Lecture 6: cohomology and the connecting map

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E/F G Galois

$V/F \rightsquigarrow V\otimes E$ w/ a G action

$(V\otimes E)^G \hookleftarrow$

V/F $\qquad\qquad$ W/E w/ "semilinear G action"

$$V \longmapsto V\otimes E$$

$$W^G \longleftarrow W$$

$V = F^n \qquad V\otimes E = E^n \qquad \mathrm{End}_E(V\otimes E) = E^{n^2}$

action of G "coordinate wise" on $\mathrm{End}_E(V\otimes E) = E^{n^2}$

can be thought of as $\quad \sigma(f) = \sigma\circ f\circ \sigma^{-1}$

$f = xe_{ij} \qquad \sigma(f)(e_k) = \sigma f \sigma^{-1}(e_k)$

$\qquad\qquad\qquad\qquad\qquad = \sigma(f(e_k)) = \sigma(xe_{ij}e_k)$

$\sigma(x)\, e_{ij}(e_k)$

$\qquad\qquad\qquad\qquad\qquad = \sigma(x\,\delta_{jk}e_i)$

$\quad = \sigma(x)\,\delta_{jk}e_i$

$\qquad\qquad\qquad\qquad\qquad = \sigma(x)\,\delta_{jk}e_i$

Given "model" algebra A_0/F $\qquad (M_n(F))$

Classify A/F's s.t. $A\otimes_F E \cong A_0\otimes_F E$ $\qquad$ (CSA/F split by E deg n)

if $\varphi : A \otimes E \longrightarrow A_0 \otimes E \circlearrowleft$

transport action of G on left to right

$$\sigma \circ x = \varphi \sigma \varphi^{-1}(x)$$

set $b(\sigma) = \varphi \sigma \varphi^{-1} \sigma^{-1} \in \mathrm{Aut}_E(A_0 \otimes E)$

$$\sigma \circ x = b(\sigma) \circ \sigma(x)$$

$$\boxed{b(\sigma\tau) = b(\sigma)\,\sigma(b(\tau))}$$

$$\sigma \circ (\tau \circ x) = \sigma\tau \circ x$$

also, modify φ by

$$A \otimes E \xrightarrow[\varphi]{\sim} A_0 \otimes E \circlearrowleft a \in \mathrm{Aut}(A_0 \otimes E)$$

set $\varphi' = a^{-1}\varphi$

$$\varphi' \sigma \varphi'^{-1} \sigma^{-1} = a^{-1} \varphi \sigma (a^{-1}\varphi)^{-1} \sigma^{-1} = a^{-1} \varphi \sigma \varphi^{-1} a \sigma^{-1}$$

$$= a^{-1} \underbrace{\varphi \sigma \varphi^{-1} \sigma^{-1}}_{b(\sigma)} \underbrace{\sigma a \sigma^{-1}}_{\sigma(a)}$$

$$= a^{-1} b(\sigma) \sigma(a)$$

$b \sim b'$ if $b'(\sigma) = a^{-1} b(\sigma) \sigma(a)$ some $a \in \mathrm{Aut}(A_0 \otimes E)$

<u>Def</u> X is a group w/ action of G

$$Z^1(G, X) = \{ \text{Maps } G \xrightarrow{b} X \mid b(\sigma\tau) = b(\sigma)\sigma(b(\tau)) \}$$

$b \sim b'$ if $\exists\, x \in X$ s.t. $b'(\sigma) = x^{-1}\, b(\sigma)\, \sigma(x)$
all $\sigma \in G$

$H^1(G, X)$ = eq. classes.

<u>In particular:</u>

$$\{ \text{CSA}/F \text{ of deg } n \text{ w/ sp. field } E/F \}/_{\text{iso}} \longleftrightarrow H^1(G, \mathrm{Aut}_E(M_n(E)))$$

$$\begin{array}{ccc} \mathrm{Aut}_E(M_n(E)) & \twoheadleftarrow & GL_n(E) \\ (x \mapsto TxT^{-1}) & \longleftarrow & T \end{array}$$

