Search and Optimization 21/22

Assignment

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Contents

1 Context

The optimisation problem chosen for this assignment delves into the design of a badminton racket to optimise its cost to the minimum, while still meeting all the design constraints. The theme of the problem is material mix optimisation.

While the author has used his best judgement for the numbers and some of the constraints in the problems, these are the typical types of trade-offs required to be made to reach optimal solutions in a product design problem.

2 Linear Programming Problem

2.1 Problem Description

The problem chosen for the linear programming section goes into the material volume optimisation between carbon fibre and titanium, which are the two materials used to manufacture badminton rackets.

Carbon fibre is more expensive than titanium, but is significantly lighter. The cost and density comparisons are shown in Table 1.

Table 1: Costs and densities of carbon fibre and titanium

Material	Cost ($/cm^3)	Density (g/cm^3)
Carbon fibre	4	1.55
Titanium	2	4.5

The objective is to manufacture the racket to the minimum cost, while still meeting the weight requirement to achieve balance between power and racket speed, and meet the volume constraint.

As per the regulation from the badminton federation, the weight of the badminton racket needs to be between 80g and 100g to achieve the optimal balance between racket speed and power

Although it is not a regulation, design best practice recommends the volume of the racket to be at least 32 cm^3. This ensures good bending stiffness during game play.

Minimum of 12 cm^3 of titanium is required to prevent the racket from being too brittle.

2.2 Decision variables

The decision variables for the linear programming problem are:

$$x = volume\ of\ carbon\ fibre\ in\ cm^3$$
$$y = volume\ of\ titanium\ in\ cm^3$$

2.3 Objective function

The objective is to minimise the material cost of the racket, which is denoted as Z in this problem.

$$Z = 4x + 2y$$

2.4 Problem constraints

The volume of the racket needs to be at least 32 cm^3.

$$x + y \geq 32$$

The weight of the badminton racket needs to be between 80g and 100g.

$$80 \leq 1.55x + 4.5y \leq 100$$

Minimum of 12 cm^3 of titanium is required, therefore by default non-negative.

$$y \geq 12$$

$$y \geq 0$$

The volume of carbon fibre should be a non-negative number.

$$x \geq 0$$

2.5 Mathematical formulation

The mathematical formulation below is shown in the standard format accepted in the MATLAB solver-based approach.

Minimise:

$$Z = 4x + 2y$$

Subject to:

$$-x - y \leq -32$$

$$1.55x + 4.5y \leq 100$$

$$-1.55x - 4.5y \leq -80$$

$$-y \leq -12$$

$$x \geq 0$$

$$y \geq 0$$

2.6 Graphical Method

The constraint lines have been plotted in Figure 1 using MATLAB. Using the plotregion function, the corresponding feasible region has been shaded green.

Since it is known that the optimal solution should lie in one of the corner points of the feasible (CPF) region, the optimal solution can be calculated by solving for the equation Z at each of the corner points.

