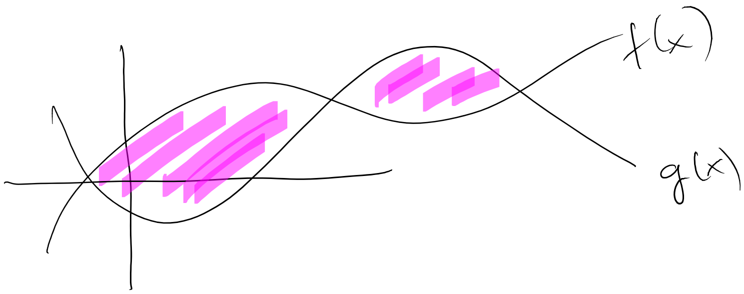
Lecture 21: Areas, volumes, lengths

Thursday, November 5, 2015 9:31 AM

Area between curves



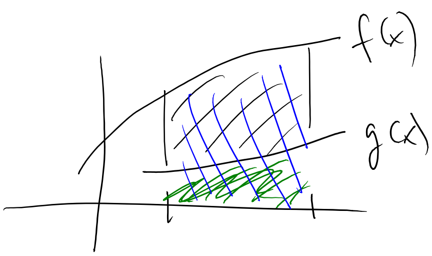
Area between curves

1. Is a positive number.

2. Area $= \int \text{fcn above} - \int \text{fcn below}$

$= \int f_{above} - f_{below}$

(if one is always above & the other below)



between = blue − green

3. Break up into intervals so that 2 applies

ex: the bounded region between $y = x$ & $y = x^2$

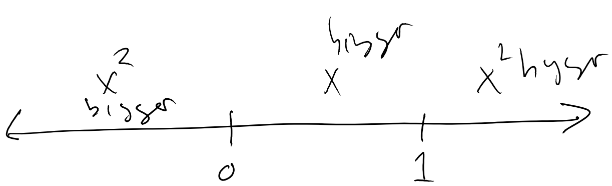
Q: where do these cross?

setting them equal: $x = x^2$

$0 = x^2 - x$

$= x(x-1)$

$x = 0$ or $x = 1$



hmm... when is $x^2 > x$? if x pos, can divide by it.

$\Rightarrow x > 1$

$\int_0^1 (x - x^2)\,dx = \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big]_0^1 = \frac{1}{2}1^2 - \frac{1}{3}1^3 - (0)$

$$\int_0^1 (x - x^2)\,dx = \frac{1}{2}x^2 - \frac{1}{3}x^3\Big]_0^1 = \frac{1}{2}1^2 - \frac{1}{3}1^3 - (0)$$
$$= \frac{1}{6}$$

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1. between $y = \sqrt{x}$ & $y = x^2$ (bounded region)

2. $y = x^2 - 2x$ & $y = x - 2$

might get extra soln's.

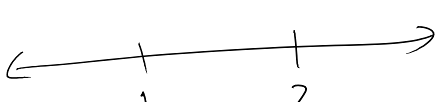
$$\sqrt{x} = x^2$$
$$x = x^4 \longrightarrow x^4 - x = 0$$
$$x(x^3 - 1) = 0$$
$$x = 0 \text{ or } x^3 = 1$$
$$x = 1$$

check these work: yes.

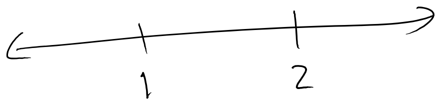
$$\int_0^1 (\sqrt{x} - x)\,dx = \dots = \frac{1}{3}$$

