

2-Dimensional Wavelet Transforms

1-d signals

typical example: sounds

$z[k]$ = signal at time k

2-d signals

typical example: images (grayscale)

$z[j, k]$ = signal at point
with coordinates
 (j, k)

1-d signals

represented as vectors $z[k] \in \mathbb{C}^N$

$$\begin{bmatrix} z[0] \\ \vdots \\ z[N-1] \end{bmatrix}$$

2-d signals

represented as matrices $z[j,k] \in M_N(\mathbb{C})$

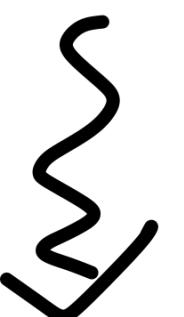
$$\begin{bmatrix} z[0,0] & z[0,1] & \cdots & z[0,N-1] \\ \vdots & & & \vdots \\ z[N-1,0] & \cdots & \cdots & z[N-1,N-1] \end{bmatrix}$$

$\mathcal{O}(n^2)$

Plot: Given a 1-dimensional wavelet transform,
how can we use it to obtain a 2-dim'l transform?

$$T_a x = \begin{bmatrix} s \\ d \end{bmatrix}$$

$$N=2m$$



$$T_a : \mathbb{C}^N \longrightarrow \mathbb{C}^N$$

$$T_a \in M_N(\mathbb{C})$$

$$T_a^{2-d} : \mathbb{C}^{N^2} \longrightarrow \mathbb{C}^{N^2}$$

Idea: We apply the wavelet transform

- vertically (to each column)
- horizontally (to each row)

$$\begin{bmatrix} z[0,0] & \dots & z[0,N-1] \\ \vdots & & \vdots \\ z[M_1] & \dots & z[N-1, N-1] \end{bmatrix} = z \rightsquigarrow \begin{bmatrix} v.\text{tr}(z) \\ \dots \\ v.\det(z) \end{bmatrix} \rightsquigarrow \begin{bmatrix} h_{\text{tr}}(v.\text{tr}(z)) & h_{\det}(v.\det(z)) \\ h_{\text{tr}}(v.\det(z)) & h_{\det}(v.\det(z)) \end{bmatrix}$$

$$\begin{bmatrix} z[0,0] & \dots & z[0,N-1] \\ \vdots & & \vdots \\ z[N-1] & \dots & z[N-1,N-1] \end{bmatrix}$$

$$= z \rightsquigarrow \begin{bmatrix} v.\text{tr}(z) \\ \dots \\ v.\det(z) \end{bmatrix} \rightsquigarrow$$