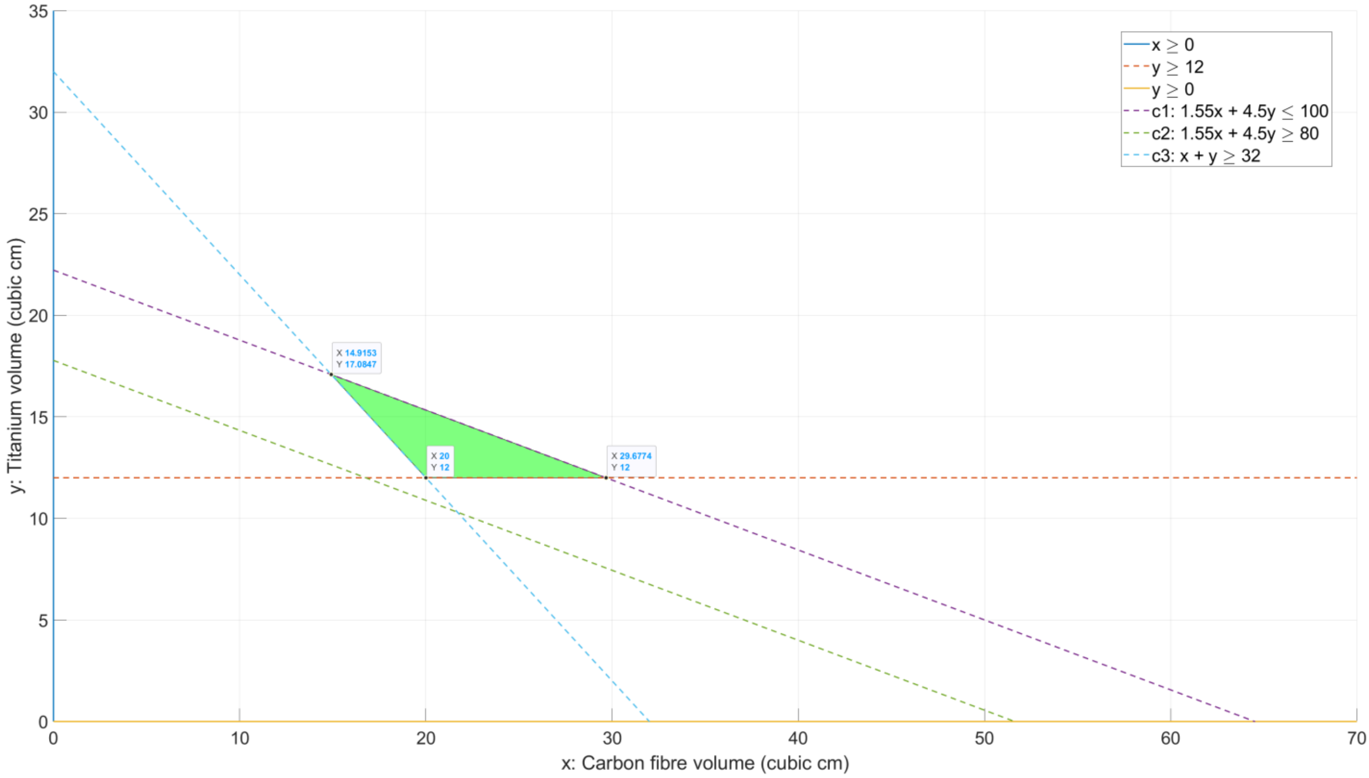

Figure 1: Graph showing the shaded feasible region and corner point feasible solutions.

The Z values at the corner points are as below:

Point 1 - (14.91, 17.08)

$$Z = 4 * 14.91 + \ 2 * 17.08 = \$93.83$$

Point 2 – (29.68, 12)

$$Z\ = 4 * 29.68 + \ \ 2 * 12 = \$142.72$$

Point 3 – (20, 12)

$$Z = 4 * 20 + \ \ 2 * 12 = \$104$$

This is a cost minimisation exercise; therefore, the objective is to find the lowest Z, which in this case is Point 1 (14.91, 17.08).

Figure 2 below shows the objective function line passing through the optimal point - Point 1.

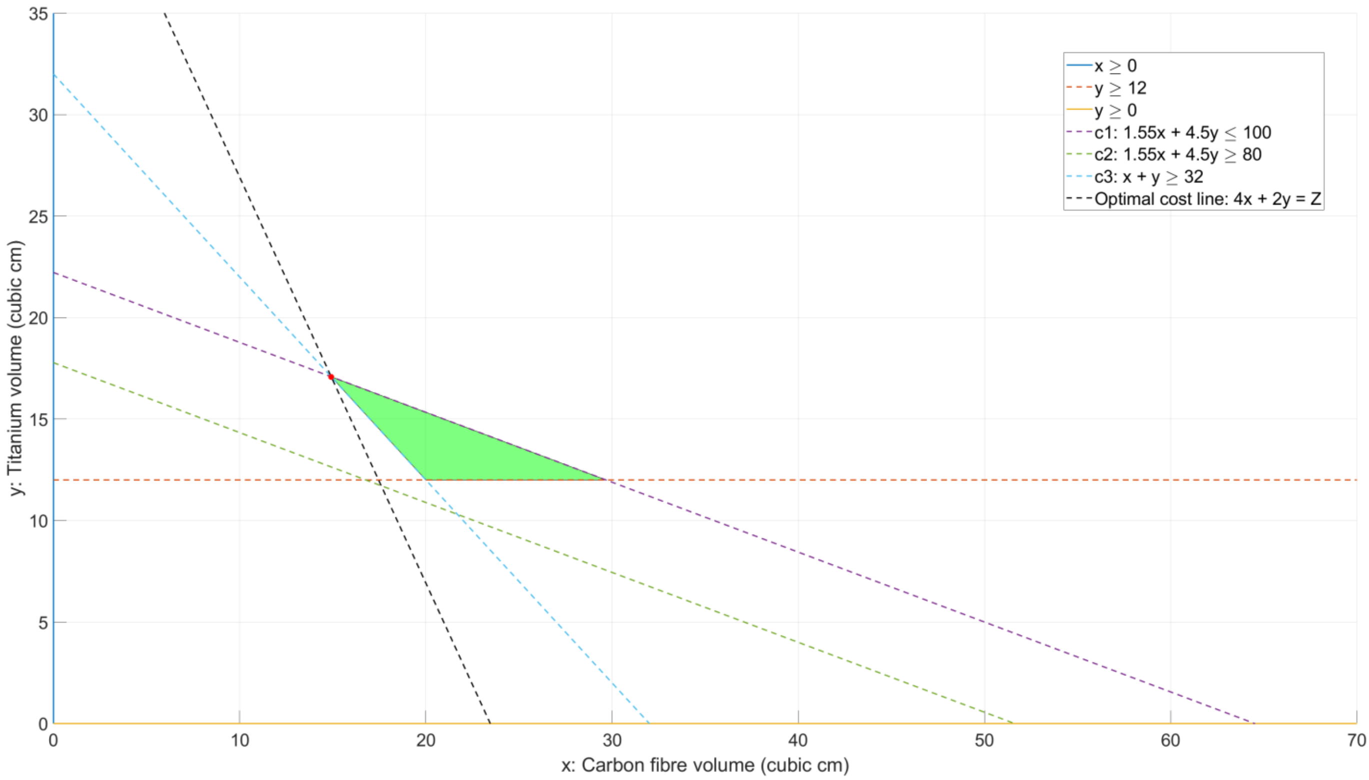

Figure 2: Objective function line passing through the Optimal Point (Point 1)

