18.s096pset 3 Dimitris Koutentakis

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1 Poblem 1

1.1 (a)

The variable yi is a Bernoulli variable with $p = 1 - e^{-\lambda}$ so

$$f(y) = (1 - e^{-\lambda})^{-y}(e^{-\lambda})^{1-y}$$

Thus the Likelihood Function will be:

$$L(Y_1, ...Y_n, \lambda) = \prod_{i=1}^{n}(1 - e^{-\lambda})^{y_i}(e^{-\lambda})^{1-y_i} \tag{1}$$
$$= e^{-\lambda \sum_{i=1}^{n}(1-y_i)}(1 - e^{-\lambda})^{\sum_{i=1}^{n} y_i} \tag{2}$$

Taking the logarithm, we get:

$$ln(Y_1, ...Y_n, \lambda) = ln(e^{-\lambda \sum_{i=1}^{n}(1-y_i)}(1 - e^{-\lambda})^{\sum_{i=1}^{n} y_i}) \tag{3}$$
$$= -\lambda(n - \sum_{i=1}^{n} y_i) + \sum_{i=1}^{n} y_i ln(1 - e^{-\lambda}) \tag{4}$$

By taking the derivative and setting it to zero, we get:

$$0 = \sum_{i=1}^{n} y_i - n + \sum_{i=1}^{n} y_i \frac{1}{e^{\lambda}_{MLE} - 1} \tag{5}$$
$$\lambda_{MLE} = ln\Big(\frac{\sum_{i=1}^{n} y_i}{n - \sum_{i=1}^{n} y_i} + 1\Big) \tag{6}$$

1.2 (b)

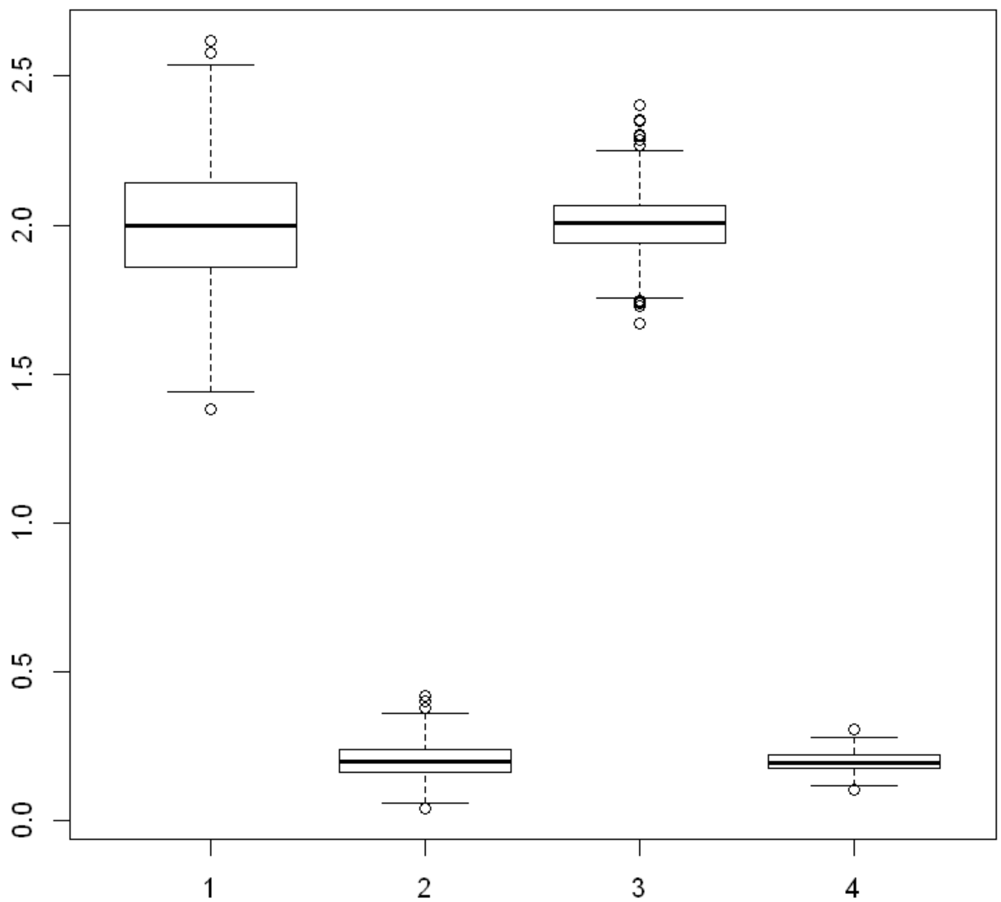
```
In [16]: options(warn=-1)

         nsamples=1000
         samplesize=50
         r2_50 = matrix(rpois(nsamples*samplesize, lambda=2), nrow=samplesize, ncol=nsamples)
         r02_50 = matrix(rpois(nsamples*samplesize, lambda=0.2), nrow=samplesize, ncol=nsample
```

