Measures of location

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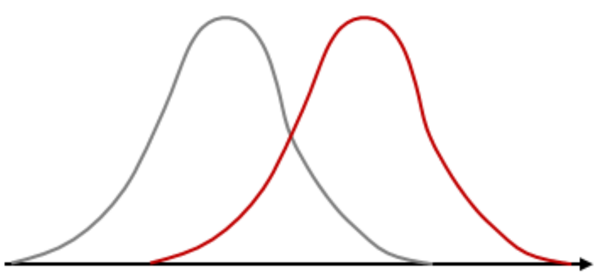
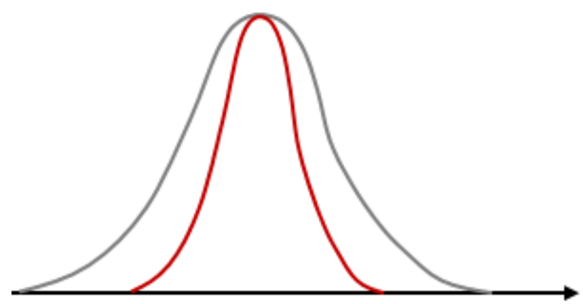
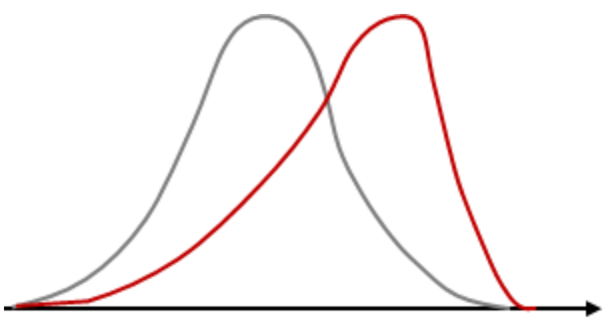
Descriptive statistics

Descriptive statistics

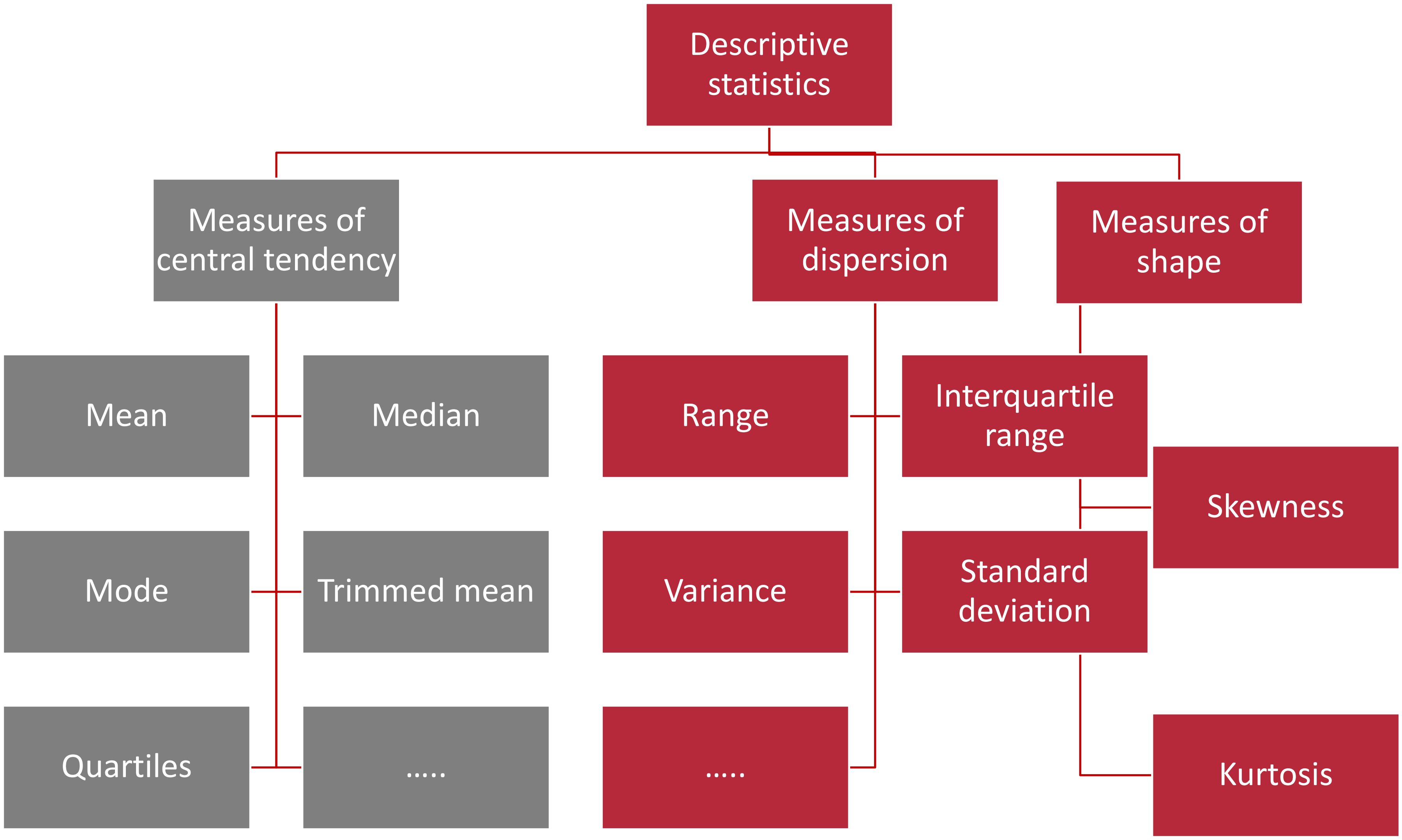
- Measures of location (central tendency/position)
- Measures of dispersion
- Measures of shape







