NOTES FOR
GEOMETRY OF DIVISION ALGEBRAS

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Contents

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1. Central simple algebras and Azumaya algebras

1.1. Central simple algebras.

Definition 1.1. We say that an algebra A over a field F is a central simple F-algebra if

- $Z(A) = F$, and $\dim_F A < \infty$ (A is F-central),
- A has no nontrivial 2-sided ideals (A is simple).

Definition 1.2. We say that an algebra A over a field F is a central division algebra if it is a central simple algebra which is a division ring. That is, if it is and F-central division algebra.

Let's recall the various characterizations we have made for central simple algebras.

Proposition 1.3. *Let A be a finite dimensional algebra over a field F. Then the following conditions are equivalent:*

1. *A is a central simple F-algebra,*
2. *$A \cong \mathrm{M}_n(D)$ where D is an F-central division algebra,*
3. *The "sandwich map" $A \otimes_F A^{\mathrm{op}} \to End_F(A)$ via*
4. *$a \otimes b \mapsto (x \mapsto axb)$ is an isomorphism,*
5. *there exists an F-algebra B such that $A \otimes B \cong \mathrm{M}_n(F)$ for some n,*
6. *there exists an F-algebra B such that $A \otimes B$ is a central simple F-algebra,*
7. *there exists a field extension E/F such that $A \otimes E \cong \mathrm{M}_n(E)$ for some n,*
8. *there exists a separable field extension E/F such that $A \otimes E \cong \mathrm{M}_n(E)$ for some n,*
9. *$A \otimes \overline{F} \cong \mathrm{M}_n(\overline{F})$ for some n.*

One more equivalent condition we didn't prove, but which is worth mentioning is that A be a projective module over the enveloping algebra $A \otimes_F A^{\mathrm{op}}$ (i.e. the multiplication map $A \otimes A^{\mathrm{op}} \to A$ splits).

1.2. Azumaya algebras.

To generalize from fields to commutative rings, we define the concept of Azumaya.

In the following proposition, for a commutative ring R and a prime $\mathfrak{p} \in \mathrm{Spec}(R)$, we will write $\kappa(\mathfrak{p})$ to denote the field $\mathrm{frac}(R/\mathfrak{p})$ (also called the residue field of $\mathfrak{p}$).

Proposition 1.4. *For an algebra A over a commutative ring R which is finitely generated and projective as a module, the following are equivalent:*

1. *for every $\mathfrak{p} \in \mathrm{Spec}(R)$, $A \otimes_R \kappa(\mathfrak{p})$ is a central simple $\kappa(\mathfrak{p})$-algebra,*
2. *the sandwich map $A \otimes_R A^{\mathrm{op}} \to End_R(A)$ is an isomorphism.*

Definition 1.5. If the equivalent conditions of Proposition 1.4 hold, we say that A is an Azumaya algebra over R (also called a central separable algebra over R).

Just as a side comment – it turns out that when A/R is Azumaya it will follow that A is finitely presented as an R module and is a generator in the category of R modules (recall that M is a generator if for every other R-module N, there is a surjective map $M^I \twoheadrightarrow N$ for some index set I). So being an Azumaya algebra imposes serious module-theoretic constraints on an algebra.

2. Galois extension of rings

Much like the story for division algebras, while we may start by wanting to construct interesting examples of (central) division algebras, it is useful to consider instead central simple algebras. There are a few natural reasons that this kind of consideration comes up:

- many natural constructions which sometimes yield division algebras will often produce central simple algebras instead,
- when we construct central simple algebras, by the Wedderburn structure theorem, we may find that we have constructed division algebras within them,
- division algebras are not "preserved by scalar extension." In other words, if D/F is a central division algebra, and E/F is a field extension, $D \otimes_F E$ will be central simple, but need not be division.

A very similar discussion arises when considering Galois extension, which leads us to consider the concept of Galois extensions of the form E/F where E need not be a field. From here we will then proceed to consider the case where both F and E are replaced by more general commutative rings (in some analogy with the concept of Azumaya algebras).

2.1. étale extensions of fields.

Let's start with the generalization of the concept of a (not necessarily Galois) separable field extension, before considering the Galois case:

Definition 2.1 (Etale extensions of fields)**.** Let F be a field. We say that a commutative F-algebra E/F is étale over F if we can write E as a finite (possibly empty) product $E = \bigtimes_{i \in I} E_i$ where each E_i is a separable field extension of F.

We note that in the literature, one also says that E/F is a separable extension of rings.

2.1.1. *A strange digression into empty rings.* Let us take just a moment to discuss the edge case in which the product is empty. By convention, an empty product is a final object in a category, and here, considering ourselves to be in the category of unital commutative rings, we find that this final object is the "zero ring," consisting of a single element $0 = 1$. While this ring is not actually a field (because, for example, its nonzero elements fail to form a group, not having an identity element), we still consider the zero ring to be a product of fields, as it is an empty product of fields. Consequently it is an étale extension of every field.

2.2. Galois étale extensions of fields.

We may or may not get to proving all these equivalent conditions, but here are some ways we can characterize what it means for an étale extension to be Galois.

Recall the following definition:

Definition 2.2. Let S be a ring and G a finite group acting on S as automorphisms. We define $(S, G, 1)$, the twisted group ring, to be the algebra generated by S and symbols u_σ for $\sigma \in G$, so that as a left S-module we have

$$(S, G, 1) = \bigoplus_{\sigma \in G} S u_\sigma,$$

with multiplication given by the rules

$$u_\sigma u_\tau = u_{\sigma\tau} \quad \text{and} \quad u_\sigma x = \sigma(x) u_\sigma, \text{ for } x \in S, \sigma, \tau \in G.$$

Definition/Lemma 2.3. *Let F be a field and E a commutative F-algebra and let $G \subset \mathrm{Aut}(E/F)$ be a group of automorphisms of E fixing F. We say that E is a G-Galois extension of F if the following equivalent conditions hold:*

(1) $|G| = \dim_F E$ *and* $E^G = F$,
(2) $(E, G, 1)$ *is a central simple F-algebra,*
(3) the natural map $(E, G, 1) \to \mathrm{End}_F(E)$ *is an isomorphism,*
(4) the natural map $(E, G, 1) \to \mathrm{End}_F(E)$ *is injective (i.e. Dedekind's Lemma holds),*
(5) we can write $E = \bigotimes_{i \in I} E_i$ *with E_i/F separable extensions, and such that the induced action of G on I is transitive and for each $i \in I$, E_i/F is $Stab_G(i)$-Galois.*

An important thing to note is that there is generally no canonical choice for the group G for a given F-algebra E. So, for example, the $\mathbb{R}$-algebra $\mathbb{C} \times \mathbb{C}$ can be regarded as Galois

- with respect to the group $C_2 \times C_2 = \langle \sigma, \tau \mid \sigma^2, \tau^2 \rangle$ via the action $\sigma(z_1, z_2) = (z_2, z_1)$ and $\tau(z_1, z_2) = (\overline{z}_1, \overline{z}_2)$, or
- with respect to the group $C_4 = \langle \gamma \mid \gamma^4 \rangle$ via the action $\gamma(z_1, z_2) = (z_2, \overline{z}_1)$.

2.3. **Etale extensions of commutative rings.** We will come back to this a bit later when considering étale cohomology and more general descent, but let's define, as we are now able to, the notions of what it means for an extension of commutative rings to be étale.

Definition 2.4. Let R be a commutative ring. We say that an R-algebra S is étale if it is finitely ***presented*** ~~generated~~ and flat as an R-module, and if, for every $\mathfrak{p} \in \mathrm{Spec}(R)$, we have $S \otimes_R \kappa(\mathfrak{p})$ is an étale extension of the field $\kappa(\mathfrak{p})$.

2.4. **Galois extensions of commutative rings.** As with the notion of Azumaya, we are now ready to present the notion of what it means for an extension of rings to be Galois.

Definition/Lemma 2.5. *Let R be a commutative ring and S a commutative R-algebra. Let $G \subset \mathrm{Aut}(S/R)$ be a group of automorphisms of S fixing R. We say that S is a G-Galois extension of R if the following equivalent conditions hold:*

(1) for every $\mathfrak{p} \in \mathrm{Spec}(R)$, $S \otimes_R \kappa(\mathfrak{p})$ *is a G-Galois extension over* $\kappa(\mathfrak{p})$,
(2) $(S, G, 1)$ *is an Azumaya algebra over R,*
(3) the natural map $(S, G, 1) \to \mathrm{End}_R(S)$ *is an isomorphism.*

While not obvious from the definitions, the condition that S/R is G-Galois also imposes strong module-theoretic constraints on S, namely that S is a finitely generated projective R-module which is a generator in the category of R-modules. These conditions also imply that $S^G = R$ (as expected from usual Galois theory).

