

Math 6020, Graduate Algebra, Fall 2024, Homework 4

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Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Recall for $H < G$, we define $\text{core}(H) = \bigcap_{g \in G} gHg^{-1}$.

Suppose that $|G| = mn$ with $n > m!$ and that $H_1, H_2 < G$ with $|H_1| = |H_2| = n$. Show that $H_1 \cap \text{core}(H_2) \neq (e)$.

hint: consider the action of H_1 on the left cosets of H_2

2. (counting problem)

Let G be a group of order 30. Show that either G has a normal subgroup of order 3 or a normal subgroup of order 5.

3. Let $\mathcal{H} : (e) = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = G$ be a chief series for a group G . Recall that this means \mathcal{H} is a composition series for G considered as an X -group with $X = G$ acting by conjugation.

Consider the homomorphism

$$\text{inn} : G \rightarrow \text{Aut}(G)$$

given by $\text{inn}(g)(h) = ghg^{-1}$.

- (a) Show that $N \triangleleft G$ with $H_i < N < H_{i+1}$ implies $N = H_i$ or $N = H_{i+1}$.

- (b) Show that we have a well defined homomorphism $\text{inn}^i : G \rightarrow \text{Aut}(H_i/H_{i-1})$ given by $\text{inn}^i(g)(hH_{i-1}) = ghg^{-1}H_{i-1}$.

- (c) Let $K_{\mathcal{H}} = \bigcap_i \ker(\text{inn}^i)$. We use this notation to emphasize that, a priori, the definition of $K_{\mathcal{H}}$ depends on the chosen chief series. Show that this is not the case: that is, if \mathcal{H}' is any other chief series for G , we have $K_{\mathcal{H}} = K_{\mathcal{H}'}$.

- (d) Show that $K_{\mathcal{H}}$ is nilpotent.

4. Suppose G is a group with $|G| = pq^a r^b$ where $p < q < r$ are prime numbers. Let $P \in \text{Syl}_p(G), Q \in \text{Syl}_q(G), R \in \text{Syl}_r(G)$. Show that if $R \subset N_G(Q)$ then $Q \triangleleft G$.

5. Suppose we are given finite G groups $A \triangleleft B$ (recall this means that G acts on each of these as automorphisms), and suppose that $|G|$ is relatively prime to $|A|$. Show that every G -fixed element of B/A is the image of a G -fixed element of B .

hint: this problem uses group cohomology and the long exact cohomology sequence

6. (practice) Let G be a finite group with $p \mid |G|$, and $P \in \text{Syl}_p(G)$. Show $n_p = [G : N_G(P)]$ where $n_p = |\text{Syl}_p(G)|$.
7. (practice) (trickier counting problem with various cases)
Let G be a group of order 90 which contains an element of order 9. Show that either G has a normal subgroup of order 3 or a normal subgroup of order 5.
8. (practice) Suppose H, K are groups and $\phi : H \rightarrow \text{Aut}(K)$ is a homomorphism.
 - (a) Show that $Z(K \times H) = Z(K) \times Z(H)$.
 - (b) Give an example of H, K, ϕ with $Z(H) \neq (e) \neq Z(K)$ but $Z(K \rtimes_\phi H) = (e)$.
9. (practice) Show that $GL_2(\mathbb{F}_2) \cong S_3$.
10. (practice) Suppose G is a group with subgroups K, H, N with $H \triangleleft K$.
 - (a) Show that $(H \cap N) \triangleleft (K \cap N)$.
 - (b) Show that if $K \subset N_G(N)$ then $HN \triangleleft KN$.
11. (practice) The icositetrahedron is a 4-dimensional polytope with 24 3-dimensional faces each of which is a 3-dimensional solid octahedra (these each have 8 2-dimensional faces). The symmetry group G of this polytope acts transitively on these 3-dimensional octahedral faces. Furthermore, for each face F and every symmetry σ of F , we can find an element $\tilde{\sigma} \in G$ such that $\tilde{\sigma}$ takes F to itself while performing σ on F . Find $|G|$.
12. (practice) (this one takes a bit of work)
Suppose G is a group with $|G| = pqr$ with $p < q < r$ prime numbers and with

$$q \equiv 1 \pmod{p} \text{ and } r \not\equiv 1 \pmod{p}.$$
 Show that either $G = P \rtimes (R \rtimes Q)$ or $G = (P \rtimes R) \rtimes Q$ or $G = R \rtimes (Q \rtimes P)$.
13. (practice) The octahedron is a 3-dimensional polytope with 8 triangular faces. It has the property that its group G of symmetries acts transitively on these triangular faces, and further, for each symmetry σ of a triangular face F , we can find an element $\tilde{\sigma} \in G$ such that $\tilde{\sigma}$ takes F to itself while performing σ on F . Find $|G|$.
14. (practice) Suppose we are given finite G modules $A \triangleleft B$ (recall this means that A and B are Abelian groups and that G acts on each of these as automorphisms), and suppose that $|G|$ is prime to $|A|$. Show that every crossed homomorphism $\phi : G \rightarrow B$ has image which lies in A .
hint: this problem uses group cohomology and the long exact cohomology sequence