NOTES ON DERIVED CATEGORIES AND MOTIVES

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1. Introduction

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On the face of it, motives and derived categories[1] seem to play similar roles. Both of these objects sit in between geometric objects, say varieties, and their cohomologies. It is therefore natural to ask what the relationship might be between them.

Out the outset, however, we should aknowledge that these sit in different worlds – in fact, there are essentially different parts of speech. While the motive of a variety is an object in a conjectural Abelian category of motives, the derived category of a variety is itself a category with additional structure (i.e. a triangulated structure). How then are we to reasonably compare these things?

One should say that it would be quite reasonable to consider, instead of the derived category, the noncommutative motive of a variety. This is a natural thing to consider which is both closer to the derived category and shares a closer philosophical similarity to the motive, however, we will not pursue this direction here.

[1]By which we mean here the bounded derived categories of coherent sheaves on algebraic varieties. More precision and definitions will be given later.

Both the category of motives and the derived category of a variety are in some sense built to resolve objects – break things up, or explain how to reconstruct our objects from simpler pieces. However, while the category of motives is designed to do this for varieties, the derived category of a variety is designed to do this for (coherent) sheaves on the variety.

One possible relationship between these is via the Chern characters. In some sense, the derived category is an enrichment of K-theory and motives an enrichment of Chow groups. Since these have a classical relationship (Riemenn-Roch) expressed in terms of the Chern character, it is natural to see the extent to which this information may be leveraged, and we look a bit at this question.

The structure of this document is as follows. To begin, in section 2 we will give a quick and somewhat dirty introduction to the bounded derived category of coherent sheaves of a scheme, particularly focusing on the notion of the Fourier-Mukai transform and mention a few results of importance. In Section 3 we give a quick overview of K-theory[2], Chow groups, the Chern character and the Riemann-Roch Theorem. In Section 4, we say a bit about the conjectural notion of the motive of a variety, looking in particular at the Zeta function of a variety over a finite field, and at Chow motives. Finally, in Section conj we describe some current open problems and partial results relating derived categories and motives.

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2. The bounded derived category of a variety

2.1. **Quick and dirty derived categories.** A derived category is a category associated to an Abelian category, which sits in between the original category and various derived and cohohomological-type functors which one might want to compute.

In fact, there are a number of standard variations on the derived category. If $\mathscr{A}$ is an Abelian category, the objects of the derived category are complexes of objects in $\mathscr{A}$, that is, sequences of objects and morphisms indexed[3] by the integers

$$\cdots \xrightarrow{d_{i-1}} A^i \xrightarrow{d_i} A^{i+1} \xrightarrow{d_{i+1}} \cdots$$

[2]really just the Grothendieck group K_0

[3]here we are using "cohomological notation" in which the degrees are written as superscripts, and where the differentials increase the degree by 1, versus the homological notation which does the opposite. Writing $A_i = A^{-i}$ switches between these two conventions.

and we may, if we wish, consider all complexes, or only complexes such that $A^n = 0$ for $n >> 0$, for $n << 0$, or for $|n| >> 0$. These result in categories which would be referred to as $D(\mathscr{A})$, $D^+(\mathscr{A})$, $D^-(\mathscr{A})$, and $D^b(\mathscr{A})$, respectively. If we wish to be ambiguous about which we are referring to, we will simply write $D^*(\mathscr{A})$ to denote any one of these possibilities.

Although we will need to refine this later, we can quickly define the derived category $D^*(\mathscr{A})$ of an Abelian category $\mathscr{A}$ as follows.

Definition 1. *For an Abelian category $\mathscr{A}$, we define the category $coCh^*(\mathscr{A})$ of cochain complexes of $\mathscr{A}$ (where $*$ is either "empty" or is one of the symbols $+, -, b$), to be the category whose objects are sequences of objects and morphisms of $\mathscr{A}$ of the form:*

$$A^\bullet = \cdots \xrightarrow{d_{i-1}} A^i \xrightarrow{d_i} A^{i+1} \xrightarrow{d_{i+1}} \cdots$$

where $A^n = 0$ if $n >> 0$ in case $* = +$, or if $n << 0$ in case $* = -$, or if $|n| >> 0$ in case $* = b$, and such that $d^{i+1}d^i = 0$ for all i. Morphisms $f^\bullet : A^\bullet \to B^\bullet$ defined to be collections of morphisms $f^i : A^i \to B^i$ such that we have commutative diagrams:

$$\begin{tikzcd} A^i \arrow[r, "d^i"] \arrow[d, "f^i"'] & A^{i+1} \arrow[d, "f^{i+1}"] \\ B^i \arrow[r, "d^i"'] & B^{i+1} \end{tikzcd}$$

We will typically omit the superscripts from the d^i, and write $d = d^i$.