3. Galois Descent – an equivalence of categories

One important consequence of Definition/Lemma 2.5 this is that the Morita theorems apply (see Proposition A.2), and we obtain an equivalence of categories as follows:

Lemma 3.1. *Let S/R be a G-Galois extension of commutative rings. Then we obtain an equivalence of categories*

$$R\text{-}modules \leftrightarrow (S, G, 1)\text{-}modules$$
$$M \mapsto S \otimes_R M$$

via the standard $(S, G, 1) \cong \mathrm{End}_R(S)$*-module structure on S.*

We can make this particularly useful by recalling the notion of semilinear actions.

Definition 3.2. Let G be a group acting on a commutative ring S and let M be an S-module. A G-semilinear action on M is an action of G on M as an Abelian group such that for each $\sigma \in G$, $m \in M$, $x \in S$, we have $\sigma(xm) = \sigma(x)\sigma(m)$.

A G-semilinear S-module is defined to be an S-module with a G-semilinear action.

We may then consider the category of such G-semilinear S-modules and observe that this category is also equipt with a tensor product (monoidal) structure. That is, if $\mathrm{M}_1, \mathrm{M}_2$ are G-semilinear S-modules, we can define $\mathrm{M}_1 \otimes_S \mathrm{M}_2$ to have a G-semilinear action via

$$\sigma(m_1 \otimes m_2) = \sigma(m_1) \otimes \sigma(m_2).$$

With this notion, we can then define the notion of a G-semilinear S-algebra (via its structural maps such as $A \otimes_S A \to A$ satisfying various axioms).

We note the following fact, which is easily verified via the definitions:

Lemma 3.3. *Let S be a ring with an action of a group G. Then there is an equivalence (actually an isomorphism) of categories between $(S, G, 1)$-modules G-semilinear S-modules.*

Combining Lemma 3.3 with Lemma 3.1, we obtain the following:

Theorem 3.4 (Galois descent)**.** *Let S/R be a G-Galois extension of commutative rings. Then we obtain an equivalence of categories*

$$\begin{aligned} R\text{-}modules &\leftrightarrow G\text{-}semilinear\ S\text{-}modules \\ M &\mapsto S \otimes_R M \\ N^G &\leftarrow\!\shortmid N. \end{aligned}$$

Furthermore, this equivalence respects tensor products.

We verified implicitly that one of these directions gives an equivalence (at least, by quoting Morita theory). The other direction is given in the exercises.

4. Galois Descent – twisted forms and obstructions

The fundamental question of Galois descent is the following: given a G-Galois extension of commutative rings S/R, how can one go between algebraic structures over R and algebraic structures over S? We can phrase this in terms of two concrete questions:

Question 4.1. *Given an R algebra A, how can we describe all R algebras A' such that $A \otimes S \cong A' \otimes S$?*

Question 4.2. *Given an S algebra B, when can we find an R algebra A such that $A \otimes S \cong B$?*

Twisted forms and H^1

Question 4.1 is in large part the subject of the exercises, and we recall here the conclusions. In the context of Theorem 3.4, we can reframe this first question as follows. Given a semilinear action of G on an S-algebra B (for example, $B = S \otimes A$), how can we describe all other semilinear actions on B. These other actions, via Theorem 3.4, would correspond to R-algebras A' such that $S \otimes A' \cong B$. Recall the following definitions:

Definition 4.3. Let X, Y be sets with action by a group G. Then we obtain a natural action on the set of maps $Map(X, Y)$ via $(\sigma \cdot f)(x) \equiv \sigma(f(\sigma^{-1}(x)))$.

Definition 4.4. Let G, A be groups, and suppose we have a homomorphism $G \to \mathrm{Aut}(A)$ providing an action of G on A. We say that a map $\alpha : G \to A$ is a crossed homomorphism, or a 1-cocycle, if

$$\alpha(\sigma\tau) = \alpha(\sigma)\sigma(\alpha(\tau)), \quad \forall \sigma, \tau \in G.$$

We write $Z^1(G, A)$ for the set of all crossed homomorphisms.

Definition 4.5. The group A acts on $Z^1(G, A)$ via $(a \cdot \alpha)(\sigma) = a\alpha(\sigma)\sigma(a)^{-1}$, and we define $\mathrm{H}^1(G, A) = Z^1(G, A)/A$ to be the set of orbits under this action.

We note that in the case A is an Abelian group, this corresponds to the standard group cohomology construction, and the sets $Z^1(G, A)$ and $\mathrm{H}^1(G, A)$ have natural group structure given by pointwise multiplication in A. In general, however, these are just sets with distinguished elements (pointed sets), where the distinguished element comes from the crossed homomorphism $G \to A$ sending all elements to the identity.

Proposition 4.6. *Let B be a G-semilinear S-algebra, with action written as $(\sigma, b) \mapsto \sigma b$. Consider the G-action on $\mathrm{Aut}_S(B)$ given by Definition 4.3. Then if we have any other G-semilinear action on B, $(\sigma, b) \mapsto \sigma \cdot b$, then we may find a crossed homomorphism $\alpha : G \to \mathrm{Aut}_S(B)$ such that*

$$\sigma \cdot b = \alpha(\sigma)\sigma b,$$

and this gives a bijection between crossed homomorphism and semilinear actions.

Further, if $\alpha, \beta \in Z^1(G, \mathrm{Aut}_S(B))$ are crossed homomorphisms, then the resulting semilinear algebras are isomorphic if and only if α and β are in the same $\mathrm{Aut}_S(B)$ orbit. In particular, we have a bijection between isomorphism classes of algebras A'/R such that $S \otimes A' \cong B$ and the pointed set $\mathrm{H}^1(G, \mathrm{Aut}_S(B))$.

Descent obstructions and H^2

We now consider the Question 4.2 – given an S-algebra B, when can we find an R-algebra A such that $S \otimes_R A \cong B$? In light of Theorem 3.4, this is equivalent to asking the question of when we are able to define a semilinear action of G on B.

To make this easier to work with, let's define a bit of language:

Definition 4.7. Let B be an S-algebra and let σ be an automorphism of S. We define a new S-algebra, denoted ${}^\sigma S$ to have underlying set ${}^\sigma x$, $x \in S$ (that is, there is a bijection between the elements of B and ${}^\sigma B$), with operations:

$${}^\sigma x + {}^\sigma y = {}^\sigma(x + y), \qquad ({}^\sigma x)({}^\sigma y) = {}^\sigma(xy), \qquad \forall x, y \in B$$

and with S-module structure given by:

$$\lambda\, {}^\sigma x = {}^\sigma(\sigma^{-1}(\lambda)x), \qquad \forall \lambda \in S, x \in B,$$

or in other words, $\sigma(\lambda)\, {}^\sigma x = {}^\sigma(\lambda x)$.

Example 4.8. *As an example, note that if B is an S-algebra with a free S-module basis e_i and with multiplication table given by*

$$e_i e_j = \sum_k c_{i,j}^k e_k,$$

then the algebra ${}^\sigma B$ has multiplication table given by

$$ {}^\sigma e_i {}^\sigma e_j = \sum_k \sigma(c_{i,j}^k)\ {}^\sigma e_k. $$

Now, back to the case of a G-Galois extension S/R and an S-algebra B, we would like to ask whether or not it is possible to define a semilinear action of G on B. This amounts to defining, for every $\sigma \in G$ a "possible action,"

$$ \phi_\sigma : B \to B $$

which will satisfy $\phi_\sigma(\lambda x) = \sigma(\lambda)\phi_\sigma(x)$ for $\lambda \in S, x \in B$, and such that $\phi_\sigma \phi_\tau = \phi_{\sigma\tau}$. One complicating factor is that such maps ϕ_σ are evidently not S-linear, but we can change our perspective by considering the corresponding maps $\psi_\sigma : {}^\sigma B \to B$ given by $\psi_\sigma({}^\sigma x) = \phi_\sigma(x)$. For this map, we find

$$ \psi_\sigma(\lambda\ {}^\sigma x) = \psi_\sigma({}^\sigma(\sigma^{-1}(\lambda)x)) = \phi_\sigma(\sigma^{-1}(\lambda)x) = \lambda\phi_\sigma(x) = \lambda\psi_\sigma({}^\sigma x), $$

