# SUMMARY

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Contents



## 1. Central simple algebras and Azumaya algebras

### 1.1. Central simple algebras.

**Definition 1.1.** We say that an algebra $A$ over a field $F$ is a central simple $F$-algebra if

- $Z(A) = F$, and $\dim_F A < \infty$ ($A$ is $F$-central),
- $A$ has no nontrivial 2-sided ideals ($A$ is simple).

**Definition 1.2.** We say that an algebra $A$ over a field $F$ is a central division algebra if it is a central simple algebra which is a division ring. That is, if it is and $F$-central division algebra.

Let's recall the various characterizations we have made for central simple algebras.

**Proposition 1.3.** *Let $A$ be a finite dimensional algebra over a field $F$. Then the following conditions are equivalent:*

*(1) $A$ is a central simple $F$-algebra,*
*(2) $A \cong M_n(D)$ where $D$ is an $F$-central division algebra,*
*(3) The "sandwich map" $A \otimes_F A^{op} \to End_F(A)$ via $a \otimes b \mapsto (x \mapsto axb)$ is an isomorphism,*
*(4) there exists an $F$-algebra $B$ such that $A \otimes B \cong M_n(F)$ for some $n$,*
*(5) there exists an $F$-algebra $B$ such that $A \otimes B$ is a central simple $F$-algebra,*
*(6) there exists a field extension $E/F$ such that $A \otimes E \cong M_n(E)$ for some $n$,*

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(7) *there exists a separable field extension $E/F$ such that $A \otimes E \cong M_n(E)$ for some $n$,*
(8) *$A \otimes \overline{F} \cong M_n(\overline{F})$ for some $n$.*

One more equivalent condition we didn't prove, but which is worth mentioning is that $A$ be a projective module over the enveloping algebra $A \otimes_F A^{op}$ (i.e. the multiplication map $A \otimes A^{op} \to A$ splits).

1.2. **Azumaya algebras.** To generalize from fields to commutative rings, we define the concept of Azumaya.

In the following proposition, for a commutative ring $R$ and a prime $\mathfrak{p} \in \mathrm{Spec}(R)$, we will write $\kappa(\mathfrak{p})$ to denote the field $\mathrm{frac}(R/\mathfrak{p})$ (also called the residue field of $\mathfrak{p}$).

**Proposition 1.4.** *For an algebra $A$ over a commutative ring $R$ which is finitely generated and projective as a module, the following are equivalent:*
(1) *for every $\mathfrak{p} \in \mathrm{Spec}(R)$, $A \otimes_R \kappa(\mathfrak{p})$ is a central simple $\kappa(\mathfrak{p})$-algebra,*
(2) *the sandwich map $A \otimes_R A^{op} \to End_R(A)$ is an isomorphism.*

**Definition 1.5.** If the equivalent conditions of Proposition 1.4 hold, we say that $A$ is an Azumaya algebra over $R$ (also called a central separable algebra over $R$).

Just as a side comment – it turns out that when $A/R$ is Azumaya it will follow that $A$ is finitely presented as an $R$ module and is a generator in the category of $R$ modules (recall that $M$ is a generator if for every other $R$-module $N$, there is a surjective map $M^I \twoheadrightarrow N$ for some index set $I$). So being an Azumaya algebra imposes serious module-theoretic constraints on an algebra.

## 2. Galois extension of rings

Much like the story for division algebras, while we may start by wanting to construct interesting examples of (central) division algebras, it is useful to consider instead central simple algebras. There are a few natural reasons that this kind of consideration comes up:

- many natural constructions which sometimes yield division algebras will often produce central simple algebras instead,
- when we construct central simple algebras, by the Wedderburn structure theorem, we may find that we have constructed division algebras within them,
- division algebras are not "preserved by scalar extension." In other words, if $D/F$ is a central division algebra, and $E/F$ is a field extension, $D \otimes_F E$ will be central simple, but need not be division.

A very similar discussion arises when considering Galois extension, which leads us to consider the concept of Galois extensions of the form $E/F$ where $E$ need not be a field. From here we will then proceed to consider the case where both $F$ and $E$ are replaced by more general commutative rings (in some analogy with the concept of Azumaya algebras).

