CHAPTER 1: One degree-of-freedom vibration
An introduction to basic vibration theory

A system may vibrate when it is possible for energy to be converted from one form to another and back again. In mechanical systems this is usually from the kinetic energy of motion to the stored energy in for example a spring. A simple vibrating system consists of a rigid mass attached by a massless spring to a fixed abutment, see figure 1.1 For one degree-of-freedom vibration the mass is constrained to move in one direction.. If there is an energy dissipation source, such as a viscous damper, then the vibration will gradually decay as energy is converted to heat.

Vibration is often considered as "transient" or "forced" vibration. A transient vibration is one that dies away with time due to energy dissipation. Usually there is some initial disturbance and following this the system vibrates without any further input. This is called transient vibration. A forced vibration is usually defined as being one that is kept going by an external excitation.

1.1 Transient vibration: Undamped

Consider the motion of the spring/mass system, shown in figure 1.1, when it is initially disturbed and then allowed to vibrate freely. The displacement of the mass with time, x(t), is measured from the static equilibrium position, ie. the rest position.

If at some time t the mass is displaced an amount x(t) in the positive direction shown. Then for a linear spring there will be a force on the mass from the spring of -kx(t).

Thus from Newton's second law of motion using a free-body diagram,

$$mx''(t) = -kx(t) \quad .... \quad (1.1)$$

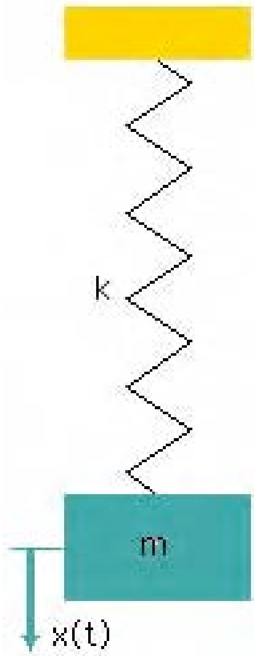
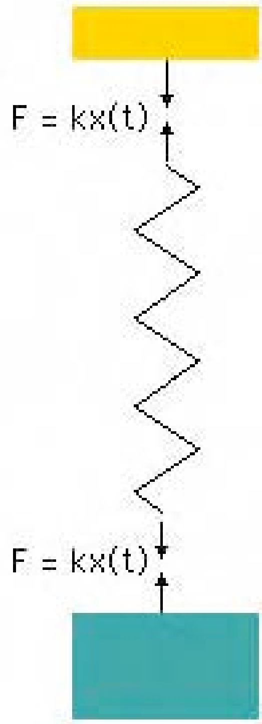


Figure 1.1



Free body diagram

Equation (1.1) is called the equation of motion. The equation is unchanged if gravity effects are included. Rearranging and dividing by m gives,

$$x''(t) + \omega_n^2 x(t) = 0 \quad .... \quad (a) \quad \text{where} \quad \omega_n = \sqrt{\frac{k}{m}}$$

the solution of (a) will be of the form,

$$x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$$

where the values of A, B, α_1 and α_2 may be found from the initial conditions x(0) and x'(0). Thus substituting $x(t) = Ae^{\alpha t}$ in equation (a) gives,

$$Ae^{\alpha t}(\alpha^2 + \omega_n^2) = 0 \quad \text{thus} \quad \alpha^2 + \omega_n^2 = 0$$

$$\text{and hence} \quad \alpha^2 = -\omega_n^2 \quad \text{so that} \quad \alpha_1 = +i\omega_n \quad \text{and} \quad \alpha_2 = -i\omega_n \quad \text{where} \quad i = \sqrt{-1}$$

Thus,

$$x(t) = Ae^{+i\omega_n t} + Be^{-i\omega_n t}$$

rearranging gives,

$$x(t) = \frac{(A+B)}{2}\left(e^{+i\omega_n t} + e^{-i\omega_n t}\right) + \frac{(A-B)}{2}\left(e^{+i\omega_n t} - e^{-i\omega_n t}\right)$$

$$\text{so that} \quad x(t) = C\cos\omega_n t + D\sin\omega_n t \quad \ldots\ldots\ldots\ldots \quad \text{(b)}$$

where C=A+B and D=A-B

when t=0 x(t)=x(0), so that substituting in (b) gives x(0) = C

also when t=0 x'(t)=x'(0), so that substituting differentiating (b) and substituting gives $x'(t) = \omega_n D$

so that finally we have,

$$x(t) = x(0)\cos\omega_n t + \frac{x'(0)}{\omega_n}\sin\omega_n t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.2)$$

You may investigate the motion by running a simulation program.

For example with an initial displacement but with no initial velocity the motion is sinusoidal with an amplitude x(0) and a frequency ω_n.

$$x(t) = x(0)\cos\omega_n t$$

This is a non decaying sinusoidal vibration of frequency ω_n that is conventionally termed the undamped natural frequency where

$$\omega_n = \sqrt{\frac{k}{m}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.3)$$

Note that the undamped natural frequency does not depend on the initial conditions or the amplitude of the motion.

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Also note that with an initial velocity alone or with both initial velocity and displacement a sinusoidal vibration of frequency ω_n would be obtained.

1.2 Transient vibration: Damped

We now add a viscous damper having a damping coefficient c to the spring/mass system previously consider, see figure 1.2.

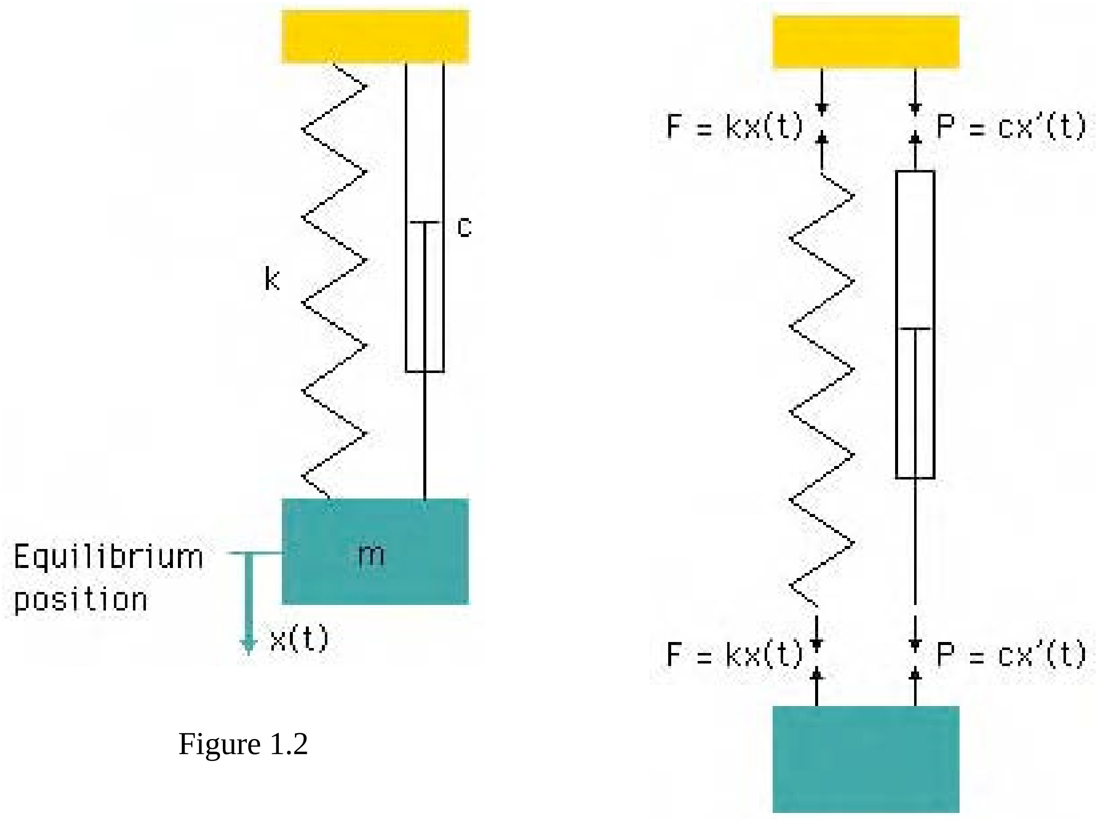

Figure 1.2

Drawing a free body diagram and applying Newton's second law of motion to gives,

$$mx''(t) = -kx(t) - cx'(t) \quad \text{...} \quad (1.4)$$

and hence

$$mx''(t) + cx'(t) + kx(t) = 0 \quad \text{.......................................} \quad (1.5)$$