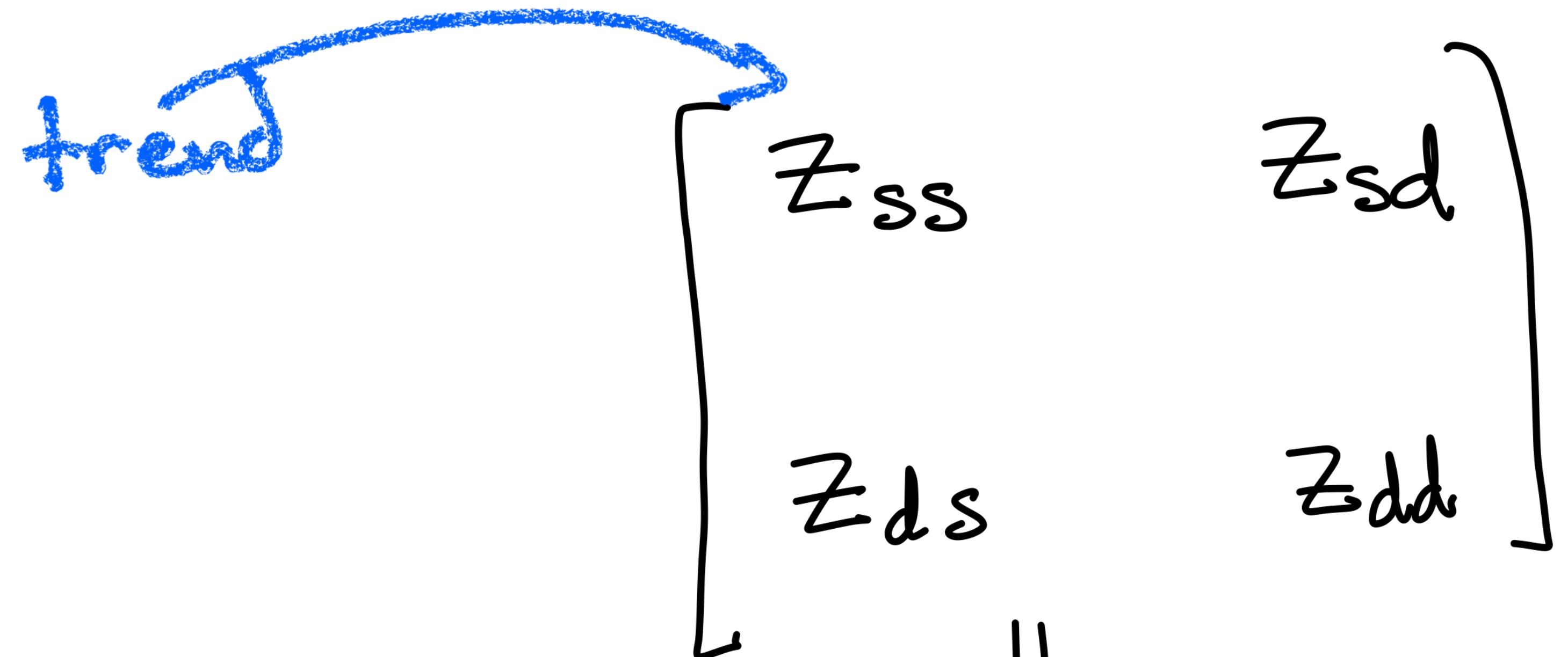
$$\begin{bmatrix} h_{\text{tr}}(v.\text{tr}(z)) & h_{\det}(v.\det(z)) \\ h_{\text{tr}}(v.\det(z)) & h_{\det}(v.\det(z)) \end{bmatrix}$$

