CHAPTER 8: Continuous Systems. Transverse vibration of bars

In the previous chapter we considered the axial and torsional vibration of bars. These had an infinite number of degrees-of-freedom and hence an infinite number of natural frequencies and associated mode shapes. At every position along a bar it was necessary to determine its axial or torsional displacement. We shall now turn to the transverse vibration of bars and it will be necessary at each point to determine the transverse displacement and the slope. The maths is consequentially more complex than for axial or torsional vibration.

8.1 Equation of motion

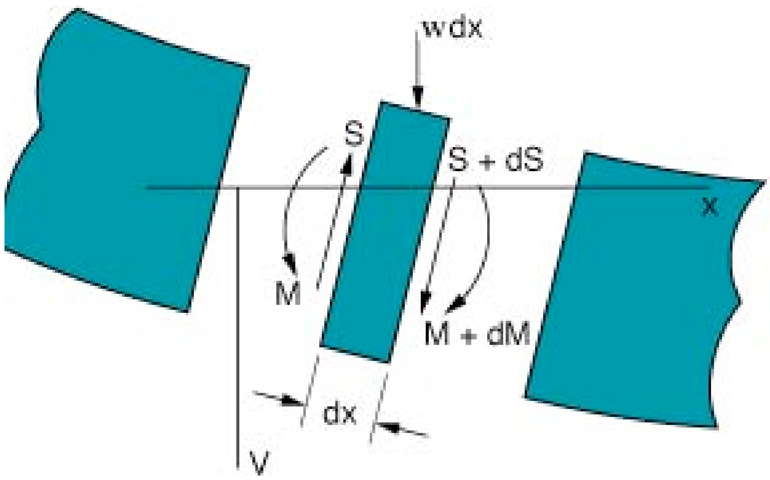

Figure 8.1 Shaft element with transverse deflection and rotation.

The analysis to be presented is based upon the Bernoulli-Euler equations and ignores shear and rotary inertia effects. An element of a shaft is shown in figure 8.1. Applying Newton II to the v direction for small deflections

$$S - (S + dS) + wdx = \rho A dx \frac{\partial^2 v}{\partial t^2} \quad \ldots\ldots \quad (8.1)$$

where w is the weight per unit length, ρ is the density, A the cross-section area and the other variables are as defined in figure 8.1. Equation (8.1) reduces to,

$$\frac{dS}{dx} + w = \rho A \frac{\partial^2 v}{\partial t^2} \quad \ldots\ldots \quad (8.2)$$

Since rotary inertia effects are ignored the sum of moments on the element are zero so that,

$$Sdx + M + dM - M = 0$$

and therefore

$$S = -\frac{dM}{dx} \quad \ldots\ldots \quad (8.3)$$

From simple bending theory

$$M = EI\frac{\partial^2 v}{\partial x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8.4)$$

where E is Young's modulus and I is the second moment of area of the section.
Substituting (8.4) in (8.3) gives

$$S = -EI\frac{\partial^3 v}{\partial x^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8.5)$$

and substituting this value for S in (8.2)

$$-EI\frac{\partial^4 v}{\partial x^4} + w = \rho A\frac{\partial^2 v}{\partial t^2}$$

and rearranging gives

$$\frac{\partial^2 v}{\partial t^2} + \frac{EI}{\rho A}\frac{\partial^4 v}{\partial x^4} = \frac{w}{\rho A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8.6)$$

The right hand term results from gravity and results in a deflection from the weight of the bar. If v is taken to be the deflection from the static equilibrium position then the right hand term becomes zero so that,

$$\frac{\partial^2 v}{\partial t^2} + \frac{EI}{\rho A}\frac{\partial^4 v}{\partial x^4} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8.7)$$

8.2 Steady-state sinusoidal motion

For small amplitudes it may be assumed that $v(x,t) = V(x)e^{i\omega t}$ which is a sinusoidal vibration with an amplitude varying along the bar.

Substituting in (8.7)

$$-\omega^2 V(x) + \frac{EI}{\rho A}\frac{\partial^4 V(x)}{\partial x^4} = 0$$

$$\text{thus} \quad \frac{\partial^4 V(x)}{\partial x^4} - \frac{\rho A\omega^2}{EI}V(x) = 0$$

and the general form of the solution is

$$V(x) = A\cos\lambda x + B\sin\lambda x + C\cosh\lambda x + D\sinh\lambda x \quad \ldots\ldots\ldots\ldots\ldots \quad (8.8)$$

where $\lambda^4 = \dfrac{\rho A\omega^2}{EI}$ and the values of A, B, C and D depend on the end conditions of the bar.