ker = central matrices
= scalars
= E^*

<u>Def</u> $PGL_n(E) = GL_n(E)/E^*$

$$H^1(G, \mathrm{Aut}_E(M_n(E))) = H^1(G, PGL_n(E))$$

<u>Recall:</u> $(E, G, c) = \bigoplus_{\sigma \in G} E u_\sigma$

$$u_\sigma u_\tau = c(\sigma,\tau)\, u_{\sigma\tau}$$

$$u_\sigma(u_\tau u_\gamma) = (u_\sigma u_\tau) u_\gamma = c(\sigma,\tau)\, u_{\sigma\tau} u_\gamma$$

$$u_\sigma(u_\tau u_\gamma) = u_\sigma\, c(\tau,\gamma)\, u_{\tau\gamma}$$

$$c(\sigma,\tau)\, u_{\sigma\tau} u_\gamma = c(\sigma,\tau)\, c(\sigma\tau,\gamma)\, u_{\sigma\tau\gamma}$$

$=$

$$\sigma(c(\tau,\gamma))\, u_\sigma u_{\tau\gamma} = \sigma(c(\tau,\gamma))\, c(\sigma,\tau\gamma)\, u_{\sigma\tau\gamma}$$

$$c(\sigma,\tau)\, c(\sigma\tau,\gamma) = c(\sigma,\tau\gamma)\, \sigma(c(\tau,\gamma))$$

$$u_\sigma \rightsquigarrow v_\sigma = b(\sigma)\, u_\sigma$$

$$\begin{aligned} v_\sigma v_\tau = b(\sigma)\, u_\sigma\, b(\tau)\, u_\tau &= b(\sigma)\,\sigma(b(\tau))\, u_\sigma u_\tau \\ &= b(\sigma)\,\sigma(b(\tau))\, c(\sigma,\tau)\, u_{\sigma\tau} \\ &= b(\sigma)\,\sigma(b(\tau))\, c(\sigma,\tau)\, b(\sigma\tau)^{-1}\, v_{\sigma\tau} \end{aligned}$$

$$c'(\sigma,\tau) = b(\sigma)\,\sigma(b(\tau))\, b(\sigma\tau)^{-1}\, c(\sigma,\tau)$$

$c \sim c'$ if $\exists\, b: G \to E^*$ st

eq. classes form a gp $\quad H^2(G,E^*) = B(E/F)$

Abstract H^2:

X Ab. gp. w/ G action

$$Z^2(G,X) = \{ c: G\times G \to X \mid c(\sigma,\tau)\, c(\sigma\tau,\gamma) = c(\sigma,\tau\gamma)\,\sigma(c(\tau,\gamma)) \}$$

$$\sim \quad C^1(G,X) = \mathrm{Maps}(G,X)$$

$$b \in C^1(G,X), \quad \partial b(\sigma,\tau) = b(\sigma)\,\sigma(b(\tau))\, b(\sigma\tau)^{-1}$$

$$B^2(G,X) = \mathrm{im}\,(C^1(G,X))$$

$$H^2(G,X) = \frac{Z^2(G,X)}{B^2(G,X)}$$

X set w/ G action

$$H^0(G,X) = Z^0(G,X) = \{ x \in X \mid \cancel{x^{-1}\sigma(x)=1} \ \sigma(x)=x \}$$
$$= X^G$$

Long exact Sequences

<u>Thm</u> Given $1 \to A \to B \to C \to 1$ groups w/ G action,

get a l.e.s.

$$1 \to H^0(G,A) \to H^0(G,B) \to H^0(G,C) \xrightarrow{\delta_0}$$
$$H^1(G,A) \to H^1(G,B) \to H^1(G,C) \xrightarrow{\delta_1}$$
$$H^2(G,A) \to H^2(G,B) \to \cdots$$

stop here unless $A \subset Z(B)$

stop here unless B is Abelian

<u>Rem</u> if X, Y, Z pointed sets, we say

$X \xrightarrow{f} Y \xrightarrow{g} Z$ exact if $\ker g = \operatorname{im} f$

inv. im. of pt $g^{-1}(\text{pt in } Z)$

in particular $1 \to X \xrightarrow{f} Y$ exact

doesent mean f is injective!

$X \xrightarrow{f} Y \to 1$ exact ,

does mean f surjective.

What are δ's when gps not Abelian?

