_3 = 1.392 \frac{\text{rad}^2}{\text{sec}^2}$	$\lambda_4 = 2.295 \frac{\text{rad}^2}{\text{sec}^2}$	$\lambda_5 = 3.212 \frac{\text{rad}^2}{\text{sec}^2}$
$\omega_{n1} = 0.2553 \frac{\text{rad}}{\text{sec}}$	$\omega_{n2} = 0.7496 \frac{\text{rad}}{\text{sec}}$	$\omega_{n3} = 1.18 \frac{\text{rad}}{\text{sec}}$	$\omega_{n4} = 1.515 \frac{\text{rad}}{\text{sec}}$	$\omega_{n5} = 1.792 \frac{\text{rad}}{\text{sec}}$
$\{u\}_1 = \begin{bmatrix} 0.3309 \\ 0.6048 \\ 0.8085 \\ 0.9389 \\ 1.0 \end{bmatrix}$	$\{u\}_2 = \begin{bmatrix} 0.9075 \\ 1.0 \\ 0.3635 \\ -0.462 \\ -0.9764 \end{bmatrix}$	$\{u\}_3 = \begin{bmatrix} 1.0 \\ -0.1116 \\ -0.944 \\ -0.2315 \\ 0.7588 \end{bmatrix}$	$\{u\}_4 = \begin{bmatrix} -0.6988 \\ 1.0 \\ -0.373 \\ -0.6843 \\ 0.5943 \end{bmatrix}$	$\{u\}_5 = \begin{bmatrix} -0.1603 \\ 0.4442 \\ -0.8289 \\ 1.0 \\ -0.4972 \end{bmatrix}$
				

Figure 5.8 Natural frequencies and mode shapes of five degree of freedom system.

The program used to calculate these values is available and the effect of varying parameter values may be investigated.

5.2.2 Forced vibration

Consider forces applied to each of the masses. This is a common approach as then if we wish to have a force applied to only one mass we may set the others to zero.



Using Newton's second law we obtain five equations of motion.

$$
\begin{gathered}
m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1) + F_1 \\
m_2 x_2'' = -k_2(x_2 - x_1) + k_3(x_3 - x_2) + F_2 \\
m_3 x_3'' = -k_3(x_3 - x_2) + k_4(x_4 - x_3) + F_3 \\
m_4 x_4'' = -k_4(x_4 - x_3) + k_5(x_5 - x_4) + F_4 \\
m_5 x_5'' = -k_5(x_5 - x_4) + +F_5
\end{gathered}
$$

and if we rearrange and write these in matrix form we have,

$$
\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 \\ 0 & 0 & 0 & 0 & m_4 \end{bmatrix}
\begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \\ x_4'' \\ x_5'' \end{bmatrix}
+
\begin{bmatrix} (k_1+k_2) & -k_2 & 0 & 0 & 0 \\ -k_2 & (k_2+k_3) & -k_3 & 0 & 0 \\ 0 & -k_3 & (k_3+k_4) & -k_4 & 0 \\ 0 & 0 & -k_4 & (k_4+k_5) & -k_5 \\ 0 & 0 & 0 & -k_5 & k_5 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}
=
\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{bmatrix}
$$

As an example consider steady state sinusoidal excitation with $F_1(t) = F_1 \sin \omega t$ and $x_1 = X_1 \sin\omega t$, $x_2 = X_2 \sin\omega t$, $x_3 = X_3 \sin\omega t$, $x_4 = X_4 \sin\omega t$ and $x_5 = X_5 \sin\omega t$. We then obtain,

$$
\begin{bmatrix} (k_1+k_2)-m_1\omega^2 & -k_2 & 0 & 0 & 0 \\ -k_2 & (k_2+k_3)-m_2\omega^2 & -k_3 & 0 & 0 \\ 0 & -k_3 & (k_3+k_4)-m_3\omega^2 & -k_4 & 0 \\ 0 & 0 & -k_4 & (k_4+k_5)-m_4\omega^2 & -k_5 \\ 0 & 0 & 0 & -k_5 & k_5-m_5\omega^2 \end{bmatrix}
\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix}
=
\begin{bmatrix} F_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

..........5.16)

After much maths it is possible to obtain the responses $\frac{X_1}{F_1}$, $\frac{X_2}{F_1}$, $\frac{X_3}{F_1}$, $\frac{X_4}{F_1}$ and $\frac{X_5}{F_1}$.