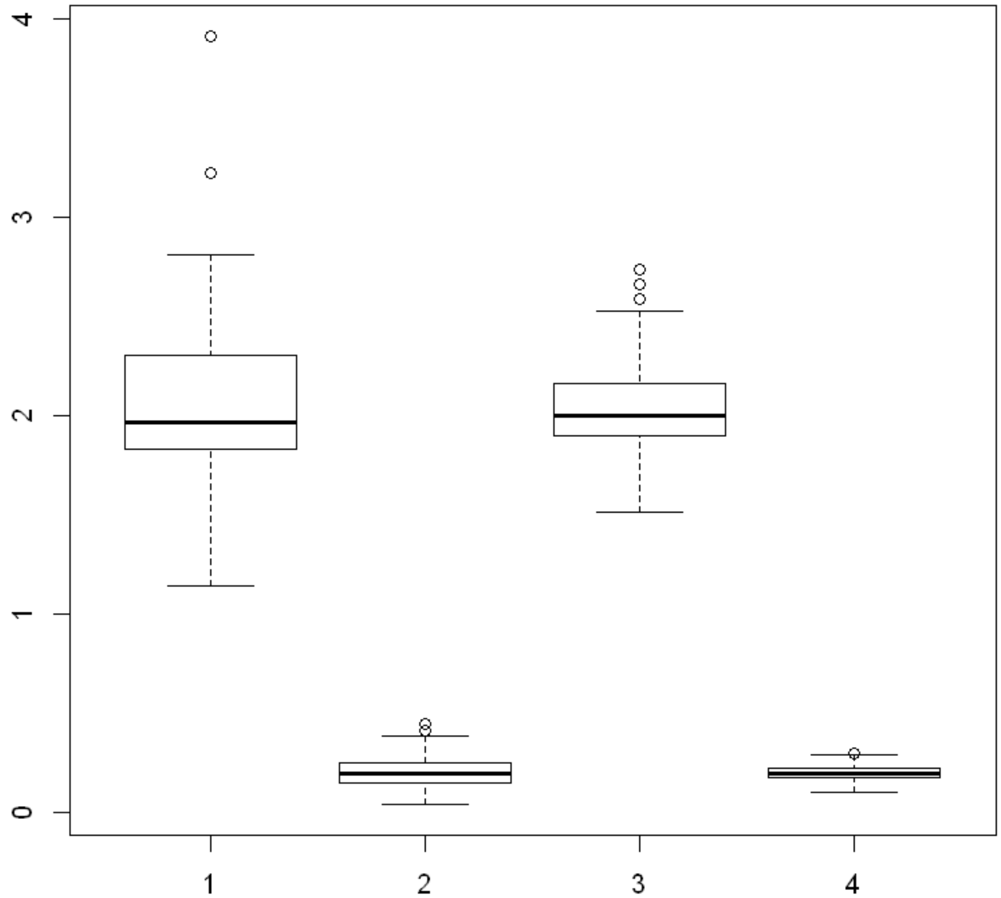
```
samplesize=200
r2_200 = matrix(rpois(nsamples*samplesize, lambda=2), nrow=samplesize, ncol=nsamples)
r02_200 = matrix(rpois(nsamples*samplesize, lambda=0.2), nrow=samplesize, ncol=nsampl

r02_50_dist = colMeans(r02_50)
r02_200_dist = colMeans(r02_200)
r2_50_dist = colMeans(r2_50)
r2_200_dist = colMeans(r2_200)
```

In [17]:
```
r2_50_t = ifelse(r2_50!=0, 1,0)
r2_200_t = ifelse(r2_200!=0, 1,0)
r02_50_t = ifelse(r02_50!=0, 1,0)
r02_200_t = ifelse(r02_200!=0, 1,0)

r2_50_t_dist = colMeans(r2_50_t)
r2_200_t_dist = colMeans(r2_200_t)
r02_50_t_dist = colMeans(r02_50_t)
r02_200_t_dist = colMeans(r02_200_t)
```

In [18]:
```
r2_50_mle=1/50*(colSums(r2_50))
r2_200_mle=1/200*(colSums(r2_200))
r02_50_mle=1/50*(colSums(r02_50))
r02_200_mle=1/200*(colSums(r02_200))

r2_50_t_mle = log(colSums(r2_50_t)/(50-colSums(r2_50_t))+1)
r2_200_t_mle = log(colSums(r2_200_t)/(200-colSums(r2_200_t))+1)
r02_50_t_mle = log(colSums(r02_50_t)/(50-colSums(r02_50_t))+1)
r02_200_t_mle = log(colSums(r02_200_t)/(200-colSums(r02_200_t))+1)
```

In [19]:
```
boxplot(r2_50_mle, r02_50_mle, r2_200_mle, r02_200_mle)
boxplot(r2_50_t_mle, r02_50_t_mle, r2_200_t_mle, r02_200_t_mle)
```


