# Graph Theory

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# Chapter 1

# Lecture 1: The language of graphs

## 1.1 Graphs

Graphs encode the idea of connections between things, for example

- networks of computers
- people and their relationships
- cities and highways
- sets and intersections
- workers and tasks

In formal mathematical terms, a graph is:

**Definition 1.1.1.** *A **graph** $G$ is an ordered triple $(V, E, \psi)$ consisting of*

- *a set $V$, whose elements are referred to as vertices,*
- *a set $E$, whose elements are referred to as edges, and*
- *an "incidence" function $\psi : E \to \mathscr{R}_2(V)$,*

*where $\mathscr{R}_2(V)$ is the set of unordered pairs of elements of $V$ (which one may also think of as two elements multisubsets of $V$ – see Definition A.1.10).*

PICTURES AND EXAMPLES HERE

**Notation 1.1.2.** *For a graph $G = (V, E, \psi)$ we write $V_G$ for $V$, $E_G$ for $E$ and $\psi_G$ for $\psi$.*

In other words, using this notational convention, if we are given graphs $G, H, K$, and have not specified letters for their sets of vertices, edges, etcetera, we may write, for example, $E_K$ for the edges of the graph $K$, $V_H$ for the vertices of $H$, and $\psi_G$ for the incidence function of $G$.

**Definition 1.1.3.** *Let $G$ be a graph, $e \in E_G$ an edge and $v \in V_G$ a vertex. We say that $e$ and $v$ are **incident** if $v \in \psi(e)$.*

**Definition 1.1.4.** *Let $G$ be a graph $v, w \in V_G$. We say that $v$ and $w$ are adjacent if there is an edge $e$ with both $v$ and $w$ incident to $e$.*

DIAGRAM

**Definition 1.1.5.** *Let $G$ be a graph. If $e \in E_G$ is an edge, we say that $e$ is a **loop**, if $e$ is incident to exactly one vertex.*

**Definition 1.1.6.** *We say that $G$ is a **simple graph** if*

- *$G$ has no loops,*
- *there is at most 1 edge incident to any pair of vertices.*

Note that the second condition is the same as requiring that the function $\psi_G$ be one-to-one.

Graphs can be drawn in many different ways:

**Definition 1.1.7.** *$G$ is called a **planar graph** if it may be drawn in the plane with no edges crossing.*

## 1.2 Real world graph problems

### 1.2.1 Scheduling

- vertices = jobs that need to be done
- edges = jobs which require conflicting resources

problem: how to decide how many "periods of work" needed to complete all jobs.

Similar problem: table arrangements at a wedding

- vertices = guests
- edges = guest that don't get along

problem: how many tables?

translation: vertex colorings, chromatic number of a graph

### 1.2.2 Tournaments

various teams need to play each other. disjoint pairs of teams can play simultaneously, but of course the same team can't play at the same time. How many rounds are needed for teams to play each other?

- vertices = teams
- edges = teams who need to play each other

problem: how many rounds?

## 1.3 The rationale behind the language

The choice of thinking of a graph as a triple $G = (V, E, \psi)$ has its advantages and disadvantages. If we were only concerned with simple graphs, we could have simplified our notation somewhat by omitting the function $\psi$, and letting $E$ itself be a subset of the set of unordered pairs of distinct elements of $V$. In the case of general graphs, however, where there can be multiple edges between two vertices, this is somewhat less convenient. We could persist with this approach by saying that $E$ be a multisubset instead of a subset, however, this is a little bit less convenient later when we wish to talk about colorings or labellings of edges.

An alternate way of defining things could be as follows: Instead of defining the function $\psi$ as the fundamental concept, one may instead define the notion of **incidence** as the fundamental concept as follows:

**Definition 1.3.1.** *A griph $G$ is an ordered triple $(V, E, \alpha)$ consisting of a set of vertices $V$, a set of edges $E$, and a set of ordered pairs $\alpha \subset V \times E$ such that*

*for every $e \in E$, there is at least one, and at most two elements $v \in V$ such that $(v, e) \in \alpha$. If $(v, e) \in \alpha$, we say that $v$ is incident to $e$. A griph is called simple if every edge is incident to exactly two vertices.*