8.2.1 For a free end S = 0 and M = 0 (ie. no excitation)

From equation (8.4)

$$M = EI\frac{\partial^2 v}{\partial x^2} \text{ so that } \frac{\partial^2 v}{\partial x^2} = 0 \text{ and hence } \frac{\partial^2 V(x)}{\partial x^2} = 0$$

and from equation (8.5)

$$S = -EI\frac{\partial^3 v}{\partial x^3} \text{ so that } \frac{\partial^3 v}{\partial x^3} = 0 \text{ and hence } \frac{\partial^3 V(x)}{\partial x^3} = 0$$

8.3 Natural frequencies of a free/free bar (ie. no excitation at either end)

equation (8.8) gives

$$V(x) = A\cos\lambda x + B\sin\lambda x + C\cosh\lambda x + D\sinh\lambda x$$

differentiating with respect to x

$$\frac{\partial V(x)}{\partial x} = -A\lambda\sin\lambda x + B\lambda\cos\lambda x + C\lambda\sinh\lambda x + D\lambda\cosh\lambda x$$

and again

$$\frac{\partial^2 V(x)}{\partial x^2} = -A\lambda^2\cos\lambda x - B\lambda^2\sin\lambda x + C\lambda^2\cosh\lambda x + D\lambda^2\sinh\lambda x$$

and again

$$\frac{\partial^3 V(x)}{\partial x^3} = A\lambda^3\sin\lambda x - B\lambda^3\cos\lambda x + C\lambda^3\sinh\lambda x + D\lambda^3\cosh\lambda x$$

x = 0 is a free end so $\frac{\partial^2 V(0)}{\partial x^2} = 0$ and $\frac{\partial^3 V(0)}{\partial x^3} = 0$ which gives

$$\frac{\partial^2 V(0)}{\partial x^2} = -A + C = 0 \quad \text{..........} \quad (a)$$

$$\frac{\partial^3 V(0)}{\partial x^3} = -B + D = 0 \quad \text{..........} \quad (b)$$

x = L is also a free end so $\frac{\partial^2 V(L)}{\partial x^2} = 0$ and $\frac{\partial^3 V(L)}{\partial x^3} = 0$ which gives

$$\frac{\partial^2 V(L)}{\partial x^2} = -A\lambda^2\cos\lambda L - B\lambda^2\sin\lambda L + C\lambda^2\cosh\lambda L + D\lambda^2\sinh\lambda L = 0 \quad \text{..........} \quad (c)$$

$$\frac{\partial^3 V(L)}{\partial x^3} = A\lambda^3\sin\lambda L - B\lambda^3\cos\lambda L + C\lambda^3\sinh\lambda L + D\lambda^3\cosh\lambda L = 0 \quad \text{..........} \quad (d)$$

As there is no excitation the four equations (a), (b), (c) and (d) will give non-zero values for A, B, C and D only at natural frequencies.

From (a) C = A and from (b) D = B so substituting in (c) and (d)

$$-A\lambda^2\cos\lambda L - B\lambda^2\sin\lambda L + A\lambda^2\cosh\lambda L + B\lambda^2\sinh\lambda L = 0$$
$$A\lambda^3\sin\lambda L - B\lambda^3\cos\lambda L + A\lambda^3\sinh\lambda L + B\lambda^3\cosh\lambda L = 0$$

and rearranging

$$A(\cos\lambda L - \cosh\lambda L) + B(\sin\lambda L - \sinh\lambda L) = 0 \quad \text{..........} \quad (e)$$

$$A(\sin\lambda L + \sinh\lambda L) - B(\cos\lambda L - \cosh\lambda L) = 0 \quad \text{..........} \quad (f)$$

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from (e)

$$B = -\frac{A(\cos\lambda L - \cosh\lambda L)}{(\sin\lambda L - \sinh\lambda L)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (g)$$

and substituting in (f)

$$A(\sin\lambda L + \sinh\lambda L) + \frac{A(\cos\lambda L - \cosh\lambda L)}{(\sin\lambda L - \sinh\lambda L)}(\cos\lambda L - \cosh\lambda L) = 0$$

$$\therefore A\left[\frac{\sin^2\lambda L - \sinh^2\lambda L + (\cos\lambda L - \cosh\lambda L)^2}{(\sin\lambda L - \sinh\lambda L)}\right] = 0$$

$$\therefore A\left[\frac{\sin^2\lambda L - \sinh^2\lambda L + \cos^2\lambda L - 2\cos\lambda L\cosh\lambda L + \cosh^2\lambda L}{(\sin\lambda L - \sinh\lambda L)}\right] = 0$$

$$\therefore A\left[\frac{2(1 - \cos\lambda L\cosh\lambda L)}{(\sin\lambda L - \sinh\lambda L)}\right] = 0$$

Thus A = 0 is a solution and there is no motion which is not unexpected as there is no excitation or when

$$1 - \cos\lambda L\cosh\lambda L = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8.9)$$

or A may have any value and vibration occurs, Thus equation (8.9) is the natural frequency equation for a free/free bar. As expected there is an infinite set of solutions since the system has an infinite number of degrees-of-freedom. The lower value solutions are,

$\lambda_0 L = 0.0$; $\lambda_1 L = 4.73$; $\lambda_2 L = 7.853$; $\lambda_3 L = 10.996$; $\lambda_4 L = 14.137$; $\lambda_5 L = 17.279$; $\lambda_6 L = 20.42$;

since $\lambda^4 = \frac{\rho A\omega^2}{EI}$ rearranging gives $\omega = \frac{(\lambda L)^2}{L^2}\sqrt{\frac{EI}{\rho A}}$. Thus the lower natural frequencies are,

$\omega_{n0} = 0.0$; $\omega_{n1} = \frac{22.37}{L^2}\sqrt{\frac{EI}{\rho A}}$; $\omega_{n2} = \frac{61.67}{L^2}\sqrt{\frac{EI}{\rho A}}$; $\omega_{n3} = \frac{120.9}{L^2}\sqrt{\frac{EI}{\rho A}}$; $\omega_{n4} = \frac{199.9}{L^2}\sqrt{\frac{EI}{\rho A}}$; $\omega_{n5} = \frac{298.6}{L^2}\sqrt{\frac{EI}{\rho A}}$; $\omega_{n6} = \frac{417}{L^2}\sqrt{\frac{EI}{\rho A}}$

The zero frequency mode is expected because the system is free/free.