2.5
2.0
1.5
1.0
0.5
0.0
1
2
3
4




```
In [20]: cat(mean(r2_50_mle), mean(r2_200_mle), mean(r02_50_mle), mean(r02_200_mle), "\n")
         cat(sd(r2_50_mle), sd(r2_200_mle), sd(r02_50_mle), sd(r02_200_mle))
         cat("\n\nTruncated MLE: \n\n")
         cat(mean(r2_50_t_mle), mean(r2_200_t_mle), mean(r02_50_t_mle), mean(r02_200_t_mle), "\
         cat(sd(r2_50_t_mle), sd(r2_200_t_mle), sd(r02_50_t_mle), sd(r02_200_t_mle))
```

```
2.00308 2.00328 0.19742 0.198165
0.1959043 0.09936405 0.06035052 0.03226296

Truncated MLE:

2.064381 2.029564 0.1992984 0.1984418
0.4124002 0.1788553 0.06417971 0.03398784
```

1.3 (c)

```
In [21]: cat(var(r2_200_mle)/var(r2_200_t_mle), "\n")
         cat(var(r02_200_mle)/var(r02_200_t_mle), "\n")
         cat(var(r2_50_mle), var(r2_200_mle), var(r02_50_mle), var(r02_200_mle),"\n")
         cat(var(r2_50_t_mle), var(r2_200_t_mle), var(r02_50_t_mle), var(r02_200_t_mle),"\n")
```

```
0.308642
0.9010759
0.03837849 0.009873215 0.003642186 0.001040899
0.1700739 0.03198921 0.004119035 0.001155173
```

We can see that the relative efficiency of the regular vs the truncated MLE estimator increases as λ drops and the absolute efficiency of both gets better (variance decreases) as the sample size increases, while it also improves whel λ decreases.

2 Problem 2

2.1 (a)

```
In [22]: library(MASS)
         set.seed(1)
         list.samplesize<-c(100,400,800,1600)
         for (samplesize in list.samplesize){assign(paste("sample.gamma",samplesize,sep="."),rg
         mlefit.sample.gamma.100<-fitdistr(sample.gamma.100,densfun="gamma")
         mlefit.sample.gamma.400<-fitdistr(sample.gamma.400,densfun="gamma")
         mlefit.sample.gamma.800<-fitdistr(sample.gamma.800,densfun="gamma")
         mlefit.sample.gamma.1600<-fitdistr(sample.gamma.1600,densfun="gamma")
```

2.2 (2b)

```
In [23]: mlefit.sample.gamma.100$sd
         mlefit.sample.gamma.400$sd
         mlefit.sample.gamma.800$sd
         mlefit.sample.gamma.1600$sd
```

shape	0.510948047695206	**rate**	0.374878929040892
shape	0.18725575766242	**rate**	0.13813760987183
shape	0.133297360770547	**rate**	0.0946917208868348
shape	0.1056492461658	**rate**	0.0757412358129735

We can see that the standard deviation decreases as the sample size increases. This is consistent with the theory.

2.3 (c)