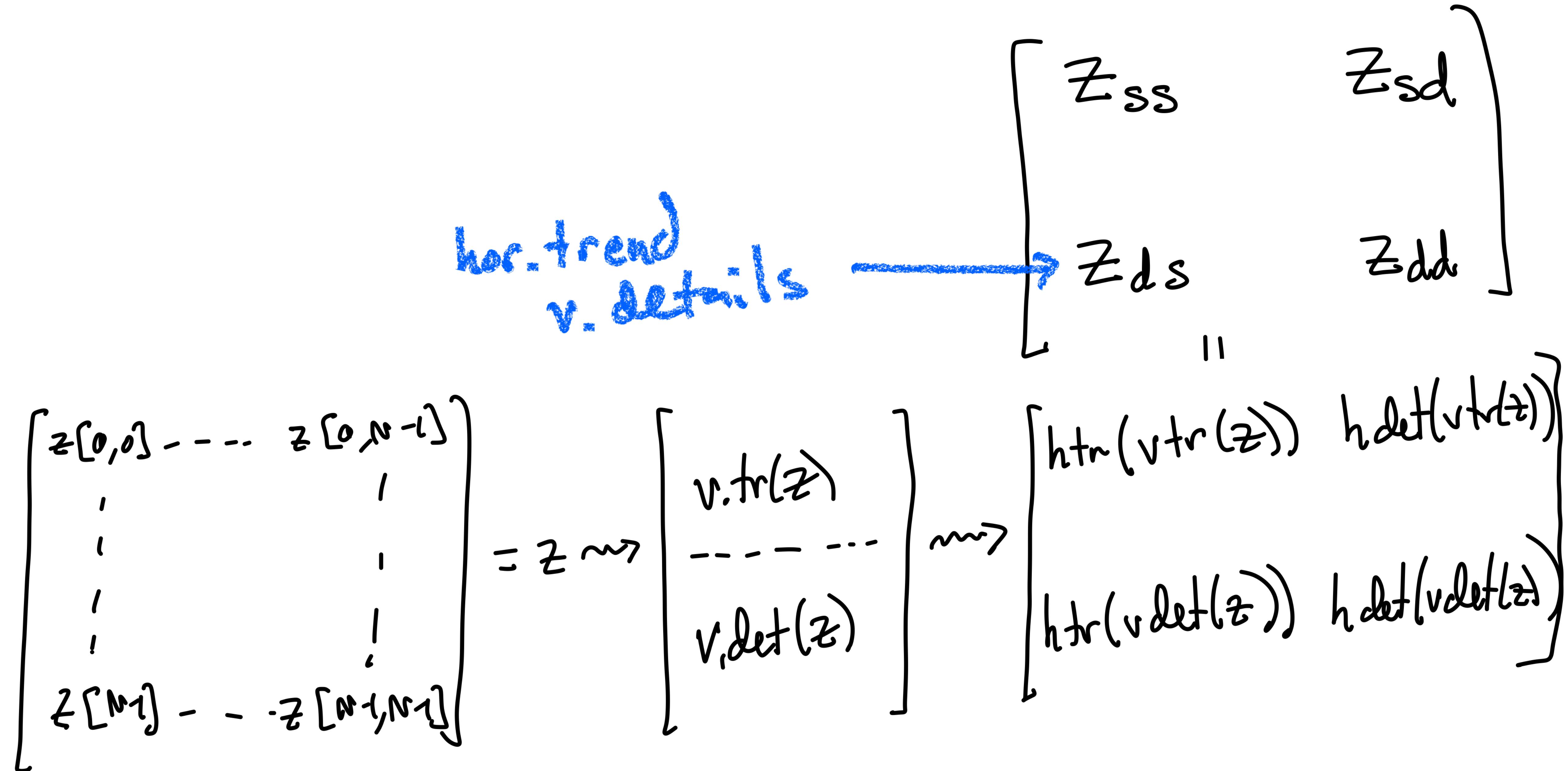
$$\begin{bmatrix} z_{ss} & z_{sd} \\ z_{ds} & " \end{bmatrix}$$

$$\begin{bmatrix} z[0,0] & \dots & z[0,N-1] \\ \vdots & & \vdots \\ z[N-1] & \dots & z[N-1,N-1] \end{bmatrix}$$

$$= z \rightsquigarrow \begin{bmatrix} v.\text{tr}(z) \\ \dots \\ v.\det(z) \end{bmatrix} \rightsquigarrow$$

