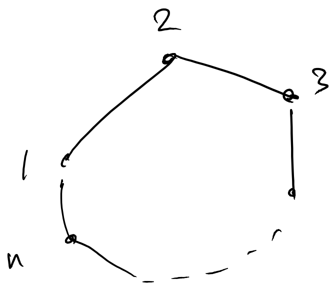
Lecture 4: walks, components, connectedness

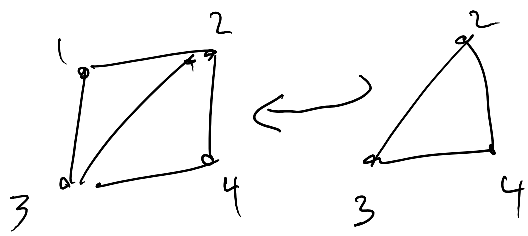
Tuesday, January 26, 2016 12:24 PM

Last time:

cyclegraph C_n



n-cycle = subgraph isomorphic to C_n

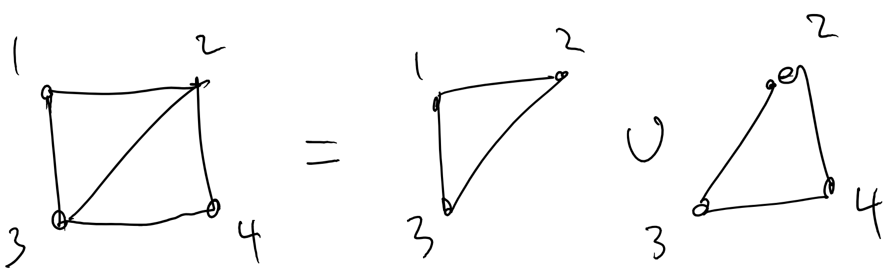


is a 3-cycle

G a graph, H_1, H_2 subgraphs of G

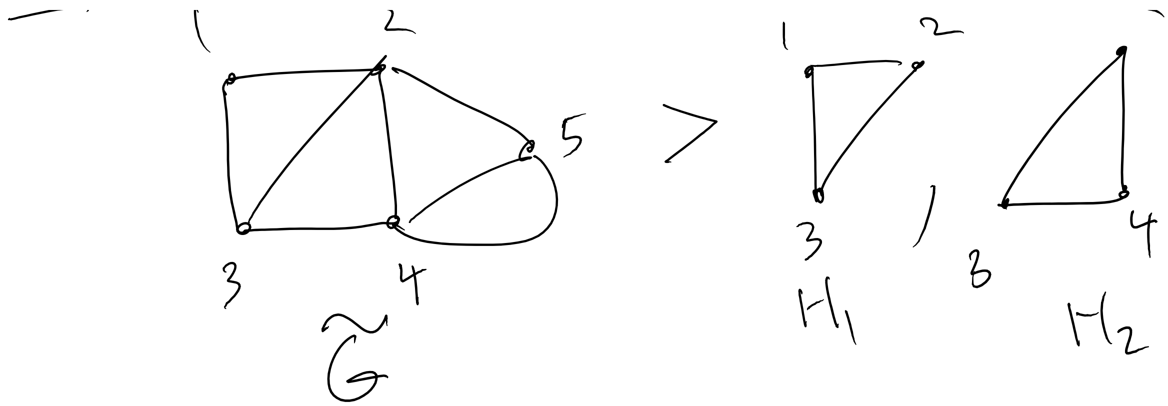
Def we say G is union of H_1 & H_2, $G = H_1 \cup H_2$

if $V_{H_1} \cup V_{H_2} = V_G$, $E_{H_1} \cup E_{H_2} = E_G$



G

ex:



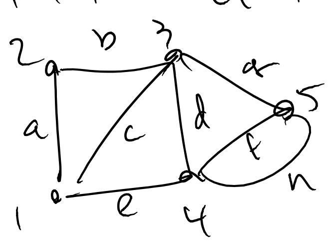
$H_1 \cup H_2 = G$

Def if $H_1, H_2 < G$, we define $H_1 \cap H_2$ to be the subgraph of G described by

$$V_{H_1 \cap H_2} = V_{H_1} \cap V_{H_2}, \quad E_{H_1 \cap H_2} = E_{H_1} \cap E_{H_2}$$