2.1. **étale extensions of fields.** Let's start with the generalization of the concept of a (not necessarily Galois) separable field extension, before considering the Galois case:

**Definition 2.1** (Etale extensions of fields)**.** Let $F$ be a field. We say that a commutative $F$-algebra $E/F$ is étale over $F$ if we can write $E$ as a finite (possibly empty) product $E = \bigtimes_{i \in I} E_i$ where each $E_i$ is a separable field extension of $F$.

We note that in the literature, one also says that $E/F$ is a separable extension of rings.

2.1.1. *A strange digression into empty rings.* Let us take just a moment to discuss the edge case in which the product is empty. By convention, an empty product is a final object in a category, and here, considering ourselves to be in the category of unital commutative rings, we find that this final object is the "zero ring," consisting of a single element $0 = 1$. While this ring is not actually a field (because, for example, its nonzero elements fail to form a group, not having an identity element), we still consider the zero ring to be a product of fields, as it is an empty product of fields. Consequently it is an étale extension of every field.

2.2. **Galois étale extensions of fields.** We may or may not get to proving all these equivalent conditions, but here are some ways we can characterize what it means for an étale extension to be Galois.

Recall the following definition:

**Definition 2.2.** Let $S$ be a ring and $G$ a finite group acting on $S$ as automorphisms. We define $(S, G, 1)$, the twisted group ring, to be the algebra generated by $S$ and symbols $u_\sigma$ for $\sigma \in G$, so that as a left $S$-module we have

$$(S, G, 1) = \bigoplus_{\sigma \in G} S u_\sigma,$$

with multiplication given by the rules

$$u_\sigma u_\tau = u_{\sigma\tau} \quad \text{and} \quad u_\sigma x = \sigma(x) u_\sigma, \text{ for } x \in S, \sigma, \tau \in G.$$

**Definition/Lemma 2.3.** *Let $F$ be a field and $E$ a commutative $F$-algebra and let $G \subset \mathrm{Aut}(E/F)$ be a group of automorphisms of $E$ fixing $F$. We say that $E$ is a $G$-Galois extension of $F$ if the following equivalent conditions hold:*

1. $|G| = \dim_F E$ *and* $E^G = F$,
2. $(E, G, 1)$ *is a central simple $F$-algebra,*
3. *the natural map* $(E, G, 1) \to \mathrm{End}_F(E)$ *is an isomorphism,*
4. *the natural map* $(E, G, 1) \to \mathrm{End}_F(E)$ *is injective (i.e. Dedekind's Lemma holds),*
5. *we can write* $E = \bigotimes_{i \in I} E_i$ *with $E_i/F$ separable extensions, and such that the induced action of $G$ on $I$ is transitive and for each $i \in I$, $E_i/F$ is $Stab_G(i)$-Galois.*

An important thing to note is that there is generally no canonical choice for the group $G$ for a given $F$-algebra $E$. So, for example, the $\mathbb{R}$-algebra $\mathbb{C} \times \mathbb{C}$ can be regarded as Galois

- with respect to the group $C_2 \times C_2 = \langle \sigma, \tau \mid \sigma^2, \tau^2 \rangle$ via the action $\sigma(z_1, z_2) = (z_2, z_1)$ and $\tau(z_1, z_2) = (\overline{z}_1, \overline{z}_2)$, or
- with respect to the group $C_4 = \langle \gamma \mid \gamma^4 \rangle$ via the action $\gamma(z_1, z_2) = (z_2, \overline{z}_1)$.

2.3. **Etale extensions of commutative rings.** We will come back to this a bit later when considering étale cohomology and more general descent, but let's define, as we are now able to, the notions of what it means for an extension of commutative rings to be étale.

**Definition 2.4.** Let $R$ be a commutative ring. We say that an $R$-algebra $S$ is étale if it is finitely generated and flat as an $R$-module, and if, for every $\mathfrak{p} \in \mathrm{Spec}(R)$, we have $S \otimes_R \kappa(\mathfrak{p})$ is an étale extension of the field $\kappa(\mathfrak{p})$.

2.4. **Galois extensions of commutative rings.** As with the notion of Azumaya, we are now ready to present the notion of what it means for an extension of rings to be Galois.

