Math 6020, Graduate Algebra, Fall 2024, Homework 2

Instructor: Danny Krashen

Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Let S be a set. Define $M(S)$ to be the set of pairs of the form (s, ϵ) where $s \in S$ and $\epsilon \in \{1, -1\}$, and define $W(S)$ to be the set of finite sequences of elements of $M(S)$ (the empty sequence is allowed). We call $W(S)$ the set of group-words in S.

   We use the notation $s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_r^{\epsilon_r}$ to denote the sequence $((s_1, \epsilon_1), (s_2\epsilon_2), \ldots, (s_r, \epsilon_r))$.

   (a) Show that with respect to the operation of concatenation, given by

   $$(s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_r^{\epsilon_r}) \cdot (t_1^{\delta_1} t_2^{\delta_2} \cdots t_k^{\epsilon_k}) = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_r^{\epsilon_r} t_1^{\delta_1} t_2^{\delta_2} \cdots t_k^{\epsilon_k}$$

   $W(S)$ forms a monoid with identity element given by the empty sequence.

   (b) Suppose that $s = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_r^{\epsilon_r}$ and $t = t_1^{\delta_1} t_2^{\delta_2} \cdots t_k^{\epsilon_k}$ are groups words in S. We say that t is a one-step reduction of s if s can be written as $s = t_1^{\delta_1} t_2^{\delta_2} \cdots t_i^{\delta_i} u^\rho u^{-\rho} t_{i+1}^{\delta_{i+1}} \cdots t_k^{\epsilon_k}$ for some $i \in \{0, \ldots k\}$. We say that $s, t \in W(S)$ are elementarily equivalent if either s is a one-step reduction of t or if t is a one-step reduction of s. Let $\sim$ be the equivalence relation generated by elementary equivalence.

   Show that concatenation of equivalence classes gives a well defined operation on $W(S)/\sim$, giving it the structure of a group.

   (c) If S is a set, G is a group, and $f : S \to G$ is a set map, note that we have a natural extension of f to $W(S)$, which we write as $W(f) : W(S) \to G$, given by

   $$W(f)((s_1, \epsilon_1), (s_2\epsilon_2), \ldots, (s_r, \epsilon_r)) = s_1^{\epsilon_1} \cdots s_r^{\epsilon_r}.$$

   (here the right hand side is meant to express the multiplication and exponentiation within the group G). We say that two words $s, t \in W(S)$ are evaluation equivalent, and write $s \equiv t$ if for every group G and every set map $f : S \to G$ we have $W(f)(s) = W(f)(t)$.

   Show that the two equivalence relations $\sim$ and $\equiv$ on $W(S)$ coincide.

   (d) If S is a set, define the set of reduced words in S to be the subset $R(S)$ of $W(S)$ consisting of those words $s_1^{\epsilon_1} \cdots s_r^{\epsilon_r}$ such that whenever $s_i = s_{i+1}$ we have $\epsilon_i = \epsilon_{i+1}$.

   Show that every equivalence class of $W(S)$ with respect to the equivalence relation $\sim$ (or equivalently $\equiv$) contains a unique element of $R(S)$. Conclude that the induced map $R(S) \to W(S)/\sim$ given by the inclusion $R(S) \to W(S)$ is a bijection.

2. Suppose P is a p-group, $H < P$ is a subgroup of index p. Show that $Z(H)$ is normal in P.

3. For a group G and a subgroup H, we define the core of H, denoted $core_G(H)$ is the intersection of the conjugates of H. That is, $core_G(H) = \bigcap_{g \in G} gHg^{-1}$.

   (a) Show that $core_G(H) \triangleleft G$, and that for any $N \triangleleft G$ with $N \subset H$, we have $N \subset core_G(H)$. In other words, $core_G(H)$ is the largest normal subgroup of G contained in H.

   (b) Suppose that we have finite groups $H < G$ with $|G| \nmid [G : H]!$. Show that $core_G(H) \neq \emptyset$.

4. Let G be a group of order $728 = 2^3 \cdot 7 \cdot 13$.

   (a) Show that G has a normal subgroup P of order 13.

   (b) Let $Q \in Syl_7(G)$ be a subgroup of order 7. Show that P must normalize Q.
   That is, show that $P \subset N_G(Q)$.

   (c) Show that G must have subgroups of order $91 = 13 \cdot 7$ and order $104 = 2^3 \cdot 13$.

   (d) Show that either G has a normal subroup of order 91 or G has a normal subgroup of order 104.

   (e) Show that G admits Sylow subgroups (of different orders) S_1, S_2, S_3 such that every element of G can be uniquely written in the form $s_1 s_2 s_3$ for $s_i \in S_i$.