2.7 MATLAB Implementation

```
clear all
close all
clc

% The code below uses the original constraints. Please adjust the code to
% carry out sensitivity studies. The code has been commented to help
% understand the constraint/function that each line signifies.

f = [4 2]; % objective function definition in the order x (carbon fibre) and y (titanium)

% inequality constraints
A = [1.55 4.5;... % 1.55x + 4.5y <= 100
    -1.55 -4.5;... % 1.55x + 4.5y >= 80
    -1 -1;... % x + y >= 32
    0 -1]; % y >= 12

b = [100 -80 -32 -12]; % corresponding RHS for the above inequality constraints

lb = [0 0] % non-negativity constraints (lower bound definition for x, y)
ub = [Inf Inf] % upper bound definition for x, y

[val, fval] = linprog(f, A, b, [], [], lb, ub) % solution optimal point, optimal solution

figure(1) % feasible region and constraint lines

lb_plt = [0 0]; % define lower bounds (just for plot axis)
ub_plt = [70 35]; % define upper bound (just for plot axis)

plotregion(-A, -b, lb, ub, 'g', 0.5) % from MATLAB File Exchange

xlabel('x: Carbon fibre volume (cubic cm)'), ylabel('y: Titanium volume (cubic cm)')

% axis limits definition
```

```
axis([lb_plt(1) ub_plt(1) lb_plt(2) ub_plt(2)]), grid
hold on

% compute constraints lines:
x1 = lb_plt(1):ub_plt(1); % x axis lb to ub
x2 = lb_plt(2):ub_plt(2); % y axis lb to ub
x2_1 = (100 - 1.55.*x1)/4.5; % constraint Y<=(100-1.55X)/4.5
x2_2 = (-1.55.*x1 + 80)/4.5; % constraint Y>=(80-1.55X)/4.5
x2_3 = (32 - x1); % constraint Y>=32-X

% compute objective function line:
x2_cf = (fval-4.*x1) / 2;

% plotting the lines
obj = plot(zeros(size(x2)), x2, x1, 12*ones(size(x1)), x1, zeros(size(x1)),...
 x1, x2_1, x1, x2_2, x1, x2_3, x1, x2_cf,'LineWidth',1.5);
set(gca,'FontSize',20);

% defining line style and colours
obj(2).LineStyle = '--';
obj(4).LineStyle = '--';
obj(5).LineStyle = '--';
obj(6).LineStyle = '--';
obj(7).LineStyle = '--';
obj(7).Color = 'k'

% highlighting optimal solution
plot(val(1), val(2),'r*', 'LineWidth', 4)
hold off % releases graph

% set axes and figure legend:
legend(obj, {'x \geq 0', 'y \geq 12', 'y \geq 0', ...
    'c1: 1.55x + 4.5y \leq 100', ...
    'c2: 1.55x + 4.5y \geq 80', ...
    'c3: x + y \geq 32', ...
    'Optimal cost line: 4x + 2y = Z'}, ...
    'Location', 'Best', 'FontSize', 20)
```

2.8 Solution

Based on the above constraints, the optimal point corresponds to 14.91 cm^3 of carbon fibre and 17.08 cm^3 of titanium.

This corresponding optimal cost of the racket is $93.83.

2.9 Sensitivity Analysis

The graphical method gives useful visual insight into the problem. The optimum solution is bound by constraints c1 (upper limit on the weight of the racket) and c3 (lower limit on the volume of the racket). These are the constraints that the company will have to evaluate to arrive at a lower optimal cost solution.

Study 1: Shadow price of volume

The lower limit on the volume of the racket is a binding constraint at the professional level. However, the company is looking at launching a new range of rackets for intermediate level players that are slightly under the 32 cm^3 volume required by the badminton federation.

Modified volume constraint (c3):

$$x + y \geq \mathbf{31}\ cm^3$$

A reduced volume of 31 cm^3 (1 cm^3 reduction) was investigated in the sensitivity analysis to analyse the shadow price and the results are presented in Figure 3.

The optimal point has moved to 13.39 cm^3 of carbon fibre and 17.61 cm^3 of titanium. The corresponding cost has reduced to $88.78. The shadow price is the difference in optimal cost from the original constraint and modified constraint, which is $5.05 in this case.

This means that each additional cm^3 reduction in racket volume will result in a $5.05 reduction in the racket cost, provided the binding constraints remain the same. Changing the problem parameters without caution may change the binding constraints. This value can be used by the company to design a range of intermediate and beginner level rackets with lower volume without violating regulations.

Shadow price of the weight constraint has **not** been investigated as it would violate the regulations of the badminton federation. Therefore, the manufacturer has to work within the current constraint of maximum weight of 100g.