**Exercise 1.3.2.** *Show that griphs are exactly in correspondence with graphs in such a way that the relationship of incidence lines up.*

# Chapter 2

# Lecture 2: Digraphs and degree formulas

## 2.1 Directed graphs

A variation on the notion of a graph is also very useful both theoretically and in applications:

**Definition 2.1.1.** *A* ***directed graph*** *or* ***digraph*** $D$ *is an ordered triple* $(V, A, \psi)$ *where* $V$ *is a set, referred to as the* ***vertices*** *of* $D$*, a set* $A$ *referred to as the* ***arrows*** *of* $D$*, and a pair of functions* $s, t : A \to V$*, taking arrows to elements of* $V$*.*

**Notation 2.1.2.** *For a digraph* $D = (V, A, \psi)$*, as before, we write* $V_D$ *for* $V$*,* $A_D$ *for* $A$*,* $s_D$ *for* $s$*, and* $t_D$ *for* $t$*.*

**Notation 2.1.3.** *For a digraph* $D$*, and an arrow* $a \in A_D$*, we call* $s(a)$ *the* ***source*** *of* $a$ *and* $t(a)$ *the* ***target*** *of* $a$*.*

EXAMPLES:

- one way street maps
- irreversible processes
- dependencies: e.g. scheduling with dependencies

**Definition 2.1.4.** *Let* $D$ *be a digraph. For a vertex* $v$*, we define the* ***outdegree*** *of* $v$*, denoted* $\operatorname{outdeg} v$*, to be the number of edges whose source is* $v$*, and the* ***indegree*** *of* $v$*, denoted* $\operatorname{indeg} v$*, to be the number of edges whose target is* $v$*. Formall, we have*

$$\operatorname{outdeg} v = \#\{a \in A_D \mid s(a) = v\}, \quad \operatorname{indeg} v = \#\{a \in A_D \mid t(a) = v\}.$$

*If it is necessary to specify the digraph, we may also write* $\operatorname{outdeg}_D(v)$ *or* $\operatorname{indeg}_D(v)$*.*

**Proposition 2.1.5** (The degree formula for digraphs)**.** *Suppose that $D$ is a digraph. Then*

$$\sum_{v \in V_D} \operatorname{outdeg}(v) = \sum_{v \in V_D} \operatorname{indeg}(v) = \# A_D.$$

*Proof.* Informally, this is clear for the following reason: every arrow in $A_D$ has its source at exactly one vertex, and so contributes exactly 1 to the first sum, and every arrow has its target at exactly one vertex and similarly contributes exactly once to the second sum.

Let us, however, for the sake of practice, give a more formal argument:

$$\sum_{v \in V_D} \operatorname{outdeg}(v) = \sum_{v \in V_D} \sum_{a \in A_D, s(a)=v} 1 = \sum_{(v,a) \in V_D \times A_D, s(a)=v} 1$$

But now, let us notice that the pairs $(v, a)$ with $s(a) = v$ are in bijection with simply the set $A_D$, since by the description, $v$ is determined by $a$. Therefore, we may rewrite this as:

$$\sum_{(v,a) \in V_D \times A_D, s(a)=v} 1 = \sum_{a \in A_D} 1 = \# A_D.$$

The rest of the proof follows in an analogous way.

□

## 2.2 From graphs to digraphs

Given a graph $G$, we may construct a digraph $dig(G)$ by definining

- $V_{dig(G)} = V_G$,
- $A_{dig(G)} = \{(v, e) \in V \times E | v \in \psi(e)\}$,
- $s_{dig(G)}(v, e) = v$, $t_{dig(G)}(v, e) = w$, where $\psi(e) = vw$.

**Definition 2.2.1.** *Suppose that $G$ is a graph. We define the* ***degree*** *of a vertex $v \in V_G$, denoted* $\deg v$ *to be the number of edges incident to $v$. If it is necessary to specify the graph, we may also write* $\deg_G v$*.*

**Proposition 2.2.2** (The degree formula for graphs)**.** *Suppose that $G$ is any graph. Then we have*

$$\sum_{v \in V_G} \deg v = 2\# E.$$

*Proof.* Intuitively, one way to see this is that if we chop each edge in the middle, making two "half edges" for every edge, then each half edge is incident to exactly one vertex, and contritutes exactly once to the degree count on the left.