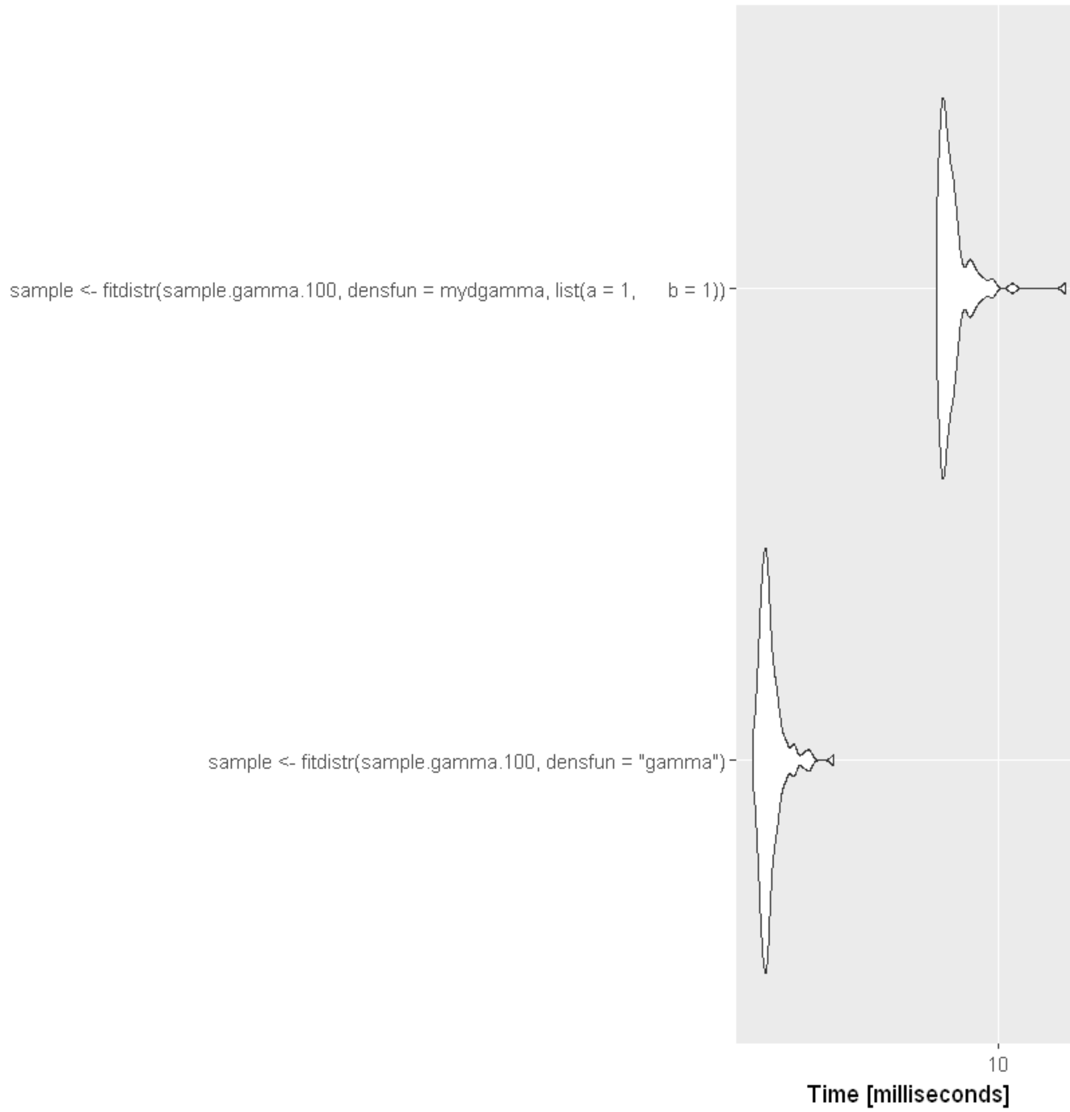
```
In [24]: mydgamma <- function(x, a, b){

             if (x>0){
```

```
        g = (1/gamma(a))*(b^a)*x^(a-1)*exp(-b*x)
    }
    else{
        print("ERROR")
    }

    return(g)

}
```

2.4 (d)

```
In [25]: library(microbenchmark)
         microbm.1<-microbenchmark(
            sample <- fitdistr(sample.gamma.100,densfun="gamma"),
             sample <- fitdistr(sample.gamma.100,densfun=mydgamma, list(a = 1, b=1))
         )
```

```
In [26]: library(ggplot2)
         autoplot(microbm.1)
```



The function `fitdistr()` is slower with `mydgamma()` mainly because it has not been more optimized. We could potentially change this behavior by choosing different start values.

3 Problem 3

3.1 (a)

$$\begin{aligned}
E[X] &= \frac{1}{2b}\int xe^{\frac{-|x-\mu|}{b}}dx && (7)\\
&= \frac{1}{2b}\int (bt+\mu)e^{-|t|}bdt && (8)\\
&= \mu\int_0^{\infty} e^{-t}dt && (9)\\
&= \mu && (10)
\end{aligned}$$

The above comes from a substitution ($t = \frac{x-\mu}{b}$) and from the fact that $\int te^{-|t|} = 0$ since it's odd and $\int_{-\infty}^{\infty} e^{-|t|} = 2\int_0^{\infty} e^{-t}$.

Similarly, for the variance we have:

$$\begin{aligned}
E[X^2] &= \int x^2 f(x)dx && (11)\\
&= \frac{1}{2b}\int x^2 e^{\frac{-|x-\mu|}{b}}dx && (12)\\
&= \frac{1}{2b}\int (bt+\mu)^2 e^{-|t|}bdt && (13)\\
&= \int_0^{\infty}((bt)^2+\mu^2)e^{-t}dt && (14)\\
&= \mu^2\int_0^{\infty} e^{-t}dt + b^2\int_0^{\infty} t^2 e^{-t}dt && (15)\\
&= \mu^2 + 2b^2 && (16)
\end{aligned}$$

3.2 (b)

For the MLE, we get:

$$\begin{aligned}
L(X_i, .., X_n, \mu, b) &= \prod_{i=1}^{n}\frac{1}{2b}e^{-\frac{|X_i-\mu|}{b}} && (17)\\
&= (\frac{1}{2b})^n e^{-\frac{\sum_{i=1}^{n}|X_i-\mu|}{b}} && (18)\\
ln(X_i, .., X_n, \mu, b) &= -nlog(2b) - \frac{\sum_{i=1}^{n}|X_i-\mu|}{b} && (19)
\end{aligned}$$

By taking the derivative with respect to μ, we see that the function is minimized for

$$\sum_{i=1}^{n}|X_i-\mu| = 0 \Rightarrow \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n}X_i$$