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Definition 2. *If $A^\bullet$ is an object of $coCh^*(\mathscr{A})$, we define $\mathcal{H}^n(A^\bullet)$ to be the cohomology of the complex $A^\bullet$ at the n-th spot. That is*

$$\mathcal{H}^n(A) = \frac{\ker\left(d : A^n \to A^{n+1}\right)}{\operatorname{im}\left(d : A^{n-1} \to A^n\right)}.$$

One can check that a morphism $f^\bullet : A^\bullet \to B^\bullet$ induces homomorphisms of Abelian groups $\mathcal{H}^n(f^\bullet) : \mathcal{H}^n(A^\bullet) \to \mathcal{H}^n(B^\bullet)$, making $\mathcal{H}^n : coCh^*(\mathscr{A}) \to \mathscr{A}$ an additive functor.

Definition 3. *Let $f^\bullet : A^\bullet \to B^\bullet$ be a morphism in $coCh^*(\mathscr{A})$. We say that $f^\bullet$ is a quasi-isomorphism if $\mathcal{H}(f^\bullet)$ is an isomorphism for all n.*

We now recall the following category-theoretic fact:

Theorem 4. *([GM03, Section III.2.2]) Let $\mathscr{B}$ be an arbitrary category and S an arbitrary class of morphisms of $\mathscr{B}$. Then there exists a category $\mathscr{B}[S^{-1}]$ and a functor $Q : \mathscr{B} \to \mathscr{B}[S^{-1}]$ with the following universal property:* YouTube

- *$Q(f)$ is an isomorphism for every $f \in S$,*
- *given any functor $F : \mathscr{B} \to \mathscr{D}$ such that $F(f)$ is an isomorphism for every $f \in S$, there exists a unique functor $G : B[S^{-1}] \to D$ such that $F = G \circ Q$.*

We may now define

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Definition 5. *For an Abelian category $\mathscr{A}$, let $QI^*(\mathscr{A})$ to be the collection of quasi-isomorphisms in $coCh^*(\mathscr{A})$. We define the derived category of $\mathscr{A}$ to be:*

$$D^*(\mathscr{A}) = coCh^*(\mathscr{A})[(QI^*(\mathscr{A})^{-1}].$$

In case $* = b$, we refer to $D^b(\mathscr{A})$ as the bounded derived category of A, and $D(\mathscr{A})$ as the (unbounded) derived category.

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Definition 6. *Let X be a scheme. We define $D^*(X)$ to be the derived category $D^*(Coh(X))$ where $Coh(X)$ is the Abelian category of coherent sheaves on X.*

Although this definition has the advantage of a compact description, it is lacking in various other important features which we will need to address:

- the derived category $D^*(\mathscr{A})$ has extra structure: it is a triangulated category, and this extra structure is not easily apparent in the above definition
- hom sets in this description are difficult to deal with – it is hard both to compute homs, and to see how they compose

For the first problem, we will need to first discuss the concept of triangualted categories, and then go over a somewhat more constructive definition of the derived category.

For the latter problem, our constructive definition will also be helpful, but there are futher problems, which can also be understood in terms of injective objects. As we have seen, the computation of many important things, cohomology, exts, etc, depend on the existence of injective resolutions, however these injective objects are essentially never coherent. To resolve our objects, we often need to go to larger categories, such as quasi-coherent sheaves, or even arbitrary sheaves of $\mathscr{O}_X$-modules. To address this, we will have to see how derived categories of different Abelian categories relate to each other.

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2.2. **Triangulated categories.** The derived category, as we have said, contains extra structure, and this is related to the way in which one obtains long exact sequences in cohomology from short exact sequences of complexes. A triangulated category is a category $\mathscr{T}$, together with two extra pieces of data: an endofunctor $T : \mathscr{T} \to \mathscr{T}$ (corresponding to shifting of indices for complexes), and a notion of distinguished triangles, which are particular objects and morphisms of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$$

satisfying a number of axioms. For the sake on convenience, and just temporarily, we will use the notation $C \overset{h}{\rightsquigarrow} A$ to say that h is a morphism from C to TA. We can then denote the above diagram as looking like

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \arrow[dl, "g"] \\ & C \arrow[ul, rightsquigarrow, "h"] & \end{tikzcd}$$

hence the term "triangle."[4]

There are many ways to define triangulated categories, and I am relying particularly on [Nee01; GM03; May] as my sources for this and related materials.

Let $\mathscr{T}$ be an additive category, and $T : \mathscr{T} \to \mathscr{T}$ an additive functor which is an equivalence (we will refer to such a functor as an additive autoequivalence).

