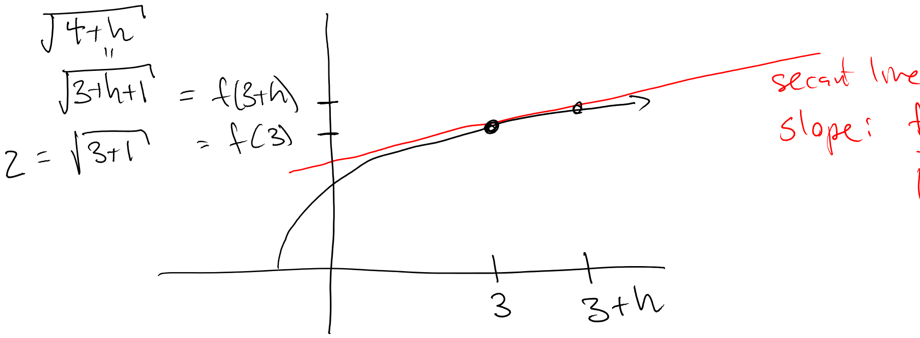
Lecture 5: tangent line practice, continuity and the intermediate value theorem

Thursday, August 27, 2015 9:40 AM

Practice Find slopes of tangent lines

1. For $f(x)=\sqrt{x+1}$ at $x=3$



slope: $\dfrac{f(3+h)-f(3)}{(3+h)-3} = \dfrac{f(3+h)-f(3)}{h} = \dfrac{\sqrt{4+h}-2}{h}$

slope of tangent line $\lim_{h\to 0}\dfrac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\dfrac{\sqrt{3+h+1}-\sqrt{3+1}}{h}$

$$=\lim_{h\to 0}\left(\frac{\sqrt{4+h}-2}{h}\right)\left(\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right) = \lim_{h\to 0}\frac{(4+h)-4}{h(\sqrt{4+h}+2)}$$

$$=\lim_{h\to 0}\frac{h}{h}\left(\frac{1}{\sqrt{4+h}+2}\right) = \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2}$$

$$=\frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

2. $f(x)=\sin x$ at $x=\pi/3$

$$\lim_{h\to 0}\frac{f(\pi/3+h)-f(\pi/3)}{h} = \lim_{h\to 0}\frac{\sin(\pi/3+h)-\sin(\pi/3)}{h}$$

$$\lim \frac{(\sin \pi/3)(\cos h)+(\sin h)(\cos \pi/3) - \sin \pi/3}{}$$

$$\lim_{h\to 0} \frac{(\sin \pi/3)(\cos h) + (\sin h)(\cos \pi/3) - \sin \pi/3}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sin \pi/3 \cos h - \sin \frac{\pi}{3}}{h} + \frac{\sin h \cos \pi/3}{h} \right)$$

$$= \lim_{h\to 0} \frac{\sin \pi/3 \cos h - \sin \pi/3}{h} + \lim_{h\to 0} \frac{\sin h \cos \pi/3}{h}$$

$$= \lim_{h\to 0} (\sin \pi/3) \frac{(\cos h - 1)}{h} + \cos \pi/3 \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \sin \pi/3 \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{?\ 0} + \cos \pi/3 \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{1} = \boxed{\cos \frac{\pi}{3}}$$

$= \frac{1}{2}$.

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)} = \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)}$$

$$= \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h + 1)} = \lim_{h\to 0} \left(\frac{\sin h}{h}\right)\left(\frac{-\sin h}{\cos h + 1}\right)$$

$$= \underbrace{\left(\lim_{h\to 0} \frac{\sin h}{h}\right)}_{1}\left(\lim_{h\to 0} \frac{-\sin h}{\cos h + 1}\right) = 1 \cdot \left(\frac{-\sin 0}{\cos 0 + 1}\right)$$

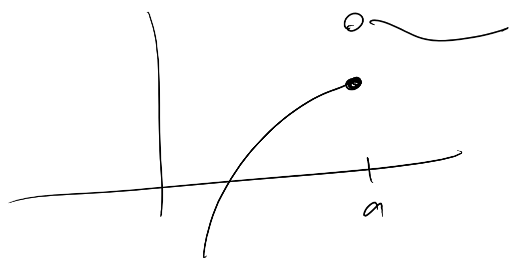
continuous!!

$= 0$

The slope of the tangent line at $x = a$ of $f(x)$

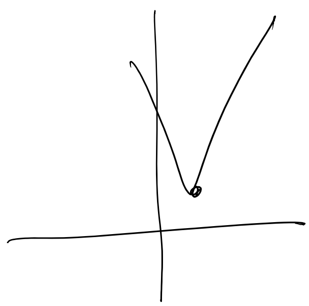
is also called "the derivative" of f(x) at a
written $f'(a)$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$



no reasonable tangent line
"not differentiable"

also, not continuous.



not diff, but continuous.

Differentiable $\Rightarrow$ Continuous.

Suppose $f'(a)$ exists, i.e., $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ exists.

$$\lim_{x\to a} f(x) \overset{?}{=} f(a)$$

$$\lim_{x\to a} \big(f(x) - f(a)\big) = \lim_{h\to 0} f(a+h) - f(a) = \lim_{h\to 0} \left(\frac{f(a+h) - f(a)}{h}\right) \cdot h$$

write $x = a+h$

set $h = x - a$

$$= \left(\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}\right)\left(\lim_{h\to 0} h\right) = f'(a) \cdot 0$$

$$\lim_{x\to a} f(x) - f(a) = 0$$

$$\lim_{x\to a} f(x) - \lim_{x\to a} f(a) = 0$$

$$\left(\lim_{x\to a} f(x)\right) - f(a) = 0$$

$$\lim_{x\to a} f(x) = f(a)$$

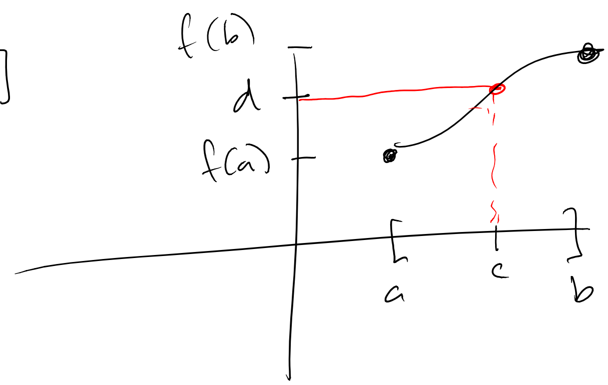
so $f(x)$ is continuous.

Intermediate Value Theorem (IVT)

Suppose $f(x)$ is a continuous function on an interval $[a,b]$

and $f(a) \leq d \leq f(b)$

then, there exists a c in $[a,b]$ such that $f(c) = d$

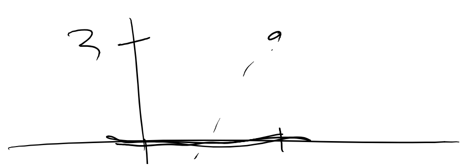


Aside:

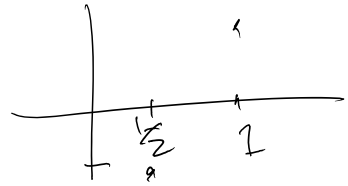
algorithm for solving eqns w/ IVT

$$f(x) = x^3 + 3x^2 + x - 2$$

$f(0) = -2$ $\qquad$ $f(1) = 3$



IVT $\Rightarrow$ have to hit 0 between.



$$f\left(\frac{1}{2}\right) = -\frac{5}{8}$$