A computer program has been written to obtain the responses. For the example considered previously and for which the natural frequencies have been found.

The five responses are shown in figure 5.9.

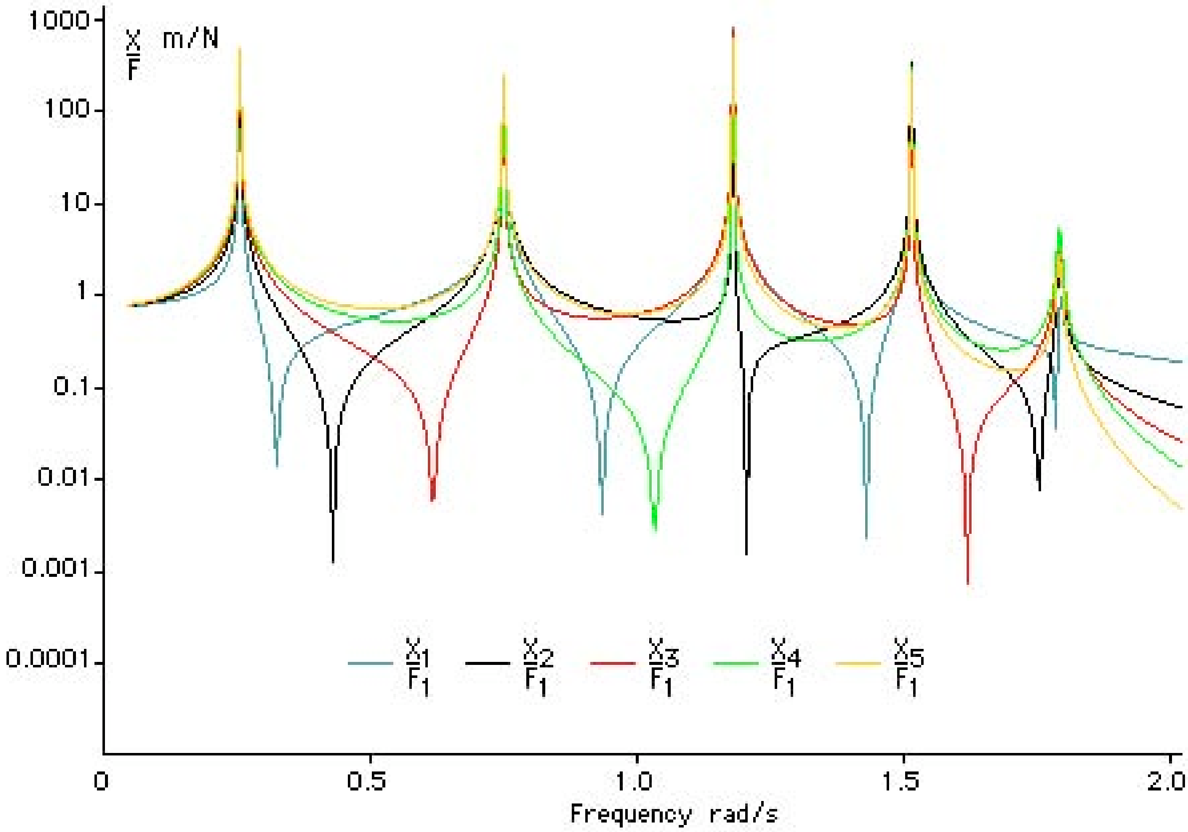


Figure 5.9 Responses of each mass caused by a force on mass 1.

The program that calculates the response may be run and the parameter values changed and also which mass is excited. It will be found that the number of anti-resonances also changes with the excitation position.

