

Today group actions (instead of equiv. relations)

- Recall we discussed isomorphisms, homomorphisms

homomorphism $\varphi: G \rightarrow H$

$$\varphi(gh) = \varphi(g)\varphi(h)$$

isomorphism = bijective homomorphism.

monomorphism = injective homomorphism

$\varphi: G \rightarrow H$ monomorphism then
 $\{ G \text{ is in bijection with } \varphi(G) \subset H \}$

$\varphi: G \rightarrow \underbrace{\varphi(G)}_{\text{im } \varphi} \subset H$ and in fact $\varphi(G) \leq H$
 and $\varphi(G)$ is isomorphic to G .

automorphism = isomorphism from a group to itself.

$$\varphi: G \rightarrow G$$

endomorphism = hom from a gp to itself

epimorphism = surj. hom.

Actions

Examples

$GL_n(\mathbb{C})$ acts on \mathbb{C}^n

\dagger

can define $T(v)$

$$T(Sv) = (TS)v \quad T \cdot v$$

$$\text{Ex } S_X = \{\text{bijective maps } \sigma: X \rightarrow X\} \quad (\sigma\tau)(x)$$

\$S_X\$ acts on \$X\$ \$\boxed{\sigma(x)}\$ \$(\sigma\tau)\cdot x\$
 \$\sigma\$ \$x\$ \$\sigma\circ x\$ \$\sigma\circ(\tau\cdot x)\$
 \$\sigma x\$ \$\sigma(\tau(x))\$ by def of function composition

A left
Def An action of a group \$G\$ on a set \$X\$ is a rule

$$\begin{array}{ll}
 G \times X \rightarrow X & \text{such that} \\
 (g, x) \mapsto g(x) & \\
 g \circ x & 1) e \cdot x = x \\
 g x & 2) \sigma \cdot (\tau \cdot x) = (\sigma\tau) \cdot x
 \end{array}$$

$$\left(\text{Alternatively } G \times X \rightarrow X \atop G \rightarrow \text{Fun}(X, X) \right)$$

Examples

$$\begin{array}{l}
 \mathbb{Z} \text{ acting on } \mathbb{R} \\
 (n, x) \mapsto nx
 \end{array}$$

$$\begin{aligned}
 n(x) &= nx \\
 (m+n)(x) &= m+n+x \\
 m(n(x)) &= m+(n(x)) \\
 &= m+n+x
 \end{aligned}$$

$$0(x) = 0+x=x.$$

\$G\$ any group,
\$G\$ acts on \$G\$, by left multiplication.

$$\begin{array}{ccc}
 g \cdot h = gh & e \cdot h = h & \\
 g \cdot (h \cdot k) = \overset{\text{"}}{g} \overset{\text{"}}{h} \overset{\text{"}}{k} = (\overset{\text{"}}{gh}) \cdot \overset{\text{"}}{k} & & \\
 g \cdot \overset{\text{"}}{hk} = \overset{\text{"}}{gh} \overset{\text{"}}{k} & & \\
 & & \overset{\text{"}}{ghk}
 \end{array}$$

G acts on G by conjugation:

$$g \cdot h = ghg^{-1}$$

$$\begin{aligned}
 g \cdot (h \cdot k) &= gh \cdot k = \overset{\text{"}}{gh} (k) (\overset{\text{"}}{gh})^{-1} = \overset{\text{"}}{gh} k \overset{\text{"}}{h}^{-1} \overset{\text{"}}{g}^{-1} \\
 &\quad \text{---} \\
 g \cdot \overset{\text{"}}{(h \cdot k)} \overset{\text{"}}{g}^{-1} &= \overset{\text{"}}{g} (h \overset{\text{"}}{k} \overset{\text{"}}{h}^{-1}) \overset{\text{"}}{g}^{-1}
 \end{aligned}$$

Notice: if $g, h \in G$ $(gh)^{-1} = h^{-1}g^{-1}$

(just check $(h^{-1}g^{-1})(gh) = e$)

ex: (of ex)

$$G = GL_n(\mathbb{R})$$

acts on G

$$T \cdot \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \end{pmatrix} = \begin{pmatrix} T v_1 & | & T v_2 & \cdots & | \end{pmatrix}$$

$$T \mapsto U T U^{-1} \text{ "change of basis action"}$$

Def if G acts on X and $x \in X$, we define

$$Gx = \{gx \mid g \in G\} \quad \text{"orbit of } x \text{ under the action"}$$

$T \in GL_n(\mathbb{C})$ $GL_n(\mathbb{C})$ conj action,
orbit of T

if $T = I_n$ $U^T I_n U^{-1} = I_n$
 arbit is just $\{I_n\}$

In G acts on itself via left mult,
 single orbit = whole group. choose any $x \in G$

$$G \cdot x = \underset{c}{G}$$

$$gx^{-1} \cdot x > g$$

Fundamental principle of group actions

- Neutral principle of group action

 - If G acts on X then X is a disjoint union of orbits.
 - orbits are either equal or non-intersecting.
 - e. orbits give an equivalence relation
 $\text{def } x \sim y \text{ if } g \cdot x = y \text{ some } g.$

