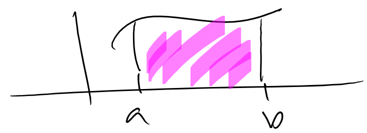
# Lecture 18: the fundamental theorem(s) of calculus

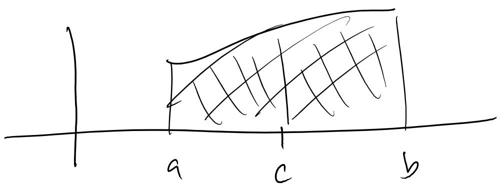
Thursday, October 29, 2015 9:32 AM

## Notice a few facts about definite integrals

Recall: $\int_a^b f(x)\,dx$ = signed area under the graph of $f(x)$ between $x=a$ & $x=b$

" (area above) − (area below)





$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^a f(x)\,dx = \int_a^a f(x)\,dx + \int_a^a f(x)\,dx$$

cancel one

$$0 = \int_a^a f(x)\,dx$$

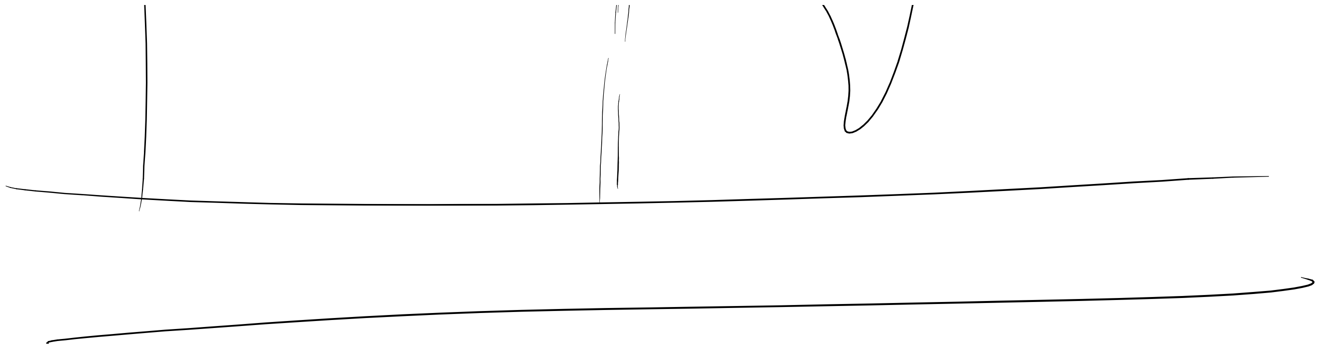
We define by convention:

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

it makes sense (is consistent) because:

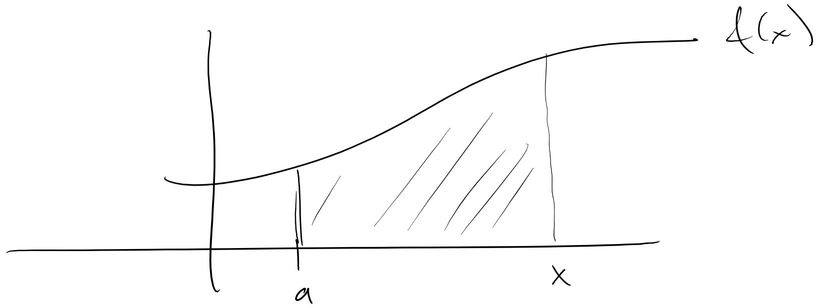
$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx = 0$$

in small intervals, ht doesn't change much



## Fundamental Theorem of Calculus (I)

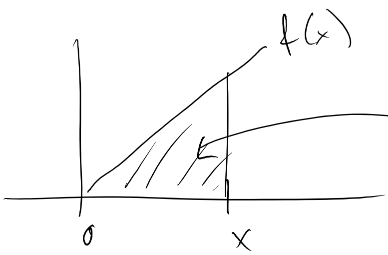
Let $f(x)$ be some function




$$F(x) = \int_a^x f(t)\,dt$$

Q: what is $F'(x)$?

ex: $f(x) = x$




$$\int_0^x f(t)\,dt = \frac{1}{2}x^2$$

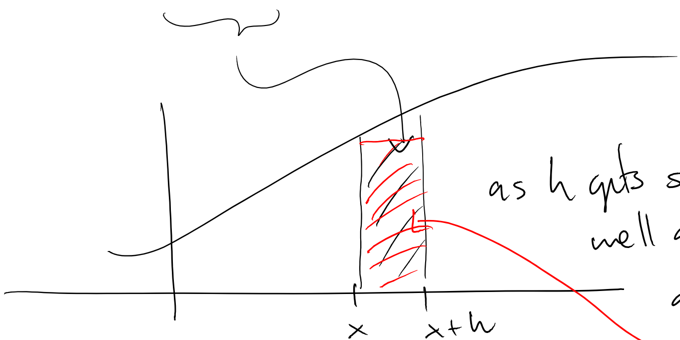
Theorem (FTC I) If $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$