which allows us to encode the information of ϕ_σ as an S-linear map ψ_σ. If we let $\sigma : B \to {}^\sigma B$ denote the map $x \mapsto {}^\sigma x$ (which we can think of as a "universal" σ-linear map), then we can consider this via the following diagram

$$ \begin{tikzcd} {}^\sigma B \arrow[r, "\psi_\sigma"] & B \\ B \arrow[u, "\sigma"] \arrow[r, "\phi_\sigma"'] & B \arrow[u, equal] \end{tikzcd} $$

as $\psi_\sigma(x) = \sigma(\phi_\sigma(\sigma^{-1}x))$. More generally, we may "twist" these to obtain maps

$$ \begin{tikzcd} {}^{\sigma\tau} B \arrow[r, "{}^\sigma\psi_\tau"] & {}^\sigma B \\ {}^\tau B \arrow[u, "\sigma"] \arrow[r, "\psi_\tau"'] & B \arrow[u, "\sigma"'] \end{tikzcd} $$

$$ \begin{aligned} {}^\sigma\psi_\tau : {}^{\sigma\tau} B &\to {}^\sigma B, \\ {}^{\sigma\tau} x &\mapsto {}^\sigma(\psi_\tau({}^\tau x)) = {}^\sigma\phi_\tau(x). \end{aligned} $$

This perspective allows us to interpret the condition $\phi_\sigma\phi_\tau = \phi_{\sigma\tau}$ in terms of S-linear maps. That is, we have

$$ \begin{aligned} \psi_{\sigma\tau} : {}^{\sigma\tau} B &\to B, \\ {}^{\sigma\tau} x &\mapsto \phi_{\sigma\tau}(x), \end{aligned} $$

and,

$$ \begin{aligned} \psi_\sigma {}^\sigma\psi_\tau : {}^{\sigma\tau} B &\to B, \\ {}^{\sigma\tau} x &\mapsto \phi_\sigma\phi_\tau(x). \end{aligned} $$

Consequently, the condition $\phi_\sigma\phi_\tau = \phi_{\sigma\tau}$ corresponds to the condition $\psi_{\sigma\tau} = \psi_\sigma {}^\sigma\psi_\tau$.

Analyzing the possibilities, we see:

Case 1: ${}^\sigma B$ and B are not isomorphic for some $\sigma \in G$.

In this case, there is no possible way that σ can act on B, and so no hope for defining a semilinear action of B. Consequently, there is no algebra A/R such that $S \otimes A \cong B$.

Case 2: There exist isomorphisms $\psi_\sigma : {}^\sigma B \xrightarrow{\sim} B$ for each $\sigma \in G$.
In this case, we need only consider whether or not these can be chosen so that $\psi_{\sigma\tau} = \psi_\sigma {}^\sigma\psi_\tau$. To measure our "distance" from this condition, and make a connection with group cohomology, we define:[1]

$$\beta(\sigma,\tau) = \psi_{\sigma\tau}^{-1}\psi_\sigma {}^\sigma\psi_\tau \in \mathrm{Aut}_S(B).$$

We are successful if we can choose ϕ_σ so as to make $\beta(\sigma,\tau) = 1$ for all σ,τ. Tracing the following diagram:

$$\begin{tikzcd}
{}^{\sigma\tau\gamma}B \arrow[r, "{}^{\sigma\tau}\psi_\gamma"] \arrow[rr, bend left, "{}^{\sigma\tau}\psi_\gamma"] \arrow[rrr, bend left=50, "\psi_{\sigma\tau\gamma}"] \arrow[rrr, bend right=50, "\psi_{\sigma\tau\gamma}"'] & {}^{\sigma\tau}B \arrow[r, "{}^\sigma\psi_\tau"] \arrow[rr, bend right, "\psi_{\sigma\tau}"'] & {}^\sigma B \arrow[r, "\psi_\sigma"] & B
\end{tikzcd}$$

we find

$$\beta(\sigma\tau,\gamma)\beta(\sigma,\tau) = \beta(\sigma,\tau\gamma)\ \psi_\sigma {}^\sigma\beta(\tau,\gamma)\psi_\sigma^{-1}.$$

which we can think of as a nonabelian version of a 2-cocycle condition, although we won't try to define this "cohomology set" precisely here.

Of course, changing our isomorphisms ϕ_σ (and hence the maps ψ_σ) will alter our choice of β's. More precisely, if $\phi'_\sigma : B \to B$ is another σ-linear isomorphism, with corresponding isomorphism $\psi'_\sigma : {}^\sigma B \to B$, we see that $\psi'_\sigma\psi_\sigma^{-1} \equiv \rho(\sigma) \in \mathrm{Aut}_S(B)$, and so $\psi'_\sigma = \rho(\sigma)\psi_\sigma$ for some unique automorphism $\rho(\sigma)$, and conversely, different choices of isomorphisms ψ correspond to arbitrary functions $\rho : G \to \mathrm{Aut}_S(B)$. Given such a ρ corresponding to ϕ', we find that the corresponding β' is given by

$$\beta'(\sigma,\tau) = (\psi'_{\sigma\tau})^{-1}\psi'_\sigma {}^\sigma\psi'_\tau = \psi_{\sigma\tau}^{-1}\rho(\sigma\tau)^{-1}\rho(\sigma)\psi_\sigma\ {}^\sigma\rho(\tau)\ {}^\sigma\psi_\tau.$$

In general, this is a difficult formula to interpret, but with sufficient commutativity, it will reduce to the standard notion of 2-cocycles and their equivalence via differing by a coboundary.

We note that the previous machinery, which was introduced in the context of ring extensions, works perfectly well for schemes as well. Let's gather these definitions and observations in this situation:

Definition 4.9. Let $X = (X, \mathcal{O}_X)$ be a scheme and let $\mathcal{A}$ be a quasicoherent sheaf of associative $\mathcal{O}_X$ algebras. We say that $\mathcal{A}$ is Azumaya if for every affine open set $\operatorname{Spec} R = U \subset X$, $\mathcal{A}(U)$ is an Azumaya algebra over R.

We see, essentially as a consequence of Proposition 1.4, that we can characterize Azumaya-ness as follows. Here for a scheme X and a point $x \in X$, we write $\kappa(x)$ for the residue field

[1] note, this is a somewhat different convention than the one we did in class

of x – that is, $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$ algebras, we write $\mathcal{F}|_x$ to mean $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$.

Proposition 4.10. *For a scheme X and $\mathcal{A}$ a locally free and finitely generated sheaf of associative $\mathcal{O}_X$ algebras, the following are equivalent:*

(1) $\mathcal{A}$ is Azumaya,
(2) for every point $x \in X$, $\mathcal{A}|_x$ is a central simple algebra over $\kappa(x)$,
(3) the sandwich map $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^{\mathrm{op}} \to \mathscr{E}\mathrm{nd}_{\mathcal{O}_X}(\mathcal{A})$ is an isomorphism (here $\mathscr{E}\mathrm{nd}$ denotes the endomorphism sheaf).

Proof. We leave the verification of this as an exercise, via Proposition 1.4. □

Next, we define the notion of an étale morphism, relying on the corresponding definition for rings from Definition 2.4.

Definition 4.11. Let $f : X \to Y$ be a morphism of schemes. We say that f is étale at $x \in X$ if there exists an affine open neighborhood $\operatorname{Spec} B = V \subset X$ of x, and an affine open neighborhood $\operatorname{Spec} A = U \subset Y$ containing $f(U)$, such that B is an étale ring extension of A.

One thing that this definition should emphasize is that this definition is local on X[2]. The following definition connects more directly to Definition 2.4.

Lemma 4.12. *Let $f : X \to Y$ be a morphism of schemes. Then f is étale if and only if it is flat, locally of finite presentation, and for every $y \in Y$, the fiber $X_y = X \times_Y y$ is the spectrum of an étale (commutative ring) extension of the field $\kappa(y)$.*

Proof. [Sta24, Tag 02GM] □

Let's now define the notion of a Galois extension of schemes. Note that, unlike the case of étale ring extensions, Galois extensions of rings are necessarily flat and locally free of finite rank. In particular, these are module finite maps. It follows that the corresponding type of map for schemes would be a finite map, and as such would be affine. Hence, we can talk about Galois extensions either as morphisms of schemes $f : X \to Y$, or as coherent sheaves of commutative $\mathcal{O}_Y$-algebras corresponding to $f_*\mathcal{O}_X$.