Definition 7. *A **triangular diagram** is a diagram of the form*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA.$$

A morphism of triangular diagrams

$$[A \to B \to C \to TA] \to [A' \to B' \to C' \to TA']$$

is a triple of morphisms $a : A \to A'$, $b : B \to B'$, $c : C \to C'$ such that we have a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & TA \arrow[d, "Ta"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & TA' \end{tikzcd}$$

Definition 8. *A **triangulated category** is an additive category $\mathscr{T}$, together with an additive autoequivalence T and a class Δ of triagular diagrams (called **distinguished triangles**), such that axioms TR1, TR2, TR3, and TR4 described below hold.*

[4]more conventionally in the literature one typically writes $\xrightarrow{[1]}$ in place of this squiggly arrow. I found the squigglies easier to typeset however.

These axioms take a bit of explanation, particularly the infamous TR4 (a.k.a. the "octahedral axiom"). Let's take a look at them:

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2.2.1. *Triangulated axioms, part one (everything but the octahedral axiom).* Throughout the section, let $(\mathscr{T}, T, \Delta)$ be an additive category with an autoequivalence and a class of triangular diagrams. We will abuse notation and refer to this entire triple simply as $\mathscr{T}$.

The first axiom is actually a three-part axiom concerining the existence of distinguished triangles:

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Axiom (TR1). *We say that $\mathscr{T}$ satisfies TR1 if:*

i. *For any object A of $\mathscr{T}$, the triangular diagram $A \xrightarrow{id} A \to 0 \to TA$ is in Δ,*
ii. *Any triangular diagram which is isomorphic to a triangle in Δ is also in Δ,*
iii. *For every morphism $A \to B$ in $\mathscr{T}$, we can find an object C and morphisms $B \to C$ and $C \to TA$ giving a triangular diagram $A \to B \to C \to TA$ which is in Δ.*

This axiom says that triangles can be "shifted" or "rotated" (depending on how the diagram is drawn):

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Axiom (TR2). *We say that $\mathscr{T}$ satisfies axiom TR2 if for any triangular diagram $A \xrightarrow{f} B \to C \to TA$ in Δ, the triangular diagram $B \to C \to TA \xrightarrow{-Tf} TB$ is also in Δ.*

The third axiom concerns the existence of morphisms between distinguished triangles:

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Axiom (TR3). *We say that $\mathscr{T}$ satisfies axiom TR3 if the following holds. Suppose $A \to B \to C \to TA$ and $A' \to B' \to C' \to TA'$ are triangular diagrams in Δ, and we are given morphisms $a : A \to A'$ and $b : B \to B'$ such that we have a commutative square*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r] & B'. \end{tikzcd}$$

Then we may find a morphism $c : C \to C'$ giving rise to a morphism of triangular diagrams in Δ

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & TA \arrow[d, "Ta"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & TA' \end{tikzcd}$$

Just for fun, let's notice a quick corollary of these axioms so far. YouTube

Lemma 9. *Suppose $\mathscr{T} = (\mathscr{T}, T, \Delta)$ satisfies axioms TR1, TR2, TR3. Then for every triangular diagram* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$, *the compositions $gf, hg, (Tf)h$ are all 0.*[5]

Proof. By Axiom TR2 we need only check $gf = 0$, since the others will follow by shifting. Now, by Axiom TR1(i), the triangular diagram $A \xrightarrow{\mathrm{id}} A \to 0 \to TA$ is in Δ, and we have the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

By Axiom TR3, we may complete this to a morphism of triangular diagrams

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & TA \arrow[d, "\mathrm{id}"] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & TA, \end{tikzcd}$$

from which it follows from the commutativity of the middle square that $gf = 0$ as claimed. □

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2.2.2. *Triangulated axioms, part two (the octahedral axiom).* Before plunging into the octahedral axiom, which one can interpret as being able to fill in morphisms between triangles a la TR3 in a compatible way when multiple triangles are involved. Let's first develop a bit of context.

Suppose that $\mathscr{T} = (\mathscr{T}, T, \Delta)$ satisfies axioms TR1, TR2, TR3 above, and suppose we are given a commutative diagram[6]:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "h"] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

[5]In [Nee01], only triangular diagrams satisfying the conclusion of this lemma are considered. These are called "candidate triangles."

[6]although this is shaped like a triangle, we are not saying that it is a triangular diagram!

From this diagram, we can consider $f : A \to B$, $g : B \to C$ and $h : A \to C$ and for each of these, use Axiom TR2 to make triangular diagrams in Δ:

$$t_1 : A \xrightarrow{f} B \xrightarrow{f'} D \xrightarrow{f''} TA,$$

$$t_2 : A \xrightarrow{h} C \xrightarrow{h'} E \xrightarrow{h''} TA,$$

$$t_3 : B \xrightarrow{g} C \xrightarrow{g'} F \xrightarrow{g''} TB.$$

From the commutative squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & B \arrow[d, "g"] \\ A \arrow[r, "h"'] & C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "\mathrm{id}"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

and Axiom TR3, we can put these together to obtain morphisms of triangles

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & B \arrow[r, "f'"] \arrow[d, "g"] & D \arrow[r, "f''"] \arrow[d, "j"] & TA \arrow[d, "\mathrm{id}"] \\ A \arrow[r, "h"'] & C \arrow[r, "h'"'] & E \arrow[r, "h''"'] & TA \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[r, "h"'] \arrow[d, "f"'] & C \arrow[r, "h'"'] \arrow[d, "\mathrm{id}"'] & E \arrow[r, "h''"'] \arrow[d, "j'"] & TA \arrow[d, "Tf"] \\ B \arrow[r, "h"'] & C \arrow[r, "h'"'] & F \arrow[r, "h''"'] & TB \end{tikzcd}$$