**Definition/Lemma 2.5.** *Let $R$ be a commutative ring and $S$ a commutative $R$-algebra. Let $G \subset \mathrm{Aut}(S/R)$ be a group of automorphisms of $S$ fixing $R$. We say that $S$ is a $G$-Galois extension of $R$ if the following equivalent conditions hold:*

*(1) for every $\mathfrak{p} \in \mathrm{Spec}(R)$, $S \otimes_R \kappa(\mathfrak{p})$ is a $G$-Galois extension over $\kappa(\mathfrak{p})$,*
*(2) $(S, G, 1)$ is an Azumaya algebra over $R$,*
*(3) the natural map $(S, G, 1) \to \mathrm{End}_R(S)$ is an isomorphism.*

While not obvious from the definitions, the condition that $S/R$ is $G$-Galois also imposes strong module-theoretic constraints on $S$, namely that $S$ is a finitely generated projective $R$-module which is a generator in the category of $R$-modules. These conditions also imply that $S^G = R$ (as expected from usual Galois theory).

One important consequence of this is that the Morita theorems apply (see Proposition A.2), and we obtain an equivalence of categories as follows:

**Lemma 2.6.** *Let $S/R$ be a $G$-Galois extension of commutative rings. Then we obtain an equivalence of categories*

$$R\text{-modules} \leftrightarrow (S, G, 1)\text{-modules}$$
$$M \mapsto S \otimes_R M$$

*via the standard $(S, G, 1) \cong \mathrm{End}_R(S)$-module structure on $S$.*

We can make this particularly useful by recalling the notion of semilinear actions.

**Definition 2.7.** Let $G$ be a group acting on a commutative ring $S$ and let $M$ be an $S$-module. A $G$-semilinear action on $M$ is an action of $G$ on $M$ as an Abelian group such that for each $\sigma \in G$, $m \in M$, $x \in S$, we have $\sigma(xm) = \sigma(x)\sigma(m)$.

A $G$-semilinear $S$-module is defined to be an $S$-module with a $G$-semilinear action.

We may then consider the category of such $G$-semilinear $S$-modules and observe that this category is also equipt with a tensor product (monoidal) structure. That is, if $M_1, M_2$ are $G$-semilinear $S$-modules, we can define $M_1 \otimes_S M_2$ to have a $G$-semilinear action via

$$\sigma(m_1 \otimes m_2) = \sigma(m_1) \otimes \sigma(m_2).$$

With this notion, we can then define the notion of a $G$-semilinear $S$-algebra (via its structural maps such as $A \otimes_S A \to A$ satisfying various axioms).

We note the following fact, which is easily verified via the definitions:

**Lemma 2.8.** *Let $S$ be a ring with an action of a group $G$. Then there is an equivalence (actually an isomorphism) of categories*

**Lemma 2.9.** *Let $S/R$ be a $G$-Galois extension of commutative rings. Then we obtain an equivalence of categories*

$$R\text{-modules} \leftrightarrow (S, G, 1)\text{-modules}$$
$$M \mapsto S \otimes_R M$$
$$N^G \leftarrow\!\shortmid N$$

We verified implicitly that one of these directions gives an equivalence (at least, by quoting Morita theory). The other direction is given in the exercises.

## 3. Galois Descent

The fundamental question of Galois descent is the following: given a $G$-Galois extension of commutative rings $S/R$, how can one go between algebraic structures over $R$ and algebraic structures over $S$? We can phrase this in terms of two concrete questions:

(1) Given an $R$ algebra $A$, how can we describe all $R$ algebras $A'$ such that $A \otimes S \cong A' \otimes S$?
(2) Given an $S$ algebra $B$, when can we find an $R$ algebra $A$ such that $A \otimes S \cong B$?

### 3.1. **Twisted forms and $H^1$.**

The first question is in large part the subject of the exercises, and we recall here the conclusions. In the context of Lemma 2.9, we can reframe this first question as follows. Given a semilinear action of $G$ on an $S$-algebra $B$ (for example, $B = S \otimes A$), how can we describe all other semilinear actions on $B$. These other actions, via Lemma 2.9, would correspond to $R$-algebras $A'$ such that $S \otimes A' \cong B$. Recall the following definitions:

**Definition 3.1.** Let $X, Y$ be sets with action by a group $G$. Then we obtain a natural action on the set of maps $Map(X, Y)$ via $()\sigma \cdot f)(x) \equiv \sigma(f(\sigma^{-1}(x)))$.