Definition/Lemma 4.13. *Let Y a scheme and $\mathcal{R}$ a sheaf of commutative $\mathcal{O}_Y$ algebras, which are locally free of finite type as $\mathcal{O}_Y$-modules. Suppose that G is a finite group of $\mathcal{O}_Y$-linear automorphisms of $\mathcal{R}$. We say that f is a G-Galois extension if the following equivalent conditions are true:*

(1) for every open affine $\operatorname{Spec} S \subset Y$, $\mathcal{R}(\operatorname{Spec} S)/S$ is G-Galois,
(2) the sheaf of algebras $(\mathcal{R}, G, 1)$ is Azumaya over $\mathcal{O}_Y$,
(3) the natural map $(\mathcal{R}, G, 1) \to \mathscr{E}\mathrm{nd}_{\mathcal{O}_Y}(\mathcal{R})$ is an isomorphism,
(4) for every $y \in Y$, $\mathcal{R}|_y$ is a G-Galois commutative ring extension of the field $\kappa(y)$ (as in Definition/Lemma 2.5, Definition/Lemma 2.3).

The machinery of Section 4 goes through as previously described, and we will work through it via an example:

4.1. Galois descent for line bundles.

need to fill this section

[2]as an illustrative example, the doubled affine line mapping to the affine line is, locally on the domain, an isomorphism and hence étale

5. Etale descent – an equivalence of categories

We would like to ask a question which is analogous to those we asked in Section 4, in the context of morphisms of schemes. Namely, for a morphism of schemes $\pi : X \to U$, how can one go between structures on U and structures on X. That is, we have a natural functor π^*, taking sheaves on U to sheaves on X. In some sense, this is a forgetful process. We would like to know how much information is lost, and what additional information is needed to "go backwards."

We will use the psychological crutch of considering the case that X is a disjoint union of schemes $X = \sqcup U_i$ so that the individual morphisms $\pi_i : U_i \to U$ are étale morphisms. But we will visualize these maps as open covers in terms of intuition. We will write $U_{i,j}$ for the fiber product $U_i \times_U U_j$, which we will think of as the analog of the intersection of two open sets. Similarly we define $U_{i,j,k}$ as the triple fiber product $U_i \times_U U_j \times_U U_k$, etcetera. We write $\pi_{i,j}$ to denote the map $U_i \times_U U_j \to U$ (via the equal morphisms induced by π_i or π_j).

In order to reduce the notational clutter, if $\mathscr{F}$ is a sheaf of $\mathcal{O}_U$-modules, we will write $\mathscr{F}|_{i,j}$ to denote the sheaf $\pi_{i,j}^* \mathscr{F}_{i,j}$, and similarly for triple fiber products, etc.

So as to eliminate all possible suspense, let us simply give the "answer:" We first recall the notion of an étale covering:

Definition 5.1. Let U be a scheme and $\mathscr{U} = \{\pi_i : U_i \to U\}$ a family of morphisms. We say that $\mathscr{U}$ is an étale covering of U if for all i, π_i is an étale morphism, and if the family is jointly surjective. That is, if for every $y \in U$ a scheme-theoretic point, there exists $u \in U_i$ for some i a scheme theoretic point, such that $\pi_i(u) = y$.

Definition 5.2. If $\mathscr{U} = \{\pi_i : U_i \to U\}$ is a family of morphisms, we define the descent category $\mathrm{Desc}(\mathscr{U}, \mathrm{QCoh})$ to be the category whose objects are pairs $((\mathcal{F}_i), (\phi_{i,j}))$ where each $\mathcal{F}_i$ is a quasicoherent sheaf over U_i and where $\phi_{i,j} : \mathcal{F}_i|_{i,j} \to \mathcal{F}_j|_{i,j}$ are isomorphisms such that for all i, j, k, we have

$$\phi_{i,k}|_{i,j,k} = \phi_{j,k}|_{i,j,k} \circ \phi_{i,j}|_{i,j,k}.$$

Note that in the case where $\mathscr{U}$ is an open covering, this is just describing gluing data for sheaves (see [Har77, Exercise II.1.22]). We say that descent holds if sheaves are exactly described by such gluing data. Note that there is always a canonical functor $\mathrm{QCoh}_U \to \mathrm{Desc}(\mathscr{U}, \mathrm{QCoh})$ taking a sheaf $\mathcal{F}$ on U to the tuple $((\mathcal{F}|_i), (1_{i,j}))$, where $1_{i,j}$ represents the canonical identification of $\mathcal{F}|_i|_{i,j}$ with $\mathcal{F}|_j|_{i,j}$ (both being canonical equal to $\mathcal{F}|_{i,j}$).

Theorem 5.3 (Etale descent)**.** *Let $\{\pi_i : U_i \to U\}$ be an étale covering. Then the natural functor $\mathrm{QCoh}_U \to \mathrm{Desc}(\mathscr{U}, \mathrm{QCoh})$ given by $\mathcal{F} \mapsto ((\mathcal{F}|_i), (1_{i,j}))$ is an equivalence of categories.*

Proof. See (for a somewhat more general context) [Sta24, Tag 023T]. □

6. Sites and Grothendieck topologies

In fact, and possibly we should have started here, descent is closely tied to the notion of a sheaf itself in the context of a Grothendieck topology.

Definition 6.1. Let $\mathcal{C}$ be a category. A Grothendieck topology τ on $\mathcal{C}$ is a set whose elements are collections of morphisms with common codomain $\{U_i \to U\}_{i \in I}$, which we call covers, with the following properties:

1) if $\{U' \to U\}$ is a family consisting of a single isomorphism, then $\{U' \to U\} \in \tau$,
2) if $\{U_i \to U\} \in \tau$ and $V \to U$ is a morphism in $\mathcal{C}$ then the fiber products $U_i \times_U V$ exist and $\{U_i \times_U V \to V\} \in \tau$,
3) if $\{U_i \to U\} \in \tau$ and if $\{V_{i,j} \to U_i\} \in \tau$ for each i, then the family obtained by compositions $\{V_{i,j} \to U\}$ is also in τ.

We define a site to be a pair $\mathcal{C} = (\mathcal{C}, \tau)$ where $\mathcal{C}$ is a category and τ is a Grothendieck topology on $\mathcal{C}$.

Definition 6.2. Let $\mathcal{C}$ be a site and $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ a presheaf (=contravariant functor) with values in some other category $\mathcal{D}$. We say that $\mathscr{F}$ is a sheaf if for every cover $\{U_i \to U\}$, the natural map $\mathscr{F}(U) \to \prod \mathscr{F}(U_i)$ realize $\mathscr{F}(U)$ as the equalizer of the diagram

$$\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{i,j}).$$

Note that in particular, if $\mathcal{C}$ has an "empty set" in the sense of an object $\varnothing_{\mathcal{C}}$ for which the empty set is a cover of $\varnothing_{\mathcal{C}}$, then it would follow that for $\mathscr{F}$ a sheaf, $\mathscr{F}(\varnothing)$ would be a terminal object in $\mathcal{D}$ (so, for example, a singleton in Sets, the zero group in Abelian groups, or the zero ring in Rings).

Theorem/Exercise 6.3. *Sheaves satisfy descent. That is, for a site $\mathcal{C}$ and a covering $\mathscr{U} = \{\pi_i : U_i \to U\}$, the natural functor $\mathscr{S}\mathrm{hv}_U \to \mathrm{Desc}(\mathscr{U}, \mathscr{S}\mathrm{hv})$ given by $\mathcal{F} \mapsto ((\mathcal{F}|_i), (1_{i,j}))$ is an equivalence of categories.*

Proof idea. Verification of the fact that this map is fully faithful is relatively straightforward. For essential surjectivity, this amounts to extending a sheaf on a cover to a sheaf on the whole space U via application of the sheaf axiom of Definition 6.2 and then checking that this indeed defines a sheaf (i.e. that the sheaf axiom continues to hold on general covers). □

Remark 6.4. This idea can be extended to sheaves with extra structure – that is to say, the same result will hold when considering sheaves of groups, sheaves of Abelian groups, sheaves of rings, sheaves of modules or algebras over a given sheaf of rings, etcetera.