We can put these together to make a diagram like this[7]:

[7]credit for the code for this and similar diagrams following go to [Gre]

$$
\begin{tikzcd}
A \arrow[rr, bend left, "h"] \arrow[dr, "f"'] & & C \arrow[rr, bend left, "g'"] \arrow[dr, "h'"] & & F \arrow[dr, "g''"] & & TD \\
& B \arrow[ur, "g"] \arrow[dr, "f'"'] & & E \arrow[ur, "j'"] \arrow[dr, "h''"] & & TB \arrow[ur, "Tf'"'] & \\
& & D \arrow[ur, "j"] \arrow[rr, bend right, "f''"'] & & TA \arrow[ur, "Tf"'] & &
\end{tikzcd}
$$

Now, there was some potential choices involved here. We note that the morphisms j, j' are known to exist by Axiom TR3, they were not assumed to be unique. Axiom TR4 will say that we can choose them to be part of a triangular diagram in Δ in a natural way:

Axiom (Axiom TR4)**.** *Suppose we are given a commutative diagram of the form*

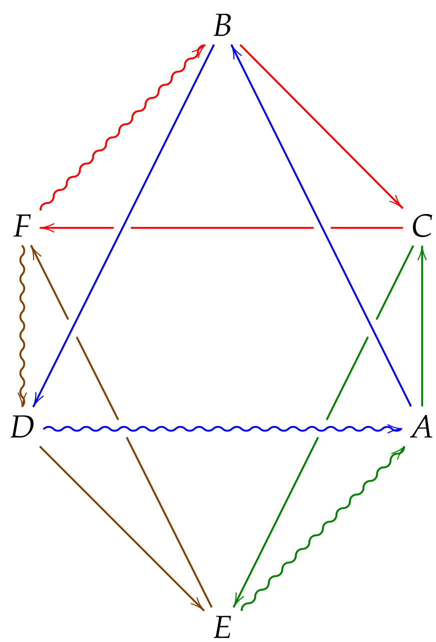
$$
\begin{tikzcd}
A \arrow[rr, bend left, "h"] \arrow[dr, "f"'] & & C \arrow[rr, bend left, "g'"] \arrow[dr, "h'"] & & F \arrow[rr, bend left, "j''"] \arrow[dr, "g''"] & & TD \\
& B \arrow[ur, "g"] \arrow[dr, "f'"'] & & E \arrow[dr, "h''"] & & TB \arrow[ur, "Tf'"'] & \\
& & D \arrow[rr, bend right, "f''"'] & & TA \arrow[ur, "Tf"'] & &
\end{tikzcd}
$$

Such that the blue, green and red subdiagrams are in Δ (so that, in particular $j'' = (Tf')g''$). Then we may find morphisms $j : D \to E$ and $j' : E \to F$ giving morphisms between the blue and green and green and red triangles respectively (as in TR3) giving a commutative diagram

$$\begin{tikzcd}
A \arrow[rr, bend left, "h"] \arrow[dr, "f"'] & & C \arrow[rr, bend left, "g'"] \arrow[dr, "h'"] & & F \arrow[rr, bend left, "j''"] \arrow[dr, "g''"] & & TD \\
& B \arrow[ur, "g"] \arrow[dr, "f'"'] & & E \arrow[ur, "j'"] \arrow[dr, "h''"] & & TB \arrow[ur, "Tf'"'] & \\
& & D \arrow[ur, "j"] \arrow[rr, bend right, "f''"'] & & TA \arrow[ur, "Tf"'] & &
\end{tikzcd}$$

and such that the brown triangular diagram is also in Δ*.*

The reason that this is called the octahedral axiom is as follows. Using the prior notation $f : C \rightsquigarrow A$ to denote an arrow $f : C \to TA$, we may draw this diagram as:



To state the octahedral formulation, we will adopt a temporary convention.