1.2.1 Laplace transform solution.

This maths approach is helpful when the transients arising at the start of forced vibration are considered. Taking Laplace transforms of equation (1.5) yields,

$$m[-x'(0) - sx(0) + s^2X(s)] + c[-x(0) + sX(s)] + kX(s) = 0$$

rearranging this gives

$$X(s) = \frac{m(x'(0) + sx(0)) + cx(0)}{ms^2 + cs + k} \quad \text{.................................} \quad (1.6)$$

For no damping (c=0) we can check that this is the same result as previously - equation (1.2)

Thus for c = 0. Equation (1.6) then becomes,

$$X(s) = \frac{m(x'(0) + sx(0))}{ms^2 + k} = \frac{sx(0)}{s^2 + k/m} + \frac{x'(0)}{s^2 + k/m}$$

Now taking inverse Laplace transforms and substituting $\omega_n = \sqrt{\frac{k}{m}}$ as before gives,

$$x(t) = x(0)\cos\omega_n t + \frac{x'(t)}{\omega_n}\sin\omega_n t \quad \text{.......................................} \quad (1.2)$$

1.2.2 Equations in non-dimensional form

It is common to write the basic equation of motion in terms of ω_n and another parameter ξ, the viscous damping ratio, which is defined as,

$$\xi = \frac{c}{2\sqrt{km}}$$ (The significance of ξ will become apparent later).

Thus if equation (1.5) is divided throughout by m we obtain,

$$x''(t) + 2\xi\omega_n x'(t) + \omega_n^2 x(t) = 0 \quad \text{.....................................} \quad (1.7)$$

If Laplace transforms are taken we obtain

$$X(s) = \frac{x'(0) + 2\xi\omega_n x(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} + \frac{sx(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad \text{.................................} \quad (1.8)$$

Now the form of solution depends on the value of ξ so that different solutions are obtained depending on whether $\xi < 1$, $\xi = 1$ or $\xi > 1$. We have already considered the case where c and hence ξ is zero. The solution was a non-decaying oscillation of frequency ω_n.

1.2.3 Case when $\xi < 1$

When $\xi < 1$ the inverse Laplace transform of equation (1.8) gives,

$$x(t) = e^{-\xi\omega_n t}\left[x(0)\cos\omega_n\sqrt{(1-\xi^2)}t + \frac{[x'(0) + \xi\omega_n x(0)]\sin\omega_n\sqrt{(1-\xi^2)}t}{\omega_n\sqrt{(1-\xi^2)}}\right] \quad \text{...........} \quad (1.9)$$

This is an exponentially decaying oscillation. If again, the specific example of an initial displacement and no initial velocity is taken, ie, $x(0) = X_0$ and $x'(t) = 0$ then

$$x(t) = X_0 e^{-\xi\omega_n t}\left[\cos\omega_n\sqrt{(1-\xi^2)}t + \frac{\xi\sin\omega_n\sqrt{(1-\xi^2)}t}{\sqrt{(1-\xi^2)}}\right] \quad \text{................} \quad (1.10)$$

and the damped natural frequency is $\omega_D = \omega_n \sqrt{(1-\xi^2)}$ .... (1.11)

A typical variation of x(t) with t is shown in figure 1.3.

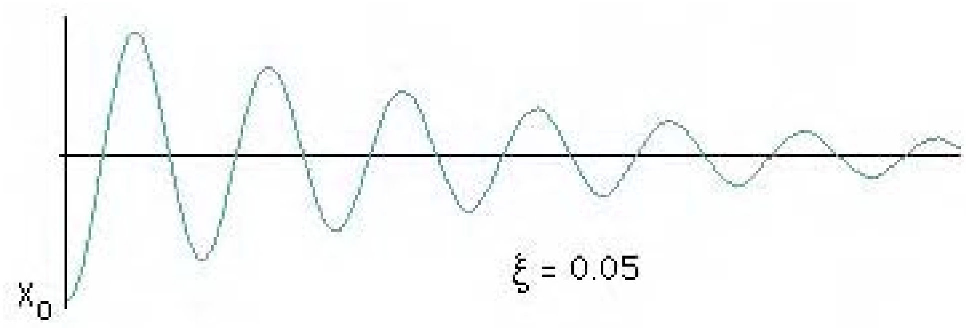

Figure 1.3 Transient vibration with $\xi < 1$

Run a simulation program and vary ξ

The greater the value of ξ the more rapid the decay of the oscillation. As ξ–>1 then the solution tends to have no oscillation. In fact $\xi = 1$ is the lowest value of ξ which does not give any oscillation and this is thus termed the critical damping ratio. From the definition of ξ (= $c / 2\sqrt{km}$) it is clear that when $\xi = 1$ the critical value of $c_c = 2\sqrt{km}$, so that $\xi = c/c_c$.

<u>1.2.4 Case when $\xi = 1$</u>

When $\xi = 1$ the inverse Laplace transform of equation (1.8) gives,

$$x(t) = x(0)e^{-\omega_n t} + [x'(0) + \omega_n x(0)]te^{-\omega_n t}$$ .... (1.12)

This is a non oscillatory motion and for the case considered previously, ie. $x(0) = X_0$ and $x'(t) = 0$ then

$$x(t) = x(0)(1 + \omega_n t)e^{-\omega_n t}$$ .... (1.13)

The variation of x(t) with t is shown in figure 1.4. If the value of ξ exceeds 1.0 then another mathematical solution is obtained.

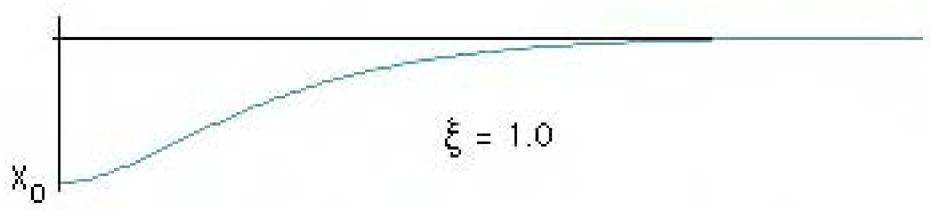

Figure 1.4 Transient vibration with $\xi = 1$

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1.2.5 Case when $\xi > 1$

When $\xi > 1$ the inverse Laplace transform of equation (1.8) gives,

$$x(t) = e^{-\xi\omega_n t}\left[x(0)\cosh\omega_n\sqrt{(\xi^2-1)}t + \frac{[x'(0)+\xi\omega_n x(0)]\sinh\omega_n\sqrt{(\xi^2-1)}t}{\omega_n\sqrt{(\xi^2-1)}}\right] \quad \text{......... (1.14)}$$

This is a non-oscillatory decaying motion and for the case considered previously, ie. $x(0) = X_0$ and $x'(t) = 0$ then

$$x(t) = X_0 e^{-\xi\omega_n t}\left[\cosh\omega_n\sqrt{(\xi^2-1)}t + \frac{\xi\sinh\omega_n\sqrt{(\xi^2-1)}t}{\sqrt{(\xi^2-1)}}\right] \quad \text{........... (1.15)}$$

Typical variations of x(t) with t for various values of ξ are shown in figure 1.5.

