Govt. Engineering College, Thrissur

Assignment –

MA202 – FOURIER INTEGRALS

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3. Find the fourier transform of $f(x) = \begin{cases} 4-x^2; & |x| \le 2 \\ 0, & \text{otherwise} \end{cases}$

Solution

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-2} 0\, dx + \int_{-2}^{2} (4-x^2) e^{-i\omega x} dx + \int_{+2}^{\infty} 0\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-2}^{2} 4 e^{-i\omega x} - \int_{-2}^{2} x^2 e^{-i\omega x} dx \right]$$

$\rightarrow 0$ (odd function)

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-2}^{2} 4\cos\omega x\, dx - i \int_{-2}^{2} 4 \sin\omega x\, dx - \int_{-2}^{2} x^2 \cos\omega x + i \int_{-2}^{2} x^2 \sin\omega x\, dx \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[4 \int_{0}^{2} \cos\omega x - \left[\left[x^2 \frac{\sin\omega x}{\omega} \right]_0^2 - \int_0^2 2x \frac{\sin\omega x}{\omega} dx \right] \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[4 \left[\frac{\sin\omega x}{\omega} \right]_0^2 - \left\{ 4 \frac{\sin 2\omega}{\omega} - \left[2x \frac{-\cos\omega x}{\omega^2} \right]_0^2 + \int_0^2 2 \frac{\cos\omega x}{\omega^2} dx \right. \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\cancel{\frac{4\sin 2\omega}{\omega}} - \cancel{\frac{4\sin 2\omega}{\omega}} + \frac{-4\cos 2\omega}{\omega^2} + 2\frac{\sin\omega 2}{\omega^3} \right]$$

$$= \frac{4}{\sqrt{2\pi}} \left[\frac{\sin 2\omega - 2\omega \cos 2\omega}{\omega^3} \right]$$

4. Find the fourier transform of $f(x) = \begin{cases} x, & |x| \le 4 \\ 0, & \text{otherwise} \end{cases}$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-4} 0\, dx + \int_{-4}^{4} x e^{-i\omega x} dx + \int_{4}^{\infty} 0\, dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \underbrace{\int_{-4}^{4} x \cos \omega x\, dx}_{\to 0 \text{ (odd funtion)}} - i \underbrace{\int_{-4}^{4} x \sin \omega x\, dx}_{\to \text{even funtion}} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ -i2 \int_{0}^{4} x \sin \omega x\, dx \right\}$$

$$= \frac{-2i}{\sqrt{2\pi}} \left[\left[x \frac{-\cos \omega x}{\omega} \right]_0^4 + \int_0^4 \frac{+\cos \omega x}{\omega} dx \right]$$

$$= \frac{-2i}{\sqrt{2\pi}} \left[-4 \frac{\cos 4\omega}{\omega} + \frac{\cos 4\omega}{\omega^2} - \frac{1}{\omega^2} \right]$$

$$= \frac{2i}{\omega^2 \sqrt{2\pi}} \left[4\omega \cos 4\omega - \cos 4\omega - 1 \right]$$

$\sqrt{2\pi}$

5. Find the fourier transform of $e^{-x^2/2}$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-i\omega x} dx \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[i\omega x + \frac{x^2}{2} - \frac{\omega^2}{2} + \frac{\omega^2}{2}\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}} + \frac{\omega i}{\sqrt{2}}\right)^2} \cdot e^{-\frac{\omega^2}{2}} dx$$

Let $t = \frac{x}{\sqrt{2}} + \frac{\omega i}{\sqrt{2}}$

$dt = \frac{1}{\sqrt{2}} dx$

$dx = \sqrt{2}\, dt$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}} \int_{-\infty}^{\infty} e^{-\left(x/\sqrt{2} + \frac{\omega i}{\sqrt{2}}\right)^2} dx$$

$$= \frac{1}{\sqrt{\pi}} e^{-\omega^2/2} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \frac{1}{\sqrt{\pi}} e^{-\omega^2/2} \left[\frac{e^{-t^2}}{-2t}\right]_{-\infty}^{\infty}$$

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

$$= \frac{1}{\cancel{\sqrt{\pi}}} e^{-\omega^2/2} \left[\cancel{\sqrt{\pi}}\right]$$

$$= e^{-\omega^2/2}$$

2. Find the fourier transform of $f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & \text{otherwise} \end{cases}$

Hence prove that $f(x) = \frac{4}{\pi} \int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3} \cos\omega x \, d\omega$