7. Etale (and general) Descent – twisted forms and obstructions

Let us now ask the same questions we asked before for étale covers, which we asked previously for Galois extensions. In fact, in light of Remark 6.4, we can really consider this in the context o a general site, perhaps with a sheaf of rings. We will ask this concretely in the context of quasicoherent sheaves of algebras over schemes with respect to the étale topology, and we will phrase things in this way, but we could also ask this for other types of algebraic structures over more general sites as well. So, suppose that $\{\pi_i : U_i \to U\}$ is an (étale) covering. We may ask the following questions:

Question 7.1. *Given a sheaf of $\mathcal{O}_U$ algebras $\mathcal{A}$, how can we describe all other sheaves of $\mathcal{O}_U$-algebras $\mathcal{A}'$ such that $\pi_i^*\mathcal{A} \cong \pi_i^*\mathcal{A}'$ for all i?*

Question 7.2. *Given sheaves of $\mathcal{O}_{U_i}$ algebras $\mathcal{B}_i$, when can we find a sheaf of $\mathcal{O}_U$ algebras $\mathcal{A}$ such that $\pi_i^*\mathcal{A} \cong B_i$ for each i?*

As before, we will write, for notational convenience, such things as $\mathcal{A}|_i$ for $\pi_i^*\mathcal{A}$.

To answer Question 7.1, we note that by Theorem 5.3, it suffices to consider the following question: if we are given descent data $\mathcal{B}_\bullet = ((\mathcal{B}_i), (\phi_{i,j}))$ for an algebra with respect to the cover $\mathcal{U} = \{U_i \to U\}$, we need to consider what other possible descent data we are able to define. Before we proceed, let's make a quick notational comment:

Clarification 7.3 (Automorphisms of sheaves versus sheaves of autmorphisms)**.** Here when we have a sheaf of algebras $\mathcal{A}$ and we write $\mathrm{Aut}(\mathcal{A})$, what we mean is the group of automorphisms of the sheaf $\mathcal{A}$. That is, such an autormophism is a natural transformation of functors (i.e. a morphism of presheaves) $\mathcal{A} \to \mathcal{A}$. One may also consider the automorphsim sheaf $\mathscr{A}\mathrm{ut}$ which on some U is defined via $\mathscr{A}\mathrm{ut}(U) = \mathrm{Aut}(\mathcal{A}|_U)$. This should not be confused with the presheaf which associates to each U the group of automorphisms of the value of the sections on U, $\mathrm{Aut}(\mathcal{A}(U))$. However, one can check that $\mathscr{A}\mathrm{ut}(\mathcal{A})$ is the sheafification of this presheaf and $\mathrm{Aut}(\mathcal{A})$ is the group of global sections of the sheaf $\mathscr{A}\mathrm{ut}(\mathcal{A})$.

Similarly, for sheaves of algebras $\mathcal{A}, \mathcal{A}'$ we can analogously define the sheaf of isomorphisms $\mathscr{I}\mathrm{so}(\mathcal{A}, \mathcal{A}')$ and its global sections $\mathrm{Iso}(\mathcal{A}, \mathcal{A}')$ consisting of "global isomorphisms."

In analogy to Definition 4.3, our descent data $\mathcal{B}_\bullet$ gives rise to descent data for its automorphisms – that is, we can define descent data $((\mathscr{A}\mathrm{ut}(\mathcal{B}_i)), (\mathscr{A}\mathrm{ut}(\phi_{i,j})))$ for a sheaf (and hence an étale sheaf by Theorem/Exercise 6.3 which we could call $\mathscr{A}\mathrm{ut}(\mathcal{B}_\bullet)$ as the sheaf $\mathscr{A}\mathrm{ut}(\mathcal{B}_i)$ on U_i and with

$$\mathrm{Iso}(\mathcal{B}_i|_{i,j}, \mathcal{B}_j|_{i,j}) \ni \mathscr{A}\mathrm{ut}(\phi_{i,j}) : \mathscr{A}\mathrm{ut}(\mathcal{B}_i)|_{i,j} \to \mathscr{A}\mathrm{ut}(\mathcal{B}_j)|_{i,j}$$

via for $V \to U_{i,j}$ and $f \in \mathrm{Aut}(\mathcal{B}_i)(V)$, we have $\mathrm{Aut}(\phi_{i,j})(V)(f) = \phi_{i,j}|_V f\ \phi_{i,j}^{-1}|_V$.

Note that in the case $\mathcal{B}_\bullet$ arises from a sheaf of $\mathcal{O}_U$-algebras $\mathcal{A}$, we would simply have that the descent data for $\mathcal{B}$ would be given by $((\mathcal{A}|_i), (\mathrm{id}_{\mathcal{A}|_{i,j}}))$ and $\mathrm{Aut}(\mathcal{B})$ would be given by $((\mathrm{Aut}(\mathcal{A}|_i)), (\mathrm{Aut}(\mathrm{id}_{\mathcal{A}|_{i,j}})))$ corresponding simply to the sheaf $\mathrm{Aut}(\mathcal{A})$. In fact, by Theorem 5.3, such an $\mathcal{A}$ always exists, so we can assume, without loss of generality, that $\mathcal{B}_\bullet$ has this form. This gives a significant notational simplification.

With this in mind, we can assume we start with a sheaf of algebras $\mathcal{A}$, and want to find all possible descent data of the form $((\mathcal{A}|_i), \psi_{i,j})$. In this context, $\psi_{i,j} \in \mathrm{Aut}(\mathcal{A}|_{i,j})$

Definition 7.4. For a sheaf of groups $\mathfrak{A}$ on a site $\mathcal{C}$ and a cover $\mathcal{U} = \{U_i \to U\}$, we define the pointed set $Z^1(\mathcal{U}, \mathfrak{A}) \equiv \{(\psi_{i,j}) \in \prod \mathfrak{A}(U_{i,j}) \mid \psi_{i,k} = \psi_{j,k}\psi_{i,j}\}$.

The following is essentially immediate from the defnitions:

Lemma 7.5. *Let $\mathcal{A}$ be a sheaf of algebras. Then we have a bijection between $Z^1(\mathcal{U}, \mathscr{A}\mathrm{ut}(\mathcal{A}))$ and descent data of the form $((\mathcal{A}_i), (\psi_{i,j}))$.*

In this way, we have parametrized all possible $\mathcal{A}'$ such that $\mathcal{A}'|_i \cong \mathcal{A}|_i$, however there is some amount of double counting. That is, we may have different descent datum $((\mathcal{A}_i), (\psi_{i,j}))$, $((\mathcal{A}_i), (\psi'_{i,j}))$ which are isomorphic (as descent data) and

8. (mostly March 27) Azumaya algebras over locally ringed spaces

In this section, we'll consider the notion of Azumaya algebras in the context of locally ringed spaces. So, suppose that X is a site (that is, a category equipped with a Grothendieck topology as in Definition 6.1), together with a sheaf of ringes $\mathcal{O}_X$. In principle there may be many ways to try to define the notion of an Azumaya algebra over X. For example, we could say that it is a sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}$ such that the natural map $\mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{A}^{\mathrm{op}} \to \mathscr{E}\mathrm{nd}_{\mathcal{O}_X}(\mathcal{A})$ is an isomorphism, or we could use one of the many other notions motivated by Proposition 1.3. In fact we will use one which was not part of our prior characterization of Azumaya algebras over rings (see Proposition 1.4), and it won't be until a bit later until we see that these notions are compatible (see **??**).

Definition 8.1 (Azumaya algebras over a ringed space)**.** Let $X = (X, \mathcal{O}_X)$ be a ringed space. We say that a sheaf of algebras $\mathcal{A}$ is Azumaya of rank n if for every object U in X, there exists a covering $\{U_i \to U\}$ such that the restriction $\mathcal{A}|_{U_i}$ is isomorphic to the sheaf of matrix algebras $\mathrm{M}_n(\mathcal{O}_X)$.

As before, we can define both a monoid structure on the collection of isomorphism classes of Azumaya algebras, and an equivalence relation which turns this monoid into a group. Recall that if X is a locally ringed space, a sheaf $\mathcal{V}$ of $\mathcal{O}_X$-modules is locally free of rank n if for every $U \in X$, there exists a covering $\{U_i \to U\}$ such that $\mathcal{V}|_{U_i} \cong \mathcal{O}_X^n$.

Definition 8.2. We define an equivalence relation on the isomorphism classes of Azumaya algebras, called Brauer equivalence to be the equivalence relation generated by the relation consisting of $\mathcal{A} \sim \mathcal{A} \otimes_{\mathcal{O}_X} \mathscr{E}\mathrm{nd}_{\mathcal{O}_X}(\mathcal{V})$ where $\mathcal{V}$ is a locally free sheaf of $\mathcal{O}_X$-modules of rank n for some n.