$y = x^2 - 2x$ $\qquad$ $y = x - 2$

$$x^2 - 2x = x - 2$$
$$x^2 - 3x + 2 = 0$$
$$(x-2)(x-1) = 0$$



1, 2



$(x-2)(x-1) = 0$

$x = 1, 2$

$\left(\frac{3}{2}\right)^2 - 2\left(\frac{3}{2}\right)$

$\frac{3}{2} - 2$

$\frac{9}{4} - \frac{6}{2} = \frac{9}{4} - 3$

$-\frac{1}{2}$

$= \frac{9}{4} - \frac{12}{4} = -\frac{3}{4}$

$$\int_1^2 \left((x-2) - (x^2-2x)\right) dx$$

$$= \int_1^2 (-x^2 + 3x - 2)\, dx$$

$$= -\frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x \Big|_1^2$$

$$= \left(-\frac{1}{3}8 + \frac{3}{2}4 - 4\right) - \left(-\frac{1}{3} + \frac{3}{2} - 2\right)$$

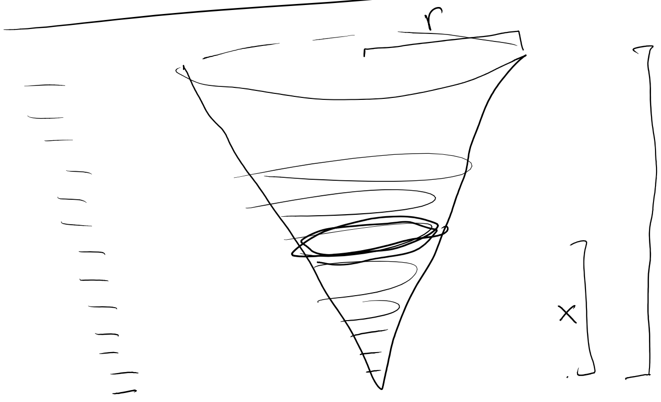
$$= -\frac{8}{3} + 6 - 4 + \frac{1}{3} - \frac{3}{2} + 2$$

$$= -\frac{7}{3} + 4 - \frac{3}{2}$$

$$= -\frac{14}{6} + \frac{24}{6} - \frac{9}{6} = \frac{1}{6}$$

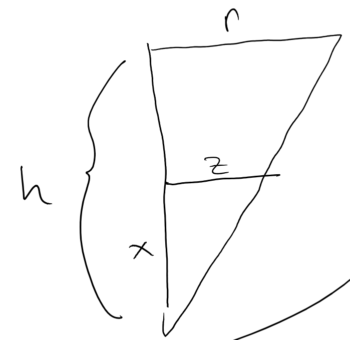
Same ideas for definite integrals can compute volumes, arclengths, surface areas.

Volume of a circular cone.



disk at a ht. of x

has radius $= \frac{r}{h} x$



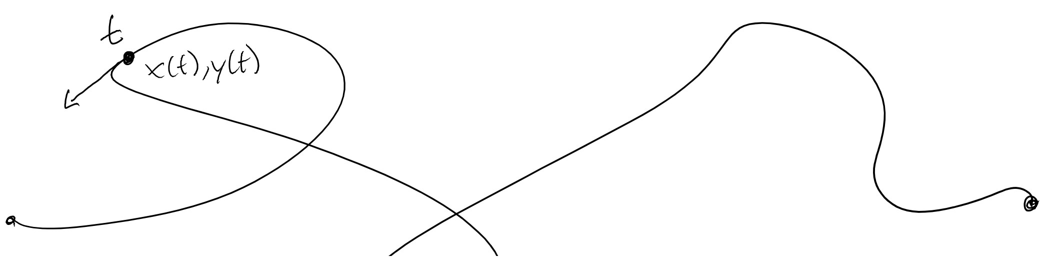
$$\frac{z}{x} = \frac{r}{h} \qquad z = \left(\frac{r}{h}\right) x$$