Descriptive statistics

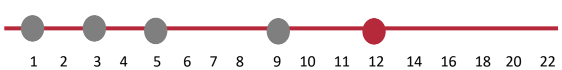


Arithmetic mean

- The **arithmetic mean** is one of the most intuitive measures of central tendency.
- It is often simply referred to as „the mean” or „the average”

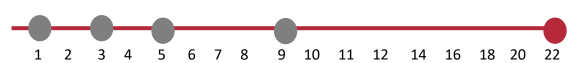
$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

- **The measure is sensitive to extreme values (outliers)!**



Sample 1: 1,3,5,9,12

$$\bar{x} = \frac{(1+3+5+9+12)}{5} = 6$$



Sample 2: 1,3,5,9,22

$$\bar{x} = \frac{(1+3+5+9+22)}{5} = 8$$

Properties of the arithmetic mean

- The sum of the deviations of each variable around the arithmetic mean is zero:

$$\sum_{i=1}^{n}(x_i-\bar{x})=\sum_{i=1}^{n}x_i-n\bar{x}=n\bar{x}-n\bar{x}=0$$

- For linear transformation of the form $y_i=a+bx_i$, where *a* and *b* are known constants, it holds that:

$$\bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i=\frac{1}{n}\sum_{i=1}^{n}(a+bx_i)=\frac{1}{n}\sum_{i=1}^{n}a+\frac{b}{n}\sum_{i=1}^{n}x_i=a+b\bar{x}$$

- **Caution! The mean is not equal to the mean of the means:**

e.g.:

$$\frac{1+2+4+5+8+10+12}{7}=6$$

$$\frac{1}{3}\left(\frac{1+2+4}{3}+\frac{5+8}{2}+\frac{10+12}{2}\right)=\frac{1}{3}(2.3+6.5+11)=6.61$$

Mean for grouped data (weighted mean)

$$\bar{x} = \frac{1}{n}\sum_{j=1}^{k} n_j m_j = \sum_{j=1}^{k} f_j m_j$$

Absolute frequencies

Relative frequencies

The mid-value of the *j* th class interval (group)

Example:

Age	<20	(20-35]	(35-50]	(50-100]
Absolute frequencies	20	25	40	55
Relative frequencies	20/140	25/140	40/140	55/140

$$\bar{x} = \frac{1}{140}(20*10 + 25*27.5 + 40*42.5 + 55*75) =$$

$$\frac{20}{140}*10 + \frac{25}{140}*27.5 + \frac{40}{140}*42.5 + \frac{55}{140}*75 = 47.95$$

Weighted mean

$$\bar{x}_w = \frac{\sum_{i=1}^{n} w_i X_i}{\sum_{i=1}^{n} w_i}$$

Weights

Example:

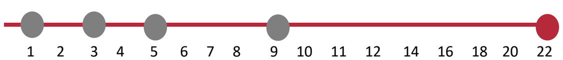
Goods in consumer's basket	Number of goods	Price of a good	Weights
Good A	200	25	0.2
Good B	300	15	0.3
Good C	500	10	0.5

$$\bar{x}_w = \frac{\sum_{i=1}^{n} w_i X_i}{\sum_{i=1}^{n} w_i} = \frac{0.2*25+0.3*15+0.5*10}{0.2+0.3+0.5} = 14.5$$

Trimmed mean (truncated mean)

- It is the arithmetic mean value computed with a specified percentage of values removed from each tail to eliminate the highest and lowest outliers and extreme values.
- For small samples a specific number of observations that represent extreme cases (e.g. 1) rather than a percentage, is simply dropped when calculating the mean.

- In our example:



Sample 2: 1,3,5,9,22

$$\bar{x} = \frac{(1+3+5+9+22)}{5} = 8$$

We would drop the extreme value of 22 to get the trimmed mean equal to:

$$\bar{x} = \frac{(1+3+5+9)}{4} = 4.5$$

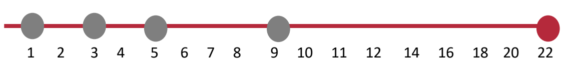
We could also calculate the trimmed 20% mean to get:

$$\bar{x} = \frac{(3+5+9)}{3} = 5.7$$

Winsorized mean

- The Winsorized mean (named after the biostatistician C P Winsor) is similar to the trimmed mean, but instead of dropping extreme values they are simply replaced with the most extreme remaining values.

- In our example:



Sample 2: 1,3,5,9,22

$$\bar{x} = \frac{(1+3+5+9+22)}{5} = 8$$

We would drop the extreme value of 22 and replace it with 9 to get the winsorized mean equal to get:

$$\bar{x} = \frac{(1+3+5+9+9)}{5} = 5.4$$

We could also calculate the 20% winsorized mean to get:

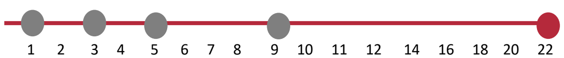
$$\bar{x} = \frac{(3+3+5+9+9)}{5} = 5.8$$

Harmonic mean

$$\bar{x}_H = \frac{n}{\sum_{1=1}^{n} \frac{1}{x_i}}$$

- It is used when there are few very large/small values
- It is often applied to averaging rates of speed.
- Intuition: 5 machines – we produce 1,3,5,9,22 notebooks per hour, we produce the same amount of notebooks for every machine (i.e. 1). We spent 1,68 hours to have 5. If we have 1 machines producing 2,96 notebooks per hour, we would have 5 notebooks in this time.

