Math 6030, Graduate Algebra, Spring 2025, Homework 5

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Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Let E/F be a Galois extension with group G. Recall that an (E,G)-semilinear vector space is an E-vector space V together with an action of G such that

   i. $\sigma(x+y) = \sigma(x) + \sigma(y)$ for $x, y \in V$, $\sigma \in G$, and

   ii. $\sigma(\lambda x) = \sigma(\lambda)\sigma(x)$ for $x \in V$, $\lambda \in E$, $\sigma \in G$.

   If V_1, V_2 are (E,G)-semilinear vector spaces, a homomorphism of semilinear vector spaces is a homomorphism $\phi : V_1 \to V_2$ of Abelian groups such that $\phi(\sigma(v)) = \sigma(\phi(v))$.

   Recall also that the algebra $(E,G,1)$ is the associative F-algebra, which can be written as an Abelian group as $(E,G,1) = \bigoplus_{\sigma \in G} Eu_\sigma$, and with the multiplication $(xu_\sigma)(yu_\tau) = x\sigma(y)u_{\sigma\tau}$.

   Show that the category of (E,G)-semilinear vector spaces is isomorphic to the category of $(E,G,1)$-modules.

2. (a.k.a Dedekind's Lemma) Let E/F be a Galois extension of fields with group G. As E is an (E,G)-semilinear vector space, we have a canonical homomorphism of F-algebras $(E,G,1) \to End_F(E)$. Show that this map is an isomorphism.

   hint: show that the map is injective by considering an element $\alpha = \sum a_\sigma u_\sigma$ in the kernel with a minimal number of nonzero a_σ's, and is $a_\tau \neq 0$, consider $\tau(b)\alpha - \alpha b$ for $b \in E$.

3. Let E/F be a Galois extension with group G. The Morita theorems show that the functor $W \mapsto E \otimes_F W$ from F-vector spaces to $End_F(E)$-modules is an equivalence of categories. Using Problem 2 to identify $End_F(E)$ with $(E,G,1)$ and the isomorphism of categories of Problem 1, we obtain an equivalence of categories

$$E \otimes_F \_ : [F - \text{vector spaces}] \to [(E,G) - \text{semilinear vector spaces}]$$
$$W \mapsto E \otimes_F W$$

   Let Fix be the functor in the other direction taking a semilinear vector space V to the F-vector space $Fix(V) = V^G = \{v \in V \mid \sigma(v) = v \text{ for all } \sigma \in G\}$.

   (a) Consider the composition of functors $Fix \circ (E \otimes_F \_)$ from F-vector spaces to itself. Show that this is naturally isomorphic to the identity functor. *hint: consider the natural map $W \to Fix(E \otimes_F W)$.*

   (b) Consider the composition of functors $(E \otimes_F \_) \circ Fix$ from (E,G)-semilinear vector spaces to itself. Show that this is naturally isomorphic to the identity functor. *hint: consider the natural map*

   $(E \otimes_F V^G) \to V$ and use the fact that $E \otimes_F \_$ is essentially surjective.

4. Let E/F be a G-Galois extension. Recall that an (E, G)-semilinear algebra is an E-semilinear vector space A with an E-algebra structure satisfying $\sigma(ab) = \sigma(a)\sigma(b)$ for $a, b \in A$, $\sigma \in G$.
   (a) If A is an (E, G)-semilinear algebra, show that A^G is an F-algebra, and if B is an F-algebra, show that $E \otimes_F B$ is an (E, G)-semilinear algebra.

   (b) If B is an F-algebra, show that the canonical map $B \to Fix(E \otimes_F B)$ is an algebra isomorphism.

   (c) If A is an (E, G)-semilinear algebra, show that the canonical map $E \otimes_F A^G \to A$ is an isomorphism of (E, G)-semilinear algebras.

   (d) Show that the functor $B \mapsto E \otimes_F B$ from F-algebras to (E, G)-semilinear algebras is an equivalence of categories.

5. Let $\sigma : \mathbb{C} \to \mathbb{C}$ denote complex conjugation.

   Consider the matrix algebra $M_2(\mathbb{C})$ with action of $\langle \sigma \mid \sigma^2 \rangle = C_2$ given by

$$\sigma\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \sigma(d) & -\sigma(c) \\ -\sigma(b) & \sigma(a) \end{bmatrix}.$$

   Show that this defines a $(\mathbb{C}, C_2)$-semilinear algebra structure on $M_2(\mathbb{C})$.

   hint: for optimal enjoyment, consider the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.