Math 6020, Graduate Algebra, Fall 2024, Homework 4

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Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Recall for $H < G$, we define $core(H) = \bigcap_{g \in G} gHg^{-1}$.

   Suppose that $|G| = mn$ with $n > m!$ and that $H_1, H_2 < G$ with $|H_1| = |H_2| = n$. Show that $H_1 \cap core(H_2) \neq (e)$.

   hint: consider the action of H_1 on the left cosets of H_2

2. (counting problem)

   Let G be a group of order 30. Show that either G has a normal subgroup of order 3 or a normal subgroup of order 5.

3. Let $\mathcal{H} : (e) = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$ be a chief series for a group G. Recall that this means $\mathcal{H}$ is a composition series for G considered as an X-group with $X = G$ acting by conjugation.

   Consider the homomorphism

   $$inn : G \to Aut(G)$$

   given by $inn(g)(h) = ghg^{-1}$.

   (a) Show that $N \lhd G$ with $H_i < N < H_{i+1}$ implies $N = H_i$ or $N = H_{i+1}$.

   (b) Show that we have a well defined homomorphism $inn^i : G \to Aut(H_i/H_{i-1})$ given by $inn^i(g)(hH_{i-1}) = ghg^{-1}H_{i-1}$.

   (c) Let $K_{\mathcal{H}} = \bigcap_i ker(inn^i)$. We use this notation to emphasize that, a priori, the definition of $K_{\mathcal{H}}$ depends on the chosen chief series. Show that this is not the case: that is, if $\mathcal{H}'$ is any other chief series for G, we have $K_{\mathcal{H}} = K_{\mathcal{H}'}$.

   (d) Show that $K_{\mathcal{H}}$ is nilpotent.

4. Suppose G is a group with $|G| = pq^a r^b$ where $p < q < r$ are prime numbers. Let $P \in Syl_p(G), Q \in Syl_q(G), R \in Syl_r(G)$. Show that if $R \subset N_G(Q)$ then $Q \lhd G$.

5. Suppose we are given finite G groups $A \lhd B$ with A Abelian (recall this means that G acts on each of these as automorphisms), and suppose that $|G|$ is relatively prime to $|A|$. Show that every G-fixed element of B/A is the image of a G-fixed element of B.

   hint: this problem uses group cohomology and the long exact cohomology sequence

6. (practice) Let G be a finite group with $p \big| |G|$, and $P \in Syl_p(G)$. Show $n_p = [G : N_G(P)]$ where $n_p = |Syl_p(G)|$.

7. (practice) (trickier counting problem with various cases)

   Let G be a group of order 90 which contains an element of order 9. Show that either G has a normal subgroup of order 3 or a normal subgroup of order 5.

8. (practice) Suppose H, K are groups and $\phi : H \to Aut(K)$ is a homomorphism.

   (a) Show that $Z(K \times H) = Z(K) \times Z(H)$.

   (b) Give an example of H, K, ϕ with $Z(H) \neq (e) \neq Z(K)$ but $Z(K \rtimes_\phi H) = (e)$.

9. (practice) Show that $GL_2(\mathbb{F}_2) \cong S_3$.

10. (practice) Suppose G is a group with subgroups K, H, N with $H \triangleleft K$.

    (a) Show that $(H \cap N) \triangleleft (K \cap N)$.

    (b) Show that if $K \subset N_G(N)$ then $HN \triangleleft KN$.

11. (practice) The icositetrachoron is a 4-dimensional polytope with 24 3-dimensional faces each of which is a 3-dimensional solid octahedra (these each have 8 2-dimensional faces). The symmetry group G of this polytope acts transitively on these 3-dimensional octahedral faces. Furthermore, for each face F and every symmetry σ of F, we can find an element $\widetilde{\sigma} \in G$ such that $\widetilde{\sigma}$ takes F to itself while performing σ on F. Find $|G|$.

12. (practice) (this one takes a bit of work)

    Suppose G is a group with $|G| = pqr$ with $p < q < r$ prime numbers and with

    $$q \equiv 1 \pmod p \text{ and } r \not\equiv 1 \pmod p.$$

    Show that either $G = P \dot{\rtimes} (R \dot{\rtimes} Q)$ or $G = (P \dot{\rtimes} R) \dot{\rtimes} Q$ or $G = R \dot{\rtimes} (Q \dot{\rtimes} P)$.

13. (practice) The octahedron is a 3-dimensional polytope with 8 triangular faces. It has the property that its group G of symmetries acts transitively on these triangular faces, and further, for each symmetry σ of a triangular face F, we can find an element $\widetilde{\sigma} \in G$ such that $\widetilde{\sigma}$ takes F to itself while performing σ on F. Find $|G|$.

14. (practice) Suppose we are given finite G modules $A \triangleleft B$ (recall this means that A and B are Abelian groups and that G acts on each of these as automorphisms), and suppose that $|G|$ is prime to $|A|$. Show that every crossed homomorphism $\phi : G \to B$ has image which lies in A.

    hint: this problem uses group cohomology and the long exact cohomology sequence