**Definition 3.2.** Let $G, A$ be groups, and suppose we have a homomorphism $G \to \mathrm{Aut}(A)$ providing an action of $G$ on $A$. We say that a map $\alpha : G \to A$ is a crossed homomorphism, or a 1-cocycle, if

$$\alpha(\sigma\tau) = \alpha(\sigma)\sigma(\alpha(\tau)), \quad \forall \sigma, \tau \in G.$$

We write $Z^1(G, A)$ for the set of all crossed homomorphisms.

**Definition 3.3.** The group $A$ acts on $Z^1(G, A)$ via $(a \cdot \alpha)(\sigma) = a\alpha(\sigma)\sigma(a)^{-1}$, and we define $H^1(G, A) = Z^1(G, A)/A$ to be the set of orbits under this action.

We note that in the case $A$ is an Abelian group, this corresponds to the standard group cohomology construction, and the sets $Z^1(G, A)$ and $H^1(G, A)$ have natural group structure given by pointwise multiplication in $A$. In general, however, these are just sets with distinguished elements (pointed sets), where the distinguished element comes from the crossed homomorphism $G \to A$ sending all elements to the identity.

**Proposition 3.4.** *Let $B$ be a $G$-semilinear $S$-algebra, with action written as $(\sigma, b) \mapsto \sigma b$. Consider the $G$-action on $\mathrm{Aut}_S(B)$ given by Definition 3.1. Then if we have any other $G$-semilinear action on $B$, $(\sigma, b) \mapsto \sigma \cdot b$, then we may find a crossed homomorphism $\alpha : G \to \mathrm{Aut}_S(B)$ such that*

$$\sigma \cdot b = \alpha(\sigma)\sigma b,$$

*and this gives a bijection between crossed homomorphism and semilinear actions.*

*Further, if $\alpha, \beta \in Z^1(G, \mathrm{Aut}_S(B))$ are crossed homomorphisms, then the resulting semilinear algebras are isomorphic if and only if $\alpha$ and $\beta$ are in the same $\mathrm{Aut}_S(B)$ orbit. In particular, we have a bijection between isomorphism classes of algebras $A'/R$ such that $S \otimes A' \cong B$ and the pointed set $H^1(G, \mathrm{Aut}_S(B))$.*

### 3.2. **Descent obstructions and $H^2$.**

We now consider the second question – given an $S$-algebra $B$, when can we find an $R$-algebra $A$ such that $S \otimes_R A \cong B$? In light of Lemma 2.9, this is equivalent to asking the question of when we are able to define a semilinear action of $G$ on $B$.

To make this easier to work with, let's define a bit of language:

**Definition 3.5.** Let $B$ be an $S$-algebra and let $\sigma$ be an automorphism of $S$. We define a new $S$-algebra, denoted ${}^{\sigma}S$ to have underlying set ${}^{\sigma}x$, $x \in S$ (that is, there is a bijection between the elements of $B$ and ${}^{\sigma}B$), with operations:

$${}^{\sigma}x + {}^{\sigma}y = {}^{\sigma}(x+y), \qquad ({}^{\sigma}x)({}^{\sigma}y) = {}^{\sigma}(xy), \qquad \forall x, y \in B$$

and with $S$-module structure given by:

$$\lambda\ {}^{\sigma}x = {}^{\sigma}(\sigma^{-1}(\lambda)x), \qquad \forall \lambda \in S, x \in B,$$

or in other words, $\sigma(\lambda)\ {}^{\sigma}x = {}^{\sigma}(\lambda x)$.

**Example 3.6.** *As an example, note that if $B$ is an $S$-algebra with a free $S$-module basis $e_i$ and with multiplication table given by*

$$e_i e_j = \sum_k c_{i,j}^k e_k,$$

*then the algebra ${}^{\sigma}B$ has multiplication table given by*

$${}^{\sigma}e_i {}^{\sigma}e_j = \sum_k \sigma(c_{i,j}^k)\ {}^{\sigma}e_k.$$

Now, back to the case of a $G$-Galois extension $S/R$ and an $S$-algebra $B$, we would like to ask whether or not it is possible to define a semilinear action of $G$ on $B$. This amounts to defining, for every $\sigma \in G$ a "possible action,"