Also deduce that $\int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3} \cos\frac{\omega}{2} \, d\omega = \frac{3\pi}{16}$

Solution

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-1} 0 \, dx + \int_{-1}^{1} (1-x^2) e^{-i\omega x} dx + \int_1^\infty 0 \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-1}^{1} (1-x^2) \cos\omega x \, dx - i \underbrace{\int_{-1}^{1} (1-x^2) \sin\omega x \, dx}_{\text{odd here integral} = 0} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ 2 \int_0^1 (1-x^2) \cos\omega x \, dx \right\}$$

$$= \frac{2}{\sqrt{2\pi}} \left[\int_0^1 \cos\omega x \, dx - \int_0^1 x^2 \cos\omega x \, dx \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\left[\frac{\sin\omega x}{\omega}\right]_0^1 - \left[x^2 \frac{\sin\omega x}{\omega}\right]_0^1 + \int_0^1 2x \frac{\sin\omega x}{\omega} dx \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\cancel{\frac{\sin\omega}{\omega}} - \cancel{\frac{\sin\omega}{\omega}} - \left[2x \frac{\cos\omega x}{\omega^2}\right]_0^1 + 2\left[\frac{\sin\omega x}{\omega^3}\right]_0^1 \right]$$

$$= \frac{4}{\sqrt{2\pi}} \left[\frac{\sin\omega - \omega\cos\omega}{\omega^3} \right]$$ Hence proved

When $x=\frac{1}{2}$, $\frac{4}{2\pi}\int_0^\infty\left(\frac{\sin w - w\cos w}{w^2}\right)\cos\left(\frac{w}{2}\right)dw = 1-\left(\frac{1}{2}\right)^2 = \frac{3}{4}$

$$\therefore \int_0^\infty\left(\frac{\sin w - w\cos w}{w^2}\right)\cos\left(\frac{w}{2}\right)dw = \frac{3\pi}{16}$$

1. Find the fourier tranform and integral representation of

$$f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & \text{otherwise} \end{cases}$$

Hence find $\int_0^\infty \frac{1-\cos x}{x^2}dx = \frac{\pi}{2}$

Solution

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1}(1-|x|)e^{-i\omega x}dx + \int_1^\infty 0\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\underbrace{\int_{-1}^{1}(1-|x|)\cos\omega x\,dx}_{\text{even funtion}} + \underbrace{i\int_{-1}^{1}(1-|x|)\sin\omega x\,dx}_{\text{odd function}\ \therefore\ \text{integral is zero}}\right]$$

$$= \frac{1}{\sqrt{2\pi}}2\int_0^1(1-|x|)\cos\omega x\,dx$$

$$= \frac{1}{\sqrt{2\pi}}2\int_0^1(1-x)\cos\omega x\,dx$$

$$= \frac{1}{\sqrt{2\pi}}2\left[\left[\frac{\sin\omega x}{\omega}\right]_0^1 - \int_0^1 x\cos\omega x\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}2\left[\frac{\sin\omega}{\omega} - \left[\frac{x\sin\omega x}{\omega} + \frac{\cos\omega x}{\omega^2}\right]_0^1\right]$$

$$= \frac{1}{\sqrt{2\pi}} 2\left[\frac{\sin w}{w} - \left(\frac{\sin w}{w} + \frac{\cos w}{w^2} - \frac{1}{w^2} \right) \right]$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \left(\frac{1 - \cos w}{w^2} \right)$$

$$\therefore f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwx} F(w)\, dw$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \left(\frac{1 - \cos w}{w^2} \right) \cancel{dw}\, e^{iwx}\, dw$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{1 - \cos w}{w^2} \right) e^{iwx}\, dw$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \left(\frac{1 - \cos w}{w^2} \right) \cos wx\, dw$$

As it is having sine part as integration of odd funtion

When $x = 0$

$$1 = \frac{2}{\pi} \int_{0}^{\infty} \left(\frac{1 - \cos w}{w^2} \right) dw$$

$$\therefore \int_{0}^{\infty} \left(\frac{1 - \cos w}{w^2} \right) dw = \frac{\pi}{2} \Rightarrow \int_{0}^{\infty} \frac{1 - \cos x}{x^2}\, dx = \frac{\pi}{2}$$