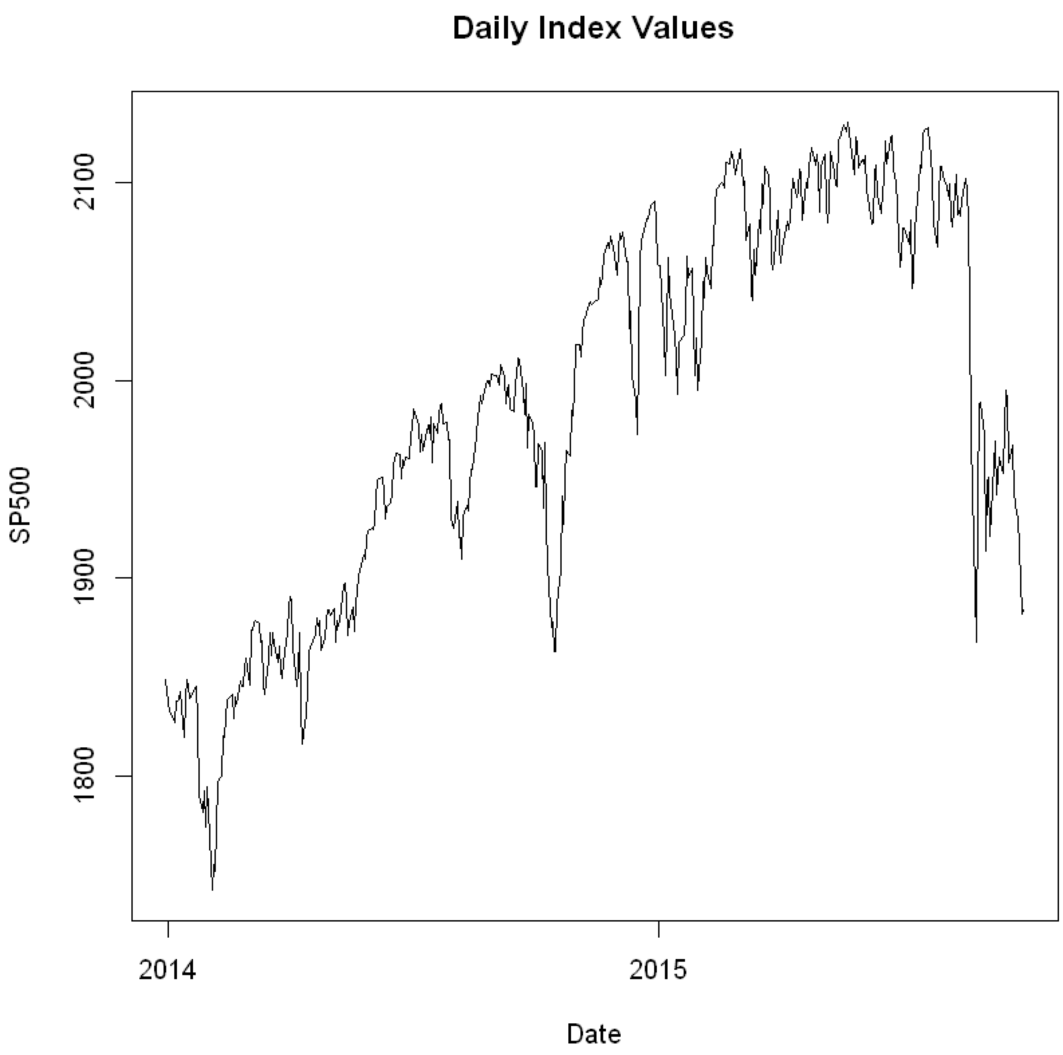
When taking the derivative with respect to b, we have:

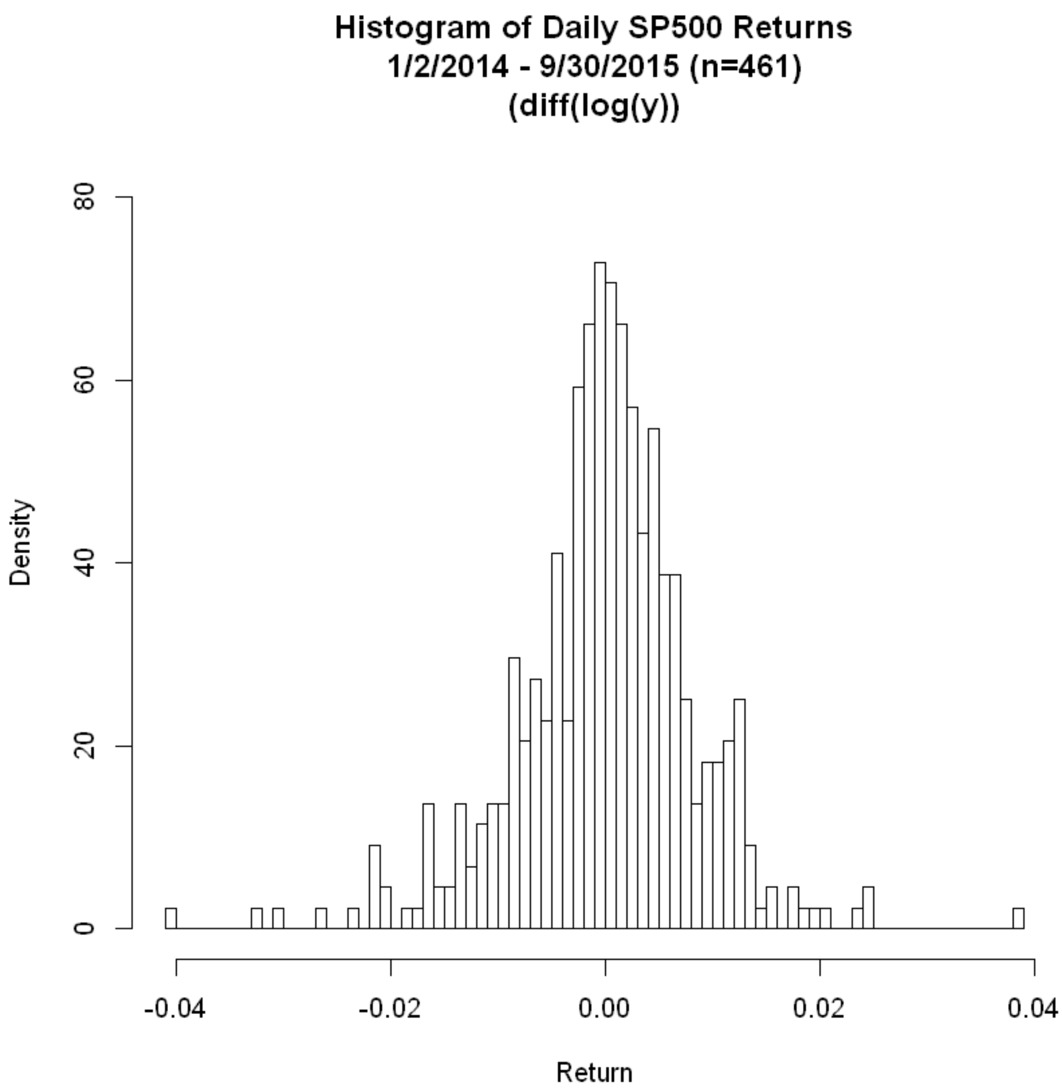
$$-\frac{n}{b} - \frac{1}{b^2}\sum_{i=1}^{n}|X_i-\mu| = 0 \Rightarrow n = \frac{1}{b}\sum_{i=1}^{n}|X_i-\mu| \Rightarrow \hat{b} = \frac{1}{n}\sum_{i=1}^{n}|X_i-\mu|$$

3.3 (c)

In [43]:
```
library(zoo)
# setwd("./Downloads")
.libPaths("C:/Users/dkout/Documents/R/win-library/3.4")
SP500 <- read.zoo(file="SP500.csv")
```

In [39]:
```
par(mfcol=c(1,1))
plot(SP500, main="Daily Index Values", xlab="Date")
y<-diff(log(SP500))
hist( y, breaks=100,ylab="Density",xlab="Return",freq=FALSE,ylim=c(0,84.),main=paste(
```