8.4 Mode shapes of a free/free bar

Each natural frequency has an associated mode shape. These are found from equation (8.8)

$$V(x) = A\cos\lambda x + B\sin\lambda x + C\cosh\lambda x + D\sinh\lambda x$$

with $C = A$ and $D = B$ and $B = -\dfrac{A(\cos\lambda L - \cosh\lambda L)}{(\sin\lambda L - \sinh\lambda L)}$ so that

$$V(x) = A(\cos\lambda x + \cosh\lambda x) + B(\sin\lambda x + \sinh\lambda x)$$

and

$$\therefore V(x) = A\left[\frac{(\sin\lambda L - \sinh\lambda L)(\cos\lambda x + \cosh\lambda x) - (\cos\lambda L - \cosh\lambda L)(\sin\lambda x + \sinh\lambda x)}{(\sin\lambda L - \sinh\lambda L)}\right]$$

since it is the "shape" we are interested in then the value of $\dfrac{A}{(\sin\lambda L - \sinh\lambda L)}$ may be any value and the mode shape is given by,

$$V(x) = K\left[(\sin\lambda L - \sinh\lambda L)(\cos\lambda x + \cosh\lambda x) - (\cos\lambda L - \cosh\lambda L)(\sin\lambda x + \sinh\lambda x)\right]$$

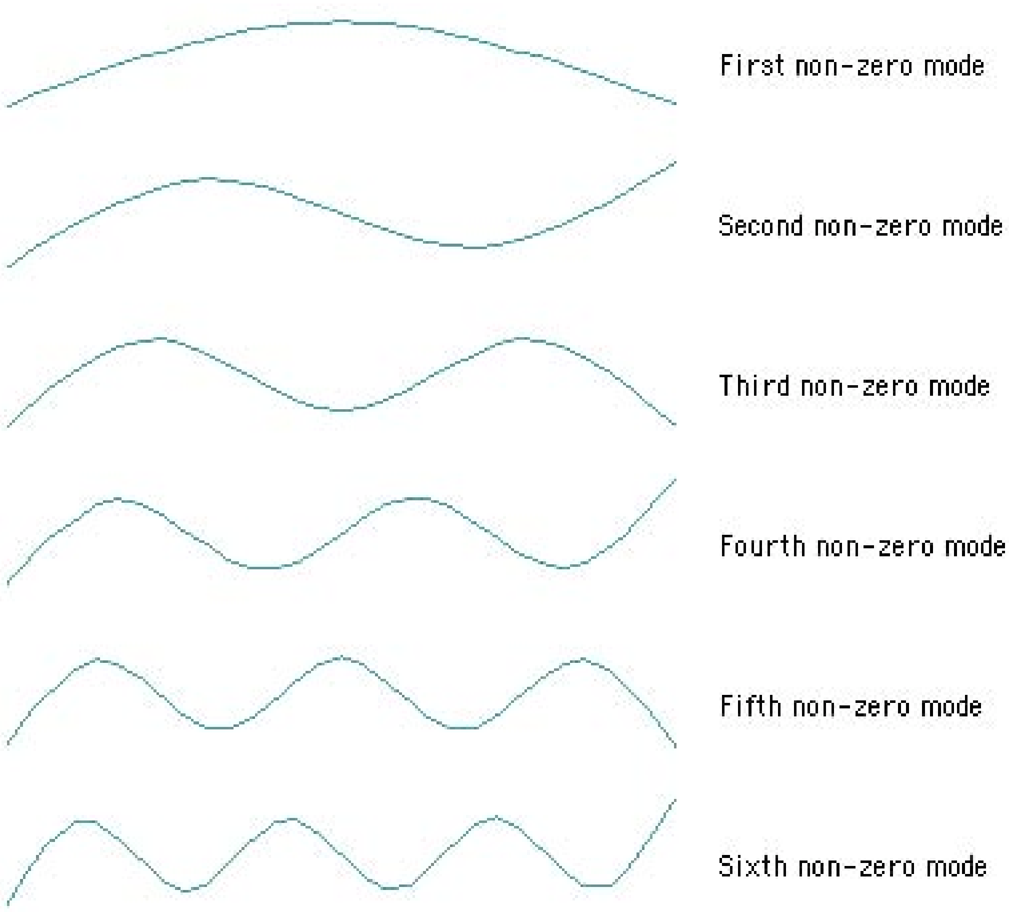

Figure 8.2 Mode shapes of free/free bar

8.5 Natural frequencies and modes of a clamped/free bar

If a similar analysis is completed for a clamped/free bar with the clamped end having no deflection and no slope it is found that the natural frequencies are given by,

$$1 + \cos\lambda L \cosh\lambda L = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8.10)$$

