Graduate Algebra, Fall 2019, Final Exam

Instructor: Daniel Krashen

Name ____

By signing below, I pledge that the work on this exam is my own,
and was done without outside help.

Signature ____

Please hand only only clearly written work, not scratch paper.

You may hand in your work either on separate pieces of paper, or written on a printed version of this exam, or TeXed up separately. You may scan or photograph your completed exam and email it to me, or simply hand in a physical copy to my office or departmental mailbox. If you can think of another way you would rather hand it in, let me know.

Due 11:59pm, Thursday December 12, 2019.

This exam has 7 questions, for a total of 70 points.

1. (10 points) (a) Show that the groups $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$ are not isomorphic.

(b) Show that the groups $\mathbb{Q}$ and $\mathbb{Q} \oplus \mathbb{Q}$ are not isomorphic.

(c) Show that the groups $\mathbb{Q}$ and $\mathbb{Q}/Z$ are not isomorphic.

2. (10 points) Suppose that T is a complex $n \times n$ matrix such that $T^n = 1$ for some $n > 0$. Show that T is diagonalizable.

   remember – we are not using any of the facts from the representation theory portion of the course!

3. (10 points) Let G be a group, and suppose that K and H are subgroups such that $K \subset H$ and $H \subset N_G(K)$. Show that H is also in the normalizer of $C_G(K)$.

4. (10 points) Let T be an $n \times n$ matrix over the field $\mathbb{F}_2$ with two elements. Suppose that $T^2 = 1$. Describe the possible Jordan forms of T.

5. (10 points) Show that no group of order 56 can be simple.

6. (10 points) Show that a group of order pqr for distinct primes p, q, r must have a non-trivial normal Sylow subgroup.

7. (10 points) Give an example of a matrix over the field of rational numbers, which cannot be put into Jordan canonical form.