It should be noted that all responses have five resonances. These resonances are at the natural frequencies found previously and hence are at the same frequencies for each response. The 'height' of the resonances varies because the computer draws the graph at discrete frequencies. These discrete frequencies are not exactly at the natural frequencies (when the response would be infinite) and the responses are not the same off resonance.

The responses have anti-resonances, when the response tends to zero. The 'depth' of these anti-resonances again would be zero if the discrete frequency used by the program was at the 'correct' value. It is of interest to note from figure 5.9 that for $\frac{X_1}{F_1}$ there are 4 anti-resonances,

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for $\frac{X_2}{F_1}$ there are 3 anti-resonances, for $\frac{X_3}{F_1}$ there are 2 anti-resonances, for $\frac{X_4}{F_1}$ there is 1 anti-resonance, and for $\frac{X_5}{F_1}$ there are no anti-resonances, It is informative to view an animation of the motion at a particular excitation frequency. A program has been written to do this.

A typical output from the program is shown with an additional drawing of the system added at the bottom. A typical response is shown - $\frac{X_4}{F_1}$. This illustrates that this program also allows the steady state motion of the system to be observed at a particular frequency. This frequency may be changed by 'dragging' the blue line across the graph. In the diagram below the animation is frozen at an extreme of its motion. By comparison with the fixed drawing of the system the amplitudes can be deduced. As the frequency of the animation is set at 0.75 rad/s (the second natural frequency) the animation shows the second mode shape.

