

Math 6030, Graduate Algebra, Spring 2025, Homework 4

Instructor: Danny Krashen

Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Let R be a commutative ring and $f_1, \dots, f_m \in R[x_1, \dots, x_n]$. Show that the ring

$$T = R[x_1, \dots, x_n]/(f_1, \dots, f_m)$$

has the following universal property: if S is any R -algebra with elements $a_1, \dots, a_n \in S$ such that $f_i(a_1, \dots, a_n) = 0$ for all i , then there is a unique homomorphism $\phi : T \rightarrow S$ with $\phi(x_i) = a_i$. Show also that this universal property characterizes T up to isomorphism.

2. Recall that the tensor product $A \otimes_R B$ for commutative R -algebras A and B is the coproduct (sum) in the category of R -algebras. Suppose $\phi : R \rightarrow S$ is a homomorphism (thereby giving S the structure of an R -algebra).

Let $f_1, \dots, f_m \in R[x_1, \dots, x_n]$. Show that there is an isomorphism

$$S \otimes_R R[x_1, \dots, x_n]/(f_1, \dots, f_m) \cong S[x_1, \dots, x_n]/(\phi(f_1), \dots, \phi(f_m)).$$

3. Show that $\mathbb{Q}(\sqrt{3}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{3}) \cong \mathbb{Q}(\sqrt{3}) \times \mathbb{Q}(\sqrt{3})$.

4. Suppose that $E = F[x]/f$, $K = F[x]/g$ are field extensions with $[E : F] = 5$ and $[K : F] = 3$. Let L be a splitting field of fg , so that L contains both E and K .

Show that the compositum $\langle E, K \rangle$ in L has degree 15 and that the extensions E and K are linearly disjoint.

5. Suppose E/F is a G -Galois extension with $|G| = 15$. Show $E \cong K \otimes_F L$ for subextensions K, L with $[K : F] = 3$ and $[L : F] = 5$.

6. Suppose E/F is a G -Galois extension. If M, L are subextensions of E , with $M = E^H$ and $L = E^K$, how can you describe the subgroup $N < G$ such that $\langle M, L \rangle = E^N$?

7. (optional) Let F be a field of characteristic p and let $a \in F \setminus F^p$. Show that $F[x]/(x^p - a)$ is a purely inseparable extension of F .

8. (optional) Show that every purely transcendental extension $F(t)/F$ with F a field of characteristic p contains subextensions L/F with $F(t)/L$ purely inseparable.

9. (optional) Let f be a degree 4 monic polynomial over a field F . Show that there exists a degree 6 polynomial \tilde{f} such that for every extension field L/F , f factors as $f = gh$ with g and h quadratic over L if and only if \tilde{f} has a root over L .

10. (optional) Show that $\mathbb{Q}(\sqrt[3]{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt[3]{2}) \cong \mathbb{Q}(\sqrt[3]{2}) \times K$ for K/\mathbb{Q} a quadratic extension.