Def A walk in a graph G is a sequence of alternate vertices & edges $\omega = v_1 e_2 v_2 e_3 v_3 e_4 \cdots e_n v_n$ $\quad v_i \in V_G$, $e_i \in E_G$

such that e_i incident to v_{i-1} & v_i



1 a 2 b 3 c 1 e 4 h 5 f 4

Note: if G is simple, walk is determined by its list of vertices.
we'll write $\omega = v_1 v_2 \cdots v_n$ in this case.

if $W = v_1 e_1 \cdots e_n v_n$ is a walk, $v_1 = v$ $\quad v_n = w$
we'll call it a (v,w) walk.

given a (v,w) walk W & a (w,u) walk W', can form a new walk WW', a (v,u)-walk (concatenation)

via $\quad W = v_1 e_2 v_2 \cdots e_n v_n \qquad W' = w_1 f_2 w_2 \cdots f_m w_m$

$\qquad v_1 = v \qquad\qquad v_n = w = w_1 \qquad w_m = u$

$$WW' = v_1 e_2 v_2 \cdots e_n \underset{\substack{\| \\ v_n \\ \| \\ w_1}}{w} f_2 w_2 \cdots f_m \underset{\substack{\| \\ u}}{w_m}$$

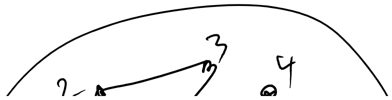
define W^{-1} as $\quad v_n e_n v_{n-1} \cdots e_2 v_1$

Lemma if we define $v \sim w$ if and only if $\exists (v,w)$-walk
then $\sim$ is an equiv. relation.

Pf D.

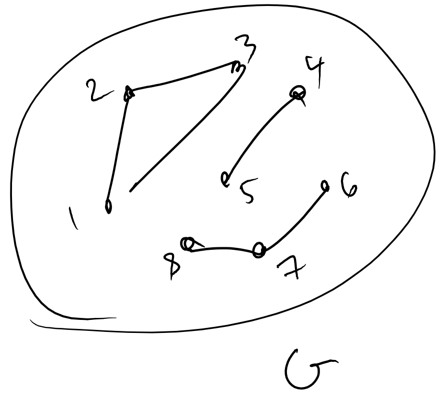
$\Longrightarrow$ can write $V_G = \bigcup_{i=1}^{r} V_i \qquad V_i$ are eq. classes.

Def $G[V_i]$ are called the components of G



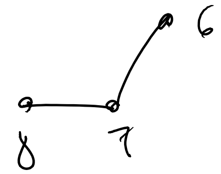
$V_1 = \{1,2,3\} \qquad V_2 = \{5,6,7,8\}$

ex:



$V_1 = \{1,2,3\}$ $\quad V_3 = \{6,7,8\}$

$V_2 = \{5,4\}$

$G[V_3] =$ 

Claim: $G = \bigcup G[V_i]$ ✓

Def: if $G = \bigcup H_i$, we say the union is disjoint if $V_{H_i} \cap V_{H_j} = \emptyset$ for $i \neq j$

Def: if $G = \bigcup H_i$, we say union is edge disjoint if $E_{H_i} \cap E_{H_j} = \emptyset$ all i,j

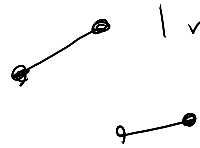
(disjoint $\Rightarrow$ edge disjoint)

Notation if $V_G = \bigcup_{i=1}^{r} V_i$ equ classes, we set $c(G) = r$

Def G is connected if $c(G) = 1$.

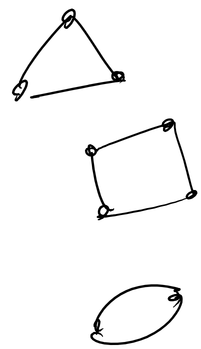
Def G is called k-regular if each vertex $v \in V_G$ has $\deg_G(v) = k$.

Def G is called … has $\deg(v) = k$.


1 regular graph.

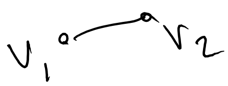
Prop if G is a simple connected 2-regular graph, then $G \cong C_n$ some n.

