Graph Theory

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Chapter 1

Lecture 1: Basic Notions

1.1 Preliminaries and notation

The substructure of the majority of modern mathematics is set theory. It therefore would behoove us to take a very slight digression into some useful concepts and notations.

Definition 1.1.1. *A **set** is a collection of elements, is defined exactly by its elements. Two sets are equal if they contain the same elements.*

Notation 1.1.2. *We will denote a set using "set notation." This consists of listing the elements of a set enclosed in braces, and separated by commas. Note that the order in which elements are written doesen't change the set. For example* $\{a, b, c\} = \{b, c, a\}$.

Definition 1.1.3. *For a set* S*, its **power set*** $\mathscr{P}(S)$*, is the set whose elements are the subsets of* S

Definition 1.1.4. *For a set* S*, we let* $\mathscr{P}_k(S)$ *denote its subsets with exactly* k *elements.*

Definition 1.1.5. *For sets* S, T *we let* $S \times T$ *denote the set whose elements are ordered pairs* (s, t) *where* $s \in S$ *and* $t \in T$.

Definition 1.1.6. *A **multiset*** $\mathscr{S}$ *is a pair* (S, m)*, where* S *is a set, and* m *is a function* $m : S \to \mathbb{Z}_{>0}$ *from* S *to the positive integers. For* $s \in S$*, we refer to* $m(s)$ *as the multiplicity of* s *in* $\mathscr{S}$*. We write* $s \in \mathscr{S}$*. We call* S *the underlying set of* $\mathscr{S}$.

For a multiset $\mathscr{S} = (S, m)$*, we will use the notation* $m_{\mathscr{S}}$ *to refer to* m.

Notation 1.1.7. *We will write a multiset* S *by writing a list of its elements, with repetition, in a string, each elements arising in the string as many times as it its multiplicity. The order in which the elements are written doesen't matter. For example,* $abbbcc = abcbcb$.

Definition 1.1.8. *Let* $\mathscr{S}$ *and* $\mathscr{T}$ *be multisets. We say that* $\mathscr{S} \subset \mathscr{T}$ *if the underlying set of* $\mathscr{S}$ *is contained in the underlying set of* $\mathscr{T}$.

If S is a set, we will also identify S with the multiset (S, m) defined by $m(s) = 1$ for each $s \in S$ (that is, S contains each of its elements exactly 1 time).

These mutisets are occasionally useful in combinatorics to think of the idea of sampling with replacement/repetition.

Definition 1.1.9. *If $\mathscr{S} = (S, m)$ is a multiset, we define the **cardinality** of $\mathscr{S}$, denoted $\#\mathscr{S}$, to be*

$$\sum_{s \in S} m(s).$$

In particular, considering a set T as a multiset as described above, we have $\#T$ is exactly the number of elements of T.

Definition 1.1.10. *Let S be a set. We write $\mathscr{R}(S)$ to denote the set of all mulisubsets of S, and $\mathscr{R}_k(S)$ the set of all multisubsets of S with cardinality exactly S.*

1.2 Graphs

Graphs encode the idea of connections between things, for example

- networks of computers
- people and their relationships
- cities and highways
- sets and intersections
- workers and tasks

In formal mathematical terms, a graph is:

Definition 1.2.1. *A **graph** G is an ordered triple (V, E, ψ) consisting of*

- *a set V, whose elements are referred to as vertices,*
- *a set E, whose elements are referred to as edges, and*
- *an "incidence" function $\psi : E \to \mathscr{R}_2(V)$,*

where $\mathscr{R}_2(V)$ is the set of unordered pairs of elements of V (which one may also think of as two elements multisubsets of V – see Definition 1.1.10).

PICTURES AND EXAMPLES HERE

Notation 1.2.2. *For a graph $G = (V, E, \psi)$ we write V_G for V, E_G for E and ψ_G for ψ.*

In other words, using this notational convention, if we are given graphs G, H, K, and have not specified letters for their sets of vertices, edges, etcetera, we may write, for example, E_K for the edges of the graph K, V_H for the vertices of H, and ψ_G for the incidence function of G.