Again as expected there is an infinite set of solutions since the system has an infinite number of degrees-of-freedom. The lower value solutions are,

$$\lambda_1 L = 1.875;\ \lambda_2 L = 4.694;\ \lambda_3 L = 7.854;\ \lambda_4 L = 11.0;\ \lambda_5 L = 14.14;\ \lambda_6 L = 17.28;$$

since $\lambda^4 = \dfrac{\rho A \omega^2}{EI}$ rearranging gives $\omega = \dfrac{(\lambda L)^2}{L^2}\sqrt{\dfrac{EI}{\rho A}}$. Thus the lower natural frequencies are,

$$\omega_{n1} = \frac{3.52}{L^2}\sqrt{\frac{EI}{\rho A}};\ \omega_{n2} = \frac{22.03}{L^2}\sqrt{\frac{EI}{\rho A}};\ \omega_{n3} = \frac{61.7}{L^2}\sqrt{\frac{EI}{\rho A}};\ \omega_{n4} = \frac{120.9}{L^2}\sqrt{\frac{EI}{\rho A}};$$
$$\omega_{n5} = \frac{199.9}{L^2}\sqrt{\frac{EI}{\rho A}};\ \omega_{n6} = \frac{298.6}{L^2}\sqrt{\frac{EI}{\rho A}}$$

Each natural frequency has an associated mode shape given by,

$$V(x) = K\left[(\sin\lambda L + \sinh\lambda L)(\cos\lambda x - \cosh\lambda x) - (\cos\lambda L + \cosh\lambda L)(\sin\lambda x - \sinh\lambda x)\right] \quad .... \quad (8.11)$$

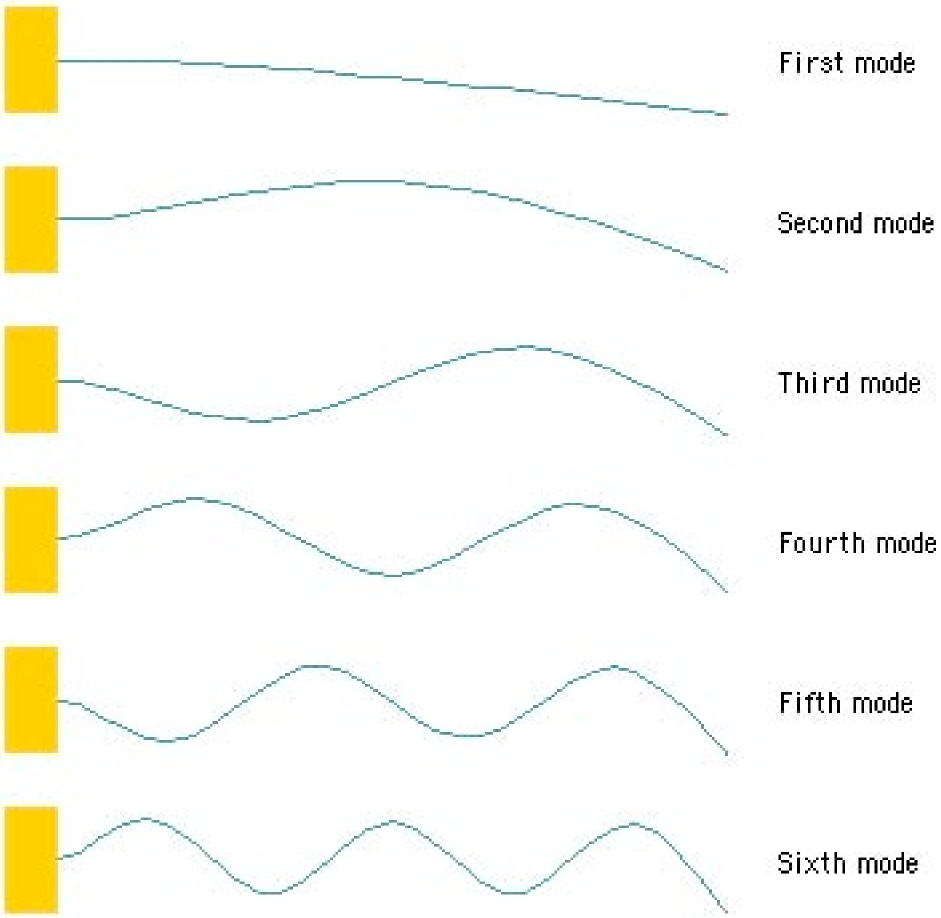

Figure 8.3 Mode shapes of a clamped/free bar

8.6 Steady state response of a free/free bar

It is informative and useful to obtain the responses of a bar when excited at one end. Consider a free/free bar with a sinusoidal force applied at one end.

For an exciting force at one end. eg $S = F_L e^{i\omega t}$ at $x = L$

If there is only an exciting force at the end then there will be no moment and so $\dfrac{\partial^2 V(x)}{\partial x^2} = 0$ as for a free end.

From figure 8.1 it should be noted that at the right hand end when x = L (the length of the bar) the force S is in the same positive direction as v.

