18.s096 pset 7 - Dimitris Koutentakis

May 8, 2018

1 Problem 1

1.1 (a)

```
In [2]: library("mcsm")
        data(challenger)
        fit.logistic<-glm(oring ~ temp, data=challenger, family=binomial(link="logit"))
        fit.logistic
        summary(fit.logistic)
```

```
Call:  glm(formula = oring ~ temp, family = binomial(link = "logit"),
    data = challenger)

Coefficients:
(Intercept)         temp
    15.0429      -0.2322

Degrees of Freedom: 22 Total (i.e. Null);  21 Residual
Null Deviance:     28.27
Residual Deviance: 20.32  AIC: 24.32



Call:
glm(formula = oring ~ temp, family = binomial(link = "logit"),
    data = challenger)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.0611  -0.7613  -0.3783   0.4524   2.2175

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  15.0429     7.3786   2.039   0.0415 *
temp         -0.2322     0.1082  -2.145   0.0320 *
---
```

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 28.267  on 22  degrees of freedom
Residual deviance: 20.315  on 21  degrees of freedom
AIC: 24.315

Number of Fisher Scoring iterations: 5
```

From the above, we get the MLE values of α and β. In specific, we have:

$$\hat{\alpha}_{MLE} = 15.0429, \hat{\beta}_{MLE} = -0.2322$$

.

The we have that the 95% confidence intervals are:

$$\begin{aligned}
I_\alpha &= [\hat{\alpha}_{MLE} - 1.96 \cdot 7.3786, \hat{\alpha}_{MLE} + 1.96 \cdot 7.3786] && (1)\\
&= [0.58, 29.505] && (2)\\
I_\beta &= [\hat{\beta}_{MLE} - 1.96 \cdot 0.1082, \hat{\beta}_{MLE} + 1.96 \cdot 0.1082] && (3)\\
&= [-0.444, -0.02] && (4)
\end{aligned}$$

1.2 (b)

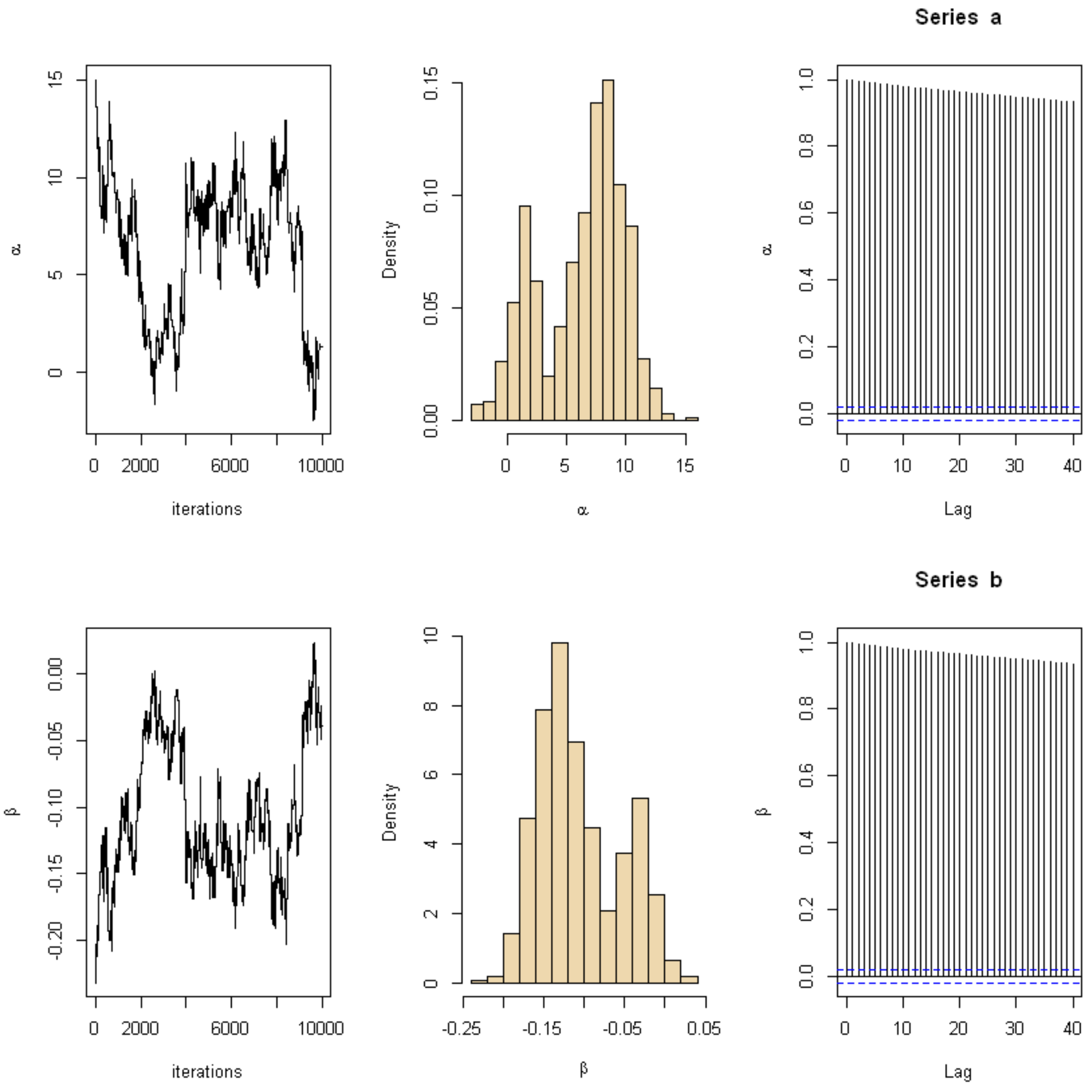
```
In [3]: set.seed(1)
        Nsim=10^4
        x=challenger$temp
        y=challenger$oring
        sigmaa=5 ; sigmab=5/sd(x)
        lpost=function(a,b){sum(y*(a+b*x)-log(1+exp(a+b*x)))+
                            dnorm(a,sd=sigmaa,log=TRUE)+dnorm(b,sd=sigmab,log=TRUE)}
        # Initialize a and b to equal the MLEs
        beta=as.vector(fit.logistic$coefficients)
        a=b=rep(0,Nsim)
        a[1]=beta[1]
        b[1]=beta[2]
        #As scale for the proposal densities consider the square root of the
        # cov.unscaled from the ml fit to the logistic model
        fit.logistic.summary<-summary(fit.logistic)
        scala=sqrt(fit.logistic.summary$cov.unscaled[1,1])
        scalb=sqrt(fit.logistic.summary$cov.unscaled[2,2])
        for (t in 2:Nsim){
            propa=a[t-1]+sample(c(-1,1),1)*rexp(1)*scala
            if (log(runif(1))<lpost(propa,b[t-1])- lpost(a[t-1],b[t-1]))
                a[t]=propa else
```

```
                a[t]=a[t-1]
            propb=b[t-1]+sample(c(-1,1),1)*rexp(1)*scalb
        if (log(runif(1))<lpost(a[t],propb)- lpost(a[t],b[t-1]))
            b[t]=propb
        else b[t]=b[t-1]
    }
```

```
In [4]: print(length(unique(a))/Nsim)
        print(length(unique(b))/Nsim)
```