Figure 1.5 Transient vibration with $\xi > 1$

Run a simulation program and vary ξ

For real engineering structures it would be extremely unusual for ξ to exceed unity and thus any transient vibration is normally oscillatory. It is possible from measured transients to calculate ξ. This approach uses the logarithmic decrement.

1.3 Logarithmic decrement.

For $\xi<1$ and free motion, ie. no exciting force, the solution is,

$$x(t) = X_0 e^{-\xi\omega_n t}\left[\cos\omega_n\sqrt{(1-\xi^2)}t + \frac{\xi\sin\omega_n\sqrt{(1-\xi^2)}t}{\sqrt{(1-\xi^2)}}\right]$$

and this may be written

$$x(t) = e^{-\xi\omega_n t}\left[A\cos\omega_n\sqrt{(1-\xi^2)}t + B\sin\omega_n\sqrt{(1-\xi^2)}t\right] \quad \text{.................... (1.15)}$$

where $A = X_0$ and $B = \dfrac{x'(0)+\xi\omega_n X_0}{\omega_n\sqrt{(1-\xi^2)}}$.

Thus for arbitrary initial conditions a solution of the form given in equation (1.15) will apply. Now equation (1.15) may be manipulated as follows.

$$x(t) = e^{-\xi\omega_n t}\sqrt{(A^2 - B^2)}\left[\frac{A}{\sqrt{(A^2 - B^2)}}\cos\omega_n\sqrt{(1-\xi^2)}t + \frac{B}{\sqrt{(A^2 - B^2)}}\sin\omega_n\sqrt{(1-\xi^2)}t\right]$$

$$= e^{-\xi\omega_n t}\sqrt{(A^2 - B^2)}\left[\sin\phi\cos\omega_n\sqrt{(1-\xi^2)}t + \cos\phi\sin\omega_n\sqrt{(1-\xi^2)}t\right]$$

$$= Ce^{-\xi\omega_n t}\sin\left[\omega_n\sqrt{(1-\xi^2)}t + \phi\right] \quad \text{......................................} \ (1.16)$$

where $\tan\phi = A/B$ and $C = \sqrt{A^2 + B^2}$

Equation (1.16) represents a decaying oscillation of frequency $\omega_n\sqrt{(1-\xi^2)}$, the damped natural frequency.

Now consider successive maximum amplitudes in the vibration as shown in figure 1.6.

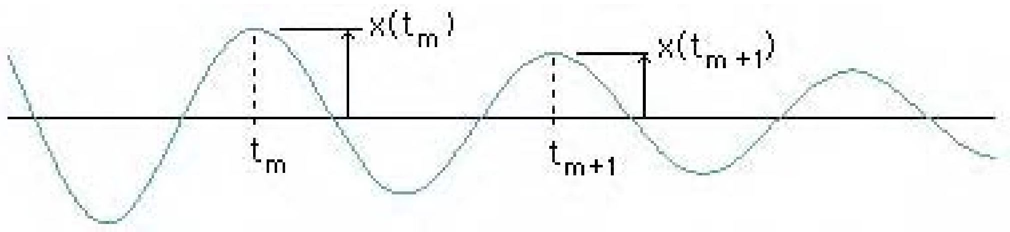

Figure 1.6 Transient vibration with $\xi < 1$

$\frac{dx(t)}{dt} = 0$ at a maximum. Differentiating equation (1.16)

$$\frac{dx(t)}{dt} = C\left[-\xi\omega_n e^{-\xi\omega_n t}\sin\left[\omega_n\sqrt{(1-\xi^2)}t + \phi\right] + \omega_n\sqrt{(1-\xi^2)}e^{-\xi\omega_n t}\cos\left[\omega_n\sqrt{(1-\xi^2)}t + \phi\right]\right]$$

this is zero when

$$-\xi\omega_n\sin\left[\omega_n\sqrt{(1-\xi^2)}t + \phi\right] + \omega_n\sqrt{(1-\xi^2)}\cos\left[\omega_n\sqrt{(1-\xi^2)}t + \phi\right] = 0$$

ie. when
$$\tan\left[\omega_n\sqrt{(1-\xi^2)}t + \phi\right] = \frac{\sqrt{(1-\xi^2)}}{\xi}$$

which is when

$$\omega_n\sqrt{(1-\xi^2)}t + \phi = \tan^{-1}\left[\frac{\sqrt{(1-\xi^2)}}{\xi}\right] + n\pi$$

This represents alternate max/min. Consider any two successive maxima or minima.

Then $$\omega_n\sqrt{(1-\xi^2)}t_m + \phi = \tan^{-1}\left[\frac{\sqrt{(1-\xi^2)}}{\xi}\right] + 2m\pi \quad \text{............................} \ (a)$$

$$\text{and} \quad \omega_n\sqrt{(1-\xi^2)}\,t_{m+1} + \phi = \tan^{-1}\left[\frac{\sqrt{(1-\xi^2)}}{\xi}\right] + 2(m+1)\pi \quad \text{..................} \quad \text{(b)}$$

$$\text{(b) - (a) gives} \qquad t_{m+1} - t_m = \frac{2\pi}{\omega_n\sqrt{(1-\xi^2)}} \quad \text{................................} \quad \text{(c)}$$

Now
$$x(t_m) = Ce^{-\xi\omega_n t_m}\sin\left[\omega_n\sqrt{(1-\xi^2)}\,t_m + \phi\right]$$

and
$$x(t_{m+1}) = Ce^{-\xi\omega_n t_{m+1}}\sin\left[\omega_n\sqrt{(1-\xi^2)}\,t_{m+1} + \phi\right]$$

$$\therefore \frac{x(t_m)}{x(t_{m+1})} = \frac{e^{-\xi\omega_n t_m}\sin\left[\omega_n\sqrt{(1-\xi^2)}\,t_m + \phi\right]}{e^{-\xi\omega_n t_{m+1}}\sin\left[\omega_n\sqrt{(1-\xi^2)}\,t_{m+1} + \phi\right]}$$

but since $\tan\left[\omega_n\sqrt{(1-\xi^2)}\,t + \phi\right] = \dfrac{\sqrt{(1-\xi^2)}}{\xi}$ for $t = t_m$ and t_{m+1}

$$\sin\left[\omega_n\sqrt{(1-\xi^2)}\,t_{m+1} + \phi\right] = \sin\left[\omega_n\sqrt{(1-\xi^2)}\,t_m + \phi\right]$$

$$\therefore \frac{x(t_m)}{x(t_{m+1})} = \frac{e^{-\xi\omega_n t_{m+1}}}{e^{-\xi\omega_n t_m}} = e^{\xi\omega_n(t_{m+1}-t_m)}$$

and substituting for $t_{m+1} - t_m$ from (c),

$$\frac{x(t_m)}{x(t_{m+1})} = e^{\left(\frac{2\pi\xi}{\sqrt{(1-\xi^2)}}\right)}$$

The log decrement δ is defined as

$$\delta = \log_e\left(\frac{x(t_m)}{x(t_{m+1})}\right) = \left(\frac{2\pi\xi}{\sqrt{(1-\xi^2)}}\right) \quad \text{....................................} \quad (1.17)$$

If δ is measured we may determine ξ in terms of δ thus

$$\delta = \left(\frac{2\pi\xi}{\sqrt{(1-\xi^2)}}\right)$$

$$\therefore \delta^2 = \frac{4\pi^2\xi^2}{1-\xi^2}$$

$$\therefore \delta^2 - \xi^2\delta^2 = 4\pi^2\xi^2$$

$$\therefore \xi^2(4\pi^2 - \delta^2) = \delta^2$$

$$\text{and hence} \qquad \therefore \xi = \frac{\delta}{\sqrt{(4\pi^2 + \delta^2)}} \quad \text{....................................} \quad (1.18)$$

Thus it is possible from a decay trace to determine the damping ratio ξ.