Thus for $x = L$ when $S = F_L e^{i\omega t}$ from (8.5) $S = -EI\dfrac{\partial^3 v}{\partial x^3}$ so that $F_L e^{i\omega t} = -EI\dfrac{\partial^3 v}{\partial x^3}$

substituting for $v(x,t) = V(x)e^{i\omega t}$ gives $F_L = -EI\dfrac{\partial^3 V(x)}{\partial x^3}$ and hence $\dfrac{\partial^3 V(x)}{\partial x^3} = -\dfrac{F_L}{EI}$

When the excitation is at $x = L$ then $x = 0$ is a free end so $\dfrac{\partial^2 V(0)}{\partial x^2} = 0$ and $\dfrac{\partial^3 V(0)}{\partial x^3} = 0$ which gives

$$\frac{\partial^2 V(0)}{\partial x^2} = -A + C = 0 \quad \text{.......} \quad (a)$$

$$\frac{\partial^3 V(0)}{\partial x^3} = -B + D = 0 \quad \text{.......} \quad (b)$$

At $x = L$ is a forced end so $\dfrac{\partial^2 V(L)}{\partial x^2} = 0$ and $\dfrac{\partial^3 V(L)}{\partial x^3} = -\dfrac{F_L}{EI}$ which gives

$$\frac{\partial^2 V(L)}{\partial x^2} = -A\cos\lambda L - B\sin\lambda L + C\cosh\lambda L + D\sinh\lambda L = 0 \quad \text{.......} \quad (h)$$

$$\frac{\partial^3 V(L)}{\partial x^3} = A\lambda^3\sin\lambda L - B\lambda^3\cos\lambda L + C\lambda^3\sinh\lambda L + D\lambda^3\cosh\lambda L = -\frac{F_L}{EI} \quad \text{.......} \quad (i)$$

The four equations (a), (b), (h) and (i) allow the constants A, B, C and D to be found.
From (a) $C = A$ and from (b) $D = B$ so substituting in (h) and (i)

$$-A\cos\lambda L - B\sin\lambda L + A\cosh\lambda L + B\sinh\lambda L = 0$$

$$A\lambda^3\sin\lambda L - B\lambda^3\cos\lambda L + A\lambda^3\sinh\lambda L + B\lambda^3\cosh\lambda L = -\frac{F_L}{EI}$$

and rearranging

$$A(\cos\lambda L - \cosh\lambda L) + B(\sin\lambda L - \sinh\lambda L) = 0 \quad \text{.......} \quad (j)$$

$$A(\sin\lambda L + \sinh\lambda L) - B(\cos\lambda L - \cosh\lambda L) = -\frac{F_L}{EI\lambda^3} \quad \ldots\ldots\ldots\ldots \quad (k)$$

from (j)

$$B = -\frac{A(\cos\lambda L - \cosh\lambda L)}{(\sin\lambda L - \sinh\lambda L)} \quad \ldots\ldots\ldots\ldots \quad (l)$$

and substituting in (k)

$$A(\sin\lambda L + \sinh\lambda L) + \frac{A(\cos\lambda L - \cosh\lambda L)}{(\sin\lambda L - \sinh\lambda L)}(\cos\lambda L - \cosh\lambda L) = -\frac{F_L}{EI\lambda^3}$$

$$\therefore A = -\frac{F_L}{EI\lambda^3}\frac{(\sin\lambda L - \sinh\lambda L)}{\left(\sin^2\lambda L - \sinh^2\lambda L + \cos^2\lambda L - 2\cos\lambda L\cosh\lambda L + \cosh^2\lambda L\right)} = \frac{F_L}{2EI\lambda^3}\frac{(\sin\lambda L - \sinh\lambda L)}{(\cos\lambda L\cosh\lambda L - 1)} = C$$

substituting in (8.l)

$$B = -\frac{(\cos\lambda L - \cosh\lambda L)}{(\sin\lambda L - \sinh\lambda L)}\frac{F_L}{2EI\lambda^3}\frac{(\sin\lambda L - \sinh\lambda L)}{(\cos\lambda L\cosh\lambda L - 1)} = -\frac{F_L}{2EI\lambda^3}\frac{(\cos\lambda L - \cosh\lambda L)}{(\cos\lambda L\cosh\lambda L - 1)} = D$$

Now substituting the constants in equation (8.8) with $C = A$ and $D = B$

$$V(x) = A(\cos\lambda x + \cosh\lambda x) + B(\sin\lambda x + \sinh\lambda x)$$

$$\therefore V(x) = \frac{F_L}{2EI\lambda^3}\frac{(\sin\lambda L - \sinh\lambda L)}{(\cos\lambda L\cosh\lambda L - 1)}(\cos\lambda x + \cosh\lambda x) - \frac{F_L}{2EI\lambda^3}\frac{(\cos\lambda L - \cosh\lambda L)}{(\cos\lambda L\cosh\lambda L - 1)}(\sin\lambda x + \sinh\lambda x)$$

$$\therefore \alpha_{xL} = \frac{V(x)}{F_L} = \frac{(\sin\lambda L - \sinh\lambda L)(\cos\lambda x + \cosh\lambda x) - (\cos\lambda L - \cosh\lambda L)(\sin\lambda x + \sinh\lambda x)}{2EI\lambda^3(\cos\lambda L\cosh\lambda L - 1)}$$

Adopting the notation of Bishop and Johnson [1]

$$\therefore \alpha_{xL} = \frac{V(x)}{F_L} = \frac{F_8(\cos\lambda x + \cosh\lambda x) - F_{10}(\sin\lambda x + \sinh\lambda x)}{2EI\lambda^3 F_3}$$

where

$$F_3 = \cos\lambda L\cosh\lambda L - 1$$
$$F_8 = \sin\lambda L - \sinh\lambda L$$
$$F_{10} = \cos\lambda L - \cosh\lambda L$$

It should be noted that the response tends to infinity when $F_3 = 0$ ie at the natural frequencies for a free/free bar.