Proof: $F'(x) = \lim_{h \to 0} \dfrac{F(x+h) - F(x)}{h}$

$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt}{h}$$

$$= \lim_{h \to 0} \frac{\int_a^{x} f(t)\,dt + \int_x^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = \lim_{h \to 0} \frac{1}{h}\left( f(x) \cdot h \right) = f(x)$$




as h gets small, this is well approximated by a rectangle

w/ area = $f(x) \cdot h$

$\quad\quad\quad\quad$ " ht $\quad$ " width

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Theorem (FTC II)

If $F(x)$ is any antiderivative for $f(x)$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

ex: $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \frac{1}{3}.$

$$\int_0^x \ldots$$

an antidr for $x^2 = \frac{1}{3}x^3$

Proof If we set $G(x) = \int_a^x f(t)\,dt$ then FTC I says $G'(x) = f(x)$

If $F(x)$ is any anti-derivative for $f(x)$ then $F(x) = G(x) + C$ some constant $C$.

$$F(b) - F(a) = \left(G(b)+C\right) - \left(G(a)+C\right) = G(b) - G(a)$$

$$= \int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{0}$$

$$= \int_a^b f(t)\,dt$$

$$= \int_a^b f(x)\,dx$$

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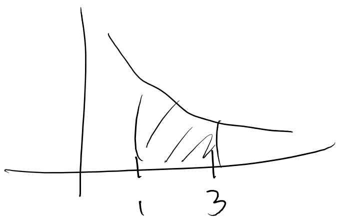
Notation: $F(b) - F(a) = F(x)\Big|_a^b = \Big[F(x)\Big]_a^b$

$$= F(x)\Big]_a^b$$

FTC II: $\int_a^b f(x)\,dx = F(x)\Big|_a^b$

ex: $\int^3 \frac{1}{x}\,dx = \ln|x|\Big|_1^3 = \ln 3 - \ln 1 = \ln 3$

ex: $\int_1 \frac{1}{x}\,dx = \ln|x|\Big|_1 = \ln 3 - \ln 1 = \ln$

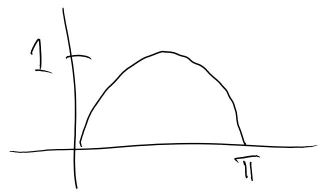



$$\int_0^{\pi} \sin x\,dx = -\cos x \Big|_0^{\pi}$$

$$= (-\cos(\pi)) - (-\cos(0))$$

$$(-(-1)) - (-1)$$

$$1 + 1 = 2$$

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Practice

1. $\int_0^6 e^x\,dx$

3. $\int_{-2}^{-3} (x-6)^2\,dx$

2. $\int_0^3 x^3\,dx$

4. $\int_1^2 \frac{1}{x\ln x}\,dx$

1. $\int_0^6 e^x\,dx = e^x\Big|_0^6 = e^6 - e^0 = e^6 - 1$

2. $\int_0^3 x^3\,dx = \frac{1}{4}x^4\Big|_0^3 = \frac{1}{4}3^4 - \frac{1}{4}0^4 = \frac{81}{4}$

3. $\int_{-2}^{-3} (x-6)^2\,dx = \frac{1}{3}(x-6)^3\Big|_{-2}^{-3} = \frac{1}{3}(-3-6)^3 - \frac{1}{3}(-2-6)^3 = -243 + \frac{512}{3}$

$$\int (x-6)^2\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(x-6)^3 + C$$

$$\int (x-6)^2\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(x-6)^3 + C$$

$u = x-6$

$du = dx$

4. $$\int_1^2 \frac{1}{x\ln x}\,dx = \ln|\ln(x)|\Big|_1^2 = \ln|\ln 2| - \ln|\ln 1| = \ln(\ln 2) - \cancel{\ln 0}$$

does not exist!

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln x| + C$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

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$$\int_2^3 \frac{1}{x\ln x}\,dx = \int_{x=2}^{x=3} \frac{1}{x\ln x}\,dx = \int_{u=\ln 2}^{u=\ln 3} \frac{1}{u}\,du = \ln|u|\Big|_{\ln 2}^{\ln 3}$$

$= \ln(\ln 3) - \ln(\ln 2)$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$x=2 \to u = \ln 2$

$x=3 \to u = \ln 3$

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1. $$\int_0^{\pi} \sin^5(3x)\cos(3x)\,dx = \frac{1}{3}\int_0^0 u^5\,du = 0.$$

$u = \sin 3x$

$du = (\cos 3x)\,3\,dx$

$\frac{1}{3}\,du = \cos 3x\,dx$

$x=0 \rightsquigarrow u = \sin 0 = 0$

$x=\pi \rightsquigarrow u = \sin(3\pi) = 0$