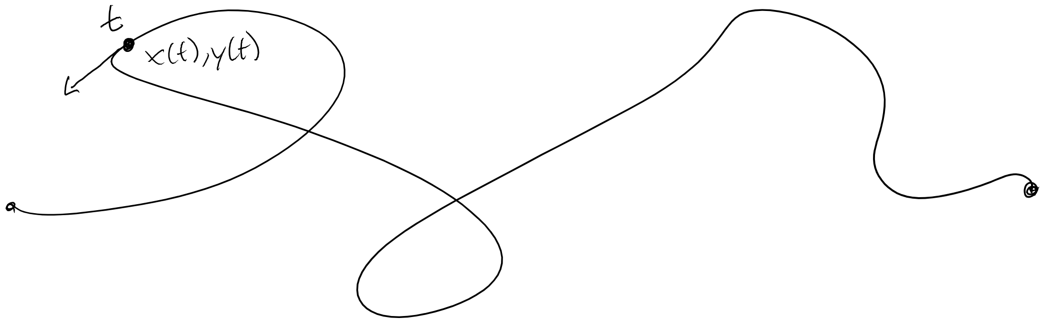
$$\text{Volume} = \int_a^b \text{area}(x)\,dx = \int_0^h \pi \left(\frac{r}{h} x\right)^2 dx$$

$$= \int_0^h \pi \frac{r^2}{h^2} x^2\,dx$$

$$= \pi \frac{r^2}{h^2} \frac{1}{3} x^3 \Big|_0^h$$

$$= \pi \frac{r^2}{h^2} \frac{1}{3} h^3 = \frac{1}{3} \pi r^2 h.$$





$x(t) \qquad y(t) \qquad a \le t \le b$

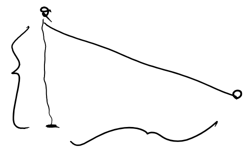
idea: if we go from t to $t+h$

x goes from $x(t)$ to $x(t)+hx'(t)$

y goes from $y(t)$ to $y(t)+hy'(t)$

new length is distance from $(x(t),y(t))$ to

$hy'(t)$



$(x(t)+hx'(t),$
$y(t)+hy'(t))$

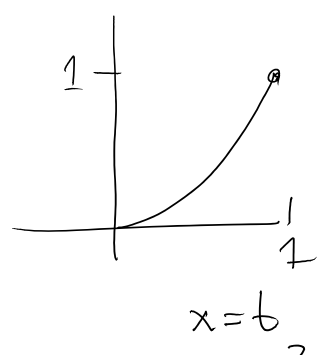
$hx'(t)$

$$\text{hyp} = \text{dist} = \sqrt{h^2x'(t)^2+h^2y'(t)^2}$$

$$\frac{dL}{dt} = \frac{\text{chge in length}}{\text{chge in } t} \approx \frac{h\sqrt{x'(t)^2+y'(t)^2}}{h} \qquad = h\underbrace{\sqrt{x'(t)^2+y'(t)^2}}_{\text{rate of chge of length}}$$

$$\approx \sqrt{x'(t)^2+y'(t)^2}$$

$$\underset{\text{from } a \text{ to } b}{L} = \int_a^b \sqrt{x'(t)^2+y'(t)^2}\,dt$$



$y=x^2$

$x=t$
$y=t^2$

$$L = \int^1 \sqrt{1^2+(2t)^2}\,dt$$

$$L = \int_0 \sqrt{1^2+(2t)^2}\; dt$$

$x = t$

$y = t^2$

$$= \int_0^1 \sqrt{1+(2t)^2}\, dt$$

$2t = \tan y$

$2\,dt = \sec^2 y\, dy$

$dt = \frac{1}{2}\sec^2 y\, dy$

$$= \frac{1}{2}\int \sqrt{1+\tan^2 y}\, \sec^2 y\, dy$$

$$= \frac{1}{2}\int \sec^3 y\, dy$$

$$= \frac{1}{2}\int \sec y(1+\tan^2 y)\, dy$$

$$= \frac{1}{2}\int (\sec y + \sec y \tan^2 y)\, dy$$

$$= \frac{1}{2}\int \sec y\, dy + \frac{1}{2}\int \sec y \tan^2 y\, dy$$

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