In our example:



$$\bar{x} = \frac{(1+3+5+9+22)}{5} = 8$$

$$\bar{x}_H = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} = \frac{5}{\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{9}+\frac{1}{22}} = 2.96$$

Or for a series of 8 speeds of: 20,40,60,100,120,180,200,10000

$$\bar{x}_H = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} = \frac{8}{\frac{1}{20}+\frac{1}{40}+\frac{1}{60}+\frac{1}{100}+\frac{1}{120}+\frac{1}{180}+\frac{1}{200}+\frac{1}{10000}} = 66.3$$

Geometric mean

$$\bar{x}_G = \sqrt[n]{\prod_{i=1}^{n} x_i}$$

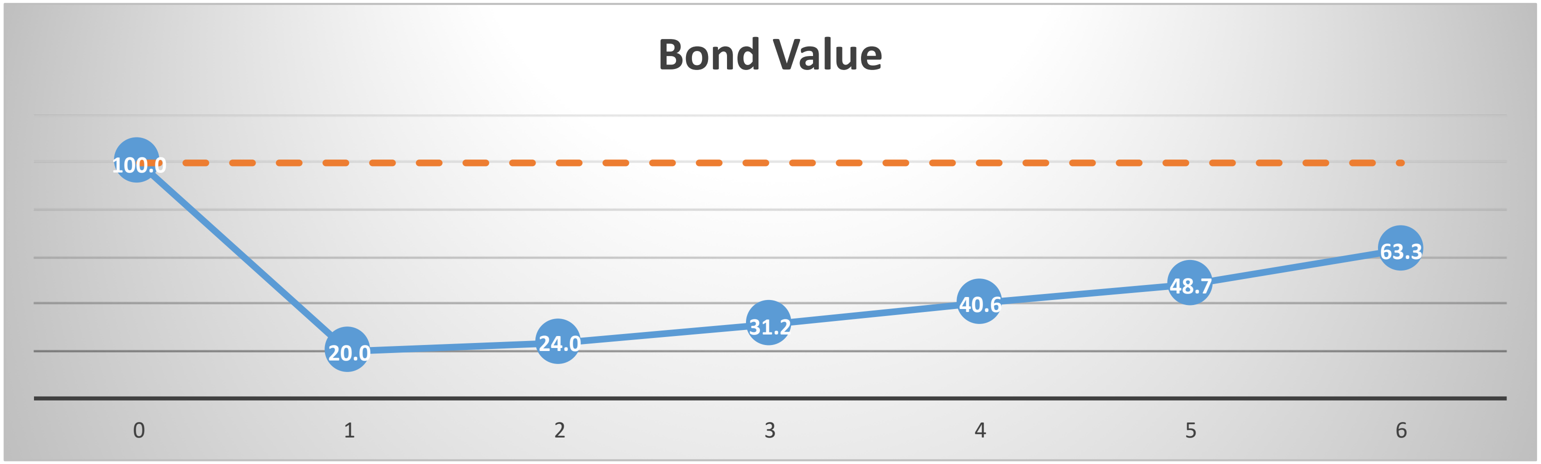
- It is used when the variable is log-normally distributed
- It is often applied to calculating average rate of change

An example:

The annual rates of return from a bond A are: -0.80, 0.20, 0.30, 0.30, 0.20, 0.30
The respective rates of change are: 0.20, 1.20, 1.30, 1.30, 1.20, 1.30

$$\bar{x} = \frac{(0.20+1.20+1.30+1.30+1.20+1.30)}{6} = 1.0833$$

$$\bar{x}_G = \sqrt[6]{0.20*1.20*1.30*1.30*1.20*1.30} = 0.926$$



Means

Exercise 1:

Use data on Apple stocks (Apple.csv) and calculate rate of change for each year. Then calculate:

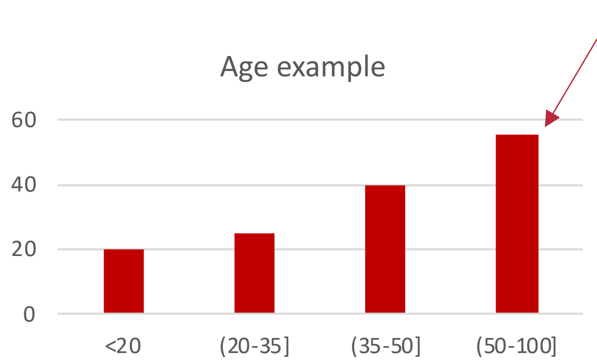
- Airthemitic mean
- Geometric mean
- Weighted mean
- Trimmed mean
- Winsorized mean

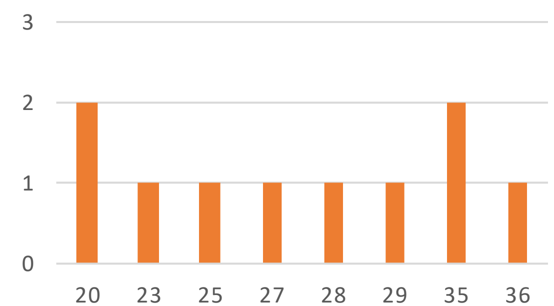
Mode

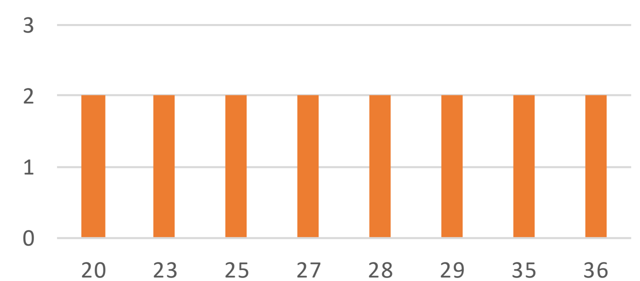
- The mode is the value that occurs most frequently
- **It is not influenced by outliers**
- It can be applied to quantitaive and qualitative variables