Definition 10. *We will say that a diagram of the form*

$$\begin{tikzcd}
X \arrow[rr, rightsquigarrow] \arrow[dr, rightsquigarrow] & & Y \arrow[dl] \\
& Z &
\end{tikzcd}$$

commutes if the corresponding diagram

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & TY \arrow[dl] \\
& TZ &
\end{tikzcd}$$

commutes, and similarly for diagram of the form

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, squiggly] & & Y \arrow[dl, squiggly] \\
& Z. &
\end{tikzcd}$$

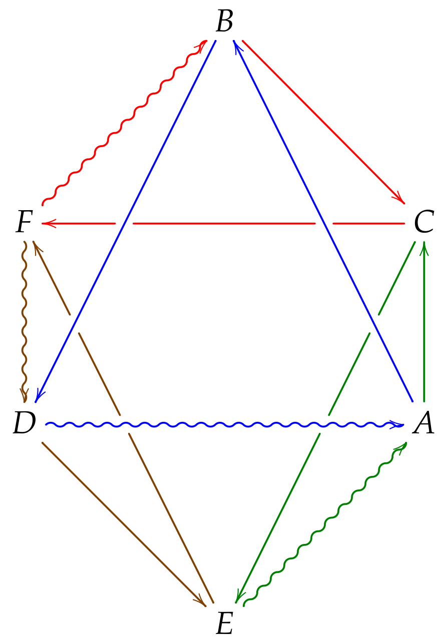
So, in this language, the octahedral formulation is:

Axiom (Octahedral formulation of TR4)**.** *Suppose we are given a diagram of the form*

$$\begin{tikzcd}
& B \arrow[dr, red] \arrow[ddl, blue] & \\
F \arrow[ur, squiggly, red] & & C \arrow[ll, red] \arrow[ddl, green] \\
D \arrow[rr, squiggly, blue] & & A \arrow[uul, blue] \arrow[u, green] \\
& E \arrow[ur, squiggly, green] &
\end{tikzcd}$$

Where the triangle on the upper right is commutative, and the three monochromatic triangular diagrams are in Δ. Then we may find morphisms $D \to$

$E \to F \to TD$ such that in the diagram



every monochromatic triangular subdiagram is in Δ and every tricolored face is commutative (literally, or commuative in the sense of Definition 10).

2.3. **Homotopy categories and derived categories.** To get to a more practical definition of the derived category, it is useful to consider a category which naturally sits in betweeen the categories $coCh^*(\mathscr{A})$ of cochain complees and its localization $D^*(\mathscr{A})$, the derived category. This is the homotopy category $K^*(\mathscr{A})$.

Definition 11. *If $A^\bullet, B^\bullet$ are cochain complexes, and $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ are morphisms between them, a* ***(cochain) homotopy*** *$h^\bullet : f^\bullet \to g^\bullet$ is a collection of maps $h^i : A^i \to B^{i-1}$ such that $g - f = dh + hd$. If such a homotopy exists, we say that f and g are* ***homotopic****. If a morphism is homotopic to the 0 map, we say it is* ***null-homotopic****.*

It is clear that homotopy gives an equivalence relation on the Abelian group of cochain homotopies between two cochain complexes. Further, the quotient by this relation can be identified with the quotient by the subgroup of null-homotopic maps.

The following Lemma is of key importance:

Lemma 12. *[GM03, Lemma III.4.3, p. 159]*
Homotopic maps of cochain complexes have equal images in the derived category.

Definition 13. *Let $\mathscr{A}$ be an additive category*[8]*. We define $K^*(\mathscr{A})$ to be the category whose objects coincide with the objects of $coCh^*(\mathscr{A})$, and with $Hom_{K^*(\mathscr{A})}(A^\bullet, B^\bullet) = Hom_{coCh^*(\mathscr{A})}(A^\bullet, B^\bullet)/N(A^\bullet, B^\bullet)$, where $N(A^\bullet, B^\bullet)$ is the subgroup of null-homotopic cochain maps.*

One does need to check that gives a well defined category – the main observation being that null-homotopic maps are like an "ideal," in the sense that they are closed under composition on either side.

The category $K^*(\mathscr{A})$ has some distinct advantages over $coCh^*(\mathscr{A})$:

- it is clearly additive and in fact it is a triangulated category,
- it's morphisms are easy to describe and compose,
- for $\mathscr{A}$ Abelian, we can construct $D^*(\mathscr{A})$ from it by a significantly simpler localization process,
- in special cases we will be able to find auxiliary additive subcategories $\mathscr{B} \subset \mathscr{A}$ such that $D^*(\mathscr{A}) = K^*(\mathscr{B})$.

2.3.1. *The homotopy category is triangulated.* Let $\mathscr{A}$ be an additive category. The fact that $K(\mathscr{A})$ is actually triangulated is pointed out in [BBD82, Section 1.1.2]. We will not give a proof of this, but simply point out a few important aspects.