Definition 1.2.3. *Let G be a graph, $e \in E_G$ an edge and $v \in V_G$ a vertex. We say that e and v are **incident** if $v \in \psi(e)$.*

DIAGRAM

Definition 1.2.4. *Let G be a graph. If $e \in E_G$ is an edge, we say that e is a **loop**, if e is incident to exactly one vertex.*

Definition 1.2.5. *We say that G is a **simple graph** if*

- *G has no loops,*
- *there is at most 1 edge incident to any pair of vertices.*

Note that the second condition is the same as requiring that the function ψ_G be one-to-one.

Graphs can be drawn in many different ways:

Definition 1.2.6. *G is called a **planar graph** if it may be drawn in the plane with no edges crossing.*

1.3 Real world graph problems

1.3.1 Scheduling

- vertices = jobs that need to be done
- edges = jobs which require conflicting resources

problem: how to decide how many "periods of work" needed to complete all jobs.

Similar problem: table arrangements at a wedding

- vertices = guests
- edges = guest that don't get along

problem: how many tables?

translation: vertex colorings, chromatic number of a graph

1.3.2 Tournaments

various teams need to play each other. disjoint pairs of teams can play simultaneously, but of course the same team can't play at the same time. How many rounds are needed for teams to play each other?

- vertices = teams
- edges = teams who need to play each other

problem: how many rounds?

1.4 The rationale behind the language

The choice of thinking of a graph as a triple $G = (V, E, \psi)$ has its advantages and disadvantages. If we were only concerned with simple graphs, we could have simplified our notation somewhat by omitting the function ψ, and letting E itself be a subset of the set of unordered pairs of distinct elements of V. In the case of general graphs, however, where there can be multiple edges between two vertices, this is somewhat less convenient. We could persist with this approach by saying that E be a multisubset instead of a subset, however, this is a little bit less convenient later when we wish to talk about colorings or labellings of edges.

An alternate way of defining things could be as follows: Instead of defining the function ψ as the fundamental concept, one may instead define the notion of **adjacency** as the fundamental concept as follows:

Definition 1.4.1. *A griph G is an ordered triple (V, E, α) consisting of a set of vertices V, a set of edges E, and a set of ordered pairs $\alpha \subset V \times E$ such that*

for every $e \in E$, there is at least one, and at most two elements $v \in V$ such that $(v, e) \in \alpha$. If $(v, e) \in \alpha$, we say that v is incident to e. A griph is called simple if every edge is adjacent to exactly two vertices.

Excercise 1.4.2. *Show that griphs are exactly in correspondence with graphs in such a way that the relationship of incidence lines up.*

Chapter 2

Lecture 2: Digraphs and degree formulas

2.1 Directed graphs

A variation on the notion of a graph is also very useful both theoretically and in applications:

Definition 2.1.1. *A* ***directed graph*** *or* ***digraph*** *D is an ordered triple (V, A, ψ) where V is a set, referred to as the* ***vertices*** *of D, a set A referred to as the* ***arrows*** *of D, and a pair of functions $s, t : A \to V$, taking arrows to elements of V.*

Notation 2.1.2. *For a digraph $D = (V, A, \psi)$, as before, we write V_D for V, A_D for A, s_D for s, and t_D for t.*

Notation 2.1.3. *For a digraph D, and an arrow $a \in A_D$, we call $s(a)$ the* ***source*** *of a and $t(a)$ the* ***target*** *of a.*

EXAMPLES:

- one way street maps
- irreversible processes
- dependencies: e.g. scheduling with dependencies

Definition 2.1.4. *Let D be a digraph. For a vertex v, we define the* ***outdegree*** *of v, denoted* $\operatorname{outdeg} v$*, to be the number of edges whose source is v, and the* ***indegree*** *of v, denoted* $\operatorname{indeg} v$*, to be the number of edges whose target is v. Formall, we have*

$$\operatorname{outdeg} v = \#\{a \in A_D \mid s(a) = v\}, \quad \operatorname{indeg} v = \#\{a \in A_D \mid t(a) = v\}.$$

If it is necessary to specify the digraph, we may also write $\operatorname{outdeg}_D(v)$ *or* $\operatorname{indeg}_D(v)$.