- Sometimes there is more than one mode
- Sometimes the mode does not exist

Mode

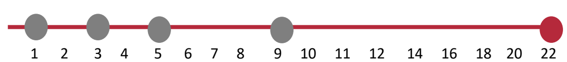






Mid-range

- It is the arithmetic mean of the maximum and minimum values in a dataset
- **It is highly sensitive to outliers** (it only takes into account the two most extreme values from a sample).
- In our example:



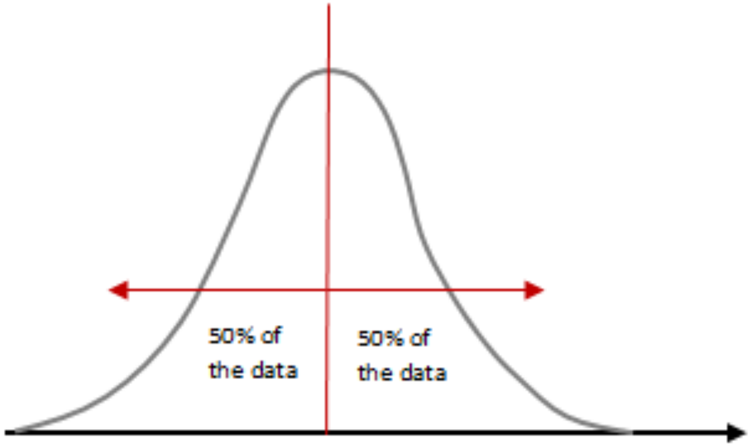
Sample 2: 1,3,5,9,22

$$\bar{x} = \frac{(1+3+5+9+22)}{5} = 8$$

The mid-range is $\frac{(1+22)}{2} = 11.5$

Median

- Median is the middle value; we will denote it as $\tilde{x}_{0.5}$
- It is the value which divides the observations into two equal parts such that at least 50% of the values are greater than or equal to the median and at least 50% of the values are less than or equal to the median.
- In terms of the empirical cumulative distribution function the median satisfies: $F(\tilde{x}_{0.5}) = 0.5$
- **Outliers do not influence median**
- There is always only one median (uniqueness)



Median

- To calculate median „by hand" we need to sort the data in an ascending order

- Then calculate the median as:
  - When n is odd: $\tilde{x}_{0.5} = x_{((n+1)/2)}$
  - When n is even: $\tilde{x}_{0.5} = \frac{1}{2}(x_{(n/2)} + x_{(n/2+1)})$

- Example:

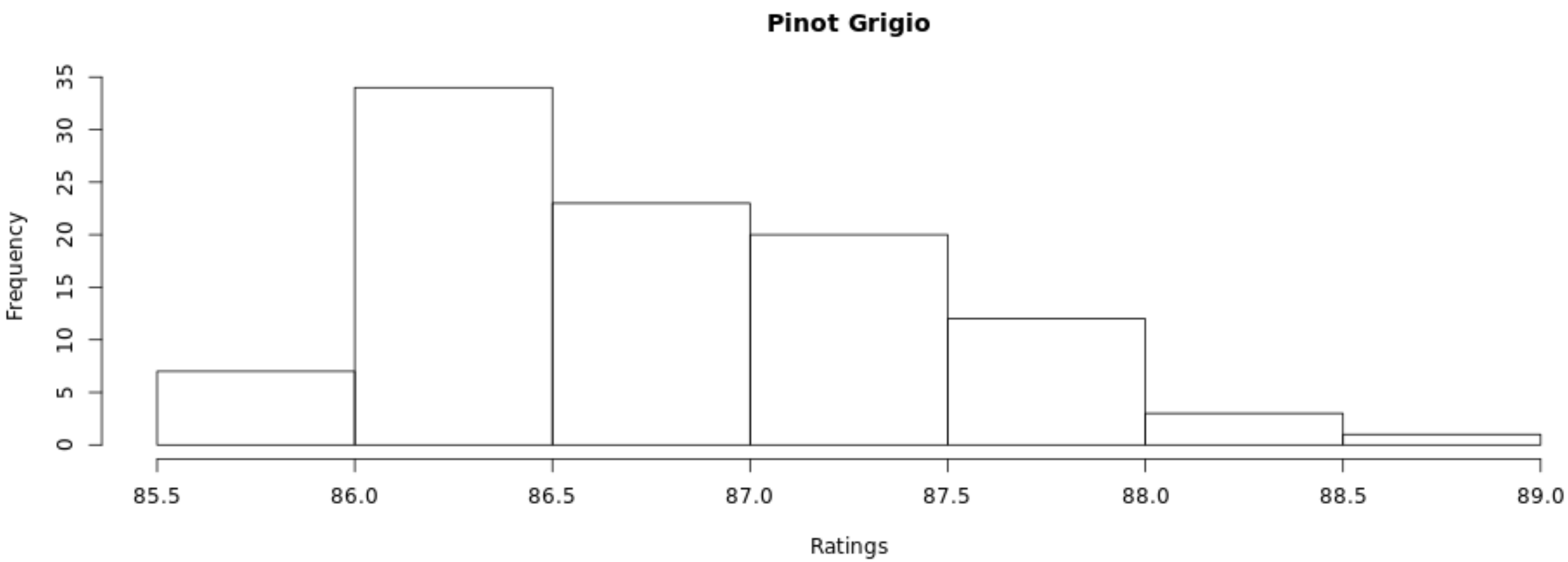
Day	Temperature
1	21
2	23
3	25
4	27
5	28
6	29
7	31
8	35
9	35
10	36