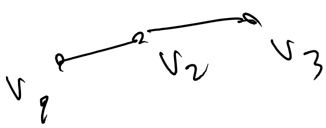
2 regular graphs

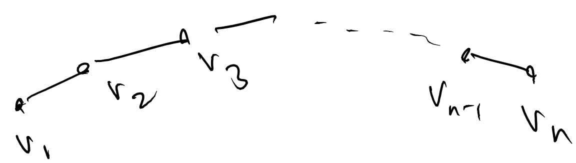

C_n

Pf: choose vertex $v = v_1$


v_1 adjacent to 2 vertices pick one: v_2


v_2 adjacent to v_1 & one other: v_3

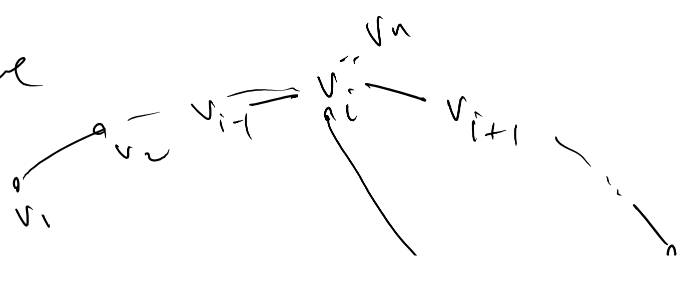

inductively get

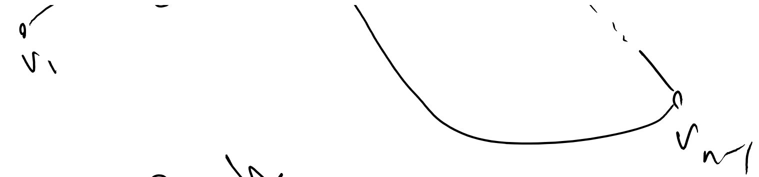
$W_n =$ 

G finite, eventually have $v_n = v_i$ some i.

know $v_i \neq v_{n-2}$, suppose $v_n \neq v_1$

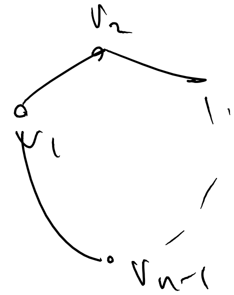
then we have





$\Rightarrow$ dg $v_i \geq 3$ $\forall i$.

$\Rightarrow$ $v_n = v_1$ $\Rightarrow$

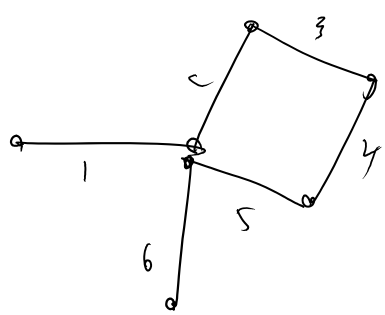


define isomorphism

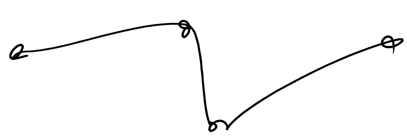
$$C_{n-1} \longrightarrow G$$

$$i \longmapsto v_i$$

<u>Def</u>: A <u>trail</u> is a walk whose edges are distinct



<u>Def</u> A <u>path</u> is a walk whose vertices are distinct.

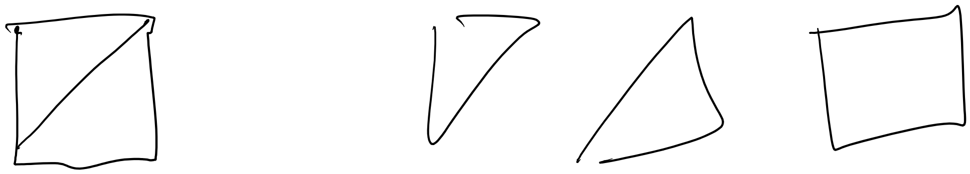


Lem In a graph G, $\exists$ a (u,w)-walk $\Longleftrightarrow$ $\exists$ (u,w)-path.