```
[1] 0.1001
[1] 0.0987
```

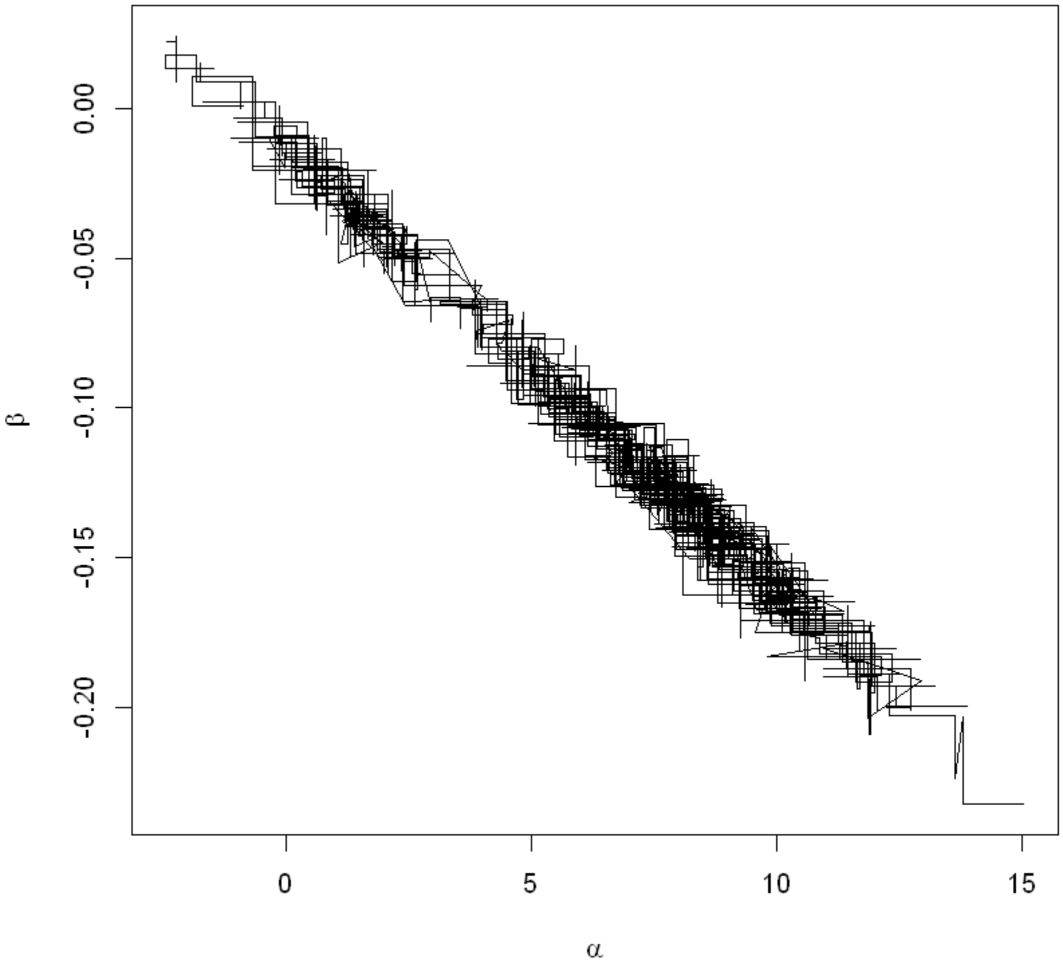
```
In [5]: par(mfrow=c(2,3))
        plot(a,type="l",xlab="iterations",ylab=expression(alpha))
        hist(a,prob=TRUE,col="wheat2",xlab=expression(alpha),main="")
        acf(a,ylab=expression(alpha))
        plot(b,type="l",xlab="iterations",ylab=expression(beta))
        hist(b,prob=TRUE,col="wheat2",xlab=expression(beta),main="")
        acf(b,ylab=expression(beta))
```



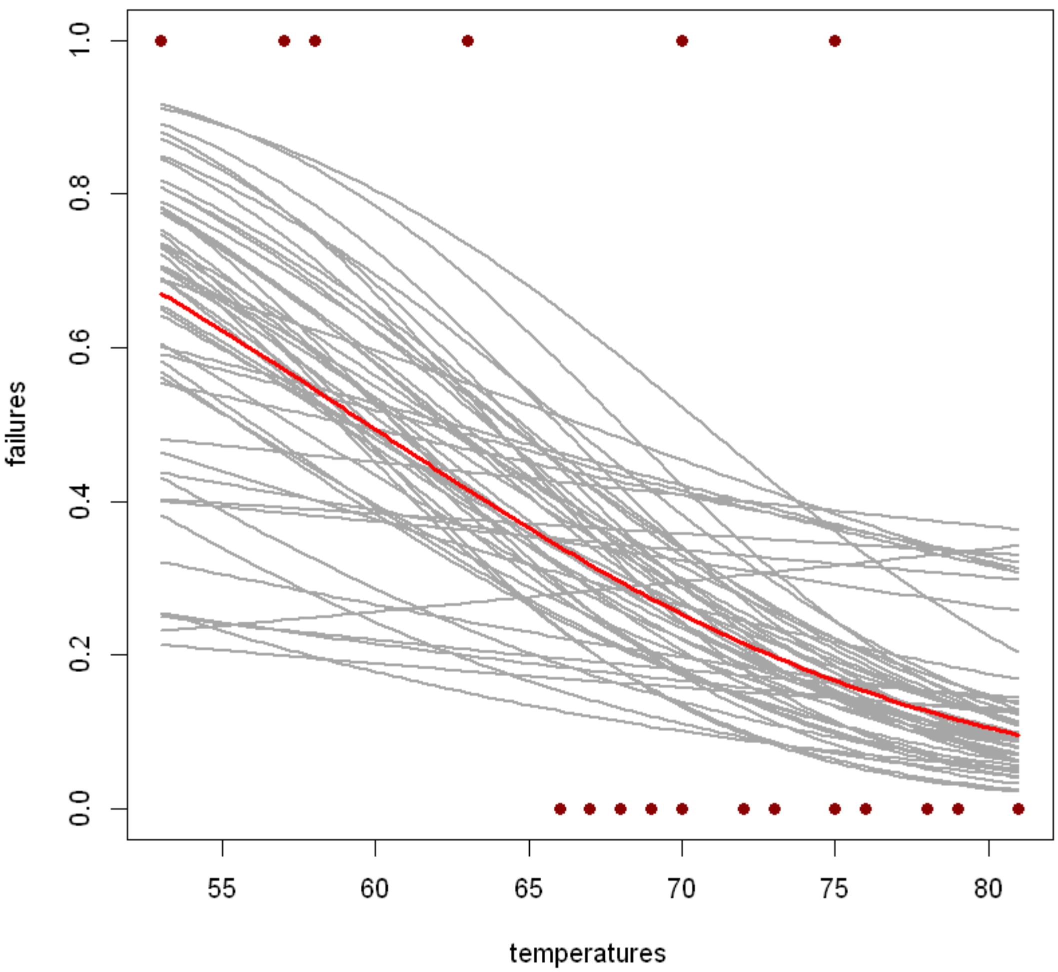

```
In [6]: print (c(mean(a), sd(a))); print (c(mean(b), sd(b)))
```

```
[1] 6.294067 3.497769
[1] -0.10533114  0.05123078
```

```
In [7]: par(mfcol=c(1,1))
        plot(a,b,type="l",xlab=expression(alpha),ylab=expression(beta))
```




```
In [8]: plot(challenger$temp,challenger$oring,pch=19,col="red4",xlab="temperatures",ylab="failu
        for (t in seq(1000,Nsim,le=50)) curve(1/(1+exp(-a[t]-b[t]*x)), add=TRUE,col="grey65",lw
        curve(1/(1+exp(-mean(a)-mean(b)*x)),add=TRUE,col="red",lwd=2.5)
        postal=rep(0,1000);i=1
```