More formally, if we let $dig(G)$ be the associated digraph to $G$, then we note that for every edge of $G$, there are two arrows of $dig(G)$. Also, for every edge $e$ incident to $v$, there is exactly one arrow, which we call $(v, e)$ which starts at $e$. That is, the map

$$\{e \in E_G | e \text{ is incident to } v\} \to \{a \in A_{dig(G)} | s(a) = v\}$$
$$e \mapsto (v, e)$$

is a bijection (its inverse being given by $(v, e) \mapsto e$). In particular, this says that $\operatorname{outdeg}_{dig(G)} v = \deg_G v$.

By the degree formula for digraphs, we therefore have

$$\sum_{v \in V_G} \deg_G v = \sum_{v \in A_{dig(G)}} \operatorname{outdeg}_{dig(G)} v = \# A_{dig(G)} = 2\# E_G$$

as desired. $\square$

A surprising conclusion here is that the sum of the degrees of the vertices of a graph must be even!

# Chapter 3

# Lecture 3: Subgraphs, isomorphisms

## 3.1 Isomorphisms

**Definition 3.1.1.** *Given two graphs $G, H$, an* ***isomorphism*** *$f$ from $G$ to $H$, written $f : G \to H$ is a pair of maps $f = (f_V, f_E)$, where $f_V : V_G \to V_H$ is a function from the vertices of $G$ to the vertices of $H$ and $f_E : E_G \to E_H$ is a function from the edges of $G$ to the edges of $H$ such that both $f_V$ and $f_E$ are bijective and such that for $v \in V_G$, $e \in E_G$, we have that $v$ and $e$ are incident if and only if $f_V(v)$ is incident to $f_E(e)$.*

**Notation 3.1.2.** *If $G$ and $H$ are graphs, we write $G \cong H$ and say that $G$ and $H$ are* ***isomorphic*** *if there exists an isomorphism $f : G \to H$.*

In other words, thinking of the sets $V_G, E_G$ as the labels for the vertices and edges of $G$ and thinking of $V_H, E_H$ as the labels for the vertices and edges of $H$, we may think of $f_V$ and $f_E$ as re-assigning the labels of the vertices and edges of a graph. The final property says that the relation of incidence is preserved. Note that the incidence relation encodes the function $\psi$, as $v$ and $e$ are incident if and only if $v \in \psi(e)$ (and since the unordered pair/two element multiset $\psi(e)$ is determined precisely by which elements it contains).

This is very useful, as our main concern is structural information about graphs, as opposed to the specific names which we have assigned to their edges and vertices.

### 3.1.1 Simplifications for simple graphs

It is perhaps useful to note that this definition may be made a bit simpler in the case of simple graphs (not suprising, I'm sure). To be precise:

**Exercise 3.1.3.** *Suppose that $G$ and $H$ are simple graphs. Suppose we have a bijective function $g : V_G \to V_H$. Then there exists a unique graph isomorphism $f = (f_V, f_E) : G \to E$ with $f_V = g$ if and only if for every two vertices $v, w \in V_G$, we have that $v$ is adjacent to $w$ if and only if $g(v)$ is adjacent to $g(w)$.*

Expanding on this idea a bit, we see that in a simple graph, the graph is determined up to isomorphism by the relationship of adjacency – that is, which vertices are joined by an edge. That is to say, although these two graphs shown below are different as graphs:

TWO GRAPHS WITH THE SAME LABELS ON VERTICES BUT DIFFERENT LABELS ON EDGES,

they are still isomorphic in a canonical way.

