Gauss Elimination Method to solve Simultaneous Linear Equations

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Matrix form of Simultaneous Linear Equations

- A system of m linear equations in n unknowns is of the form

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = k_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = k_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = k_m$$

- The matrix form of the system is $AX = K$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_m \end{bmatrix} K = \begin{bmatrix} k_1 \\ k_2 \\ \ldots \\ k_m \end{bmatrix}$$

Consistency

- $AK = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & k_1 \\ a_{21} & a_{22} & \dots & a_{2n} & k_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & k_m \end{bmatrix}$
- Here A is called the co-efficient matrix, AK is called the augmented matrix.
- If $R(A) \neq R(AK)$, the system is inconsistent, ie. have no solution.
- If $R(A) = R(AK) = n$, the system is consistent and have a unique solution.
- If $R(A) = R(AK) < n$, the system is consistent and have an infinite number of solutions.

Gauss Elimination Method- Working Rule

- Write the augmented matrix AK.
- Reduce it to an upper triangular matrix by elementary row operations.
- Find value of unknowns by back subtitution

Problem

Apply Gauss elimination method to solve the system

$x + 4y - z = -5$

$x + y - 6z = -12$

$3x - y - z = 4$

Solution

$$\begin{bmatrix} 1 & 4 & -1 & -5 \\ 1 & 1 & -6 & -12 \\ 3 & -1 & -1 & 4 \end{bmatrix} R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 4 & -1 & -5 \\ 0 & -3 & -5 & -7 \\ 0 & -13 & 2 & 19 \end{bmatrix} R_3 \to R_3 - \frac{13}{3}R_2$$

$$\begin{bmatrix} 1 & 4 & -1 & -5 \\ 0 & -3 & -5 & -7 \\ 0 & 0 & \frac{71}{3} & \frac{148}{3} \end{bmatrix}$$

By back substitution,

$\frac{71}{3}z = \frac{148}{3}; z = 2.0845$

$3y = 7 - 5z; y = -1.1408$

$x = -5 - 4y + z; x = 1.6479$

Problem

Apply Gauss elimination method to solve the system
$2x + z = 3$
$x - y - z = 1$
$3x - y = 4$

Solution

$$\begin{bmatrix} 2 & 0 & 1 & 3 \\ 1 & -1 & -1 & 1 \\ 3 & -1 & 0 & 4 \end{bmatrix} R_2 \to R_2 - \frac{1}{2}R_1 \text{ and } R_3 \to R_3 - \frac{3}{2}R_1$$

$$\begin{bmatrix} 2 & 0 & 1 & 3 \\ 0 & -1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -1 & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 2 & 0 & 1 & 3 \\ 0 & -1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here $R(A) = R(AK) = 2 <$no. of unknowns. Hence there are infinite number of solutions.

By back substitution,

$2x + z = 3$

$-y - \frac{3}{2}z = -\frac{1}{2}$

$z = t$, an arbitary value; $y = \frac{1-3t}{2}; x = \frac{3-t}{2}$

Problem

Apply Gauss elimination method to solve the system

$4y + 3z = 8$

$2x - z = 2$

$3x + 2y = 5$

Solution

$$\begin{bmatrix} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{bmatrix} R_2 \leftrightarrow R_1 \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{bmatrix} R_3 \to R_3 - \frac{3}{2}R_1$$

$$\begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & 2 & \frac{3}{2} & 2 \end{bmatrix} R_3 \to R_3 - \frac{2}{4}R_2$$

$$\begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Here $R(A) \neq R(AK)$. Hence the system is inconsistent.

AssignmentProblems

1. Apply Gauss elimination method to solve the system
   $$10x - 7y + 3z + 5u = 6$$
   $$-6x + 8y - z - 4u = 5$$
   $$3x + y + 4z + 11u = 2$$
   $$5x - 9y - 2z + 4u = 7$$
2. Apply Gauss elimination method to solve the system
   $$x - 2y + 3z - u = 10$$
   $$2x + 3y - 3z - u = 5$$
   $$3x + 2y - 4z + 3u = 2$$
   $$2x - y + 2z + 3u = 7$$
3. Apply Gauss elimination method to solve the system
   $$x - 2y + 3z = -2$$
   $$-x + y - 2z = 3$$
   $$2x - y + 3z = 1$$

Thank You