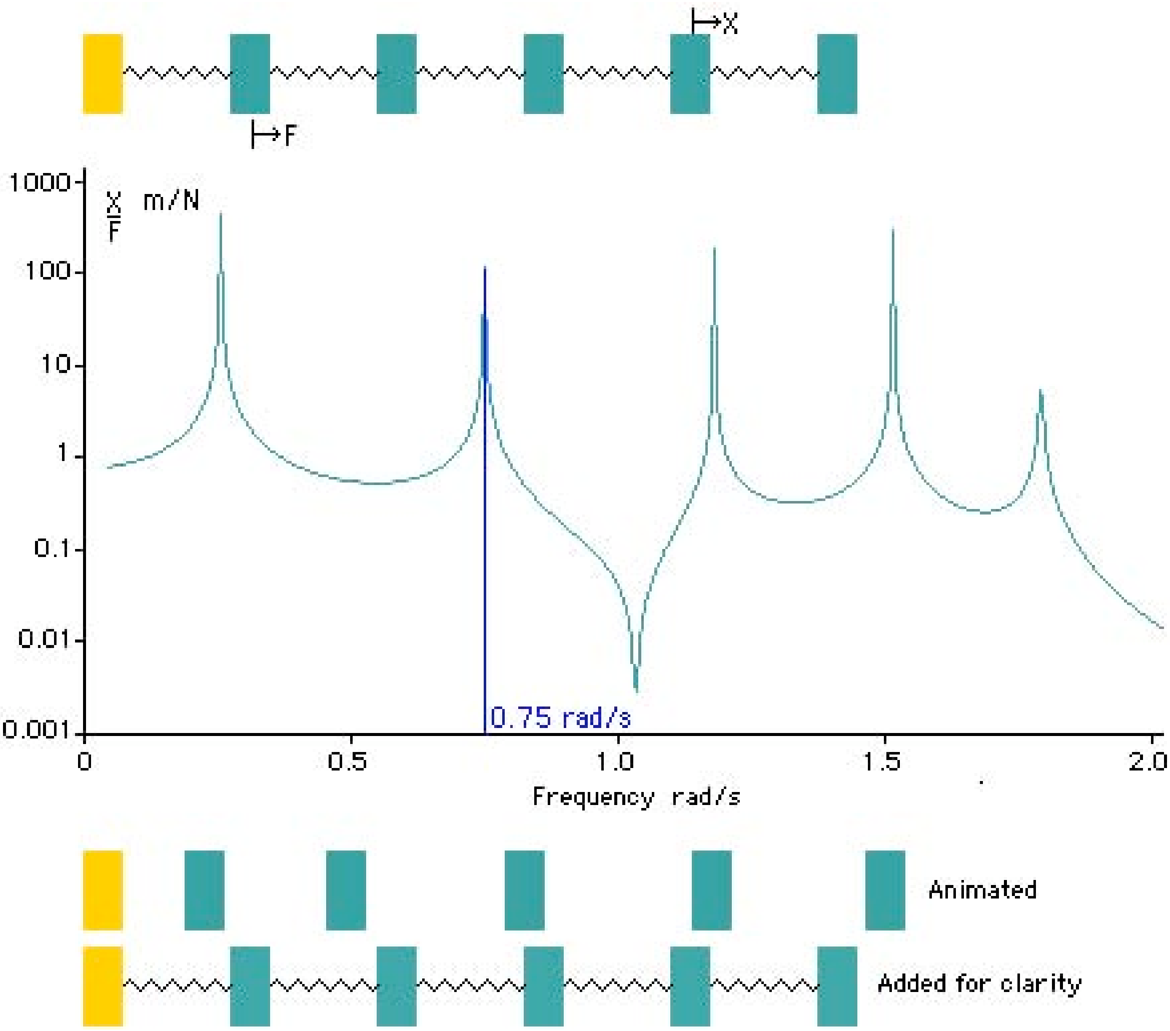

Figure 5.10 Output from animation program 'frozen' at extreme of amplitudes.

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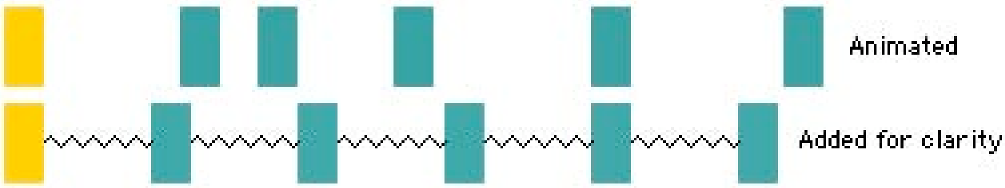
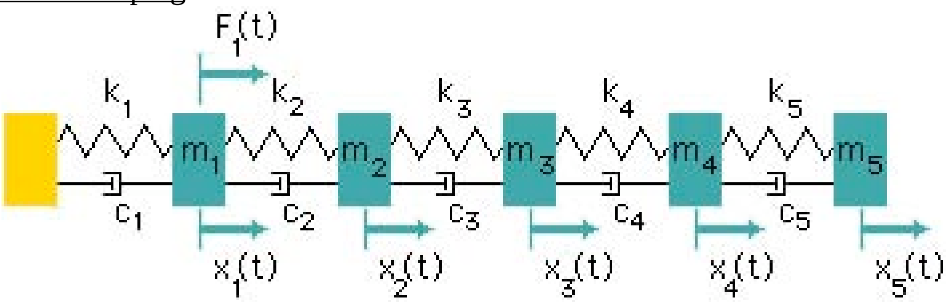
We may begin to understand the anti-resonance effect when the blue line is dragged to the appropriate frequency (1.035 rad/s). At this frequency the mass m_4 is not moving. However the mass m_5 to the right is moving with no excitation on it. Thus the mass m_5 and the stiffness k_5 act as a detuner. The detuned frequency is $\sqrt{k_5 / m_5} = \sqrt{1.6 / 1.5} = 1.033 \text{rad/s}$. The discrepancy with 1.035 rad/s is because the program uses discrete frequencies.



5.2.3 Effects of damping



The transient response involves a decaying vibration and is a major maths exercise to determine. For the steady state solution for sinusoidal excitation of mass m_1 the previous undamped solution (equation (5.16)) is easily modified. When viscous damping is included in parallel with the springs, wherever there was a kx term we will now have $kx + cx'$ so that equation (5.16) becomes,

$$\begin{bmatrix} (k_1^* + k_2^*) - m_1\omega^2 & -k_2^* & 0 & 0 & 0 \\ -k_2^* & (k_2^* + k_3^*) - m_2\omega^2 & -k_3^* & 0 & 0 \\ 0 & -k_3^* & (k_3^* + k_4^*) - m_3\omega^2 & -k_4^* & 0 \\ 0 & 0 & -k_4^* & (k_4^* + k_5^*) - m_4\omega^2 & -k_5^* \\ 0 & 0 & 0 & -k_5^* & k_5^* - m_5\omega^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $k_1^* = k_1 + i\omega c_1$; $k_2^* = k_2 + i\omega c_2$; $k_3^* = k_3 + i\omega c_3$; $k_4^* = k_4 + i\omega c_4$ and $k_5^* = k_5 + i\omega c_5$.