As an example of the receptance consider the response shown in figure 8.4. A log scale is used for the plot as the range is very large. The magnitude of the peaks and troughs is a function of the frequency step used by the program in calculating the response. If the steps were infinitely small then the infinite resonances would be shown. Also the anti-resonances would have a zero response.

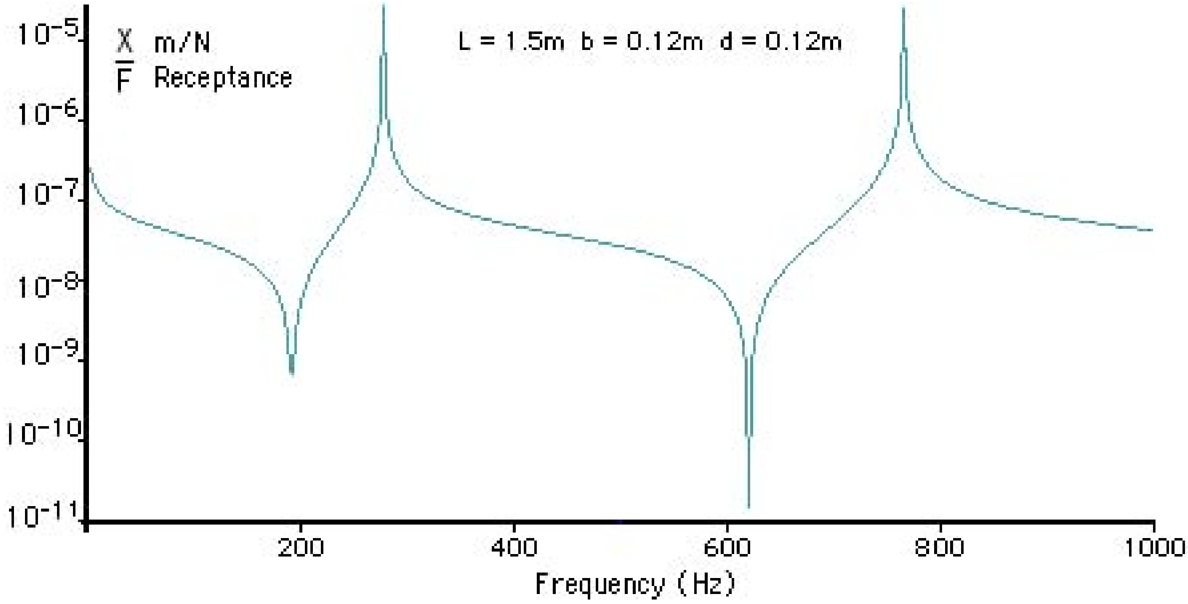

Figure 8.4 End response of square section beam.

The program shows an animation at a selected frequency.

Note that as shown below in figure 8.5 the solid model is schematic, the end faces would not stay vertical.

Figure 8.5 Deflected shape of free/free bar excited at one end.

In the same manner as above it is possible to determine the other relevant receptances of a clamped/free bar. Again using the notation of Bishop and Johnson [1].

$$\therefore \alpha_{xo} = \frac{V(x)}{F_o} = \frac{F_5(\cos\lambda x + \cosh\lambda x) + (F_1 + F_3)\sin\lambda x + (F_1 - F_3)\sinh\lambda x}{-2EI\lambda^3 F_3}$$

$$\therefore \alpha_{xo'} = \frac{V(x)}{M_o} = \frac{(F_1 - F_3)\cos\lambda x + (F_1 + F_3)\cosh\lambda x - F_6(\sin\lambda x + \sinh\lambda x)}{-2EI\lambda^2 F_3}$$

$$\therefore \alpha_{xL'} = \frac{V(x)}{M_L} = \frac{F_{10}(\cos\lambda x + \cosh\lambda x) + F_7(\sin\lambda x + \sinh\lambda x)}{-2EI\lambda^2 F_3}$$

where the ' indicates a moment or slope and

$F_1 = \sin\lambda L \sinh\lambda L$ $\quad$ $F_4 = \cos\lambda L \cosh\lambda L + 1$

$F_5 = \cos\lambda L \sinh\lambda L - \sin\lambda L \cosh\lambda L$ $\quad$ $F_6 = \cos\lambda L \sinh\lambda L + \sin\lambda L \cosh\lambda L$

$F_7 = \sin\lambda L + \sinh\lambda L$ $\quad$ $F_{10} = \cos\lambda L - \cosh\lambda L$

When building up complex systems the so called "tip" responses will be required. These are all the responses that involve the ends of the bar, ie x = 0 or L. These can be shown to be,

$$\therefore \alpha_{LL} = \frac{V(L)}{F_L} = \alpha_{oo} = \frac{V(0)}{F_o} = \frac{-F_5}{EI\lambda^3 F_3} \qquad \therefore \alpha_{o'o} = \frac{V'(0)}{F_o} = \alpha_{oo'} = \frac{V(0)}{M_o} = \frac{-F_1}{EI\lambda^2 F_3}$$

$$\therefore \alpha_{L'L} = \frac{V'(L)}{F_L} = \alpha_{LL'} = \frac{V(L)}{M_L} = \frac{F_1}{EI\lambda^2 F_3} \qquad \therefore \alpha_{oL} = \frac{V(0)}{F_L} = \alpha_{Lo} = \frac{V(L)}{M_o} = \frac{F_8}{EI\lambda^3 F_3}$$