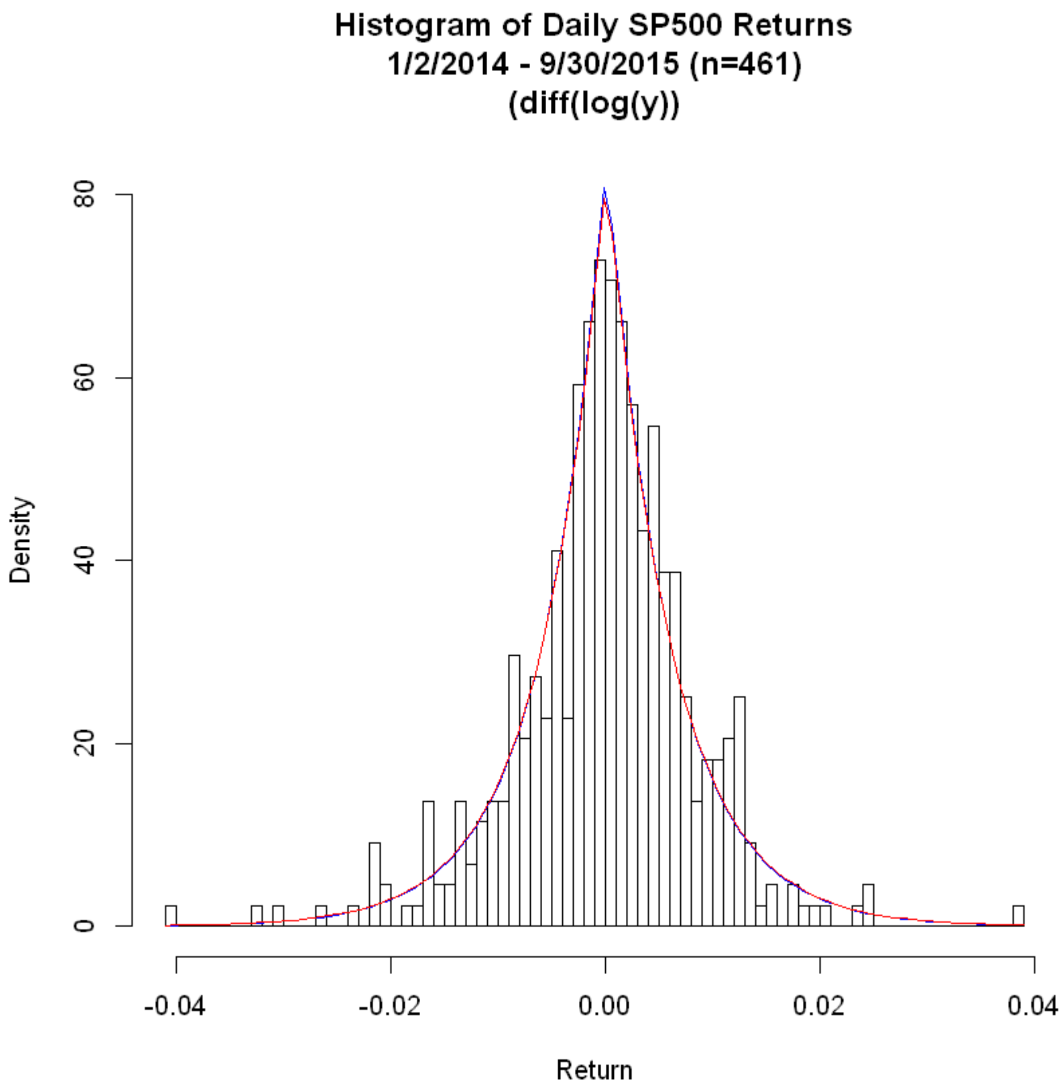


In [40]:
```
#Moments:

mu_mom=mean(y)
b_mom = sqrt((var(y)+mean(y)^2-mu_mom^2)/2)
mu_mle = mean(y)
b_mle = sum(abs(y-mu_mle))/length(y)

cat(mu_mom, b_mom, mu_mle, b_mle)
```

```
4.361316e-05 0.005948328 4.361316e-05 0.006034661
```

3.4 (d)

```
In [46]: library(rmutil)
         hist( y, breaks=100,ylab="Density",xlab="Return",freq=FALSE,ylim=c(0,84.),main=paste(
         curve(dlaplace(x, mu_mom, b_mom), add = TRUE, col = "blue")
         curve(dlaplace(x, mu_mle, b_mle), add = TRUE, col = "red")
```



Both methods yield very similar and quite accurate results. We can see the MLE in red and the method of moments in blue.

3.5 (e)

```
In [47]: mydlaplace<-function(x, location=0,scale=1){
         dx=(0.5/scale)* exp(-abs(x-location)/scale)
         }
```

3.6 (f)