$\delta_0 : H^0(G,C) \longrightarrow H^1(G,A)$

$\quad \| $

$C^G \ni c$, chose $b \mapsto c$, $\partial b(\sigma) = b^{-1}\sigma(b)$

know $\partial b(\sigma) \longrightarrow 1 \in C$

$\Rightarrow \partial b(\sigma) \in A$ $\quad$ define $a(\sigma) \equiv \partial b(\sigma) = b^{-1}\sigma(b)$

$a(\sigma\tau) = b^{-1}\sigma\tau(b) = b^{-1}\sigma(b)\sigma(b^{-1})\sigma\tau(b)$

$= b^{-1}\sigma(b)\sigma\left(b^{-1}\tau(b)\right) = a(\sigma)\sigma\left(a(\tau)\right)$ ✓

δ^1: Assume $A \subset Z(B)$ $\quad$ choose $c \in Z^1(G,C)$ $\quad$ $H^2(G,C)$

pick $b \in C^1(G,B)$ $\quad$ $b(\sigma) \in B \mapsto c(\sigma) \in A$

$\partial b(\sigma,\tau) = b(\sigma)\sigma(b(\tau))b(\sigma\tau)^{-1} \in C^2(G,B)$

$\longmapsto 1$ in $C^2(G,C)$

$\Rightarrow \partial b(\sigma,\tau) \in C^2(G,A)$

$\quad \|$

$a(\sigma,\tau)$

$a(\sigma,\tau)\, a(\sigma\tau,\gamma) \overset{?}{=} a(\sigma,\tau\gamma)\,\sigma\left(a(\tau,\gamma)\right)$

$\quad \|$

$b(\sigma)\sigma(b(\tau))\,b(\sigma\tau)^{-1}\, b(\sigma\tau)\,\sigma\tau(b(\gamma))\,b(\sigma\tau\gamma)^{-1}$

$\quad \|$

$b(\sigma)\sigma(b(\tau))\,\sigma\tau(b(\gamma))\,b(\sigma\tau\gamma)^{-1}$

$$b(\sigma)\underbrace{\sigma(b(\tau))\,\sigma\tau(b(\gamma))\,\sigma(b(\tau\gamma))^{-1}}\sigma(b(\tau\gamma))\;b(\sigma\tau\gamma)$$

$$b(\sigma)\;\sigma\Big[b(\tau)\underbrace{\tau(b(\gamma))\,b(\tau\gamma)^{-1}}_{a(\tau,\gamma)}\Big]\,\sigma(b(\tau\gamma))\,b(\sigma\tau\gamma)^{-1}$$

$$= b(\sigma)\,\sigma(b(\tau\gamma))\,b(\sigma\tau\gamma)^{-1}\;\sigma(a(\tau,\gamma))$$

$$= a(\sigma,\tau\gamma)\;\sigma(a(\tau,\gamma)) \quad \square$$

Specialize to our friend:

$$1 \to E^* \to GL(V\otimes E) \to PGL(V\otimes E) \to 1$$

we get $H^1(G, PGL(V\otimes E)) \longrightarrow H^2(G, E^*)$

$H^1(G,PGL(V\otimes E))$ = CSA's $\deg = \dim V$, split by E

$H^2(G,E^*) = Br(E/F)$

Fix $n = [E:F] = \dim V$

Claim: this is surjective

Pick $c \in Z^2(G, E^*)$ let e_σ be a basis for V (indexed by G)

define $b \in C^1(G, GL(V\otimes E))$ via $b(\sigma)(e_\tau) = c(\sigma,\tau)e_{\sigma\tau}$

$$b(\sigma)\sigma(b(\tau))(e_\gamma) = b(\sigma)(\sigma\, b(\tau)\,\sigma^{-1}(e_\gamma))$$

$$= b(\sigma)(\sigma(b(\tau)e_\gamma)) = b(\sigma)(\sigma(c(\tau,\gamma)e_{\tau\gamma}))$$

$$= b(\sigma)\;\sigma(c(\tau,\gamma))\,e_{\tau\gamma}$$

$$= \sigma(c(\tau,\gamma))\;c(\sigma,\tau\gamma)\;e_{\sigma\tau\gamma}$$

$= c(\sigma,\tau)\, c(\sigma\tau,\gamma)\, e_{\sigma\tau\gamma}$

$= c(\sigma,\tau)\, b(\sigma\tau)\, e_{\gamma}$

$b(\sigma)\sigma(b(\tau)) = c(\sigma,\tau) b(\sigma\tau)$ $\quad b(\sigma)\sigma(b(\tau))b(\sigma\tau)^{-1} \sim c(\sigma,\tau)$