```
In [9]: for (x0 in c(60,50,40,30)){
        print(c(x0, mean(1/(1+exp(-a-b*x0))),
        sd(1/(1+exp(-a-b*x0)))))
        }
```

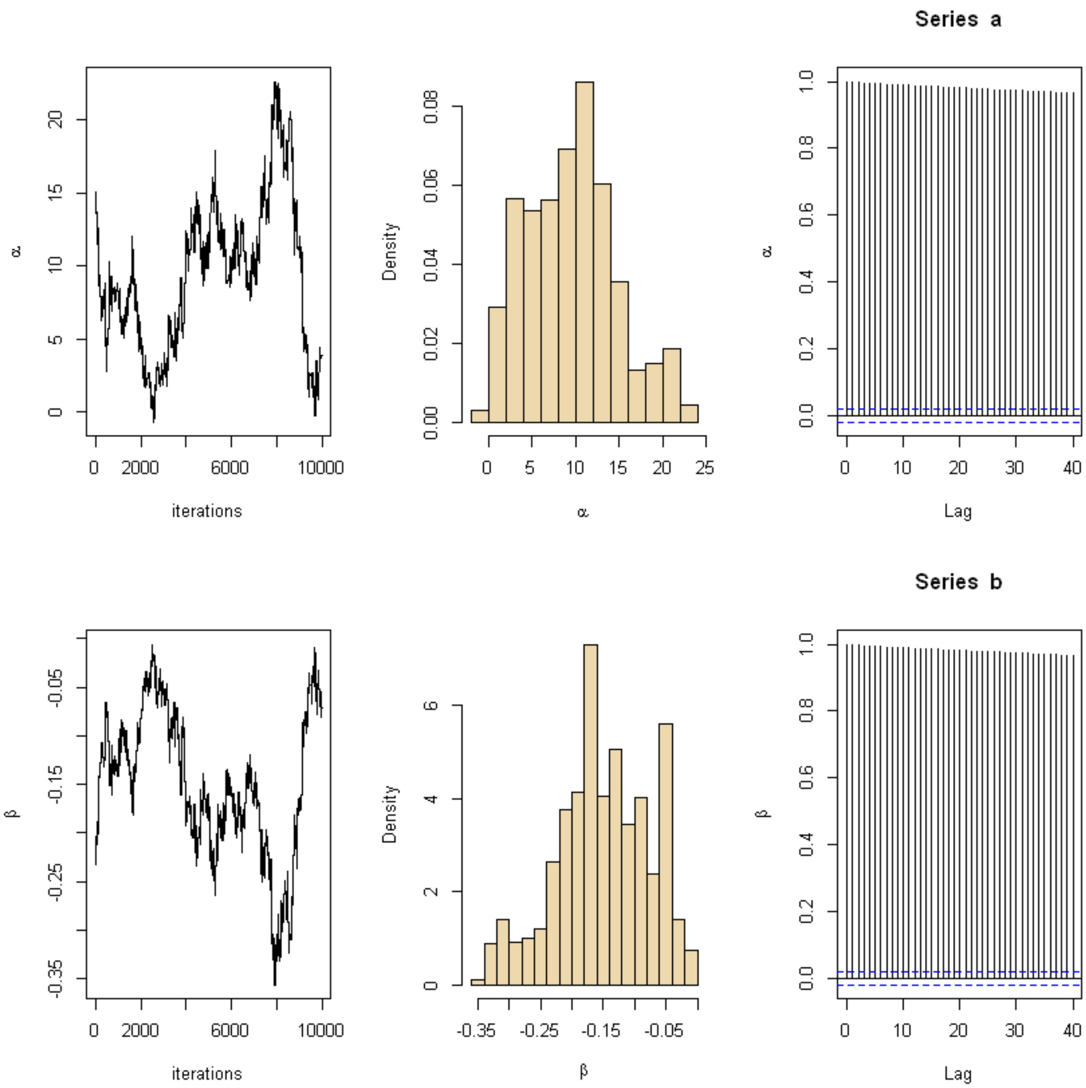
```
[1] 60.0000000  0.4944781  0.1431809
[1] 50.0000000  0.7012076  0.1959169
[1] 40.0000000  0.8134438  0.2077082
[1] 30.0000000  0.8673593  0.1983737
```

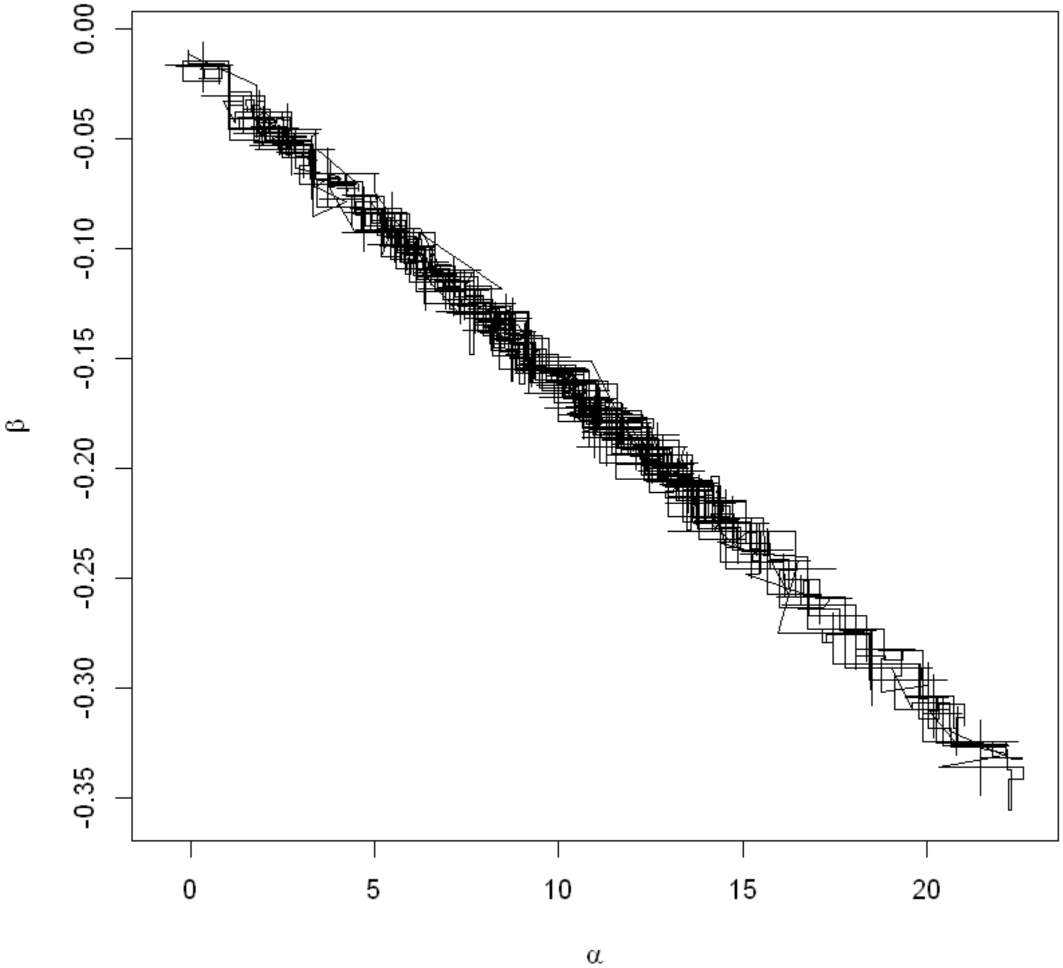
1.3 (c)

We want a variance of 100 the previous one, thus we multiply the standard deviation σ by 10.

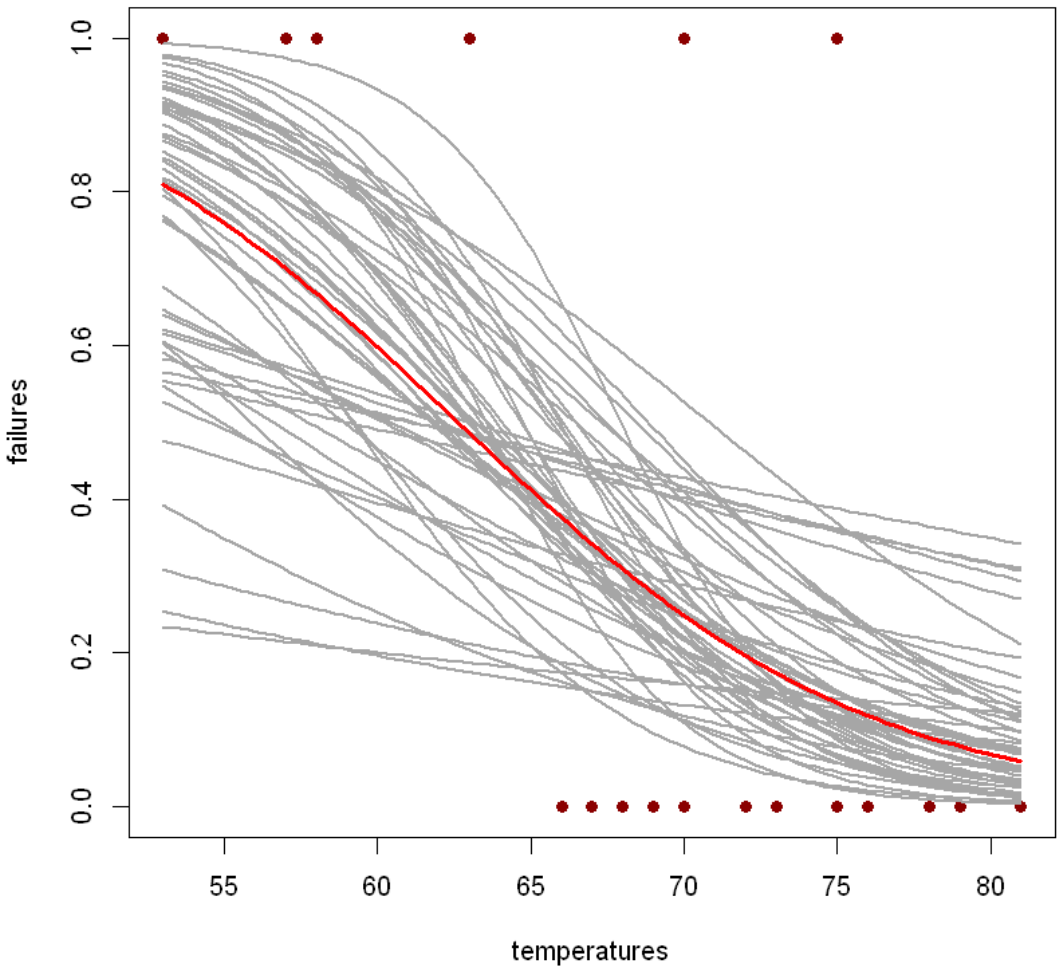
In [10]:

```
set.seed(1)
Nsim=10^4
x=challenger$temp
y=challenger$oring
sigmaa=5*10 ; sigmab=5*10/sd(x)
lpost=function(a,b){sum(y*(a+b*x)-log(1+exp(a+b*x)))+
                    dnorm(a,sd=sigmaa,log=TRUE)+dnorm(b,sd=sigmab,log=TRUE)}
# Initialize a and b to equal the MLEs
beta=as.vector(fit.logistic$coefficients)
a=b=rep(0,Nsim)
a[1]=beta[1]
b[1]=beta[2]
#As scale for the proposal densities consider the square root of the
# cov.unscaled from the ml fit to the logistic model
fit.logistic.summary<-summary(fit.logistic)
scala=sqrt(fit.logistic.summary$cov.unscaled[1,1])
scalb=sqrt(fit.logistic.summary$cov.unscaled[2,2])
for (t in 2:Nsim){
    propa=a[t-1]+sample(c(-1,1),1)*rexp(1)*scala
    if (log(runif(1))<lpost(propa,b[t-1])- lpost(a[t-1],b[t-1]))
        a[t]=propa else
            a[t]=a[t-1]
        propb=b[t-1]+sample(c(-1,1),1)*rexp(1)*scalb
    if (log(runif(1))<lpost(a[t],propb)- lpost(a[t],b[t-1]))
        b[t]=propb
    else b[t]=b[t-1]
}



par(mfrow=c(2,3))
plot(a,type="l",xlab="iterations",ylab=expression(alpha))
hist(a,prob=TRUE,col="wheat2",xlab=expression(alpha),main="")
acf(a,ylab=expression(alpha))
plot(b,type="l",xlab="iterations",ylab=expression(beta))
hist(b,prob=TRUE,col="wheat2",xlab=expression(beta),main="")
acf(b,ylab=expression(beta))

par(mfcol=c(1,1))
plot(a,b,type="l",xlab=expression(alpha),ylab=expression(beta))

plot(challenger$temp,challenger$oring,pch=19,col="red4",xlab="temperatures",ylab="fail
for (t in seq(1000,Nsim,le=50)) curve(1/(1+exp(-a[t]-b[t]*x)), add=TRUE,col="grey65",l
curve(1/(1+exp(-mean(a)-mean(b)*x)),add=TRUE,col="red",lwd=2.5)
postal=rep(0,1000);i=1
```


Series a
Series b
α
β
iterations
Density
Lag



0.00
-0.05
-0.10
-0.15
-0.20
-0.25
-0.30
-0.35
β
0
5
10
15
20
α




