Still to go: • Ring structure theory
• Categories of modules / Morita theory
• Homological Algebra bits (ext, tor)

Rings are associative, unital, modules are unital

Last time:

Def R is $\overset{\text{right}}{\vee}$ semisimple if R_R is completely reducible.

(Note: submodules of $R_R \longleftrightarrow$ right ideals of R)

Showed: R right semisimple $\Rightarrow R \cong \overset{m}{\underset{i=1}{\times}} M_{n_i}(D_i)$

D_i is a division ring.

Aside: Module theory over division rings

Punchline: Lots of vague thing carries over, same proofs.

if M is a left D-module (D a division ring)

then $M \cong D^N$ some N (i.e. any left D-vector space has a basis)

$\overset{\shortparallel}{\bigoplus_{i=1}^{N} D}$

and N is uniquely defined (maybe infinite)

in particular $\exists!$ simple left D-module (namely D)

$\mathrm{End}_{\text{left } D}(D^n) = M_n(D^{op})$ via HW.

$\mathrm{End}_{\text{right } D}(D) = \mathrm{Hom}_{r.D}(D,D) = D$ via left mult.

$\mathrm{End}_{\text{left}}(D) = D$ as sets via right mult.

$= D^{op}$ as alg.

R right semisimple $\Rightarrow R \cong \times M_{n_i}(D_i)$

then as an R-module, R is a product of D_i-modules

D_i R-submodules are also D_i submods.

$\Rightarrow$ each factor finite length R-modules $\Rightarrow$

R is right & left Art & Noeth.

$D_i \hookrightarrow M_{n_i}(D_i)$ via diagonal.

R semisimple & commutative $\Rightarrow R \cong \times F_i$ fields
right

<u>Notation:</u>

If $M \in R\text{-Mod}$ $X \subset M$, can define $l.\mathrm{ann}_R(X)$
$= \{r \in R \mid rx = 0 \text{ all } x \in X\}$

Similarly right annihilators $r.\mathrm{ann}_R(X)$ $X \subset N \in \text{Mod-}R$.

$l.\mathrm{ann}_R(X) \trianglelefteq_l R \qquad r.\mathrm{ann}_R(X) \trianglelefteq_r R$

Note: $l\mathrm{ann}_R(M)$ $\quad M \in R\text{-mod}$

$$l\text{-}\mathrm{ann}_R(M) \trianglelefteq R$$

$$\underset{r}{\ni} \quad rsx = r(\underset{M}{sx}) = 0$$

in particular, if $I \trianglelefteq_l R$ $\quad l.\mathrm{ann}_R(I) \trianglelefteq R$

Q: What information can we learn from simple modules?

Remark: $M \in R\text{-Mod}$ $\;M$ simple then for $m \in M \setminus \{0\}$

$$R \twoheadrightarrow M \qquad \text{image } Rm < M \implies Rm = M$$
$$r \mapsto rm$$

$$M \cong R/\mathrm{ann}_R(m)$$

simple module, $\forall m \in M$ $\;\mathrm{ann}_R(m) \trianglelefteq_l R$ max'l.

and $R/\mathrm{ann}_R(m) \cong M$

Def $\;J^l(R) = \{r \in R \mid r \in \mathrm{ann}_R(m) \text{ all } m \in M, M \text{ simple}\}$

$$= \bigcap_{\substack{m \in M \\ M \text{ simple}}} \mathrm{ann}_R(m) \trianglelefteq_l R$$

$= \bigcap_{M \text{ simple}} \operatorname{ann}_R(M) \triangleleft R.$

$I \triangleleft_\ell R$ max'l $\Rightarrow R/I$ simple

$= \bigcap_{\substack{I \triangleleft_\ell R \\ \text{max'l}}} I \quad \supset \bigcap_{\substack{m \in M \\ M \text{ simple}}} \operatorname{ann}(m) \quad \subset$