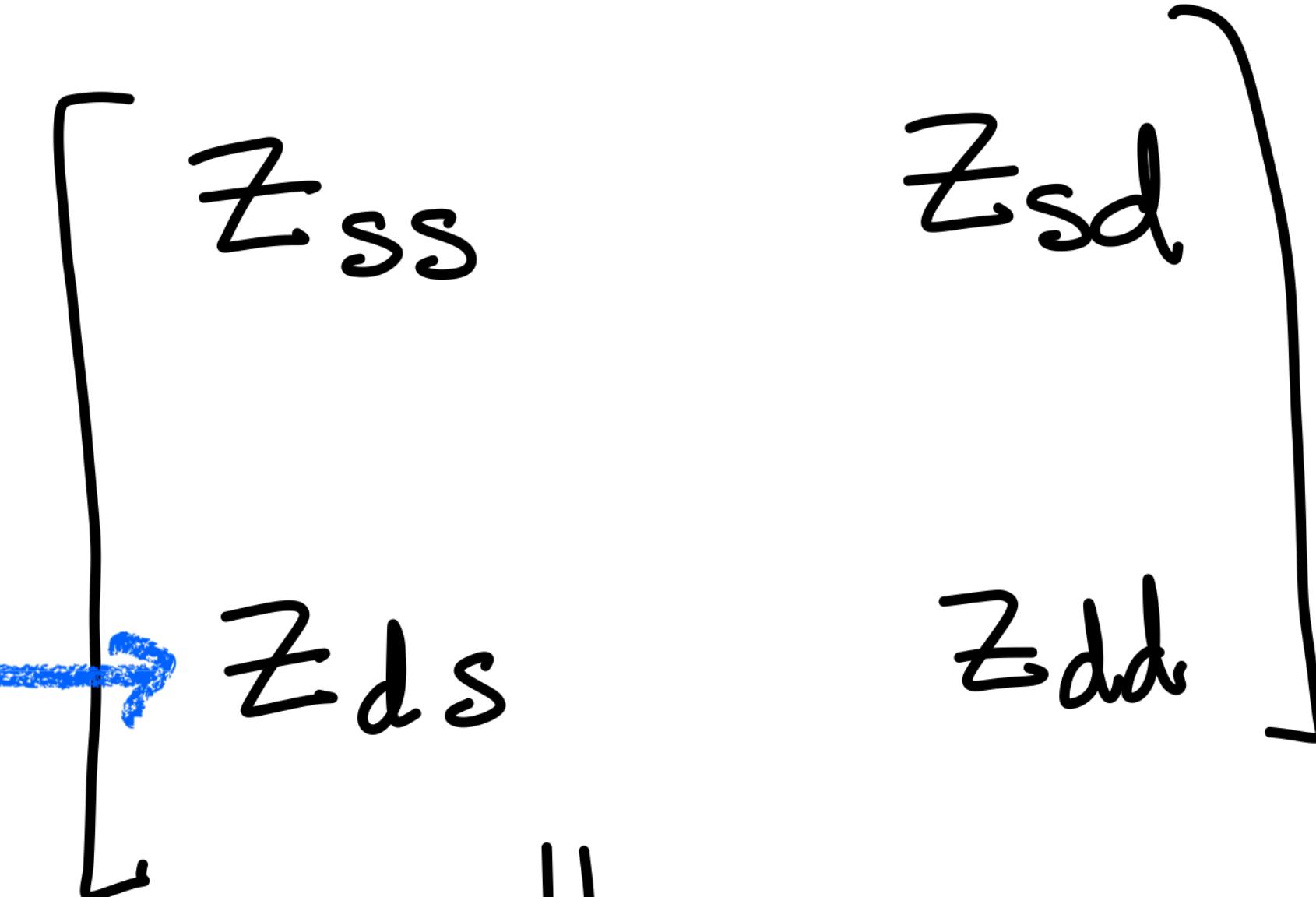
$$\begin{bmatrix} h_{\text{tr}}(v_{\text{tr}}(z)) & h_{\det}(v_{\det}(z)) \\ h_{\text{tr}}(v_{\det}(z)) & h_{\det}(v_{\det}(z)) \end{bmatrix}$$





See changes under
v. movement

hor. trend
v. details



"

$$\begin{bmatrix} z[0,0] & \dots & z[0,N-1] \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ z[N-1] & \dots & z[N-1,N-1] \end{bmatrix} = z \rightsquigarrow \begin{bmatrix} v.tr(z) \\ \dots \\ v.det(z) \end{bmatrix} \rightsquigarrow \begin{bmatrix} htr(vtr(z)) & hdet(vtr(z)) \\ htr(vdet(z)) & hdet(vdet(z)) \end{bmatrix}$$

horizontal
features

= See changes under
v. movement

hor. trend
v. details

z_{ss}

z_{sd}

z_{ds}

z_{dd}

"

$$\begin{bmatrix} z[0,0] & \dots & z[0,N-1] \\ \vdots & & \vdots \\ z[N-1] & \dots & z[N-1,N-1] \end{bmatrix}$$

$$= z \rightsquigarrow \begin{bmatrix} v.tr(z) \\ \dots \\ v.det(z) \end{bmatrix} \rightsquigarrow \begin{bmatrix} htr(vtr(z)) \\ \dots \\ htr(vdet(z)) \end{bmatrix}$$