$$\tilde{x}_{0.5} = \frac{1}{2}(x_5 + x_6) = \frac{1}{2}(28+29) = 28.5$$

Day	Temperature
1	21
2	23
3	25
4	27
5	28
6	29
7	31
8	35
9	35

$$\tilde{x}_{0.5} = x_5 = 28$$

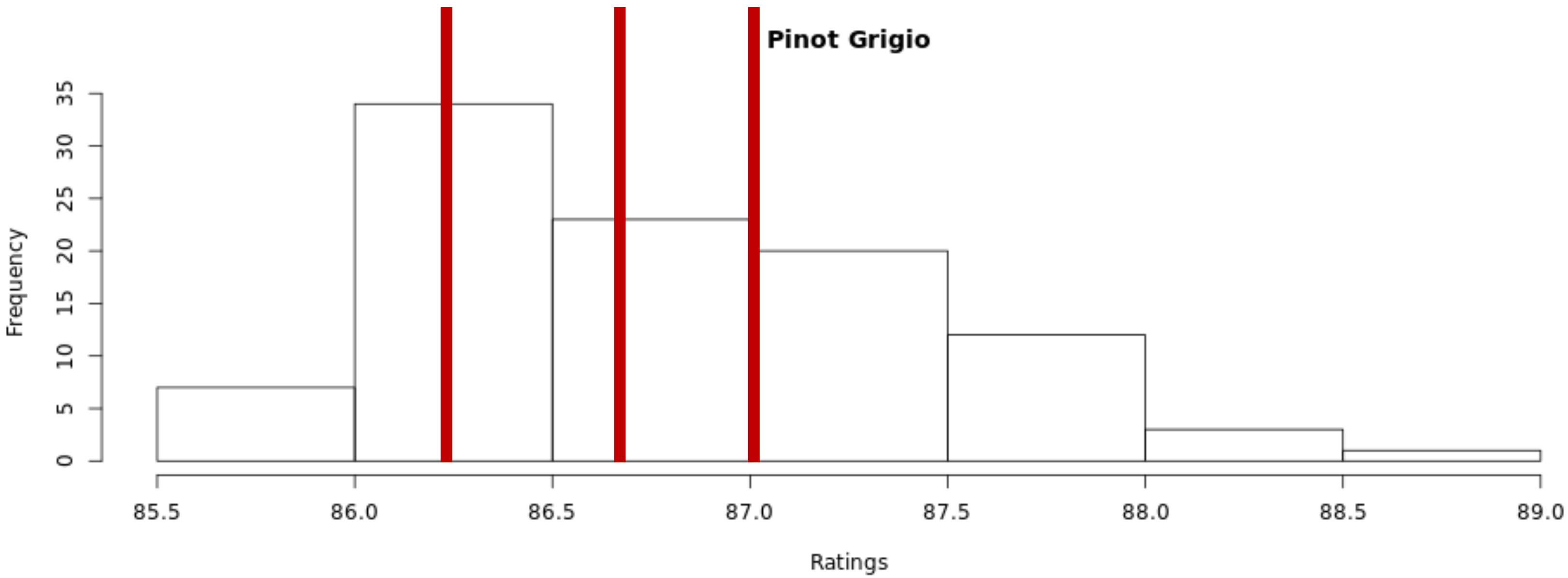
Mean, median, mode



By looking at the graph can you guess what is the relationship between the mean, median, and mode of the Pinot Grigio ratings distribution displayed here?

Source:https://campus.datacamp.com

Mean, median, mode



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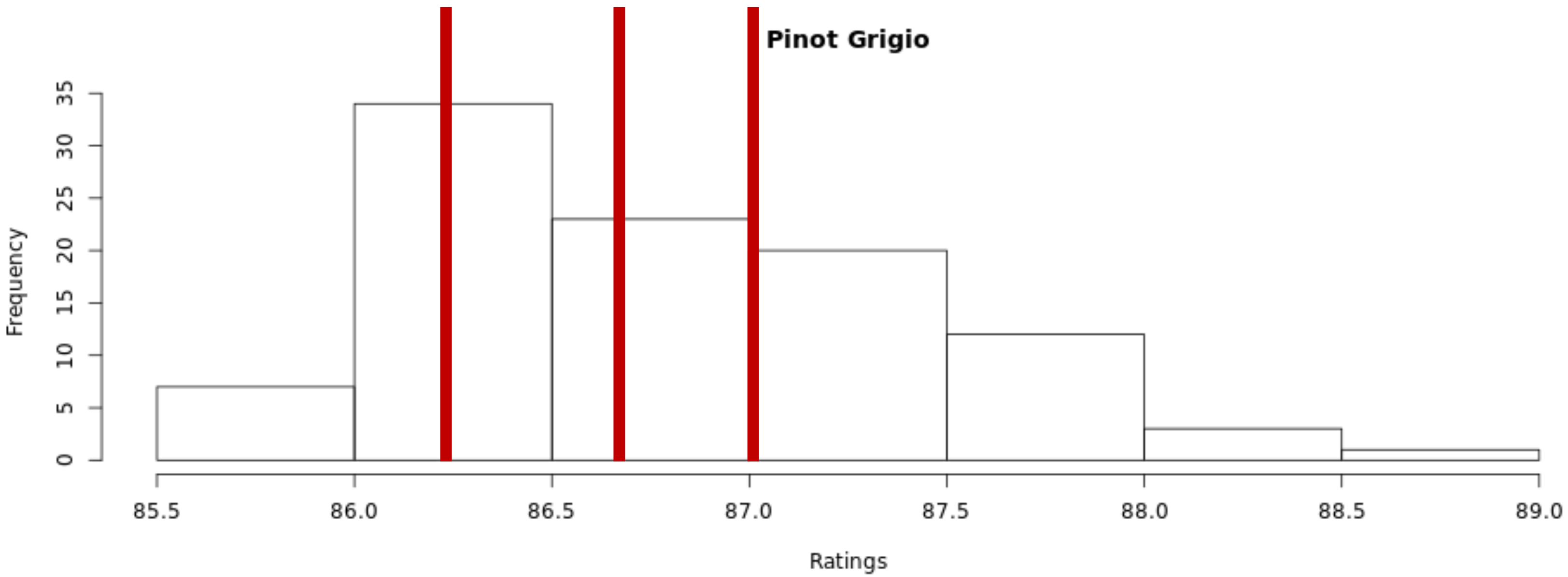
Mean, median, mode