**Notation 3.1.4.** *Along the same lines, for a simple graph $G = (V, E, \psi)$, if $e$ is an edge with $\psi(e) = vw$, we will abuse language and refer to $vw$ as $e$. In other words, the statement that for a pair of vertices $v, w \in V$, $vw \in E$ means that there is some edge $e$ with $\psi(e) = vw$, or equivalently $v$ and $w$ are adjacent.*

## 3.2 Subgraphs

**Definition 3.2.1.** *Let $G$ be a graph. We say that a graph $H$ is a* ***subgraph*** *of a graph $G$ if $V_H \subset V_G$, $E_H \subset E_G$ and $\psi_H = \psi_G|_{E_H}$. If $H$ is a subgraph of $G$, we write $H \subset G$.*

**Definition 3.2.2.** *Let $G$ be a graph, and $W \subset V_G$ a subset of its vertices. We define a new graph $G[W]$, called* ***the subgraph of*** *$G$* ***induced by*** *$W$, to be the graph whose vertex set is $W$ and whose edge set $E_{G[W]}$ consists of all the edges of $G$ which are only incident to vertices in $W$, together with the same incidence relations. Formally, we set*

$$V_{G[W]} = W, \; E_{G[W]} = \{e \in E_G \mid \psi(e) \subset W\}, \; \psi_{G[W]} = \psi|_{E_{G[W]}},$$

*where $\psi|_{E_{G[W]}}$ denotes the restriction of the function $\psi$ to the edges of $G[W]$.*

Given a graph $G$, we will often want to understand its structure by looking at which subgraphs it has, and their properties. For example, consider the following famous statement:

> In every group of 6 people, there is either a group of 3 people all of whom know each other, or a group of 3 people, none of whom know each other.

One convenient way of conceptualizing this question within the framework of graph theory is as follows: consider the two special graphs below

TRIANGLE DISCRETE GRAPH WITH 3 VERTICES

We may now formally state the previous result as follows:

**Exercise 3.2.3.** *Show that every simple graph with 6 vertices must contain an induced subgraph isomorphic to one of the above graphs*[1]*.*

In generalizing this result, which we will look into later in Chapter **??**, it is natural to want to consider groups of $n$ vertices in a graph, all of which are connected. The correponding subgraphs of interest are called the complete graphs:

[1] hint: as a first step, considering the friends of one particular person $A$, partitions the other 5 people into two groups (friends of $A$ and not friends of $A$), it follows that one of these two groups must have at least 3 people

**Definition 3.2.4.** *The* ***complete graph*** *on $n$ vertices, denoted $K_n$ is the simple graph with vertices consisting of the set $\{1, \dots, n\}$, and where every two vertices are adjacent.*

**Definition 3.2.5.** *Let $G$ be a simple graph. An $n$-**clique** in $G$ is a collection of vertices $v_1, \dots, v_n \in V_G$ such that the induced subgraph $G[\{v_1, \dots, v_n\}]$ is isomorphic to $K_n$.*

It is very natural to ask, for a given graph, about the existence or non-existence of cliques of a given size.

**Exercise 3.2.6.** *Find the smallest number $n$ such that every simple graph with $n$ edges and 6 vertices has a 3-clique.*

Another way to generalize the graph $K_3$ is in the notion of a cycle graph:

**Definition 3.2.7.** *The* ***cycle graph*** *on $n$ vertices ($n \geq 3$)[2], denoted $C_n$ is the simple graph with vertices consisting of the set $\{1, \dots, n\}$, and where vertices $i$ and $j$ are adjacent if and only if $|i - j|$ is 1 or $n - 1$.*

In other words, there are edges between vertices which are 1 unit apart, and one additional edge connecting 1 and $n$.

**Definition 3.2.8.** *A cycle in a (not necessarily simple) graph $G$ is a subgraph $C \subset G$ such that $C \cong C_n$ for some $n \geq 3$.*

**Exercise 3.2.9.** *Give an example of a simple graph with 4 vertices and exactly 3 cycles, and a graph with 3 vertices and exactly 2 cycles.*

As we will see, cycles and cliques are interesting for a variety of reasons, both practically and theoretically. Intuitively, one should view the existence of cycles and cliques as helping to describe how highly connected a graph is, somehow encapsulating "redundancies of connections." We will explore these ideas more in Chapters **??**.

[2] we let bigons be bigons, as they say

# Appendix A

# Foundational notions

## A.1 Sets and multisets

The substructure of the majority of modern mathematics is set theory. It therefore would behoove us to take a very slight digression into some useful concepts and notations.

**Definition A.1.1.** *A* ***set*** *is a collection of elements, is defined exactly by its elements. Two sets are equal if they contain the same elements.*

**Notation A.1.2.** *We will denote a set using "set notation." This consists of listing the elements of a set enclosed in braces, and separated by commas. Note that the order in which elements are written doesn't change the set. For example* $\{a, b, c\} = \{b, c, a\}$.

