Graph Theory, Spring 2016, Homework 6

1. We showed in class that if G is a bipartite graph with bipartition $V_G = X \cup Y$ and $|X| = |Y|$, then G has a perfect matching if and only if for every subset $S \subset X$, we have $|N(S)| \geq |S|$. Show that in fact, for any bipartite graph G, G has a perfect matching if and only if for every subset $S \subset V_G$, we have $|N(S)| \geq |S|$.

2. Show that if G is a connected simple k-regular graph with $k \geq 2$ and $\kappa'(G) = k$, then G is Hamiltonian.

3. Assuming the result of the last problem show that a connected simple cubic (3-regular graph) is Hamiltonian if and only if $\kappa'(G) = 3$.

4. Define the complete bipartite graph $K_{n,m}$ to be the simple graph whose vertex set is a disjoint union of two sets X and Y with $|X| = n$ and $|Y| = m$, where every vertex of X adjacent to every vertex of Y, no two vertices in X are adjacent to each other, and no two vertices in Y are adjacent to each other. Assuming the results to the above problems, show that $K_{n,m}$ is Hamiltonian if and only if $n = m \geq 2$.

5. For simple graphs G and H, define a new simple graph $G \times H$ as follows: the vertices of $G \times H$ are $V_{G \times H} = V_G \times V_H$ and where (v, w) and (v', w') are adjacent exactly when $v = v'$ and w is adjacent to w' or $w = w'$ and v is adjacent to v'.

   (a) Draw $K_2 \times K_2$

   (b) Draw $K_3 \times K_2$

   (c) Let P_n be the path graph, the simple graph consisting of vertices $\{1, 2, \ldots, n\}$ and where i, j are adjacent if they differ by 1. Draw $P_n \times K_2$.

   (d) Show that if G has a Hamiltonian path (a spanning path), then $G \times K_2$ has a Hamiltonian cycle.