Definition 8.3 (Azumaya Brauer group)**.** For a ringed space X, we define the Azumaya Brauer group $\mathrm{Br}^{Az}(X)$ of X to be the set of Brauer equivalence classes $[\mathcal{A}]$ of Azumaya algebras $\mathcal{A}$ over X, with the operation $[\mathcal{A}] + [\mathcal{B}] = [\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B}]$.

Verifying that this operation is associative and that $[\mathcal{O}_X]$ provides an additive identity element is straightforward. To see that we have inverses, we note that there is a canonical map

$$\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \to \mathscr{E}\mathrm{nd}(\mathcal{A})$$

as before given by $a \otimes b \mapsto (x \mapsto axb)$. To finish, we need only check that this is an isomorphism of sheaves of algebras, which is to say that for every U, there exists a cover $\{U_i \to U\}$ such that the restriction of this map to U_i is an isomorphism. But by definition Definition 8.1, restricting to U_i allows us to assume that $\mathcal{A} \cong \mathscr{E}\mathrm{nd}(\mathcal{V})$ for some free $\mathcal{O}_X$-module $\mathcal{V}$. The result then follows from the observation that for any commutative ring R, the natural map

$$\mathrm{M}_n(R) \otimes \mathrm{M}_n(R) \to \mathrm{End}_R(\mathrm{M}_n(R)) = \mathrm{M}_{n^2}(R)$$

is an isomorphism. This in turn can be seen by observing the map on matrix units

$$e_{i,j} \otimes e_{k,\ell} \mapsto (e_{p,q} \mapsto \delta_{j,p}\delta_{k,q}e_{i,\ell})$$

which is to say that if we regard $M_{n^2}(R)$ as having matrix units $e_{(a,b),(c,d)}$ relative to a basis indexed by $\{1, \dots, n\}^2$, we see this is described by

$$e_{i,j} \otimes e_{k,\ell} \mapsto e_{(i,\ell),(k,j)},$$

and hence is an isomorphism (as it takes an R-module basis to a R-module basis).

Definition 8.4 (Cohomological Brauer group)**.** Let X be a ringed space. The cohomological Brauer group $\mathrm{Br}^{Coh}(X)$ of X is defined to be the group $\mathrm{H}^2(X, \mathbb{G}_m)^{tors}$, that is, the torsion part of the second cohomology with coefficients in the multiplicative group.

To see how these groups relate to each other, we will need to consider the exact sequence

$$1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1,$$

and cohomology sequence

$$\mathrm{H}^1(X, \mathbb{G}_m) \to \mathrm{H}^1(X, GL_n) \to \mathrm{H}^1(X, PGL_n) \to \mathrm{H}^2(X, \mathbb{G}_m). \tag{1}$$

These sheaves of groups are defined as follows.

Definition 8.5. Let $X = (X, \mathcal{O}_X)$ be a ringed space. We define the sheaf of groups GL_n on X by $U \mapsto GL_n(\mathcal{O}_X(U))$, and $\mathbb{G}_m = GL_1$. We have a natural "diagonal" map $\mathbb{G}_m \to GL_n$ and we let PGL_n be the sheafification of the presheaf $U \mapsto GL_n(\mathcal{O}_X(U))/\mathbb{G}_m(\mathcal{O}_X(U))$. That is, PGL_n is the sheaf quotient $GL_n/\mathbb{G}_m$.

8.1. **(Not from lecture) When is the projective general linear group a quotient?** Somewhat unintuitively, it need not be the case that $PGL_n(U) = GL_n(U)/\mathbb{G}_m(U)$. Indeed, we can understand this from examining aspects of the sequence (1). This will take us a bit to unpack though:

unpacking the exact sequence (1)**.** Via descent, we may interpret the pointed sets $\mathrm{H}^1(X, \mathbb{G}_m)$, $\mathrm{H}^1(X, GL_n)$ and $\mathrm{H}^1(X, PGL_n)$ by considering the groups $\mathbb{G}_m$, GL_n and PGL_n as sheaves of automorphisms. In particular, we find that $\mathbb{G}_m$ is the sheaf of automorphisms of $\mathcal{O}_X$ as a sheaf of modules over itself and GL_n is the sheaf of automorphisms of $\mathcal{O}_X^n$ as a sheaf of modules. Consequently, $\mathrm{H}^1(X, GL_n)$ is in bijection with isomorphism classes of sheaves of modules over $\mathcal{O}_X$ which are locally isomorphic to $\mathcal{O}_X^n$ – that is, locally free sheaves of rank n. In particular, $\mathrm{H}^1(X, \mathbb{G}_m)$ corresponds to locally free sheaves of modules of rank 1. The natural map $\mathbb{G}_m \to GL_n$ diagonally then can be interpreted as taking a locally free sheaf N of rank 1 to N^n, a locally free sheaf of rank n.

Let M_n denote the sheaf of matrix algebras given by $\mathrm{M}_n(U) = \mathrm{M}_n(\mathcal{O}_X(U))$. In favorable circumstances (for example for X a locally ringed space, as we will describe in Lemma 8.12 and Proposition 8.13), we will find that conjugation induces an identification of sheaves $PGL_n \cong \mathrm{Aut}(\mathrm{M}_n)$. We think about the map $GL_n \to PGL_n$ as taking an automorphism of R^n to the corresponding "change of basis" on its ring of linear transformations $\mathrm{M}_n(R)$. We can then show that the map from GL_n to PGL_n is given by associating to a locally free sheaf M of rank n, its endomorphism sheaf of algebras $\mathscr{E}\mathrm{nd}(M)$.

Definition 8.6. Let N be a sheaf of $\mathcal{O}_X$-modules. We say that N is n-free if $N^n \cong \mathcal{O}_X^n$.

The n-free line bundles form a subgroup of the Picard group – if P, Q are n-free then

$$(P \otimes Q)^n \cong P \otimes Q \otimes \mathcal{O}_X^n \cong P \otimes (Q \otimes \mathcal{O}_X^n) \cong P \otimes \mathcal{O}_X^n \cong \mathcal{O}_X^n,$$

and consequently, $P \otimes Q$ is n-free as well.

Definition 8.7. Let X be a locally ringed space. We let $Pic_{(n)}(X)$ denote the subgroup of $Pic(X)$ consisting of those locally free sheaves of rank 1 which are n-free. If R is a commutative ring, we similarly write $Pic_{(n)}(R)$ to denote $Pic_{(n)}(\mathrm{Spec}\, R)$. That is, isomorphism classes of projective R-modules N of rank 1 such that $N^n \cong R^n$.

Lemma 8.8. *For any ringed space X, $Pic_{(n)}(X)$ is n-torsion.*

Proof. Let $N \in Pic_{(n)}(X)$. Then $N^{\otimes n} \cong \Lambda^n N^n \cong \Lambda^n \mathcal{O}_X^n \cong \mathcal{O}_X$. □

Remark 8.9. As a partial converse to Lemma 8.8, it follows from the structure theory of modules over a Dedekind domain that $Pic_{(n)}(R)$ is exactly the n-torsion subgroup of $Pic(R)$ in the case that R is a Dedekind domain. Indeed, for a Dedekind domain, every projective module M is of the form $M \cong R^m \oplus P$ for some rank 1 projective module P. In particular, if $N \in Pic(R)$ is n-torsion, then if we write $N^n \cong R^{n-1} \oplus P$ and we find

$$R \cong N^{\otimes n} \cong \Lambda^n N^n \cong \Lambda^n(R^{n-1} \oplus P) \cong P,$$

and so $N^n \cong R^n$ which tells us that $N \in Pic_{(n)}(R)$ as claimed.

$$\mathbb{G}_m(U) \to GL_n(U) \to PGL_n(U) \to$$

We may attempt to define a map $\mathrm{Br}^{Az}(X) \to \mathrm{Br}^{Coh}(X)$ as follows. For an Azumaya algebra $\mathcal{A}$, we may consider $\mathcal{A}$ as a twisted form of the sheaf of $\mathcal{O}_X$-algebras $\mathrm{M}_n(\mathcal{O}_X)$. We would like to say that this is represented by a class in $\mathrm{H}^1(X, PGL_n)$ as would follow from the logic of Lemma 7.5. However, for this to work, we would need to know that the sheaf $\mathscr{A}\mathrm{ut}_{\mathcal{O}_X}(\mathrm{M}_n(\mathcal{O}_X))$ of automorphisms of matrix algebras is given by $PGL_n(\mathcal{O}_X)$ – that is, by conjugation. We knew that this was true in the case of fields by the Noether-Skolem theorem, however in general this is an extra assumption. For the purposes of the present conversation, we will make the following ad-hoc definitions:

Definition 8.10 (Hilbert 90 spaces)**.** We say that a ringed space X is a Hilbert 90 space if the presheaf $Pic(\mathcal{O}_X)$ given by $U \mapsto Pic(\mathcal{O}_X(U))$ is locally trivial (i.e. has trivial sheafification).