__Lem__ In a graph G, $\exists$ a (v,w)-walk $\Leftrightarrow$ $\exists$ (v,w) path.

__Def__ if G is a graph, $e \in E_G$ is a bridge if $c(G) < c(G-e)$

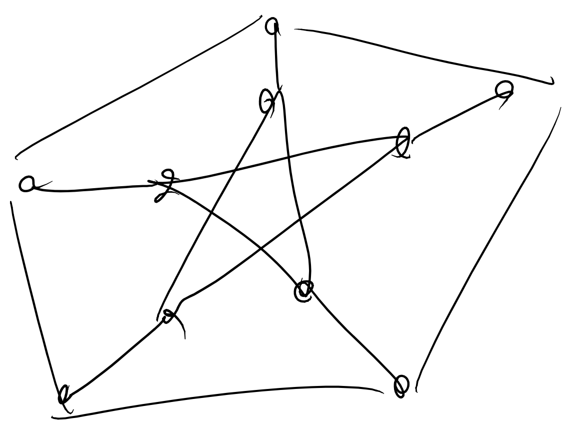
__Conjecture__ (Szerkers-Seymour)
every simple bridgeless graph has a collection of cycles such that every edge is in exactly 2 cycles.



"cycle double cover conjecture"

can reduce problem to the case of __snarks__

snark = connected simple 3-regular graph which is not 3-edge colorable.

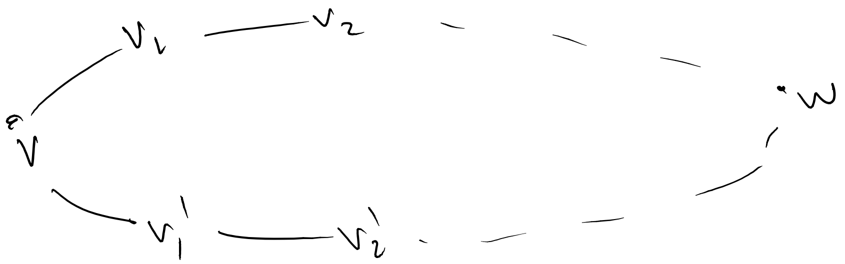


Def A forest is a simple graph with no cycles.

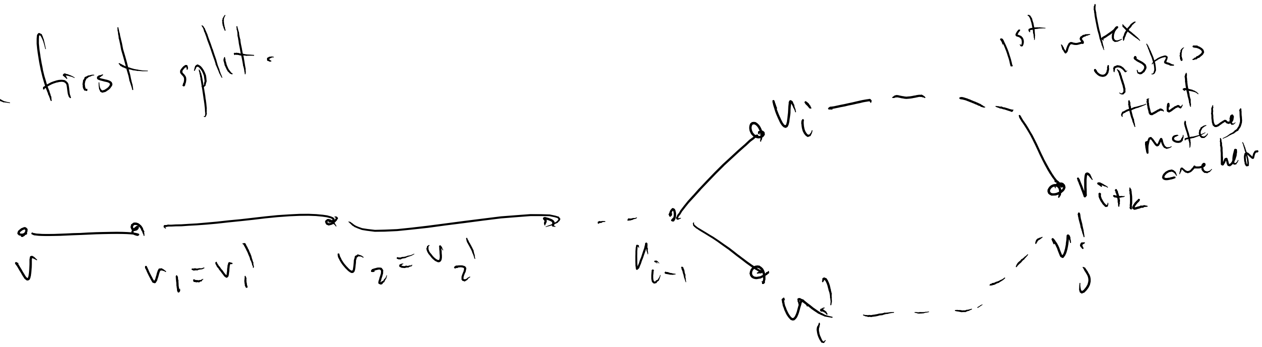
Def A tree is a connected forest.

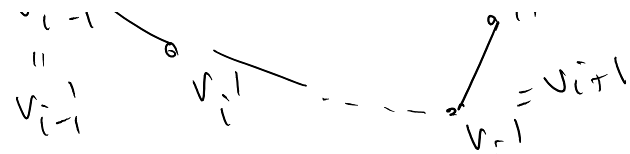
Thm A graph is a tree if it is connected and if there is a unique path between any 2 vertices.

Pf: if there are two paths v to w



same first split.





is a cycle in G.