$$\phi_\sigma : B \to B$$

which will satisfy $\phi_\sigma(\lambda x) = \sigma(\lambda)\phi_\sigma(x)$ for $\lambda \in S, x \in B$, and such that $\phi_\sigma\phi_\tau = \phi_{\sigma\tau}$. One complicating factor is that such maps $\phi_\sigma$ are evidently not $S$-linear, but we can change our perspective by considering the corresponding maps $\psi_\sigma : {}^{\sigma}B \to B$ given by $\psi_\sigma({}^{\sigma}x) = \phi_\sigma(x)$. For this map, we find

$$\psi_\sigma(\lambda\ {}^{\sigma}x) = \psi_\sigma({}^{\sigma}(\sigma^{-1}(\lambda)x)) = \phi_\sigma(\sigma^{-1}(\lambda)x) = \lambda\phi_\sigma(x) = \lambda\psi_\sigma({}^{\sigma}x),$$

which allows us to encode the information of $\phi_\sigma$ as an $S$-linear map $\psi_\sigma$. If we let $\sigma : B \to {}^{\sigma}B$ denote the map $x \mapsto {}^{\sigma}x$ (which we can think of as a "universal" $\sigma$-linear map), then we can consider this via the following diagram

$$\begin{tikzcd} {}^{\sigma}B \arrow[r, "\psi_\sigma"] & B \arrow[d, equal] \\ B \arrow[u, "\sigma"] \arrow[r, "\phi_\sigma"'] & B \end{tikzcd}$$

as $\psi_\sigma(x) = \sigma(\phi_\sigma(\sigma^{-1}x))$. More generally, we may "twist" these to obtain maps

$$\begin{tikzcd} {}^{\sigma\tau}B \arrow[r, "{}^{\sigma}\psi_\tau"] & {}^{\sigma}B \\ {}^{\tau}B \arrow[u, "\sigma"] \arrow[r, "\psi_\tau"'] & B \arrow[u, "\sigma"'] \end{tikzcd}$$

$$\begin{aligned} {}^{\sigma}\psi_\tau : {}^{\sigma\tau}B &\to {}^{\sigma}B, \\ {}^{\sigma\tau}x &\mapsto {}^{\sigma}(\psi_\tau({}^{\tau}x)) = {}^{\sigma}\phi_\tau(x). \end{aligned}$$

This perspective allows us to interpret the condition $\phi_\sigma\phi_\tau = \phi_{\sigma\tau}$ in terms of $S$-linear maps. That is, we have

$$\begin{aligned} \psi_{\sigma\tau} : {}^{\sigma\tau}B &\to B, \\ {}^{\sigma\tau}x &\mapsto \phi_{\sigma\tau}(x), \end{aligned}$$

and,

$$\begin{aligned} \psi_\sigma {}^{\sigma}\psi_\tau : {}^{\sigma\tau}B &\to B, \\ {}^{\sigma\tau}x &\mapsto \phi_\sigma\phi_\tau(x). \end{aligned}$$

Consequently, the condition $\phi_\sigma\phi_\tau = \phi_{\sigma\tau}$ corresponds to the condition $\psi_{\sigma\tau} = \psi_\sigma {}^{\sigma}\psi_\tau$.

Analyzing the possibilities, we see:

**Case 1**: ${}^{\sigma}B$ and $B$ are not isomorphic for some $\sigma \in G$.
In this case, there is no possible way that $\sigma$ can act on $B$, and so no hope for defining a semilinear action of $B$. Consequently, there is no algebra $A/R$ such that $S \otimes A \cong B$.

**Case 2**: There exist isomorphisms $\psi_\sigma : {}^{\sigma}B \xrightarrow{\sim} B$ for each $\sigma \in G$.
In this case, we need only consider whether or not these can be chosen so that $\psi_{\sigma\tau} = \psi_\sigma {}^{\sigma}\psi_\tau$. To measure our "distance" from this condition, we define:[1]

$$\beta(\sigma,\tau) = \psi_{\sigma\tau}^{-1}\psi_\sigma {}^{\sigma}\psi_\tau \in \mathrm{Aut}_S(B).$$