Definition 14. *We say that a sequence of morphisms $0 \to A \to B \to C \to 0$ in $\mathscr{A}$ is split exact if it is isomorphic to $0 \to A \to A \oplus C \to C \to 0$ via the identity maps on A and C. We say that a sequence of morphsims of complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet$ is termwise split exact if $0 \to A^i \to B^i \to C^i \to 0$ is split exact for all i.*

Definition 15. *For $A^\bullet \in coCh(\mathscr{A})$, we define $A[1]^\bullet$ to be the cochain complex with $A[1]^i = A^{i+1}$ and with boundary maps the negative of the boundary maps for $A^\bullet$.*

Similarly, we write $A[n]^\bullet$ for the above process iterated n times. Note that $T : coCh(\mathscr{A}) \to coCh(\mathscr{A})$ is an additive autoequivalence.

Suppose that $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is a short termwise split short exact sequence of cochain complexes. If we choose termwise splittings $s^n : C^n \to B^n$, we find that $sd - ds$ defines a map of complexes $C^\bullet \to A[1]^\bullet$.

[8]Note that we will not need $\mathscr{A}$ to be Abelian in this defnition, but just an additive category[9]. This will be important in application later on...

[9]Of course, if $\mathscr{A}$ is not Abelian, we won't be able to make sense of such things as kernels and images, and quasi-isomorphisms and the derived category of such things can't be constructed. The homotopy category has no such issues however.

Definition 16. *We say that a sequence of morphisms*
$$A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$$
is a distinguished triangle if $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ *is termwise split exact, and if the morphism* $C^\bullet \to A[1]^\bullet$ *is of the form* $sd - ds$ *for some* s *a termwise splitting* $C^\bullet \to B^\bullet$*.*

Proposition 17. *[BBD82, Section 1.1.2] Let* $\mathscr{A}$ *be an additive category, and consider the category* $K(\mathscr{A})$ *with the additive autoequivalence* $T : A^\bullet \mapsto A[1]^\bullet$ *and* Δ *the class of triangular diagrams represented by the distinguished triangles of Definition 16. Then* $K(\mathscr{A})$ *is a triangulated category.*

Let's consider, just for fun, Axiom TR3. The main point here is that, using the "mapping cone" and "mapping cylinder" constructions, we can complete an arbitrary morphism to such a distinguished triangle in the homotopy category.

Definition 18 (Mapping cones)**.** *Let* $f^\bullet : A^\bullet \to B^\bullet$ *be a cochain map. We define the mapping cone of* $f^\bullet$*, denoted* $C(f)^\bullet$ *to be the cochain complex defined by*
$$C(f)^i = A^{i+1} \oplus B^i, \textit{ and with differential } d(a,b) = (-da, f(a) + db)$$

Definition 19 (Mapping cylinder)**.** *Let* $f^\bullet : A^\bullet \to B^\bullet$ *be a cochain map. We define the mapping cylinder of* $f^\bullet$*, denoted* $Cyl(f)^\bullet$ *to be the cochain complex defined by*
$$Cyl(f)^i = A^i \oplus A^{i+1} \oplus B^i, \textit{ and with differential } d(a', a, b) = (da' - a, -da, f(a) + db)$$

Note that we have a canonical termwise split exact sequence
$$0 \to A^\bullet \to Cyl(f) \to C(f) \to 0$$
and morphisms $B^\bullet \xrightarrow{\alpha} Cyl(f) \xrightarrow{\beta} B^\bullet$ defined by
$$\alpha(b) = (0, 0, b), \quad \beta(a', a, b) = f(a') + b.$$
One may check [GM03, Lemma III.3.3, p. 155] that $\alpha\beta$ is homotopic to 1 and $\beta\alpha = 1$. In particular, we may extend f to a distinguished triangle:
$$A^\bullet \to B^\bullet \to C(f)^\bullet \to A[1]$$
where $B^\bullet \to C(f)$ is the composition $B \xrightarrow{\alpha} Cyl(f) \to C(f)$, and where the map $C(f)^\bullet \to A[1]^\bullet$ is given by the choice of a termwise splitting in the termwise split exact sequence
$$0 \to A^\bullet \to Cyl(f)^\bullet \to C(f)^\bullet \to 0.$$

Of course there are a number of other things to check to see that this is triangulated. You can find the necessary arguments in, for example [Wei94][10].

2.3.2. *The homotopy category is not good enough.* Although it is nice that the homotopy category is triangulated, we really do need to go further to get to the derived category in general. One very nice feature that the derived category has for us, is the fact that arbitrary short exact sequences of complexes give rise to distinguished triangles in the derived category. This is not necessarily the case in the homotopy category, unless the sequence happened to be termwise split exact. Before moving on, let's record this useful feature:

Also, I should mention that I'm getting tired of typing all the bullets (like this one: $\bullet$). From here on out, I'll just drop them.

Proposition 20. *[GM03, Prop. III.3.5, p. 157] Let* $0 \to A \xrightarrow{f} B \to C \to 0$ *be a short exact sequences of complexes. Then we have a morphism of short exact sequences:*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r] & C \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, equal] & Cyl(f) \arrow[r] \arrow[u] & C(f) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

with the vertical maps all quasi-isomorphisms.