1.4 Forced vibration: External force.

If the mass is subjected to a force F(t) acting in the positive x(t) as shown in figure 1.6 then the equation of motion (1.5) becomes,

$$mx''(t) + cx'(t) + kx(t) = F(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1.19)$$

Taking Laplace Transforms

$$m[-x'(0) - sx(0) + s^2X(s)] + c[-x(0) + sX(s)] + kX(s) = LF(t)$$

$$X(s) = \frac{LF(t)}{ms^2 + cs + k} + \frac{m(x'(0) + sx(0)) + cx(0)}{ms^2 + cs + k} \quad \ldots\ldots\ldots \quad (1.20)$$

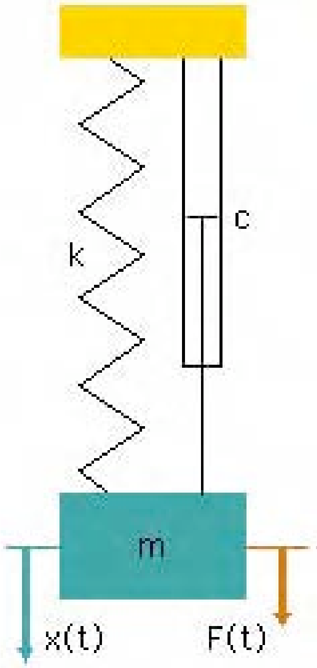

Figure 1.6

The latter term is identical to that already examined, ie. there is a transient component of the response which is dependent on the initial conditions.

The solution of x(t) is thus the superposition of a free vibration arising from the initial conditions plus a solution resulting from the applied force. We shall now examine this component of the solution.

$$X(s) = \frac{LF(t)}{ms^2 + cs + k}$$

Frequently F(t) is cyclic in nature. Thus the solution when F(t) = Fsinωt is commonly examined. This is also relevant since many excitation forces may be separated into sinusoidal components.

Now $$LF(t) = LF\sin\omega t = \frac{F}{s^2 + \omega^2}$$

therefore $$X(s) = \frac{F}{(s^2 + \omega^2)(ms^2 + cs + k)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.21)$$

taking partial fractions

$$X(s) = \frac{As + B}{(s^2 + \omega^2)} + \frac{Cs + D}{(ms^2 + cs + k)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.22)$$

where the second term will yield an exponential decaying motion of the kind already encountered. This is a transient motion arising from the fact that taking the Laplace Transform implies that the forcing function is zero when t < 0. Thus the sinusoidal excitation commences at t = 0 and there is a transient associated with the 'start up' of the motion. A typical start up transient is shown in figure 1.7. In some real applications these transients can be very significant. For example with road drills and rotating machinery.

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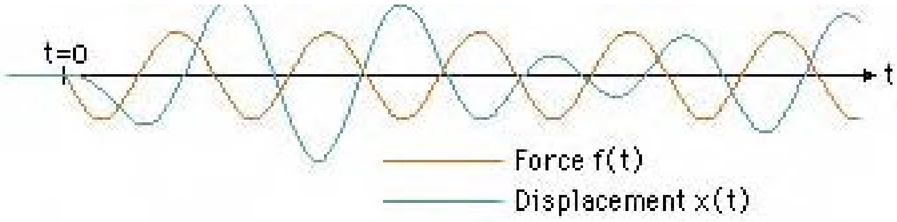

Figure 1.7 Forced vibration showing initial transient.


The 'start up' transient may be examined by using the program

When considering sinusoidal excitation we are normally interested in the steady state motion that results and is given by the first term in equation (1.22). Thus we are interested in A and B. From equations (1.21) and (1.22) if the coefficients of s^3, s^2, s and the constant term are compared it is possible to show that,

$$A = \frac{-\omega c F}{(k - m\omega^2)^2 + \omega^2 c^2}$$
$$B = \frac{\omega(k - m\omega^2)F}{(k - m\omega^2)^2 + \omega^2 c^2}$$

The steady state component of x(t) is given by taking the inverse Laplace transform of,

$$X(s) = \frac{As + B}{(s^2 + \omega^2)}$$

thus

$$X(t) = A\cos\omega t + \frac{B}{\omega}\sin\omega t$$

and substituting for A and B gives

$$x(t) = \frac{F\left(-\omega c\cos\omega t + (k - m\omega^2)\sin\omega t\right)}{(k - m\omega^2)^2 + \omega^2 c^2}$$

so that $$x(t) = \frac{F\sin(\omega t + \phi)}{\left((k - m\omega^2)^2 + \omega^2 c^2\right)^{1/2}} \qquad \text{..................................} \quad (1.23)$$

$$\text{where } \tan\phi = \frac{-\omega c}{(k - m\omega^2)}$$

It is conventional to represent x(t) as $X\sin(\omega t+\phi)$, where X is the amplitude of the response and ϕ the phase angle. Thus from (1.23)

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$$X = \frac{F}{\left((k - m\omega^2)^2 + \omega^2 c^2\right)^{1/2}}$$

The results may be non-dimensionalised by multiplying throughout by k and dividing by F.

$$\text{Thus, } \frac{kX}{F} = \frac{1}{\left(\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2 \frac{\omega^2}{\omega_n^2}\right)^{1/2}} \quad \text{and} \quad \tan\phi = \frac{-2\xi\omega / \omega_n}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2} \quad \text{.................. (1.24)}$$

The variation of kX/F and ϕ as functions of ω/ω_n are shown in figure 1.8 for $\xi = 0.1$.

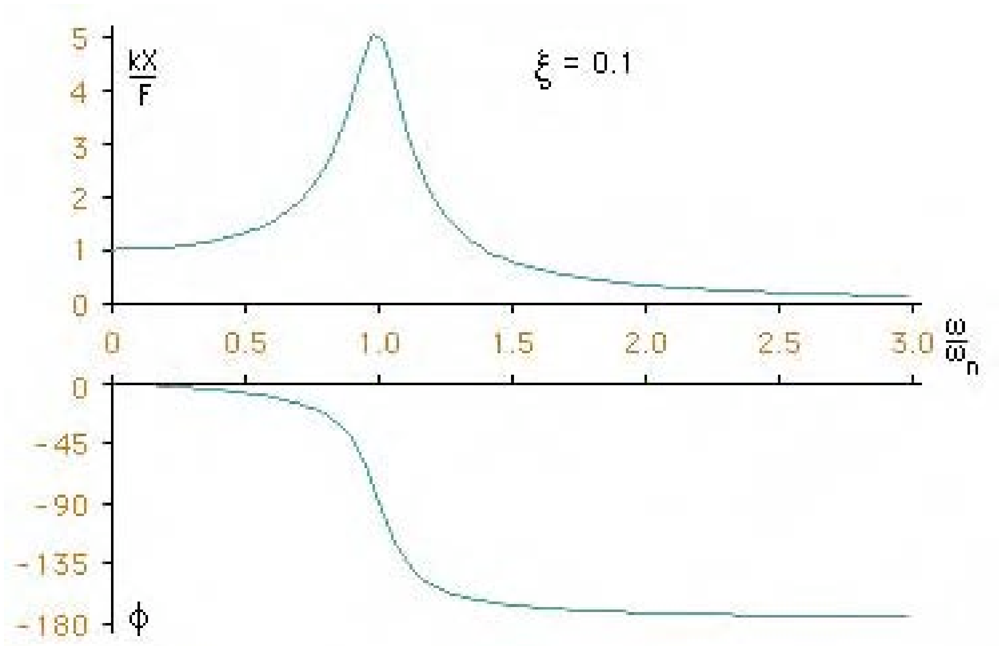


Figure 1.8 Steady state response and phase for external force excitation.