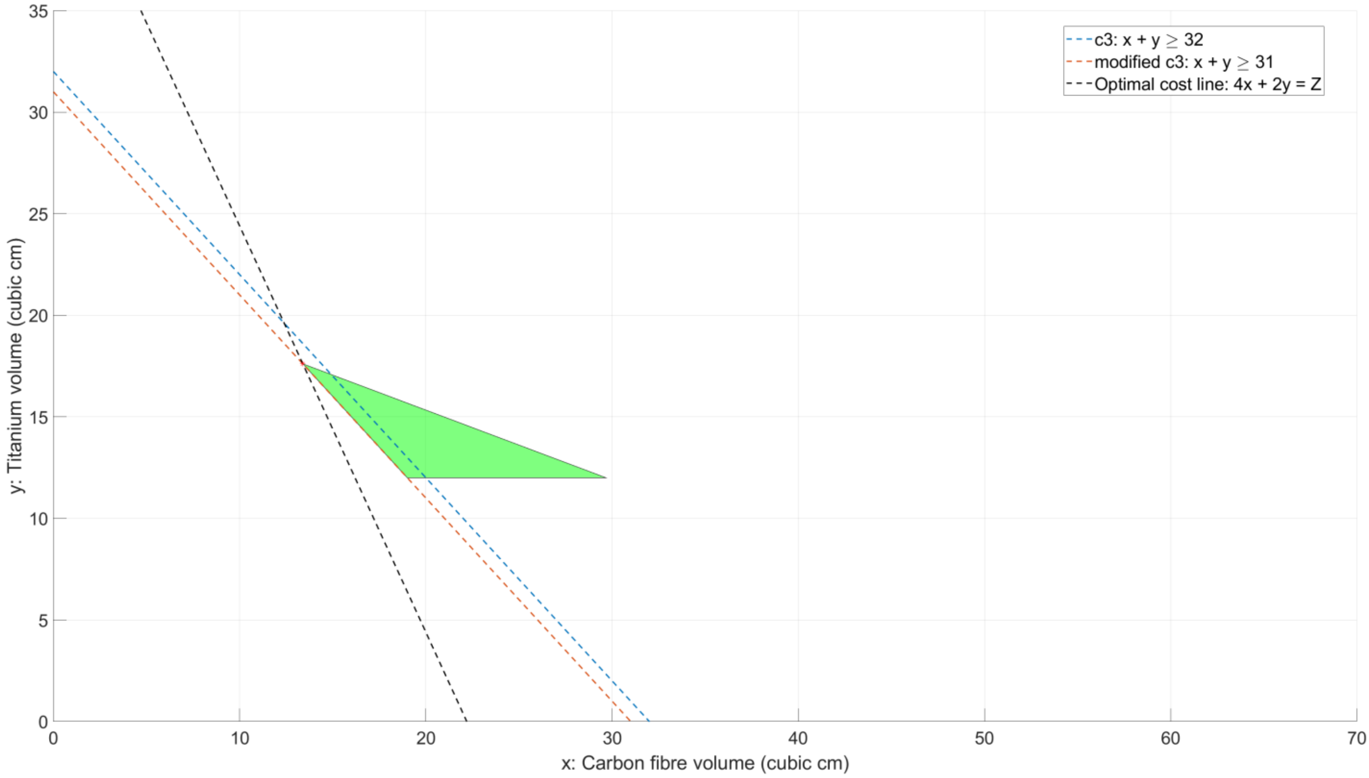

Figure 3: Sensitivity of reducing the overall volume to 31 cm^3

Study 2: Objective function coefficient

The company is considering negotiating a better price from the supplier by leveraging its volumes.

The company wants to model the new cost of $3/cm^3 ($1/cm^3 reduction). The results are shown in Figure 4. The figure shows that the two optimal cost lines still intersect at Point 1 of the feasible region, meaning that the change in unit cost does not affect the optimal point. It only changes the slope of the line and therefore the cost, which has reduced from $93.83 to $78.91.

Modified objective function for the material cost of the racket shown in Figure 4:

$$Z = \mathbf{3}x + 2y$$

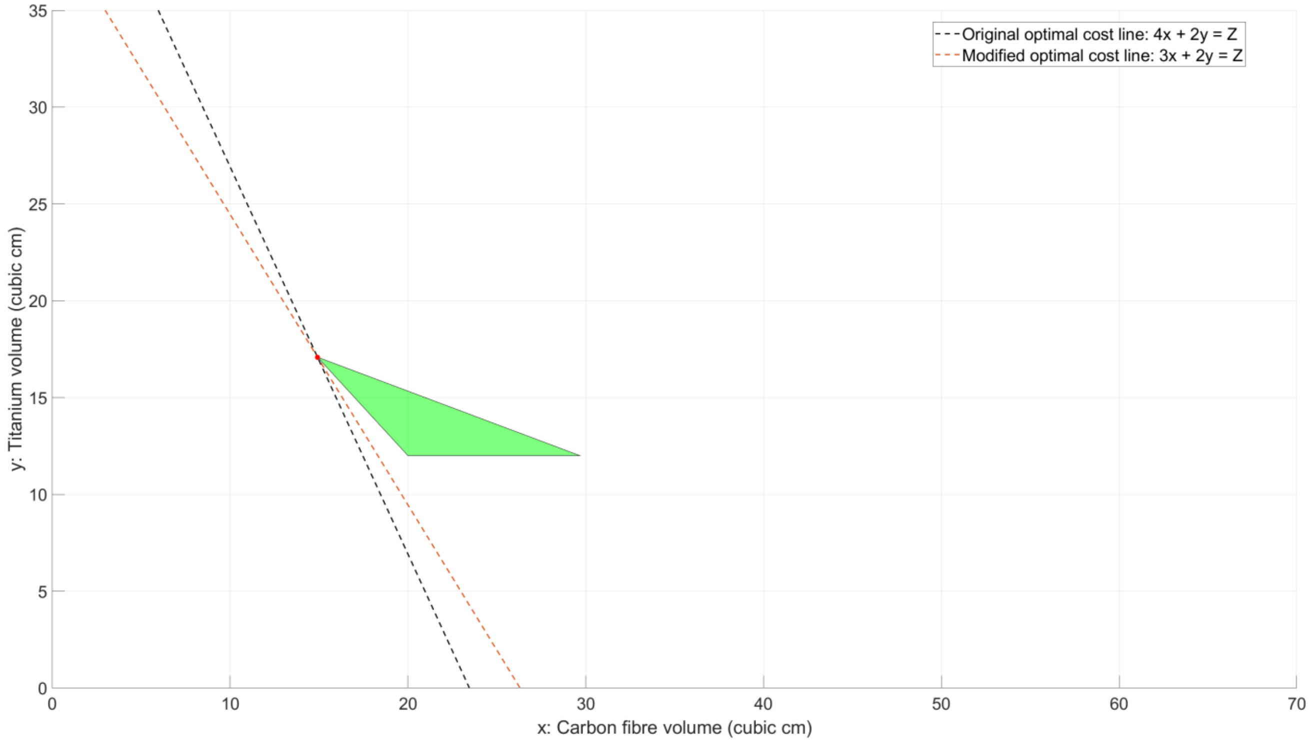

Figure 4: Sensitivity of changing the unit price of carbon fibre

This trend does not continue beyond a certain limit, reducing the cost further to $2/cm^3 makes the objective function line coincident with constraint line c3 and the new optimal solution shifts to the point (20, 12), which uses 20 cm^3 of carbon fibre and 12 cm^3 of titanium.

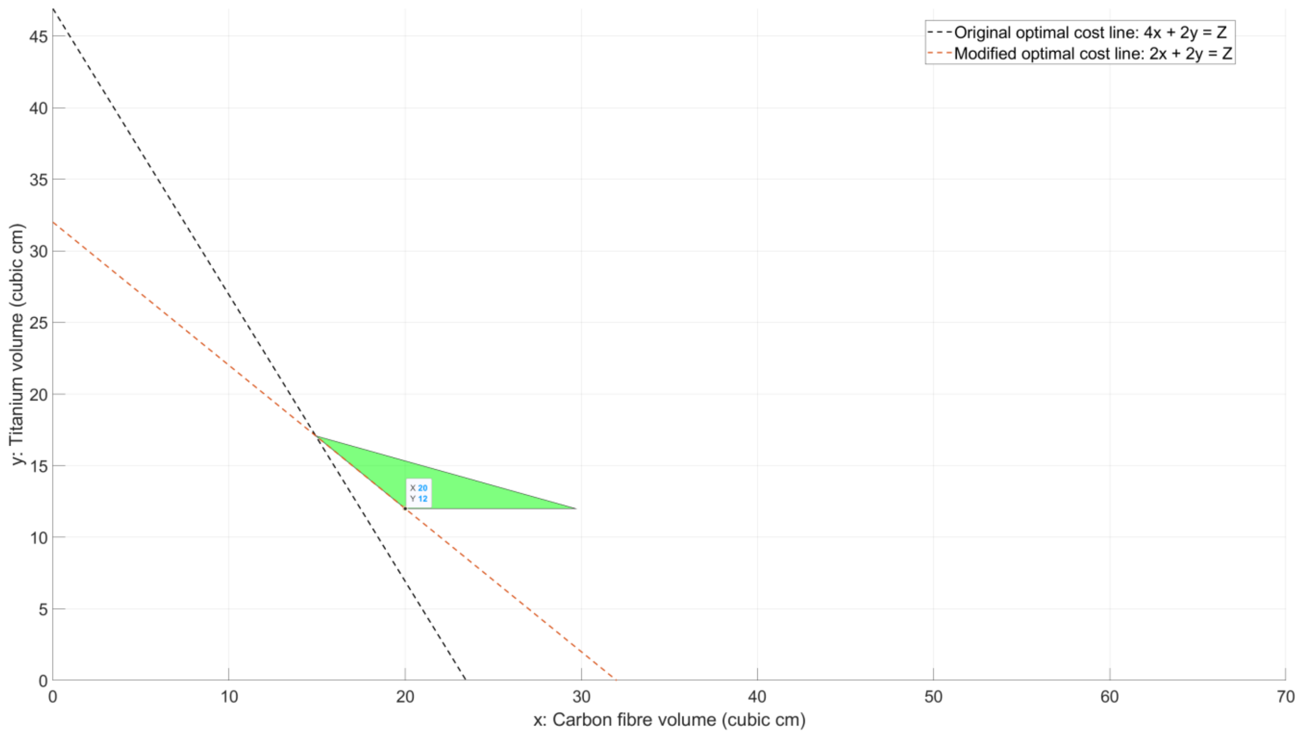

Figure 5: New optimal point at (20,12) on further reduction of the price of carbon fibre to $2/cm^3

Therefore, the cost per cm^3 of carbon fibre can be changed between the range $\$2/cm^3$ and $\$4/cm^3$ without changing the optimal point. The sensitivity study should be used by the manufacturer in this manner to build awareness of the point at which the optimal point shifts from one feasible point to another.

3 Mixed-integer Programming Problem

3.1 Problem Description

The linear programming problem dived into the lower end of professional rackets. The top-end of the professional badminton rackets used in international tournaments require heat treatment to achieve the best strength and durability properties.

The racket core is made from carbon fibre and a titanium shell is moulded over the core. The titanium shell requires a heat treatment agent (HTA) to be added as a percentage of the titanium volume, this data is presented in Table 2. The HTA is procured as pellets of volume 1 cm^3 and have to be used in integer numbers on each racket.

Table 2: Volume of HTA required as a percentage of the material volumes

Material	**Min volume of HTA as % of material volume**	**Max volume of HTA as % of material volume**
Carbon fibre	0	0
Titanium	15	20

The costs and densities of the materials are shown in Table 3.

