Graph Theory, Spring 2016, Practice Sheet for Exam 2

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1. Suppose that G and H are graphs and G has a perfect matching. Show that $G \times H$ also has a perfect matching.

2. Suppose that G is edge 2-colorable. If $\delta(G) \geq 2$, show that G has an even number of vertices (note that G need not be connected).

3. Suppose that G is a 3-regular connected graph with a bridge e. Show that each of the components of $G - e$ has an odd number of vertices.

4. Suppose that G is 3-regular and has a bridge. Show that $\chi'(G) = 4$ (hint: use the previous problems).

5. Suppose that G is a tree. Show that G has a Eulerian trail (a trail passing through every edge) if and only if $\Delta(G) \leq 2$.

6. Let G be the graph whose vertices are the vertices of a cube, and whose edges are the edges of the cube. Calculate the chromatic polynomial of G.

7. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$. Let D be the digraph whose vertices are $\{x, y\} \cup X \cup Y$ and with arrows a_i from x to x_i for each i, arrows $b_{i,j}$ from x_i to y_j for each i, j, and arrows c_j from y_j to y. Let N be the network with digraph D, source x, target y, and with capacity function c with $c(b_{i,j}) = 1$ for each i, j, with $c(a_i) = p_i$ and $c(c_j) = q_j$ for some non-negative integers p_i, q_j.
   i. Draw the network for $n = 3, m = 3, p_1 = 2, p_2 = 1, p_3 = 0, q_1 = 1, q_2 = 1, q_3 = 1$.
   ii. Draw a max flow for this example.
   iii. Show that a max flow correponds to a bipartite graph with vertices $x_1, x_2, x_3, y_1, y_2, y_3$ with degrees $2, 1, 0, 1, 1, 1$ respectively.
   iv. In general, suppose that $\sum p_i = \sum q_j$. Show that there is a bijection between flows with value $\sum p_i$ and bipartite graphs with bipartition $X \cup Y$, where $deg(x_i) = p_i$ and $deg(y_j) = q_j$. Show that any such flow is a max flow.