Investigate the effect of changing ξ by running the program.

The main points to note are that:-

i) $\frac{kX}{F} \to 1$ as $\frac{\omega}{\omega_n} \to 0$ this is generally known as the quasi-static condition as $\frac{X}{F} \to \frac{1}{k}$.

ii) $\frac{kX}{F} \to 0$ as $\frac{\omega}{\omega_n} \to \infty$.

iii) Resonance occurs, ie. kX/F is a maximum, when d(kX/F)/dω is zero. This can be shown to be when ,

$$\omega_r = \omega_n\sqrt{(1 - 2\xi^2)}$$

However, note that there is no real solution for ω_r when $\xi > 1/\sqrt{2}$, ie. the response continuously falls with frequency.

iv) The final point of interest is the response amplitude at resonance, ie.

$$\frac{kX}{F} = \frac{1}{\left(\left(1-\left(1-2\xi^2\right)\right)^2 + 4\xi^2\left(1-2\xi^2\right)\right)^{1/2}}$$

$$\frac{kX}{F} = \frac{1}{\left(4\xi^4 + 4\xi^2 - 8\xi^4\right)^{1/2}}$$

which for small values of ξ is equal to 1/(2ξ).

The method of approach thus far has been to use Laplace Transforms. However when the steady state response is required it is possible to obtain this more directly using an exciting force $Fe^{i\omega t}$, and assuming a steady state response $Xe^{i\omega t}$. This method does not provide the transient solutions due to the initial conditions at the commencement of the excitation. Consider again the equation of motion (1.19), this becomes

$$mx''(t) + cx'(t) + kx(t) = Fe^{i\omega t} \quad \text{......................................} \quad (1.25)$$

substituting $x(t) = Xe^{i\omega t}$ gives

$$-m\omega^2 Xe^{i\omega t} + i\omega cXe^{i\omega t} + kXe^{i\omega t} = Fe^{i\omega t}$$

and thus

$$\frac{X}{F} = \frac{1}{k - m\omega^2 + i\omega c}$$

This is a complex expression the amplitude of which is the response amplitude and the phase of which indicates the phase between displacement and force. If the equation is non-dimensionalised by multiplying throughout by k we obtain,

$$\frac{kX}{F} = \frac{1}{1 - \frac{\omega^2}{\omega_n^2} + i2\xi\frac{\omega}{\omega_n}}$$

and this has the same response amplitude and phase lag as before,

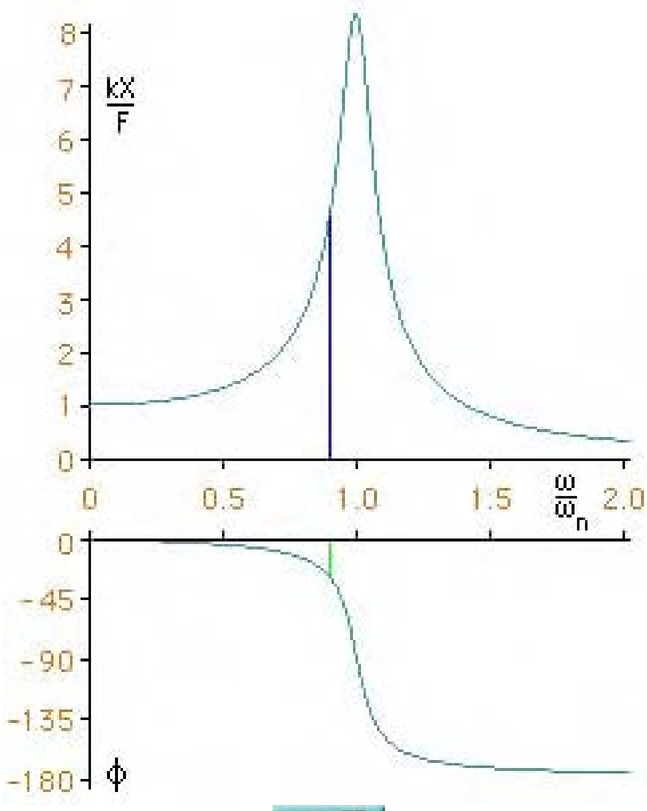
$$\frac{kX}{F} = \frac{1}{\left(\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2 \frac{\omega^2}{\omega_n^2}\right)^{1/2}} \quad \text{and } \tan\phi = \frac{-2\xi\omega / \omega_n}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)} \quad \text{.................. (1.24)}$$

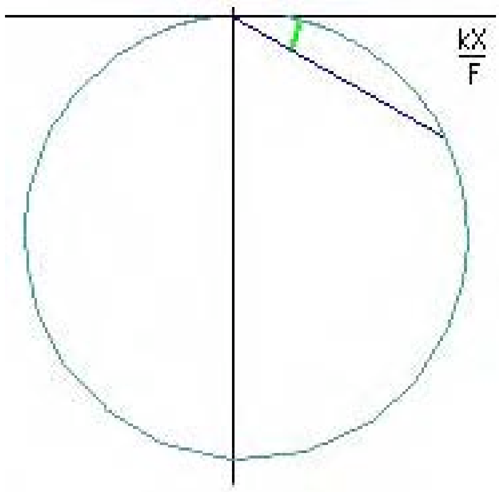
It is clear that this method allows the steady state response to be obtained far more simply and thus it will be the preferred method for steady state analysis.

So far the excitation has been limited to an oscillating force of constant amplitude. There are, however, other types of excitation. Those normally considered are excitation by a rotating out-of-balance mass and also abutment (or floor) excitation. However before considering these excitations we will briefly consider the definition of response locus.

1.4.1 Response locus

For some applications the response is presented as a response locus that combines the response amplitude and phase in one diagram. This is illustrated in figure 1.9.





At each frequency of excitation the response amplitude (blue) and the phase (green) is drawn as a vector with phase angles positive anti-clockwise from the kX/F axis. The end of the this vector traces out a locus as the excitation frequency is varied.

Figure 1.9. Response as amplitude and phase and also response locus.

1.5 Forced vibration: Out-of-balance excitation

The excitation by an out-of-balance is shown diagrammatically in figure 1.10. The out-of-balance mass m' is at a radius r and is rotating at an angular frequency ω, so that $\theta = \omega t$. The acceleration of the mass m' in the x(t) direction is then $-\omega^2 r \sin\omega t$ relative to the mass m. As this mass has an acceleration x"(t) the mass m' has an acceleration $x''(t) - \omega^2 r \sin\omega t$. There is therefore a force $m'(x''(t) - \omega^2 r \sin\omega t)$ on m' to provide this acceleration. From Newton III there is an opposite reaction force on m given by,

$$-m'x''(t) + m'\omega^2 r \sin\omega t$$

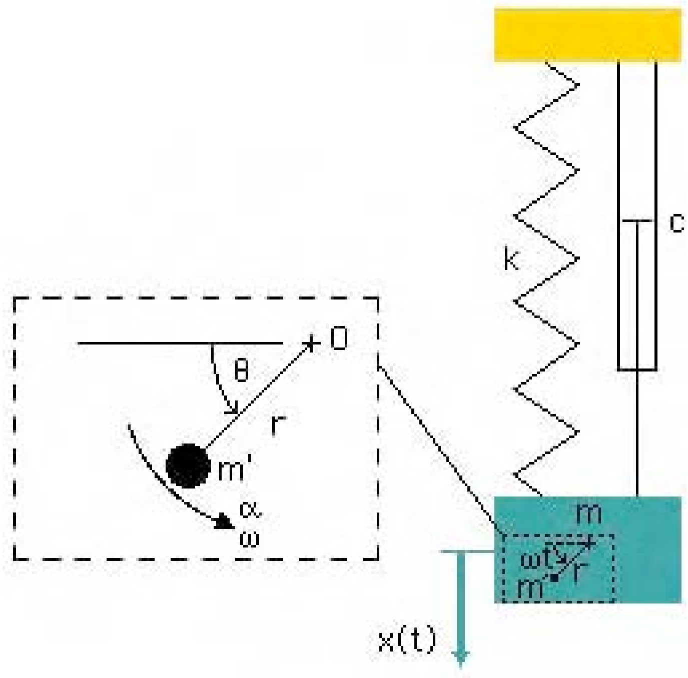


Figure 1.10 Excitation by an out-of-balance mass.