```
In [11]: print(length(unique(a))/Nsim)
         print(length(unique(b))/Nsim)
         print (c(mean(a), sd(a))); print (c(mean(b), sd(b)))
         print(c( mean(1/(1+exp(-a-b*30))),sd(1/(1+exp(-a-b*30)))))
```

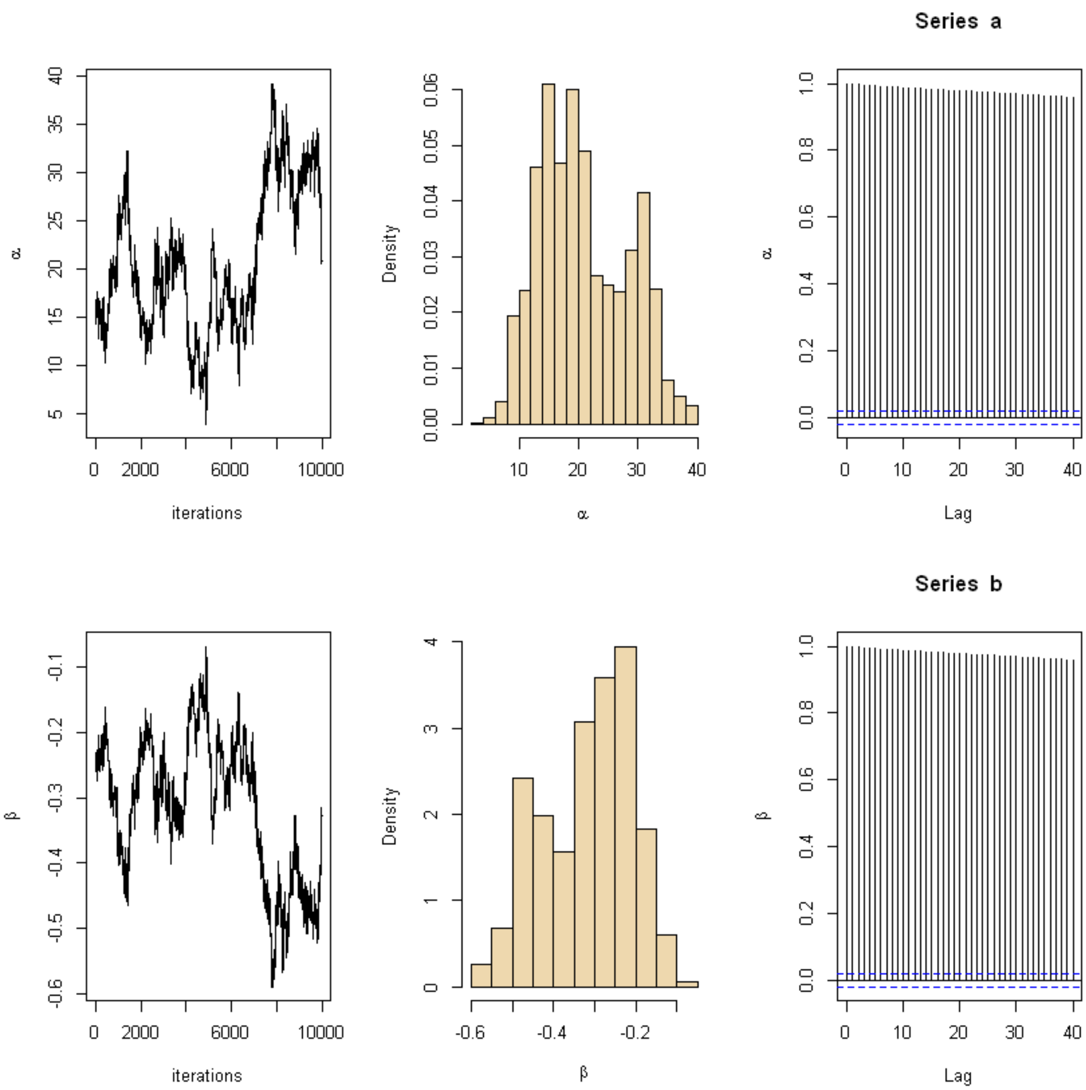
```
[1] 0.1062
[1] 0.1039
[1] 9.414120 5.119201
[1] -0.1502712  0.0746917
[1] 0.9311312 0.1273459
```

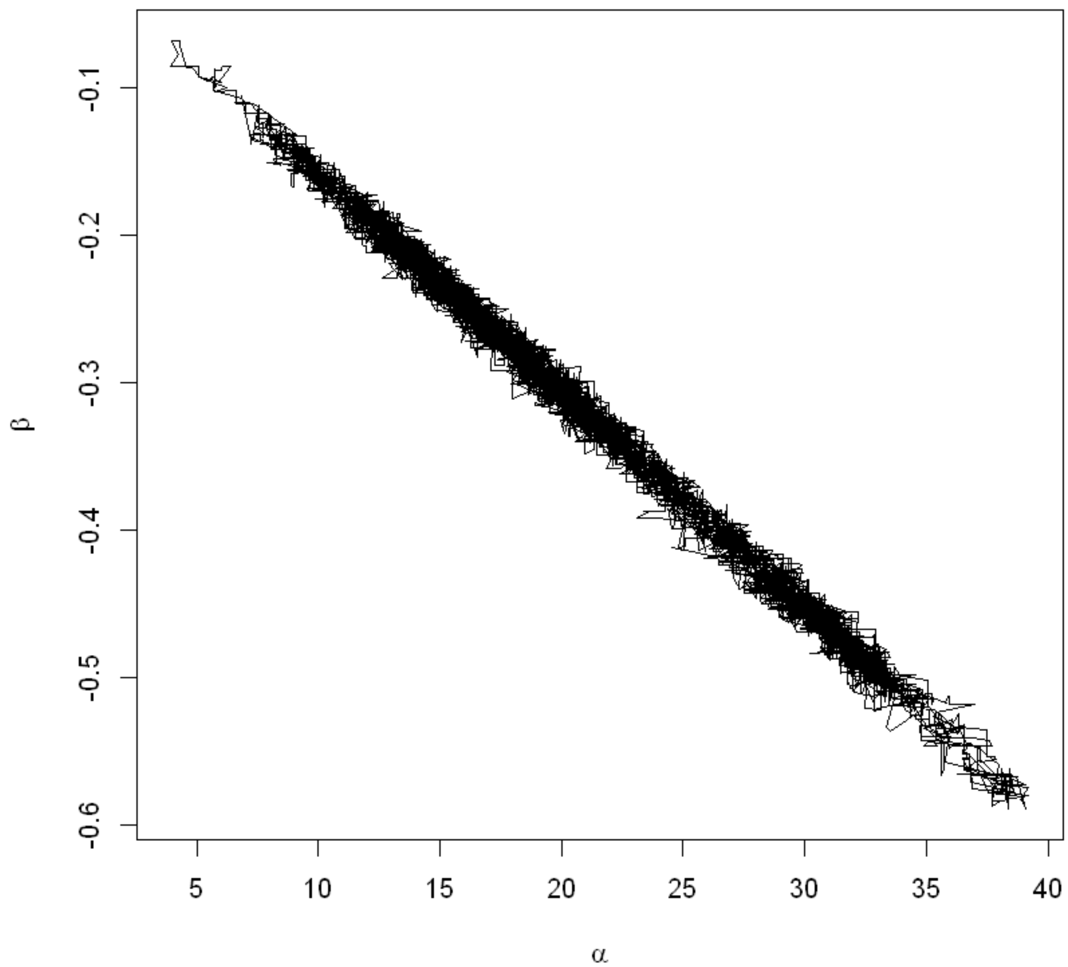
From the results above we get:

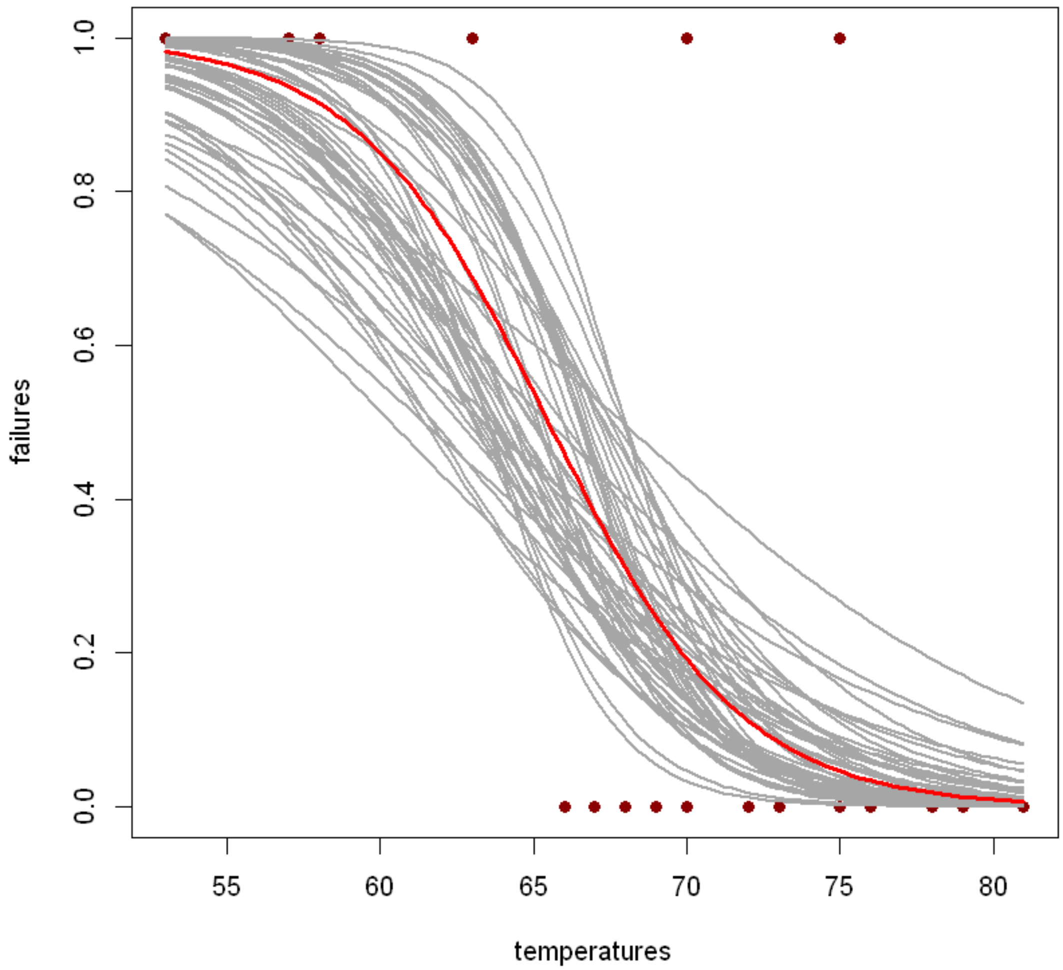
|
$\sigma' = \sigma_0$
$\sigma' = 10\sigma_0$ |---| :---:|:---:|:---: | :---: | | α | β | α | β | | acceptance rates | 0.1001 | 0.0987 | 0.1062 | 0.1039 | means | 6.294067 | -0.10533114 | 9.414120 | -0.1502712 | standard deviations | 3.497769 | 0.05123078 | 5.119201 | 0.0746917 | P(error) at 30 F
0.8673593
0.9311312 | standard error
0.1983737
0.1273459

In [12]:
```
set.seed(1)
Nsim=10^4
x=challenger$temp
y=challenger$oring
sigmaa=5*10 ; sigmab=5*10/sd(x)
lpost=function(a,b){sum(y*(a+b*x)-log(1+exp(a+b*x)))+
                     dnorm(a,sd=sigmaa,log=TRUE)+dnorm(b,sd=sigmab,log=TRUE)}
# Initialize a and b to equal the MLEs
beta=as.vector(fit.logistic$coefficients)
a=b=rep(0,Nsim)
a[1]=beta[1]
b[1]=beta[2]
#As scale for the proposal densities consider the square root of the
# cov.unscaled from the ml fit to the logistic model
fit.logistic.summary<-summary(fit.logistic)
scala=sqrt(fit.logistic.summary$cov.unscaled[1,1])/10
scalb=sqrt(fit.logistic.summary$cov.unscaled[2,2])/10
for (t in 2:Nsim){
    propa=a[t-1]+sample(c(-1,1),1)*rexp(1)*scala
    if (log(runif(1))<lpost(propa,b[t-1])- lpost(a[t-1],b[t-1]))
        a[t]=propa else
            a[t]=a[t-1]
        propb=b[t-1]+sample(c(-1,1),1)*rexp(1)*scalb
    if (log(runif(1))<lpost(a[t],propb)- lpost(a[t],b[t-1]))
        b[t]=propb
    else b[t]=b[t-1]
}



par(mfrow=c(2,3))
plot(a,type="l",xlab="iterations",ylab=expression(alpha))
hist(a,prob=TRUE,col="wheat2",xlab=expression(alpha),main="")
acf(a,ylab=expression(alpha))
plot(b,type="l",xlab="iterations",ylab=expression(beta))
hist(b,prob=TRUE,col="wheat2",xlab=expression(beta),main="")
acf(b,ylab=expression(beta))
```

