

Math 6030, Graduate Algebra, Spring 2025, Homework 6  
(and study guide)

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Discussing the problems with other people is encouraged,  
but you must write up your own work independently!

1. Let  $R$  be a commutative ring. We say that an element  $e \in R$  is an idempotent if  $e^2 = e$ . We say that  $e$  is a primitive idempotent if for every other idempotent  $f$ , we have either  $ef = e$  or  $ef = 0$ .

(a) Show that an idempotent  $e$  is primitive if and only if it cannot be written as  $e = e' + e''$  where  $e'$  and  $e''$  are nonzero idempotents with  $e'e'' = 0$ .

(b) Let  $F$  be a field and let  $E = F \times \cdots \times F$  be a product of  $n$  copies of  $F$ . Show that  $\text{Aut}_F(E) = S_n$

*hint: note that automorphisms send primitive idempotents to primitive idempotents.*

2. Recall that for a finite group  $G$ , a  $G$ -Galois algebra over  $F$  is a separable  $F$ -algebra  $E$  with  $G$  action such that  $|G| = \dim_F E$  and  $E^G = F$ .

Suppose that  $E = E_1 \times \cdots \times E_r$  is a  $G$ -Galois algebra over  $F$  with each  $E_i$  a separable extension of  $F$ .

Show that if  $H = \{g \in G \mid g(E_1) \subseteq E_1\}$ , then  $|H| = |G|/r$  and  $E_1/F$  is a Galois extension with group  $H$ .

3. Let  $E/F$  be a finite field extension (not necessarily Galois) and let  $G \subseteq \text{Gal}(E/F)$ .

Show that  $|G| \mid [E : F]$ .

4. Let  $F$  be a field of characteristic  $p$ . If  $E/F$  is a purely inseparable extension (recall, this means that for every  $a \in E$ , we have  $a^{p^n} \in F$  for some  $n$ ), is  $E/F$  necessarily normal? Either prove it must be normal, or give a counterexample.

5. Suppose  $E/F$  and  $K/F$  are Galois extensions with Galois group  $C_7$ . Suppose  $L/F$  is a field extension of degree 5.

(a) Show that  $L \otimes_F E$  and  $L \otimes_F K$  are fields,

(b) Show that if  $L \otimes_F E \cong L \otimes_F K$  as field extensions of  $L$ , then  $E \cong K$  as field extensions of  $F$ .

*Hint: you can use the fact that  $G$ -Galois algebras which are split by a  $\Gamma$ -Galois extension  $\tilde{F}/F$  correspond to elements of  $H^1(\Gamma, G) = \text{Hom}(\Gamma, G)/\sim$  where  $\sim$  corresponds to conjugation in  $G$ . Choose your field  $\tilde{F}$  wisely...*

6. (optional/review) Suppose  $E/F$  is a purely inseparable extension and let  $f \in E[x]$  be a monic irreducible polynomial.

(a) Show that  $g = f^{p^n}$  is a monic irreducible polynomial in  $F[x]$  for some  $n$ .

(b) Let  $L/E$  be a splitting field of  $f$ . Show that  $g$  factors as a product of linear polynomials in  $L[x]$ .

(c) What is the relationship between the roots of  $f$  and the roots of  $g = f^{p^n}$  in  $L$ ?

7. (optional/review) Consider  $\alpha = \sqrt{2 + \sqrt{5}} \in \mathbb{C}$ .

(a) What is the minimal polynomial of  $\alpha$  over  $\mathbb{Q}$ ?

(b) What is the Galois group of the splitting field of this minimal polynomial?

8. (optional/review) Suppose  $E/F$  is a Galois extension with group  $G$ . Let  $L, K$  be subextensions with  $[L : F]$  and  $[K : F]$  relatively prime and suppose that  $E$  is the smallest subfield containing both  $L$  and  $K$ . Suppose that  $L/F$  is Galois. Show that  $G = \text{Gal}(E/L) \rtimes \text{Gal}(E/K)$ .

9. (optional/review) Suppose  $E/F$  is a separable and normal (but not necessarily finite) algebraic extension. Show that if  $F \subseteq L \subseteq E$  with  $L/F$  Galois, then

(a)  $\sigma(L) \subseteq L$  for  $\sigma \in \text{Gal}(E/F)$ ,

(b) the induced map  $\text{Gal}(E/F) \rightarrow \text{Gal}(L/F)$  is surjective.

(c) we have a bijection

$$\text{Gal}(E/F) \longleftrightarrow \left\{ (\sigma_L)_L \in \prod_{\substack{F \subseteq L \subseteq E \\ [L:F] < \infty \\ L/F \text{ Galois}}} \left| \sigma_L|_K = \sigma_K \text{ when } F \subseteq K \subseteq L \subseteq E \right. \right\}$$

10. (optional/review) Suppose  $E/F$  is a Galois extension with group  $G$ . Let  $K$  be an intermediate subfield with  $[K : F] = n$ . Show that there exists an intermediate field  $L$  containing  $K$  with  $L/F$  Galois and with  $[L : F] | n!$   
*hint: depending on how you do this, it may be useful to remember that if  $a + b = n$ , then  $n!$  is divisible by  $a!b!$  (because of binomial coefficients)*
11. (optional/review) Let  $G$  be a finite group. Show that there exists some field extension  $E/F$  which is  $G$ -Galois.
12. (optional/review) Let  $G$  be a finite group and  $H$  a subgroup of  $G$ . Show that there exists a normal subgroup  $N \subset H$  with  $[G : N] \mid [G : H]!$ .
13. (optional/review) Prove the following statement or give a counterexample:  
 If  $F \subseteq K \subseteq E$  are fields with  $E/K$  and  $K/F$  Galois then  $E/F$  is Galois.
14. (optional/review) Describe the splitting field of  $x^4 - 5$  over  $\mathbb{Q}$  and compute its Galois group.
15. (optional/review) Give an example of a UFD which is not a PID (and justify your answer).
16. (optional/review) Consider the  $F$ -algebra  $A = F[x]/x^2$ .
  - (a) Compute the automorphism group of  $A$  as an  $F$ -algebra.
  - (b) Suppose  $E/F$  is a Galois extension. Show that if  $B$  is any  $F$ -algebra such that  $E \otimes_F A \cong E \otimes_F B$  as  $E$ -algebras, then  $A \cong B$  as  $F$ -algebras.