The equation of motion thus becomes,

$$mx''(t) + cx'(t) + kx(t) = -m'x''(t) + m'\omega^2 r \sin\omega t \quad \text{......................} \quad (1.26)$$

which may be rearranged to give

$$(m + m')x''(t) + cx'(t) + kx(t) = m'\omega^2 r \sin\omega t$$

This is basically the same as for the excitation by F$\sin\omega t$, but F is replaced by $m'\omega^2 r$ and the mass of the system is increased to m+m'. If a Laplace Transform solution was completed it would be found that there is an initial 'start up' transient. If the running speed is above the resonant frequency than the angular acceleration up to the running speed becomes important. If the acceleration is to small then severe vibration may occur even though at the top speed vibration should not be a problem. For some large rotating systems it is also important for the run down to occur quickly to avoid resonance effects if the running speed is above resonance. Examine the start up transients.

Instant running at selected speed.

Constant angular acceleration to top speed.

The steady state response is thus given by comparison with equation (1.23), so that

$$x(t) = \frac{m'\omega^2 r \sin(\omega t - \phi)}{\left([k-(m+m')\omega^2]^2 + \omega^2 c^2\right)^{1/2}} \quad \text{where } \tan\phi = \frac{-\omega c}{[k-(m+m')\omega^2]}$$

These results may also be non-dimensionalised thus,

$$X = \frac{\dfrac{m' r}{m+m'}\dfrac{\omega^2}{\omega_n^2}}{\left(\left(1-\dfrac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2\dfrac{\omega^2}{\omega_n^2}\right)^{1/2}} \quad \text{and } \tan\phi = \frac{-2\xi\omega/\omega_n}{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)} \quad \text{.......................} \quad (1.27)$$

$$\text{where } \omega_n = \sqrt{\frac{k}{(m+m')}} \quad \text{and } \xi = \frac{c}{2\sqrt{(m+m')k}}$$

Typical response curves are shown in figure 1.11. The main points to note are that:-

i) $X \to 0$ as $\dfrac{\omega}{\omega_n} \to 0$ this is because the excitation force tends to zero.

ii) $X \to \dfrac{-m' r}{m+m'}$ as $\dfrac{\omega}{\omega_n} \to \infty$ This is anti-phase with the excitation and the centre of mass of m and m' does not move.

iii) Resonance occurs, ie. X is a maximum, when $dX/d\omega$ is zero. This can be shown to be when $\omega_r = \dfrac{\omega_n}{\sqrt{(1-2\xi^2)}}$. However, note that there is no real solution for ω_r when $\xi > 1/\sqrt{2}$, that is the response continuously rises with frequency and approaches $\dfrac{-m' r}{m+m'}$ asymptotically. When there is a resonance it should be noted that ω_r is greater than ω_n.

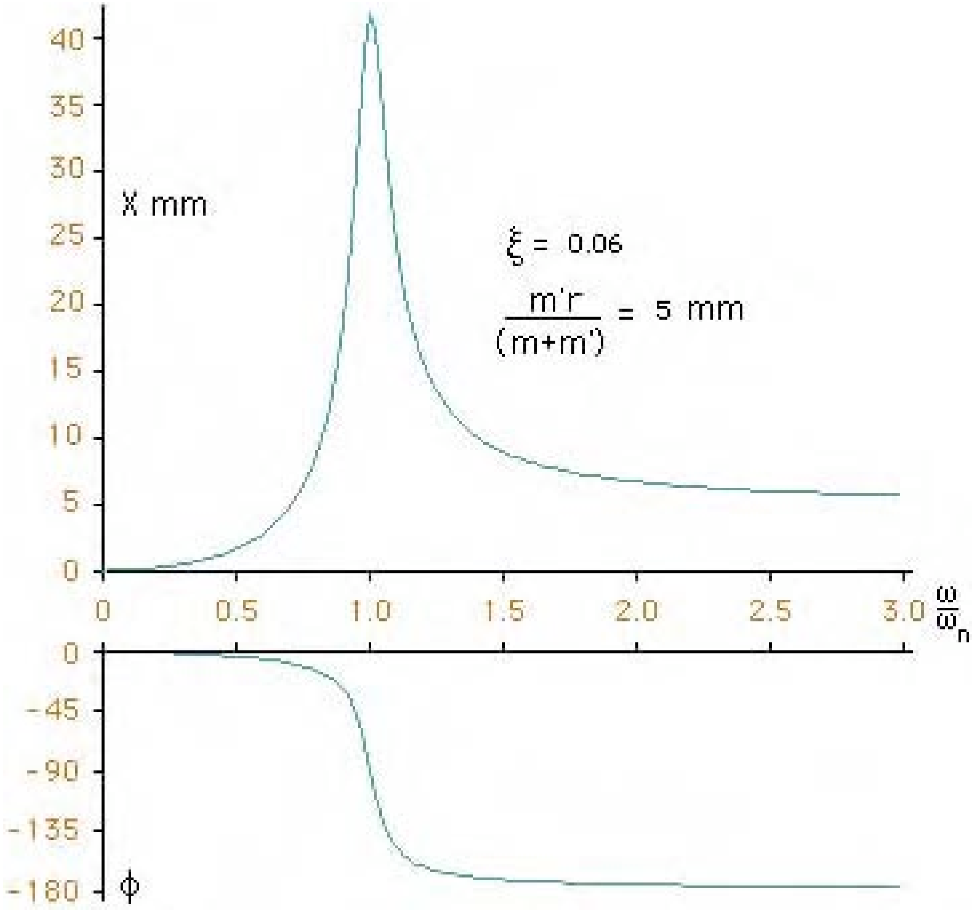


Figure 1.11 Steady state response and phase for out-of-balance excitation.

Investigate the effect of changing ξ. Note that $\frac{m' r}{(m + m')}$ has dimensions of length.

1.6 Forced vibration: Abutment excitation

Sinusoidal excitation by the abutment is shown in the figure 1.12.

Applying Newton's second law of motion gives,

$$mx''(t) = k\left(x_0 - x(t)\right) + c(x'_0 - x'(t)) \quad \ldots\ldots\ldots\ldots \quad (1.27)$$

If a Laplace Transform solution was completed it would be found that there is an initial 'start up' transient.

Examine the start up transient.