$$\therefore \alpha_{L'o} = \frac{V'(L)}{F_o} = \alpha_{oL'} = \frac{V(0)}{M_L} = \frac{F_{10}}{EI\lambda^2 F_3} \qquad \therefore \alpha_{Lo'} = \frac{V(L)}{M_o} = \alpha_{o'L} = \frac{V'(0)}{F_L} = \frac{-F_{10}}{EI\lambda^2 F_3}$$

$$\therefore \alpha_{o'o'} = \frac{V'(0)}{M_o} = \alpha_{L'L'} = \frac{V'(L)}{M_L} = \frac{F_6}{EI\lambda F_3} \qquad \therefore \alpha_{L'o'} = \frac{V'(L)}{M_o} = \alpha_{o'L'} = \frac{V'(0)}{M_L} = \frac{F_7}{EI\lambda F_3}$$

8.7 Steady state response of a clamped/free bar

In the same manner as for the free/free condition it is possible to determine the relevant receptances of a clamped/free bar.

$$\alpha_{LL} = \frac{-F_5}{EI\lambda^3 F_4} \qquad \alpha_{LL'} = \alpha_{L'L'} = \frac{F_1}{EI\lambda^2 F_4} \qquad \alpha_{L'L'} = \frac{F_6}{EI\lambda F_4}$$

An example of the tip receptance α_{LL} of a clamped/free bar is shown in figure 8.6.

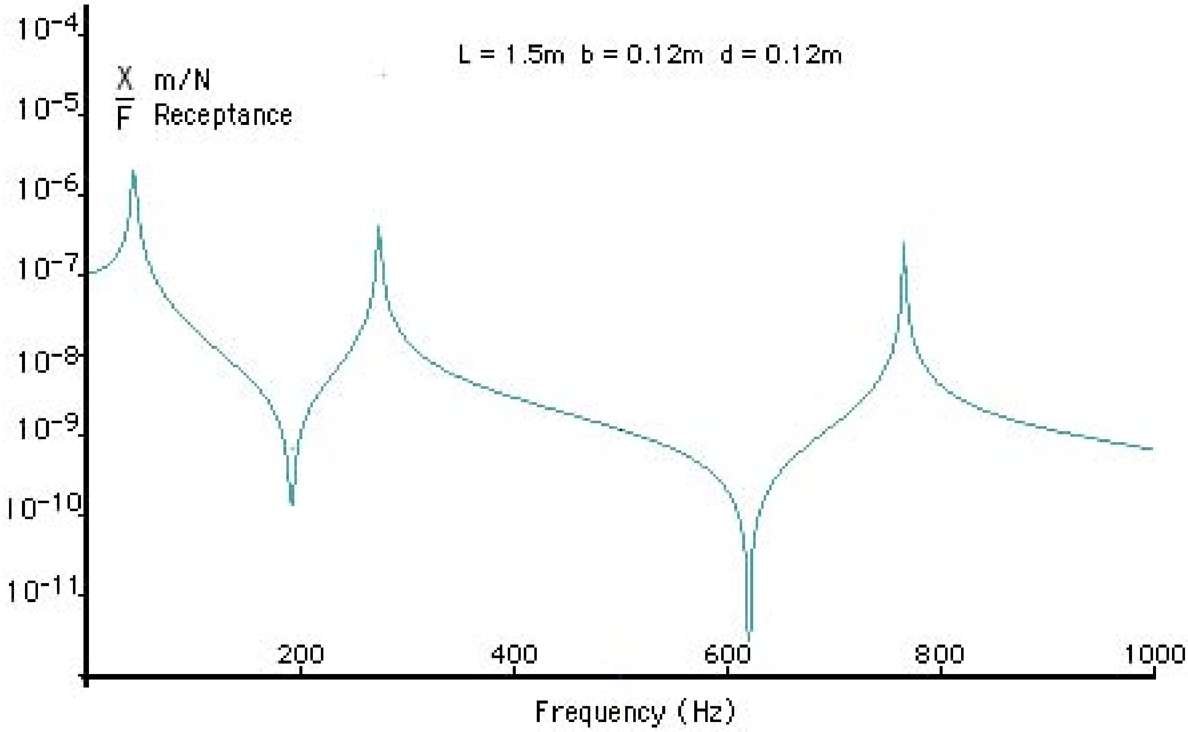

Figure 8.6 Response of clamped/free bar

The program shows an animation at a selected frequency.

8.7 Shear and rotary inertia effects

The transverse responses including shear and rotary inertia effects, are taken from Stone [2]. The complete set of equations is not included here. The transverse receptances are given for the case when,

$$\omega^2 > \frac{\lambda G A}{\rho I}$$

these are,

$$\alpha_{oo} = \alpha_{LL} = \frac{\left[F_5 \left(\frac{r\eta^2}{s\xi} - \eta \right) + F_4 \left(\frac{s\xi^2}{r\eta} - \xi \right) \right]}{\Delta}$$

$$\alpha_{Lo} = \alpha_{oL} = \frac{\left[\sin\eta L \left(\frac{r\alpha^2}{s\xi} - \eta \right) + \sin\zeta L \left(\frac{s\xi^2}{r\eta} - \xi \right) \right]}{\Delta}$$