$\Downarrow$

$\overline{b(\sigma)}\ \overline{\sigma(b(\tau))} = \overline{b(\sigma\tau)}$ $\quad / E^*$

$\partial b = c$

b a lift of $\overline{b} \in Z^1(G, PGL)$

tweaky stand. gal action on $\operatorname{End}_E(V\otimes E)$ by $\overline{b}$

$f \longmapsto \overline{b} \circ \sigma(f) = b(\sigma)\circ\sigma(f)\circ b(\sigma)^{-1}$

fixed stuff is a CSA, want to compare w/ (E, G, c)

$\operatorname{End}_E(V\otimes E)^{G,\overline{b}} = \{ f \mid b(\sigma)\sigma(f) = f\, b(\sigma) \text{ all } \sigma\in G\}$

if $\sigma \in G$, define $y_\sigma \in \operatorname{End}_E(V\otimes E)$ via

$y_\sigma(e_\tau) = c(\tau,\sigma)\, e_{\tau\sigma}$

if $x \in E$, $v_x \in \operatorname{End}_E(V\otimes E)$ via $v_x(e_\tau) = \tau(x)\, e_\tau$

these are fixed! $\quad b(\sigma)\sigma(v_x) = v_x b(\sigma)$

$v_x b(\sigma)(e_\tau) = v_x(c(\sigma,\tau) e_{\sigma\tau}) = c(\sigma,\tau) v_x(e_{\sigma\tau})$

$= c(\sigma,\tau)\, \sigma\tau(x)\, e_{\sigma\tau}$ ✓

$b(\sigma)\sigma(v_x)(e_\tau) = b(\sigma)\left(\sigma(v_x(\sigma^{-1}e_\tau))\right) = b(\sigma)(\sigma(v_x e_\tau))$ $\quad \|$

$b(\sigma)(\sigma(\tau(x) e_\tau))$

$$b(\sigma)\sigma(v_x)(e_\tau) = b(\sigma)(\sigma v_x \sigma^{-1}(e_\tau)) = b(\sigma)(\sigma v_x e_\tau)$$

$$= b(\sigma)(\sigma(\tau(x) e_\tau))$$

$$= b(\sigma)(\sigma\tau(x) e_\tau)$$

$$= \sigma\tau(x)\, b(\sigma)\, e_\tau =$$

$$\sigma\tau(x)\, c(\sigma,\tau)\, e_{\sigma\tau} \quad \checkmark$$

$$y_\tau b(\sigma) \overset{?}{=} b(\sigma)\sigma(y_\tau)$$

$$y_\tau\, b(\sigma)\, (e_\gamma) = y_\tau(c(\sigma,\gamma)\, e_{\sigma\gamma})$$

$$= c(\sigma,\gamma)\, c(\sigma\gamma,\tau)\, e_{\sigma\gamma\tau} \quad \checkmark$$

$$b(\sigma)\sigma(y_\tau)(e_\gamma) = b(\sigma)(\sigma y_\tau \sigma^{-1}(e_\gamma))$$

$$= b(\sigma)(\sigma y_\tau(e_\gamma)) = b(\sigma)(\sigma(c(\gamma,\tau)\, e_{\gamma\tau}))$$

$$= b(\sigma)(\sigma(c(\gamma,\tau))\, e_{\gamma\tau})$$

$$= \sigma(c(\gamma,\tau))\, b(\sigma)\, e_{\gamma\tau} = \sigma(c(\gamma,\tau))\, c(\sigma,\gamma\tau)\, e_{\sigma\gamma\tau} \quad \checkmark$$

$$(E,G,c)^{op} \xrightarrow{\sim} (\mathrm{End}(V \otimes E))^{G,b}$$

$$x u_\sigma \longmapsto v_x \circ y_\sigma$$

$$H^1(G, PGL_n) \longrightarrow H^2(G, E^\times) \cong Br(E/F) \subset Br(F)$$

$$A \text{ csa} \rightsquigarrow [A^{op}]$$

Operations

Given $a \in GL(V)$ $\quad b \in GL(W)$

define $a \otimes b \in GL(V \otimes W)$ by $a \otimes b(v \otimes w) = a(v) \otimes b(w)$

