COMBINATORICS, SUPPLEMENTARY

DANIEL KRASHEN

Contents

1. Distribution problems and multisets

1.1. **definitions.** Let's start with some examples. Here are some sets with elements taken from $1, 2, 3$:

$$\{1,2\} \qquad \{1,3\} \qquad \{1\} \qquad \{1,2,3\}$$

and here are some multisets with elements taken from $1, 2, 3$:

$$\{1,1,2\} \qquad \{3\} \qquad \{1,2,2,3,3,3,3,3,3,3\}$$

all of these are generally referred to as "multi-subsets" of the set $\{1, 2, 3\}$.

To move towards a precise definition, consider an alternative way of representing the multisets above. Let's try the informal notation

$$\{1,1,2\} = \{1(2 \text{ times}), 2(1 \text{ time})\}$$

or

$$\{1,2,2,3,3,3,3,3,3,3\} = \{1(1 \text{ time}), 2(2 \text{ times}), 3(7 \text{ times})\}$$

In this way, it makes sense to think of a multiset as a set, together with an assignment of a "multiplicity" to each object (how many times the object occurs).

We can therefore formally define

Definition 1.1. *A multiset is a pair (S, m) consisting of a set S and a function $m : S \to \mathbb{Z}_{>0}$.*

For example, the multiset $\{1, 1, 2\}$ we think of as the pair $(\{1, 2\}, m)$, where $m(1) = 2$ and $m(2) = 1$. Since notationally it is easier to write $\{1, 1, 2\}$, we will generally stick to this notation for multisets.

Note that just as for sets, the order in which we write our elements doesn't matter. That is to say, $\{1, 1, 2\} = \{1, 2, 1\} = \{2, 1, 1\}$.

1.2. **counting trick example.** Let's count the number of multisets having 10 elements, taken from the set $\{1, 2, 3\}$. For example, here are a few:

$$\{1, 1, 1, 2, 2, 2, 2, 3, 3, 3\}, \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}, \{1, 1, 2, 3, 3, 3, 3, 3, 3, 3\}$$

How do we count these? Here's a simple trick: if we write the numbers which occur within the multiset in order (as we have done above), We can easily fill in which numbers are which, if we know the "transition points in the sequence." That is, if we imagine putting "separators" into our sequences above, to yield:

$$(111s2222s333), (1111111111ss), (11s2s33333333)$$

(where s stands for "separator"). To be clear, the rule here is:

(1) the items to the left of all separators are 1's
(2) the items between the separators are 2's
(3) the items to the right of all separators are 3's

Note that with these rules, we really didn't need to specify our numbers at all. That is to say, if n stands for "number," we could represent the above sequences as:

$$(nnnsnnnnsnnn), (nnnnnnnnnnss), (nnsnsnnnnnnn)$$

and we can easily fill in the numbers as above. This tells us that what we are really doing is choosing which 2 or our 12 symbols should be s. Therefore, the number of ways of choosing these are $\binom{12}{2}$.