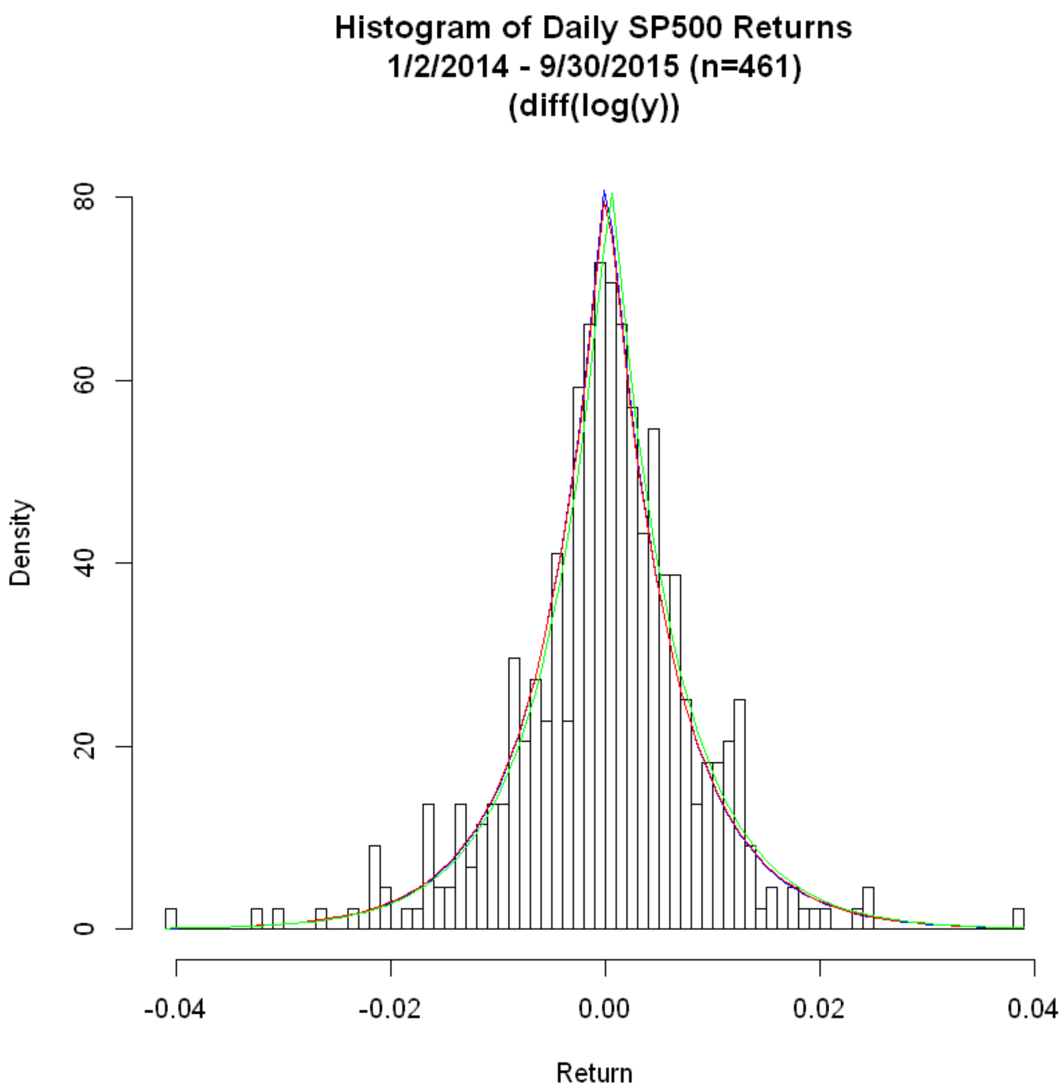
```
In [48]: mlefit <- fitdistr(y,densfun=mydlaplace, list(location = 0, scale=1))
         mlefit$estimate
         cat("Scale difference with MLE in sd", (mlefit$estimate["scale"] - b_mom)/mlefit$sd["s
         cat("Scale difference with Moments in sd",(mlefit$estimate["scale"] - b_mle)/mlefit$sc
         cat("Location difference with Moments in sd",(mlefit$estimate["location"] - mu_mom)/ml
         cat("Location difference with MLE in sd",(mlefit$estimate["location"] - mu_mle)/mlefi
```

location 0.000429915620975948 **scale** 0.00604623669376146

```
Scale difference with MLE in sd 0.3686043
Scale difference with Moments in sd 0.04358039
Location difference with Moments in sd 1.235272
Location difference with MLE in sd 1.235272
```

The difference is about 0.4 standard deviations for the scale variable (b) and 1.2 for the mean. This is not a huge difference as can be seen in the next graph (green corresponds to the fitdistr).

```
In [49]: library(rmutil)
         hist( y, breaks=100,ylab="Density",xlab="Return",freq=FALSE,ylim=c(0,84.),main=paste(c
         curve(dlaplace(x, mu_mom, b_mom), add = TRUE, col = "blue")
         curve(dlaplace(x, mu_mle, b_mle), add = TRUE, col = "red")

         curve(dlaplace(x, mlefit$estimate["location"], mlefit$estimate["scale"]), add = TRUE,
```



4 Problem 5

4.1 (a)

By applying the Newton Method to minimize the error $\vec{\epsilon} = \vec{y} - X\vec{\beta}$, we have:

$$\vec{\beta_1} = \vec{\beta_0} - \frac{f(\vec{\beta_0})}{f'(\vec{\beta_0})} \tag{20}$$

$$= \vec{\beta_0} + \frac{\vec{y} - X\vec{\beta_0}}{X} \tag{21}$$

$$= \vec{\beta_0} - \vec{\beta_0} + X^{-1}\vec{y} \tag{22}$$

$$= X^{-1}\vec{y} \tag{23}$$

4.2 (b)

$$\vec{\beta_1} = X^{-1}\vec{y} \tag{24}$$

$$= X^{-1} \cdot I\vec{y} \tag{25}$$

$$= X^{-1} \cdot (X^T)^{-1}X^T\vec{y} \tag{26}$$

$$= X^{-1}(X^T)^{-1} \cdot X^T\vec{y} \tag{27}$$

$$= (X^T X)^{-1} \cdot X^T\vec{y} = \hat{\beta} \tag{28}$$

4.3 (c)

Since our final solution $\beta_1 = \hat{\beta}$ does not depend on ϵ, or σ, this will not depend on the value of σ^2 and thus will work for any uknown variance.