We can refine this slightly as follows:

Definition 8.11 (Hilbert 90(n) spaces)**.** We say that a ringed space X is Hilbert 90(n) space if the presheaf $Pic_{(n)}(\mathcal{O}_X)$ given by $U \mapsto Pic_{(n)}(\mathcal{O}_X(U))$ is locally trivial (i.e. has trivial sheafification).

Now, if X is a locally ringed space, for example – that is, a topological space with a sheaf of rings $\mathcal{O}_X$ such that $O_{X,x}$ is a local ring for every point $x \in X$, then it is also a Hilbert 90 space, since projective modules over a local ring are free.

This will be a particularly useful concept for understanding the extent to which Noether-Skolem will apply for us, as the following Lemma illustrates:

Lemma 8.12. *Suppose $Pic(R) = 0$. Then the natural map $PGL_n(R) \to \mathrm{Aut}(\mathrm{M}_n(R))$ is an isomorphism.*

Proof. For the commutative ring R, the concept of rank of a projective module defines a function $\mathrm{Spec}(R) \to \mathbb{N}$.

Morita theory tells us that since R^n is a projective generator in the category of R-modules, we have an equivalence of categories between the category of R-modules and the category of $\mathrm{End}_R(R^n) = \mathrm{M}_n(R)$-modules, and this equivalence takes R to R^n. Let $\phi \in \mathrm{Aut}(\mathrm{M}_n(R))$. We see that if N is an R-module which is projective of rank r, then its image $R^n \otimes_R N$ is a projective $\mathrm{M}_n(R)$-module which, viewed as an R-module via the R-algebra structure of $\mathrm{M}_n(R)$, is a projective R-module of rank rn.

Precomposition with ϕ gives an auto-equivalence on the category of $\mathrm{M}_n(R)$-modules, where an $\mathrm{M}_n(R)$-module P is taken to a new module with structure given by $T \cdot p \equiv \phi(T)p$. As this is a categorical equivalence, it preserves categorical notions such as projectives and generators.

Note that as every automorphism of $\mathrm{M}_n(R)$ preserves the R-algebra structure by definition, the R-module structure of modules is left unchanged.

In particular, we obtain two different $\mathrm{M}_n(R)$-module structures on R^n, the first being the standard one, and the second given by $T \cdot v = \phi(T)v$. Correspondingly, this second structure corresponds to an R-module N which is also a projective generator. Suppose N has rank r. Then it follows that R^n has rank rn as an R-module, which tells us that $r = 1$, or that N is a rank one projective module. As $Pic(R) = 0$, it follows $N \cong R$ which implies these two $\mathrm{M}_n(R)$-module structures determine isomorphic modules. Therefore we have an isomorphism $\psi : R^n \to R^n$ of R-modules such that

$$T \cdot \psi(v) = \psi(Tv)$$

or in other words, $\phi(T)\psi(v) = \psi(Tv)$ or $\phi(T) = \psi T \psi^{-1}$ as desired. $\square$

There is actually a bit more one could say here:

Proposition 8.13. *Let R be a commutative ring. Then the natural map $PGL_n(R) \to \mathrm{Aut}(\mathrm{M}_n(R))$ is an isomorphism if and only $Pic_{(n)}(R) = 0$.*

Proof sketch. Looking more carefully at the proof, one can see that the two $\mathrm{M}_n(R)$-module structures on R^n yield modules which are isomorphic as R-modules (by construction). Hence, by the explicit Morita equivalence, if the latter corresopnds to N as an R-module, then we must have an isomorphism $N^n \cong R^n$. So, in fact, we find the stronger conclusion that $PGL_n(R) \to \mathrm{Aut}(\mathrm{M}_n(R))$ is an isomorphism as long as there are no rank 1 projective R-modules N such that $N^n \cong R^n$.

Conversely, if we have such an N, and we choose an isomorphism $N^n \cong R^n$, we find that we obtain two corresponding $\mathrm{M}_n(R)$ module structures on R^n where via Morita theory, one corresponds to R and the other to N as R-modules. Hence these are different $\mathrm{M}_n(R)$-modules. However the isomorphism of R-modules $N^n \cong R^n$ induces an isomorphism of their endomorphism groups, which then gives an automorphism of $\mathrm{M}_n(R)$ which is cannot be given by conjugation. $\square$

Proposition 8.14. *Let X be a Hilbert 90(n) space. Then we have an isomorphism of sheaves of groups:*

$$PGL_n(\mathcal{O}_X) \to \mathrm{Aut}(\mathrm{M}_n(\mathcal{O}_X))$$

The following Lemma now follows immediately from descent:

Lemma 8.15. *Suppose X is a Hilbert 90(n) space. Then we have a bijection between isomorphism classes of Azumaya algebras of rank n and the pointed set $\mathrm{H}^1(X, PGL_n)$.*

to explain
this lands
rsion still!

In this case, we obtain a map $\mathrm{Br}^{Az}(X) \to \mathrm{Br}^{Coh}(X)$ via the boundary map

$$\delta : \mathrm{H}^1(X, PGL_n) \to \mathrm{H}^2(X, \mathbb{G}_m).$$

Lemma 8.16. *Let $\mathcal{A}, \mathcal{B}$ be Azumaya algebras over X. Then $\delta(\mathcal{A} \otimes \mathcal{B}) = \delta(\mathcal{A}) + \delta(\mathcal{B})$.*

It follows that the map is injective – if $\mathcal{A}$ has trivial class in $\mathrm{H}^2(X, \mathbb{G}_m)$, then it must be in the image of $\mathrm{H}^1(X, GL_n)$. But by our description of the sequence, it follows that we then would have $\mathcal{A} \cong \mathscr{E}\mathrm{nd}(\mathcal{V})$ for some locally free sheaf $\mathcal{V}$ of rank n. Hence $[\mathcal{A}] = 0$ in $\mathrm{Br}^{Az}(X)$.

Proposition 8.17. *Suppose X is a Hilbert 90(n) space for all n. Then we have an injective group homomorphism*

$$\mathrm{Br}^{Az}(X) \to \mathrm{Br}^{Coh}(X).$$

9. Spectral sequences: from Cech to Artin-Leray

There are many different spectral sequences we find in life, but in many ways, there are only a few from which all others are derived. Or perhaps there is only one. In any case, one candidate for such a "mother" spectral sequence is the Čech sequence. Let X be a site and $\mathscr{F}$ a sheaf of Abelian groups on $\mathscr{F}$ (or a sheaf in some appropriate Abelian category). This spectral sequence works as follows:

9.1. **Čech combinatorics and simplicial objects.**

Given a covering $\{U_i \to U\}_{i\in I}$ in X, we can consider, for every ordered tuple of indices $i_\bullet = (i_0, i_1, \ldots, i_p)$ the iterated fiber product

$$U_{i_\bullet} = U_{i_0} \times_U U_{i_1} \times_U \cdots \times_U U_{i_p}$$

if we write $|i_\bullet| = p+1$ in the above situation, we can then set

$$U_p = \coprod_{|i_\bullet|=p+1} U_{i_\bullet}.$$

This collection comes with a natural collection of maps. For example, if

$$f : [p] = \{0, 1, \ldots, p'\} \to \{0, 1, \ldots p\} = [p]$$

is any map which preserves the partial order $\leqslant$, we see that for any tuple $i_\bullet$ with $|i_\bullet| = p+1$, if we let $f(i_\bullet) = (i_{f(0)}, i_{f(1)}, \ldots i_{f(p')})$, then there is a corresponding map on the fiber products in the other direction

$$U_{i_\bullet} \to U_{f(i_\bullet)}$$

(given by the universal property of fiber products). Proceeding this way for each index $i_\bullet$ with $|i_\bullet| = p+1$, we may put these together to obtain a map:

$$f^* : U_p \to U_{p'}.$$

In other words, if Δ is the category of finite, linearly ordered sets and order preserving maps (which can be taken, up to equivalence, to consist exactly of the objects $[p]$ and maps between them), then the rule

$$[p] \mapsto U_p$$

extends to a contravariant functor

$$U_\bullet : \Delta \to X^{\mathrm{op}}.$$

Composing this with the functor $\mathscr{F}$, we obtain a covariant functor

$$\mathscr{F}(U_\bullet) : \Delta \to \underline{\mathrm{Ab}}$$

Definition 9.1. Let $\mathscr{C}$ be a category. A simplicial object in $\mathscr{C}$ is a contravariant functor $\Sigma : \Delta \to \mathscr{C}$. A cosimplicial object in $\mathscr{C}$ is a covariant functor $\Xi : \Delta \to \mathscr{C}$.