Similarly can define $J^r(R)$

Correspondence thm ideal $I \triangleleft R$

$$\left\{ R/I\text{-modules} \right\} \xleftrightarrow{\text{"identification"}} \left\{ \begin{matrix} R\text{-modules } M \text{ s.t.} \\ IM = 0 \end{matrix} \right\}$$

$\varphi : R \to S \rightsquigarrow$ S-Mods $\longrightarrow$ R-Modules

$M \rightsquigarrow {}_\varphi M = M$ as a set

$r \cdot m = \varphi(r) m$

R-module M

$R \longrightarrow \operatorname{End}_{Ab}(M)$

R/I-modules N

$$\begin{matrix} & R/I \longrightarrow \operatorname{End}_{Ab}(M) \\ R \nearrow & \nearrow \end{matrix}$$

Rem $J^\ell(R/J^\ell(R)) = 0$

if $\bar{r} \in J(R/J)$ then consider $r \in R$ lift.

Claim: $r \in J(R)$. consider M simple module

note by def. $JM = 0$ so M is an R/J-module.

and r acts on M via $\bar{r}$

M simple as an R/J module $\Rightarrow \bar{r}M = 0$ b
$\bar{r} \in J(R/J)$

$\Rightarrow \bar{r}M = rM = 0 \Rightarrow r \in J(R)$

Def $r \in R$ is right quasiregular if $1-r$ has a right inverse
" left — " — left "

$r \in R$ is quasi regular if $1-r$ is invertible.

Lem $r \in J^\ell(R) \Rightarrow r$ is quasiregular.

Pf: if $R(1-r) = R$ then $(1-r)$ is l. quasi reg

or $R(1-r) \subseteq M \trianglelefteq^{*}_{\ell} R$ $\quad 1-r \in M \quad r \in J^\ell(R) \subseteq M$ (all max'l left)

max'l proper ideal

$\Rightarrow 1 \in M + r = M$ $\;$ $\nRightarrow$.

$\Rightarrow 1-r$ is left q. regular. $s(1-r) = 1$ some s

let $y = 1-s$ $\quad s = 1-y$ & $(1-y)(1-r) = 1$

$$1 - y - r + yr = 1$$

$$y = (y-1)r \in J^l(R)$$

$\Rightarrow y \in J^l(R) \Rightarrow 1-y$ has a left inverse

but $1-r$ is also its right inverse

$\Rightarrow 1-y$ is invertible $\Rightarrow 1-r$ is its inverse

$\Rightarrow 1-y$ is the inverse of $1-r$.

$\Rightarrow r$ is r. quasiregular. $\square$.

<u>Lem</u>: If $I \trianglelefteq_r R$ & every element of I is r. quasiregular then $I \subset J^r(R)$

if $K \trianglelefteq_l R$ & every element is l. quasireg $\Rightarrow K \subset J^l(R)$.

$\Big(\Rightarrow$ $J^l(R)$ consists of q. reg. elements & so does $J^r(R)$

$$J^l(R) \underset{\text{r. q. reg}}{\subset} J^r(R) \underset{\text{l. q. reg}}{\subset} J^l(R) \Big)$$

<u>Nakayama's Lemma</u>:

If $M \in R\text{-mod}$ and M is finitely generated.

$J(R)M = M \Rightarrow M = 0$.

Pf: let $\{m_1, \ldots, m_n\}$ a min'l generating set for M.

$$M = J(R)M = J(R) \sum_{i=1}^{n} Rm_i$$

$$= \sum_{i=1}^{n} J(R) \cdot Rm_i = \sum J(R)m_i$$

$$m_1 = \sum_{i=1}^{n} x_i m_i \quad x_i \in J(R)$$

$$(1-x_1)m_1 = \sum_{i=2}^{n} x_i m_i \quad \text{but } x_1 \in J(R) \text{ is quasireg}$$

$$\Rightarrow 1-x_1 \text{ invertible } y(1-x_1)=1$$

$$\Rightarrow m_1 = \sum_{i=2}^{n} \underbrace{(yx_i)}_{\in R} m_i \Rightarrow m_1 \in \langle m_2, \ldots, m_n \rangle$$

$\Rightarrow$ generating set not min'l. $\square$

Def R is prime if $I, J \trianglelefteq R$ w/ $IJ = 0 \Rightarrow I = 0$ or $J = 0$.