By looking at the graph can you guess what is the relationship between the mean, median, and mode of the Pinot Grigio ratings distribution displayed here?

Source:https://campus.datacamp.com

mode < median < mean

Mean, median, mode

Exercise 2:

Create 1000 random sampling from binomial distribution with n=10 and p=0.6.

Calculate mean, median and mode of your sample.

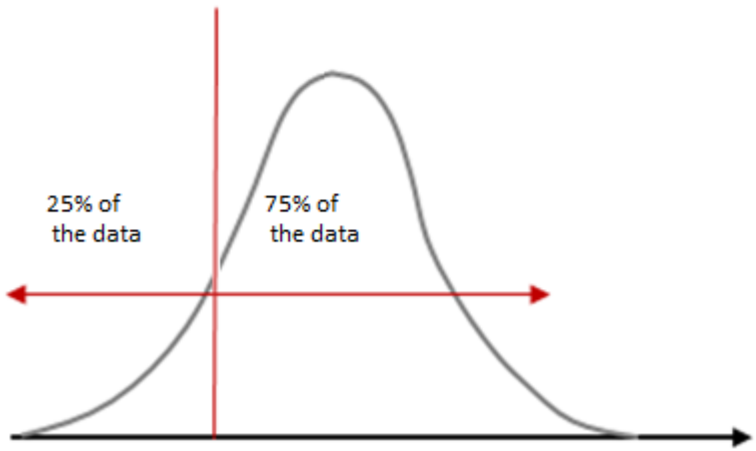
Exercise 3:

Create 1000 random sampling from log-normal distribution.
Calculate mean and median of your sample.

Interpret the relations between the measures - what do they imply?

Quantiles

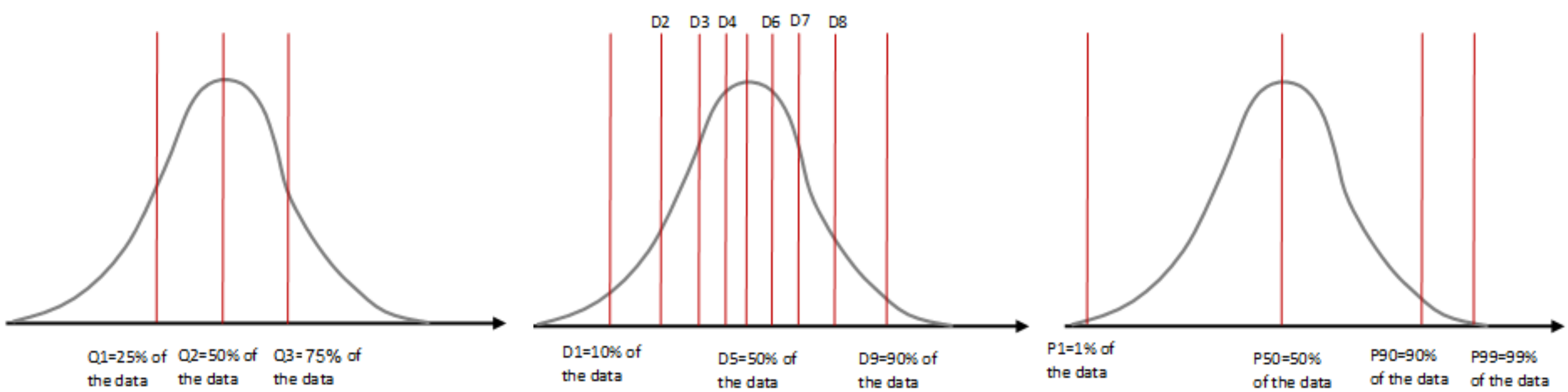
- Quantiles are a generalization of the median idea
- Median splits the data into two equal parts; quantiles split the data into other proportions
- Let's denote a numer from 0 to 1 as α
- The (α *100)% quantile is denoted as $\tilde{x}_\alpha$ and it is defined as the value that divides the data in proportions of (α *100)% and ((1-α) *100)% such that at least (α *100)% values are less than or equal to the quantile and at least ((1-α) *100)% values are greater than or equal to the quantile.