**Definition A.1.3.** *For a set* $S$*, its* ***power set*** $\mathscr{P}(S)$*, is the set whose elements are the subsets of* $S$

**Definition A.1.4.** *For a set* $S$*, we let* $\mathscr{P}_k(S)$ *denote its subsets with exactly* $k$ *elements.*

**Definition A.1.5.** *For sets* $S, T$ *we let* $S \times T$ *denote the set whose elements are ordered pairs* $(s, t)$ *where* $s \in S$ *and* $t \in T$.

**Definition A.1.6.** *A* ***multiset*** $\mathscr{S}$ *is a pair* $(S, m)$*, where* $S$ *is a set, and* $m$ *is a function* $m : S \to \mathbb{Z}_{>0}$ *from* $S$ *to the positive integers. For* $s \in S$*, we refer to* $m(s)$ *as the multiplicity of* $s$ *in* $\mathscr{S}$*. We write* $s \in \mathscr{S}$*. We call* $S$ *the underlying set of* $\mathscr{S}$.

*For a multiset* $\mathscr{S} = (S, m)$*, we will use the notation* $m_{\mathscr{S}}$ *to refer to* $m$.

**Notation A.1.7.** *We will write a multiset* $S$ *by writing a list of its elements, with repetition, in a string, each elements arising in the string as many times as it its multiplicity. The order in which the elements are written doesn't matter. For example,* $abbbcc = abcbcb$.

**Definition A.1.8.** *Let* $\mathscr{S}$ *and* $\mathscr{T}$ *be multisets. We say that* $\mathscr{S} \subset \mathscr{T}$ *if the underlying set of* $\mathscr{S}$ *is contained in the underlying set of* $\mathscr{T}$.

If $S$ is a set, we will also identify $S$ with the multiset $(S, m)$ defined by $m(s) = 1$ for each $s \in S$ (that is, $S$ contains each of its elements exactly 1 time).

These mutisets are occasionally useful in combinatorics to think of the idea of sampling with replacement/repetition.

**Definition A.1.9.** *If $\mathscr{S} = (S, m)$ is a multiset, we define the* ***cardinality*** *of $\mathscr{S}$, denoted $\#\mathscr{S}$, to be*

$$\sum_{s \in S} m(s).$$

In particular, considering a set $T$ as a multiset as described above, we have $\#T$ is exactly the number of elements of $T$.

**Definition A.1.10.** *Let $S$ be a set. We write $\mathscr{R}(S)$ to denote the set of all mulisubsets of $S$, and $\mathscr{R}_k(S)$ the set of all multisubsets of $S$ with cardinality exactly $S$.*

## A.2 Relations

**Definition A.2.1.** *Recall that if $X, Y$ are sets, a* ***relation*** *(from $X$ to $Y$) is a subset $R \subset X \times Y$ of the product of $X$ and $Y$. If an ordered pair $(x, y) \in R$ we say that $x$ is related to $y$ and write $xRy$.*

**Notation A.2.2.** *Frequently we will talk about a relation on a set $X$. This this is shorthand for a relation $R \subset X \times X$ from $X$ to iteself.*

Two important types of relations are **functions** and **equivalence relations**, which we now describe.

### A.2.1 Functions

**Definition A.2.3.** *A function from $X$ to $Y$ is a relation $f \subset X \times Y$ such that for every $x \in X$ there is exactly one $y \in Y$ such that $xfy$.*

**Notation A.2.4.** *If $f$ is a function from $x$ to $y$, we write $f(x)$ to denote the unique $y \in Y$ such that $xfy$.*

### A.2.2 Equivalence Relations

**Definition A.2.5.** *Let $R$ be a relation on $X$. We say that $R$ is an* ***equivalence relation****, if the following properties hold*

1. *(reflexivity) for every $x \in X$, we have $xRx$,*
2. *(symmetry) for every $x, y \in X$, we have $xRy$ if and only if $yRx$,*
3. *(transitivity) for every $x, y, z \in X$, whenever $xRy$ and $yRz$ we must also have $xRz$.*

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