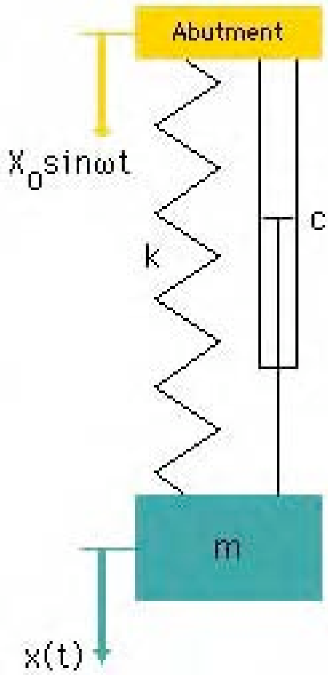


Figure 1.12

For this example it is much simpler to find the steady state response by taking the input excitation to be $x_o = X_o e^{i\omega t}$ and the response to be $x(t) = X e^{i\omega t}$. On substituting in (1.27) this gives,

$$(k - m\omega^2 + i\omega c)Xe^{i\omega t} = (k + i\omega c)X_0 e^{i\omega t}$$

rearranging,

$$\frac{X}{X_0} = \frac{(k + i\omega c)}{(k - m\omega^2 + i\omega c)}$$

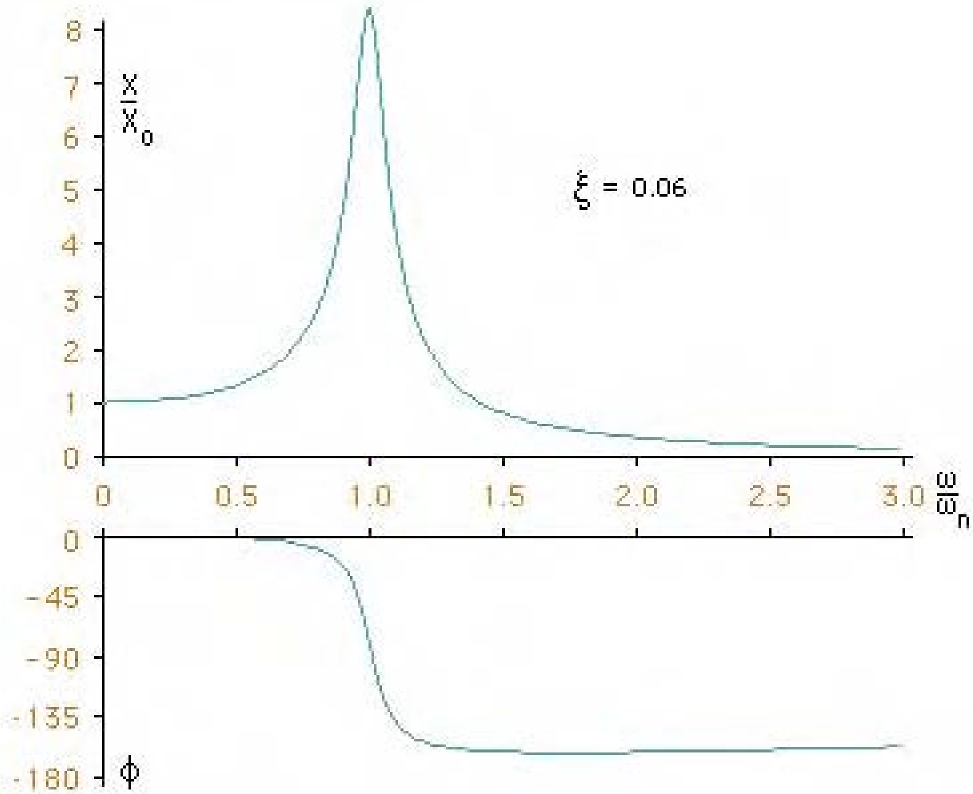
The phase relationship is very complex but normally we would be interested in the amplitude of the response and this may be found by finding the amplitude of the 'top' and 'bottom' vectors. If the result is also non-dimensionalised by dividing throughout by k, the following result is obtained,

$$\frac{X}{X_0} = \frac{\left(1 + 4\xi^2 \dfrac{\omega^2}{\omega_n^2}\right)^{1/2}}{\left(\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2 \dfrac{\omega^2}{\omega_n^2}\right)^{1/2}} \qquad \text{.......................................} \quad (1.28)$$

and the phase is given by,

$$\tan\phi = \frac{-2\xi \dfrac{\omega^3}{\omega_n^3}}{\left(1 - \dfrac{\omega^2}{\omega_n^2}\right) + 4\xi^2 \dfrac{\omega^2}{\omega_n^2}}$$

Typical response and phase curves are shown in figure 1.13.



Investigate the effect of changing ξ.

Figure 1.13 Steady state response and phase for abutment excitation.

The main points to note are that:-

i) $\dfrac{X}{X_o} \to 1$ as $\dfrac{\omega}{\omega_n} \to 0$, that is there is no deflection across the spring.

ii) $\dfrac{X}{X_o} = 1$ when $\dfrac{\omega}{\omega_n} = \sqrt{2}$ independent of the value of ξ.

iii) $\dfrac{X}{X_o} \to 0$ as $\dfrac{\omega}{\omega_n} \to \infty$, this is termed vibration isolation.

iv) Resonance occurs, ie. X/X_o is a maximum, when $d(X)/d\omega$ is zero. This can be shown to be when

$$\omega_r = \omega_n \sqrt{\frac{-1 + \sqrt{1 + 8\xi^2}}{4\xi^2}}$$

Note that this resonance occurs for all values of ξ.

1.7 Transmissibility

When the system shown in figure 1.14 is excited by an oscillating force it is often of interest to determine the force on the abutment that results from the motion. This is important since it is this force that may induce floor borne vibrations and cause problems elsewhere.

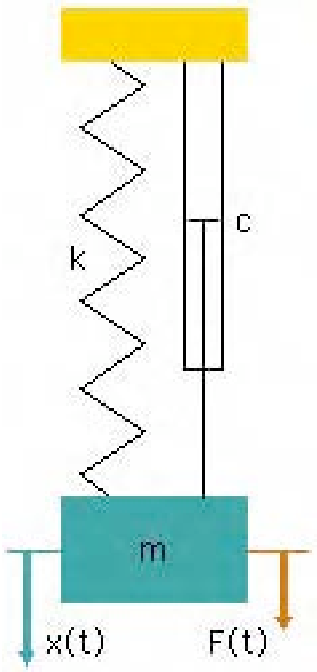


Figure 1.14

Transmissibility is defined as the amplitude of the force on the abutment divided by the exciting force amplitude. The force on the abutment is given by,

$$F_T e^{i\omega t} = kx(t) + cx'(t)$$

when the excitation force is $Fe^{i\omega t}$ then $x(t) = Xe^{i\omega t}$ and hence $F_T = (k + i\omega c)X$.

Substituting for X from equation (1.24) gives,

$$\frac{F_T}{F} = \frac{\left(1 + 4\xi^2 \dfrac{\omega^2}{\omega_n^2}\right)^{1/2}}{\left(\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2 \dfrac{\omega^2}{\omega_n^2}\right)^{1/2}} \quad \text{.....................................} \quad (1.29)$$

This is the same result as for abutment motion given in equation (1.28) ie. $\frac{F_T}{F} = \frac{X}{X_o}$

1.8 Measures of Damping

One of the areas of confusion in the vibration area is the multiplicity of methods of expressing damping. So far we have encountered:-

i) c the viscous damping coefficient.

ii) ξ the viscous damping ratio defined as $\xi = c/2\sqrt{km}$

iii) δ the logarithmic decrement, defined as $\delta = 2\pi/\sqrt{(1-\xi^2)}$

There are also several others which we will need to consider.

1.8.1 Dynamic magnification factor (Q)

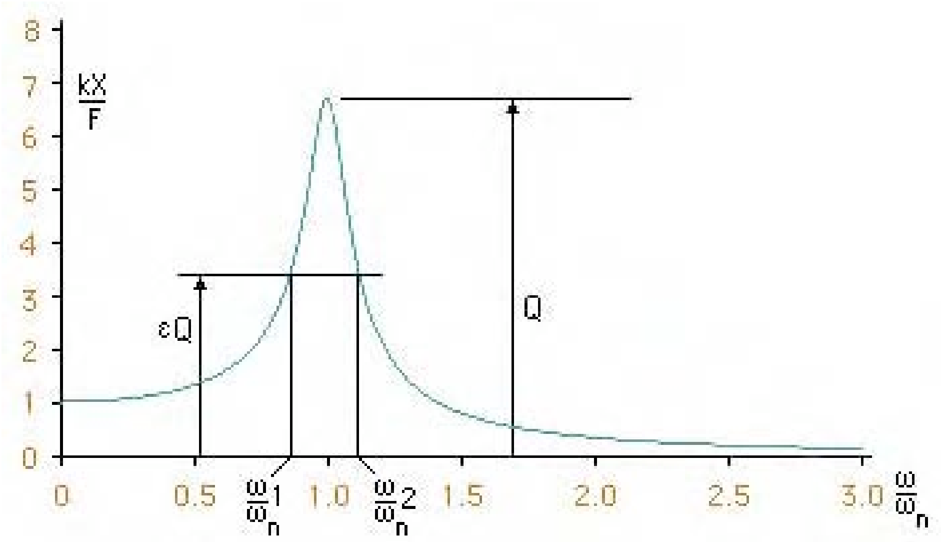


Figure 1.15 Typical response curve.