Quartiles, deciles, percentiles

- For specific value of α quantiles have different names:

Name	Proportions in which data are split	α
Quartiles (Q)	4	0.25; 0.5; 0.75
Deciles (D)	10	0.1; 0.2; 0.3;…;0.9
Percentiles (P)	100	0.01;0.02;….;0.99



Quartiles, deciles, percentiles

- Quartiles, deciles and percentiles are calculated mannually in a similar manner to median

- Example:

Day	Temperature
1	21
2	23
3	25
4	27
5	28
6	29
7	31
8	35
9	35
10	36

$$Q1 = 25$$

$$Q2 = \frac{1}{2}(28 + 29) = 28.5$$

$$Q3 = 35$$

Day	Temperature
1	21
2	23
3	25
4	27
5	28
6	29
7	31
8	35
9	35
10	36

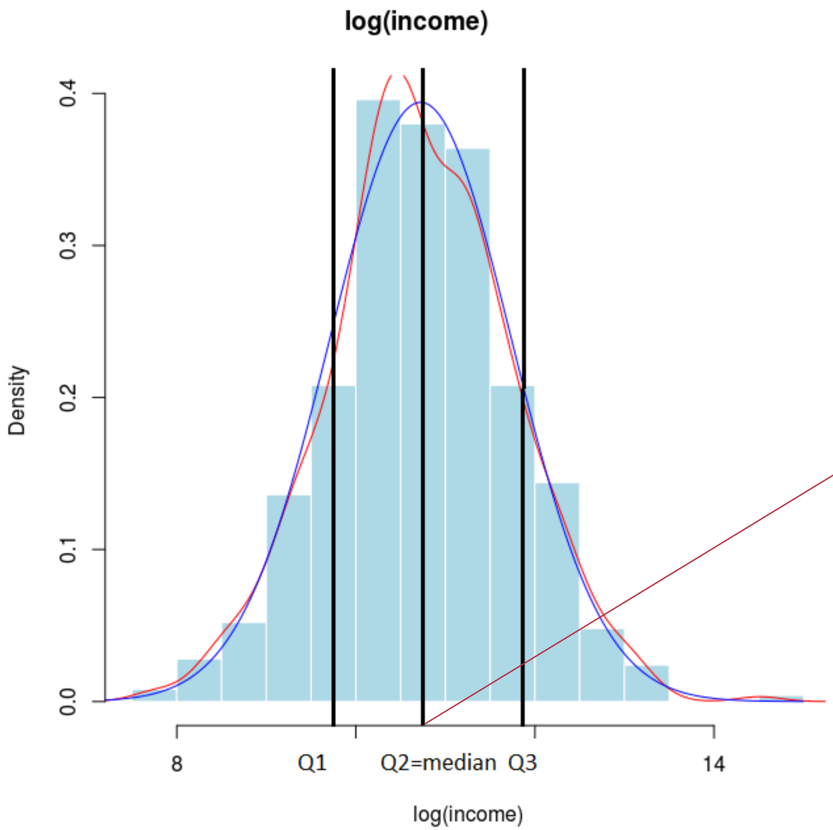
$$D1 = \frac{1}{2}(21 + 23) = 22$$

$$D2 = \frac{1}{2}(23 + 25) = 24$$

$$D5 = \frac{1}{2}(28 + 29) = 2.5$$

$$D9 = \frac{1}{2}(35 + 36) = 35.5$$

Quartiles, deciles, percentiles



Which decile is this??

Which percentile is this??

Quartiles, deciles, percentiles

Exercise 4:

Use data on wages from the NLSY dataset for the US (data for 2010); (NLSY_EDA_class.csv).

- Calculate mean and median wages by sex, by race and by education and interpret the values
- Calculate the value of the 1st, 2nd and 3rd quartile, 1st and 9th decile and 1st, 90th, 99th percentile for full sample and by sex. Interpret the values.

Trimean

- It is sometimes referred to as Tukey's trimean aftern John Tukey - its inventor (1977)

- It is defined as the weighted average of the median and upper and lower quartiles:

$$TM = \frac{Q1 + 2 * Q2 + Q3}{4}$$

- Unlike median it also utilizes information on the first and the third quartiles, which makes it more likely to be representative for the data.

- Example:

Day	Temperature
1	21
2	23
3	25
4	27
5	28
6	29
7	31
8	35
9	35
10	36

$$\mathrm{Q1} = 25$$

$$\mathrm{Q2} = \frac{1}{2}(28 + 29) = 28.5$$

$$\mathrm{Q3} = 35$$

$$\mathrm{TM} = \frac{1}{4}(25 + 2 * 28.5 + 35) = 29.25$$

Midmean / Interquartile mean

- It is the mean of the middle 50% of the data
- By dropping from the calculations 25% of extreme values from above and below of the disrtibution, the measure is more resistant to outliers than arithmetic mean
- Example:

We have n=12, so in each quartile there are 3 observations and in the interquartile there are 6 observations.

Day	Temperature
1	21
2	23
3	25
4	26
5	27
6	28
7	29
8	31
9	35
10	35
11	36
12	36

$$Q1 = \frac{1}{2}(25 + 26) = 25.5$$

$$IQM = \frac{1}{6}(26 + 27 + 28 + 29 + 31 + 35) = 29.(3)$$

$$Q3 = \frac{1}{2}(35 + 35) = 35$$

We have n=10, so in each quartile there are 2.5 observations and in the interquartile there are 5 observations: 4 that contribute in 100% and 2 that make up 1 remaining observation, i.e. contribute in 50% each.

