Lecture 2: u-substitution for indefinite and definite integrals

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$$\int e^{(x^2)} 2x\,dx$$

$u = x^2 \longrightarrow \dfrac{du}{dx} = 2x \longrightarrow \int e^u \dfrac{du}{\cancel{dx}} \cancel{dx} = \int e^u du$

$du = 2x\,dx$

$$\int e^{\overbrace{(x^2)}^{u}} \underbrace{2x\,dx}_{du} = \int e^u du = e^u + C$$
$$= e^{x^2} + C$$

$$\int \sin(x^2)\,dx = \int \sin u \cdot \frac{du}{2x} = \int \frac{\sin u}{2\sqrt{u}} du$$

$u = x^2 \qquad du = 2x\,dx$

$\dfrac{du}{dx} = 2x \qquad \dfrac{du}{2x} = dx$

$x = \sqrt{u}$

Method of u-substitution:

1. $\int \text{STUFF}(x)\,dx$

look in here, find a chunk, call it u. $u = \text{chunk}$

2. say $du = \dfrac{d}{dx}(\text{chunk})\,dx$

3. Try to rewrite everything in terms of u.

4. Hope we can integrate it.

$$\int \frac{2x}{(x^2-4)^3}\,dx^{du} = \int \frac{1}{u^3}\,du = \int u^{-3}\,du$$

$u = x^2 - 4 \qquad du = 2x\,dx$

$$= \frac{-1}{2} u^{-2} + C$$

$$= \boxed{-\frac{1}{2}(x^2-4)^{-2} + C}$$

$$\frac{1}{-3+1} u^{-3+1} + C$$

$$\int \frac{1}{\sqrt{3x-4}}\,dx = \int (3x-4)^{-1/2}\,dx$$

$u = 3x-4$

$du = 3\,dx$

$$= \int u^{-1/2} \cdot \frac{1}{3}\,du = \frac{1}{3}\int u^{-1/2}\,du = \frac{1}{3}\,\frac{1}{-1/2+1}\,u^{-1/2+1} + C$$

$$= \frac{1}{3}\,2\,u^{1/2} + C$$

$$= \frac{2}{3}\sqrt{3x-4} + C$$

$$\int\left(\frac{4\ln\sqrt{x}}{x}+\cos x\right)dx$$

$$=\int\frac{4\ln\sqrt{x}}{x}\,dx+\int\cos x\,dx$$

$$=4\int\frac{\ln\sqrt{x}}{x}\,dx+\int\cos x\,dx$$

$$=4\int\frac{\ln x^{1/2}}{x}\,dx+\sin x+C$$

$$=4\cdot\frac{1}{2}\int\frac{\ln x}{x}\,dx+\sin x+C$$

$$\left[\int\frac{\ln x}{x}\,dx=\int u\,du=\frac{1}{2}u^2+C=\frac{1}{2}(\ln x)^2+C\right]$$

$u=\ln x$

$du=\frac{1}{x}dx$

$$=2\cdot\frac{1}{2}(\ln x)^2+\sin x+C$$

$$=\boxed{(\ln x)^2+\sin x+C}$$

Practice

1. $\int 4\sin x\sqrt{3+\cos x}\,dx$

2. $\int(11\sin x\,\sin(\cos x)-2x)\,dx$

2. $\int (\sin x \sin(\cos x) - 2x)\, dx$

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1. $\int 4\sin x \sqrt{3+\cos x}\, dx = \int u^{1/2} (-4)\, du$

$u = 3+\cos x$

$du = -\sin x\, dx$

$-4\, du = 4\sin x\, dx$

$= \frac{2}{3}(-4)\, u^{3/2} + C$

$= -\frac{8}{3}(3+\cos x)^{3/2} + C$

2. $\int (\sin x \sin(\cos x) - 2x)\, dx$

$(-1)\,du$

$\int \sin x \sin(\underbrace{\cos x}_{u})\, dx - \int 2x\, dx$

$u = \cos x$

$du = -\sin x\, dx$

$(-1)\, du = \sin x\, dx \rightarrow \int \sin(u)(-1)\, du - x^2 + C$

$= -(-\cos u) - x^2 + C$

$= \boxed{\cos(\cos x) - x^2 + C}$

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u-substitution for definite $\int$'s.

$$\int_0^1 x\sqrt{1-x^2}\,dx$$

1. Find an anti-dv

$$\int x\sqrt{1-x^2}\,dx = \int \sqrt{1-x^2}\;x\,dx$$

$u = 1-x^2$ $\qquad = \int u^{1/2}\left(-\tfrac{1}{2}\right)du$

$du = -2x\,dx$

$-\tfrac{1}{2}du = x\,dx$ $\qquad = -\tfrac{1}{2}\,\tfrac{2}{3}u^{3/2} + C$

$$= \underbrace{-\tfrac{1}{3}\left(1-x^2\right)^{3/2} + C}_{\text{all anti-dv's.}}$$

$\Downarrow$ same antiderivative

$$2.\ \int_0^1 x\sqrt{1-x^2}\,dx = \left[-\tfrac{1}{3}\left(1-x^2\right)^{3/2}\right]_0^1 =$$

$$\left(-\tfrac{1}{3}(1-1)^{3/2}\right) - \left(-\tfrac{1}{3}(1-0^2)^{3/2}\right)$$

$$= \boxed{\tfrac{1}{3}}$$

Short way

$x=1$ $\qquad u=0$

Shot way

$$\int_{x=0}^{x=1} x\sqrt{1-x^2}\,dx = \int_{x=0}^{x=1} u^{1/2}\left(-\tfrac{1}{2}\right)du = -\frac{1}{2}\int_{u=1}^{u=0} u^{1/2}\,du$$

$u = 1-x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$x=0 \rightsquigarrow u = 1-0^2 = 1$

$x=1 \rightarrow u = 1-1^2 = 0$

pointless.

$$\frac{1}{2}\int_{u=0}^{u=1} u^{1/2}\,du = \left[\frac{1}{2}\,\frac{2}{3}\,u^{3/2}\right]_0^1$$

$$= \left[\frac{1}{3}1^{3/2} - \frac{1}{3}0^{3/2}\right] = \frac{1}{3}$$

$$\int_0^{\pi/2} \sin x\, e^{\cos x}\,dx = -\int_{x=0}^{x=\pi/2} e^u\,du = -\int_{u=1}^{u=0} e^u\,du$$

$u = \cos x$

$du = -\sin x\,dx$

$(-1)\,du = \sin x\,dx$

$x=0 \to u = \cos 0 = 1$

$x = \pi/2 \to u = \cos\frac{\pi}{2} = 0$

$$-\left[e^u\right]_1^0 = -\left(e^0 - e^1\right)$$

$$= -(1-e)$$

$$= \boxed{e-1}$$