The ratio of the amplitude at resonance to the quasi-static amplitude when excited by an oscillating force is termed the dynamic magnification factor Q. For lightly damped structures Q = $1/(2\xi)$. This is a particularly useful measure as it may be determined from the steady state response curve using the bandwidth of the resonance peak. Figure 1.15 shows a typical resonance curve and at the frequencies ω_1 and ω_2 the response is ε times the response at resonance. Now $\omega_2 - \omega_1$ is called the bandwidth ($\Delta\omega$) and may be used to determine Q. As has been shown previously,

$$\frac{kX}{F} = \frac{1}{\left(\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2 \frac{\omega^2}{\omega_n^2}\right)^{1/2}}$$

and at resonance this is $1/(2\xi)$. Thus to find ω_1 and ω_2 when the response is ε times that at resonance we need the condition that,

$$\frac{\varepsilon}{2\xi} = \frac{1}{\left(\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2\frac{\omega^2}{\omega_n^2}\right)^{1/2}}$$

rearranging gives

$$\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2\frac{\omega^2}{\omega_n^2} = \frac{4\xi^2}{\varepsilon^2}$$

if we put $q = \omega/\omega_n$ then

$$q^4 + q^2(4\xi^2 - 2) + 1 - \frac{4\xi^2}{\varepsilon^2} = 0$$

$$q^2 = \frac{-(4\xi^2-2) \pm \sqrt{(4\xi^2-2)^2 - 4\left(1-\frac{4\xi^2}{\varepsilon^2}\right)}}{2}$$

$$\therefore q^2 = 1 - 2\xi^2 \pm 2\xi\sqrt{\xi^2 - 1 + \frac{1}{\varepsilon^2}}$$

Now if ξ is small and ε is not too near unity,

$$\therefore q^2 = 1 \pm 2\xi\sqrt{\frac{1}{\varepsilon^2} - 1}$$

thus $$\frac{\omega_1^2}{\omega_n^2} = 1 - 2\xi\sqrt{\frac{1}{\varepsilon^2} - 1} \quad \text{...} \quad (1.30)$$

and $$\frac{\omega_2^2}{\omega_n^2} = 1 + 2\xi\sqrt{\frac{1}{\varepsilon^2} - 1} \quad \text{...} \quad (1.31)$$

Now $$\frac{\omega_2 - \omega_1}{\omega_n} = \frac{\omega_2^2 - \omega_1^2}{\omega_n(\omega_2 + \omega_1)}$$

and $\omega_2 + \omega_1$ is approximately equal to $2\omega_n$ therefore (31) - (32) gives,

$$\frac{\Delta\omega}{\omega_n} = \frac{\omega_2 - \omega_1}{\omega_n} = \frac{\omega_2^2 - \omega_1^2}{2\omega_n^2} = \frac{\left(1 + 2\xi\sqrt{\frac{1}{\varepsilon^2} - 1}\right) - \left(1 - 2\xi\sqrt{\frac{1}{\varepsilon^2} - 1}\right)}{2} = 2\xi\sqrt{\frac{1}{\varepsilon^2} - 1}$$

It is common to measure $\Delta\omega$ when $\varepsilon = 1/\sqrt{2}$ (this is called the 3dB point as $20\log(1/\sqrt{2}) = -3$) and also to approximate $\omega_r = \omega_n$ therefore

$$\frac{\Delta\omega}{\omega_r} = 2\xi \quad \text{...} \quad (1.32)$$

This is a very useful measure of damping as it may be applied to a response with several resonances and will give the equivalent damping ratio for each resonance.

1.8.2 Specific damping capacity(φ) and loss factor (η)

Another common basis for measuring damping involves energy loss and involves the determination of the energy lost per cycle. In all cases involving damping the force - displacement relationship when plotted graphically will enclose an area, commonly called the hysteresis loop. The work done during a cycle is given by,

$$W_d = \int F_d dx$$

which for viscous damping gives (since the other force components do no net work during a cycle)

$$W_d = \int cx'\,dx = \int c\frac{dx}{dt}dx = \int c\frac{dx}{dt}\frac{dx}{dt}dt = \int cx'^2\,dt$$

Now for cyclic motion $x = X\sin(\omega t - \phi)$ and therefore $x' = \omega X\cos(\omega t - \phi)$ and thus for one cycle,

$$W_d = \int_0^{2\pi/\omega} c(\omega X\cos(\omega t - \phi))^2 dt = c\omega^2 X^2 \int_0^{2\pi/\omega} (\cos(\omega t - \phi))^2 dt$$

$$= c\omega^2 X^2 \int_0^{2\pi/\omega} \left(\frac{1 + \cos 2(\omega t - \phi)}{2}\right) dt$$

$$= c\omega^2 X^2 \left|\frac{t}{2} + \frac{\sin 2(\omega t - \phi)}{4\omega}\right|_0^{2\pi/\omega}$$

therefore $$W_d = \pi c\omega X^2$$

The main interest involves the energy lost per cycle at resonance, which for small levels of damping requires, $\omega = \sqrt{k/m}$ and $c = 2\xi\sqrt{km}$. This gives

$$W_d = 2\pi\xi kX^2 \quad \text{..} \quad (1.33)$$

Now the specific damping capacity φ is defined as the energy loss per cycle divided by the peak energy stored, which for the spring mass system being considered is the energy stored in the spring at maximum deflection, ie. $kX^2/2$.

$$\text{Therefore } \varphi = \frac{4\pi\xi kX^2}{kX^2} = 4\pi\xi \quad \text{..} \quad (1.34)$$

The main drawback to the above is that the damping force is dependent on frequency and thus a specific frequency is required in order to evaluate the energy loss. In practice the energy loss per cycle is often not too dependent on frequency and a hysteretic damping factor is introduced, which is defined as $h = \omega c$ so that at any frequency

$$W_d = \pi h X^2$$

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In this case, the specific damping capacity φ is then

$$\varphi = \frac{2\pi h X^2}{k X^2} = 2\pi \frac{h}{k} = 2\pi\eta \quad \text{...} \quad (1.35)$$

where η=h/k and is defined as the loss factor.

If we combine all the definitions of damping the following relationships are obtained,

$$\varphi = 2\pi\eta = 2\pi\xi = 2\pi / Q = 2\delta = 2\pi\Delta\omega / \omega_r$$

where $\Delta\omega$ is measured at the 3dB point

1.9 Conclusions

It has been shown that vibration may occur because of some initial conditions/disturbances and also because of excitation by cyclic forces, rotating out-of-balance and abutment motion. Cures may be achieved by,

a. Removing the initial conditions/disturbance that cause the transient vibration.
b. Removing the cyclic excitation.
c. Increasing the damping to reduce the response at resonance.
d. Avoiding resonant conditions by either changing the resonant frequency or the frequency of excitation.
e. Employing vibration isolation, ie. making the resonant frequency much less than the excitation frequency. Also note that for abutment excitation increasing damping is not good for vibration isolation.

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