reads as:

$$\check{\mathrm{H}}^n(X, \mathcal{F}) \Rightarrow \mathrm{H}$$

9.2. **The Brauer group and Picard group.**

10. (mostly April 1) The Brauer group of a local ring

(1) purity – Brauer group of punctured spectra in dimension > 1

11. The Brauer group of a complete discretely valued field (tame case)

(1) Hensel's lemma and the correspondence between finite étale algebras (unramified extensions) over a Henselian dvr and its residue field
(2) Existence of unramified splitting fields in the perfect case (and mention Kato cohomology / differential forms / crystalline ideas for the bad characteristic case)
(3) The short exact ramification sequence

more topics

12. Severi-Brauer schemes

13. Formal smoothness, etaleness

Appendix A. semilinear spaces (the descent data category) with exercises

Exercise 1. If E/F is a G-Galois extension of fields, show that the natural map

$$\begin{aligned} (E, G, 1) &\to \mathrm{End}_F(E) \\ x &\mapsto [y \mapsto xy], \quad x, y \in E \\ u_\sigma &\mapsto [y \mapsto \sigma(y)], \quad \sigma \in G, y \in E \end{aligned}$$

gives an isomorphism of algebras.

Definition A.1. Recall that if E/F is a G-Galois extension, an E/F-semilinear vector space is an E-vector space V together with an action of G on V such that for every $x \in E$, $v \in V$, we have

$$\sigma(xv) = \sigma(x)\sigma(v).$$

A homomorphism of E/F semilinear vector spaces $\phi : V \to W$ consists of an E-linear map ϕ which commutes with the G-action in the sense that $\phi(\sigma v) = \sigma(\phi v)$.

Exercise 2. If V is an F-vector space then $E \otimes_F V$ is naturally an E/F semilinear vector space, where the action of G is via the first factor.

Exercise 3. Show that we have an equivalence of categories between $(E, G, 1)$-modules and E/F-semilinear vector spaces.

Recall the following result which we claimed in the last lecture:

Proposition A.2 (Morita)**.** *Let R be a ring and P a right R-progenerator (i.e. finitely generated, projective generator in the category of right R-modules). Let $S = \mathrm{End}_R(P)$. Then the functor from R-modules to S-modules given by*

$$N \mapsto P \otimes_R N$$

is an equivalence of categories. Further, if $P^\star = Hom_R(P, R)$ then $P^\star$ is an $R-S$ bimodule, and

$$M \mapsto P^\star \otimes_S M$$

gives the (homotopy) inverse equivalence.

Exercise 4. Show that the functor from F-vector spaces to E/F-semilinear vector spaces given by

$$V \mapsto V_E \equiv E \otimes_F V$$

is an equivalence of categories.

Now, if we are interested in talking about algebraic objects (such as central simple algebras), we need more than just vector spaces and linear maps, but we also need the concept of the tensor product (for multiplicative structures).

Definition A.3. Suppose V, W are E/F semilinear vector spaces. Then $V \otimes_E W$ is also a semilinear vector space with respect to the action:

$$\sigma(v \otimes w) = \sigma(v) \otimes \sigma(w).$$

Exercise 5. Show that the above definition gives a well defined E/F semilinear space and that this commutes with the functor given above.

That is, show that if V, W are F-vector spaces, then we have a natural isomorphism of E/F semilinear vector spaces

$$V_E \otimes_E W_E \cong (V \otimes_F W)_E.$$

More formally (if you like), this means you are showing that the two functors

$$(V, W) \mapsto (V_E \otimes_E W_E) \quad (V_W) \mapsto (V \otimes_F W)_E$$

from $Vec/F \times Vec/F$ to the category of E/F semilinear vector spaces are naturally isomorphic.

From this point of view it makes sense to talk about E/F semilinear algebras.

Definition A.4. An E/F semilinear algebra is an E/F semilinear vector space A, together with an E/F-semilinear map

$$m : A \otimes_E A \to A$$

and an E/F-semilinear map

$$\iota : E \to A$$

which gives A the structure of an algebra (where $\iota(1) = 1$ is the multiplicative identity of A).

Exercise 6. Show that an E/F semilinear algebra is just an E-algebra A with a semilinear action of G on A as a vector space such that $\sigma(ab) = \sigma(a)\sigma(b)$ (i.e. such that G acts via ring isomorphisms).

Exercise 7. Show that we have an equivalence of categories between F-algebras and E/F-semilinear algebras given by $A \mapsto E \otimes_F A$.

Exercise 8. It follows from the above exercise that if we let $F = \mathbb{R}$ and $E = \mathbb{C}$, then we have an equivalence between $\mathbb{R}$-algebras and $\mathbb{C}$-algebras with a notion of conjugation (action by $\mathcal{G}\mathrm{al}(\mathbb{C}/\mathbb{R})$). In particular, if we consider the $\mathbb{R}$-algebras $\mathbb{H}$ and $\mathrm{M}_2(\mathbb{R})$, we see that

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathrm{M}_2(\mathbb{C}) \cong \mathbb{C} \otimes_{\mathbb{R}} \mathrm{M}_2(\mathbb{R})$$

and so as $\mathbb{C}/\mathbb{R}$ semilinear algebras, both of these algebras are given as $\mathrm{M}_2(\mathbb{C})$ with two different notions of conjugation. What are these notions of conjugation?

Appendix B. twisted forms (the gluing problem) with exercises

Throughout the section, let us fix E/F a G-Galois extension.

Definition B.1. Let A be an F-algebra. We say that an F-algebra B is a(n E/F-)twisted form of A if there is an isomorphism of E-algebras, $A_E \cong B_E$.

Note that we are not assuming here that we have an isomoprhism of E/F semilinear algebras (which would imply they were isomorphic over F), but just as E-algebras.

As we saw in the previous section, we can recover the structure of B from B_E by specifying a semilinear action. As we are able to identify A_E and B_E, our quest to understanding the possible B's we may have then reduces to understanding all possible semilinear actions of G on A_E.

Definition B.2. Suppose V is a vector space with an action of G. We define an action of G on $\mathrm{Aut}(V)$ by $(\sigma\phi)(v) = \sigma(\phi(\sigma^{-1}(v))$.

Exercise 9. Show that in the case $V = E^n$, with component-wise action, the action of the Galois group $G = \mathcal{G}\mathrm{al}(E/F)$ on $Aut(V) = GL_n(E)$ is given by the standard action on the matrix entries.

Exercise 10. Suppose $\phi, \psi : G \to \mathrm{Aut}(A_E)$ are two different semilinear actions of G on A_E. That is, for $\sigma \in G$, we have $\sigma(a) \equiv \phi(\sigma)(a)$ and $\sigma(a) \equiv \psi(\sigma)(a)$ define semilinear actions (note here that ϕ and ψ need not have values in E-automorphisms, but in general just F-linear automorphisms).

Show that $\phi(\sigma) = \alpha(\sigma)\psi(\sigma)$ for a map $\alpha : G \to \mathrm{Aut}(A_E)$ and α is a crossed homomorphism (where the action of G on $\mathrm{Aut}(A_E)$ here is given by the previous excercise via ψ).

Exercise 11. Show that the above correspondence gives, after fixing an algebra A/F a bijection between semilinear actions on A_E and crossed homomorphisms $G \to \mathrm{Aut}_F(A_E)$.

From this we see so far that for B/F a twisted form of A, given an isomorphism $\phi : B_E \to A_E$, we obtain a new semilinear action on A_E which corresponds to the algebra B/F via the equivalence of categories previously described. This semilinear action, in turn gives rise to crossed homomorphism $G \to \mathrm{Aut}(A_E)$.

It therefore is natural to ask: in what way does this semilinear action depend on the isomorphism ϕ?

References

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