Table 3: Costs and densities of the materials and HTA

Material	**Cost ($\$/cm^3$)**	**Density (g/cm^3)**
Carbon fibre	4	1.55
Titanium	2	4.5
Heat treatment agent	8	3

The objective is to manufacture the racket to the minimum cost, while still meeting the weight requirement in order to achieve the balance between power and racket speed.

As per the regulation from the badminton federation, the weight of the badminton racket needs to be between 80g and 100g to achieve the optimal balance between racket speed and power

Although it is not a regulation, design best practice recommends the volume of the racket to be at least 32 cm^3. This ensures good bending stiffness during game play.

Minimum of 12 cm^3 of titanium is required to prevent the racket from being too brittle.

3.2 Decision variables

The decision variables for the mixed integer programming problem are:

$$x = volume\ of\ carbon\ fibre$$
$$y = volume\ of\ titanium$$
$$z = number\ of\ pellets\ (1\ cm^3\ each)\ of\ heat\ treatment\ agent$$

3.3 Objective function

The objective function is to reduce the material cost of the racket, which is denoted as Z.

$$Z = 4x + 2y + 8z$$

3.4 Problem constraints

The volume of the racket needs to be at least 32 cm^3.

$$x + y + z \geq 32$$

The weight of the badminton racket needs to be between 80g and 100g.

$$80 \leq 1.55x + 4.5y + 3z \leq 100$$

Minimum limit on the volume of heat treatment agent

$$z \geq 0.15y$$

Maximum limit on the volume of heat treatment agent

$$z \leq 0.2y$$

Minimum of 12 cm^3 of titanium is required, therefore by default also non-negative.

$$y \geq 12$$

$$y \geq 0$$

The volume of carbon fibre should be a non-negative number.

$$x \geq 0$$

The number of 1 cm^3 pellets of HTA should be a non-negative integer.

$$z \geq 0$$

3.5 Mathematical formulation

The mathematical formulation below is shown in the standard format accepted in the MATLAB solver-based approach.

Minimise:

$$Z = 4x + 2y + 8z$$

Subject to:

$$-x - y - z \leq -32$$

$$1.55x + 4.5y + 3z \leq 100$$

$$-1.55x - 4.5y - 3z \leq -80$$

$$0x - 0.2y + z \leq 0$$

$$0x + 0.15y - z \leq 0$$

$$-y \leq -12$$

$$x \geq 0$$

$$y \geq 0$$

$$z \geq 0$$

$$z \text{ is an integer}$$

3.6 MATLAB Implementation

```
clear all
close all
clc

% The code below uses the original constraints. Please adjust the code to
% carry out sensitivity studies. The code has been commented to help
% understand the constraint/function each line signifies.

f = [4 2 8]; % objective function definition x (carbon fibre), y (titanium), z (heat
treatment agent)

% inequality constraints as described in mathematical formulation
A = [1.55 4.5 3;... % c1: 1.55x + 4.5y \leq 100
    -1.55 -4.5 -3;... % c2: 1.55x + 4.5y \geq 80
    -1 -1 -1;...% c3: x + y \geq 32
    0 -0.20 1;...% c4: z ≤ 0.2y
    0 0.15 -1;...% c5: z ≥ 0.15y
    0 -1 0]; % c6: y ≥ 12

b = [100 -80 -32 0 0 -12]; % RHS for above constraints

intcon = 3; % integer condition for HTA

lb = [0 0 0]; % non-negativity constraints (lower bound definition for x - carbon fibre, y -
titanium, z - heat treatment agent)

ub = [Inf Inf Inf]; % upper bound definition for x - carbon fibre, y - titanium, z - heat
treatment agent

[val_s2, fval_s2] = intlinprog(f, intcon, A, b, [], [], lb, ub) % solution optimal point,
optimal solution
```

3.7 Solution

Based on the above-described constraints, the optimally designed racket (lowest cost) uses the below mix of materials:

- 13.38 cm^3 of carbon fibre
- 15.61 cm^3 of titanium
- 3 pellets of HTA

The corresponding optimal minimum cost of the racket is $108.78.

3.8 Sensitivity Analysis

Substituting the optimal solution into the constraint equations, it was found that the solution is bound by the following constraints:

Volume constraint:

$$x + y + z \geq 32$$

Upper limit on the weight of the racket:

$$1.55x + 4.5y + 3z \leq 100$$

In this sensitivity study, the effect of changing the cost per cm^3 of the HTA in the objective function has been analysed to find the point of inflection, where the optimal point shifts from one point to another. The results are summarised in Table 4.

$$Z = 4x + 2y + \mathbf{8}z$$

Table 4: Effect of changing the cost of carbon fibre on the optimal solution

Cost of HTA ($)	**x, y, z (cm^3)**	**HTA as % of volume of titanium**
0 - 8	13.39, 15.61, 3	19.2%
8 - 8.55	13.39, 15.61, 3	19.2%
8.56 - ∞	16.67, 13.33, 2	15%

For cost of the HTA from $0/$cm^3$ to $8.55/$cm^3$ for, the optimal solution is bound by the volume constraint and the upper limit of the weight constraint, and the solution remains at the same optimal point.

