

# Math 6020, Graduate Algebra, Fall 2024, Homework 4

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Discussing the problems with other people is encouraged,  
but you must write up your own work independently!

1. Suppose  $H, K$  are groups and  $\phi : H \rightarrow \text{Aut}(K)$  is a homomorphism.

(a) Show that  $Z(K \times H) = Z(K) \times Z(H)$ .

(b) Give an example of  $H, K, \phi$  with  $Z(H) \neq (e) \neq Z(K)$  but  $Z(K \rtimes_{\phi} H) = (e)$ .

2. Show that  $GL_2(\mathbb{F}_2) \cong S_3$ .

3. Suppose  $G$  is a group with subgroups  $K, H, N$  with  $K \triangleleft H$ .

(a) Show that  $(H \cap N) \triangleleft (K \cap N)$ .

(b) Show that if  $K \subset N_G(N)$  then  $HN \triangleleft KN$ .

4. Let  $G$  be a finite group with  $p \mid |G|$ , and  $P \in \text{Syl}_p(G)$ . Show  $n_p = [G : N_G(P)]$  where  $n_p = |\text{Syl}_p(G)|$ .

5. Recall for  $H < G$ , we define  $\text{core}(H) = \bigcap_{g \in G} gHg^{-1}$ .

Suppose that  $|G| = mn$  with  $n > m!$  and that  $H_1, H_2 < G$  with  $|H_1| = |H_2| = n$ . Show that  $H_1 \cap \text{core}(H_2) \neq (e)$ .

*hint: consider the action of  $H_1$  on the left cosets of  $H_2$*

6. (easier counting problem)

Let  $G$  be a group of order 30. Show that either  $G$  has a normal subgroup of order 3 or a normal subgroup of order 5.

7. (trickier counting problem with various cases)

Let  $G$  be a group of order 90 which contains an element of order 9. Show that either  $G$  has a normal subgroup of order 3 or a normal subgroup of order 5.

8. Let  $\mathcal{H} : (e) = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = G$  be a chief series for a group  $G$ . Recall that this means  $\mathcal{H}$  is a composition series for  $G$  considered as an  $X$ -group with  $X = G$  acting by conjugation.

Consider the homomorphism

$$\text{inn} : G \rightarrow \text{Aut}(G)$$

given by  $\text{inn}(g)(h) = ghg^{-1}$ .

- (a) Show that  $N \triangleleft G$  with  $H_i < N < H_{i+1}$  implies  $N = H_i$  or  $N = H_{i+1}$ .
- (b) Show that we have a well defined homomorphism  $\text{inn}^i : G \rightarrow \text{Aut}(H_i/H_{i-1})$  given by  $\text{inn}^i(g)(hH_{i-1}) = ghg^{-1}H_{i-1}$ .
- (c) Let  $K_{\mathcal{H}} = \bigcap_i \ker(\text{inn}^i)$ . We use this notation to emphasize that, a priori, the definition of  $K_{\mathcal{H}}$  depends on the chosen chief series. Show that this is not the case: that is, if  $\mathcal{H}'$  is any other chief series for  $G$ , we have  $K_{\mathcal{H}} = K_{\mathcal{H}'}$ .
- (d) Show that  $K_{\mathcal{H}}$  is nilpotent.

9. Suppose  $G$  is a group with  $|G| = pq^a r^b$  where  $p < q < r$  are prime numbers. Let  $P \in \text{Syl}_p(G), Q \in \text{Syl}_q(G), R \in \text{Syl}_r(G)$ . Show that if  $R \subset N_G(Q)$  then  $Q \triangleleft G$ .

10. (this one takes a bit of work)

Suppose  $G$  is a group with  $|G| = pqr$  with  $p < q < r$  prime numbers and with

$$q \equiv 1 \pmod{p} \text{ and } r \not\equiv 1 \pmod{p}.$$

Show that either  $G = P \rtimes (R \rtimes Q)$  or  $G = (P \rtimes R) \rtimes Q$  or  $G = R \rtimes (Q \rtimes P)$ .

11. The octahedron is a 3-dimensional polytope with 8 triangular faces. It has the property that its group  $G$  of symmetries acts transitively on these triangular faces, and further, for each symmetry  $\sigma$  of a triangular face  $F$ , we can find an element  $\tilde{\sigma} \in G$  such that  $\tilde{\sigma}$  takes  $F$  to itself while performing  $\sigma$  on  $F$ . Find  $|G|$ .
12. The icositetrahedron is a 4-dimensional polytope with 24 3-dimensional faces each of which is a 3-dimensional solid octahedra (these each have 8 2-dimensional faces). The symmetry group  $G$  of this polytope acts transitively on these 3-dimensional octahedral faces. Furthermore, for each face  $F$  and every symmetry  $\sigma$  of  $F$ , we can find an element  $\tilde{\sigma} \in G$  such that  $\tilde{\sigma}$  takes  $F$  to itself while performing  $\sigma$  on  $F$ . Find  $|G|$ .