In particular, every short exact sequence is quasi-isomorphic to a distinguished triangle, and hence becomes equal to one, once we pass to the derived category.

2.3.3. *Localizing classes of morphisms.* The main advantage of the localization process to the derived category from the homotopy category as opposed to the category of cochain complexes is that the morphisms have a much nicer presentation, and composition is easier to describe. The reson for this is that the class of quasi-isomorphisms in the homotopy category is a "localizing class" of morphisms (described below). One the other hand, one can also consider the localization process as defined by sending all the acyclic complexes to 0 in the derived category. Viewed from this perspective, the acyclic complexes form what is referred to as a "thick subcategory," from which it also follows that the derived category is triangulated.

[10]Although the category $\mathscr{A}$ is assumed in this reference to be Abelian, the proofs only use that it is additive!

Although we won't dwell on it overly much, this dual perspective on quotients in categories is a common occurence in the additive context – one has two distinct perspectives on localizing a particular category $\mathcal{C}$ – one can either declare a certain class of morphisms to be invertible by formally adjoining their inverses, or one can declare a certain class of objects to be 0[11].

Definition 21. *Suppose $\mathscr{C}$ is a category, and S is a class of morphisms. We say that S is* ***localizing*** *if*

(1) *S is closed under composition and contains all identity morphisms*
(2) *For any $f : A \to B$, $s : C \to B$ in $\mathscr{C}$ with $s \in S$, there exists an object D and morphisms $g : D \to C$ and $t : D \to B$ such that $sg = ft$.*[12]
(3) *For any $f : B \to A$, $s : B \to C$ in $\mathscr{C}$ with $s \in S$, there exists an object D and morphisms $g : C \to D$ and $t : B \to D$ such that $gs = tf$.*[13]
(4) *For morphisms $f, g : A \to B$, there exists $s \in S$ such that $fs = gs$ if and only if there exists $t \in S$ such that $tf = tg$.*[14]

The nice feature, possibly somethat apparent from the definition is that if one has a localizing class of morphisms, then when constructing the category $\mathscr{C}[S^{-1}]$, one can represent an arbitrary morphism in the resulting category both in the form fs^{-1} or $t^{-1}g$, as opposed to having to resort to long compositions of such expressions. Such morphisms can be represented by diagrams refered to as "roofs:"

$$\begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A \arrow[rr, dashed] & & B \end{tikzcd}$$

[11] This is something like how when one localizes a ring, elements which are killed by things which become invertible become zero.

[12] We think of this as saying that the "left quotient" $s^{-1}f$ can also be written as a right quotient gt^{-1}.

[13] see previous footnote, *mutatis mutandis*.

[14] i.e. right and left cancellation are the same

where when $s \in S$, we think of this diagram as represnting the morphism $fs^{-1} : A \to B$. Of course in the commutative diagram,

$$\begin{tikzcd} & & C' \arrow[dl, "t"'] \arrow[dd, "g"] \\ & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A \arrow[rr, dashed] & & B \end{tikzcd}$$

one would want $fs^{-1} = ftt^{-1}s^{-1} = g(st)^{-1}$, and so we consider an equivalence relation on roofs to ensure that this happens. It is not hard to see that the equivalence relation generated by the above turns out to be exactly as follows. One declares the roofs

$$\begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

and

$$\begin{tikzcd} & C' \arrow[dl, "s'"'] \arrow[dr, "f'"] & \\ A' & & B' \end{tikzcd}$$

to be equivalent, if we can find a commutative diagram of the form

$$\begin{tikzcd} & & C'' \arrow[dl, "s''"'] \arrow[dr, "f''"] & & \\ & C' \arrow[dl, "s'"'] \arrow[drrr, "f'"' near end] & & C \arrow[dlll, "s" near end] \arrow[dr, "f"] & \\ A & & & & B \end{tikzcd}$$

Finally, composition of roofs is defined by filling in a diagram like this:

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & & R' \arrow[dl] \arrow[dr] & \\ A & & B & & C \end{tikzcd}$$

to one like this:

$$\begin{tikzcd}[column sep=small] & & R'' \arrow[dl] \arrow[dr] & & \\ & R \arrow[dl] \arrow[dr] & & R' \arrow[dl] \arrow[dr] & \\ A & & B & & C \end{tikzcd}$$

and declaring the composition to be the (equivalence class of) the roof

$$\begin{tikzcd}[column sep=small] & R'' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A & & C \end{tikzcd}$$

We record the following lemma:

Theorem 22. *[GM03, III.§4, Thm. 4, p. 160] The class of quasi-isomorphisms form a localizing class in $K^*(\mathscr{A})$ for $\mathscr{A}$ an Abelian category.*

With this in mind, we note/recall:

Definition 23. *Let $\mathscr{A}$ be an Abelian category, $K^*(\mathscr{A})$ the homotopy category of cochain complexes (of type $*$). Then $D^*(\mathscr{A})$ is the localization of $K^*(\mathscr{A})$ with respect to the class of quasi-isomorphisms.*

This is very nice in that it makes the morphisms in the derived category more transparent. We will also, however, consider the other viewpoint to this localization:

2.3.4. *Thick subcategories of triangulated categories.* Since the homotopy category is triangulated, it makes sense to ask for a localization process/perspective which respects a triangulated structures. This is captured by the concept of *Verdier Localization*, which we now describe.