$$\begin{bmatrix} hdet(vtr(z)) \\ \dots \\ hdet(vdet(z)) \end{bmatrix}$$

vertical trend
 horizontal detail
 = vertical features

z_{ss}

z_{sd}

z_{ds}

z_{dd}

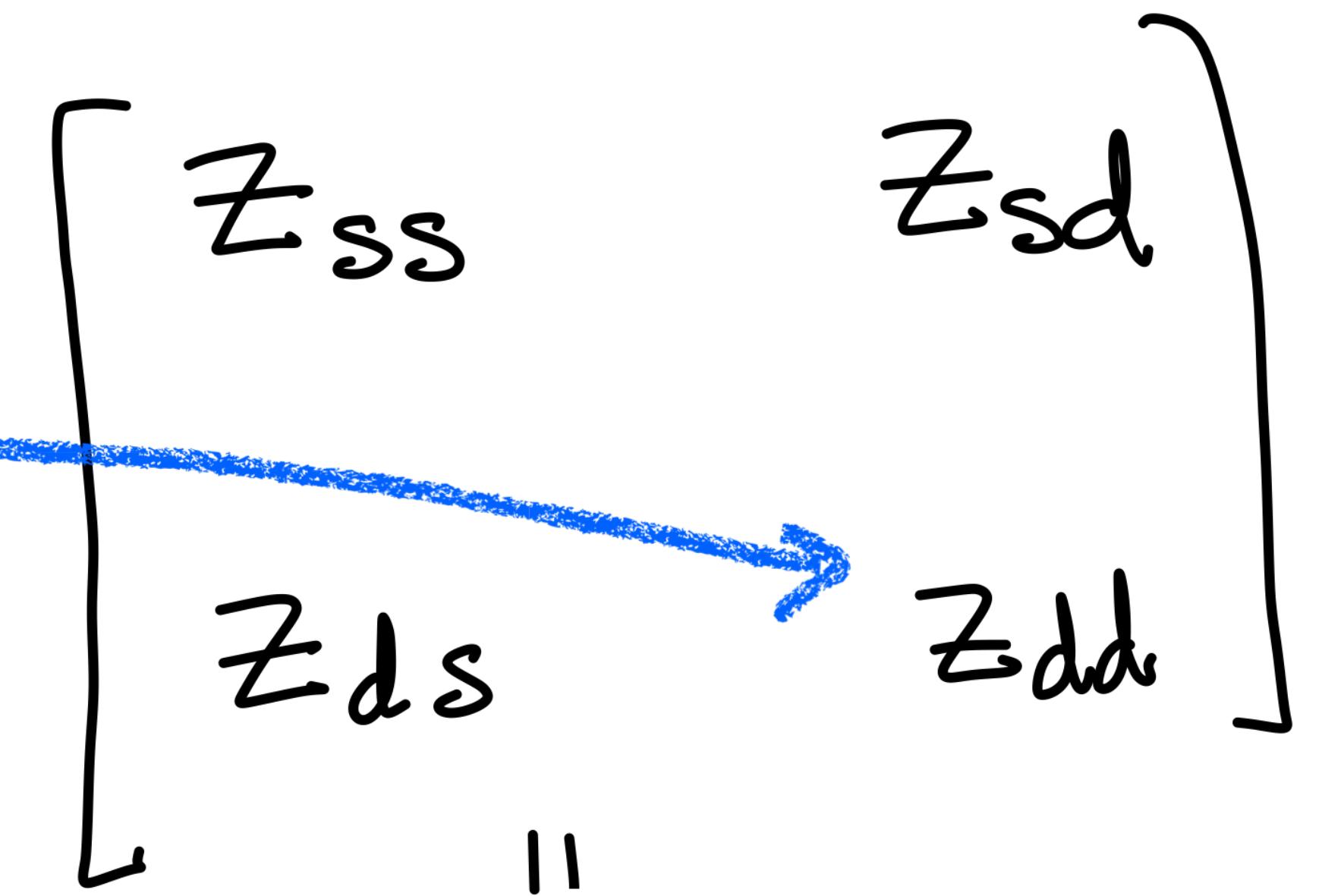
"

$$\begin{bmatrix} z[0,0] & \dots & z[0,N-1] \\ \vdots & & \vdots \\ z[M] & \dots & z[N-1,N-1] \end{bmatrix}$$