```
par(mfcol=c(1,1))
plot(a,b,type="l",xlab=expression(alpha),ylab=expression(beta))

plot(challenger$temp,challenger$oring,pch=19,col="red4",xlab="temperatures",ylab="fail
for (t in seq(1000,Nsim,le=50)) curve(1/(1+exp(-a[t]-b[t]*x)), add=TRUE,col="grey65",l
curve(1/(1+exp(-mean(a)-mean(b)*x)),add=TRUE,col="red",lwd=2.5)
postal=rep(0,1000);i=1
```




-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
β
5
10
15
20
25
30
35
40
α




```
In [13]: print(length(unique(a))/Nsim)
         print(length(unique(b))/Nsim)
         print (c(mean(a), sd(a))); print (c(mean(b), sd(b)))
         print(c(mean(1/(1+exp(-a-b*30))),sd(1/(1+exp(-a-b*30)))))
```

```
[1] 0.6104
[1] 0.6141
[1] 20.807518  7.358899
[1] -0.3177214  0.1082698
[1] 0.998901915 0.005897289
```

From the results above we get:

|
one-tenth scales
original |---| :---:|:---:|:---: | :---: | | α | β | α | β | |acceptance rates | 0.6104|0.6141| 0.1062 |0.1039 |means |20.807518|-0.3177214|9.414120|-0.1502712 |standard deviations |7.358899|0.1082698|5.119201|0.0746917 |P(error) at 30 F
0.998901915
0.9311312 |standard error
0.005897289
0.1273459
Problem 2

1.4 (a)

The joint posterior distribution simplifies to:

$$\Big[\prod_{i=1}^{10}(\lambda_i t_i)^{x_i}e^{-\lambda_i t_i}\lambda_i^{a-1}e^{-\beta\lambda_i}\Big]\beta^{10\alpha}\beta^{\gamma-1}e^{-\delta\beta} = \tag{5}$$

$$\Big[\prod_{i=1}^{10}\lambda_i^{x_i+\alpha-1}t_i^{x_i}e^{-\lambda_i(t_i+\beta)}\Big]\beta^{10\alpha+\gamma-\delta\beta-1} = \tag{6}$$

$$\Big[\prod_{i=1}^{10}\lambda_i^{x_i+\alpha-1}t_i^{x_i}\Big]e^{-\delta\beta-\sum_{i=1}^{10}\lambda_i(t_i+\beta)}\beta^{10\alpha+\gamma-1} \tag{7}$$

1.5 (b)

The conditional distribution of λ is defined as:

$$\pi(\lambda_i|\beta,t_i,x_i) = \frac{\pi(\lambda_i,\beta|t_i,x_i)}{\pi(\beta)} \tag{8}$$

$$= \frac{\lambda_i^{x_i+\alpha-1}t_i^{x_i}e^{-\delta\beta-\lambda_i(t_i+\beta)}\beta^{10\alpha+\gamma-1}}{} = \Gamma(x_i+\alpha,t_i+\beta) \tag{9}$$

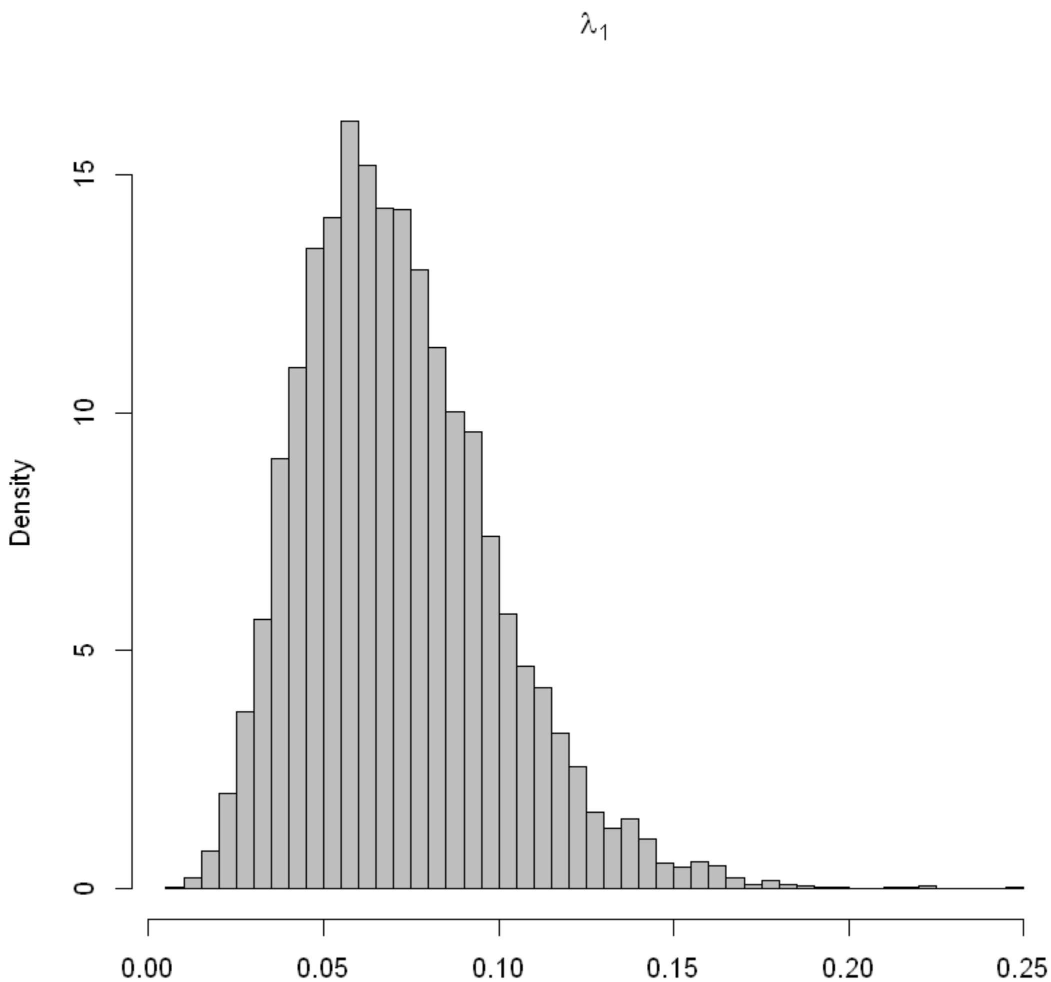
The conditional distribution of β will be defined as:

$$\pi(\beta|\lambda_1,...,\lambda_{10}) = \Gamma(\gamma+10\alpha,\delta+\sum_{i=1}^{10}\lambda_i) \tag{10}$$

1.6 (c)

```
In [14]: xdata=c(5, 1, 5, 14, 3, 19, 1, 1, 4, 22)
         Time=c(94.32, 15.72, 62.88, 125.76, 5.24, 31.44, 1.05, 1.05, 2.10, 10.48)
         nx=length(xdata)
         nsim=10^4;alpha = 1.8;gamma= 0.01;delta=1

         beta=rgamma(1, shape= alpha, rate = delta)
         l1=rgamma(nsim,shape=xdata[1]+alpha,rate=Time[1]+beta)
```

```
In [15]: hist(l1,breaks=50,col="grey",xlab="",main=expression(lambda[1]), freq = F)
```




```
In [16]: summary(l1)
         mean(l1)
         median(l1)
         (var(l1))^0.5
```