Where,

$$F_1 = \sin\xi L \sin\eta L$$

$$F_2 = \cos\xi L \cos\eta L$$

$$F_4 = \sin\xi L \cos\eta L$$

$$F_5 = \cos\xi L \sin\eta L$$

$$s = \frac{\rho\omega^2}{\lambda G \xi} - \xi$$

$$r = \frac{\rho\omega^2}{\lambda G \eta} - \eta$$

$$\eta = \left[\frac{-\rho I\omega^2 \left(1 + \frac{E}{\lambda G} \right) + \sqrt{\rho I\omega^2 \left[\rho I\omega^2 \left(1 - \frac{E}{\lambda G} \right)^2 + 4EA \right]}}{2EI} \right]^{1/2}$$

$$\xi = \left[\frac{\rho I\omega^2 \left(1 + \frac{E}{\lambda G} \right) + \sqrt{\rho I\omega^2 \left[\rho I\omega^2 \left(1 - \frac{E}{\lambda G} \right)^2 + 4EA \right]}}{2EI} \right]^{1/2}$$

$$\Delta = \rho\omega^2 A \left[2 - 2F_2 - \left(\frac{r\eta^2}{s\xi^2} + \frac{s\xi^2}{r\eta^2} \right) F_1 \right]$$

Where A - cross-section area; E - Young's modulus; ω - excitation frequency; L - length of bar, I - second moment of area; ρ - density; λ - shear factor.

An example of the effect of including shear and rotary inertia is shown in figure 8.7. This shows the responses with and without shear and rotary inertia. For the dimensions shown, ie with a length to diameter ratio of 0.7/0.15 (= 4.67) the first natural frequency is not significantly changed but the frequencies of the higher modes are changed. As the length to diameter ratio is reduced the difference in natural frequency becomes much more significant.

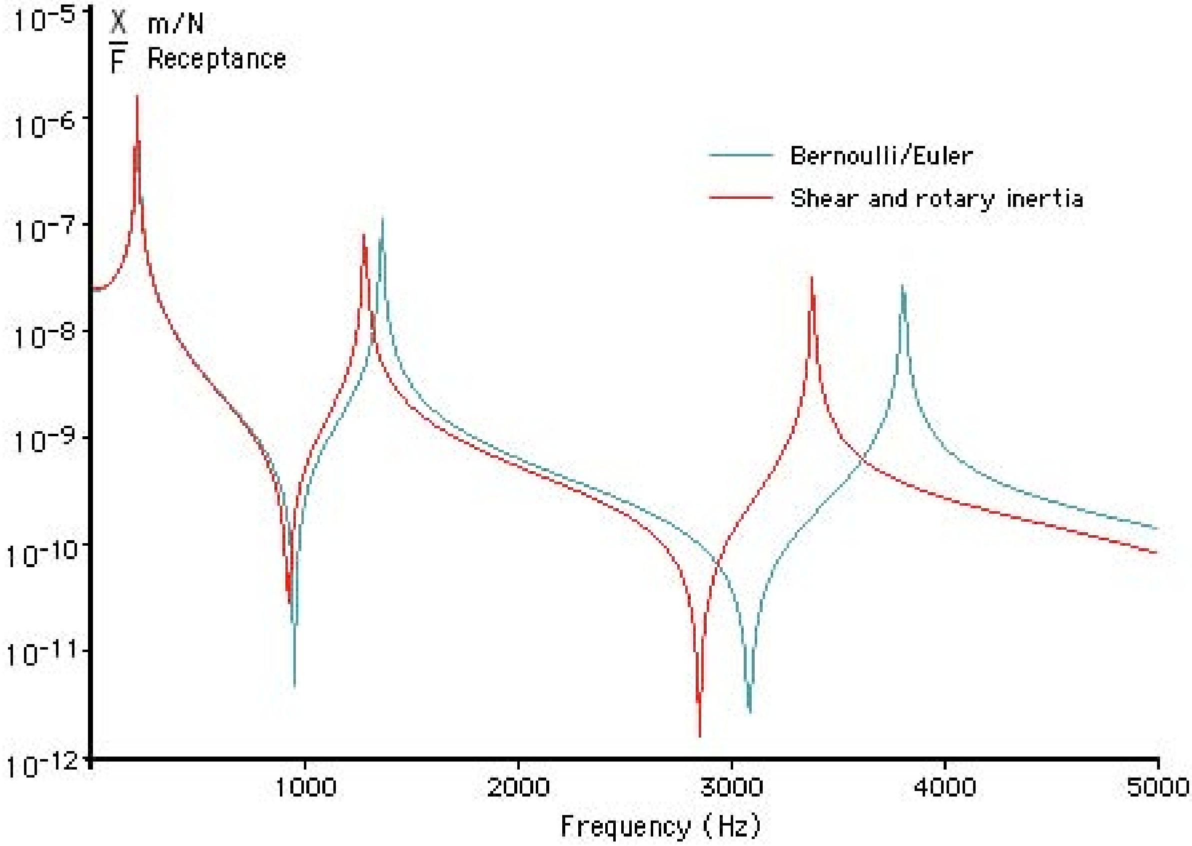

Figure 8.7 Responses of clamped/free bar of length 0.7m and diameter 0.15m

8.8 References

1. Bishop, R E D, and Johnson, D C, 1960, The Mechanics of Vibration, Cambridge University Press.

2. Stone, B. J., 'The receptances of beams, in closed form, including the effects of shear and rotary inertia.', Proc. I. Mech. E., 1992, Vol 206, 87-94.

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