Definition 24. *[Nee01, Def. 1.5.1, Rem. 2.1.2] Let $\mathscr{T}$ be a triangulated category. We say that a full additive subcategory $\mathscr{K} \subset \mathscr{T}$ is a triagulated subcategory if*

- *every object of $\mathscr{T}$ isomorphic to an object of $\mathscr{K}$ is an object of $\mathscr{K}$,*
- *$\mathscr{K}$ is preserved by the shift functor T,*
- *a triangle of $\mathscr{K}$ is distinguished if and only if it is distinguished in $\mathscr{T}$.*

Definition 25. *Let $\mathscr{T}$ be a triangulated category and $\mathscr{K}$ a triangulated subcategory. We say that $\mathscr{K}$ is **thick** if it is closed under direct summands – that is, if $A \oplus B$ is in $\mathscr{T}$*

We will see that one can define a quotient triangulated category $\mathscr{T}/\mathscr{K}$ where $\mathscr{K}$ is a thick subcategory which satisfies a universal property. To describe the universal property we first need the notion of a triangulated functor.

At first inspection, it would seem that this should be an additive functor $F : \mathscr{T} \to \mathscr{T}$ between triangulated categories such that F commutes with the shift T, and which takes distinguished triangles to distinguished triangles. On the other hand, since we are talking about categories and not sets, asking that F commutes on the nose with T is too restrictive. Instead it is natural to ask that FT and TF should be naturally isomorphic. This leads to the following definition:

Definition 26. *[Nee01, Def. 2.1.1] Let $\mathscr{T}$, $\mathscr{T}'$ be triangulated categories. A triangulated functor from $\mathscr{T}$ to $\mathscr{T}'$ is an additive functor $F : \mathscr{T} \to \mathscr{T}'$ together with a natural isomorphism $\phi : FT \to TF$ such that for any distinguished triangle*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$$

in $\mathscr{T}$, we have a distinguished triangle in $\mathscr{T}'$:

$$\begin{tikzcd} FA \arrow[r, "Ff"] & FB \arrow[r, "Fg"] & FC \arrow[rr, "\phi_A \circ Fh"] \arrow[dr, "Fh"'] & & TFA \\ & & & FTA \arrow[ur, "\phi_A"'] & \end{tikzcd}$$

Now, we can state the main result of Verdier Localization.

Theorem 27. *[Nee01, Lem 2.1.33,and some other things...] [**Weibel**] Let $\mathscr{K}$ be a thick subcategory of a triangulated category $\mathscr{T}$. Then there exists a triangulated category $\mathscr{T}/\mathscr{K}$ together with a triangulated functor $F : \mathscr{T} \to \mathscr{T}/\mathscr{K}$ such that:*

- *$FK \simeq 0$ if and only if K is an object of $\mathscr{K}$,*
- *Given any other triangulated functor $G : \mathscr{T} \to \mathscr{T}'$ with $GK \simeq 0$ for all $K \in \mathscr{K}$, there is a unique triangulated functor $H : \mathscr{T}/\mathscr{K} \to \mathscr{T}'$ such that $G = HF$.*

We can now also describe the derived category as a Verdier localization:

Proposition 28. *Let $\mathscr{A}$ be an Abelian category and $\mathscr{E} \subset K^*(\mathscr{A})$ the full subcategory of acyclic complexes. Then $\mathscr{E}$ is a thick subcategory and $K^*(\mathscr{A})/\mathscr{E} = D^*(\mathscr{A})$.*

2.4. **Comparison of derived categories.** coherent sheaves, perfect complexes, injective complexes, Fourier-Mukai transforms

3. K-theory, Chow groups and the Chern character

3.1. **K-theory.** The K group[15] of a variety X is designed to be the universal repository of invariants of vector bundles which are additive. That is, associated to every vector bundle E on X, there is a class $[E] \in K(X)$ such that whenever we have an exact sequence

$$0 \to E \to F \to G \to 0$$

of vector bundles, we have $[F] = [E] + [G]$.

Definition 29. *Let X be a variety. We define $K(X)$ to be the free Abelian group on the set of vector bundles, modulo the relation $[E] - [F] + [G]$ for every short exact sequence*

$$0 \to E \to F \to G \to 0.$$

4. The motive of a variety

5. Relations and conjectures

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