Def R is a domain if $a, b \in R$, $ab = 0 \Rightarrow a = 0$ or $b = 0$

Lem For R comm, domain $\Leftrightarrow$ prime.

Pf: if domain $\Rightarrow$ prime $\quad IJ = 0 \quad I \neq 0$ then $x \in I \setminus \{0\}$

$\Rightarrow \forall y \in J \; xy = 0 \Rightarrow y = 0$

$\Rightarrow J = 0$.

If R prime, $ab = 0 \Rightarrow (Ra)(Rb) = 0 \Rightarrow Ra = 0$ or $Rb = 0$

$\Rightarrow a=0$ or $b=0$.

Def R is semiprime if $I \triangleleft R$, $I^n=0 \Rightarrow I=0$

Def R reduced if $a\in R$, $a^n=0 \Rightarrow a=0$.

Lem R comm., R reduced $\Leftrightarrow R$ semiprime.

Def $a\in R$ is nilpotent if $a^n=0$ some n.

Def $I\subset R$ is nil if $\forall x\in I$, x is nilpotent

Def $I\subset R$ is nilpotent if $I^n=0$ some n.

I is an additive subgp. of R

Rem: I Nilpotent $\Rightarrow I$ Nil converse generally not true.

Lem: If R is left Artinian $\Rightarrow J(R)$ nilpotent.

Pf: $J\supset J^2\supset\cdots\supset J^n=J^{n+1}=\cdots$

$I=J^n$ then $JI=I$

$\underset{\text{Nak}}{\Rightarrow} I=0$ $\square$.

Cor: R l. Artinian $I\subset J(R) \Leftrightarrow I$ nilp $\Leftrightarrow I$ nil.

Pf: $x^n=0 \Rightarrow x$ q. regular since $(1-x)(1+x+x^2+\cdots x^{n-1})$

$=1-x^n=1$

$\Rightarrow x\in J(R)$

I nil $\Rightarrow I \subset J(R)$

I nilp $\Rightarrow I$ nil $\Rightarrow I \subset J(R) \Rightarrow I$ nilp. $\square$

So: if R is left Artinian then

R semiprime $\Rightarrow J(R) = 0$

if $J(R) = 0 \Rightarrow \underbrace{(I^n = 0 \Rightarrow I \subset J(R) \Rightarrow I = 0)}_{\text{semiprime}}$

__Def__ R is semiprimitive if $J(R) = 0$.

__Def__ R is left Wedderburn if R is left Artinian & semiprimitive.

$\boxed{\text{Thm: } R \text{ left Wedd} \Leftrightarrow R \text{ semisimple} \Leftrightarrow R \text{ right Wedd.}}$

__Thm (Hopkins)__ R Artinian $\Rightarrow R$ Noeth.

__Pf:__ R/J is Wederburn. $\Rightarrow$ ss. finite length $\Rightarrow R/J$ Noeth.

$J/J^2 \quad J^2/J^3 \quad \dots$ each Artinian R/J-modules

R/J semisimple $\Rightarrow$ f. length modules

$N \subset M \qquad M$ Noeth $\Leftrightarrow M/N$ & N noeth

R Noeth $\Leftrightarrow J$ & R/J Noeth.

J Noet $\Leftrightarrow J/J^2$ & J^2 Noet

⋮

J^i/J^{i+1}, $\underset{0}{\underline{J^{i+1}}}$ Noeth.

$\Rightarrow R$ finite length as an R-mod $\Rightarrow$ Art & Noeth. $\square$