Proposition 2.1.5 (The degree formula for digraphs)**.** *Suppose that D is a digraph. Then*

$$\sum_{v \in V_D} \operatorname{outdeg}(v) = \sum_{v \in V_D} \operatorname{indeg}(v) = \# A_D.$$

Proof. Informally, this is clear for the following reason: every arrow in A_D has its source at exactly one vertex, and so contributes exactly 1 to the first sum, and every arrow has its target at exactly one vertex and similarly contributes exactly once to the second sum.

Let us, however, for the sake of practice, give a more formal argument:

$$\sum_{v \in V_D} \operatorname{outdeg}(v) = \sum_{v \in V_D} \sum_{a \in A_D, s(a)=v} 1 = \sum_{(v,a) \in V_D \times A_D, s(a)=v} 1$$

But now, let us notice that the pairs (v, a) with $s(a) = v$ are in bijection with simply the set A_D, since by the description, v is determined by a. Therefore, we may rewrite this as:

$$\sum_{(v,a) \in V_D \times A_D, s(a)=v} 1 = \sum_{a \in A_D} 1 = \# A_D.$$

The rest of the proof follows in an analogous way.

□

2.2 From graphs to digraphs

Given a graph G, we may construct a digraph $dig(G)$ by definining

- $V_{dig(G)} = V_G$,
- $A_{dig(G)} = \{(v, e) \in V \times E | v \in \psi(e)\}$,
- $s_{dig(G)}(v, e) = v$, $t_{dig(G)}(v, e) = w$, where $\psi(e) = vw$.

Definition 2.2.1. *Suppose that G is a graph. We define the* ***degree*** *of a vertex $v \in V_G$, denoted* $\deg v$ *to be the number of edges incident to v. If it is necessary to specify the graph, we may also write* $\deg_G v$*.*

Proposition 2.2.2 (The degree formula for graphs)**.** *Suppose that G is any graph. Then we have*

$$\sum_{v \in V_G} \deg v = 2\# E.$$

Proof. Intuitively, one way to see this is that if we chop each edge in the middle, making two "half edges" for every edge, then each half edge is incident to exactly one vertex, and contritutes exactly once to the degree count on the left.

More formally, if we let $dig(G)$ be the associated digraph to G, then we note that for every edge of G, there are two arrows of $dig(G)$. Also, for every edge e incident to v, there is exactly one arrow, which we call (v, e) which starts at e. That is, the map

$$\begin{gathered}\{e \in E_G | e \text{ is incident to } v\} \to \{a \in A_{dig(G)} | s(a) = v\} \\ e \mapsto (v, e)\end{gathered}$$

is a bijection (its inverse being given by $(v, e) \mapsto e$). In particular, this says that $\operatorname{outdeg}_{dig(G)} v = \deg_G v$.

By the degree formula for digraphs, we therefore have

$$\sum_{v \in V_G} \deg_G v = \sum_{v \in A_{dig(G)}} \operatorname{outdeg}_{dig(G)} v = \#A_{dig(G)} = 2\#E_G$$

as desired. □

A surprising conclusion here is that the sum of the degrees of the vertices of a graph must be even!

Chapter 3

Lecture 2.5: Specific Graphs, Subgraphs, Isomorphisms

Definition 3.0.1. *The* ***complete graph*** *on n vertices, denoted K_n is the simple graph with vertices consisting of the set $\{1, \ldots, n\}$, edges consisting of the set of unordered pairs*

Definition 3.0.2. *Let G be a graph. We say that a graph H is a* ***subgraph*** *of a graph G if $V_H \subset V_G$, $E_H \subset E_G$ and $\psi_H = \psi_G|_{E_H}$.*

Given a graph G, we will often want to understand its structure by looking at which subgraphs it has, and their properties.

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