We are successful if we can choose $\phi_\sigma$ so as to make $\beta(\sigma,\tau) = 1$ for all $\sigma,\tau$. Tracing the following diagram:

$$\begin{tikzcd}
{}^{\sigma\tau\gamma}B \arrow[r, "{}^{\sigma\tau}\psi_\gamma"] \arrow[rr, bend left, "{}^{\sigma\tau}\psi_\gamma"] \arrow[rrr, bend left=40, "\psi_{\sigma\tau\gamma}"] \arrow[rrr, bend right=40, "\psi_{\sigma\tau\gamma}"'] & {}^{\sigma\tau}B \arrow[r, "{}^{\sigma}\psi_\tau"] \arrow[rr, bend right, "\psi_{\sigma\tau}"'] & {}^{\sigma}B \arrow[r, "\psi_\sigma"] & B
\end{tikzcd}$$

[1]note, this is a somewhat different convention than the one we did in class

## Appendix A. semilinear spaces (the descent data category) with exercises

**Exercise 1.** If $E/F$ is a $G$-Galois extension of fields, show that the natural map

$$\begin{aligned}(E, G, 1) &\to \mathrm{End}_F(E)\\ x &\mapsto [y \mapsto xy], \quad x, y \in E\\ u_\sigma &\mapsto [y \mapsto \sigma(y)], \quad \sigma \in G, y \in E\end{aligned}$$

gives an isomorphism of algebras.

**Definition A.1.** Recall that if $E/F$ is a $G$-Galois extension, an $E/F$-semilinear vector space is an $E$-vector space $V$ together with an action of $G$ on $V$ such that for every $x \in E$, $v \in V$, we have

$$\sigma(xv) = \sigma(x)\sigma(v).$$

A homomorphism of $E/F$ semilinear vector spaces $\phi : V \to W$ consists of an $E$-linear map $\phi$ which commutes with the $G$-action in the sense that $\phi(\sigma v) = \sigma(\phi v)$.

**Exercise 2.** If $V$ is an $F$-vector space then $E \otimes_F V$ is naturally an $E/F$ semilinear vector space, where the action of $G$ is via the first factor.

**Exercise 3.** Show that we have an equivalence of categories between $(E, G, 1)$-modules and $E/F$-semilinear vector spaces.

Recall the following result which we claimed in the last lecture:

**Proposition A.2** (Morita)**.** *Let $R$ be a ring and $P$ a right $R$-progenerator (i.e. finitely generated, projective generator in the category of right $R$-modules). Let $S = \mathrm{End}_R(P)$. Then the functor from $R$-modules to $S$-modules given by*

$$N \mapsto P \otimes_R N$$

*is an equivalence of categories. Further, if $P^\star = Hom_R(P, R)$ then $P^\star$ is an $R - S$ bimodule, and*

$$M \mapsto P^\star \otimes_S M$$

*gives the (homotopy) inverse equivalence.*

**Exercise 4.** Show that the functor from $F$-vector spaces to $E/F$-semilinear vector spaces given by

$$V \mapsto V_E \equiv E \otimes_F V$$

is an equivalence of categories.

Now, if we are interested in talking about algebraic objects (such as central simple algebras), we need more than just vector spaces and linear maps, but we also need the concept of the tensor product (for multiplicative structures).

**Definition A.3.** Suppose $V, W$ are $E/F$ semilinear vector spaces. Then $V \otimes_E W$ is also a semilinear vector space with respect to the action:

$$\sigma(v \otimes w) = \sigma(v) \otimes \sigma(w).$$

**Exercise 5.** Show that the above definition gives a well defined $E/F$ semilinear space and that this commutes with the functor given above.

That is, show that if $V, W$ are $F$-vector spaces, then we have a natural isomorphism of $E/F$ semilinear vector spaces

$$V_E \otimes_E W_E \cong (V \otimes_F W)_E.$$

More formally (if you like), this means you are showing that the two functors

$$(V, W) \mapsto (V_E \otimes_E W_E) \quad (V_W) \mapsto (V \otimes_F W)_E$$

from $Vec/F \times Vec/F$ to the category of $E/F$ semilinear vector spaces are naturally isomorphic.

From this point of view it makes sense to talk about $E/F$ semilinear algebras.