$$= z \rightsquigarrow \begin{bmatrix} v.\text{tr}(z) \\ \dots \\ v.\text{det}(z) \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} h.\text{tr}(v.\text{tr}(z)) & h.\text{det}(v.\text{tr}(z)) \\ h.\text{tr}(v.\text{det}(z)) & h.\text{det}(v.\text{det}(z)) \end{bmatrix}$$

detects changes when
both vect.-& horiz.-
 move



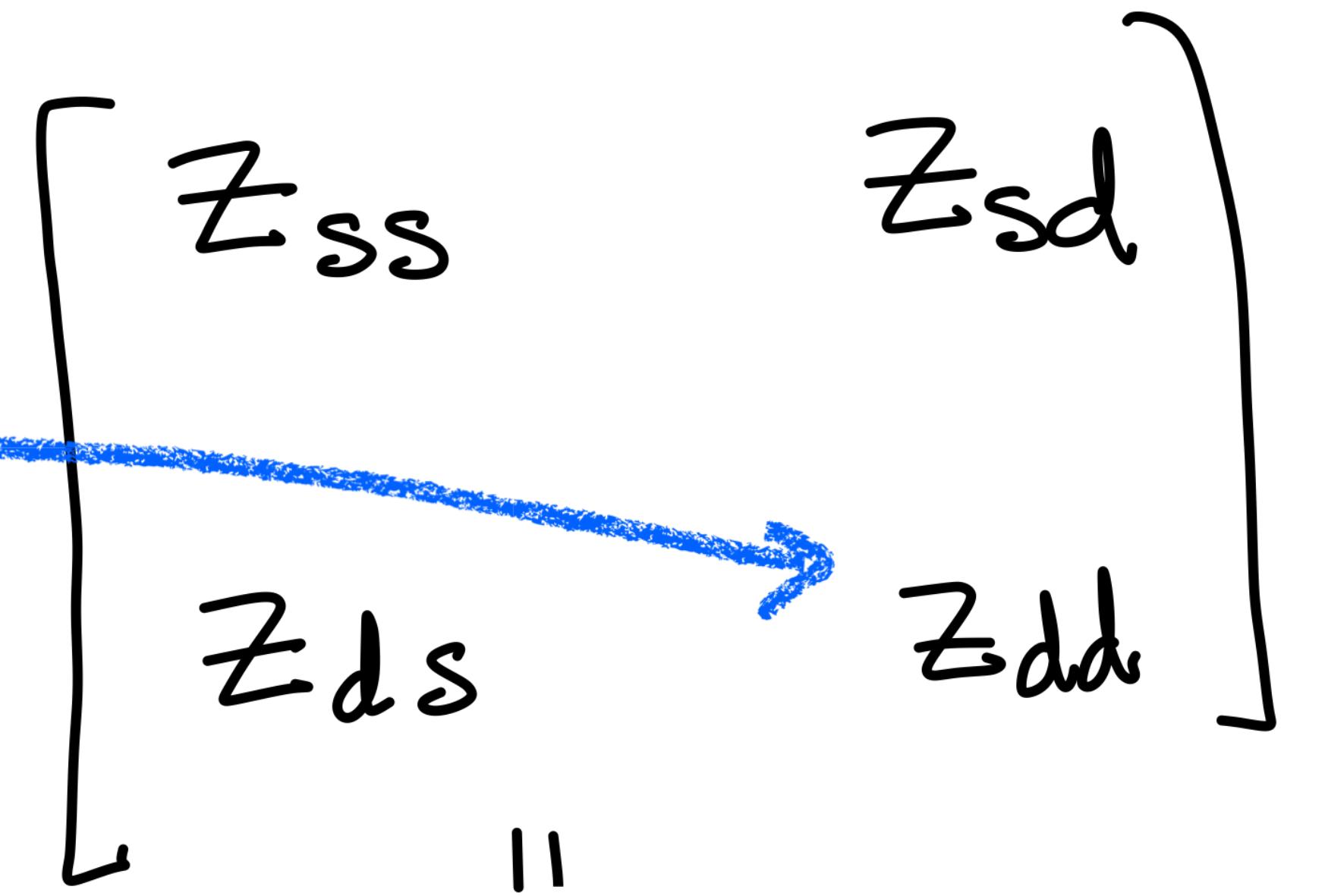
$$\begin{bmatrix}
 z[0,0] & \dots & z[0,N-1] \\
 & \vdots & \\
 & \vdots & \\
 & \vdots & \\
 z[N-1] & \dots & z[N-1,N-1]
 \end{bmatrix}$$

