Lecture 2: limits, continuity, squeeze

Thursday, August 20, 2015 9:32 AM

By the way, there is another textbook resource as well "Whitman Calculus"

Continuity:

Def a function $f(x)$ is continuous at $x=a$ if

$$f(a) = \lim_{x\to a} f(x)$$

Q: which functions are continuous?

Limit laws actually tells us that many functions are continuous. (1.3 Moodle)

Suppose $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$

Constant Law: $\lim_{x\to a} C f(x) = C \lim_{x\to a} f(x) = CL$

Sum Law: $\lim_{x\to a} f(x)+g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

Product $\lim_{x\to a} f(x) g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$

Quotient $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$ $\quad M \neq 0$

Power law: $\lim_{x\to a} f(x)^n = \left(\lim_{x\to a} f(x)\right)^n = L^n$

Root Law: $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$ is L is in domain of $\sqrt[n]{x}$

($L \geq 0$ if n even)

Composition: If $\lim_{x\to a} g(x) = M$ &

$\lim_{x\to M} f(x) = f(M)$ then $\lim_{x\to a} f(g(x)) = f(M)$

ex: $$\lim_{x\to 2}\sqrt{\frac{x^2-3}{2x+4}} \underset{\text{root law}}{=} \sqrt{\lim_{x\to 2}\frac{x^2-3}{2x+4}} \underset{\text{quotient law}}{=} \sqrt{\frac{\lim_{x\to 2} x^2-3}{\lim_{x\to 2} 2x+4}}$$

$$\underset{\substack{\text{sum law}\\ \text{sum law}}}{=} \sqrt{\frac{\lim_{x\to 2} x^2 + \lim_{x\to 2}(-3)}{\lim_{x\to 2} 2x + \lim_{x\to 2} 4}} \underset{\substack{\text{power law}\\ \text{const. law}}}{=} \sqrt{\frac{\left(\lim_{x\to 2} x\right)^2 + \lim_{x\to 2}(-3)}{2\left(\lim_{x\to 2} x\right) + \lim_{x\to 2} 4}}$$

(limit fact)

$$= \sqrt{\frac{\left(\lim_{x\to 2} x\right)^2 - 3}{2\left(\lim_{x\to 2} x\right) + 4}} \xrightarrow{\text{limit fact}} \underset{\text{plug in!}}{=} \sqrt{\frac{2^2-3}{2(2)+4}} = \sqrt{\frac{1}{8}}$$

really shown: $f(x) = \sqrt{\frac{x^2-3}{2x+4}}$ is continuous at $x=2$.

Punchline is "(pf by example)": any function "like this" made of quotients, polynomials, roots, ... is going to be continuous

practice show that $f(x) = \frac{x^2-\sqrt{x}}{?}$ is continuous at $x=2$

practice show that $f(x) = \dfrac{x^2 - \sqrt{x}}{2x-3}$ is continuous at $x=2$

$$\lim_{x\to 0} \frac{x}{x} = 1$$

if in domain of function!

In practice: will typically need to manipulate limit before we can plug in

$$\lim_{x\to 1} \frac{x^2+2x-3}{x-1} = \lim_{x\to 1} \frac{(x+3)(x-1)}{x-1} = \lim_{x\to 1} (x+3) = 4$$

can't plug in.

is continuous (and defined at $x=1$)

example

$$\lim_{x\to 1} \frac{\sqrt{x+8}-3}{x-1}$$

$$\lim_{x\to 1} \left(\frac{\sqrt{x+8}-3}{x-1}\right)\left(\frac{\sqrt{x+8}+3}{\sqrt{x+8}+3}\right) = \lim_{x\to 1} \frac{(x+8)-3^2}{(x-1)(\sqrt{x+8}+3)}$$

$$= \lim_{x\to 1} \frac{x+8-9}{(x-1)(\sqrt{x+8}+3)} = \lim_{x\to 1} \frac{x-1}{x-1} \cdot \frac{1}{\sqrt{x+8}+3}$$

$$= \lim_{x\to 1} \frac{1}{\sqrt{x+8}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

continuous at $x=1$

Some other continuous functions:

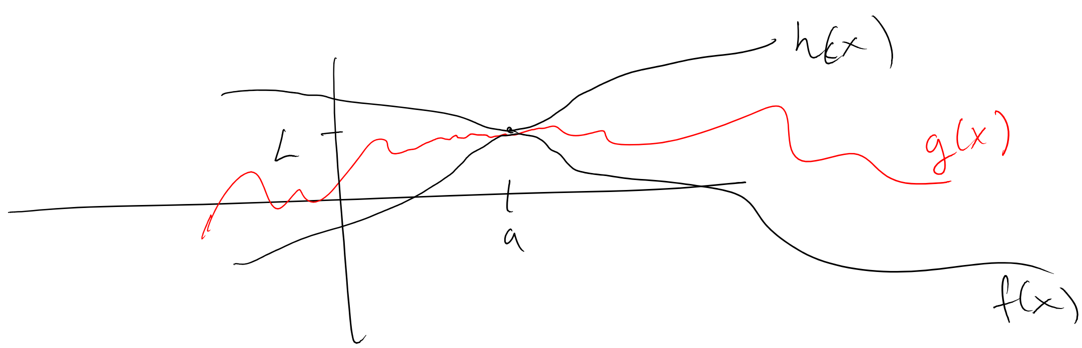
exponential, trigonometric, logarithmic (within domains)

One more tool:

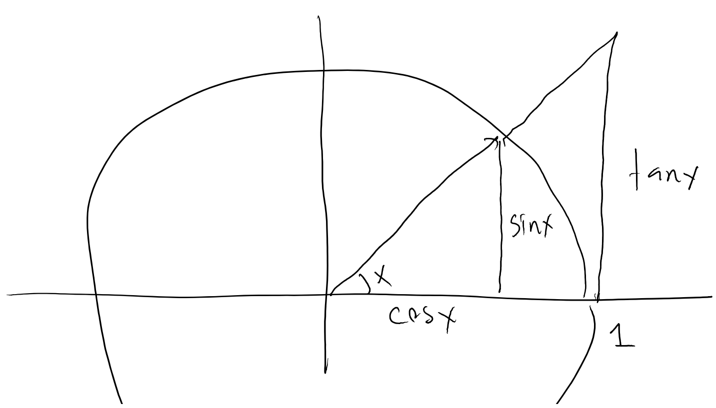
Squeeze theorem (1.3.5 Mooculus)

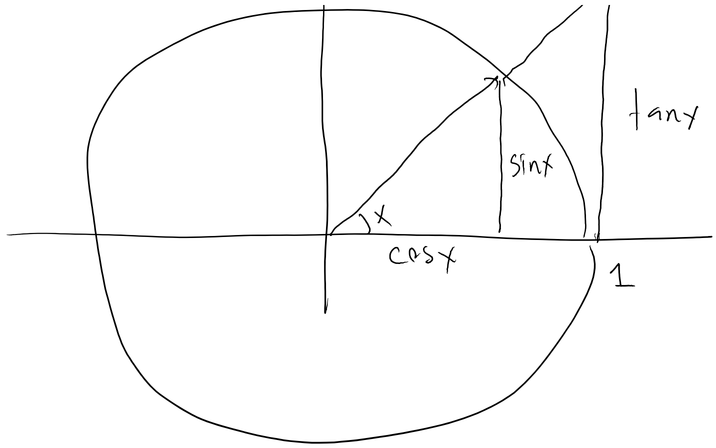
If we have functions $f(x), g(x), h(x)$ with $f(x) \le g(x) \le h(x)$ for x near a, and if

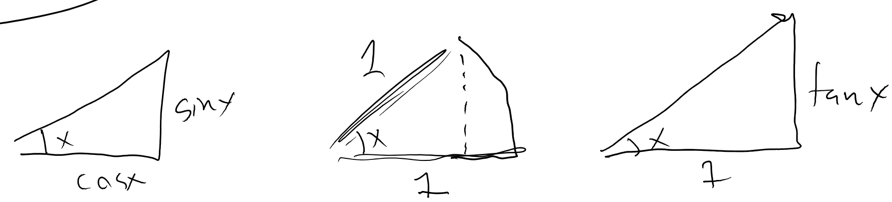
$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$ then $\lim_{x\to a} g(x) = L$



ex $\lim_{x\to 0} \frac{\sin x}{x}$







$$\text{Area} \le \text{Area} \le \text{Area}$$

$$\frac{1}{2}(\cos x)(\sin x) \qquad \frac{x}{2} \qquad \frac{1}{2}\,1(\tan x)$$

$$\frac{1}{2}(\cos x)(\sin x) \le \frac{x}{2} \le \frac{1}{2}\,\frac{\sin x}{\cos x}$$

$$\cos x \le \frac{x}{\sin x} \le \frac{1}{\cos x}$$

$$\frac{1}{\cos x} \geq \boxed{\frac{\sin x}{x}} \geq \cos x \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{\cos 0} \qquad \lim_{x \to 0} \cos x = \cos 0 = 1$$

$$\frac{1}{\cos 0} = 1$$