Gives a hom $GL(V) \times G(W) \longrightarrow G(V \otimes W)$ of groups.

$$(a,b) \quad (a',b') \quad (a \otimes b)(a' \otimes b')(v \otimes w)$$
$$= aa'v \otimes bb'w$$

if $\bar{a} \in PGL(V)$ $\quad \bar{b} \in PGL(W)$

define $\bar{a} \otimes \bar{b} = \overline{a \otimes b} \in PGL(V \otimes W)$

$$GL(V) \xrightarrow{\subset \Delta} \underbrace{GL(V) \times GL(V) \times \cdots \times GL(V)}_{k \text{ times}} \to GL(V^{\otimes k})$$

induce a hom $PGL(V) \longrightarrow PGL(V^{\otimes k})$

$$\bar{a} \longmapsto \overline{a \otimes a \otimes \cdots \otimes a}$$
$$[A] \longmapsto k[A]$$

Given $\bar{a} \in Z^1(G, PGL(V \otimes E))$

$A \longleftrightarrow \bar{a}$

$B \longleftrightarrow \bar{b} \in Z^1(G, PGL(W \otimes E))$

define $\bar{a} \otimes \bar{b} \in Z^1(G, PGL(V \otimes W \otimes E))$

$$\bar{a} \otimes \bar{b}(\sigma) = \bar{a}(\sigma) \otimes \bar{b}(\sigma)$$

Observation: $\bar{a} \otimes \bar{b}$ is a cocycle, and describes the action of G on $A \otimes B$

so

$$[A] \longleftrightarrow a \in H^1(G, PGL\,V)$$
$$\rightsquigarrow a \otimes b \in H^1(G, PGL(V \otimes W))$$
$$[B] \longleftrightarrow b \in H^1(G, PGL\,W)$$
$$[A \otimes B]$$

Suppose we have $\bar{b} \in Z^1(G, PGL(V \otimes E))$

and $V_E = (W_1)_E \oplus (W_2)_E$ s.t. $b(\sigma) = \begin{bmatrix} b_1(\sigma) & 0 \\ 0 & b_2(\sigma) \end{bmatrix}$

$$b_i(\sigma) \in GL(W_i \otimes E)$$

$$\partial b(\sigma,\tau) = \begin{bmatrix} \partial b_1(\sigma,\tau) & 0 \\ 0 & \partial b_2(\sigma,\tau) \end{bmatrix} = \begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix}$$

$\quad \parallel$

scalar

$\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}$

$\lambda \in E^*$

$$\lambda = \partial b_i$$

$$\partial b_i = \partial b$$

$\Rightarrow \bar{b}_i \in H^1(G, PGL(W_i))$ rep. something Br eq. to b.

$$\Lambda^k V \subset \otimes^k V \supset \mathrm{Rest}^k V$$

$$T \circlearrowright V \qquad T(\Lambda^k V) \subset \Lambda^k V$$

$$PGL(V) \longrightarrow PGL(\otimes^k V)$$

$$PGL(\Lambda^k V \oplus \mathrm{Rest}^k V)$$

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \xrightarrow{\partial} \begin{bmatrix} \boxed{\ddots} & \\ & \ddots \end{bmatrix}$$

$$\begin{tikzcd}
H^1(G, PGL(V)) \arrow[r] \arrow[dr, squiggly] & H^1(G, PGL(\otimes^k V)) \\
& H^1(G, PGL(\Lambda^k V))
\end{tikzcd}$$

$\bar{a} \in H^1(G, PGL(V))$, $\bar{a} \leftarrow A$

$\bar{a} \otimes \cdots \otimes \bar{a} \leftrightarrow k[A]$

i.e. kth power is rep by somthg in

$$H^1(G, PGL(\Lambda^k V))$$

if $n = \dim V \Rightarrow$ nth power rep $H^1(G, PGL(\Lambda^n V))$

$$H^1(G, PGL_1(E))$$

$\{1\}$

$\{F\}$

$\Rightarrow n[A] = 0.$

per $A \mid$ ind A.