```
    Min.  1st Qu.   Median     Mean  3rd Qu.     Max.
0.008929 0.051490 0.067961 0.071503 0.087785 0.247411
```

0.0715025667320052
0.0679608729590416

0.0276862333499326

The maximum likelihood estimates of $\lambda_1, ..., \lambda_{10}$ will be derived from the MLE for the parameter of the poisson random variable X_i. For a Poisson random variable $Y_i \sim Pois(\lambda^*)$ the MLE estimate is found as following.

$$\hat{\lambda^*}_{MLE} = \frac{\sum_{i=1}^{n} Y_i}{n}$$

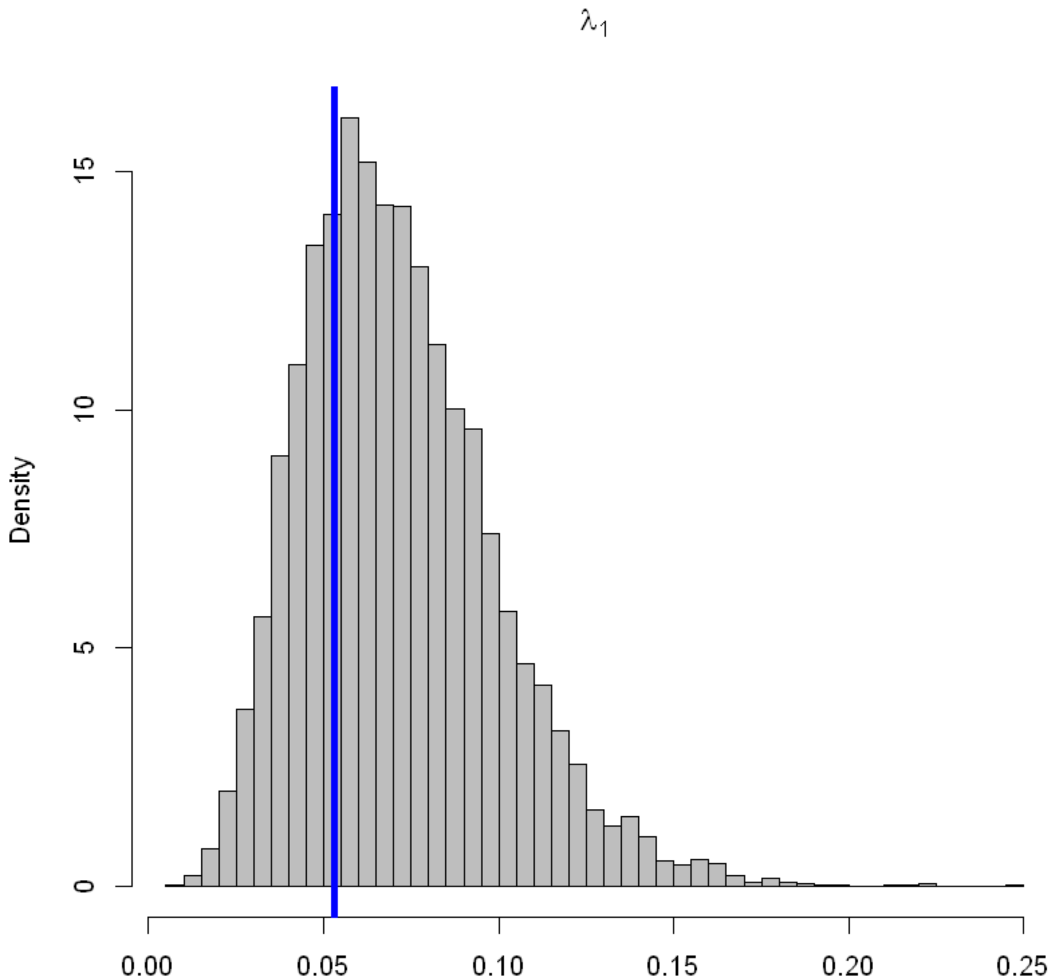
Thus, for the data we have, we get:

$$\hat{\lambda_i t}_{i_{MLE}} = X_i \hat{\lambda}_{i_{MLE}} = \frac{X_i}{t_i}$$

```
In [17]: xdata=c(5, 1, 5, 14, 3, 19, 1, 1, 4, 22)
         Time=c(94.32, 15.72, 62.88, 125.76, 5.24, 31.44, 1.05, 1.05, 2.10, 10.48)

         l_mle = xdata/Time
         l_mle
```

1. 0.053011026293469 2. 0.0636132315521628 3. 0.0795165394402036 4. 0.111323155216285 5. 0.572519083969466 6. 0.604325699745547 7. 0.952380952380952 8. 0.952380952380952 9. 1.9047619047619 10. 2.09923664122137

```
In [18]: hist(freq = FALSE, l1,breaks=50,col="grey",xlab="",main=expression(lambda[1]))
         abline(v=l_mle[1], lw=4, col = 'blue')
```



2 (d)

```
In [19]: nx=length(xdata)
         nsim=10^4;alpha = 1.8;gamma= 0.01;delta=1
         lambda=array(xdata*Time/sum(Time),dim=c(nsim,nx))
         beta=rep(gamma*delta,nsim)
         for (i in 2:nsim){
            for (j in 1:nx){
               lambda[i,j]=rgamma(1,shape=xdata[j]+alpha,rate=Time[j]+beta[i-1])
               }
            beta[i]=rgamma(1,shape=gamma+nx*alpha,rate=delta+sum(lambda[i,]))
            }
```

3 (e)

The 95% credible intervals for the parameters will be calculated as $I_\lambda = [\hat{\lambda}_{MLE} - 1.96\sigma, \hat{\lambda}_{MLE} + 1.96\sigma]$. Thus, for all of the parameters, we have:

```
In [43]: l_int = matrix(nrow = 10, ncol = 2)
         for (i in 1:length(l_mle)){
             sigma = sqrt(var(lambda[,i]))
             l_int[i,1] = l_mle[i]-1.96*sigma
             l_int[i,2] = l_mle[i]+1.96*sigma
         }
```

```
In [45]: l_int
```

-0.0055884076	0.1116105
-0.1181394103	0.2453659
-0.0031878778	0.1622210
0.0462782423	0.1763681
-0.0002468046	1.1452850
0.3382452347	0.8704062
-0.0732285818	1.9779905
-0.0823859472	1.9871479
0.7829220751	3.0266017
1.3384232977	2.8600500

It seems that pump 10 is one of the least reliable ones and pumps 1 and 2 are very reliable.

3.1 (f)

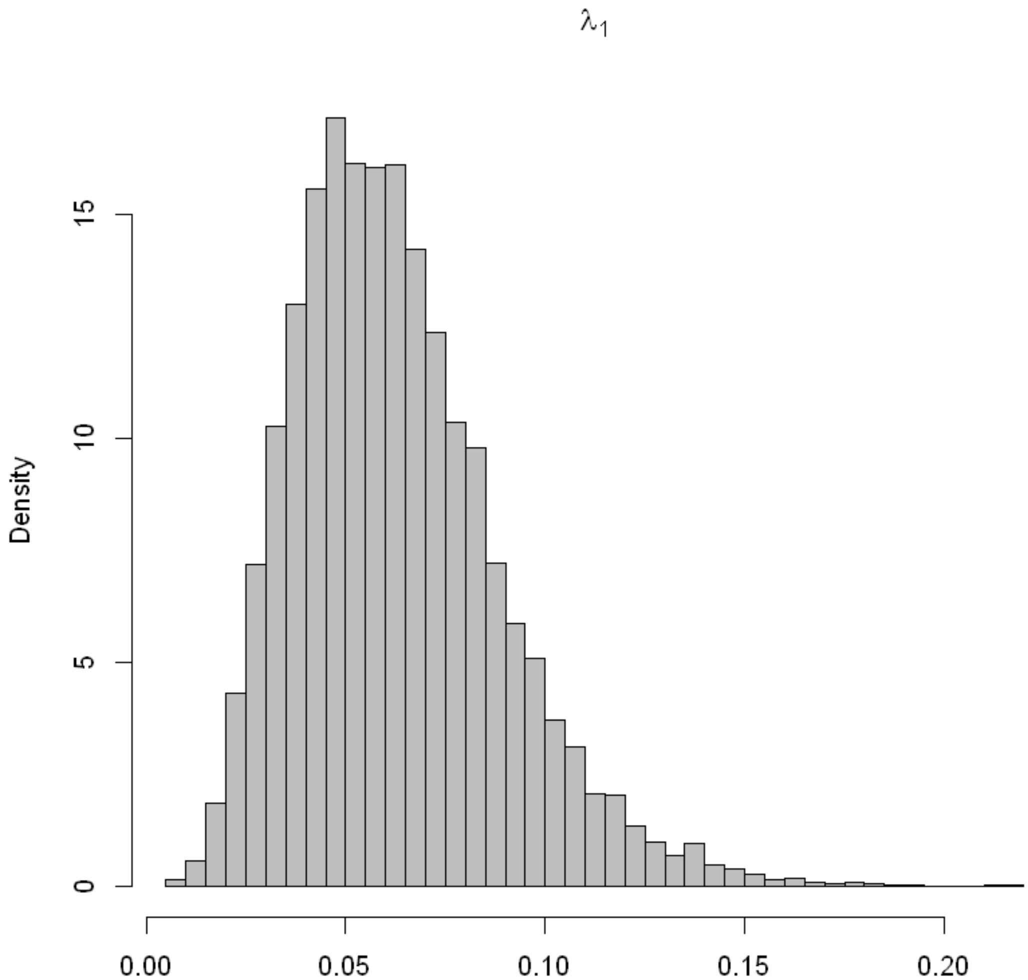
Trying with a prior with parameters computed from the MLE estimates. For gamma distributions $X \sim \Gamma(k, \theta)$, we have:

$$\hat{\theta}_{MLE} = \frac{1}{kN}\sum_{i=1}^{N} x_i s = ln(\frac{1}{N}\sum_{i=1}^{N} x_i) - \frac{1}{N}\sum_{i=1}^{N} ln(x_i) k \simeq \frac{3 - s + \sqrt{(s-3)^2 + 24s}}{12s}$$