Pf: \sim equiv rel

sym. $g \circ g \checkmark$ because $e \circ g = g$

reflex: if $x \sim y$ then by def $g \circ x = y$

$$g^{-1} \circ (g \circ x) = g^{-1} \circ y$$

$$\begin{array}{c} g^{-1} \circ y = x \\ \Downarrow \\ y \sim x \end{array}$$
$$\begin{array}{c} (g^{-1}g) \circ y \\ \Downarrow \\ e \circ x = x \end{array}$$

transitivity:

if $x \sim y$ & $y \sim z$ then by def $g \circ x = y$ some g
 $h \circ y = z$ some h

$$g \circ x = y$$

$$h \circ (g \circ x) = h \circ y$$

$$(hg) \circ x = h \circ y = z$$

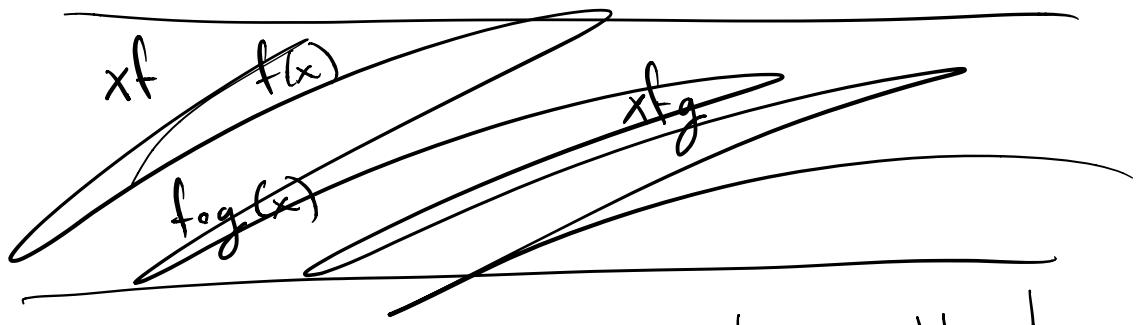
$$x \sim z$$

equiv. classes

$$\begin{aligned} [x] &= \{y \in X \mid x \sim y\} = \{y \in X \mid g \circ x = y \text{ some } g\} \\ &= \{gx \mid g \in G\} = G \circ x = Gx \end{aligned}$$

orbit of x .

$$\text{if } y = gx \Rightarrow [x] = [y]$$
$$Gx = Gy$$



Def If G is a group, X is a set, a right action of G on X is a rule

$$\begin{array}{ccc} X \times G & \longrightarrow & X \\ (x, g) & \longmapsto & x \cdot g \\ x & \longmapsto & xg \end{array} \quad \text{such that} \quad \begin{array}{l} x \cdot e = x \\ x \cdot (gh) = (x \cdot g) \cdot h \end{array}$$

$GL_n(\mathbb{C})$ acts on \mathbb{C}^n (on left)

$$T \cdot v = T v$$

$$\underline{\text{Define:}} \quad v \cdot T = T v$$

$$v \cdot (ST) = ST v$$

$$\begin{array}{c} \text{?} \\ (v \cdot S) \cdot T \\ \text{?} \\ (Sv) \cdot T = TSv \end{array} \quad \cancel{\text{X}}$$

$GL_n(\mathbb{C})$ acts on \mathbb{C}^n on right
in row vectors.

$$v^t \cdot T = v^t T$$

(right) orbits $\times G$

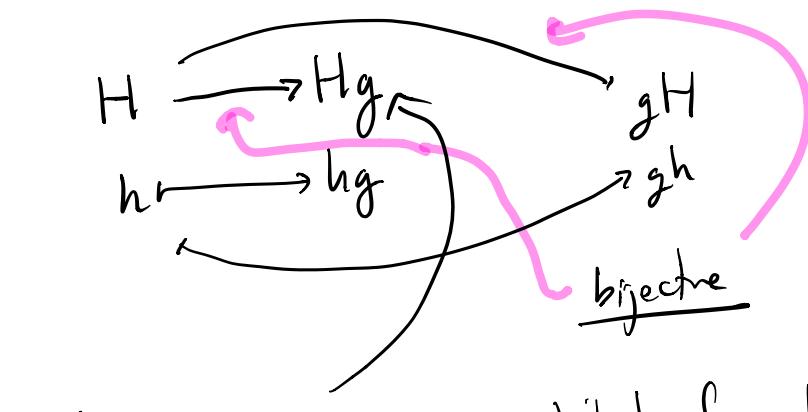
Example: if $H < G$ then H acts on G on right & left

$$H \times G \rightarrow G$$
$$(h, g) \mapsto hg$$

$$G \times H \rightarrow G$$
$$(g, h) \mapsto gh$$

orbits: Hg , $\{hg | h \in H\}$ called right
 gH , $\{gh | h \in H\}$ called left cosets of H in G .

Observation: for these actions all orbits have same size



$xg^{-1} \longleftrightarrow x$ each orbit has form Hg
and each Hg is in bijection
w/ H .

In particular, if G is a finite group and $m = \#$ of cosets of H in G ^{left}

then each coset has $\#H$ elements.

$$\Rightarrow \#G = (\#H)m$$

Theorem (Lagrange's theorem)

$$\text{if } G \text{ is a finite group, } H \subset G \Rightarrow \#H / \#G$$

Def if G acts on X

$$\text{the kernel of the action} = \{g \in G \mid g \cdot x = x \text{ all } x \in X\}$$

ex: if $T = \lambda I_n \in GL_n(\mathbb{C})$ then $T S T^{-1} = S$
 $\Rightarrow T$ is in kernel of conj. action.

if $H \subset G$, H acts on left by m^H .

then kernel is only identity:

$$h \cdot g = g \Rightarrow hg = g \Rightarrow hg^{-1} = gg^{-1} \\ \Rightarrow h = e.$$