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[dkrashen.org/algebra1](http://dkrashen.org/algebra1)

HW due each Monday

Assumed background: groups, rings, fields,  
some modules, various linear alg stuff.

W 3:20-4:40 (Review) HILL 425

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Algebra - sets & operations

Binary operation - typical axioms

- associativity
- commutativity
- existence of units
- cancellation / inverses.

Ex: Magma = Set w/ binary operation  $(M, \cdot)$

→ Monoid = Set w/ bin. op, associative, unit

Group = Set w/ bin op, unit, inverses.

→ Group = loop w/ associativity

Ab. group = group w/ commutativity.

$(2\mathbb{Z}, \cdot)$

Often, multiple operations

(mgs, etc)

Def An  $n$ -ary operation on a set  $S$  is

a map  $S^n \rightarrow S$

$\underbrace{S \times \dots \times S}_{n \text{ times}}$

0-ary

$\{\emptyset\} \rightarrow S$

Monoid

Set  $M$ ,

$\text{id}_M: M \rightarrow M$   
 $a \mapsto a$

0-ary operation:  $1: \{\emptyset\} \rightarrow M$

2-ary op... ;  $m: M \times M \rightarrow M$

s.t.  $M \times M \times M \rightarrow M$

$m(\text{id}_M \times m) = m(m \times \text{id}_M)$

$$\text{assoc} \rightarrow m(x, m(y, z)) = m(m(x, y), z) \quad \forall x, y, z \in M$$

$$x(yz) = (xy)z \quad xy \equiv m(x, y)$$

$$\text{unit} \rightarrow m(id_M \times 1) = m(1 \times id_M) = id_M$$

$$\forall x \in M \quad m(x, 1(\emptyset)) = m(1(\emptyset), x) = x$$

$$1 \equiv 1(\emptyset) \quad x1 = 1x = x$$

$$xy \equiv m(x, y)$$

### Notational Aside

given a product  $A \times B$  to define a map

$$C \xrightarrow{f} A \times B$$

$$f(c) = (a, b) \quad a = "f_1(c)"$$

$$b = "f_2(c)"$$

$$\text{we write } f = f_1 \times f_2$$

Similarly define groups

$$0\text{-ary op } e: \{\emptyset\} \rightarrow G \quad e \equiv e(\emptyset)$$

$$1\text{-ary op } \iota: G \rightarrow G \quad g^{-1} \equiv \iota(g)$$

$$2\text{-ary op } m: G \times G \rightarrow G \quad gh \equiv m(g, h)$$

same axioms.

$$\text{Rings } (R, \overset{0\text{ary}}{\iota}, \overset{2\text{ary}}{+}, \overset{1\text{ary}}{-}, \overset{0\text{ary}}{0}, \overset{2\text{ary}}{\cdot}, \overset{1\text{ary}}{(-)})$$

$\Omega$ -algebras:

$\Omega$  a set of symbols w/ "arities"

$$\Omega: \{n, 2, 1\} \rightarrow \mathbb{Z}_{\geq 0}$$

$$n \mapsto 2$$

$$2 \mapsto 1$$

$$1 \mapsto 0$$

Def an  $\Omega$ -algebra is a set  $S$  w/  
maps  $\lambda: S^n \rightarrow S$

for each  $\lambda \in \Sigma$  w/ arity  $n$ .

Def Homomorphisms of  $\Sigma$  alg's.

are fns  $S \rightarrow T$  s.t.  $\forall \lambda \in \Sigma$

$$f(\lambda(s_1, \dots, s_n)) = \lambda(f(s_1), \dots, f(s_n))$$

ex:  $\mathbb{R} \xrightarrow{f} \mathbb{R} \times \mathbb{R}$   
 $\searrow \text{is } \mathbb{R}^2$

$$x \longmapsto (0, x)$$

$$x+y \longmapsto (0, x+y) = (0, x) + (0, y)$$

$$xy \longmapsto (0, xy) = (0, x)(0, y)$$

$$1 \longmapsto (0, 1) \neq 1$$

$$\lambda = 1 \quad f(\lambda(\emptyset)) = \lambda(\emptyset)$$

$$f(1) = 1$$

Def (Imprecise) A Variety = the collection of  $\Sigma$  algebras for a given  $\Sigma$ , satisfying a set of identities.

Ex:  $\Omega$  as above  $m, 2, 1$   
 w/ identities  $\left. \begin{array}{l} (xy)z = x(yz) \\ xx^{-1} = x^{-1}x = 1 \\ 1x = x1 = x \end{array} \right\}$

Variety defined by these is called "groups"

Fun: Def A congruence on an  $\Omega$ -algebra  $A$   
 is an  $\Omega$ -subalgebra of  $A \times A$

- Consider  $G$  a group  $H \triangleleft G$

$$C = \{(g_1, g_2) \in G \times G \mid g_1 g_2^{-1} \in H\}$$

Show  $C$  is a congruence  $\Leftrightarrow H \triangleleft G$

- Consider a ring  $R$ ,  $I \triangleleft (R, +)$

$$C = \{(r_1, r_2) \in R \times R \mid r_1 - r_2 \in I\}$$

congruence  $\Leftrightarrow I \triangleleft R$ .

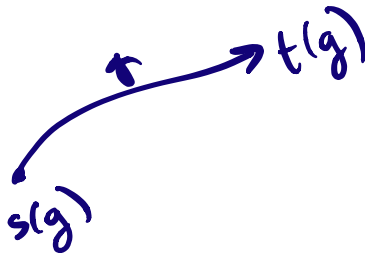
In general, if  $A \xrightarrow{f} B$  a hom. of  $\mathcal{A}$ -algs  
can define  $\ker f = \{ (a_1, a_2) \mid f(a_1) = f(a_2) \}$   
congruences are kernels.

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Doesn't capture all types of structures we  
care about

Groupoid:

$$G_1 \xrightleftharpoons[t]{s} G_0$$



ex: Pick a collection of sets  $S$

$G_1 =$  bijective maps between these sets.

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