When the cost of HTA is increased further to $8.56/$cm^3$, the solution reaches the point of inflection, where it becomes less attractive to use titanium when compared to carbon fibre. This is because the use of the more expensive HTA is driven by the titanium in the racket, and increased cost of titanium makes the racket more expensive beyond this limit.

From $8.56/$cm^3$ and above, the solution is bound by an additional constraint - the 15% minimum limit of HTA required for the volume of titanium.
$8.56/$cm^3$ is the cost the manufacturer needs to bear in mind if the cost of carbon fibre increases from its current price of $8/$cm^3$, so that they can switch to a different feasible point with a lower optimal cost.

4 Non-Linear Programming Problem

4.1 Problem Description

Building on the problem from section 3, the manufacturer has negotiated a deal with the carbon fibre supplier to leverage their economies of scale. The deal gives them a discount on incremental volume of carbon fibre used in each racket. This changes the objective function into a non-linear cost objective function.

The cost-volume relationship fibre is given by:

$$Cost\ of\ carbon\ fibre = 3.5x + \sqrt{x}$$

Where, x is the volume of carbon fibre used per racket in cm^3.

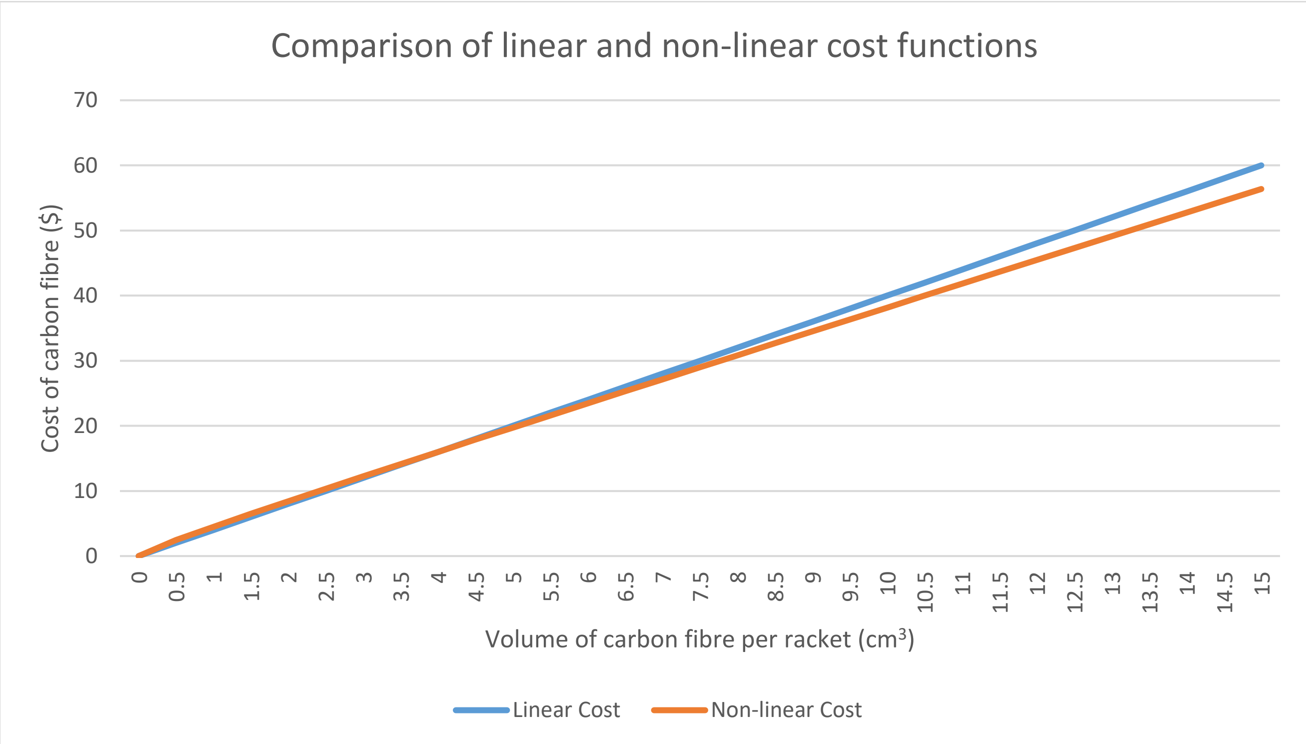

Figure 6: Comparison of linear and non-linear cost functions

Additionally, the company has sourced HTA in powdered form from a new supplier, which allows it to be used as continuous variable in cm^3.

The objective is to manufacture the racket to the minimum cost, while still meeting the weight requirement in order to achieve the balance between power and racket speed.

As per the regulation from the badminton federation, the weight of the badminton racket needs to be between 80g and 100g to achieve the optimal balance between racket speed and power

Although it is not a regulation, design best practice recommends the volume of the racket to be at least 32 cm^3. This ensures good bending stiffness during game play.

Minimum of 12 cm^3 of titanium is required to prevent the racket from being too brittle.