Day	Temperature
1	21
2	23
3	25
4	27
5	28
6	29
7	31
8	35
9	35
10	36

$$Q1 = 25$$

$$IQM = \frac{1}{5}(27 + 28 + 29 + 31 + 0.5 * 25 + 0.5 * 35) = 29$$

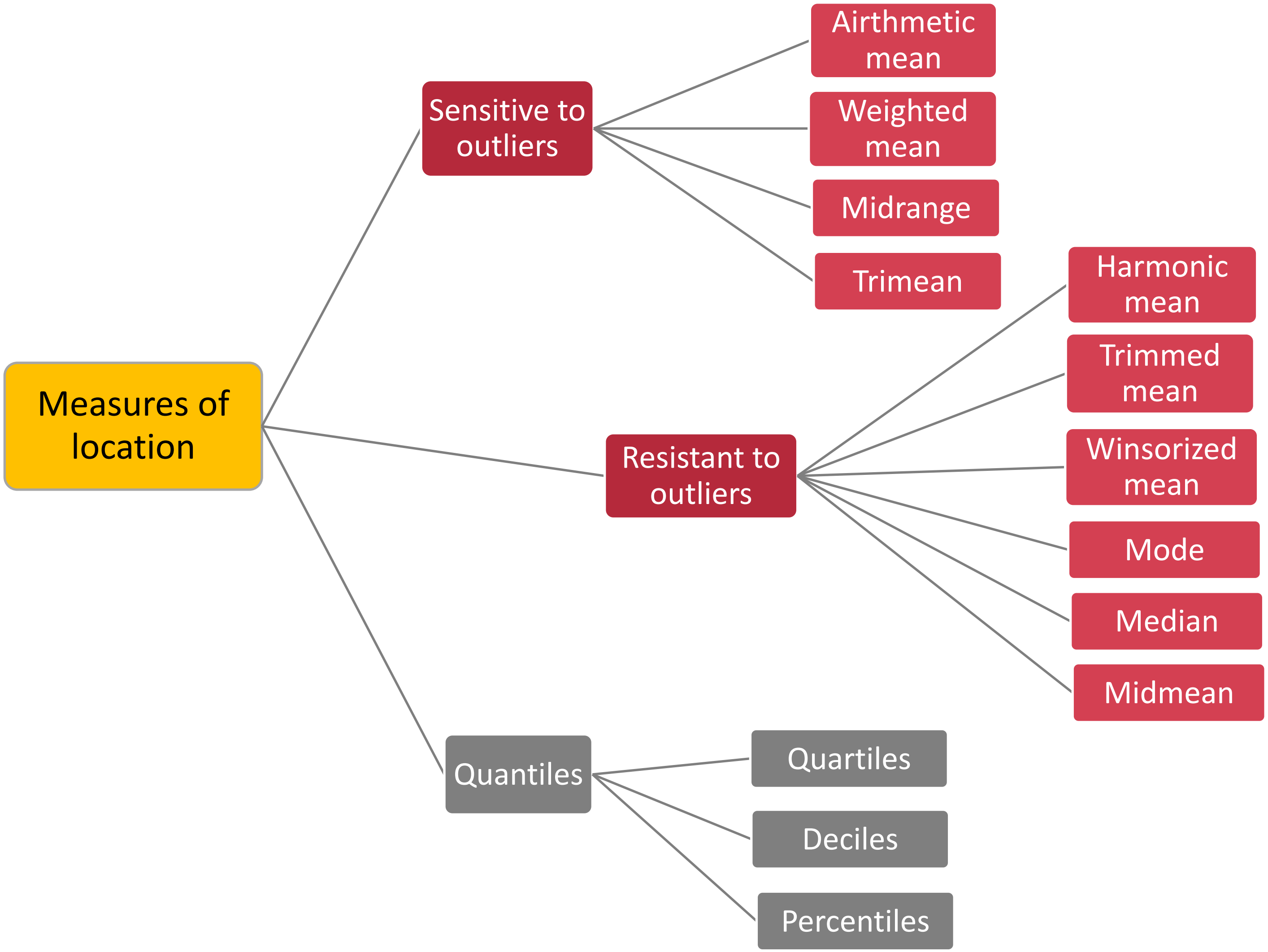
$$Q3 = 35$$

Midmean / Interquartile mean

Exercise 5:

Calculate the interquartile mean for the following temperature data: 28, 43, 32, 18, 7, 15, 22, 23, 29, 9, 11, 16.

Review and summary



Measures of location in R

Measure	Function in R	Alternative function
Arithmetic mean	mean()	
Harmonic mean	1/(mean(1/()))	library(psych) harmonic.mean()
Geometric mean	prod()^(1/length())	library(psych) geometric.mean()
Weighted mean	weighted.mean()	
Midrange	(min()+max())/2	
Trimean	TMH() – it is modified and based on hinges not quartiles	
Trimmed mean	mean(, trim=)	
Winsorized mean	winsor.mean	
Mode	table( ) or: names(sort(-table()))[1]	For continous data: d <- density(x) d$x[d$y==max(d$y)]
Median	median()	
Midmean	Calculate step-by-step using quantiles with specified type=2	
Quartiles	quantile(, probs=c(0.25, 0.5, 0.75)	
Deciles	quantile(, probs=c(0.1,..., 0.9)	
Percentiles	quantile(, probs=c(0.01, ..., 0.99)	

Bibliography

Christian Heumann, Michael Schomaker Shalabh „Introduction to Statistics and Data Analysis With Exercises, Solutions and Applications in R”, Springer 2016: Chapter 3.2.

http://www.statisticshowto.com/probability-and-statistics/statistics-definitions/ for helpful examples

Thank you for your attention

Time for practice!