(YOU COULD HAVE INVENTED) GALOIS DESCENT

1. SEMILINEAR SPACES (THE DESCENT DATA CATEGORY)

Exercise 1. If E/F is a G-Galois extension of fields, show that the natural map

$$\begin{aligned}(E, G, 1) &\to \mathrm{End}_F(E)\\ x &\mapsto [y \mapsto xy], \quad x, y \in E\\ u_\sigma &\mapsto [y \mapsto \sigma(y)], \quad \sigma \in G, y \in E\end{aligned}$$

gives an isomorphism of algebras.

Definition 1.1. Recall that if E/F is a G-Galois extension, an E/F-semilinear vector space is an E-vector space V together with an action of G on V such that for every $x \in E$, $v \in V$, we have

$$\sigma(xv) = \sigma(x)\sigma(v).$$

A homomorphism of E/F semilinear vector spaces $\phi : V \to W$ consists of an E-linear map ϕ which commutes with the G-action in the sense that $\phi(\sigma v) = \sigma(\phi v)$.

Exercise 2. If V is an F-vector space then $E \otimes_F V$ is naturally an E/F semilinear vector space, where the action of G is via the first factor.

Exercise 3. Show that we have an equivalence of categories between $(E, G, 1)$-modules and E/F-semilinear vector spaces.

Recall the following result which we claimed in the last lecture:

Proposition 1.2 (Morita)**.** *Let R be a ring and P a right R-progenerator (i.e. finitely generated, projective generator in the category of right R-modules). Let $S = \mathrm{End}_R(P)$. Then the functor from R-modules to S-modules given by*

$$N \mapsto P \otimes_R N$$

is an equivalence of categories. Further, if $P^\star = Hom_R(P, R)$ then $P^\star$ is an $R-S$ bimodule, and

$$M \mapsto P^\star \otimes_S M$$

gives the (homotopy) inverse equivalence.

Exercise 4. Show that the functor from F-vector spaces to E/F-semilinear vector spaces given by

$$V \mapsto V_E \equiv E \otimes_F V$$

is an equivalence of categories.

Now, if we are interested in talking about algebraic objects (such as central simple algebras), we need more than just vector spaces and linear maps, but we also need the concept of the tensor product (for multiplicative structures).

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Definition 1.3. Suppose V, W are E/F semilinear vector spaces. Then $V \otimes_E W$ is also a semilinear vector space with respect to the action:

$$\sigma(v \otimes w) = \sigma(v) \otimes \sigma(w).$$

Exercise 5. Show that the above definition gives a well defined E/F semilinear space and that this commutes with the functor given above.

That is, show that if V, W are F-vector spaces, then we have a natural isomorphism of E/F semilinear vector spaces

$$V_E \otimes_E W_E \cong (V \otimes_F W)_E.$$

More formally (if you like), this means you are showing that the two functors

$$(V, W) \mapsto (V_E \otimes_E W_E) \quad (V_W) \mapsto (V \otimes_F W)_E$$

from $Vec/F \times Vec/F$ to the category of E/F semilinear vector spaces are naturally isomorphic.

From this point of view it makes sense to talk about E/F semilinear algebras.

Definition 1.4. An E/F semilinear algebra is an E/F semilinear vector space A, together with an E/F-semilinear map

$$m : A \otimes_E A \to A$$

and an E/F-semilinear map

$$\iota : E \to A$$

which gives A the structure of an algebra (where $\iota(1) = 1$ is the multiplicative identity of A).

Exercise 6. Show that an E/F semilinear algebra is just an E-algebra A with a semilinear action of G on A as a vector space such that $\sigma(ab) = \sigma(a)\sigma(b)$ (i.e. such that G acts via ring isomorphisms).

Exercise 7. Show that we have an equivalence of categories between F-algebras and E/F-semilinear algebras given by $A \mapsto E \otimes_F A$.

Exercise 8. It follows from the above exercise that if we let $F = \mathbb{R}$ and $E = \mathbb{C}$, then we have an equivalence between $\mathbb{R}$-algebras and $\mathbb{C}$-algebras with a notion of conjugation (action by $\mathfrak{Gal}(\mathbb{C}/\mathbb{R})$). In particular, if we consider the $\mathbb{R}$-algebras $\mathbb{H}$ and $M_2(\mathbb{R})$, we see that

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong M_2(\mathbb{C}) \cong \mathbb{C} \otimes_{\mathbb{R}} M_2(\mathbb{R})$$

and so as $\mathbb{C}/\mathbb{R}$ semilinear algebras, both of these algebras are given as $M_2(\mathbb{C})$ with two different notions of conjugation. What are these notions of conjugation?

2. Twisted forms (the gluing problem)

Throughout the section, let us fix E/F a G-Galois extension.

Definition 2.1. Let A be an F-algebra. We say that an F-algebra B is a(n E/F-)twisted form of A if there is an isomorphism of E-algebras, $A_E \cong B_E$.

Note that we are not assuming here that we have an isomoprhism of E/F semilinear algebras (which would imply they were isomorphic over F), but just as E-algebras.

As we saw in the previous section, we can recover the structure of B from B_E by specifying a semilinear action. As we are able to identify A_E and B_E, our quest to understanding the possible B's we may have then reduces to understanding all possible semilinear actions of G on A_E.

Definition 2.2. Suppose V is a vector space with an action of G. We define an action of G on $\mathrm{Aut}(V)$ by $(\sigma\phi)(v) = \sigma(\phi(\sigma^{-1}(v))$.

Exercise 9. Show that in the case $V = E^n$, with component-wise action, the action of the Galois group $G = \mathcal{G}\mathrm{al}(E/F)$ on $Aut(V) = GL_n(E)$ is given by the standard action on the matrix entries.

Exercise 10. Suppose $\phi, \psi : G \to \mathrm{Aut}(A_E)$ are two different semilinear actions of G on A_E. That is, for $\sigma \in G$, we have $\sigma(a) \equiv \phi(\sigma)(a)$ and $\sigma(a) \equiv \psi(\sigma)(a)$ define semilinear actions (note here that ϕ and ψ need not have values in E-automorphisms, but in general just F-linear automorphisms).

Show that $\phi(\sigma) = \alpha(\sigma)\psi(\sigma)$ for a map $\alpha : G \to \mathrm{Aut}(A_E)$ and α is a crossed homomorphism (where the action of G on $\mathrm{Aut}(A_E)$ here is given by the previous excercise via ψ).

Exercise 11. Show that the above correspondence gives, after fixing an algebra A/F a bijection between semilinear actions on A_E and crossed homomorphisms $G \to \mathrm{Aut}_F(A_E)$.

From this we see so far that for B/F a twisted form of A, given an isomorphism $\phi : B_E \to A_E$, we obtain a new semilinear action on A_E which corresponds to the algebra B/F via the equivalence of categories previously described. This semilinear action, in turn gives rise to crossed homomorphism $G \to \mathrm{Aut}(A_E)$.

It therefore is natural to ask: in what way does this semilinear action depend on the isomorphism ϕ?