**Definition A.4.** An $E/F$ semilinear algebra is an $E/F$ semilinear vector space $A$, together with an $E/F$-semilinear map

$$m : A \otimes_E A \to A$$

and an $E/F$-semilinear map

$$\iota : E \to A$$

which gives $A$ the structure of an algebra (where $\iota(1) = 1$ is the multiplicative identity of $A$).

**Exercise 6.** Show that an $E/F$ semilinear algebra is just an $E$-algebra $A$ with a semilinear action of $G$ on $A$ as a vector space such that $\sigma(ab) = \sigma(a)\sigma(b)$ (i.e. such that $G$ acts via ring isomorphisms).

**Exercise 7.** Show that we have an equivalence of categories between $F$-algebras and $E/F$-semilinear algebras given by $A \mapsto E \otimes_F A$.

**Exercise 8.** It follows from the above exercise that if we let $F = \mathbb{R}$ and $E = \mathbb{C}$, then we have an equivalence between $\mathbb{R}$-algebras and $\mathbb{C}$-algebras with a notion of conjugation (action by $\mathcal{G}\mathrm{al}(\mathbb{C}/\mathbb{R})$). In particular, if we consider the $\mathbb{R}$-algebras $\mathbb{H}$ and $M_2(\mathbb{R})$, we see that

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong M_2(\mathbb{C}) \cong \mathbb{C} \otimes_{\mathbb{R}} M_2(\mathbb{R})$$

and so as $\mathbb{C}/\mathbb{R}$ semilinear algebras, both of these algebras are given as $M_2(\mathbb{C})$ with two different notions of conjugation. What are these notions of conjugation?

## Appendix B. twisted forms (the gluing problem) with exercises

Throughout the section, let us fix $E/F$ a $G$-Galois extension.

**Definition B.1.** Let $A$ be an $F$-algebra. We say that an $F$-algebra $B$ is a(n $E/F$-)twisted form of $A$ if there is an isomorphism of $E$-algebras, $A_E \cong B_E$.

Note that we are not assuming here that we have an isomoprhism of $E/F$ semilinear algebras (which would imply they were isomorphic over $F$), but just as $E$-algebras.

As we saw in the previous section, we can recover the structure of $B$ from $B_E$ by specifying a semilinear action. As we are able to identify $A_E$ and $B_E$, our quest to understanding the possible $B$'s we may have then reduces to understanding all possible semilinear actions of $G$ on $A_E$.

**Definition B.2.** Suppose $V$ is a vector space with an action of $G$. We define an action of $G$ on $\mathrm{Aut}(V)$ by $(\sigma\phi)(v) = \sigma(\phi(\sigma^{-1}(v))$.

**Exercise 9.** Show that in the case $V = E^n$, with component-wise action, the action of the Galois group $G = \mathscr{G}\mathrm{al}(E/F)$ on $Aut(V) = GL_n(E)$ is given by the standard action on the matrix entries.

**Exercise 10.** Suppose $\phi, \psi : G \to \mathrm{Aut}(A_E)$ are two different semilinear actions of $G$ on $A_E$. That is, for $\sigma \in G$, we have $\sigma(a) \equiv \phi(\sigma)(a)$ and $\sigma(a) \equiv \psi(\sigma)(a)$ define semilinear actions (note here that $\phi$ and $\psi$ need not have values in $E$-automorphisms, but in general just $F$-linear automorphisms).

Show that $\phi(\sigma) = \alpha(\sigma)\psi(\sigma)$ for a map $\alpha : G \to \mathrm{Aut}(A_E)$ and $\alpha$ is a crossed homomorphism (where the action of $G$ on $\mathrm{Aut}(A_E)$ here is given by the previous excercise via $\psi$).

**Exercise 11.** Show that the above correspondence gives, after fixing an algebra $A/F$ a bijection between semilinear actions on $A_E$ and crossed homomorphisms $G \to \mathrm{Aut}_F(A_E)$.

From this we see so far that for $B/F$ a twisted form of $A$, given an isomorphism $\phi : B_E \to A_E$, we obtain a new semilinear action on $A_E$ which corresponds to the algebra $B/F$ via the equivalence of categories previously described. This semilinear action, in turn gives rise to crossed homomorphism $G \to \mathrm{Aut}(A_E)$.

It therefore is natural to ask: in what way does this semilinear action depend on the isomorphism $\phi$?