$$z \rightsquigarrow \begin{bmatrix}
 v.\text{tr}(z) \\
 \dots \\
 v.\det(z)
 \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix}
 h_{\text{tr}}(v.\text{tr}(z)) & h_{\det}(v.\det(z)) \\
 h_{\text{tr}}(v.\det(z)) & h_{\det}(v.\det(z))
 \end{bmatrix}$$

[Diagonal Features]

"
 detects changes when
both vect.-& horiz.-
 move



$$\begin{bmatrix} z[0,0] & \dots & z[0,N-1] \\ \vdots & & \vdots \\ z[N-1] & \dots & z[N-1,N-1] \end{bmatrix}$$

$$= z \rightsquigarrow \begin{bmatrix} v.\text{tr}(z) \\ \dots \\ v.\det(z) \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} h_{\text{tr}}(v.\text{tr}(z)) & h_{\det}(v.\det(z)) \\ h_{\text{tr}}(v.\det(z)) & h_{\det}(v.\det(z)) \end{bmatrix}$$

Q: How to express T_a^{ω} in terms of matrix operations

?

e

Linear Algebra

$$\bar{T} \cdot \begin{bmatrix} & & & & \\ v_0 & | & v_1 & | & \cdots & | & v_{N-1} \\ & & & & & & \\ & & & & & & \end{bmatrix} = \begin{bmatrix} & & & & \\ \bar{T}v_0 & | & \bar{T}v_1 & | & \cdots & | & \bar{T}v_{N-1} \\ & & & & & & \\ & & & & & & \end{bmatrix}$$

$$\begin{bmatrix} w_0 \\ -\frac{w_1}{i} \\ \vdots \\ -\frac{w_{N-1}}{i} \end{bmatrix} \xrightarrow{\bar{T}^t} = \bar{T} \begin{bmatrix} w_0 \\ \vdots \\ -w_{N-1} \end{bmatrix}^t = \begin{bmatrix} & & & & \\ \bar{T}w_0 & | & \cdots & | & \bar{T}w_{N-1} \\ & & & & \\ & & & & \end{bmatrix}^t$$

Linear Algebra

$$\overline{T} \cdot \begin{bmatrix} & & & & & \\ & | & | & | & | & | \\ v_0 & v_1 & v_2 & \cdots & \vdots & v_{N-1} \\ & | & | & | & | & | \end{bmatrix} = \begin{bmatrix} & & & & & \\ & | & | & | & | & | \\ \overline{T}v_0 & \overline{T}v_1 & \overline{T}v_2 & \cdots & \vdots & \overline{T}v_{N-1} \\ & | & | & | & | & | \end{bmatrix}$$

$$\begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix} \xrightarrow{\overline{T}^t} = \overline{T} \begin{bmatrix} & & & & & \\ & | & | & | & | & | \\ w_0 & w_1 & w_2 & \cdots & \vdots & w_{N-1} \\ & | & | & | & | & | \end{bmatrix}^t = \begin{bmatrix} \overline{T}w_0 \\ \vdots \\ \overline{T}w_{N-1} \end{bmatrix}$$

$z \rightsquigarrow T_a z$ = vertical/column wavelet transform

$z \rightsquigarrow z(T_a)^t$ = horizontal/row wavelet transform

$z \rightsquigarrow T_a z(T_a)^t$ = 2-D wavelet transform

$$T_a z(T_a)^t = \begin{bmatrix} z_{ss} & \vdots & z_{sd} \\ \cdots & + & \cdots \\ z_{ds} & \vdots & z_{dd} \end{bmatrix}$$