4.2 Decision variables

The decision variables for the non-linear programming problem are:

$$x = volume\ of\ carbon\ fibre$$
$$y = volume\ of\ titanium$$
$$z = volume\ of\ heat\ treatment\ agent\ (HTA)$$

4.3 Objective function

The objective is to minimise the cost of the racket frame, which is denoted by Z.

$$Z = 3.5x + \sqrt{x} + 2y + 8z$$

4.4 Problem constraints

The volume of the racket needs to be at least 32 cm^3.

$$x + y + z \geq 32$$

The weight of the badminton racket needs to be between 80g and 100g.

$$80 \leq 1.55x + 4.5y + 8z \leq 100$$

Minimum limit on the volume of HTA is 15% of the volume of titanium.

$$z \geq 0.15y$$

Maximum limit on the volume of HTA is 20% of the volume of titanium.

$$z \leq 0.2y$$

Minimum of 12 cm^3 of titanium is required, therefore by default also non-negative.

$$y \geq 12$$

$$y \geq 0$$

The volume of carbon fibre should be a non-negative number.

$$x \geq 0$$

The volume of HTA should be a non-negative number (continuous variable).

$$z \geq 0$$

4.5 Mathematical formulation

The mathematical formulation below is shown in the standard format accepted in the MATLAB solver-based approach.

Minimise:

$$Z = 3.5x + \sqrt{x} + 2y + 8z$$

Subject to:

$$-x - y - z \leq -32$$

$$1.55x + 4.5y + 3z \leq 100$$

$$-1.55x - 4.5y - 3z \leq -80$$

$$0x - 0.2y + z \leq 0$$

$$0x + 0.15y - z \leq 0$$

$$-y \leq -12$$

$$x \geq 0$$

$$y \geq 0$$

$$z \geq 0$$

4.6 MATLAB Implementation

```
clear all
close all
clc

% The code below uses the original constraints. Please adjust the code to
% carry out sensitivity studies. The code has been commented to help
% understand the constraint/function each line signifies.

fun = @(x) 3.5*x(1) + x(1)^(1/2) + 2*x(2) + 8*x(3) %objective function definition Z = 3.5x +
√x + 2y + 8z, where x(1) represents x (carbon fibre), x(2) represents y (titanium), x(3)
represents z (heat treatment agent)

% inequality constraints as described in mathematical formulation
A = [1.55 4.5 3;... % c1: 1.55x + 4.5y ≤ 100
    -1.55 -4.5 -3;... % c2: 1.55x + 4.5y ≥80
    -1 -1 -1;... % c3: x + y ≥ 32
    0 -0.20 1;...% c4: z ≤ 0.2y
    0 0.15 -1;...% c5: z ≥ 0.15y
    0 -1 0]; % c6: y ≥ 12

b = [100 -80 -32 0 0 -12]; % RHS for above constraints

lb = [0 0 0]; % non-negativity constraints (lower bound definition for x - carbon fibre, y -
titanium, z - heat treatment agent)
```

```
ub = [Inf Inf Inf]; % upper bound definition for x - carbon fibre, y - titanium, z - heat treatment agent

x0 = [0.5,0,0]; % initial condition

[val_s3, fval_s3] = fmincon(fun, x0, A, b, [], [], lb, ub) % solution optimal point, optimal solution
```

4.7 Solution

The racket with the lowest optimal cost uses 13.7 cm^3 of carbon fibre, 15.91 cm^3 of titanium and 2.39 cm^3 of HTA.

The corresponding optimal cost of the racket is $102.57.

4.8 Sensitivity Analysis

The optimal solution is bound by the following constraints:

Volume constraint:

$$x + y + z \geq 32$$

Upper limit on the weight of the racket:

$$1.55x + 4.5y + 3z \leq 100$$

Lower limit on the volume of HTA:

$$z \geq 0.15y$$

The only constraint that the manufacturer can adjust is the volume constraint, as the weight constraint is a badminton federation regulation and volume of HTA required is bound by the material chemistry.

Although shadow pricing principles **do not** apply to non-linear programming, a study has been conducted to look at the degree of non-linearity. The results in Table 5 highlight that the degree of non-linearity in this objective function line is not highly pronounced. Therefore, the optimal cost follows a trend quite similar to shadow pricing.

The manufacturer can therefore make an assumption for the objective function line to be linear to simplify the sensitivity study, as this makes it a more intuitive problem to work with. This assumption should only be used in initial stages for approximating the problem to make it easier to understand. Final design should still consider the non-linearity of the problem.

Table 5: Change in optimal cost with volume constraint – A study to examine the degree of non-linearity.

Volume constraint (cm^3)	Optimal cost ($)	Δ optimal cost ($)
32	102.57	-
31	98.45	4.12
30	94.32	4.13
29	90.16	4.16

5 Conclusion

This report investigates a material mix problem for a badminton racket in order to optimise the cost to its lowest, while still meeting all the design constraints.

The constraints on the design include an upper and lower weight limit on the racket, which are 100 g and 80 g respectively to achieve balance of power and racket speed. There is also a constraint (more of a design best practice) on the minimum overall volume of the racket of 32 cm^3 to achieve good bending stiffness.

The optimal cost and material mix were found for 3 different versions of the problem – linear programming, mixed-integer programming and non-linear programming.

From the studies conducted, it can be seen that the optimal solution is bound by the upper limit of 100 g on the weight of the racket. Increasing the weight limit would allow a lower optimal cost, which can help boost the profitability of the company. A higher weight limit can be looked into for launching a new range of rackets for intermediate and beginner level players, for whom the badminton regulations will not apply. These rackets can be offered at a slightly lower cost to compensate for the higher weight.

The solution is also bound by the lower limit on the volume of the racket of 32 cm^3. This is not a hard constraint, but a design best practice. Therefore, some flexibility can be used to reduce the volume marginally and launch a new range of rackets for intermediate and beginner level players. This will allow for reduced use of materials in the racket and thereby lower the cost and profitability of the company.