The maths is now even more prohibitive. A computer program similar to that for the undamped case allows the various responses to be checked.

A typical response is shown in figure 5.11. The effect of low levels of damping is to reduce the 'height' of the resonance peaks and to raise the anti-resonance 'depth'. If the animation is observed for high damping it will be found that the masses do not just vibrate in-phase and out of phase, as is the case with undamped systems.

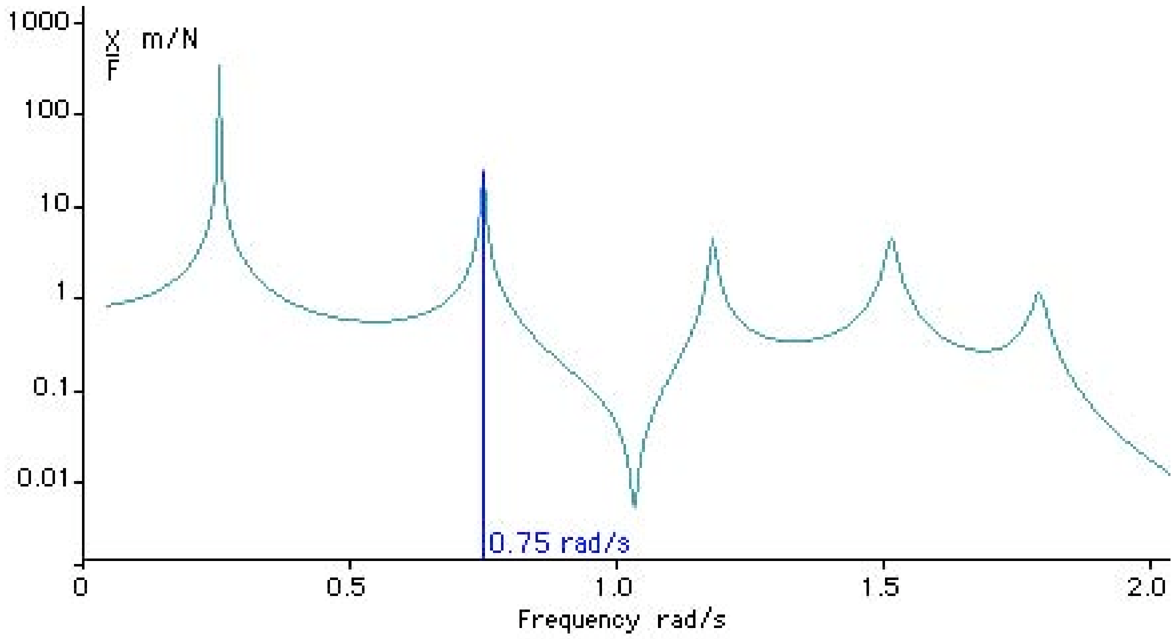


Figure 5.11 Response of damped five degree of freedom system.

5.4 Higher numbers of degrees of freedom

The axial vibration program that has an animation at a selected frequency can also be used to increase and decrease the number of masses. The system may be made free/free or clamped/clamped. The damping may be either viscous or hysteretic. The instructions for running the program are available from it.

5.5 Conclusions

As the number of degrees-of-freedom increases the maths to solve the equations increases significantly. In this chapter much of that maths has been omitted and the major trends have been highlighted. It should be clear that an 'N' degree of freedom system will have 'N' natural frequencies and associated mode shapes. When excited, resonance normally occurs at (undamped) or close to (damped) the natural frequencies. The response may have anti-resonances when a detuner comes into play.

Next chapter

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