```
In [62]: xdata=c(5, 1, 5, 14, 3, 19, 1, 1, 4, 22)
         Time=c(94.32, 15.72, 62.88, 125.76, 5.24, 31.44, 1.05, 1.05, 2.10, 10.48)
         nx=length(xdata)
         nsim=10^4;alpha = 1.8;gamma= 0.01;delta=1

         s = log(sum(xdata)/nx)-sum(log(xdata))/nx
         alpha = (3-s+sqrt((s-3)^2+24*s))/(12*s)


         beta=rgamma(1, shape= alpha, rate = delta)
         l1=rgamma(nsim,shape=xdata[1]+alpha,rate=Time[1]+beta)
         alpha
```

0.99116678629947

```
In [58]: hist(l1,breaks=50,col="grey",xlab="",main=expression(lambda[1]), freq = F)
```




```
In [59]: summary(l1)
         mean(l1)
         median(l1)
         (var(l1))^0.5
```

```
    Min.  1st Qu.   Median     Mean  3rd Qu.     Max.
0.008391 0.044136 0.059198 0.062665 0.077269 0.217807
```

0.0626651229839855
0.0591984331256458
0.0256608479223233

The maximum likelihood estimates of $\lambda_1, ..., \lambda_{10}$ will be derived from the MLE for the parameter of the poisson random variable X_i. For a Poisson random variable $Y_i \sim Pois(\lambda^*)$ the MLE estimate is found as following.

$$\hat{\lambda^*}_{MLE} = \frac{\sum_{i=1}^{n} Y_i}{n}$$

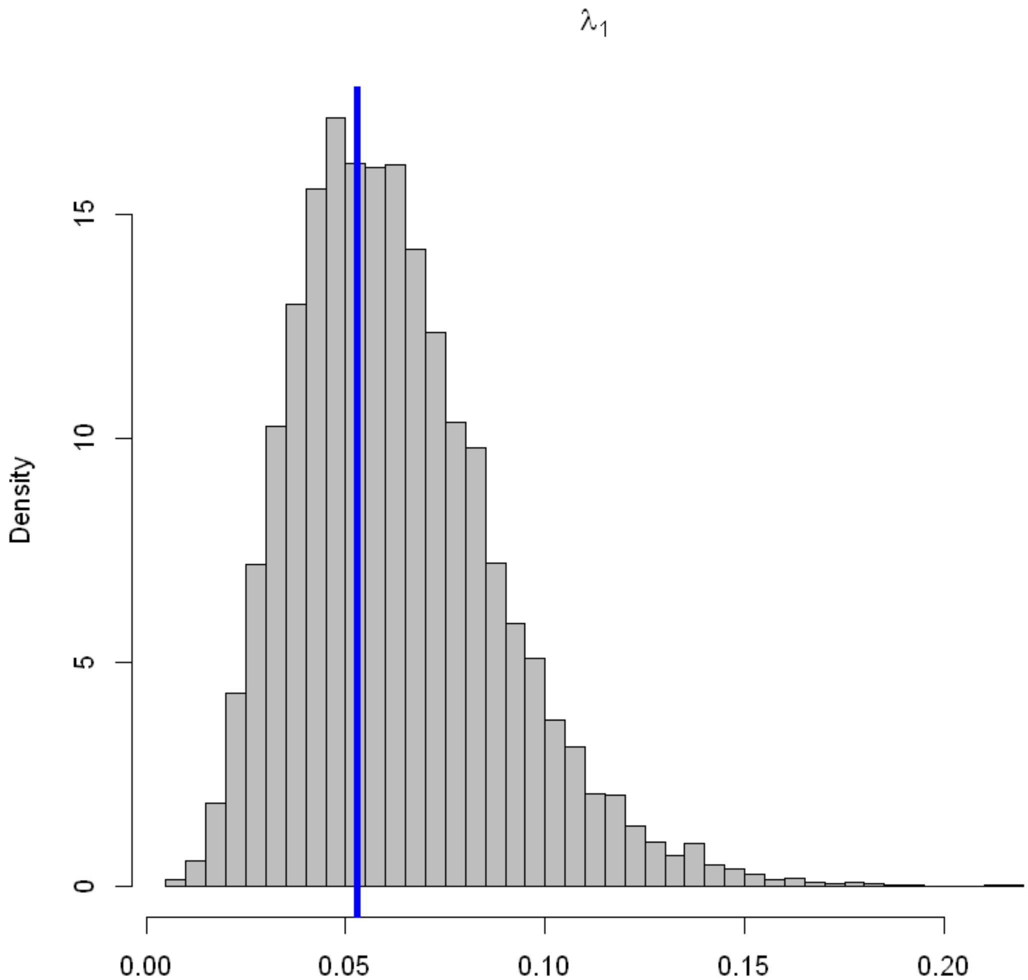
Thus, for the data we have, we get:

$$\hat{\lambda_i t}_{i_{MLE}} = X_i \hat{\lambda}_{i_{MLE}} = \frac{X_i}{t_i}$$

```
In [60]: xdata=c(5, 1, 5, 14, 3, 19, 1, 1, 4, 22)
         Time=c(94.32, 15.72, 62.88, 125.76, 5.24, 31.44, 1.05, 1.05, 2.10, 10.48)

         l_mle = xdata/Time
         l_mle
```

1. 0.053011026293469 2. 0.0636132315521628 3. 0.0795165394402036 4. 0.111323155216285 5. 0.572519083969466 6. 0.604325699745547 7. 0.952380952380952 8. 0.952380952380952 9. 1.9047619047619 10. 2.09923664122137

```
In [61]: hist(freq = FALSE, l1,breaks=50,col="grey",xlab="",main=expression(lambda[1]))
         abline(v=l_mle[1], lw=4, col = 'blue')
```



4 (d)

```
In [63]: nx=length(xdata)
         nsim=10^4;alpha = 1.8;gamma= 0.01;delta=1
         lambda=array(xdata*Time/sum(Time),dim=c(nsim,nx))
         beta=rep(gamma*delta,nsim)
         for (i in 2:nsim){
            for (j in 1:nx){
               lambda[i,j]=rgamma(1,shape=xdata[j]+alpha,rate=Time[j]+beta[i-1])
               }
            beta[i]=rgamma(1,shape=gamma+nx*alpha,rate=delta+sum(lambda[i,]))
            }
```

The 95% credible intervals for the parameters will be calculated as $I_\lambda = [\hat{\lambda}_{MLE} - 1.96\sigma, \hat{\lambda}_{MLE} + 1.96\sigma]$. Thus, for all of the parameters, we have:

```
In [64]: l_int = matrix(nrow = 10, ncol = 2)
         for (i in 1:length(l_mle)){
             sigma = sqrt(var(lambda[,i]))
             l_int[i,1] = l_mle[i]-1.96*sigma
             l_int[i,2] = l_mle[i]+1.96*sigma
         }

In [45]: l_int
```

-0.0055884076	0.1116105
-0.1181394103	0.2453659
-0.0031878778	0.1622210
0.0462782423	0.1763681
-0.0002468046	1.1452850
0.3382452347	0.8704062
-0.0732285818	1.9779905
-0.0823859472	1.9871479
0.7829220751	3.0266017
1.3384232977	2.8600500